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# On controlled mode discernibility for nonlinear hybrid systems with unknown exogenous input ☆

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ABSTRACT

This paper considers the mode discernibility and controlled mode discernibility for nonlinear hybrid systems with unknown switchings and unknown exogenous input, motivated by its potential applications to fault detection and isolation, cyber-security, and robust control. We formally define the mode discernibility and controlled mode discernibility, and then characterize these properties from a geometric perspective. Specifically, we review and complement the concept of controlled invariance for nonlinear systems, based on which a new concept called strongly controlled invariance is proposed as a tool to characterize the controlled mode discernibility. Besides the general characterization of controlled mode discernibility, we discuss some commonly considered special nonlinearities, and demonstrate that the discerning control design problem can be formulated as a quadratic game for some cases.



## 1. Introduction

A hybrid system is a dynamical system with interacting continuous and discrete state dynamics (Goebel, Sanfelice, & Teel, 2009; Van Der Schaft & Schumacher, 2000), which has been widely considered for modeling complex industrial plants and cyber–physical systems to investigate system resilience (Sayed-Mouchaweh, 2018). For fault detection and isolation (FDI) of plants, the discrete states of the hybrid system can represent fault modes, and the continuous dynamics associated with each discrete state can describe the system behavior under the corresponding fault mode (Tanwani, Domínguez-García, & Liberzon, 2010; Zhao, Koutsoukos, Haussecker, Reich, & Cheung, 2005). Meanwhile, the logical behavior and physical behavior of a cyber–physical system can be mapped to the discrete state dynamics and continuous state dynamics, respectively, and then the abnormal logical behavior due to cyber-attacks can be modeled as unexpected discrete state switchings (Sun & Hwang, 2019; Zhu & Basar, 2015). Besides, the discrete states of the hybrid system can also describe different external/internal operational environments of a plant, so robust and adaptive control techniques can be used to assure the system performance (e.g., $L_2$-gain) under the nonpredetermined time-varying environment (Anderson et al., 2000; Hespanha et al., 2001; Hu, Shen, & Putta, 2016; Xiang & Xiao, 2012; Zhang & Shi, 2009). In the context of either FDI, cyber-security, or robust control, not only the discrete state is, in general, assumed to be unavailable directly, but also the continuous dynamics often involves unknown input, which can represent actuator faults (Yang, Cocquempot, & Jiang, 2009), deceptive cyber-attacks (Yong, Zhu, & Frazzoli, 2018), or disturbance (Yang, Li, Jiang, & Cocquempot, 2018). Although many resilient control schemes (e.g., fault-tolerant control (Allerhand & Shaked, 2014), attack-mitigation methods (Feng & Tesi, 2017; Hu, Shen, & Lee, 2017; Lu & Yang, 2017), robust control (Xiang, Tran, & Johnson, 2017, 2018)) are designed to be effective without using discrete state information or exogenous input information, identifying the unknown discrete state of the hybrid system is still crucial for situational awareness (Diene, Moreira, Silva, Alvarez, & Nascimento, 2017; Vento, Travé-Massuyès, Puig, & Sarrate, 2014).

*Mode discernibility* (or *mode distinguishability* or *mode observability*) of a hybrid system, in general, stands for the property of whether or not the discrete state of the hybrid system can be recovered from available information (Küsters & Trenn, 2018). We classify the existing works on mode discernibility into three categories based on available information: recovering discrete state information from (i) the output of the discrete state dynamics (Lafortune, Lin, & Hadjicostis, 2018; Ramadge, 1986; Sampath, Sengupta, Lafortune, Sinnamohideen, & Teneketzis, 1995), (ii) the output of the continuous state dynamics (Alessandri, Baglietto, & Battistelli, 2005; Battistelli, 2013; De Santis, 2011; De Santis, Di Benedetto, & Pola, 2009; Halimi, Millérioux, & Daafouz,

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2014; Motchon, Pekpe, & Cassar, 2017; Motchon, Pekpe, Cassar, & De Bièvre, 2015; Ramdani, Travé-Massuyès, & Jauberthie, 2018; Rosa & Silvestre, 2011), and (iii) the output of the continuous state dynamics without knowing exogenous inputs. Note that the third class is of interest from the perspectives of FDI, cyber-security, and robust control, but currently, not many works have been done: the mode distinguishability for continuous-time switched linear systems with partially unknown input has been discussed by Gómez-Gutiérrez, Ramírez-Treviño, Ruiz-León, and Di Gennaro (2011); mode distinguishability for cyber-security issues that address sparse attack inputs is considered by Fiore, De Santis, and Di Benedetto (2017); the mode discernibility for a switched structured linear system with unknown input has been studied by Boukhobza and Hamelin (2011) via a graph-theoretic approach; the invertibility of continuous-time switched systems with unknown input, which describes whether the discrete state and the unknown input can be simultaneously estimated using the output and initial state information, has been studied by Kaba and Camlibel (2010), Tanwani and Liberzon (2010), Vu and Liberzon (2008). It should be remarked that the requirement of mode discernibility could be restrictive, which induces the concept of *controlled mode discernibility* (Baglietto, Battistelli, & Scardovi, 2007; Lou & Yang, 2011; Motchon & Pekpe, 2018; Motchon, Pekpe, & Cassar, 2018; Ramdani et al., 2018): whether some special control inputs can be designed and applied to the system such that the behaviors of different modes are distinguishable.

From the literature, we find that nonlinearity has rarely been considered for the mode discernibility (Motchon & Pekpe, 2018; Motchon et al., 2017; Ramdani et al., 2018), and the controlled mode discernibility has not been extensively studied (Baglietto et al., 2007; Lou & Yang, 2011; Motchon & Pekpe, 2018; Motchon et al., 2018), especially for systems with unknown exogenous input. Therefore, we consider the mode discernibility and controlled mode discernibility for nonlinear hybrid systems with unknown switchings and unknown exogenous input, motivated by its potential applications to FDI, cyber-security, and robust control for systems with unnegligible nonlinearities. It is worth mentioning that by constructing an augmented system (De Santis, 2011; Gómez-Gutiérrez et al., 2011), the mode discernibility for linear switched systems can be related to the triviality of the zero dynamics (or zero-output-constrained dynamics, the subsystem of the original system that can result in identically zero output (Isidori, 2013b)) of the augmented system. Note that the zero dynamics of nonlinear system has been extensively studied and related to system inversions (Hirschorn, 1979; Nijmeijer, 1982, 1986; Silverman, 1969; Singh, 1981), decoupling control (Descusse & Moog, 1985; Isidori, Krener, Gori-Giorgi, & Monaco, 1981; Nijmeijer & Schumacher, 1985), output tracking (Albrecht, Grasse, & Wax, 1981; Devasia, Chen, & Paden, 1996; Isidori & Byrnes, 1990; Li & Feng, 1987; Singh, 1982), etc. It is well-known that for a linear system, the set of all initial states that can generate identically zero output is equivalent to the largest controlled invariance subspace (Basile & Marro, 1969) contained in the kernel of output matrix, and this characterization can be extended to nonlinear systems by generalizing the concept of controlled invariant subspace to controlled invariant submanifold (Berger, 2016; Isidori & Moog, 1988) and controlled invariant distribution (Hirschorn, 1981; Isidori et al., 1981). In this paper, we use the concept of controlled invariant submanifold to tackle the challenges introduced by nonlinearities in the characterization of mode discernibility. However, for the characterization of controlled mode discernibility, the existing concept of controlled invariant submanifold is insufficient in the sense that the problem considered in this paper is more complicated due to the presence of both control input and unknown exogenous input, which motivates us to propose a new concept called strongly controlled invariant submanifold from a geometric perspective, which is similar to but different from the concept of robust controlled invariant subspace considered by Fiacchini, Alamo, and Camacho (2010).

To the best of our knowledge, the only works considering controlled mode discernibility for nonlinear systems are (Motchon & Pekpe, 2018; Ramdani et al., 2018) which, however, do not consider both control input and unknown input simultaneously. To provide insights and tools for investigating the mode identification problem of hybrid systems in the context of FDI, cyber-security, and robust control, this paper is focused on the characterization of (controlled) mode discernibility and discussions on the idea of discerning control design for nonlinear hybrid systems with unknown exogenous input. Specifically, the contribution of this paper is threefold. First, we formally define the mode discernibility and the controlled mode discernibility for nonlinear hybrid systems subject to unknown exogenous input. Second, we review and complement the concept of controlled invariant submanifold from a geometric perspective, and further propose a new concept called strongly controlled invariant submanifold. Then, we characterize the mode discernibility and controlled mode discernibility based on the study of controlled invariant submanifold and strongly controlled invariant submanifold, respectively. A class of systems with a special structure is discussed as an example. Furthermore, the formulation of discerning control design is proposed for a special case.

The paper is organized as follows. The system description and problem statement are presented in Section 2. In Section 3, the concept of controlled invariant submanifold is reviewed, and the concept of strongly controlled invariant is proposed. The characterizations of mode discernibility and controlled mode discernibility are provided in Section 4. In addition, the hybrid systems whose subsystems have special structures are discussed, and a formulation of discerning control design is proposed. Finally, Section 5 concludes this paper.

## 2. Problem statement

Consider the hybrid system with unknown switchings and unknown exogenous input given as:

$$\Sigma_{\mathcal{S}}:\begin{cases}\dot{x}=f_{\sigma_t}(x)+g^a_{\sigma_t}(x)u^a+g^c_{\sigma_t}(x)u^c\\ y=h_{\sigma_t}(x)\end{cases}, \tag{1}$$

where the continuous state $x$ takes values in $\mathbb{R}^n$; the output $y$ takes values in $\mathbb{R}^l$; the control input $u^c$ and the unknown exogenous input $u^a$ take values in $\mathbb{R}^{m^c}$ and $\mathbb{R}^{m^a}$, respectively; $\sigma_t \in Q = \{1, 2, \dots, n_Q\}$ is the discrete state (or mode) of the system at time $t$, which is assumed to be unavailable, and $n_Q$ is the number of modes; for each $\sigma \in Q$, $f_\sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g^a_\sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m^a}$, $g^c_\sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m^c}$, consisting of vector fields associated with mode $\sigma$, are assumed to be $C^\infty$; and $h_\sigma : \mathbb{R}^n \mapsto \mathbb{R}^l$ consists of $l$ output functions which are $C^\infty$. Since the discrete state dynamics is not explicitly considered, the hybrid system (1) is also referred as a hidden mode nonlinear switched system with unknown inputs (Yong et al., 2018). We denote the following system as the subsystem of mode $q \in Q$ (or simply mode $q$):

$$\Sigma_q:\begin{cases}\dot{x}=f_q(x)+g^a_q(x)u^a+g^c_q(x)u^c\\ y=h_q(x)\end{cases}. \tag{2}$$

For the subsystem of mode $q$, we denote the input-to-state map and the input-to-output map by $\mathcal{X}^q_t(x^q_0, u^c, u^a)$ and $\mathcal{Y}^q_t(x^q_0, u^c, u^a)$, respectively. To facilitate the analysis, we assume the control input and the unknown exogenous input are admissible to ensure the existence and uniqueness of the state trajectory (at least on an open time interval containing $t = 0$) for each pair of inputs.

Note that all the above requirements of $C^\infty$ are just mathematical techniques to avoid counting the exact order of differentiability.

We propose the following definitions for studying mode discernibility and controlled mode discernibility:

**Definition 1** (*Mode Discernibility with Unknown Input*). Modes $p$ and $q$ are called discernible at $(x_0^p, x_0^q)$ if for any $\epsilon > 0$ and for any admissible $u^p$ and $u^q$, there is a $t \in (-\epsilon, \epsilon)$ such that $\mathcal{Y}_t^p(x_0^p, 0, u^p) \neq \mathcal{Y}_t^q(x_0^q, 0, u^q)$. Otherwise, modes $p$ and $q$ are called indiscernible at $(x_0^p, x_0^q)$, and $(x_0^p, x_0^q)$ is called an indiscernible pair for modes $p$ and $q$.

**Definition 2** (*Controlled Mode Discernibility with Unknown Input*). Modes $p$ and $q$ are called controlled discernible at $(x_0^p, x_0^q)$ if for any $\epsilon > 0$ and $\bar{u}^c > 0$, there is a discerning control for $u^c$ with magnitude smaller than $\bar{u}^c$ such that for any admissible $u^p$ and $u^q$, $\mathcal{Y}_t^p(x_0^p, u^c, u^p) \neq \mathcal{Y}_t^q(x_0^q, u^c, u^q)$ for some $t \in (-\epsilon, \epsilon)$. Otherwise, we say modes $p$ and $q$ are not controlled discernible at $(x_0^p, x_0^q)$, and $(x_0^p, x_0^q)$ is called a strongly indiscernible pair for modes $p$ and $q$.

Roughly speaking, modes $p$ and $q$ are discernible if all trajectories generated by mode $p$ cannot be reproduced by mode $q$, even in a very short time period. Equivalently, there is no intersection between the set of all possible output trajectories generated by mode $p$ and the set of trajectories generated by mode $q$, no matter how short the time horizon is. It is clear that if modes are indiscernible, whenever we observe a trajectory that can be generated by both modes $p$ and $q$, we cannot immediately tell which one the current mode is. For the controlled mode discernibility, we emphasize the role of control in distinguishing the behaviors of different modes. It should be remarked that it is required the discerning control to be arbitrarily small in order to restrict the discussion in local regions.

In this paper, we are interested in the following problems:

**Problem 1.** How to characterize generic indiscernible pairs and strongly indiscernible pairs?

**Problem 2.** What type of structure for the nonlinearity could facilitate the characterization?

**Problem 3.** How to formulate the problem of discerning control design?

## 3. Geometric tools

In the literature, by introducing an augmented system, the mode discernibility is shown to be associated with the output-nulling problem, i.e., whether there are initial states and inputs that can result in identically zero output for a while. Hence, to characterize the indiscernible pairs and strongly indiscernible pairs, we need the concept of controlled invariant submanifold, which is an important geometric tool for discussing the output-nulling problem. For reviewing the concept of controlled invariant submanifold and inducing the concept of strongly controlled invariant submanifold, we consider the following nonlinear system whose dynamics is affine in controls:

$$\Sigma : \begin{cases} \dot{x} = f(x) + g^a(x)u^a + g^c(x)u^c \\ y = h(x) \end{cases}, \tag{3}$$

where $x$, taking values in $\mathbb{R}^n$, is the state; $y$, taking values in $\mathbb{R}^l$, is the output; the inputs $u^a$ and $u^c$ take values in $\mathbb{R}^{m^a}$ and $\mathbb{R}^{m^c}$, respectively; and $f$, $g^c$, $g^a$, and $h$ are $C^\infty$. We restrict the inputs to be admissible. Additional assumptions for the regularity of $f$, $g^c$, and $g^a$ will be discussed later. For this nonlinear system, we denote the input-to-output map by $\mathcal{Y}_t(x_0, u^c, u^a)$, and the input-to-state map by $\mathcal{X}_t(x_0, u^c, u^a)$.

For conciseness, we shall not strictly consider the geometric objects in coordinate-free settings (Boothby, 1986), but it does not really matter since we mainly consider the local properties of nonlinear systems (note that the global properties are typically reduced to local properties by excluding singular points). For convenience, we consider the following definition for a submanifold:

**Definition 3** (*Regular Submanifold (Lee, 2013)*). $N$ is a $n'$-dimensional regular submanifold of $\mathbb{R}^n$ if for any $x \in N$, there is a neighborhood $O$ of $x$ such that

$$N \cap O = \{x \in O | H(x) = 0\},$$

for a smooth map $H : O \mapsto \mathbb{R}^{n-n'}$ with constant rank $n - n'$. $H$ is called a local defining map of $N$ in $O$.

In this paper, a neighborhood of $x$ means an open set containing $x$. Note that a singleton can be considered as a 0-dimensional submanifold. With Definition 3, the tangent space at a point $x$ of $N \cap O$ can be characterized using the local defining map:

$$\mathcal{T}_x N = \left\{ v \in \mathbb{R}^n | \frac{\partial H}{\partial x}(x) v = 0 \right\}. \tag{4}$$

### 3.1. Review of controlled invariant submanifold

In general, a controlled invariant submanifold is a submanifold that can be rendered invariant by applying some feedback control. Although most of the materials provided in this subsection are adapted from Isidori (2013a), Nijmeijer and Van der Schaft (1990) and other resources, to be self-contained, we carefully summarize materials and complement the proofs of some important statements.

**Definition 4** (*Controlled Invariant Submanifold*). A regular submanifold $N$ is $\langle f, g^a \rangle$-controlled invariant in an open connected subset $U$ if for any $x \in N \cap U$, there is a $v^a \in \mathbb{R}^{m^a}$ satisfying

$$f(x) + g^a(x)v^a \in \mathcal{T}_x N. \tag{5}$$

Note that the required property above for a submanifold $N$ to be controlled invariant in $U$ is equivalent to:

$$f(x) \in \mathcal{T}_x N + \mathrm{Im}\{g^a(x)\}, \quad \forall x \in N \cap U, \tag{6}$$

where $\mathrm{Im}\{\cdot\}$ represents the column space of the matrix in this paper. In addition, recalling that the tangent space can be expressed as (4), (5) is also equivalent to

$$\frac{\partial H}{\partial x}(x)\left(f(x) + g^a(x)v^a\right) = 0, \tag{7}$$

if $H$ is the local defining map.

**Definition 5** (*Controlled Invariant Output-Nulling Submanifold*). If a regular submanifold $N$ is $\langle f, g^a \rangle$-controlled invariant in $U$ and

$$\forall x \in N \cap U, \ \ h(x) = 0, \tag{8}$$

then $N$ is called a $\langle f, g^a \rangle$-controlled invariant output-nulling submanifold in $U$ for the nonlinear system (3).

In general, given a submanifold $M$ (e.g., $M = \{x | h(x) = 0\}$ could be locally a submanifold), it is not clear whether or not there exists a controlled invariant submanifold contained in $M$ and whether or not there exists the maximal one (for "maximal one", we mean all other controlled invariant submanifolds are contained in it). However, if some regularity conditions (or called constant-rank conditions in some context) hold in an open connected subset $U$, the existence of the maximal one contained in $M \cap U$ is guaranteed by considering the following algorithm for nonlinear systems described by (3):

**Algorithm 1** (*$\langle f, g^a|M\rangle$-algorithm*)**.**

**Step 0:** $N_0 = M \cap U$;

**Step** $k+1$: Assume $N_k$ is a regular submanifold; let

$$N_{k+1} = \left\{x \in N_k | f(x) \in \mathcal{T}_x N_k + \mathrm{Im}\left\{g^a(x)\right\}\right\}; \tag{9}$$

**Terminate if** $N_{k+1} = N_k$ or $N_{k+1} = \emptyset$.

$\langle f, g^a|M\rangle$-algorithm is called regular on an open connected subset $U$ if i) at each step $k \geq 0$, $N_k$ is a regular submanifold or empty and ii) for $k$ satisfying $N_k = N_{k+1} \neq \emptyset$, both $\mathrm{Im}\{g^a(x)\}$ and $\mathcal{T}_x N_k \cap \mathrm{Im}\{g^a(x)\}$ have constant dimension for all $x \in N_k$. If $\langle f, g^a|M\rangle$-algorithm is regular on $U$, then $U$ is called a regular set of the algorithm, and a point $x$ is called a regular point if it is contained in a regular set. By the assumption of smoothness, we know the collection of all regular points is open and dense in $\mathbb{R}^n$ because in general the collection of regular points can be written as a set where some matrices hold constant rank (Nijmeijer & Van der Schaft, 1990).

The convergence property of $\langle f, g^a|M\rangle$-algorithm is summarized in the following lemma:

**Lemma 1** ((*Isidori, 2013a; Nijmeijer & Van der Schaft, 1990*))**.** *Suppose $\langle f, g^a|M\rangle$-algorithm is regular on an open connected subset $U$, then the algorithm terminates in a finite number of steps on $U$, i.e., there is an integer $k^*$ such that $N_{k^*} = N_{k^*+1}$. If $N_{k^*}$ is nonempty, $N_{k^*}$ is the maximal $\langle f, g^a\rangle$-controlled invariant submanifold in $M \cap U$. Otherwise, any nonempty submanifold in $M \cap U$ is not $\langle f, g^a\rangle$-controlled invariant.*

Since $\langle f, g^a|M\rangle$-algorithm will converge to the maximal $\langle f, g^a\rangle$-controlled invariant submanifold in $M \cap U$ if it is regular on $U$, we shall denote

$$N^* = \left\langle f, g^a|M \cap U\right\rangle \tag{10}$$

to indicate that $\langle f, g^a|M\rangle$-algorithm converges to $N^*$ in $U$. Clearly, if we take $M = \{x|h(x) = 0\}$ and assume the algorithm is regular on an open connected subset $U$, then $N^* = \langle f, g^a|M \cap U\rangle$ is the maximal $\langle f, g^a\rangle$-controlled invariant output-nulling submanifold in $U$.

**Remark 1.** The existence of feedback rendering a submanifold locally invariant is equivalent to the existence of smooth feedback rendering the submanifold locally invariant under the regularity condition. Formally speaking, for any $x$ in $N^*$, there is a neighborhood $O$ of $x$ such that $N^* \cap O$ can be rendered invariant by some smooth feedback defined on $O$. To see this, suppose for any $x \in N^*$, there is a neighborhood $O$ of $x$ such that $N^* \cap O = \{x \in O|H(x) = 0\}$ ($H$ is the local defining map) without loss of generality. The regularity condition that both $\mathrm{Im}\{g^a(x)\}$ and $\mathcal{T}_x N^* \cap \mathrm{Im}\{g^a(x)\}$ have constant dimension on $N_k$ implies the matrix $\frac{\partial H}{\partial x}(x) g^a(x)$ has constant rank on $N_k \cap O$. To find a feedback $\alpha : N^* \cap O \mapsto \mathbb{R}^{m^a}$ that solves $f(x) + g^a(x)\alpha(x) \in \mathcal{T}_x N^*$ for each $x \in N^* \cap O$, i.e.,

$$\frac{\partial H}{\partial x}(x)(f(x) + g^a(x)\alpha(x)) = 0, \quad \forall x \in N^* \cap O, \tag{11}$$

we may consider

$$\alpha(x) = -\left[\frac{\partial H}{\partial x}(x) g^a(x)\right]^{\dagger}\left[\frac{\partial H}{\partial x}(x) f(x)\right], \tag{12}$$

where "$\dagger$" denotes the pseudo-inverse. Then, $\alpha(x)$ satisfies $f(x) + g^a(x)\alpha(x) \in \mathcal{T}_x N^*$ for each $x \in N^* \cap O$ by the fact that $f(x) \in \mathcal{T}_x N^* + \mathrm{Im}\{g^a(x)\}$ for each $x \in N^* \cap U$. In addition, $\alpha : N_k \cap O \mapsto \mathbb{R}^{m^a}$ is smooth because $\left[\frac{\partial H}{\partial x}(x) g^a(x)\right]^{\dagger}$ is smooth when $\frac{\partial H}{\partial x}(x) g^a(x)$ has constant rank on $N^* \cap O$. Furthermore, $\alpha$ can be smoothly extended to $O$ since $N^*$ is a regular submanifold.

The maximal $\langle f, g^a\rangle$-controlled invariant submanifold contained in $\{x \in U|h(x) = 0\}$, as a geometric object, is related to the control problem by the following definition and lemmas:

**Definition 6** (*Output-Nulling State*)**.** A state $x_0$ in $\mathbb{R}^n$ is a $\langle f, g^a\rangle$-output-nulling state of system (3) if there is an $\epsilon > 0$ and an admissible $u^a$ such that

$$\mathcal{Y}_t(x_0, 0, u^a) = 0, \forall t \in (-\epsilon, \epsilon). \tag{13}$$

**Lemma 2.** *Suppose $\langle f, g^a|\ker\{h\}\rangle$-algorithm ($\ker\{h\} = \{x|h(x) = 0\}$) is regular on an open connected subset $U$ and $N^* = \langle f, g^a|\ker\{h\} \cap U\rangle$. An $x_0 \in U$ is a $\langle f, g^a\rangle$-output-nulling state of system (3) if and only if $x_0 \in N^*$.*

**Proof.** The proof is given in Appendix A. □

According to this statement, we know that near regular points, if there are output-nulling states, they form a submanifold which is the maximal controlled invariant output-nulling submanifold.

Note that the above result can be extended by considering $M$ that is a more general regular submanifold. If $M$ is a regular submanifold, then $M$ can be locally expressed as $\{x \in U|H(x) = 0\}$ with a local defining map $H$. By replacing the output function $h$ with $H$ for system (3), all discussions for the locally controlled invariant output-nulling submanifold can be applied here. Similar to Lemma 2, now we have:

**Corollary 1.** *Suppose $\langle f, g^a|M\rangle$-algorithm is regular on $U$ and $N^* = \langle f, g^a|M \cap U\rangle$. For any $x_0 \in U$, the following two statements are equivalent:*

*(1) there is an $\epsilon > 0$ and an admissible $u^a$ such that*

$$\mathcal{X}_t(x_0, 0, u^a) \in M, \forall t \in (-\epsilon, \epsilon); \tag{14}$$

*(2) $x_0 \in N^*$.*

**Remark 2.** According to Corollary 1, $x_0 \in U \setminus N^*$ is equivalent to: for any $\epsilon > 0$ and for any admissible input $u^a$, there is $t \in (-\epsilon, \epsilon)$ such that $\mathcal{X}_t(x_0, 0, u^a) \notin M$. In other words, if an initial state of system (3) is contained in $M$ but not contained in the maximal $\langle f, g^a\rangle$-controlled invariant submanifold in $M$, then no matter what $u^a$ is applied, the system state will leave from $M$ immediately.

### 3.2. Strongly controlled invariant submanifold

With the concept of controlled invariant submanifold, the indiscernible pairs could be characterized. However, for controlled mode discernibility, we will need a new concept that is defined as follows:

**Definition 7** (*Strongly Controlled Invariant Submanifold*)**.** A regular submanifold $\hat{N}$ is $\langle f, g^a; g^c\rangle$-strongly controlled invariant in an open connected subset $U$ if for any $x \in \hat{N} \cap U$ and any $v^c \in \mathbb{R}^{m^c}$, there is a $v^a \in \mathbb{R}^{m^a}$ satisfying

$$f(x) + g^c(x)v^c + g^a(x)v^a \in \mathcal{T}_x \hat{N}. \tag{15}$$

Recall (4), (15) is also equivalent to

$$\frac{\partial H}{\partial x}(x)\left(f(x) + g^c(x)v^c + g^a(x)v^a\right) = 0, \tag{16}$$

if $H$ is the local defining map of $\hat{N}$. Note that the above requirement for a submanifold $\hat{N}$ to be strongly controlled invariant is equivalent to

$$f(x) + \mathrm{Im}\{g^c(x)\} \subseteq \mathcal{T}_x \hat{N} + \mathrm{Im}\{g^a(x)\}, \forall x \in \hat{N} \cap U. \tag{17}$$

Or equivalently, both

$$f(x) \in \mathcal{T}_x\hat{N} + \mathrm{Im}\{g^a(x)\}, \forall x \in \hat{N} \cap U \tag{18}$$

and

$$\mathrm{Im}\{g^c(x)\} \subseteq \mathcal{T}_x\hat{N} + \mathrm{Im}\{g^a(x)\}, \forall x \in \hat{N} \cap U \tag{19}$$

hold.

**Definition 8** (*Strongly Controlled Invariant Output-Nulling Submanifold*)**.** If a regular submanifold $\hat{N}$ is $\langle f, g^a; g^c\rangle$-strongly controlled invariant in $U$ and

$$\forall x \in \hat{N} \cap U, h(x) = 0, \tag{20}$$

then $\hat{N}$ is called a $\langle f, g^a; g^c\rangle$-strongly controlled invariant output-nulling submanifold in $U$ for system (3).

Intuitively speaking, if a submanifold is $\langle f, g^a; g^c\rangle$-strongly controlled invariant for system (3), then no matter what admissible input is injected through the channel $g^c$, there is a choice of input for the channel $g^a$ such that the submanifold can be rendered invariant.

Again, we are interested in whether there exists a $\langle f, g^a; g^c\rangle$-strongly controlled invariant submanifold in a given submanifold $M$ and whether there is a maximal one. We consider the following algorithm that can generate the maximal strongly controlled invariant output-nulling submanifold in an open connected subset $U$ if some regularity conditions hold:

**Algorithm 2** (*$\langle f, g^a; g^c|M\rangle$-algorithm*)**.**

**Step** 0: $\hat{N}_0 = M \cap U$;

**Step** $k+1$: Assume $\hat{N}_k$ is a regular submanifold; let

$$\hat{N}_{k+1} = \Big\{ x \in \hat{N}_k | f(x) + \mathrm{Im}\{g^c(x)\} \subseteq \mathcal{T}_x\hat{N}_k + \mathrm{Im}\{g^a(x)\} \Big\}; \tag{21}$$

**Terminate if** $\hat{N}_{k+1} = \hat{N}_k$, or $\hat{N}_{k+1} = \emptyset$.

Similarly, $\langle f, g^a; g^c|M\rangle$-algorithm is called regular on an open connected subset $U$, if i) at each step $k \geq 0$, the assumptions required in the algorithm are satisfied, and ii) for $k$ satisfying $\hat{N}_k = \hat{N}_{k+1}$, both $\mathrm{Im}\{g^a(x)\}$ and $\mathcal{T}_x\hat{N}_k \cap \mathrm{Im}\,\{g^a(x)\}$ have constant dimension for $x \in \hat{N}_k \cap U$. If $\langle f, g^a; g^c|M\rangle$-algorithm is regular on $U$, then $U$ is called a regular set of the algorithm, and $x$ is called a regular point if $x$ is contained in a regular set. The convergence of the algorithm is claimed in the following proposition:

**Proposition 1.** *Suppose $\langle f, g^a; g^c|M\rangle$-algorithm is regular on an open connected subset $U$ for system (3), then the algorithm terminates in a finite number of steps, i.e., there is an integer $k^*$ such that $\hat{N}_{k^*} = \hat{N}_{k^*+1}$. If $\hat{N}_{k^*}$ is nonempty, $\hat{N}_{k^*}$ is the maximal $\langle f, g^a; g^c|M\rangle$-strongly controlled invariant output-nulling submanifold in $U$. Otherwise, there is no $\langle f, g^a; g^c|M\rangle$-strongly controlled invariant output-nulling submanifold in $U$.*

**Proof.** The proof is given in Appendix B. □

Since $\langle f, g^a; g^c|M\rangle$-algorithm will converge to the maximal $\langle f, g^a; g^c\rangle$-strongly controlled invariant submanifold in $M \cap U$ if it is regular on $U$, we shall denote

$$\hat{N}^* = \big\langle f, g^a; g^c|M \cap U\big\rangle \tag{22}$$

to indicate that $\langle f, g^a; g^c|M\rangle$-algorithm converges to $\hat{N}^*$ in $U$. If we take $M = \{x|h(x) = 0\}$ and assume the algorithm is regular on an open connected subset $U$, then $\hat{N}^* = \langle f, g^a; g^c|M \cap U\rangle$ is the maximal $\langle f, g^a; g^c\rangle$-strongly controlled invariant output-nulling submanifold in $U$.

The $\langle f, g^a; g^c\rangle$-strongly controlled invariant output-nulling submanifold for system (3), as a geometric object, can be related to the control problem by the following definition and theorem.

**Definition 9** (*Strongly Output-Nulling State*)**.** A state $x_0$ in $\mathbb{R}^n$ is a $\langle f, g^a; g^c\rangle$-strongly output-nulling state of system (3) if there is an $\epsilon > 0$ and $\bar{u}^c > 0$ such that for any admissible control of $u^c$ with magnitude smaller than $\bar{u}^c$, there is an admissible $u^a$ satisfying

$$\mathcal{Y}_t(x_0, u^c, u^a) = 0, \forall t \in (-\epsilon, \epsilon)\,. \tag{23}$$

**Theorem 1.** *Suppose $\langle f, g^a; g^c|ker\,\{h\}\rangle$-algorithm ($ker\,\{h\} = \{x|h(x) = 0\}$) is regular on an open connected subset $U$, and the algorithm converges to $\hat{N}^*$ on $U$. Then, a state $x_0 \in U$ is $\langle f, g^a; g^c\rangle$-strongly output-nulling for system (3) if and only if $x_0 \in \hat{N}^*$.*

**Proof.** The proof is given in Appendix C. □

**Remark 3.** In general, for a nonlinear system in the form of (3), the collection of output-nulling states contains, but not necessarily equivalent to, the set of strongly output-nulling states. To see this, consider the following illustrative nonlinear system:

$$\begin{cases} \dot{x}_1 = {x_2}^2 u^c \\ \dot{x}_2 = x_1 + e^{{x_1}^2} u^a \\ y = x_1 \end{cases}. \tag{24}$$

The collection of output-nulling states is given by:

$$N^* = \{x|x_1 = 0\}. \tag{25}$$

The set of strongly output-nulling states is given by:

$$\hat{N}^* = \{x|x_1 = x_2 = 0\}, \tag{26}$$

which is neither empty nor equal to $N^*$.

**Remark 4.** In Remark 3, an example has been given to show that for a general nonlinear system in form of (3), the set of strongly output-nulling states is contained in, but not necessarily equivalent to, the set of output-nulling states. However, if the system is linear, the set of strongly output-nulling states is either empty or equivalent to the set of output-nulling states. To see this, consider the linear system given as:

$$\begin{cases} \dot{x} = Ax + B^a u^a + B^c u^c \\ y = Cx \end{cases}. \tag{27}$$

where $A \in \mathbb{R}^{n\times n}$, $B^a \in \mathbb{R}^{n\times m^a}$, $B^c \in \mathbb{R}^{n\times m^c}$, $C \in \mathbb{R}^{l\times n}$ are system matrices. According to the literature (e.g., Basile and Marro (1969)), $N^*$ for the linear system (27) is the largest $\langle A, B^a\rangle$-controlled invariant subspace contained in $ker\;C$, and we already know $\hat{N}^*$ is a subset of $N^*$. Consider two cases: (1) $\mathrm{Im}\{B^c\}$ is contained in $N^* + \mathrm{Im}\{B^a\}$; and (2) $\mathrm{Im}\{B^c\}$ is not contained in $N^* + \mathrm{Im}\{B^a\}$. If (1) is the case, for any $x \in N^*$ and any $u^c$, there are $\hat{u}^a$ and $\tilde{u}^a$ such that

$$\begin{aligned} Ax + B^a\hat{u}^a &\in N^*, \\ B^c u^c + B^a\tilde{u}^a &\in N^*. \end{aligned} \tag{28}$$

Then, if we take $u^a = \hat{u}^a + \tilde{u}^a$, then

$$Ax + B^c u^c + B^a u^a \in N^*, \tag{29}$$

which reveals $N^*$ is contained in $\hat{N}^*$, and thus $N^* = \hat{N}^*$. For case (2), we claim $\hat{N}^* = \emptyset$. Otherwise, for any $x \in \hat{N}^*$, by the definition of $\hat{N}^*$,

$$\begin{aligned} \mathrm{Im}\{B^c\} &\subseteq \mathcal{T}_x\hat{N}^* + \mathrm{Im}\{B^a\} \\ &\subseteq \mathcal{T}_x N^* + \mathrm{Im}\{B^a\} = N^* + \mathrm{Im}\{B^a\}, \end{aligned} \tag{30}$$

which is contradictory to the condition.

## 4. Main results

### 4.1. Characterization of indiscernible pairs

To investigate the indiscernible pairs and strongly indiscernible pairs for modes $p$ and $q$, we may consider the augmented system (De Santis, 2011; Gómez-Gutiérrez et al., 2011) for convenience:

$$\Sigma_{pq}: \begin{cases} \dot{x}^{pq} = F_{pq}(x^{pq}) + G^a_{pq}(x^{pq})u^{pq} + G^c_{pq}(x^{pq})u^c \\ \delta y^{pq} = H_{pq}(x^{pq}) \end{cases}, \tag{31}$$

where

$$x^{pq} = \begin{bmatrix} x^p \\ x^q \end{bmatrix}, \qquad u^{pq} = \begin{bmatrix} u^p \\ u^q \end{bmatrix},$$
$$F_{pq}(x^{pq}) = \begin{bmatrix} f_p(x^p) \\ f_q(x^q) \end{bmatrix}, \quad G^a_{pq}(x^{pq}) = \begin{bmatrix} g^a_p(x^p) & 0 \\ 0 & g^a_q(x^q) \end{bmatrix},$$
$$G^c_{pq}(x^{pq}) = \begin{bmatrix} g^c_p(x^p) \\ g^c_q(x^q) \end{bmatrix}, \quad H_{pq}(x^{pq}) = h_p(x^p) - h_q(x^q).$$

By constructing the augmented system, the indiscernible pair for modes $p$ and $q$ is related to the initial state of the augmented system $\Sigma_{pq}$ that can generate identically zero output (at least for a while) by some $u^{pq}$, and the strongly indiscernible pair for modes $p$ and $q$ is equivalent to the initial state of the augmented system $\Sigma_{pq}$ such that no matter what admissible $u^c$ is injected, there is an admissible $u^{pq}$ making the output identically zero at least for a while. Formally, we have:

**Theorem 2.** *Suppose $\langle F_{pq}, G^a_{pq} | ker\left\{H_{pq}\right\}\rangle$-algorithm ($ker\left\{H_{pq}\right\} = \left\{x^{pq} | H_{pq}(x^{pq}) = 0\right\}$) is regular on $U$, an open connected subset of the state space of (31), and the algorithm converges to $N^*$ on $U$. $[x_0^{pT} \ \ x_0^{qT}]^T$ is a point in $U$. The following statements are equivalent:*

*(1) $(x_0^p, x_0^q)$ is an indiscernible pair for modes $p$ and $q$;*
*(2) $[x_0^{pT} \ \ x_0^{qT}]^T$ is a $\langle F_{pq}, G^a_{pq}\rangle$-output-nulling state of system (31);*
*(3) $[x_0^{pT} \ \ x_0^{qT}]^T \in N^*$.*

**Theorem 3.** *Suppose $\langle F_{pq}, G^a_{pq}; G^c_{pq} | ker\left\{H_{pq}\right\}\rangle$-algorithm ($ker\left\{H_{pq}\right\} = \left\{x^{pq} | H_{pq}(x^{pq}) = 0\right\}$) is regular on $U$, an open connected subset of the state space of (31), and the algorithm converges to $\hat{N}^*$ on $U$. $[x_0^{pT} \ \ x_0^{qT}]^T$ is a point in $U$. The following statements are equivalent:*

*(1) $(x_0^p, x_0^q)$ is a strongly indiscernible pair for modes $p$ and $q$;*
*(2) $[x_0^{pT} \ \ x_0^{qT}]^T$ is a $\langle F_{pq}, G^a_{pq}; G^c_{pq}\rangle$-strongly output-nulling state of system (31);*
*(3) $[x_0^{pT} \ \ x_0^{qT}]^T \in \hat{N}^*$.*

Note that the proofs of Theorems 2 and 3 are straightforward using the tools introduced in Section 3, and thus they are omitted for brevity. Let us introduce the following examples to demonstrate Theorems 2 and 3 as well as Algorithms 1 and 2.

**Example 1.** Consider a hybrid system with the following two nonlinear subsystems:

$$\begin{aligned} &\Sigma_1: \begin{cases} \dot{x}_1 = \cos x_2 + (\sin x_1)u^c \\ \dot{x}_2 = \sin x_1 - (\cos x_2)u^a \\ y = x_1 \end{cases}, \\ &\Sigma_2: \begin{cases} \dot{x}_1 = \cos x_2 - (\sin x_1)u^c \\ \dot{x}_2 = \sin x_1 + (\cos x_2)u^a \\ y = x_1 \end{cases}. \end{aligned} \tag{32}$$

We can use Theorems 2 and 3 to find indiscernible and strongly indiscernible pairs in

$$U = (-\frac{\pi}{4}, \frac{\pi}{4}) \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times (-\frac{\pi}{4}, \frac{\pi}{4}).$$

We can first construct an augmented system $\Sigma_{12}$ in the form of (31) with: $x^{12} = [x_1 \ x_2 \ x_3 \ x_4]^T$, $u^{12} = [u_1 \ u_2]^T$,

$$F_{12} = \begin{bmatrix} \cos x_2 \\ \sin x_1 \\ \cos x_4 \\ \sin x_3 \end{bmatrix}, G^a_{12} = \begin{bmatrix} 0 & 0 \\ -\cos x_2 & 0 \\ 0 & 0 \\ 0 & \cos x_4 \end{bmatrix}, G^c_{12} = \begin{bmatrix} \sin x_1 \\ 0 \\ -\sin x_3 \\ 0 \end{bmatrix},$$

and $H_{12} = x_1 - x_3$. To find indiscernible pairs, $\langle F_{12}, G^a_{12} | ker\left\{H_{12}\right\}\rangle$-algorithm can be applied. Although we do not know whether the algorithm is regular on $U$ or not in advance, we could attempt to apply it on $U$ and check the regularity conditions during the iterations. According to the algorithm, we start with

$$N_0 = \left\{x^{12} \in U | H_{12}(x^{12}) = x_1 - x_3 = 0\right\} \tag{33}$$

which is a well-defined submanifold in $U$ with defining map $H_{12}$. By (4) and (9), we know an $x_{12} \in N_0$ is contained in $N_1$ if and only if there is a $u^{12}$ such that

$$\frac{\partial H_{12}}{\partial x^{12}}(x^{12})\left(F_{12}(x^{12}) + G^a_{12}(x^{12})u^{12}\right) = 0. \tag{34}$$

Since the left-hand-side of the above equation is $\cos x_2 - \cos x_4$, we have

$$N_1 = \left\{x^{12} \in N_0 | x_2 = x_4 \ \text{ or } \ x_2 = -x_4\right\}. \tag{35}$$

Note that $N_1$ is not a regular submanifold by Definition 3 unless we exclude the shy set $\left\{x^{12} | x_2 = x_4 = 0\right\}$. In other words, $\langle F_{12}, G^a_{12} | ker\left\{H_{12}\right\}\rangle$-algorithm is not regular on $U$, but it might be regular on $\tilde{U} = U \setminus \left\{x^{12} | x_2 = x_4 = 0\right\}$.

By applying $\langle F_{12}, G^a_{12} | ker\left\{H_{12}\right\}\rangle$-algorithm on $\tilde{U}$, we have

$$\begin{aligned} N_0 &= \left\{x^{12} \in \tilde{U} | x_1 - x_3 = 0\right\}, \\ N_1 &= \left\{x^{12} \in \tilde{U} | x_1 - x_3 = x_2 - x_4 = 0\right\} \\ &\quad \cup \left\{x^{12} \in \tilde{U} | x_1 - x_3 = x_2 + x_4 = 0\right\}, \\ N_2 &= N_1, \end{aligned} \tag{36}$$

and the regularity conditions hold. By Theorem 2, the set of indiscernible pairs in $\tilde{U}$ is equivalent to $N^* = N_1$, which is coincident with our observation that if the initial states of $\Sigma_1$ and $\Sigma_2$ form a element in $N_1$, then the outputs and derivatives of the outputs from $\Sigma_1$ and $\Sigma_2$ can be made identical.

To find strongly indiscernible pairs, we attempt to apply $\langle F_{12}, G^a_{12}; G^c_{12} | ker\left\{H_{12}\right\}\rangle$-algorithm on $\tilde{U}$, and we can obtain:

$$\begin{aligned} \hat{N}_0 &= \left\{x^{12} \in \tilde{U} | x_1 - x_3 = 0\right\}, \\ \hat{N}_1 &= \left\{x^{12} \in \tilde{U} | x_1 = x_3 = x_2 - x_4 = 0\right\} \\ &\quad \cup \left\{x^{12} \in \tilde{U} | x_1 = x_3 = x_2 + x_4 = 0\right\}, \\ \hat{N}_2 &= \emptyset. \end{aligned} \tag{37}$$

Therefore, by Theorem 3, there is no strongly indiscernible pairs in $\tilde{U}$. By observation, we can see that whenever the outputs from $\Sigma_1$ and $\Sigma_2$ are identical and nonzero, the derivatives of the outputs can be made different by applying nonzero control input. In addition, whenever the outputs from $\Sigma_1$ and $\Sigma_2$ are zero, the derivatives of the outputs are nonzero.

We would like to note that the local characterization considered in this paper has its limitations. Indeed, all elements in $N_1$ given in (35) are indiscernible pairs by observation, but since the regularity condition fails, we cannot apply Theorem 2. On the other hand, although we may not know whether the given set is regular or not in advance, we can still attempt to apply the algorithms and make the regularity conditions hold typically by excluding some shy sets or partitioning the given set.

**Example 2.** Consider a hybrid system with the following two linear subsystems:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_2 + u^c \\ \dot{x}_2 = x_3 + u^c \\ \dot{x}_3 = x_1 + x_2 + u^a \\ \ y = x_1 \end{cases} \quad \Sigma_2 : \begin{cases} \dot{x}_1 = x_2 + u^c \\ \dot{x}_2 = x_3 - u^c \\ \dot{x}_3 = x_1 - x_2 + u^a \\ \ y = x_1 \end{cases} \tag{38}$$

The augmented system takes the following form:

$$\Sigma_{12} : \begin{cases} \dot{x}^{12} = Ax^{12} + B^a u^{12} + B^c u^c \\ \ y = Cx^{12} \end{cases}, \tag{39}$$

where $C = [1\ 0\ 0\ \ -1\ 0\ 0]$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \quad B^a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B^c = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

For finding all strongly indiscernible pairs, we can apply Algorithm 2 to the augmented system, and then we can obtain:

$$\begin{aligned} \hat{N}_0 &= \left\{x^{12} | x_1 - x_4 = 0\right\}, \\ \hat{N}_1 &= \left\{x^{12} | x_1 - x_4 = x_2 - x_5 = 0\right\}, \\ \hat{N}_2 &= \emptyset, \end{aligned} \tag{40}$$

which implies that there is no indiscernible pairs by Theorem 3. In stead of applying Algorithm 2, we can also apply Algorithm 1 to obtain:

$$\begin{aligned} N_0 &= \left\{x^{12} | x_1 - x_4 = 0\right\}, \\ N_1 &= \left\{x^{12} | x_1 - x_4 = x_2 - x_5 = 0\right\}, \\ N_2 &= \left\{x^{12} | x_1 - x_4 = x_2 - x_5 = x_3 - x_6 = 0\right\}, \\ N_3 &= N_2 = N^*, \end{aligned} \tag{41}$$

and check with the criterion in Remark 4. Since we have

$$\mathrm{Im}\{B^c\} \not\subseteq N^* + \mathrm{Im}\{B^a\}, \tag{42}$$

it follows that $\hat{N}^*$ is empty. The result is coincident with the observation that the first and the second order derivatives of the outputs from the two subsystems directly depend on $u^c$ rather than $u^a$, and the relationship between $\ddot{y}$ and $u^c$ for mode 1 is different from that of mode 2, and thus the two modes are controlled discernible.

We would like to note that the main result of Gómez-Gutiérrez et al. (2011) is coincident with our characterization of (controlled) mode discernibility for linear cases. The mode discernibility in our Definition 1 and the controlled mode discernibility in our Definition 2 are partially related to Problems 3 and 4 in Gómez-Gutiérrez et al. (2011), respectively. Furthermore, our Theorem 2 associates the indiscernible pairs with the maximal controlled invariant submanifold in the kernel of output function for the augmented system, which becomes the maximal controlled invariant subspace in the null space of output matrix for linear cases, and thus our Theorem 2 can imply Lemma 8 in Gómez-Gutiérrez et al. (2011). The strongly indiscernible pairs defined in our work is associated with the maximal strongly controlled invariant submanifold in the kernel of output function for the augmented system by Theorem 3. It is justified in Remark 4 that this submanifold is either identical to the largest controlled invariant subspace in the null space of the output matrix or empty for linear cases. Note that the condition for it to be nonempty is equivalent to the condition from Proposition 14 in Gómez-Gutiérrez et al. (2011). Our result is more general in the sense that we have shown that the strongly controlled invariant submanifold is not equivalent to the controlled invariant submanifold for nonlinear cases.

### *4.2. Augmented systems with relative orders*

Although we know that near regular points, the set of (strongly) indiscernible pairs is equivalent to the (strongly) controlled invariant output-nulling submanifold, it is, in general, still difficult to implement Algorithms 1 and 2. However, for nonlinear systems with well-defined relative orders, which have been commonly considered for study of nonlinear systems, Algorithms 1 and 2 can provide some clean results. Let us consider the following systems related to (31):

$$\begin{aligned} \Sigma_{pq}^a &: \begin{cases} \ \dot{x}^{pq} = F_{pq}(x^{pq}) + G_{pq}^a(x^{pq})u^{pq} \\ \delta y^{pq} = H_{pq}(x^{pq}) \end{cases}, \\ \Sigma_{pq}^c &: \begin{cases} \ \dot{x}^{pq} = F_{pq}(x^{pq}) + G_{pq}^c(x^{pq})u^{c} \\ \delta y^{pq} = H_{pq}(x^{pq}) \end{cases}. \end{aligned} \tag{43}$$

Recall that $\Sigma_{pq}^a$ in (43) is said to have globally defined relative orders (Nijmeijer & Van der Schaft, 1990) if there is a set of integers $\{r_1^a, \dots, r_l^a\}$ such that for each $j \in \{1, 2, \dots, l\}$,

$$L_{G_{pq}^a} L_{F_{pq}}^k H_{pq,j}(x^{pq}) = 0, \forall x^{pq}, \forall k \in \{0, \dots, r_j^a - 1\}, \tag{44}$$

and

$$\exists x^{pq}, L_{G_{pq}^a} L_{F_{pq}}^{r_j^a} H_{pq,j}(x^{pq}) \neq 0, \tag{45}$$

where $H_{pq,j}$ is the $j$th component of $H_{pq}$ and $L$ denotes the commonly used Lie derivative. The following result is well-known:

**Proposition 2** (Nijmeijer & Van der Schaft, 1990)**.** *Suppose $\Sigma_{pq}^a$ has finite relative orders $\{r_1^a, \dots, r_l^a\}$, and $A = [A_{ij}] : \mathbb{R}^{2n} \mapsto \mathbb{R}^{l \times 2m^a}$, defined by*

$$A_{ij}(x^{pq}) \triangleq L_{G_{pq,i}^a} L_{F_{pq}}^{r_j^a} H_{pq,j}(x^{pq}), \tag{46}$$

*($G_{pq,i}^a$ and $H_{pq,j}$ are the $i$th column of $G_{pq}^a$ and the $j$th component of $H_{pq}$, respectively), has full row rank for any $x^{pq} \in N = \{x^{pq} | H_N(x^{pq}) = 0\}$, where $H_N : \mathbb{R}^{2n} \mapsto \mathbb{R}^{\Sigma_{j=1}^l (r_j^a + 1)}$ is given by*

$$\left[H_{pq,1} \quad \cdots \quad L_{F_{pq}}^{r_1^a} H_{pq,1} \quad \cdots \quad H_{pq,l} \quad \cdots \quad L_{F_{pq}}^{r_l^a} H_{pq,l}\right]^T. \tag{47}$$

*If $N$ is nonempty, $N$ is a $\{2n - \Sigma_{j=1}^l (r_j^a + 1)\}$-dimensional submanifold and also the maximal $\langle F_{pq}, G_{pq}^a \rangle$-controlled invariant output-nulling submanifold for $\Sigma_{pq}$.*

The above proposition is useful to find the indiscernible pairs for a class of systems. To extend the result, let us assume $\Sigma_{pq}^c$ also has well-defined relative orders: there is a set of integers $\{r_1^c, \dots, r_l^c\}$ such that for each $j \in \{1, 2, \dots, l\}$,

$$L_{G_{pq}^c} L_{F_{pq}}^k H_{pq,j}(x^{pq}) = 0, \forall x^{pq}, \forall k \in \{0, \dots, r_j^c - 1\}, \tag{48}$$

and

$$\exists x^{pq}, L_{G_{pq}^c} L_{F_{pq}}^{r_j^c} H_{pq,j}(x^{pq}) \neq 0. \tag{49}$$

According to this structure, we have:

**Corollary 2.** *Suppose $\Sigma_{pq}^c$ and $\Sigma_{pq}^a$ have finite relative orders $\{r_1^c, \dots, r_l^c\}$ and $\{r_1^a, \dots, r_l^a\}$, respectively. The following statements are true:*

*(1) if $r_j^a \le r_j^c$ for all $j \in \{1, \dots, l\}$ and $A$ defined by (46) has full row rank on $N$ defined in Proposition 2, then $N$ is also the maximal $\langle F_{pq}, G_{pq}^a; G_{pq}^c \rangle$-strongly controlled invariant output-nulling submanifold for $\Sigma_{pq}$;*

*(2) if there is a $j \in \{1, \ldots, l\}$ such that $r_j^a > r_j^c$ and $L_{G_{pq}^c} L_{F_{pq}}^{r_j^c} H_{pq,j}(x^{pq})$ is non-vanishing for all $x^{pq} \in N$, where $N$ is defined in Proposition 2, then there is no nonempty $\langle F_{pq}, G_{pq}^a; G_{pq}^c \rangle$-strongly controlled invariant output-nulling submanifold for $\Sigma_{pq}$.*

**Proof.** For the first statement, we already know that $N$ is the maximal $\langle F_{pq}, G_{pq}^a \rangle$-controlled invariant output-nulling submanifold with dimension $2n - \Sigma_{j=1}^l (r_j^a + 1)$, and thus for each $x^{pq} \in N$, there is a $u^{pq}$ such that $F_{pq}(x^{pq}) + G_{pq}^a(x^{pq})u^{pq} \in \mathcal{T}_{x^{pq}} N$. Since $r_j^a \le r_j^c$ for each $j \in \{1, \ldots, l\}$, for each $j \in \{1, \ldots, l\}$,

$$L_{G_{pq}^c} L_{F_{pq}}^k H_{pq,j}(x^{pq}) = 0, \forall x^{pq}, \forall k \in \{0, \ldots, r_j^a - 1\}, \tag{50}$$

which implies that $\mathrm{Im}\{G_{pq}^c(x^{pq})\} \subseteq \mathcal{T}_{x^{pq}} N$ for each $x^{pq} \in N$. Therefore, $N$ is also a $\langle F_{pq}, G_{pq}^a; G_{pq}^c \rangle$-strongly controlled invariant output-nulling submanifold. $N$ is also the maximal one because it is already the maximal $\langle F_{pq}, G_{pq}^a \rangle$-controlled invariant output-nulling submanifold. So, we conclude the first statement.

For the second statement, we prove it by contradiction. Suppose there is a nonempty $\langle F_{pq}, G_{pq}^a; G_{pq}^c \rangle$-strongly controlled invariant output-nulling submanifold for $\Sigma_{pq}$, denoted by $N'$. Necessarily, $N' \subseteq N$. For each $x^{pq} \in N'$ and for each $v^c \in \mathbb{R}^{m^c}$, there is a $v^{pq} \in \mathbb{R}^{2m^a}$ such that $G_{pq}^c(x^{pq})v^c + G_{pq}^a v^{pq} \in \mathcal{T}_{x^{pq}} N' \subseteq \mathcal{T}_{x^{pq}} N$. However, there is $j \in \{1, \ldots, l\}$ such that $r_j^a > r_j^c$ and $L_{G_{pq}^c} L_{F_{pq}}^{r_j^c} H_{pq,j}(x^{pq})$ is non-vanishing for all $x^{pq} \in N$, which implies

$$\left[ \frac{\partial}{\partial x^{pq}} L_{F_{pq}}^{r_j^c} H_{pq,j}(x^{pq}) \right] G_{pq}^a(x^{pq}) = 0, \forall x^{pq} \in N \tag{51}$$

and

$$\left[ \frac{\partial}{\partial x^{pq}} L_{F_{pq}}^{r_j^c} H_{pq,j}(x^{pq}) \right] G_{pq}^c(x^{pq}) \neq 0, \forall x^{pq} \in N. \tag{52}$$

It means, for each $x^{pq} \in N$, there is a $v^c \in \mathbb{R}^{m^c}$ such that for any $v^{pq} \in \mathbb{R}^{2m^a}$, $G_{pq}^c(x^{pq})v^c + G_{pq}^a v^{pq} \notin \mathcal{T}_{x^{pq}} N$, which is contradictory to what we have derived. $\square$

Example 2 in Section 4.1 can be considered as a linear example for demonstrating Corollary 2. By observing the augmented system (39) in Example 2, we have $r_1^c = 2$ and $r_1^a = 3$. Since $r_1^c < r_1^a$ and the conditions required by the second statement of Corollary 2, we can conclude that there is no strongly indiscernible pair for these two modes, which is coincident with the conclusion that we get in Example 2 using Theorem 3. Let us also consider the following nonlinear example.

**Example 3.** Consider a hybrid system with the following two bilinear subsystems:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_2 + u^c \\ \dot{x}_2 = x_3 + u^a \\ \dot{x}_3 = x_1 + x_3 u^c \\ y = x_1 \end{cases} \quad \Sigma_2 : \begin{cases} \dot{x}_1 = x_2 - u^c \\ \dot{x}_2 = x_3 + x_2 u^a \\ \dot{x}_3 = x_1 + x_3 u^c + u^a \\ y = x_1 \end{cases} \tag{53}$$

To find strongly indiscernible pairs of these two subsystems, we first construct the augmented system, and then apply Algorithm 2 or check the relative orders. If we apply Algorithm 2, we obtain that $\hat{N}^* = \hat{N}_1 = \emptyset$, and thus mode 1 and mode 2 are controlled discernible. Alternatively, since we have $r_1^c = 1$ and $r_1^a = 2$ and the conditions required by the second statement of Corollary 2 are satisfied, we can conclude that there is no strongly indiscernible pairs for modes 1 and 2.

Note that the controlled mode distinguishability for bilinear systems and mode distinguishability for single-output control-affine systems are considered in Motchon and Pekpe (2018), Motchon et al. (2017) (but without addressing unknown input), respectively. The relative degrees (relative orders) are used to check the discernibility, which is similar to our result but derived from an algebraic perspective. Applying Theorem 21 from (Motchon & Pekpe, 2018), we can obtain that the two subsystems in this example are controlled discernible if $u^a$ is set to identically zero, which is consistent with the result from our approach.

### 4.3. Full-state feedback discerning control

In the previous subsections, we discuss the cases where only the output information is available. However, it should be remarked that the controlled mode discernibility introduced in Definition 2 only implies the existence of discerning control, but to see whether or not there is a discerning output feedback control requires more discussions. In fact, the proof of Theorem 1 shows that the discerning controls form an open and dense subset in the collection of all admissible controls (if, locally, all pairs of initial states are controlled discernible), which means even if we do not know the initial states, we are likely to obtain a discerning control input by random selection. In this subsection, let us consider cases where full state information is directly observed, i.e., $y = x$ for both mode $p$ and mode $q$, such that we can consider how to use the state information to effectively avoid a control input that is not discerning. Note that the discussion here can be applied to cases where the state can be recovered directly from the output without knowing inputs for each subsystem (e.g., each mode is strongly observable).

Before we move to the discussion of discerning control design, let us investigate the collection of (strongly) indiscernible pairs in this special case. Suppose for both mode $p$ and mode $q$, $y = x$. Then, for the augmented system (31), $H_{pq}(x^{pq}) = [I \quad -I]x^{pq}$, where $I$ is the identity matrix with a proper dimension. It is clear that $(x_0^p, x_0^q)$ is a (strongly) indiscernible pair only if $x_0^p = x_0^q$. For this reason, we say $x_0$ is a *(strongly) indiscernible initial state* for mode $p$ and mode $q$ if $(x_0, x_0)$ is a (strongly) indiscernible pair for modes $p$ and $q$. To characterize the (strongly) indiscernible initial states, instead of using Theorems 2 and 3, we have the following alternative approach. Suppose

$$E \triangleq \left\{ x | f_p(x) - f_q(x) \in \mathrm{Im}\{[g_p^a(x) \;\; g_q^a(x)]\} \right\} \tag{54}$$

is a regular submanifold in an open connected subset $U$ and $\mathrm{Im}\{g_p^a\} + \mathrm{Im}\{g_q^a\}$ has constant rank on $U$. Then, there are smooth $\alpha_p : U \mapsto \mathbb{R}^{m^a}$ and $\alpha_q : U \mapsto \mathbb{R}^{m^a}$ such that for any $x \in E \cap U$,

$$f_{pq}(x) \triangleq f_p(x) + g_p^a(x)\alpha_p(x) = f_q(x) + g_q^a(x)\alpha_q(x). \tag{55}$$

Suppose $\mathrm{Im}\{g_p^a\} \cap \mathrm{Im}\{g_q^a\}$ also has constant dimension on $U$, then there is a smooth map $g_{pq}^a : U \mapsto \mathbb{R}^{n \times dim(\mathrm{Im}\{g_p^a\} \cap \mathrm{Im}\{g_q^a\})}$ such that

$$\mathrm{Im}\{g_{pq}^a(x)\} = \mathrm{Im}\{g_p^a(x)\} \cap \mathrm{Im}\{g_q^a(x)\} \tag{56}$$

for all $x \in U$. Based on such $f_{pq}$ and $g_{pq}^a$, we can characterize the indiscernible initial states according to the following corollary:

**Corollary 3.** *Suppose both $Im\{g_p^a\} + Im\{g_q^a\}$ and $Im\{g_p^a\} \cap Im\{g_q^a\}$ have constant dimension on $U$ such that $f_{pq}$ and $g_{pq}^a$ are defined on $E \cap U$, where $E$ is given by (54). Assume $\langle F_{pq}, G_{pq}^a | ker \{[I \;\; -I]\} \rangle$-algorithm is regular on $U \times U$ and $\langle f_{pq}, g_{pq}^a | E \rangle$-algorithm is regular on $U$. Then, $\left[ x^T \;\; x^T \right]^T \in N^*$ if and only if $x \in N^{pq}$, where*

$$\begin{aligned} N^* &= \left\langle F_{pq}, G_{pq}^a | ker \{[I \;\; -I]\} \cap U \times U \right\rangle, \\ N^{pq} &= \left\langle f_{pq}, g_{pq}^a | E \cap U \right\rangle. \end{aligned} \tag{57}$$

Similarly, we can characterize the strongly indiscernible initial states by introducing $\hat{E}$ and $g^a_{pq}$. $\hat{E}$ is defined as:

$$\hat{E} \triangleq \left\{ x \in E | \mathrm{Im}\{g^c_p(x) - g^c_q(x)\} \in \mathrm{Im}\{[g^a_p(x) \;\; g^a_q(x)]\} \right\}, \tag{58}$$

where $E$ is defined in (54). Again, if both $\mathrm{Im}\{g^a_p\} + \mathrm{Im}\{g^a_q\}$ and $\mathrm{Im}\{g^a_p\} \cap \mathrm{Im}\{g^a_q\}$ have constant dimension on $U$, then $f_{pq}$ and $g^a_{pq}$ in Corollary 3 can be obtained. In addition, there are smooth maps $\beta_p : U \mapsto \mathbb{R}^{m^a \times m^c}$ and $\beta_q : U \mapsto \mathbb{R}^{m^a \times m^c}$ such that for any $x \in \hat{E} \cap U$,

$$g^c_{pq}(x) \triangleq g^c_p(x) + g^a_p(x)\beta_p(x) = g^c_q(x) + g^a_q(x)\beta_q(x). \tag{59}$$

Based on $\hat{E}$, $f_{pq}$, $g^a_{pq}$, and $g^c_{pq}$, we can characterize the strongly indiscernible initial states according to the following corollary:

**Corollary 4.** *Suppose both $\mathrm{Im}\{g^a_p\} + \mathrm{Im}\{g^a_q\}$ and $\mathrm{Im}\{g^a_p\} \cap \mathrm{Im}\{g^a_q\}$ have constant dimension on $U$ such that $f_{pq}$, $g^a_{pq}$, and $g^c_{pq}$ are well-defined on $\hat{E} \cap U$, where $\hat{E}$ is given by (58). Assume $\langle F_{pq}, G^a_{pq}; G^c_{pq} | \ker \{[I \;\; -I]\}\rangle$-algorithm is regular on $U \times U$ and $\left\langle f_{pq}, g^a_{pq}; g^c_{pq} | \hat{E} \right\rangle$-algorithm is regular on $U$. Then, $\left[x^T \;\; x^T\right]^T \in \hat{N}^*$ if and only if $x \in \hat{N}^{pq}$, where*

$$\begin{aligned} \hat{N}^* &= \left\langle F_{pq}, G^a_{pq}; G^c_{pq} | \ker \{[I \;\; -I]\} \cap U \times U \right\rangle, \\ \hat{N}^{pq} &= \left\langle f_{pq}, g^a_{pq}; g^c_{pq} | \hat{E} \cap U \right\rangle. \end{aligned} \tag{60}$$

We prove only Corollary 4 since the proof of Corollary 3 can be considered as a special case of that of Corollary 4.

**Proof.** The proof is given in Appendix D □

Corollaries 3 and 4 indeed match the intuition. For the case where $u^c$ is identically zero, if a state trajectory $x(t)$ generated by mode $p$ using admissible $u^p$ can also be generated by mode $q$ using admissible $u^q$, then there is an $\epsilon$ such that $x(t)$ should be contained in $E$ for any $t \in (-\epsilon, \epsilon)$, and $u^p$ and $u^q$ must satisfy

$$u^p(t) = \alpha_p(x(t)) + \tilde{u}^p(t), \quad u^q(t) = \alpha_q(x(t)) + \tilde{u}^q(t), \tag{61}$$

for some $\tilde{u}^p$ and $\tilde{u}^q$ that satisfy $g^a_p(x(t))\tilde{u}^p(t) = g^a_q(x(t))\tilde{u}^q(t)$ for all $t \in (-\epsilon, \epsilon)$. For this reason,

$$\dot{x} = f_{pq}(x) + g^a_{pq}(x)u^a \tag{62}$$

can be considered as the constrained dynamics for indiscernible $x(t)$, which means any $x(t)$ that can be generated by both mode $p$ and mode $q$ must be a trajectory of (62) and contained in $E$ (at least on a small time interval containing 0). Similarly, for any $u^c$, if there are $u^p$ and $u^q$ such that

$$x(t) = \mathcal{X}^p_t(x_0, u^c, u^p) = \mathcal{X}^q_t(x_0, u^c, u^q), \forall t \in (-\epsilon, \epsilon), \tag{63}$$

then there is an $\epsilon'$ such that $x(t)$, $t \in (-\epsilon', \epsilon')$, is contained in $\hat{E}$ and is a trajectory of the following constrained dynamics:

$$\dot{x} = f_{pq}(x) + g^c_{pq}(x)u^c + g^a_{pq}(x)u^a. \tag{64}$$

Therefore, it is natural to conclude Corollaries 3 and 4. For convenience, we shall denote the input-to-state map of the constrained dynamics (64) by $\mathcal{X}^{pq}_t(x_0, u^c, u^a)$.

With the knowledge that the strongly indiscernible initial states in a regular set form a $\langle f_{pq}, g^a_{pq}; g^c_{pq}\rangle$-strongly controlled invariant submanifold, we can conceptually construct a piecewise constant discerning control input under some regularity conditions, assuming all states in $U$ are controlled discernible.

Assume $\left\langle f_{pq}, g^a_{pq}; g^c_{pq} | \hat{E} \right\rangle$-algorithm is regular on $U$ and generates a sequence of submanifolds:

$$\hat{N}^{pq}_0 \supseteq \hat{N}^{pq}_1 \supseteq \ldots \supseteq \hat{N}^{pq}_K = \emptyset. \tag{65}$$

Let us partition $\hat{N}^{pq}_0$ into the following sets:

$$C_0 \triangleq \hat{N}^{pq}_0 \setminus \hat{N}^{pq}_1, \ldots, C_{K-1} \triangleq \hat{N}^{pq}_{K-1} \setminus \hat{N}^{pq}_K = \hat{N}^{pq}_{K-1}. \tag{66}$$

Note that $\cup_{k=0}^{K-1} C_k = \hat{N}^{pq}_0$ and $C_i \cap C_j = \emptyset$ for any $i \neq j \in \{0, 1, \ldots, K-1\}$, and thus for each given $x_0 \in \hat{N}^{pq}_0$, there is a unique $k \in \{0, 1, \ldots, K-1\}$ such that $x_0 \in C_k$.

Suppose $x_0$ is in $C_k$, and $H_k$ is the local defining map of $\hat{N}^{pq}_k$ in a neighborhood $O$ of $x_0$. Define $P_k : \hat{N}^{pq}_0 \times \mathbb{R}^{m^c} \times \mathbb{R}^{m^a} \mapsto \mathbb{R}$ as:

$$\begin{aligned} &P_k(x, v^c, v^a) \\ &\triangleq \left\| \frac{\partial H_k}{\partial x}(x) \left( f_{pq}(x) + g^c_{pq}(x)v^c + g^a_{pq}(x)v^a \right) \right\|, \end{aligned} \tag{67}$$

where $\|\cdot\|$ is the standard Euclidean norm. Since $x_0 \in C_k$ implies $x_0 \notin \hat{N}^{pq}_{k+1}$, then by the definition of $\hat{N}^{pq}_{k+1}$, we have

$$\max_{v^c \in V^c} \min_{v^a} P_k(x_0, v^c, v^a) > 0, \tag{68}$$

where $v^c \in V^c$ is used to indicate that $\|v^c\| \le \bar{u}^c$. Define $v^{c*}(x_0)$ as

$$v^{c*}(x_0) \triangleq \arg \max_{v^c \in V^c} \min_{v^a} P_k(x_0, v^c, v^a). \tag{69}$$

The following proposition is to state that if a constant input $v^{c*}(x_0)$ is applied, then modes $p$ and $q$ cannot generate the same trajectory, or the system state is driven away from $\hat{N}^{pq}_k$ and stays in the complement of $\hat{N}^{pq}_k$ for a while:

**Proposition 3.** *Suppose $\left\langle f_{pq}, g^a_{pq}; g^c_{pq} | \hat{E} \right\rangle$-algorithm converges to an empty set in a neighborhood $O$ of $x_0$ and $x_0 \in C_k$. Assume $H_k$ is the local defining map of $\hat{N}^{pq}_k$ in $O$ and $\frac{\partial H_k}{\partial x} g^a_{pq}$ has constant rank on $O$. Let $v^{c*}(x_0)$ be computed via the quadratic game (69). Then, for any $u^a$ that satisfies*

$$\left\| f_{pq}(x) + g^a_{pq}(x)u^a(t) + g^c_{pq}(x)v^{c*}(x_0) \right\| \le \bar{F}, \forall t, \tag{70}$$

*and for any $x$ in $O$, where $\bar{F}$ is a positive constant, there is a positive constant $T$ such that $x(t) \notin \hat{N}^{pq}_k$ for all $t \in (-T, 0) \cup (0, T)$, where $x(t) = \mathcal{X}^{pq}_t(x_0, v^{c*}(x_0), u^a)$.*

**Proof.** Since $x_0$ is in $C_k$, (68) holds. Consider $P^*_k$ defined as

$$P^*_k(x) = \min_{v^a} P_k(x, v^{c*}(x_0), v^a). \tag{71}$$

By the given constant rank condition, there is a neighborhood $O' \subseteq O$ of $x_0$ where $P^*_k$ is smooth and strictly positive. Let $R$ be a positive constant that satisfies $0 < R < \inf_{x \in \mathbb{R}^n \setminus O'} \|x_0 - x\|$. Then, for any $u^a$ satisfying (70), $x(t) = \mathcal{X}^{pq}_t(x_0, v^{c*}(x_0), u^a) \in O'$ for all $t \in (-R/\bar{F}, R/\bar{F})$. Since $P^*_k$ is strictly positive for all $x \in O'$, $\frac{d}{dt}H(x(t)) \neq 0$ for any $u^a$, which implies $x(t) \notin \hat{N}^{pq}_k$ for all $t \in (-R/\bar{F}, 0) \cup (0, R/\bar{F})$. □

**Remark 5.** In the proof of Proposition 3, we note that (69) could be conservative. In fact, to drive the state of the constrained dynamics away from $\hat{N}^{pq}_k$, any constant input with value $v^c$ that satisfies $\min_{v^a} P_k(x_0, v^c, v^a) > 0$ can be applied, but (69) considers a specific solution to the min–max problem, which reveals that we have certain degree of freedom to modify the formulation such that additional design criteria can be addressed.

By Proposition 3, we know it is possible to concatenate a finite number of constant functions of time to obtain the control input that can drive the trajectory of the constrained dynamics away from $\hat{E}$ within an arbitrarily small time interval. In other words, no matter what unknown exogenous input is injected into the system, even if the induced trajectory can be generated by both mode $p$ and mode $q$ within some small time interval, the system state will leave $\hat{E}$ due to the discerning control input. Once the system state $x$ is not in $\hat{E}$, by definition of $\hat{E}$ in (58), there is a $v^c$ such that for any $v^p$ and $v^q$,

$$f_p(x) + g_p^c(x)v^c + g_p^a v^p \neq f_q(x) + g_q^c(x)v^c + g_q^a v^q,$$

and thus the overall trajectory cannot be generated by both mode $p$ and mode $q$. We conclude this subsection with the following example to demonstrate Corollaries 3 and 4 as well as the idea of discerning control design.

**Example 4.** Consider a hybrid system including the following two subsystems with full-state observation:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_2 + x_3 u_1^c \\ \dot{x}_2 = \sin x_1 + u_1^a \\ \dot{x}_3 = x_4 + x_5 u_2^c \\ \dot{x}_4 = \tanh x_3 + u_2^a \\ \dot{x}_5 = u_3^c \end{cases} \quad \Sigma_2 : \begin{cases} \dot{x}_1 = x_2 - x_3 u_1^c \\ \dot{x}_2 = \cos x_1 + u_1^a \\ \dot{x}_3 = x_4 + x_5 u_2^c \\ \dot{x}_4 = \sinh x_3 + u_2^a \\ \dot{x}_5 = u_3^c \end{cases} \tag{72}$$

Let us first investigate the indiscernible and strongly indiscernible pairs (states) of these two modes. If we directly construct the augmented system and apply Algorithm 1, we can obtain:

$$N_1 = N_0 = \left\{ x^{12} \in \mathbb{R}^{10} \middle| \begin{bmatrix} I & -I \end{bmatrix} x^{12} = 0 \right\}, \tag{73}$$

which means all states are indiscernible states. Alternatively, we may construct $E$, $f_{12}$, and $g_{12}^a$ using (54), (55), (56): $E = \mathbb{R}^5$,

$$f_{12} = \begin{bmatrix} x_2 & 0 & x_4 & 0 & 0 \end{bmatrix}^T, \quad g_{12}^a = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T. \tag{74}$$

Using $\langle f_{12}, g_{12}^a | E \rangle$-algorithm, we can immediately obtain that $N^{12} = E = \mathbb{R}^5$, which is coincident with the result obtained from the augmented system approach.

For finding strongly indiscernible pairs, we can obtain the following by applying Algorithm 2 to the augmented system: $\hat{N}_0 = N_0$,

$$\begin{aligned} \hat{N}_1 &= \left\{ x^{12} \in \hat{N}_0 | x_3 = 0 \right\}, \\ \hat{N}_2 &= \left\{ x^{12} \in \hat{N}_1 | x_5 = 0 \right\}, \\ \hat{N}_3 &= \emptyset, \end{aligned} \tag{75}$$

which implies that there is no strongly indiscernible state. Alternatively, we can use $f_{12}$ and $g_{12}^a$ in (74), and construct $\hat{E}$ and $g_{12}^c$ using (58) and (59):

$$\hat{E} = \left\{ x \in \mathbb{R}^5 | x_3 = 0 \right\}, \quad g_{12}^c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T. \tag{76}$$

Applying $\left\langle f_{12}, g_{12}^a; g_{12}^c | \hat{E} \right\rangle$-algorithm, we have:

$$\begin{aligned} \hat{N}_0^{12} &= \hat{E} = \left\{ x \in \mathbb{R}^5 | x_3 = 0 \right\}, \\ \hat{N}_1^{12} &= \left\{ x \in \mathbb{R}^5 | x_3 = x_4 = x_5 = 0 \right\}, \\ \hat{N}_2^{12} &= \emptyset, \end{aligned} \tag{77}$$

which also shows that there is no strongly indiscernible state. It is worth noting that these two modes are controlled discernible, but the linearized subsystems around the origin are not controlled discernible.

Now, suppose that we observe $x(0) = 0$, but we do not know which mode is triggered. Our objective is to find a discerning control input $u^c : [0, T] \mapsto \mathbb{R}^3$ (subject to $\|u^c(t)\| \leq 1$ for any $t \in [0, T]$) to distinguish the behaviors of these two modes. Since $x(0) \in C_1 = \hat{N}_1^{12} \setminus \hat{N}_2^{12}$, and according to Proposition 3, we can solve (69) to get a constant input that can at least drive the system state away from $\hat{N}_1^{12}$. Specifically, the objective function (67) in the quadratic gaming (69) is given by

$$P_1(x = 0, v^c, v^a) = \left\| \begin{bmatrix} 0 & v_2^a & v_3^c \end{bmatrix}^T \right\|, \tag{78}$$

and thus

$$v^{c*}(0) = \arg \max_{\|v^c\| \leq 1} \min_{v^a} P_1(0, v^c, v^a) \quad = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T. \tag{79}$$

Let $u^c(t) = [0 \ \ 0 \ \ 1]^T$ for $t \in [0, t_1)$, where $t_1$ can be any real number in $(0, T)$. Suppose such a $u^c$ is applied to the system. Note that it is still possible that there are $u^1$ and $u^2$ such that

$$\mathcal{X}_t^1(0, u^c, u^1) = \mathcal{X}_t^2(0, u^c, u^2), \forall t \in [0, t_1), \tag{80}$$

for example, $u^1 = 0$ and $u^2 = 0$. However, no matter which mode is triggered and what unknown input is injected into the system, we know $x_5(t_1) = t_1 > 0$, which means $x(t_1) \notin \hat{N}_1^{12}$. Suppose we observe that $x(t_1) = [0 \ \ 0 \ \ 0 \ \ 0 \ \ t_1]^T$ (it can be resulted by $u^a = 0$ for either mode $p$ or mode $q$). Again, by Proposition 3, since $x(t_1) \in C_0 = \hat{N}_0^{12} \setminus \hat{N}_1^{12}$, we can solve (69) to get a constant control input that can drive the system state away from $\hat{N}_0^{12}$ if the induced trajectory can be generated by both modes $p$ and $q$. The objective function (67) in the quadratic gaming (69) is now given by

$$\begin{aligned} &P_0(x(t_1), v^c, v^a) \\ &= \left\| \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v_1^a & t_1 v_2^c & v_2^a & v_3^c \end{bmatrix}^T \right\| = \left\| t_1 v_2^c \right\|, \end{aligned} \tag{81}$$

and thus

$$v^{c*}(t_1) = \arg \max_{\|v^c\| \leq 1} \min_{v^a} P_0(x(t_1), v^c, v^a) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T. \tag{82}$$

Let $u^c(t) = [0 \ \ 1 \ \ 0]^T, t \in [t_1, t_2)$ for some $t_2 < T$. If such a $u^c$ is applied to the system, no matter which mode is triggered and no matter what unknown input is injected into the system, there is a time $t_2$ such that $x_3(t_2)$ is non-zero, and thus the system state is no longer contained in $\hat{E}$. Then, by finding an admissible $u^c(t), t \in [t_2, T)$ such that for any $v^1$, $v^2$,

$$\begin{aligned} &f_p(x(t_2)) + g_p^c(x(t_2))u^c(t_2) + g_p^a v^1 \\ &\neq f_q(x(t_2)) + g_q^c(x(t_2))u^c(x(t_2)) + g_q^a v^2, \end{aligned} \tag{83}$$

the resulting trajectory from the system cannot be generated by both mode 1 and mode 2. Note that a detailed systematic design approach for discerning control, especially for the general cases where the system states are not fully observable, is challenging. In addition, the design of a mode identification scheme integrated with the discerning control generation for hybrid systems with unknown exogenous input is worth further discussions.

## 5. Conclusions

In this paper, we considered the characterization of controlled mode discernibility of the hybrid system whose modes have the nonlinear continuous dynamics with unknown exogenous inputs. A concept of strongly controlled invariance was proposed for the characterization of controlled mode discernibility from a geometric perspective. Besides the general input-affine nonlinear cases, the systems with relative orders was discussed. In addition, for full-state observable cases, it was shown that the discerning control design can be conceptually formulated as a quadratic game in the tangent space.

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## Appendix A. Proof of Lemma 2

If $x_0 \in N^*$, then there is a neighborhood $O$ of $x_0$ and a smooth feedback $\alpha : O \mapsto \mathbb{R}^{m^a}$ such that

$$f(x) + g^a(x)\alpha(x) \in \mathcal{T}_x N^*, \forall x \in N \cap O. \tag{A.1}$$

Smoothness of the left-hand-side implies the existence of $\epsilon > 0$ and $x : (-\epsilon, \epsilon) \mapsto O$ with $x(0) = x_0$ that satisfies $\dot{x}(t) = f(x(t)) + g^a(x(t))\alpha(x(t))$ for any $t \in (-\epsilon, \epsilon)$. Let $u^a : (-\epsilon, \epsilon) \mapsto \mathbb{R}^{m^a}$ be given as $u^a(t) = \alpha(x(t))$, then $\mathcal{Y}_t(x_0, 0, u^a) = 0, \forall t \in (-\epsilon, \epsilon)$.

If $x_0$ is an output-nulling state, then there is a positive constant $\epsilon$ and a smooth input $u^a : (-\epsilon, \epsilon) \mapsto \mathbb{R}^{m^a}$ such that $\mathcal{Y}_t(x_0, 0, u^a) = 0, \forall t \in (-\epsilon, \epsilon)$. Note that $x_0$ is necessarily in $\{x|h(x) = 0\}$. If $x_0$ can be made an equilibrium, i.e., there is a $v_0^a \in \mathbb{R}^{m^a}$ such that $f(x_0) + g^a(x_0)v_0^a = 0$, then $x_0$ as a singleton is a submanifold that is both controlled invariant and output-nulling. Otherwise, $F : (-\epsilon, \epsilon) \mapsto \mathbb{R}^n$ defined as $F(t) = f(x(t)) + g^a(x(t))u^a(t)$ is nonzero on some open interval $(-\epsilon', \epsilon')$, where $x(t) = \mathcal{X}_t(x_0, 0, u^a)$. Thus the image of $\mathcal{X}_t(x_0, 0, u^a)$, $t \in (-\epsilon', \epsilon')$ is a local one-dimensional submanifold. Clearly, this submanifold is both controlled invariant and output-nulling, and it should be contained in $N^*$.

## Appendix B. Proof of Proposition 1

Under regularity conditions, the algorithm generates a sequence of inclusions:

$$\hat{N}_0 \supseteq \hat{N}_1 \supseteq \hat{N}_2 \supseteq \ldots \supseteq \hat{N}_k \supseteq \hat{N}_{k+1} \supseteq \ldots \tag{B.1}$$

Since they are submanifolds, according to their dimensions, the sequence converges in a finite number of steps. For a $k^* > 0$, if $\hat{N}_{k^*+1} = \hat{N}_{k^*} \neq \emptyset$, then for any $x \in \hat{N}_{k^*}$,

$$f(x) + \mathrm{Im}\{g^c(x)\} \subseteq \mathcal{T}_x \hat{N}^* + \mathrm{Im}\{g^a(x)\}, \tag{B.2}$$

which implies $\hat{N}_{k^*}$ is $\langle f, g^a; g^c|M\rangle$-strongly controlled invariant under the regularity conditions.

To show $\hat{N}_{k^*}$ is also the maximal one in $\hat{N}_0$, consider an arbitrary strongly controlled invariant submanifold $\hat{N}'$ contained in $\hat{N}_0$. Since $\hat{N}'$ is strongly controlled invariant, $f(x) + \mathrm{Im}\{g^c(x)\} \subseteq \mathcal{T}_x\hat{N}' + \mathrm{Im}\{g^a(x)\} \subseteq \mathcal{T}_x\hat{N}_0 + \mathrm{Im}\{g^a(x)\}$ for any $x \in \hat{N}' \subseteq \hat{N}_0$. Therefore, $N' \subseteq \hat{N}_1$. Next, $\hat{N}' \subseteq \hat{N}_1$ implies $\mathcal{T}_x\hat{N}' \subseteq \mathcal{T}_x\hat{N}_1$ for any $x \in \hat{N}'$, and thus $f(x) + \mathrm{Im}\{g^c(x)\} \subseteq \mathcal{T}_x\hat{N}' + \mathrm{Im}\{g^a(x)\} \subseteq \mathcal{T}_x\hat{N}_1 + \mathrm{Im}\{g^a(x)\}$ for any $x \in \hat{N}' \subseteq \hat{N}_1$. Hence, $\hat{N}'$ is contained in $\hat{N}_2$. By repeating the argument, we have $\hat{N}' \subseteq \hat{N}_k$ for any $k$, and thus $\hat{N}' \subseteq \hat{N}_{k^*}$.

## Appendix C. Proof of Theorem 1

Suppose $x_0 \in \hat{N}^*$. Since $\hat{N}^*$ is a submanifold by the regularity condition, there is a neighborhood $O \subseteq U$ of $x_0$ such that $\hat{N}^*$ can be locally expressed as

$$\hat{N}^* \cap O = \{x \in O | H(x) = 0\}, \tag{C.1}$$

where $H$ is the local defining map. Since $\hat{N}^*$ is $\langle f, g^a\rangle$-strongly controlled invariant, we know for each $x$ in $\hat{N}^* \cap O$ and for each $v^c$, there is a $v^a$ such that,

$$\frac{\partial H}{\partial x}(x)\left(f(x) + g^c(x)v^c + g^a(x)v^a\right) = 0. \tag{C.2}$$

Consider a smooth mapping $\beta : \hat{N}^* \cap O \times \mathbb{R}^{m^c} \mapsto \mathbb{R}^{m^a}$ defined as

$$\beta(x, v^c) = -\left[\frac{\partial H}{\partial x}(x)g^a(x)\right]^{\dagger}\frac{\partial H}{\partial x}(x)\left[f(x) + g^c(x)v^c\right]. \tag{C.3}$$

Then, $v^a = \beta$ solves (C.2) for all $x \in \hat{N}^* \cap O$ and all $v^c$. In addition, $\beta$ is smooth by the regularity condition (see Remark 1). Note that $\beta$ can be smoothly extended to $O \times \mathbb{R}^{m^c}$.

By the smoothness, for any $x_0 \in \hat{N}^* \cap O$, there are $\epsilon > 0$ and $\bar{u}^c > 0$ such that the initial value problem

$$\dot{x} = f(x) + g^c(x)u^c + g^a(x)\beta(x, u^c) \tag{C.4}$$

with the initial condition $x(0) = x_0$ has a unique solution $x : (-\epsilon, \epsilon) \mapsto O$ for any admissible $u^c$ with magnitude smaller than $\bar{u}^c$. Take $u^a(t) = \beta(x(t), u^c(t))$ (such a $u^a$ is clearly admissible if $u^c$ is admissible), then $\mathcal{X}_t(x_0, u^c, u^a) = x(t)$ for all $t \in (-\epsilon, \epsilon)$. $x(t) \in \hat{N}^* \cap O$ for any $t \in (-\epsilon, \epsilon)$ since $x_0 \in \hat{N}^*$ and the right-hand-side of (C.4) is contained in $\mathcal{T}_x\hat{N}^*$ for each $x \in \hat{N}^*$. Hence, we conclude the "if" part.

For "only if" part, let us consider a sequence of statements $\mathcal{S}_k$. For each $k \geq 0$, $\mathcal{S}_k$ is given as: for any strongly output-nulling state $x_0$ in $U$ for system (3), there is an $\epsilon > 0$ and $\bar{u}^c$ such that for any admissible control $u^c$ with magnitude smaller than $\bar{u}^c$, there is an admissible $u^a$ such that

$$\mathcal{X}_t(x_0, u^c, u^a) \in \hat{N}_k, \quad \forall t \in (-\epsilon, \epsilon), \tag{C.5}$$

where $\hat{N}_k$'s are defined in $\langle f, g^a; g^c|ker\,\{h\}\rangle$-algorithm.

According to the definition of strongly output-nulling states, $\mathcal{S}_k$ is clearly true for $k = 0$.

We claim $\mathcal{S}_k$ is true for all $k \geq 0$. If it is not the case, there is an integer $k' \geq 0$ such that $\mathcal{S}_{k'}$ is true, but $\mathcal{S}_{k'+1}$ is not true. If $\mathcal{S}_{k'+1}$ is not true, there are a strongly output-nulling state and an admissible $u^c$ with arbitrarily small magnitude such that for any admissible $u^a$, $\mathcal{X}_t(x_0, u^c, u^a)$ cannot be contained in $\hat{N}_{k'+1}$ over any small interval including $t = 0$. On the other hand, if $\mathcal{S}_{k'}$ is true, for such $x_0$ and $u^c$, there is an admissible $u^a$ such that $x(t) = \mathcal{X}_t(x_0, u^c, u^a)$ could be contained in $\hat{N}_{k'}$ over some intervals $(-\epsilon, \epsilon)$. Thus, there is a $\tilde{t} \in (-\epsilon, \epsilon)$ such that $\tilde{x}_0 = x(\tilde{t}) \notin \hat{N}_{k'+1}$. Now we claim that for $\tilde{x}_0 \in \hat{N}_{k'} \setminus \hat{N}_{k'+1}$, there is an admissible $\tilde{u}^c$ such that for any admissible $u^a$, $\tilde{x}(t) = \mathcal{X}_t(\tilde{x}_0, \tilde{u}^c, u^a)$ cannot be contained in $\hat{N}_{k'}$ over any small interval containing $t = 0$. To see this, by the definition of $\hat{N}_{k'+1}$, $\tilde{x}_0 \in \hat{N}_{k'} \setminus \hat{N}_{k'+1}$ reveals that

$$f(\tilde{x}_0) + \mathrm{Im}\{g^c(\tilde{x}_0)\} \not\subseteq \mathcal{T}_{\tilde{x}_0}\hat{N}_{k'} + \mathrm{Im}\{g^a(\tilde{x}_0)\}. \tag{C.6}$$

Let $O$ be a neighborhood of $\tilde{x}_0$ such that there is a local defining map $H$ for $\hat{N}_{k'} \cap O$. Condition (C.6) together with (4) can imply that there is a $v^c$ with arbitrarily small magnitude such that for any $v^a$,

$$\frac{\partial H}{\partial x}(\tilde{x}_0)\left(f(\tilde{x}_0) + g^c(\tilde{x}_0)v^c + g^a(\tilde{x}_0)v^a\right) \neq 0. \tag{C.7}$$

Hence, if $\tilde{u}^c(0) = v^c$, then $\frac{d}{dt}H(\tilde{x}(0)) \neq 0$, which means $H(\tilde{x}(\cdot))$ cannot be identically zero in any interval containing 0. Now if we construct $\hat{u}^c$ by concatenating $u^c$ and $\tilde{u}^c$ at $t = \tilde{t}$, there is no admissible $u^a$ to make $\mathcal{X}_t(x_0, \hat{u}^c, u^a)$ contained in $\hat{N}_{k'}$ over any small interval containing $t = 0$, which is contradictory to that $\mathcal{S}_{k'}$ is true. It is clear that the statement $\mathcal{S}_k$'s are true implies $x_0 \in \hat{N}_k$'s . Hence we conclude the "only if" part.

## Appendix D. Proof of Corollary 4

Since $\hat{N}^*$ is contained in $ker\,\{[I \quad -I]\}$, then there exists a submanifold $\hat{M}$ in $U$ such that $\begin{bmatrix} x^T & x^T \end{bmatrix}^T \in \hat{N}^*$ if and only if $x \in \hat{M}$. Thus, what we need to show is $\hat{M} = \hat{N}^{pq}$.

If $\begin{bmatrix} x^T & x^T \end{bmatrix}^T \in \hat{N}^*$,

$$\begin{bmatrix} f_p(x) \\ f_q(x) \end{bmatrix} \in \mathcal{T}_{\left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right]}\hat{N}^* + \mathrm{Im}\left\{\begin{bmatrix} g_p^a(x) & 0 \\ 0 & g_q^a(x) \end{bmatrix}\right\}, \tag{D.1}$$

and

$$\mathrm{Im}\left\{\begin{bmatrix} g_p^c(x) \\ g_q^c(x) \end{bmatrix}\right\} \subseteq \mathcal{T}_{\left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right]}\hat{N}^* + \mathrm{Im}\left\{\begin{bmatrix} g_p^a(x) & 0 \\ 0 & g_q^a(x) \end{bmatrix}\right\}. \tag{D.2}$$

Hence, for $\begin{bmatrix} x^T & x^T \end{bmatrix}^T \in \hat{N}^*$, there are $v^p$, $v^q \in \mathbb{R}^{m^a}$ such that

$$f_p(x) + g_p^a(x)v^p = f_q(x) + g_q^a(x)v^q \in \mathcal{T}_x\hat{M}, \tag{D.3}$$

and for any $v^c \in \mathbb{R}^{m^c}$, there are $\tilde{v}^p$, $\tilde{v}^q \in \mathbb{R}^{m^a}$ such that

$$g_p^c(x)v^c + g_p^a(x)\tilde{v}^p = g_q^c(x)v^c + g_q^a(x)\tilde{v}^q \in \mathcal{T}_x\hat{M}. \tag{D.4}$$

Therefore, $x$ is necessarily contained in $\hat{E}$. In addition, recalling the definitions of $f_{pq}$, $g_{pq}^a$ and $g_{pq}^c$, it can be inferred that $\hat{M} \subseteq \hat{E}$ is $\langle f_{pq}, g_{pq}^a; g_{pq}^c\rangle$-strongly controlled invariant. Since $\hat{N}^{pq}$ is the maximal $\langle f_{pq}, g_{pq}^a; g_{pq}^c\rangle$-strongly controlled invariant submanifold contained in $\hat{E}$, we know $x \in \hat{M} \subseteq \hat{N}^{pq}$.

If $x \in \hat{N}^{pq}$, then there is a $v^a$ such that

$$f_{pq}(x) + g_{pq}^a(x)v^a \in \mathcal{T}_x\hat{N}^{pq}, \tag{D.5}$$

and for any $v^c$, there is a $\tilde{v}^a$ such that

$$g_{pq}^c(x)v^c + g_{pq}^a(x)\tilde{v}^a \in \mathcal{T}_x\hat{N}^{pq}. \tag{D.6}$$

According to the definitions of $f_{pq}$, $g_{pq}^a$ and $g_{pq}^c$, the above expressions imply that for any $x \in \hat{N}^{pq}$, there are $v^p$ and $v^q$ such that

$$f_p(x) + g_p^a(x)v^p = f_q(x) + g_q^a(x)v^q \in \mathcal{T}_x\hat{N}^{pq}, \tag{D.7}$$

and for any $v^c$, there are $\tilde{v}^p$ and $\tilde{v}^q$ such that

$$g_p^c(x)v^c + g_p^a(x)\tilde{v}^p = g_q^c(x)v^c + g_q^a(x)\tilde{v}^q \in \mathcal{T}_x\hat{N}^{pq}, \tag{D.8}$$

which implies $\hat{N}^{pq} \times \hat{N}^{pq} \cap ker\{[I \quad -I]\}$ is $\langle F_{pq}, G_{pq}^a; G_{pq}^c\rangle$-strongly controlled invariant. Since $\hat{N}^*$ is the largest $\langle F_{pq}, G_{pq}^a; G_{pq}^c\rangle$-strongly controlled invariant submanifold contained in the kernel of $[I \quad -I]$, then $x \in \hat{N}^{pq}$ implies $\begin{bmatrix} x^T & x^T \end{bmatrix}^T \in \hat{N}^*$.

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