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BETWEEN YAMABE AND PANEITZ**

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This paper is devoted to several existence results for a generalized version of the Yamabe problem. First, we prove the remaining global cases for the range of powers $\gamma \in (0, 1)$ for the generalized Yamabe problem introduced by Gonzalez and Qing. Second, building on a new approach by Case and Chang for this problem, we prove that this Yamabe problem is solvable in the Poincaré-Einstein case for $\gamma \in (1, \min\{2, \frac{n}{2}\})$ provided the associated fractional GJMS operator satisfies the strong maximum principle.

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1. Introduction

The resolution of the Yamabe conjecture, i.e., the problem of finding a constant scalar curvature metric in a given conformal class on closed manifolds, has been a landmark in geometric analysis after the work of Yamabe [1960], Trudinger [1968], Schoen [1984] and Aubin [1998]. Several generalizations to different ambient manifolds appeared after this series of works, e.g., [Gamara 2001; Gamara and Yacoub 2001; Ahmedou 2004; González and Qing 2013].

We consider here some rather recent development whose foundation can be found in a seminal paper by Graham and Zworski [2003] about a new and fruitful approach to the realization of the GJMS operators. Suppose that X^{n+1} is a smooth

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manifold with smooth boundary M^n for $n \geq 3$. A function ρ is a defining function of the boundary M^n in X^{n+1} if

$$\rho > 0 \text{ in } X^{n+1}, \quad \rho = 0 \text{ on } M^n, \quad d\rho \neq 0 \text{ on } M^n.$$

We say that g_+ is conformally compact if, for some defining function ρ , the metric $\bar{g} = \rho^2 g_+$ extends to $\bar{X}^{n+1}$ so that $(\bar{X}^{n+1}, \bar{g})$ is a compact Riemannian manifold. This induces a conformal class of metrics $\hat{h} = \bar{g}|_{TM^n}$ on M^n when defining functions vary. The conformal manifold $(M^n, [\hat{h}])$ is called the *conformal infinity* of (X^{n+1}, g_+) . A metric g_+ is said to be *asymptotically hyperbolic* (AH) if it is conformally compact and the sectional curvature approaches -1 at infinity. (X^{n+1}, g_+) is called a *Poincaré–Einstein* (P-E) manifold, if $\text{Ric}(g_+) = -ng_+$.

Graham and Zworski [2003] introduced the meromorphic family of scattering operators $S(s)$, which is a family of pseudodifferential operators, for a given asymptotically hyperbolic manifold (X^{n+1}, g_+) and a choice of the representative $\hat{h}$ of the conformal infinity $(M^n, [\hat{h}])$. Instead one often considers the normalized scattering operators

$$P_h^\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S\left(\frac{n}{2} + \gamma\right)$$

The normalized scattering operators P_h^γ are conformally covariant

$$P_h^\gamma(uf) = u^{(n+2\gamma)/(n-2\gamma)} P_{\tilde{h}}^\gamma(f)$$

where $\tilde{h} = u^4/(n-2\gamma)h$. Then one can define the so called Q^γ -curvature as $Q_h^\gamma = P_h^\gamma(1)$. These operators P_h^γ appear to be the higher-order generalizations (for $\gamma > 1$) of the conformal Laplacian (including the Paneitz operator for $\gamma = 2$). They coincide with the GJMS operators [Graham et al. 1992] for suitable integer values of γ . Specifically, Q_h^γ is just the scalar curvature for $\gamma = 1$, and the Q -curvature for $\gamma = 2$. This new notion of curvature has been investigated in [Qing and Raske 2006; Chang and González 2011; González et al. 2012; González and Qing 2013; Kim et al. 2018]. When $\gamma = \frac{1}{2}$, Q_h^γ is just the mean curvature of (M, h) in (X, g) .

Keeping in mind the purpose of the Yamabe conjecture, one aims at finding a conformal metric $h \in [h]$ such that Q_h^γ is constant. Since the parameter γ ranges from 0 to $\frac{n}{2}$, this provides a 1-parameter family of metrics and sheds some new light on classical constant curvature prescription problems. Following [González and Qing 2013], solving the problem is equivalent to find a critical point of the following Euler–Lagrange functional

$$(1-1) \quad \mathcal{E}_h^\gamma[u] = \frac{\oint_M u P_h^\gamma u d\mu_h}{\left(\oint_M u^{2n/(n-2\gamma)} d\sigma_h\right)^{(n-2\gamma)/n}} \quad \text{for } u \in W_+^{\gamma,2}(M, h) \setminus \{0\},$$

where $W^{\gamma,2}(M, h)$ denotes the usual fractional Sobolev space on M with respect to Riemannian metric h , and $W_+^{\gamma,2}(M, h) = W^{\gamma,2}(M, h) \cap \{u \geq 0\}$. The infimum is called the γ -Yamabe constant

$$\mathcal{Y}^\gamma(M, [h]) = \inf_{W^{\gamma,2}(M, h) \setminus \{0\}} \mathcal{E}_h^\gamma[u].$$

The critical points of $\mathcal{E}_h^\gamma$ satisfy

$$(1-2) \quad P_h^\gamma u = cu^{(n+2\gamma)/(n-2\gamma)}, \quad u \geq 0$$

for some constant c . If P_h^γ satisfies the strong maximum principle, or its Green's function is positive, then u is strictly positive and satisfy the above equality. Hence, $u^{4/(n-2\gamma)}h$ is a conformal metric whose fractional curvature is constant. González and Qing [2013] proved that P_h^γ has a strong maximum principle when $\gamma \in (0, 1)$. For higher γ , in the setting of Poincaré-Einstein (X^{n+1}, g_+) with conformal infinity $(M^n, [h])$, Case and Chang [2016] proved that if $(M, [h])$ has scalar curvature $R_h \geq 0$ and $Q_h^\gamma \geq 0$ and $Q_h^\gamma \not\equiv 0$ for $1 < \gamma < \min\{2, n/2\}$, then P_h^γ has a strong maximum principle.

The present paper is two-fold. First, we complete the work started in [González and Qing 2013; Mayer and Ndiaye 2017b; González and Wang 2018; Kim et al. 2018] providing existence results in some range of dimensions depending on $\gamma \in (0, 1)$. Our arguments also apply to the general asymptotically hyperbolic (AH) manifolds. Second, for the higher order $1 < \gamma < \min\{2, \frac{n}{2}\}$, when X is a Poincaré-Einstein manifold, we completely solve the fractional Yamabe problem under the assumption of the strong maximum principle.

In the present contribution, we consider two types of situations, denoted below *Type I* and *Type II*.

First, we consider *Type I*; that is $\gamma \in (0, 1)$. Assume that (X^{n+1}, g_+) is a P-E manifold with conformal infinity $(M, [h])$. Kim et al. [2018] and Kim [2017] showed that if $n \geq 4 + 2\gamma$ and M is nonlocally conformally flat then γ -Yamabe problem is solvable. Mayer and Ndiaye [2017b] proved the solvability for M being locally conformally flat. Hence, the remaining case of Type I in P-E setting is the low dimensional case.

Case (I-1): (X^{n+1}, g_+) is P-E with $(M, [h])$ and $n < 2\gamma + 4$.

That is, $n = 3, 4$ when $\gamma \in (0, 1)$ and $n = 5$ when $\gamma \in (0, \frac{1}{2})$. If (X^{n+1}, g_+) is just AH, the second fundamental form of (M, h) will come into play. One needs consider whether (M, h) is umbilic or not, which induce many different cases. Readers are directed to [Kim et al. 2018] with additional assumptions. Nevertheless, our method also apply to the lower dimensional case in AH setting.

Case (I-2): (X^{n+1}, g_+) is AH with $(M, [h])$ and $n < 2 + 2\gamma$.

Second, we consider *Type II*; that is $\gamma \in (1, \min\{2, \frac{n}{2}\})$. The main contribution of the present paper is to deal with the higher order fractional Yamabe type problems. Assume that (X^{n+1}, g_+) is a P-E manifold with conformal infinity $(M, [h])$ for the following cases:

Case (II-1): Low dimension, $n < 2\gamma + 4$.

Case (II-2): $(M, [h])$ is locally conformally flat.

Case (II-3): $n > 2\gamma + 4$ and $(M, [h])$ is nonlocally conformally flat.

Case (II-4): $n = 2\gamma + 4$ and $(M, [h])$ is nonlocally conformally flat.

To attack the above cases, we need to notice the distinctive nature of them: (I-1), (I-2), (II-1), (II-2) are “global” cases and (II-3) and (II-4) are “local” cases. Let us recall what is commonly called local and global cases in the geometric analysis community. Take the classical Yamabe problem for example, that is, $\gamma = 1$. With this agreement in mind and recalling that the functional is $\mathcal{E}_h^1$ and the standard bubble is $U_{a,\varepsilon}$ (see (3-2) and (4-11), we omit δ for simplicity), then by a standard Taylor expansion, and using the explicit form (decay) of $U_{a,\varepsilon}$, one has the following formula:

$$(1-3) \quad \mathcal{E}_h^1(U_{a,\varepsilon}) = \mathcal{Y}_{\mathbb{S}^n}^1 - \sum_{i=1}^{n-3} \mathcal{L}_i(a) \varepsilon^i - \mathcal{L}_{n-2}(a) \varepsilon^{n-2} \ln \varepsilon - \mathcal{M}_{n-2}(a) \varepsilon^{n-2} + o(\varepsilon^{n-2}).$$

Here $\mathcal{L}_i(a)$ and $\mathcal{M}_{n-2}(a)$ are some coefficients in the expansion of $\mathcal{E}_h^1(U_{a,\varepsilon})$ around a , which are related to the derivatives of the Weyl tensor and the ADM mass. The case is called *local* if $\exists a \in M, i \in \{1, \dots, n-2\} : \mathcal{L}_i(a) \neq 0$ and it is referred to *global* if $\forall a \in M, \forall i \in \{1, \dots, n-2\} : \mathcal{L}_i(a) = 0$. The coefficient $\mathcal{M}_{n-2}$ is associated to the “mass” at a . The global case means the terms higher than mass should all vanish. When $\gamma \neq 1$, the mass term should have order $\varepsilon^{n-2\gamma}$ in (1-3). Roughly speaking, in P-E setting, since the first term in the above expansion is ε^4 with coefficient the norm square of the Weyl tensor (up to a nonzero factor), and that when the Weyl tensor is identically zero automatically all the coefficients in the above expansion until the logarithmic term vanish, then one can see how the property of being locally conformally flat and the competition between $\varepsilon^{n-2\gamma}$ and ε^4 describe fully the local and global cases. However, in AH setting, on top of the latter considerations one has additional terms starting at ε^2 with coefficient the norm of the trace free part of the second fundamental form of $(M, h) \subset (X, g)$ up to a nonzero factor. If M is umbilical, then the expansion is the same as in the case of P-E. Hence, in AH, the umbilicity, the locally conformally flatness, the size $\varepsilon^2, \varepsilon^4$, and $\varepsilon^{n-2\gamma}$ describe the global and local cases.

To solve the local cases, it is enough, in most of the arguments, to use the local $U_{a,\varepsilon}$; see (4-11). For the global cases, besides the work of Schoen [1984], there

is also an indirect method through algebraic topological arguments by Bahri and Coron [1988] (also called the barycenter technique). Later Bahri [1989] developed the theory of critical points at infinity for Yamabe problems on Euclidean domains. We refer the reader to its applications in the locally conformally flat case in [Bahri 1993] and in the low dimensional case in [Bahri and Brezis 1996]. Adapting the barycenter technique to the fractional Yamabe problem, we achieve the following theorem:

Theorem 1.1. *Suppose that $\gamma \in (0, 1)$ and $n \geq 2$. Assume (X^{n+1}, g_+) is a Poincaré-Einstein manifold with conformal infinity $(M^n, [h])$ with $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - \gamma^2$. If $\mathcal{Y}^\gamma(M, [h]) > 0$, then there exists $h \in [h]$ such that Q_h^γ is a constant.*

The previous theorem solves completely the Yamabe problem for the Q^γ -curvature, complementing the works [González and Qing 2013; Mayer and Ndiaye 2017b; González and Wang 2018; Kim et al. 2018] in the Poincaré-Einstein setting. We will provide an additional result on the more general framework of AH manifolds in the last section.

Theorem 1.2. *Suppose that $1 < \gamma < \min\{2, \frac{n}{2}\}$ and $n \geq 3$. Assume (X^{n+1}, g_+) is a Poincaré-Einstein manifold with conformal infinity $(M^n, [h])$ with $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - (2 - \gamma)^2$. If $R_h \geq 0$ and $Q_h^\gamma \geq 0$ and $Q_h^\gamma \not\equiv 0$ for some $h \in [h]$, then there exists some $\tilde{h} \in [h]$ such that $Q_{\tilde{h}}^\gamma$ is a constant.*

To prove our results in the local cases we employ the minimizing technique of Aubin [1998] and Schoen [1984]. In the global cases, we use the algebraic topological argument of Bahri and Coron [1988]. Since most of this work is concerned with Global cases, and moreover to find excellent exposition of the Aubin and Schoen minimizing technique seems not to be difficult (see for example [Lee and Parker 1987]), then we decide to discuss how the barycenter technique of Bahri and Coron works in finding a critical point. We point out that in our application of the minimizing technique of Aubin [1998] and Schoen [1984], we took a short-cut by bringing into play the Eckeland variational principle. We chose this approach not only to shorten the exposition, but to also emphasize the common point between the Aubin and Schoen minimizing technique and algebraic topological argument to Bahri and Coron.

The algebraic topological argument of Bahri and Coron [1988] is based on two fundamental facts: the quantization of $(\mathcal{E}_h^\gamma)^{n/(2\gamma)}$ (see Lemma 5.1) and the strong interaction phenomenon (see Lemma 5.9). Readers can find a detailed explanation of barycenter technique in [Mayer and Ndiaye 2017a]. Here we just sketch the main idea behind it.

On one hand, the argument needs a starting point, which is the existence of a topological class X_1 which is nonzero in the $\mathbb{Z}_2$ -homology of some lower sublevel set $L_c := \{u : (\mathcal{E}_h^\gamma[u])^{n/(2\gamma)} \leq c\}$. Here one starts with $c = (\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$ for some

$\varepsilon_1 > 0$, and the existence of X_1 is ensured by $H_n(M, \mathbb{Z}_2) \neq 0$ and bubbling; see Lemma 6.4. This is obtained by embedding M into L_c via the bubbles $v_{a,\lambda}$ and by using the quantization phenomenon that $\mathcal{E}_h^\gamma$ enjoys, via the deformation lemma (Lemma 6.1), to get that M survives topologically in L_c .

Then, the next step is to start *piling up masses* $v_{a,\varepsilon,\delta}$ (see its definition (5-1)) over X_1 , thereby moving from the level $(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$ to the level $2(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$, from the level $2(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$ to the level $3(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1, \dots$, from the level $p(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$ to the level $(p+1)(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$, so on. At each step, as one moves from the level $p(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$ to the level $(p+1)(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$, one constructs a nonzero topological class X_{p+1} which reads $(1-t)u + tv_{a,\varepsilon}$, $u \in X_p$, $t \in [0, 1]$; see Lemma 6.5. This is done by realizing $B_{p+1}(M)$ as a cone over $B_p(M)$ with top M and using the quantization phenomenon that $\mathcal{E}_h^\gamma$ enjoys, via the deformation lemma (Lemma 6.1), to get that by embedding $B_{p+1}(M)$ as a cone over $B_p(M)$ with top M into L_{p+1} via $(p+1)$ -convex combination of the bubbles $v_{a,\lambda}$, it survives as a nontrivial cone in (L_{p+1}, L_p) with $L_p = \{(\mathcal{E}_h^\gamma)^{n/(2\gamma)} \leq p(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1\}$.

However, because of the strong interaction phenomenon, for p_0 large, we are passing from the level $p_0(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1$ to the level $(p_0+1)(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} - \bar{\varepsilon}_1$ for some $\bar{\varepsilon}_1 > 0$. Then, assuming that there is no solution, we reach a contradiction to the fact that X_{p_0+1} is nontrivial in $(\hat{L}_{p_0+1}, L_{p_0})$ since, as a result of the quantization phenomenon, $(PS)_c$ holds $\forall c$ such that

$$p_0(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + \varepsilon_1 \leq c \leq (p_0+1)(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} - \bar{\varepsilon}_1.$$

This implies that X_{p_0+1} is trivial in $(\hat{L}_{p_0+1}, L_{p_0})$ with $\hat{L}_p = \{(\mathcal{E}_h^\gamma)^{n/(2\gamma)} \leq p(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} - \bar{\varepsilon}_1\}$.

We were assuming $R_h \geq 0$ and $Q_h^\gamma \geq 0$ and $Q_h^\gamma \not\equiv 0$ in the Theorem 1.2, because we need that P_h^γ satisfies the strong maximum principle, which is proved by Case and Chang [2016] under these assumptions. We conjecture that our results hold for all $\gamma \in (0, \frac{n}{2})$ provided P_h^γ satisfies the strong maximum principle.

This article is organized as follows. In Section 2, we recall some basic notions of smooth metric measure spaces and the fractional GJMS operators, which are contained in [Case 2017]. We define the standard bubbles for $\gamma \in (0, 1)$ and $\gamma \in (1, \min\{2, \frac{n}{2}\})$ respectively and list their properties need for the remaining sections. In Sections 3 and 4, we define some test function $U_{a,\varepsilon,\delta}$ and calculate their energy $\bar{\mathcal{E}}_h^\gamma[U_{a,\varepsilon,\delta}]$ for different cases respectively. In Section 5, we state the profile decomposition for the Palais–Smale sequences of $\mathcal{E}_h^\gamma$ and proved all the local cases. The crucial interaction estimate between bubbles are also established in this section. In Section 6, the algebraic topological argument is applied to all global cases. Section 7 illustrate the adaption to asymptotically hyperbolic case. Some necessary estimates are established in the Appendix at the end.

2. Preliminaries

In this section, we shall first describe the notions of smooth metric measure spaces and the fractional GJMS operators. After that we will define the standard bubbles and state their properties.

2A. Smooth metric measure spaces and fractional GJMS operators. A triple (X^{n+1}, M^n, g_+) is a *Poincaré–Einstein* manifold if:

- (1) X^{n+1} is (diffeomorphic to) the interior of a compact manifold $\bar{X}^{n+1}$ with boundary $\partial X = M^n$.
- (2) (X^{n+1}, g_+) is complete with $\text{Ric}(g_+) = -ng_+$.
- (3) There exists a nonnegative $\rho \in C^\infty(X)$ such that $\rho^{-1}(0) = M^n$, $d\rho \neq 0$ along M , and the metric $g := \rho^2 g_+$ extends to a smooth metric on $\bar{X}^{n+1}$.

A function ρ satisfying these properties is called a *defining function*. Since ρ is only determined up to multiplication by a positive smooth function on $\bar{X}$, it is clear that only the conformal class $[h] := [g|_{TM}]$ on M is well-defined for a Poincaré–Einstein manifold. We call the pair $(M^n, [h])$ the conformal boundary of the Poincaré–Einstein manifold (X^{n+1}, M^n, g_+) , and we call a metric $h \in [h]$ a representative of the conformal boundary. To each such representative there is a defining function ρ , unique in a neighborhood of M and called the *geodesic defining function*. Moreover, g_+ has normal form $g_+ = \rho^{-2}(d\rho^2 + h_\rho)$ near M , where h_ρ is a one-parameter family of metrics on M satisfying $h_0 = h$. There is an asymptotic expansion of h_ρ which contains only even powers of ρ , at least up to degree n . For a more intrinsic discussion of these topics, we refer the reader to [Graham and Zworski 2003].

A smooth metric measure space (SMMS) is a four-tuple $(\bar{X}^{n+1}, g, \rho, m)$ formed from a smooth manifold $\bar{X}^{n+1}$ with (possibly empty) boundary $M^n = \partial \bar{X}$, a Riemannian metric g on $\bar{X}$, a nonnegative function $\rho \in C^\infty(\bar{X})$ with $\rho^{-1}(0) = M$, and a dimensional constant $m \in (1 - n, \infty)$. Formally, the interior of $\bar{X}$, denoted as X , represents the base of a warped product

$$(2-1) \quad (X^{n+1} \times \mathbb{S}^m, g \oplus \rho^2 d\theta^2)$$

where $(\mathbb{S}^m, d\theta^2)$ the m -sphere with the metric of constant sectional curvature one. The geometric invariants defined on a SMMS are obtained by considering their Riemannian counterparts on (2-1) while restricting to the base X , and then extend the definition to general $m \in (1 - n, \infty)$ by treating m as a formal variable. The weighted Laplacian $\Delta_\rho^m : C^\infty(X) \rightarrow C^\infty(X)$ is defined as

$$\Delta_\rho^m U := \Delta_g U + m\rho^{-1} \langle \nabla \rho, \nabla U \rangle_g, \quad U \in C^\infty(X)$$

which is a formally self-adjoint operator with respect to the measure $\rho^m d\mu_g$. Here $d\mu_g$ is the volume element of g . The weighted Schouten scalar J_ρ^m and weighted Schouten tensor P_ρ^m are

$$J_\rho^m := \frac{1}{2(m+n)}(R - 2m\rho^{-1}\Delta\rho - m(m-1)\rho^{-2}(|\nabla\rho|^2 - 1)),$$

$$P_\rho^m := \frac{1}{m+n-1}(\text{Ric} - m\rho^{-1}\nabla^2\rho - J_\rho^m).$$

We shall confine ourselves to a special type of SMMS,

Definition 2.1. A *geodesic SMMS* $(\bar{X}, g := \rho^2 g_+, \rho, m)$ is generated by a Poincaré–Einstein manifold (X^{n+1}, M^n, g_+) and a geodesic defining function ρ near M , that is $|\nabla\rho|_g = 1$ near M .

For a geodesic SMMS, the weighted Schouten scalar and tensor take simpler forms. By [Case and Chang 2016, Lemma 3.2], we have $J_\rho^m = J$ the Schouten scalar of $(\bar{X}, g)$ and $P_\rho^m = P$ the Schouten tensor of $(\bar{X}, g)$. On a geodesic SMMS, the weighted conformal Laplacian $L_{2,\rho}^m$ and weighted Paneitz operator $L_{4,\rho}^m$ are defined as

$$L_{2,\rho}^m U := -\Delta_\rho^m U + \frac{1}{2}(m+n-1)J \cdot U,$$

$$L_{4,\rho}^m U := (-\Delta_\rho^m)^2 U + \delta_\rho((4P - (m+n-1)Jg)(\nabla U)) + \frac{1}{2}(m+n-3)Q_\rho^m U,$$

where $\delta_\rho X = \text{tr}_g \nabla X + m\rho^{-1}\langle X, \nabla\rho \rangle$ is the negative of the formal adjoint of the gradient with respect to $\rho^m d\mu_g$,

$$Q_\rho^m := -\Delta_\rho^m J - 2|P|^2 + \frac{m+n-1}{2}J^2$$

is the weighted Q -curvature. If two SMMS $(\bar{X}, g, \rho, m)$ and $(\bar{X}, \hat{g}, \hat{\rho}, m)$ are pointwise conformally equivalent, that is $\hat{g} = e^{2\sigma}g$ and $\hat{\rho} = e^\sigma \rho$ for some σ , it holds

$$(2-2) \quad \begin{aligned} \widehat{L_{2,\rho}^m}(U) &= e^{-(m+n+3)/2\sigma} L_{2,\rho}^m(e^{m+n-1/(2)\sigma} U), \\ \widehat{L_{4,\rho}^m}(U) &= e^{-(m+n+5)/2\sigma} L_{2,\rho}^m(e^{m+n-3/(2)\sigma} U). \end{aligned}$$

for all $U \in C^\infty(X)$.

The point of working with SMMS is that there are weighted GJMS operators defined on it, which incorporate the fractional GJMS operators on M as the Dirichlet-to-Neumann maps.

Suppose $\gamma \in (0, 1)$. Set $m_0 = 1 - 2\gamma$. Denoted by $\mathcal{C}^\gamma$ be the set of all $U \in C^\infty(X) \cap C^0(\bar{X})$, asymptotically near M ,

$$(2-3) \quad U = f + \psi\rho^{2\gamma} + o(\rho^{2\gamma})$$

for some $f, \psi \in C^\infty(M)$. We shall also use $\mathcal{C}_f^\gamma = \{U \in \mathcal{C}^\gamma : U|_M = f\}$. The Sobolev spaces $W^{1,2}(\bar{X}, \rho^{m_0} d\mu_g)$ are completion of $\mathcal{C}^\gamma$ with respect to the norm

$$\|U\|_{W^{1,2}}^2 := \int_X (|\nabla U|^2 + U^2) \rho^{m_0} d\mu_g.$$

Define

$$(2-4) \quad \mathcal{Q}_\gamma(U, V) = \int_X (\langle \nabla U, \nabla V \rangle + \frac{1}{2}(n - 2\gamma)JUUV) \rho^{m_0} d\mu_g.$$

Proposition 2.2 [Case 2017]. *Suppose that $\gamma \in (0, 1)$ and $(\bar{X}, g, \rho, m_0)$ is a geodesic SMMS. For any $U, V \in \mathcal{C}^\gamma$,*

$$(2-5) \quad \int_X V L_{2,\rho}^{m_0} U \rho^{m_0} d\mu_g + \oint_M V (\lim_{\rho \rightarrow 0} \rho^{m_0} (-\partial_\rho U)) d\sigma_h = \mathcal{Q}_\gamma(U, V).$$

If $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - \gamma^2$, then $\mathcal{Q}_\gamma(U, U)$ is bounded below in $\mathcal{C}_f^\gamma$. It holds that

$$(2-6) \quad \kappa_\gamma \mathcal{Q}_\gamma(U, U) \geq \oint_M f P_h^\gamma f d\sigma_h$$

for all $U \in W^{1,2}(\bar{X}, \rho^{m_0} d\mu_g)$ with $\text{Tr } U = f$. Equality holds if and only if $L_{2,\rho}^{m_0} U = 0$.

According to [Mayer and Ndiaye 2017b, Corollary 4.6], there exists a Green's function $G_g^\gamma(x, \xi)$ of $L_{2,\rho}^{m_0}$ satisfying

$$(2-7) \quad L_{2,\rho}^{m_0} G_g^\gamma(\cdot, \xi) = 0, \quad \text{and for all } \xi \in M, \quad -\kappa_\gamma \lim_{\rho \rightarrow 0} \rho^{m_0} \partial_\rho G_g^\gamma(x, \xi) = \delta_\xi(x).$$

Here $\delta_\xi(x)$ is the Dirac function at ξ . The following estimates hold for G_g^γ ,

$$(2-8) \quad \begin{aligned} |G_g^\gamma(x, \xi) - d_g(x, \xi)^{2\gamma-n}| &\leq C \max\{1, d_g(x, \xi)^{2\gamma-n+1}\}, \\ |\nabla(G_g^\gamma(x, \xi) - d_g(x, \xi)^{2\gamma-n})|_g &\leq C d_g(x, \xi)^{2\gamma-n}. \end{aligned}$$

Moreover, if $\mathcal{Y}^\gamma(M, [h]) > 0$, then $G_g^\gamma > 0$ by [González and Qing 2013].

Suppose $\gamma \in (1, 2)$. Set $m_1 = 3 - 2\gamma$. Denoted by $\mathcal{C}^\gamma$ be the set of all $U \in C^\infty(X) \cap C^0(\bar{X})$, asymptotically near M ,

$$(2-9) \quad U = f + \psi_1 \rho^2 + \psi_2 \rho^{2\gamma} + o(\rho^{2\gamma})$$

for some $f, \psi_1, \psi_2 \in C^\infty(M)$. We shall also use $\mathcal{C}_f^\gamma = \{U \in \mathcal{C}^\gamma : U|_M = f\}$. The Sobolev space $W^{2,2}(\bar{X}, \rho^{m_1} d\mu_g)$ is the completion of $\mathcal{C}^\gamma$ with respect to the norm

$$\|U\|_{W^{2,2}}^2 := \int_X (|\nabla^2 U + m_1 \rho^{-1} (\partial_\rho U)^2 d\rho \otimes d\rho|^2 + |\nabla U|^2 + U^2) \rho^{m_1} d\mu_g.$$

Define

$$(2-10) \quad \mathcal{Q}_\gamma(U, V) = \int_X [(\Delta_\rho^{m_1} U)(\Delta_\rho^{m_1} V) - (4P - (n - 2\gamma + 2)Jg)(\nabla U, \nabla V) + \frac{1}{2}(n - 2\gamma)\mathcal{Q}_\rho^{m_1} UV] \rho^{m_1} d\mu_g.$$

Proposition 2.3 [Case 2017]. *Suppose that $\gamma \in (1, 2)$ and $(\bar{X}, g, \rho, m_1)$ is a geodesic SMMS. For any $U, V \in \mathcal{C}^\gamma$,*

$$(2-11) \quad \int_X V L_{4,\rho}^{m_1} U \rho^{m_1} d\mu_g + \oint_M V (\lim_{\rho \rightarrow 0} \rho^{m_1} \partial_\rho \Delta_\rho^{m_1} U) d\sigma_h = \mathcal{Q}_\gamma(U, V).$$

If $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - (2 - \gamma)^2$, then $\mathcal{Q}_\gamma(U, U)$ is bounded below in $\mathcal{C}_f^\gamma$. It holds that

$$(2-12) \quad \kappa_\gamma \mathcal{Q}_\gamma(U, U) \geq \oint_M f P_h^\gamma f d\sigma_h$$

for all $U \in W^{2,2}(\bar{X}, \rho^{m_1} d\mu_g)$ with $\text{Tr } U = f$. Equality holds if and only if $L_{4,\rho}^{m_1} U = 0$.

Similarly, for $\gamma \in (1, 2)$, one can mimic the approach in [Mayer and Ndiaye 2017b, Corollary 4.6] to get a Green's function of $L_{4,\rho}^{m_1}$ satisfying

$$(2-13) \quad \begin{cases} L_{4,\rho}^{m_1} G_g^\gamma(\cdot, \xi) = 0 & \text{in } X, \text{ for all } \xi \in M, \\ \lim_{\rho \rightarrow 0} \rho^{m_1} \partial_\rho G_g^\gamma(\cdot, \xi) = 0 & \text{on } M \setminus \{\xi\}, \\ \kappa_\gamma \lim_{\rho \rightarrow 0} \rho^{m_1} \Delta_\rho^{m_1} G_g^\gamma(x, \xi) = \delta_\xi(x) & \text{on } M. \end{cases}$$

The Green's function has the following estimates

$$(2-14) \quad \begin{aligned} |G_g^\gamma(x, \xi) - d_g(x, \xi)^{2\gamma-n}| &\leq C \max\{1, d_g(x, \xi)^{2\gamma-n+1}\}, \\ |\nabla(G_g^\gamma(x, \xi) - d_g(x, \xi)^{2\gamma-n})|_g &\leq C d_g(x, \xi)^{2\gamma-n}, \\ |\nabla^2(G_g^\gamma(x, \xi) - d_g(x, \xi)^{2\gamma-n})|_g &\leq C d_g(x, \xi)^{2\gamma-n-1}, \\ |\nabla^3(G_g^\gamma(x, \xi) - d_g(x, \xi)^{2\gamma-n})|_g &\leq C d_g(x, \xi)^{2\gamma-n-2}. \end{aligned}$$

Moreover, if $R_h \geq 0$ and $\mathcal{Q}_h^\gamma \geq 0$ and $\mathcal{Q}_h^\gamma \not\equiv 0$ for some $h \in [h]$, then $G_g^\gamma > 0$ by [Case and Chang 2016].

2B. Energy and bubble for Type I. Suppose that $\gamma \in (0, 1)$ and $(\bar{X}^{n+1}, g, \rho, m_0)$ is a geodesic SMMS, where ρ is the geodesic defining function for a representative metric h . Define a Yamabe energy on $\bar{X}$ as

$$(2-15) \quad \bar{\mathcal{E}}_h^\gamma[U] = \frac{\kappa_\gamma \mathcal{Q}_\gamma(U, U)}{(\oint_M |U|^{2n/(n-2\gamma)} d\sigma_h)^{(n-2\gamma)/n}}$$

for any $U \in W^{1,2}(X, \rho^{m_0} d\mu_g)$ such that $U \not\equiv 0$ on M . See the precise value of κ_γ in Section 2D. Then $\bar{\mathcal{E}}_h^\gamma[f] \leq \bar{\mathcal{E}}_h^\gamma[U]$ for any U having the expansion (2-3). Denote

$N = n + 1$ and $\mathbb{R}_+^N = \{x = (\bar{x}, x_N) \mid \bar{x} \in \mathbb{R}^n, x_N > 0\}$. Recall the Sobolev trace inequality on $\mathbb{R}_+^N$ (see [Lieb 1983; Cotsiolis and Tavoularis 2004])

$$(2-16) \quad \left(\int_{\mathbb{R}^n} |U(\bar{x}, 0)|^{2n/(n-2\gamma)} d\bar{x} \right)^{(n-2\gamma)/n} \leq S_{n,\gamma} \int_0^\infty \int_{\mathbb{R}^n} x_N^{1-2\gamma} |\nabla U(\bar{x}, x_N)|^2 d\bar{x} dx_N$$

where $S_{n,\gamma}$ denotes the optimal constant; for instance, see [González and Qing 2013, Corollary 5.3]. Check our notations, Section 2D, for precise value.

It is known that the above equality is attained by $U = c W_{\varepsilon,\sigma}$ for any $c \in \mathbb{R}$, $\varepsilon > 0$ and $\sigma \in \mathbb{R}^n = \partial\mathbb{R}_+^N$, where $W_{\varepsilon,\sigma}$ are the *bubbles* defined as

$$(2-17) \quad W_{\varepsilon,\sigma}(\bar{x}, x_N) = p_{n,\gamma} \int_{\mathbb{R}^n} \frac{x_N^{2\gamma}}{(|\bar{x} - \bar{y}|^2 + x_N^2)^{(n+2\gamma)/2}} w_{\varepsilon,\sigma}(\bar{y}) d\bar{y}$$

with

$$w_{\varepsilon,\sigma}(\bar{x}) := \alpha_{n,\gamma} \left(\frac{\varepsilon}{\varepsilon^2 + |\bar{x} - \sigma|^2} \right)^{(n-2\gamma)/2} = W_{\varepsilon,\sigma}(\bar{x}, 0).$$

Here $p_{n,\gamma}$ is some constant such that

$$p_{n,\gamma} \int_{\mathbb{R}^n} \frac{x_N^{2\gamma}}{(|\bar{x} - \bar{y}|^2 + x_N^2)^{(n+2\gamma)/2}} d\bar{y} = 1.$$

We choose $\alpha_{n,\gamma}$ such that the fractional curvature of $w_{\varepsilon,\sigma}^{4/(n-2\gamma)} |dx|^2$ is 1. The precise values $p_{n,\gamma}$ and $\alpha_{n,\gamma}$ can be found in (2-26) in the following. We know that $W_{\varepsilon,\sigma}$ satisfies

$$(2-18) \quad \begin{cases} \Delta_{m_0} W_{\varepsilon,\sigma} = 0 & \text{in } \mathbb{R}_+^N, \\ -\kappa_\gamma \lim_{x_N \rightarrow 0+} x_N^{1-2\gamma} \partial_N W_{\varepsilon,\sigma} = (-\Delta)^\gamma w_{\varepsilon,\sigma} = w_{\varepsilon,\sigma}^{(n+2\gamma)/(n-2\gamma)} & \text{on } \mathbb{R}^n. \end{cases}$$

Here $\Delta_{m_0} = \Delta + m_0 x_N^{-1} \partial_N$ is the weighted Laplacian on $\mathbb{R}_+^N$ and κ_γ is a harmless constant; see (2-26). For simplicity, let us denote $W_\varepsilon = W_{\varepsilon,0}$ and $w_\varepsilon = w_{\varepsilon,0}$. Then it is easy to see

$$w_\varepsilon(\varepsilon \bar{x}) = \varepsilon^{-(n-2\gamma)/2} w_1(\bar{x}), \quad W_\varepsilon(\varepsilon \bar{x}, \varepsilon x_N) = \varepsilon^{-(n-2\gamma)/2} W_1(\bar{x}, x_N).$$

Using Lemma A.1 in the Appendix, for any nonnegative integer $k \geq 0$, one can calculate

$$(2-19) \quad \int_{B_+^N(0,\delta)} x_N^{1-2\gamma} [|x|^{k+2} |\nabla W_\varepsilon|^2 + |x|^k W_\varepsilon^2] dx \leq C_{n,\gamma} \begin{cases} \varepsilon^{k+2} & \text{if } n - 2\gamma - k - 2 > 0, \\ \varepsilon^{k+2} \log(\delta/\varepsilon) & \text{if } n - 2\gamma - k - 2 = 0, \\ \varepsilon^{k+2} (\delta/\varepsilon)^{2\gamma+2+k-n} & \text{if } n - 2\gamma - k - 2 < 0, \end{cases}$$

for any $0 < 2\varepsilon \leq \delta < 1$.

2C. Energy and bubble for Type II. Suppose that $\gamma \in (1, \min\{2, \frac{n}{2}\})$ and that $(\bar{X}^{n+1}, g, \rho, m_1)$ is a geodesic SMMS, where ρ is the geodesic defining function for a representative metric h . Define a Yamabe energy on $\bar{X}$ as

$$(2-20) \quad \bar{\mathcal{E}}_h^\gamma[U] = \frac{\kappa_\gamma Q_\gamma(U, U)}{(\oint_M |U|^{2n/(n-2\gamma)} d\sigma_h)^{(n-2\gamma)/n}}.$$

for any $U \in W^{2,2}(X, \rho^{m_1} d\mu_g)$ such that $U \not\equiv 0$ on M . Then $\mathcal{E}_h^\gamma[f] \leq \bar{\mathcal{E}}_h^\gamma[U]$ for any U has the expansion (2-9).

We also have the Sobolev trace inequality for $\gamma \in (1, \min\{2, \frac{n}{2}\})$; see [Chang and González 2011; Case 2017]:

$$(2-21) \quad \left(\int_{\mathbb{R}^n} |U(\bar{x}, 0)|^{2n/(n-2\gamma)} d\bar{x} \right)^{(n-2\gamma)/n} \leq S_{n,\gamma} \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} |\Delta_{m_1} U(\bar{x}, x_N)|^2 d\bar{x} dx_N$$

where $S_{n,\gamma}$ is the optimal constant. It is also known that the equality is achieved by the *bubbles* (2-17). In this case, however, $W_{\varepsilon,\sigma}$ satisfies

$$(2-22) \quad \begin{cases} \Delta_{m_1}^2 W_{\varepsilon,\sigma} = 0 & \text{in } \mathbb{R}_+^N, \\ W_{\varepsilon,\sigma} = w_{\varepsilon,\sigma} & \text{on } \mathbb{R}^n, \\ \lim_{x_N \rightarrow 0+} x_N^{m_1} \partial_N W_{\varepsilon,\sigma} = 0 & \text{on } \mathbb{R}^n, \\ (-\Delta)^\gamma w_{\varepsilon,\sigma} = \kappa_\gamma \lim_{x_N \rightarrow 0+} x_N^{m_1} \partial_N \Delta_{m_1} W_{\varepsilon,\sigma} = w_{\varepsilon,\sigma}^{(n+2\gamma)/(n-2\gamma)} & \text{on } \mathbb{R}^n. \end{cases}$$

Here $\Delta_{m_1} = \Delta + m_1 x_N^{-1} \partial_N$ is the weighted Laplacian on $\mathbb{R}_+^N$ and κ_γ can be seen in (2-26). Moreover it also satisfies $\Delta_{m_0} W_{\varepsilon,\sigma} = 0$, which is

$$(2-23) \quad \Delta_{m_1} W_{\varepsilon,\sigma} = 2x_N^{-1} \partial_N W_{\varepsilon,\sigma} \quad \text{in } \mathbb{R}_+^N.$$

Using Lemma A.1, for any integer $k \geq 0$ and $0 < 2\varepsilon \leq \delta < 1$, one has

$$(2-24) \quad \int_{B_+^N(0,\delta)} x_N^{3-2\gamma} [|x|^k W_\varepsilon^2 + |x|^{k+2} |\nabla W_\varepsilon|^2 + |x|^{k+4} |\partial_{ij} W_\varepsilon|^2] dx \leq C_{n,\gamma} \begin{cases} \varepsilon^{k+4} & \text{if } n - 2\gamma - k - 4 > 0, \\ \varepsilon^{k+4} \log(\delta/\varepsilon) & \text{if } n - 2\gamma - k - 4 = 0, \\ \varepsilon^{k+4} (\delta/\varepsilon)^{2\gamma+4+k-n} & \text{if } n - 2\gamma - k - 4 < 0. \end{cases}$$

2D. Notations. The following notations are used throughout this paper:

- (1) Let $N = n + 1$. For $x \in \mathbb{R}_+^N := \{(x_1, \dots, x_n, x_N) \in \mathbb{R}^N : x_N > 0\}$, we write $\bar{x} = (x_1, \dots, x_n, 0) \in \partial\mathbb{R}_+^N \simeq \mathbb{R}^n$ and $r = |\bar{x}|$. The indices i, j, k run from 1 to n .
- (2) $B_+^N(0, \delta)$ is an open ball in $\mathbb{R}_+^N$ and $D(0, \delta)$ is an open ball in $\mathbb{R}^n$.
- (3) $m_0 = 1 - 2\gamma$ and $m_1 = 3 - 2\gamma$.

(4) Some positive constants for $0 < 2\gamma < n$ (see [Chang and González 2011])

$$(2-25) \quad d_\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}, \quad \kappa_\gamma = \frac{\Gamma(\gamma - \lfloor \gamma \rfloor)}{\Gamma(\gamma + 1)} \frac{(-1)^{\lfloor \gamma \rfloor + 1} d_\gamma}{2^{2\lfloor \gamma \rfloor + 1} (\lfloor \gamma \rfloor)!} > 0.$$

here $\lfloor \gamma \rfloor$ is the greatest integer less than or equal to γ . One can see that

$$\kappa_\gamma = -\frac{d_\gamma}{2\gamma} \text{ if } \gamma \in (0, 1) \quad \text{and} \quad \kappa_\gamma = \frac{d_\gamma}{8\gamma(\gamma - 1)} \text{ if } \gamma \in (1, \min\{2, \frac{n}{2}\}).$$

The following positive constant are also used for $0 < 2\gamma < n$

$$(2-26) \quad S(n, \gamma) = \kappa_\gamma \frac{\Gamma((n - 2\gamma)/2)}{\Gamma((n + 2\gamma)/2)} |\text{vol}(\mathbb{S}^n)|^{-2\gamma/n}, \quad p_{n,\gamma} = \frac{\Gamma((n + 2\gamma)/2)}{\pi^{n/2} \Gamma(\gamma)}$$

$$\alpha_{n,\gamma} = [S(n, \gamma)^{-1} \kappa_\gamma]^{(n-2\gamma)/(4\gamma)} \left(2^{n-1} \pi^{-(n+1)/2} \Gamma\left(\frac{n+1}{2}\right) \right)^{(n-2\gamma)/(2n)}.$$

(5) The fractional Yamabe constant for sphere

$$(2-27) \quad \mathcal{Y}_{\mathbb{S}^n}^\gamma = \mathcal{Y}^\gamma(\mathbb{S}^n, [g_c]) = S_{n,\gamma}^{-1} \kappa_\gamma = \left(\int_{\mathbb{R}^n} w_{\varepsilon,\sigma}^{(2n)/(n-2\gamma)} d\bar{x} \right)^{2\gamma/n}.$$

Equivalently,

$$\int_{\mathbb{R}^n} w_{\varepsilon,\sigma}^{2n/(n-2\gamma)} d\bar{x} = (\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)}.$$

(6) χ is a cut-off function has support in $B_+^N(0, 2\delta)$ and $\chi = 1$ in $B_+^N(0, \delta)$ and

$$(2-28) \quad \chi_\delta = \chi(|x|^2/\delta)$$

(7) Volume element on X is $d\mu_g$ and on M is $d\sigma_h$.

3. Energy estimates for the Case (I-1)

In this section, we will derive the energy estimates for (I-1). This type of estimates will be used in Lemma 5.5 in the following.

Assume that $(\bar{X}^{n+1}, g, \rho, m_0)$ is a geodesic SMMS, where ρ is the geodesic defining function for a representative metric h . Given any $a \in M$, there exists a Fermi coordinates $\Psi_a : \mathcal{O}(a) \rightarrow B_+^N(0, 2\delta)$ on some neighborhood $\mathcal{O}(a) \subset X$. One can identify $\mathcal{O}(a)$ and $B_+^N(0, 2\delta)$ through $\Psi_a = (\bar{x}, x_N)$. It follows from [Kim et al. 2018, Lemma 2.2 and 2.4] that the following expansion of metric holds near 0:

$$(3-1) \quad g^{ij}(x) = \delta_{ij} + \frac{1}{3} R_{ikjl}[h] x_k x_l + R_{iNjN}[g] x_N^2 + O(|x|^3).$$

$$\sqrt{|g|}(x) = 1 + O(|x|^3) \quad \text{in } B_+^N(0, 2\delta).$$

Here $R_{ikjl}[h]$ is a component of the Riemannian curvature tensor on M , $R_{iNjN}[g]$ is that of the Riemannian curvature tensor in X . Every tensor in the expansions is computed at $a = 0$. Here we implicitly use the fact that $(M, h) \subset (\bar{X}, g)$ is totally geodesic. Let $C_0\varepsilon < \delta \leq \delta_0 \leq 1$. Denote

$$(3-2) \quad U_{a,\varepsilon,\delta}(x) = \chi_\delta W_\varepsilon(\Psi_a(x)) + (1 - \chi_\delta(\Psi_a(x)))\varepsilon^{(n-2\gamma)/2} G_g^\gamma,$$

where χ_δ is defined in (2-28), G_g^γ is the Green's function.

Proposition 3.1. *Suppose that $\gamma \in (0, 1)$ and $n < 4 + 2\gamma$. For $U_{a,\varepsilon,\delta}$ defined in (3-2), if δ_0 small enough and C_0 large enough, there exists a constant $C_1 > 0$ such that*

$$\mathcal{E}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \bar{\mathcal{E}}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \mathcal{Y}_{\mathbb{S}^n}^\gamma + \varepsilon^{n-2\gamma} C_1(n, \gamma, g, \delta) + o(\varepsilon^{n-2\gamma}).$$

Proof. The first inequality follows from the fact that $U_{a,\varepsilon,\delta}$ has the right expansion (2-3). Therefore we just need to justify the second inequality. Notice the above inequality echos the fact that this is a global case.

We adopt the notation $\mathcal{Q}(U : \Omega)$ is (2-4) meaning the integration over some set $\Omega \subset X$. Then

$$\begin{aligned} \mathcal{Q}_\gamma(U_{a,\varepsilon,\delta}) &= \mathcal{Q}_\gamma(W_\varepsilon : B_+^N(0, \delta)) + \mathcal{Q}_\gamma(U_{a,\varepsilon,\delta} : B_+^N(0, 2\delta) \setminus B_+^N(0, \delta)) \\ &\quad + \mathcal{Q}_\gamma(\varepsilon^{(n-2\gamma)/2} G_g^\gamma : X \setminus \mathcal{O}_a). \end{aligned}$$

Using the estimates in (2-8), one obtains

$$\begin{aligned} \mathcal{Q}_\gamma(\varepsilon^{(n-2\gamma)/2} G_a^\gamma : X \setminus \mathcal{O}_a) &= \varepsilon^{n-2\gamma} \int_{X \setminus \mathcal{O}_a} \left(|\nabla G_a^\gamma|_g^2 + \frac{n-2\gamma}{2} J(G_a^\gamma)^2 \right) \rho^{m_0} d\mu_g \\ &\leq C \varepsilon^{n-2\gamma} \delta^{2\gamma-n}. \end{aligned}$$

Here $C = C(n, \gamma, g)$. Similarly, by the estimates of W_ε in Lemma A.1, we also get

$$\mathcal{Q}_\gamma(U_{a,\varepsilon,\delta} : B_+^N(0, 2\delta) \setminus B_+^N(0, \delta)) \leq C \varepsilon^{n-2\gamma} \delta^{2\gamma-n}.$$

For the first term in $\mathcal{Q}_\gamma(U)$, applying (2-19)

$$\begin{aligned} (3-3) \quad \mathcal{Q}_\gamma(W_\varepsilon : B_+^N(0, \delta)) &= \int_{B_+^N(0, \delta)} x_N^{1-2\gamma} \left(|\nabla W_\varepsilon|_g^2 + \frac{n-2\gamma}{2} J W_\varepsilon^2 \right) d\mu_g \\ &\leq \int_{B_+^N(0, \delta)} x_N^{1-2\gamma} |\nabla W_\varepsilon|_g^2 dx + C \varepsilon^{n-2\gamma} \delta^{2\gamma-n}. \end{aligned}$$

The first term in the last inequality can be estimated by (3-1) and (2-19)

$$\begin{aligned}
 (3-4) \quad & \int_{B_+^N(0,\delta)} x_N^{1-2\gamma} |\nabla W_\varepsilon|_g^2 dx \\
 &= \int_{B_+^N(0,\delta)} x_N^{1-2\gamma} |\nabla W_\varepsilon|^2 dx + \varepsilon^2 R_{iNjN}[g] \int_{B_+^N(0,\delta/\varepsilon)} x_N^{3-2\gamma} \partial_i W_1 \partial_j W_1 dx \\
 &\quad + O(\varepsilon^3 (\delta/\varepsilon)^{2\gamma+3-n}) \\
 &\leq \int_{B_+^N(0,\delta)} x_N^{1-2\gamma} |\nabla W_\varepsilon|^2 dx + C \delta^{2\gamma+2-n} \varepsilon^{n-2\gamma},
 \end{aligned}$$

where $n < 2\gamma + 2$ is used. It follows from (2-18) and $x \cdot \nabla W_1 \leq 0$ for $x \in \mathbb{R}_+^N$ that

$$\begin{aligned}
 \int_{B_+^N(0,\delta)} x_N^{1-2\gamma} |\nabla W_\varepsilon|^2 dx &\leq k_\gamma^{-1} \int_{D(0,\delta)} w_\varepsilon^{(2n)/(n-2\gamma)} d\bar{x} \\
 &\leq S_{n,\gamma}^{-1} \left(\int_{D(0,\delta)} w_\varepsilon^{(2n)/(n-2\gamma)} d\bar{x} \right)^{(n-2\gamma)/n}
 \end{aligned}$$

where the last inequality follows from (2-27). On the other hand,

$$\begin{aligned}
 (3-5) \quad & \oint_M U_{a,\varepsilon,\delta}^{(2n)/(n-2\gamma)} d\sigma_h \geq \int_{D(0,\delta)} w_\varepsilon^{(2n)/(n-2\gamma)} d\sigma_h \\
 &\geq \int_{D(0,\delta)} w_\varepsilon^{(2n)/(n-2\gamma)} d\bar{x} - C \varepsilon^n \delta^{-n}.
 \end{aligned}$$

Putting all estimates back to the expression of (2-15), one could get the conclusion by taking ε small enough. $\square$

4. Energy estimates for Type II

In this section, we will study the energy estimates for $\gamma \in (1, \min\{2, \frac{n}{2}\})$. Again, we need the expansion of metric.

Lemma 4.1. *Suppose (X^{n+1}, M^n, g_+) is a Poincaré–Einstein manifold with conformal infinity $(M, [h])$. For a fixed point $a \in M$, there exist a representative $h = h_a$ of the class $[h]$, and the geodesic defining function ρ_a near M such that the metric $g = \rho_a^2 g_+$ in terms of Fermi coordinates around a has the following expansions*

$$\begin{aligned}
 (4-1) \quad \sqrt{|g|}(\bar{x}, x_N) &= 1 - \frac{1}{2} \text{Ric}[g]_{NN;i} x_N^2 x_i - \frac{1}{4} \text{Ric}[g]_{NN;ij} x_N^2 x_i x_j \\
 &\quad - \frac{1}{6} \text{Ric}[g]_{NN;Ni} x_N^3 x_i + O(|x|^5)
 \end{aligned}$$

and

$$(4-2) \quad g^{ij}(\bar{x}, x_N) = \delta_{ij} + \frac{1}{3}R[h]_{ikjl}x_kx_l + \frac{1}{6}R[h]_{ikjl;m}x_kx_lx_m + R[g]_{iNjN;k}x_N^2x_k \\ + \left(\frac{1}{20}R[h]_{ikjl;m} + \frac{1}{15}R[h]_{iksl}R[h]_{jmsq}\right)x_kx_lx_mx_q \\ + \frac{1}{2}R[g]_{iNjN;kl}x_N^2x_kx_l + \frac{1}{12}R[g]_{iNjN;NN}x_N^4 + O(|x|^5)$$

near a . Here all tensors are computed at a and the indices i, j, k, m, q, s run from 1 to n . Moreover, one has the following relations of the curvature:

- (1) $\text{Ric}[h]_{ij;k}(a) + \text{Ric}[h]_{jk;i}(a) + \text{Ric}[h]_{ki;j}(a) = 0$.
- (2) $\pi = 0$ on M , $\text{Sym}_{ijkl}(\text{Ric}[h]_{ij;kl} + \frac{2}{9}R[h]_{miqj}R[h]_{mkql})(a) = 0$.
- (3) $\text{Ric}[g]_{NN;N}(a) = \text{Ric}[g]_{aN}(y) = \text{Ric}[g]_{NN;NN}(a) = R[g]_{;NN}(a) = 0$.
- (4) $R[g]_{iNjN}(a) = \text{Ric}[g]_{ij}(a) = 0$.
- (5) $R[g]_{;ii}(a) = -n\|W[h]\|^2/(6(n-1))$, $\text{Ric}[g]_{NN;ii}(a) = R[g]_{iNjN;ij}(a) = -\|W[h]\|^2/(12(n-1))$.

Here $\|W[h]\|$ is the norm of the Weyl tensor of (M, h) at a .

Proof. The expansion (4-1) and (4-2) were first found by Marques [2005] in the boundary Yamabe problem. González and Wang [2018] and Kim et al. [2018] adapted them to the fractional case. Here we are just simplifying their expansion by using the fact that (X^{n+1}, M^n, g_+) is a P-E manifold. $\square$

The expansion of Ricci tensor in Fermi coordinates:

Lemma 4.2. *Suppose that $(M^n, h) \subset (\bar{X}^{n+1}, g)$ is a totally geodesic. In the Fermi coordinates around $a \in M$, the Ricci tensor $\text{Ric}[g]_{ij}$ has the following expansion,*

$$\text{Ric}[g]_{ij}(\bar{x}, x_N) \\ = \text{Ric}[g]_{ij} + (\text{Ric}[h]_{ij;k} + \text{Rm}[g]_{iNjN;k})x_k + \text{Ric}[g]_{ij;N}x_N \\ + \text{Ric}[g]_{ij;Nk}x_kx_N + \left(\frac{1}{2}\text{Ric}[g]_{ij;NN} - 2\text{Sym}_{ij}(\text{Ric}[g]_{jl}\text{Rm}[g]_{iNlN})\right)x_N^2 \\ + \left(\frac{1}{2}\text{Ric}[g]_{ij;kl} - \frac{1}{3}\text{Sym}_{ij}(\text{Rm}[h]_{iksl}\text{Rm}[g]_{sNjN})\right)x_kx_l + O(|x|^3)$$

where the tensor on the right hand side are all evaluated at 0 and $1 \leq i, j, k, l, s \leq n$. For the other component of $\text{Ric}[g]$, we have $\text{Ric}[g]_{iN}(\bar{x}, x_N) = 0$ and

$$\text{Ric}[g]_{NN}(\bar{x}, x_N) = \text{Ric}[g]_{NN} + \text{Ric}[g]_{NN;i}x_i + \text{Ric}[g]_{NN;N}x_N + \frac{1}{2}\text{Ric}[g]_{NN;ij}x_ix_j \\ + \text{Ric}[g]_{NN;Ni}x_ix_N + \frac{1}{2}\text{Ric}[g]_{NN;NN}x_N^2 + O(|x|^3)$$

Proof. It follows from the Taylor expansion that

$$\text{Ric}[g]_{ij}(\bar{x}, x_N) \\ = \text{Ric}[g]_{ij}(\bar{x}, 0) + \partial_N \text{Ric}[g]_{ij}(\bar{x}, 0)x_N + \frac{1}{2}\partial_{NN}^2 \text{Ric}[g]_{ij}(\bar{x}, 0)x_N^2 + O(|x|^3)$$

For the first term, we have $\text{Ric}[g]_{ij}(\bar{x}, 0) = \text{Ric}[h]_{ij}(\bar{x}, 0) + R[g]_{iNjN}(\bar{x}, 0)$. Since $(\bar{x}, 0)$ is a geodesic normal coordinates of a on M , then $\text{Ric}[h]_{ij}(0) = 0$ and [Marques 2005, Lemma 2.1] imply

$$\text{Ric}[h]_{ij}(\bar{x}, 0) = \text{Ric}[h]_{ij;k}(0)x_k + \frac{1}{2} \text{Ric}[h]_{ij;kl}(0)x_k x_l + O(|\bar{x}|^3).$$

Thanks to the fact that M is totally geodesic

$$\begin{aligned} \text{Rm}[g]_{iNjN}(\bar{x}, 0) &= \text{Rm}[g]_{iNjN}(0) + \text{Rm}[g]_{iNjN;k}(0)x_k \\ &\quad + \left(\frac{1}{2}R[g]_{iNjN;kl} - \frac{1}{3}\text{Sym}_{ij}\text{Rm}[h]_{iksl}\text{Rm}[g]_{sNjN}\right)x_k x_l \\ &\quad + O(|\bar{x}|^3), \end{aligned}$$

$$\begin{aligned} \partial_N \text{Ric}[g]_{ij}(\bar{x}, 0) &= \text{Ric}[g]_{ij,N}(\bar{x}, 0) \\ &= \text{Ric}[g]_{ij;N}(0) + \text{Ric}[g]_{ij;Nk}(0)x_k + O(|\bar{x}|^2). \end{aligned}$$

For the same reason that M is totally geodesic,

$$\begin{aligned} \partial_{NN}^2 \text{Ric}[g]_{ij}(\bar{x}, 0) &= \text{Ric}[g]_{ij;NN}(\bar{x}, 0) - 2\text{Sym}_{ij}(\text{Ric}[g]_{jk}\text{Rm}[g]_{iNkN}(\bar{x}, 0)) + O(|\bar{x}|). \end{aligned}$$

Collecting all the above expansion, one can get the expansion of $\text{Ric}[g]_{ij}$. It follows from Codazzi equation that

$$\text{Ric}[g]_{iN} = \pi_{jj;i} - \pi_{ij;j} = 0.$$

For $\text{Ric}[g]_{NN}$, one can do the expansion as $\text{Ric}[g]_{ij}$. □

4A. Case (II-1): Low dimension and Case (II-2): Locally conformally flat. Suppose $C_0\epsilon \leq \delta < \delta_0 \leq 1$. Define

$$(4-3) \quad U_{a,\epsilon,\delta}(x) = \chi_\delta W_\epsilon(\Psi_a(x)) + (1 - \chi_\delta(\Psi_a(x)))\epsilon^{(n-2\gamma)/2}G_a^\gamma$$

where χ_δ is defined in (2-28) and $G_a^\gamma = G_{g_a}^\gamma$ is defined in (2-13).

Proposition 4.3. *Suppose $\gamma \in (1, \min\{2, \frac{n}{2}\})$ and $n < 4 + 2\gamma$. If δ_0 small enough and C_0 large enough, then there exist a constant $C_2 > 0$ such that*

$$(4-4) \quad \mathcal{E}_h^\gamma[U_{a,\epsilon,\delta}] \leq \bar{\mathcal{E}}_h^\gamma[U_{a,\epsilon,\delta}] \leq \mathcal{Y}_{\mathbb{S}^n}^\gamma + \epsilon^{n-2\gamma}C_2(n, \gamma, g, \delta) + o(\epsilon^{n-2\gamma}).$$

Proof. Suppose ρ is the geodesic defining function for h , then

$$\lim_{\rho \rightarrow 0} \rho^{m_1} \partial_\rho U_{a,\epsilon,\delta} = 0.$$

Then $U_{a,\epsilon,\delta}$ satisfies (2-9). It follows from Proposition 2.3 that $\mathcal{E}_h^\gamma[U_{a,\epsilon,\delta}] \leq \bar{\mathcal{E}}_h^\gamma[U_{a,\epsilon,\delta}]$. Therefore we just need to prove the second inequality. Using the

estimates of W_ε in Lemma A.1 and G_a^γ in (2-14), one can get

$$(4-5) \quad \begin{aligned} \mathcal{Q}_\gamma(U_{a,\varepsilon,\delta}) &\leq \int_{B_+^N(0,\delta)} (\Delta_\rho^{m_1} W_\varepsilon)^2 x_N^{m_1} dx + \frac{n-2\gamma}{2} \int_{B_+^N(0,\delta)} \mathcal{Q}_\rho^{m_1} W_\varepsilon^2 x_N^{m_1} dx \\ &\quad - \int_{B_+^N(0,\delta)} 4(P - (n-2\gamma+2)Jg)(\nabla W_\varepsilon, \nabla W_\varepsilon) x_N^{m_1} dx \\ &\quad + C\varepsilon^{n-2\gamma} \delta^{2\gamma-n} \end{aligned}$$

similar to the argument in Proposition 3.1. Noticing that

$$\Delta_\rho^{m_1} W_\varepsilon = \Delta_g W_\varepsilon + m_1 x_N^{-1} \partial_N W_\varepsilon = \Delta_{m_1} W_\varepsilon + (\Delta_g - \Delta_{\mathbb{R}^{n+1}}) W_\varepsilon$$

and it follows from the expansion of metric (4-2) that

$$(4-6) \quad (\Delta_g - \Delta_{\mathbb{R}^{n+1}}) W_\varepsilon = O(|x|^2) |\nabla_{\bar{x}}^2 W_\varepsilon| + O(|x|) |\nabla W_\varepsilon|.$$

From the estimates in Lemmas A.1, A.5 and (2-24)

$$\begin{aligned} \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 dx &\leq \int_{B_+^N(0,\delta)} x_N^{m_1} [(\Delta_{m_1} W_\varepsilon)^2 + C|x|^2 |\nabla W_\varepsilon|^2] dx \\ &\leq \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_{m_1} W_\varepsilon)^2 dx + C\varepsilon^{n-2\gamma} \delta^{2\gamma+4-n} \end{aligned}$$

where $n < 2\gamma + 4$ is used. It follows from (2-22) and integration by parts that

$$\begin{aligned} \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_{m_1} W_\varepsilon)^2 dx &= \int_{D(0,\delta)} \lim_{x_N \rightarrow 0} x_N^{m_1} [(\partial_N \Delta_{m_1} W_\varepsilon) W_\varepsilon - \Delta_{m_1} W_\varepsilon \partial_N W_\varepsilon] \\ &\quad - \int_{\partial^+ B_+^N(0,\delta)} x_N^{m_1} (\partial_\nu \Delta_{m_1} W_\varepsilon) W_\varepsilon \\ &\quad + \int_{\partial^+ B_+^N(0,\delta)} x_N^{m_1} \Delta_{m_1} W_\varepsilon \partial_\nu W_\varepsilon \end{aligned}$$

where ν is the outer unit normal of $\partial^+ B_+^N(0, \delta) = \partial B_+^N(0, \delta) \cap \mathbb{R}_+^N$. One can get from (2-17) that $\partial_\nu W_\varepsilon < 0$ and $\partial_\nu \Delta_{m_1} W_\varepsilon > 0$. Then the above equality implies

$$(4-7) \quad \begin{aligned} \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_{m_1} W_\varepsilon)^2 dx &\leq \kappa_\gamma^{-1} \int_{D(0,\delta)} w_\varepsilon^{(n+2\gamma)/(n-2\gamma)} d\bar{x} \\ &\leq S_{n,\gamma}^{-1} \left(\int_{D(0,\delta)} w_\varepsilon^{2n/(n-2\gamma)} d\bar{x} \right)^{(n-2\gamma)/n}. \end{aligned}$$

The following fact of scalar curvature at 0 can be derived from Lemma 4.1

$$(4-8) \quad R[g] = R[g]_{;i} = R[g]_{;N} = R[g]_{;NN} = 0, \quad R[g]_{;ii} = -\frac{n \|W[h]\|^2}{6(n-1)},$$

then

$$(4-9) \quad \int_{B_+^N(0,\delta)} x_N^{m_1} R[g] |\nabla W_\varepsilon|_g^2 dx \leq C \varepsilon^{n-2\gamma} \delta^{2\gamma+6-n}.$$

Using the symmetry of W_ε and (4-8) and $\text{Ric}[g]_{NN;N}(0) = 0$, and Lemma 4.2

$$\begin{aligned} \int_{B_+^N(0,\delta)} x_N^{m_1} \text{Ric}[g](\nabla W_\varepsilon, \nabla W_\varepsilon) dx &= \int_{B_+^N(0,\delta)} x_N^{m_1} O(|x|^2 |\nabla W_\varepsilon|^2) dx \\ &\leq C \varepsilon^{n-2\gamma} \delta^{2\gamma+4-n}. \end{aligned}$$

Notice

$$J[g] = \frac{1}{2n} R[g] \quad \text{and} \quad P[g] = \frac{1}{n-1} (\text{Ric}[g] - J[g]).$$

We obtain

$$\int_{B_+^N(0,\delta)} 4(P - (n-2\gamma+2)Jg)(\nabla W_\varepsilon, \nabla W_\varepsilon) x_N^{m_1} dx \leq C \varepsilon^{n-2\gamma} \delta^{2\gamma+4-n}.$$

It is easy to see that

$$\int_{B_+^N(0,\delta)} x_N^{m_1} Q_\rho^{m_1} W_\varepsilon^2 dx \leq C \int_{B_+^N(0,\delta)} x_N^{m_1} W_\varepsilon^2 dx \leq C \varepsilon^{n-2\gamma} \delta^{2\gamma+4-n}.$$

Putting everything back to (2-20) and using (3-5) obtains

$$\begin{aligned} \bar{\mathcal{E}}_h^\gamma[U_{a,\varepsilon,\delta}] &\leq \kappa_\gamma S_{n,\gamma}^{-1} - C \varepsilon^{n-2\gamma} \delta^{2\gamma-n} + o(\varepsilon^{n-2\gamma}) \\ &= \mathcal{Y}_{\mathbb{S}^n}^\gamma + \varepsilon^{n-2\gamma} \mathcal{C}_2(n, \gamma, g, \delta) + o(\varepsilon^{n-2\gamma}). \end{aligned} \quad \square$$

Now suppose $(M^n, [h])$ is locally conformally flat. Then pick any point $a \in M$, there exists a neighborhood of a in M that can be identify with a Euclidean ball $D(0, \delta)$, that is $h_{ij} = \delta_{ij}$ in $D(0, \delta)$. Then in a neighborhood of a in X , identified with $B_+^N(0, \delta)$, the metric reads (see [Mayer and Ndiaye 2017b; Kim et al. 2018])

$$(4-10) \quad g_{ij}(\bar{x}, x_N) = \delta_{ij} + O(x_N^n) \quad \text{and} \quad |g| = 1 + O(x_N^n)$$

for $(\bar{x}, x_N) \in B_+^N(0, \delta)$.

Proposition 4.4. *Suppose that $(M^n, [h])$ is locally conformally flat and $\gamma \in (1, \min\{2, \frac{n}{2}\})$. If δ_0 small enough and C_0 large enough, then there exists some $C_3 > 0$ such that*

$$\mathcal{E}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \bar{\mathcal{E}}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \mathcal{Y}_{\mathbb{S}^n}^\gamma + \varepsilon^{n-2\gamma} \mathcal{C}_3(n, \gamma, g, \delta) + o(\varepsilon^{n-2\gamma}).$$

where $U_{a,\varepsilon,\delta}$ is defined in (4-3) for $0 < C_0 \varepsilon \leq \delta \leq \delta_0 \leq 1$.

Proof. The proof is similar to the one of Proposition 4.3, but the calculation is much more simpler because g_{ij} is almost Euclidean. We just highlight some differences. For the same reason we can obtain (4-5). However, (4-6) will be replaced by

$$(\Delta_g - \Delta_{\mathbb{R}^{n+1}})W_\varepsilon = O(|x|^n)|\nabla^2 W| + O(|x|^{n-1})|\nabla W_\varepsilon|,$$

since (4-10). This implies,

$$\begin{aligned} \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 dx &\leq \int_{B_+^N(0,\delta)} x_N^{m_1} [(\Delta_{m_1} W_\varepsilon)^2 + C|x|^{2n-2}|\nabla W_\varepsilon|^2] dx \\ &\leq \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_{m_1} W_\varepsilon)^2 dx + C\varepsilon^{n-2\gamma}\delta^{-2\gamma-n}. \end{aligned}$$

Here we have used (2-24). The rest of the proof will be the same. $\square$

4B. Case (II-3): Nonlocally conformally flat and $n > 2\gamma + 4$. We are going to use a local test function

$$(4-11) \quad U_{a,\varepsilon,\delta}(x) = \chi_\delta W_\varepsilon(\Psi_a(x)).$$

where χ_δ is defined in (2-28) and Ψ_a is the Fermi coordinates.

Theorem 4.5. *Suppose that $\gamma \in (1, \min\{2, \frac{n}{2}\})$ and $n > 4 + 2\gamma$. If the Weyl tensor $W[h]$ at a does not vanish, then there exist $C_4 > 0$ such that*

$$\mathcal{E}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \bar{\mathcal{E}}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \mathcal{Y}_{\mathbb{S}^n}^\gamma - \varepsilon^4 C_4(n, \gamma, g, \delta) \|W[h]\|^2 + o(\varepsilon^4)$$

provided $C_0\varepsilon \leq \delta \leq \delta_0 \leq 1$ for δ_0 small enough and C_0 large enough.

Proof. For the same reason as before, we just need to show the second inequality. Adopting the notation $\mathcal{Q}_\gamma(U : \Omega)$ in (2-10), one has

$$\mathcal{Q}_\gamma(U_{a,\varepsilon,\delta}) = \mathcal{Q}_\gamma(W_\varepsilon : B_+^N(0, \delta)) + \mathcal{Q}_\gamma(U_{a,\varepsilon,\delta} : B_+^N(0, 2\delta) \setminus B_+^N(0, \delta)).$$

To make our proof more clear, we use the following notation:

$$\mathcal{Q}_\gamma(W_\varepsilon : B_+^N(0, \delta)) = \mathcal{T}_1 - \mathcal{T}_2 + \mathcal{T}_3,$$

where

$$\begin{aligned} \mathcal{T}_1 &= \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 d\mu_g, \\ \mathcal{T}_2 &= \int_{B_+^N(0,\delta)} (4P - (n - 2\gamma + 2)J[g]g)(\nabla W_\varepsilon, \nabla W_\varepsilon) d\mu_g, \\ \mathcal{T}_3 &= \frac{n-2\gamma}{2} \int_{B_+^N(0,\delta)} \mathcal{Q}_\rho^{m_1} W_\varepsilon^2 d\mu_g. \end{aligned}$$

Step 1: Consider $\mathcal{T}_1$. Noticing (4-1), one gets

$$\begin{aligned}
 \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 d\mu_g \\
 &= \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 \sqrt{|g|} d\mu_g \\
 &= \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 dx \\
 &\quad - \frac{1}{4n} \text{Ric}[g]_{NN;ii} \int_{B_+^N(0,\delta)} x_N^{m_1+2} r^2 (\Delta_{m_1} W_\varepsilon)^2 dx + o(\varepsilon^4).
 \end{aligned}$$

Since $4 + 2\gamma < n$

$$\begin{aligned}
 \int_{B_+^N(0,\delta)} x_N^{m_1+2} r^2 (\Delta_{m_1} W_\varepsilon)^2 dx &= 4 \int_{B_+^N(0,\delta)} x_N^{m_1} r^2 (\partial_N W_\varepsilon)^2 dx \\
 &= 4\varepsilon^4 \int_{\mathbb{R}_+^N} x_N^{m_1} r^2 (\partial_N W_1)^2 dx + o(\varepsilon^4).
 \end{aligned}$$

Introduce the notation (see [Kim et al. 2018, Lemma B.6])

$$(4-12) \quad \mathcal{F}_5 = \int_{\mathbb{R}_+^N} x_N^{m_1} r^2 |\nabla W_\varepsilon|^2 dx, \quad \mathcal{F}_6 = \int_{\mathbb{R}_+^N} x_N^{m_1} r^2 (\partial_r W_\varepsilon)^2 dx.$$

Thus

$$(4-13) \quad \mathcal{T}_1 = \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 dx - \frac{\varepsilon^4}{n} \text{Ric}[g]_{NN;ii} (\mathcal{F}_5 - \mathcal{F}_6) + o(\varepsilon^4).$$

To handle the first term on the RHS, straightforward computation shows

$$\begin{aligned}
 \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 dx &= \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_{m_1} W_\varepsilon + (\Delta_g - \Delta_{\mathbb{R}_+^N}) W_\varepsilon)^2 dx \\
 &\leq \int_{B_+^N(0,\delta)} x_N^{m_1} [(\Delta_{m_1} W_\varepsilon)^2 + 2\Delta_{m_1} W_\varepsilon (\Delta_g - \Delta_{\mathbb{R}_+^N}) W_\varepsilon \\
 &\quad + ((\Delta_g - \Delta_{\mathbb{R}_+^N}) W_\varepsilon)^2] dx \\
 &= \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_{m_1} W_\varepsilon)^2 dx + I_1 + I_2.
 \end{aligned}$$

Applying (4-2), one can notice

$$\begin{aligned}
 (\Delta_g - \Delta_{\mathbb{R}_+^N}) W_\varepsilon &= (g^{ab} - \delta_{ab}) \partial_{ab}^2 W_\varepsilon + \partial_a g^{ab} \partial_b W_\varepsilon + g^{ab} \partial_a \log \sqrt{|g|} \partial_b W_\varepsilon \\
 &= \left[\frac{1}{3} R[h]_{ikjl} x_k x_l \right] \partial_{ij}^2 W_\varepsilon + O(|x|^3) |\nabla_x^2 W_\varepsilon| + O(|x|^2) |\nabla W_\varepsilon|
 \end{aligned}$$

Notice the following fact:

$$(4-14) \quad \partial_{ij}^2 W_1 = \partial_{rr}^2 W_1 \frac{x_i x_j}{r^2} + \partial_r W_1 \left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \right).$$

Using the symmetry of $\partial_{ij}^2 W_\varepsilon$ and the properties in Lemma 4.1, $R[h]_{ikjl} x_k x_l \partial_{ij}^2 W_\varepsilon = 0$. Consequently

$$I_2 = \int_{B_+^N(0, \delta)} x_N^{m_1} [(\Delta_g - \Delta_{\mathbb{R}_+^N}) W_\varepsilon]^2 dx = o(\varepsilon^4).$$

Now consider I_2 . Let $(g^{ij})^{(4)}$ be the fourth-order terms in the expansion (4-2) of g^{ij} .

$$\begin{aligned} I_1 &= 2 \int_{B_+^N(0, \delta)} x_N^{m_1} \Delta_{m_1} W_\varepsilon (g^{ij} - \delta^{ij}) \partial_{ij}^2 W_\varepsilon dx \\ &= 4\varepsilon^4 \int_{\mathbb{R}_+^N} x_N^{m_1-1} \partial_N W_1 (g^{ij})^{(4)} \partial_{ij}^2 W_1 dx + o(\varepsilon^4) \\ &= 2\varepsilon^4 R[g]_{iNjN;kl} \int_{\mathbb{R}_+^N} x_N^{m_1+1} x_k x_l \partial_N W_1 \partial_{ij}^2 W_1 dx \\ &\quad + \frac{\varepsilon^4}{3} R[g]_{iNiN;NN} \int_{\mathbb{R}_+^N} x_N^{m_1+3} \partial_N W_1 \Delta W_1 dx + o(\varepsilon^4). \end{aligned}$$

It follows from Bianchi identity and $R[g]_{NN;NN}(a) = 0$ that $R[g]_{iNiN;NN}(a) = 0$. Therefore the second term in I_1 is equal to 0. Using (4-14) and [Brendle 2008, Corollary 29], one could simplify I_1 as

$$I_1 = \frac{2\varepsilon^4}{n(n+2)} (R[g]_{iNiN;jj} + 2R[g]_{iNjN;ij})(\mathcal{A}_3 - \mathcal{A}_1) + \frac{2\varepsilon^4}{n} \text{Ric}[g]_{iNiN;jj} \mathcal{A}_1 + o(\varepsilon^4),$$

where we have used the notation of Lemma A.2 in the Appendix. Moreover, Lemma 4.1 implies $R[g]_{iNiN;jj} = \text{Ric}[g]_{NN;jj}$. Therefore

$$I_1 = \frac{2\varepsilon^4}{n} \text{Ric}_{NN;ii}[g] \mathcal{A}_1 + \frac{6\varepsilon^4}{n(n+2)} \text{Ric}_{NN;ii}[g](\mathcal{A}_3 - \mathcal{A}_1) + o(\varepsilon^4).$$

Collecting the computation of I_1 and I_2 and inserting to (4-13), we obtain

$$(4-15) \quad \begin{aligned} \mathcal{T}_1 &= \int_{B_+^N(0, \delta)} x_N^{m_1} (\Delta_{m_1} W_\varepsilon)^2 dx \\ &\quad + \frac{\varepsilon^4 \text{Ric}[g]_{NN;ii}}{n} \left[-\mathcal{F}_5 + \mathcal{F}_6 + 2\mathcal{A}_1 + \frac{6(\mathcal{A}_3 - \mathcal{A}_1)}{n+2} \right] + o(\varepsilon^4). \end{aligned}$$

Step 2: Let us deal with $\mathcal{T}_2$ and $\mathcal{T}_3$ in $\mathcal{Q}_\gamma(W_\varepsilon : B_+^N(0, \delta))$. Using Lemma 4.2, we get

$$\begin{aligned}
& \int_{B_+^N(0, \delta)} x_N^{m_1} \text{Ric}[g](\nabla W_1, \nabla W_1) dx \\
&= \int_{B_+^N(0, \delta)} x_N^{m_1} \text{Ric}[g]_{ij}(x) \partial_i W_1 \partial_j W_1 dx \\
&\quad + \int_{B_+^N(0, \delta)} x_N^{m_1} \text{Ric}[g]_{NN}(x) \partial_N W_1 \partial_N W_1 dx \\
&= \frac{\varepsilon^4}{2n(n+2)} [\text{Ric}[h]_{kk;ii} + 2\text{Ric}[h]_{ik;ik}] \int_{\mathbb{R}_+^N} x_N^{m_1} r^2 (\partial_r W_1)^2 dx \\
&\quad + \frac{1}{2n} \text{Ric}[g]_{NN;ii} \int_{\mathbb{R}_+^N} x_N^{m_1} r^2 (\partial_N W_1)^2 dx + o(\varepsilon^4).
\end{aligned}$$

Since Lemma 4.1 implies

$$\text{Ric}[h]_{kk;ii} = R[h]_{;ii} = 2(n-1) \text{Ric}[g]_{NN;ii}$$

and by the contracted Bianchi identity $2 \text{Ric}[h]_{ik;ik} = R[h]_{;kk}$, one can simplify the above equation to

$$\begin{aligned}
& \int_{B_+^N(0, \delta)} x_N^{m_1} \text{Ric}[g](\nabla W_1, \nabla W_1) dx \\
&= \left[\frac{2n-1}{2n(n+2)} \mathcal{F}_6 + \frac{1}{2n} (\mathcal{F}_5 - \mathcal{F}_6) \right] \varepsilon^4 \text{Ric}[g]_{NN;ii} + o(\varepsilon^4).
\end{aligned}$$

We also have

$$\begin{aligned}
& \int_{B_+^N(0, \delta)} x_N^{m_1} J[g] |\nabla W_\varepsilon|_g^2 d\mu_g \\
&= \frac{1}{2n} \int_{B_+^N(0, \delta)} x_N^{m_1} R[g] |\nabla W_\varepsilon|^2 dx + o(\varepsilon^4) \\
&= \frac{1}{4n} \int_{B_+^N(0, \delta)} x_N^{m_1} (R[g]_{;ij} x_i x_j + R[g]_{;NN} x_N^2) |\nabla W_\varepsilon|^2 dx + o(\varepsilon^4) \\
&= \frac{R[g]_{;ii} \varepsilon^4}{4n^2} \int_{\mathbb{R}_+^N} x_N^{m_1} r^2 |\nabla W_1|^2 dx + o(\varepsilon^4) \\
&= \frac{R[g]_{;ii} \varepsilon^4}{4n^2} \mathcal{F}_5 + o(\varepsilon^4) \\
&= \frac{\text{Ric}[g]_{NN;ii} \varepsilon^4}{2n} \mathcal{F}_5 + o(\varepsilon^4).
\end{aligned}$$

Since the Schouten tensor $P = \frac{1}{n-1}(\text{Ric} - Jg)$,

$$\begin{aligned}
 (4-16) \quad \mathcal{T}_2 &= \frac{4}{n-1} \int_{B_+^N(0,\delta)} x_N^{m_1} \text{Ric}[g](\nabla W_1, \nabla W_1) dx \\
 &\quad - \left(\frac{4}{n-1} + n - 2\gamma + 2 \right) \int_{B_+^N(0,\delta)} x_N^{m_1} J[g] |\nabla W_\varepsilon|_g^2 d\mu_g \\
 &= \left[\frac{2}{n-1} \frac{n-3}{n(n+2)} \mathcal{F}_6 - \frac{n-2\gamma+2}{2n} \mathcal{F}_5 \right] \varepsilon^4 \text{Ric}[g]_{NN;ii} + o(\varepsilon^4)
 \end{aligned}$$

Also

$$\begin{aligned}
 (4-17) \quad \mathcal{T}_3 &= \frac{n-2\gamma}{2} \int_{B_+^N(0,\delta)} x_N^{m_1} \mathcal{Q}_{\rho_1}^{m_1} W_\varepsilon^2 d\mu_g \\
 &= \frac{n-2\gamma}{2} \int_{B_+^N(0,\delta)} x_N^{m_1} (-\Delta_{m_1} J[g](0)) W_\varepsilon^2 dx + o(\varepsilon^4) \\
 &= \frac{-(n-2\gamma)\varepsilon^4}{4n} \int_{\mathbb{R}_+^N} x_N^{m_1} (R[g]_{;ii} + R[g]_{;NN} + m_1 x_N^{-1} \partial_N R[g]) W_1^2 dx + o(\varepsilon^4) \\
 &= -\frac{(n-2\gamma)\varepsilon^4}{4n} R[g]_{;ii} \mathcal{F}_1 + o(\varepsilon^4) \\
 &= -\frac{n-2\gamma}{2} \varepsilon^4 \text{Ric}[g]_{NN;ii} \mathcal{F}_1 + o(\varepsilon^4).
 \end{aligned}$$

Here $\mathcal{F}_1 = \int_{\mathbb{R}_+^N} x_N^{m_1} W_1^2 dx$; see the notation in [Kim et al. 2018, Lemma B.6]. Inserting (4-15), (4-16), and (4-17) together back into (2-10),

$$(4-18) \quad \mathcal{Q}_\gamma(W_\varepsilon : B_+^N(0, \delta)) = \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_{m_1} W_\varepsilon)^2 dx + \varepsilon^4 R[g]_{NN;ii} \mathcal{C}_4 + o(\varepsilon^4)$$

where

$$\begin{aligned}
 (4-19) \quad \mathcal{C}_4 &= \frac{1}{n} \left[-\mathcal{F}_5 + \mathcal{F}_6 + 2\mathcal{A}_1 + \frac{6(\mathcal{A}_3 - \mathcal{A}_1)}{n+2} \right] \\
 &\quad - \left[\frac{2}{n-1} \frac{n-3}{n(n+2)} \mathcal{F}_6 - \frac{n-2\gamma+2}{2n} \mathcal{F}_5 \right] - \frac{n-2\gamma}{2} \mathcal{F}_1.
 \end{aligned}$$

It can be checked that $\mathcal{C} > 0$ when $\gamma \in (1, 2)$ and $n > 4 + 2\gamma$. See Lemma A.3 in the Appendix.

Step 3: It is standard to get

$$\mathcal{Q}_\gamma(\chi_\delta W_\varepsilon : B_+^N(0, 2\delta) \setminus B_+^N(0, \delta)) = o(\varepsilon^4).$$

Combining Step 1–3 and (4-7), we obtain

$$\mathcal{Q}_\gamma(U_{a,\varepsilon,\delta}) \leq S_{n,\gamma}^{-1} \left(\int_{D(0,\delta)} w_\varepsilon^{2n/(n-2\gamma)} d\bar{x} \right)^{(n-2\gamma)/n} + \varepsilon^4 R[g]_{NN;ii} \mathcal{C}_4 + o(\varepsilon^4).$$

Since we always have (3-5) and $R[g]_{NN;ii} = -\|W[h]\|^2/[12(n-1)]$ by Lemma 4.1,

$$\bar{\mathcal{E}}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \mathcal{Y}_{\mathbb{S}^n}^\gamma - \varepsilon^4 \mathcal{C}_4(n, \gamma, g, \delta) \|W[h]\|^2 + o(\varepsilon^4). \quad \square$$

4C. Case (II-4): Nonlocally conformally flat and $n = 2\gamma + 4$. In this case we will have $n = 4 + 2\gamma$. Since $\gamma \in (1, 2)$, then it means $\gamma = \frac{3}{2}$ and $n = 7$. The bubble has the following explicit form [Sun and Xiong 2016]

$$(4-20) \quad W_{\varepsilon,\sigma} = \alpha_{7,3/2} \left[\left(\frac{\varepsilon}{(\varepsilon + x_N)^2 + |\bar{x} - \sigma|^2} \right)^2 + 4x_N \left(\frac{\varepsilon}{(\varepsilon + x_N)^2 + |\bar{x} - \sigma|^2} \right)^3 \right]$$

where $\alpha_{7,3/2}$ is defined in (2-26). We also have $m_1 = 0$ in this case.

Theorem 4.6. *Suppose that $\gamma = \frac{3}{2}$ and $n = 7$. If the Weyl tensor at a does not vanish, define*

$$U_{a,\varepsilon,\delta}(x) = \chi_\delta W_\varepsilon(\Psi_a(x))$$

for $0 < C_0\varepsilon \leq \delta \leq \delta_0 \leq 1$. Then there exists $\mathcal{C}_5 > 0$ such that

$$\mathcal{E}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \bar{\mathcal{E}}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \mathcal{Y}_{\mathbb{S}^n}^\gamma - \varepsilon^4 \log(\delta/\varepsilon) \mathcal{C}_5(n, \gamma, g, \delta) \|W[h]\|^2 + O(\varepsilon^4)$$

provided δ_0 small enough and C_0 large enough.

Proof. Using the explicit form of $W_{\varepsilon,\sigma}$, one can calculate as in the previous section.

Step 1: Consider the leading term in $\mathcal{Q}_\gamma(W_\varepsilon : B_+^N(0, \delta))$.

$$\begin{aligned} (4-21) \quad & \int_{B_+^N(0,\delta)} (\Delta_\rho^{m_1} W_\varepsilon)^2 d\mu_g \\ &= \int_{B_+^N(0,\delta)} x_N^{m_1} (\Delta_\rho^{m_1} W_\varepsilon)^2 \sqrt{|g|} d\mu_g \\ &= \int_{B_+^N(0,\delta)} (\Delta_\rho^{m_1} W_\varepsilon)^2 dx - \frac{\varepsilon^4}{n} \text{Ric}[g]_{NN;ii} \int_{B_+^N(0,\delta/\varepsilon)} x_N^2 r^2 (\partial_N W_1)^2 dx + o(\varepsilon^4) \\ &= \int_{B_+^N(0,\delta)} (\Delta_\rho^{m_1} W_\varepsilon)^2 dx - \frac{\pi}{32} \alpha_{7,3/2}^2 |\mathbb{S}^6| \varepsilon^4 \log\left(\frac{\delta}{\varepsilon}\right) \text{Ric}[g]_{NN;ii} + O(\varepsilon^4). \end{aligned}$$

where we have used the formula of Lemma A.4 in the Appendix. Similarly

$$\int_{B_+^N(0,\delta)} (\Delta_\rho^{m_1} W_\varepsilon)^2 dx = \int_{B_+^N(0,\delta)} (\Delta_{m_1} W_\varepsilon)^2 dx + I_1 + I_2.$$

It is easy to see $I_2 = o(\varepsilon^4)$ and

$$(4-22) \quad I_1 = -\frac{\pi}{32} \alpha_{7,3/2}^2 |\mathbb{S}^6| \varepsilon^4 \log\left(\frac{\delta}{\varepsilon}\right) \text{Ric}[g]_{NN;ii} + O(\varepsilon^4).$$

$$(4-23) \quad \mathcal{T}_1 = \int_{B_+^N(0,\delta)} (\Delta_{\rho}^{m_1} W_{\varepsilon})^2 dx - \frac{\pi}{16} \alpha_{7,3/2}^2 |\mathbb{S}^6| \varepsilon^4 \log\left(\frac{\delta}{\varepsilon}\right) \text{Ric}[g]_{NN;ii} + O(\varepsilon^4).$$

Step 2: We have

$$\int_{B_+^N(0,\delta)} \text{Ric}[g](\nabla W_1, \nabla W_1) dx = \frac{7\pi}{32} \alpha_{7,3/2}^2 |\mathbb{S}^6| \varepsilon^4 \log\left(\frac{\delta}{\varepsilon}\right) \text{Ric}[g]_{NN;ii} + O(\varepsilon^4).$$

and

$$\int_{B_+^N(0,\delta)} J |\nabla W_{\varepsilon}|_g^2 d\mu_g = \frac{5\pi}{32} \alpha_{7,3/2}^2 |\mathbb{S}^6| \varepsilon^4 \log\left(\frac{\delta}{\varepsilon}\right) \text{Ric}[g]_{NN;ii} + O(\varepsilon^4).$$

Hence

$$(4-24) \quad \mathcal{T}_2 = -\frac{43}{48} \alpha_{7,3/2}^2 |\mathbb{S}^6| \varepsilon^4 \log\left(\frac{\delta}{\varepsilon}\right) \text{Ric}[g]_{NN;ii} + O(\varepsilon^4).$$

It is not hard to see that

$$\mathcal{T}_3 = \frac{n-2\gamma}{2} \int_{B_+^N(0,\delta)} Q_{\rho_1}^{m_1} W_{\varepsilon}^2 d\mu_g = -\frac{10\pi}{32} \alpha_{7,3/2}^2 |\mathbb{S}^6| \varepsilon^4 \log\left(\frac{\delta}{\varepsilon}\right) \text{Ric}[g]_{NN;ii} + O(\varepsilon^4).$$

Putting $\mathcal{T}_i$ back into $\mathcal{Q}_{\gamma}(W_{\varepsilon} : B_+^N(0, \delta))$, one gets

$$\begin{aligned} \mathcal{Q}_{\gamma}(W_{\varepsilon} : B_+^N(0, \delta)) \\ = \int_{B_+^N(0,\delta)} (\Delta_{m_1} W_{\varepsilon})^2 dx + \frac{19\pi}{48} \alpha_{7,3/2}^2 |\mathbb{S}^6| \varepsilon^4 \log\left(\frac{\delta}{\varepsilon}\right) \text{Ric}[g]_{NN;ii} + O(\varepsilon^4). \end{aligned}$$

The rest of proof will be the same as the last part of the proof of Theorem 4.5. We shall omit it here. $\square$

5. Interaction estimates on bubbles

In this section, we will state the asymptotic analysis of the Palais–Smale sequence of $\mathcal{E}_h^{\gamma}$. The local cases then follow from the Ekeland variational principle. Next we shall derive interaction estimates of bubbles which is crucial for the algebraic topological argument in the next section.

5A. Asymptotic analysis and local cases. Suppose (X^{n+1}, M^n, g^+) is a P-E manifold with conformal infinity $(M, [h])$. Assume ρ is the unique geodesic defining function for a representative metric h . Then $(\bar{X}^{n+1}, g = \rho^2 g^+, \rho, m_1)$ is a geodesic SMMS. Given any point $a \in M$, there is a “good” conformal Fermi coordinates by

Lemma 4.1. More precisely, there exists a conformal metric $h_a \in [h]$ and ρ_a the associated unique geodesic definition function such that

$$g_a = \rho_a^2 g_+, \quad g_a|_M = h_a, \quad g_a = d\rho_a^2 + h_{\rho_a} \text{ near } M$$

Since $h_a \in [h]$, one may assume $h_a = \phi_a^{4/(n-2\gamma)} h$. One can see that $g_a = (\rho_a/\rho)^2 \rho^2 g_+$. Letting $\rho \rightarrow 0$, we get

$$h_a = \lim_{\rho \rightarrow 0} \left(\frac{\rho_a}{\rho} \right)^2 h \quad \text{on } M.$$

So we may naturally extend $\phi_a = (\rho_a/\rho)^{(n-2\gamma)/2}$ onto X . It is known that the map $a \rightarrow \phi_a$ and g_a is C^0 . By the expansion of metric (4-2) near a , one knows $\phi_a(a) = 1$. Therefore $|\rho_a/\rho - 1| \leq C\delta$ near a .

Suppose $\Psi_a : \mathcal{O}(a) \rightarrow B_+^N(0, 2\delta)$ is the Fermi coordinates map, where $\mathcal{O}(a)$ is a open neighborhood of a in X . Recall the definition of $U_{a,\varepsilon,\delta}$ in (4-3). Define

$$(5-1) \quad u_{a,\varepsilon,\delta} = U_{a,\varepsilon,\delta}|_M, \quad V_{a,\varepsilon,\delta} = \left(\frac{\rho_{a_i}}{\rho} \right)^{(n-2\gamma)/2} U_{a,\varepsilon,\delta}, \quad v_{a,\varepsilon,\delta} = V_{a,\varepsilon,\delta}|_M.$$

By the works of Palatucci and Pisante [2015] and Fang and González [2015], it is not hard to see the following profile decomposition:

Lemma 5.1. *Suppose $\{u_v\} \subset W_+^{\gamma,2}(M, h)$ is a Palais–Smale sequence for $\mathcal{E}_h^\gamma$, that is $d\mathcal{E}_h^\gamma[u_v] \rightarrow 0$ and $\mathcal{E}_h^\gamma[u_v] \rightarrow c_*$ as $v \rightarrow \infty$. After some normalization, we may assume*

$$\oint_M u_v^{2n/(n-2\gamma)} d\sigma_h = c_*^{n/(2\gamma)}.$$

Then after passing to subsequence if necessary, there exists a $u_\infty \in W_+^{\gamma,2}(M, h)$, an integer $m \geq 0$ and a sequence $(a_{j,v}, \varepsilon_{j,v})$ for $1 \leq j \leq m$ with the following properties:

(i) u_∞ satisfies $P_h^\gamma u_\infty = u_\infty^{(n-2\gamma)/(n+2\gamma)}$.

(ii) As $v \rightarrow \infty$,

$$\left\| u_v - u_\infty - \sum_{j=1}^m v_{a_{j,v}, \varepsilon_{j,v}, \delta} \right\|_{W^{\gamma,2}(M, h)} \rightarrow 0,$$

$$(\mathcal{E}_h^\gamma[u_v])^{n/(2\gamma)} \rightarrow (\mathcal{E}_h^\gamma[u_\infty])^{n/(2\gamma)} + m(\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)}.$$

(iii) For $i \neq j$

$$(5-2) \quad \frac{\varepsilon_{i,v}}{\varepsilon_{j,v}} + \frac{\varepsilon_{j,v}}{\varepsilon_{i,v}} + \frac{d_h^2(a_{i,v}, a_{j,v})}{\varepsilon_{i,v} \varepsilon_{j,v}} \rightarrow \infty,$$

where d_h is the distance function on (M, h) .

It follows from the Ekeland variational principle [Ekeland 1974] that:

Lemma 5.2. *There exists a Palais–Smale sequence at level $\mathcal{Y}^\gamma(M, [h])$.*

After the existence of Palais–Smale sequence at level $\mathcal{Y}^\gamma(M, [h])$, the next ingredient in this approach is the same one as in the subcritical approximations. Precisely it is the existence of a *variational barrier* at infinity due to the presence of local information and is the content of the following proposition.

Proposition 5.3 (local information helps). *Under the assumption of case (II-3) and (II-4), we have that there exists $a \in M$, ε and δ small enough such that*

$$\mathcal{E}_h^\gamma(v_{a,\varepsilon,\delta}) < \mathcal{Y}_{\mathbb{S}^n}^\gamma.$$

Proof. It follows directly from Theorem 4.5 and 4.6. $\square$

Proof of Local case (II-3) and (II-4). By a contradiction argument, it follows directly from Lemma 5.2, and Proposition 5.3. $\square$

Remark 5.4. As in the case of the subcritical approximation technique, here also the solution obtained is a minimizer.

5B. Estimates for global cases. For the rest of this paper, we focus on the global cases, which are (I-1), (II-1) and (II-2). For every $p \in \mathbb{N}^*$ and $A := (a_1, \dots, a_p) \in M^p = M \times \dots \times M$, $\varepsilon_i, \varepsilon_j$, we define the following quantities

$$(5-3) \quad \varepsilon_{i,j} = \left(\frac{\varepsilon_i}{\varepsilon_j} + \frac{\varepsilon_j}{\varepsilon_i} + \frac{d_h^2(a_i, a_j)}{\varepsilon_i \varepsilon_j} \right)^{(2\gamma-n)/2},$$

$$(5-4) \quad e_{i,j} = \kappa_\gamma \mathcal{Q}_\gamma(V_{a_i,\varepsilon_i,\delta}, V_{a_j,\varepsilon_j,\delta}),$$

$$(5-5) \quad \epsilon_{i,j} = \oint_M (v_{a_i,\varepsilon_i,\delta})^{(n+2\gamma)/(n-2\gamma)} v_{a_j,\varepsilon_j,\delta} d\sigma_h,$$

for $i, j = 1, \dots, p$. Here and the following we always assume that δ and ε_0 are fixed numbers which will be chosen later, and $\varepsilon_i \leq \varepsilon_0$ are small comparable to δ .

Lemma 5.5 (self-action). *Under the assumptions of Propositions 3.1, 4.3 and 4.4, there exist ε_0 small enough and $C > 0$ such that for any $v_{a,\varepsilon,\delta}$ with $\varepsilon \leq \varepsilon_0$:*

- (i) $\mathcal{E}_h^\gamma(v_{a_i,\varepsilon_i,\delta}) \leq \mathcal{Y}_{\mathbb{S}^n}^\gamma + C\delta^{2\gamma-n}\varepsilon^{n-2\gamma}.$
- (ii) $\oint_M v_{a_i,\varepsilon_i,\delta}^{2n/(n-2\gamma)} d\sigma_h = (\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{n/(2\gamma)} + O(\varepsilon^n \delta^{-n}).$

Proof. These are just the results of the corresponding propositions. $\square$

Lemma 5.6 (higher exponent interaction estimates). *There exists $\mu_0 > 0$ small enough so that the following estimates hold provided $\varepsilon_{i,j} < \mu_0$ for $i \neq j$:*

- (i) $\oint_M v_{a_i,\varepsilon_i,\delta}^\alpha v_{a_j,\varepsilon_j,\delta}^\beta d\sigma_h = O(\varepsilon_{i,j}^\beta)$ for $\alpha + \beta = \frac{2n}{n-2\gamma}$ and $\alpha > \frac{n}{n-2\gamma} > \beta > 0$.
- (ii) $\oint_M v_{a_i,\varepsilon_i,\delta}^{(n)/(n-2\gamma)} v_{a_j,\varepsilon_j,\delta}^{(n)/(n-2\gamma)} d\sigma_h = O(\varepsilon_{i,j}^{(n)/(n-2\gamma)} \ln \varepsilon_{i,j}).$

Proof. These are just local estimates which does not involve any fractional derivative of v . One can borrow the proof in [Mayer and Ndiaye 2017b, Lemma 5.4]. $\square$

Lemma 5.7. *Suppose that $\gamma \in (0, 1) \cup (1, \min\{2, \frac{n}{2}\})$ and $U_{a,\varepsilon,\delta}$ is defined in (4-3) for $C_0\varepsilon \leq \delta \leq \delta_0$. If δ_0 small enough and C_0 large enough, there exist $C > 0$ such that the following hold:*

$$|L_{2,\rho_a}^{m_0}(U_{a,\varepsilon,\delta})| \leq Cd_{g_a}(x,a)\chi_\delta W_\varepsilon(\Psi_a) + C\varepsilon^{(n-2\gamma)/2}\delta^{2\gamma-n-1}\mathbf{1}_{\{\delta/2 \leq d_{g_a}(x,a) \leq 4\delta\}}.$$

$$\lim_{\rho_{a_i} \rightarrow 0} \rho_a^{m_0} \partial_{\rho_a}(U_{a,\varepsilon,\delta}) = -\kappa_\gamma^{-1} \chi_\delta w_\varepsilon^{(n+2\gamma)/(n-2\gamma)}.$$

Where $\mathbf{1}_\Omega$ is the characteristic function for a set Ω .

Proof. Using the map Ψ_a , we can consider the problem on $B_+^N(0, 2\delta)$ with metric g_a having expansions (4-1) and (4-2). Under this coordinates we have $\rho_a = x_N$. It is easy to see

$$\lim_{\rho_a \rightarrow 0} \rho_a^{m_0} \partial_{\rho_a}(U_{a,\varepsilon,\delta}) = -\kappa_\gamma^{-1} \chi_\delta w_\varepsilon^{(n+2\gamma)/(n-2\gamma)}.$$

For the one of $L_{2,\rho_a}^{m_0}(U_{a,\varepsilon,\delta})$, similar type of estimates were derived in [Brendle 2005, Proposition B.1], [Almaraz 2015, Proposition 3.13], and [Almaraz and Sun 2016, Proposition 3.14]. By the definition in (4-3) and (2-7), we have

$$(5-6) \quad L_{2,\rho_a}^{m_0}(U_{a,\varepsilon,\delta}) = \chi_\delta L_{2,\rho_a}^{m_0} W_\varepsilon + 2\langle \nabla \chi_\delta, \nabla(W_\varepsilon - \varepsilon^{(n-2\gamma)/2} G_a^\gamma) \rangle_{g_a} \\ + (\Delta_{\rho_a}^{m_0} \chi_\delta)(W_\varepsilon - \varepsilon^{(n-2\gamma)/2} G_a^\gamma) \\ = I_1 + I_2 + I_3.$$

To handle the first term in the above equality, notice

$$L_{2,\rho_a}^{m_0} W_\varepsilon = \Delta_{\rho_a}^{m_0}(W_\varepsilon) + \frac{m_0+n-1}{2} J[g_a] W_\varepsilon.$$

We only need to calculate the above in $B_+^N(0, 2\delta)$. Since $W_\varepsilon = W_\varepsilon(|\bar{x}|, x_N) = W_\varepsilon(r, x_N)$, where $r^2 = x_1^2 + \dots + x_n^2$, we have (write g_a as g for short temporarily)

$$\Delta_{\rho_a}^{m_0}(W_\varepsilon) = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \frac{x_j}{r} \partial_r W_\varepsilon) + \frac{1}{\sqrt{|g|}} \partial_N (\sqrt{|g|} \partial_N W_\varepsilon) + m_0 x_N^{-1} \partial_N W_\varepsilon \\ = \frac{g^{ij} x_i x_j}{r^2} \partial_{rr}^2 W_\varepsilon + \left[g^{ij} \partial_i \ln \sqrt{|g|} \frac{x_j}{r} + \partial_i \left(\frac{g^{ij} x_j}{r} \right) \right] \partial_r W_\varepsilon \\ + \partial_N \ln \sqrt{|g|} \partial_N W_\varepsilon + \partial_{NN}^2 W_\varepsilon + m_0 x_N^{-1} \partial_N W_\varepsilon.$$

Using $\Delta_{m_0} W_\varepsilon = 0$, the above equality leads to

$$\Delta_{\rho_a}^{m_0}(W_\varepsilon) = \left(\frac{g^{ij} x_i x_j}{r^2} - 1 \right) \partial_{rr}^2 W_\varepsilon \\ + \left[g^{ij} \partial_i \ln \sqrt{|g|} \frac{x_j}{r} + \partial_i \left(\frac{g^{ij} x_j}{r} \right) - \frac{n-1}{r} \right] \partial_r W_\varepsilon + \partial_N \ln \sqrt{|g|} \partial_N W_\varepsilon.$$

Using the expansion of g_a in (4-1) and (4-2), in $B_+^N(0, 2\delta)$, we have

$$\begin{aligned}
 (5-7) \quad \Delta_{\rho_a}^{m_0}(W_\varepsilon) &= O(|x|^3)\partial_{rr}^2 W_\varepsilon + O(|x|^2)(\partial_r W_\varepsilon + \partial_N W_\varepsilon) \\
 &= O(|x|\varepsilon^{(n-2\gamma)/2}(\varepsilon^2 + |x|^2)^{-(n-2\gamma)/2}) \\
 &= O(|x|W_\varepsilon),
 \end{aligned}$$

where in the second and last equality, Lemma A.1 is used. Consequently $|I_1| \leq C|x|\chi_\delta W_\varepsilon$.

For I_2 and I_3 in (5-6), we only need to bound them in $B_+^N(0, 2\delta) \setminus B_+^N(0, \delta)$. In this region, one can use (2-8), (2-14) and [Mayer and Ndiaye 2017b, Corollary 5.3]

$$|W_\varepsilon - \varepsilon^{(n-2\gamma)/2}G_a^\gamma| + |x \cdot \nabla(W_\varepsilon - \varepsilon^{(n-2\gamma)/2}G_a^\gamma)| \leq C\varepsilon^{(n-2\gamma)/2}\delta^{2\gamma-n+1}.$$

Therefore

$$|I_2| + |I_3| \leq C\varepsilon^{(n-2\gamma)/2}\delta^{2\gamma-n-1}\mathbf{1}_{\{\delta \leq |x| \leq 2\delta\}},$$

where $\mathbf{1}_\Omega$ is the characteristic function for a set Ω . Taking $\delta < \delta_0$ small enough such that $|x|$ and $d_g(x, a)$ are comparable, one can get the conclusion. $\square$

Remark 5.8. Since $(\bar{X}, g_{a_i}, \rho_{a_i}, m_0)$ and $(\bar{X}, g, \rho, m_0)$ are two geodesic SMMS which are conformal to each other, then by the conformal change property (2-2)

$$\begin{aligned}
 L_{2,\rho}^{m_0}(V_{a_i,\varepsilon_i,\delta}) &= \left(\frac{\rho_{a_i}}{\rho}\right)^{(n+4-2\gamma)/2} L_{2,\rho_{a_i}}^{m_0}(U_{a_i,\varepsilon_i,\delta}) \\
 &= O(d_g(x, a)\chi_\delta W_{\varepsilon_i}(\Psi_{a_i}(x))) + O(\varepsilon_i^{(n-2\gamma)/2}\delta^{2\gamma-n-1}\mathbf{1}_{\{\delta/2 \leq d_g(x, a) \leq 4\delta\}}).
 \end{aligned}$$

It follows from [Case 2017, Theorem 3.2] that $\lim_{\rho \rightarrow 0} \rho^{m_0} \partial_\rho$ is also conformally covariant. Then

$$\begin{aligned}
 \lim_{\rho \rightarrow 0} \rho^{m_0} \partial_\rho V_{a_i,\varepsilon_i,\delta} &= \phi_{a_i}^{(n+2\gamma)/(n-2\gamma)} \lim_{\rho_{a_i} \rightarrow 0} \rho_{a_i}^{m_0} \partial_{\rho_{a_i}}(U_{a_i,\varepsilon_i,\delta}) \\
 &= -\kappa_\gamma^{-1} \phi_{a_i}^{(n+2\gamma)/(n-2\gamma)} \chi_\delta w_{\varepsilon_i}^{(n+2\gamma)/(n-2\gamma)}(\Psi_{a_i}).
 \end{aligned}$$

Lemma 5.9 (interaction). *For $\gamma \in (0, 1) \cup (1, \min\{2, \frac{n}{2}\})$, and $C_0 \max\{\varepsilon_i, \varepsilon_j\} \leq \delta \leq \delta_0$ for some sufficiently small δ_0 and large C_0 . Assume $\varepsilon_{i,j} \leq \mu_0$ for some small μ_0 , then:*

- (i) $e_{i,j} = (1 + O(\delta))\varepsilon_{i,j} + O(\max\{\varepsilon_i, \varepsilon_j\}^{2\gamma}\delta^{-2\gamma})\varepsilon_{i,j}$.
- (ii) $\varepsilon_{i,j} = (\mathcal{Y}_{\mathbb{S}^n}^\gamma)^{\frac{n}{2\gamma}}(1 + O(\delta) + O(\max\{\varepsilon_j, \varepsilon_i\}^{2\gamma}\delta^{-2\gamma}))\varepsilon_{i,j}$.

Proof. For (ii), there is no fractional derivative involved. One can use the proof from [Mayer and Ndiaye 2017b, Lemma 5.5]. Now consider (i). Let us use abbreviation $V_i = V_{a_i,\varepsilon_i,\delta}$, $\varphi_i = \varphi_{a_i,\varepsilon_i,\delta}$ and $W_i = W_{\varepsilon_i}(\Psi_{a_i}(x))$.

Suppose $\gamma \in (0, 1)$. It follows from (2-5) that

$$e_{i,j} = \kappa_\gamma \int_X L_{2,\rho}^{m_0}(V_i) V_j \rho^{m_0} d\mu_g - \kappa_\gamma \oint_M \lim_{\rho \rightarrow 0} \rho^{m_0} \partial_\rho (V_i) V_j d\sigma_h.$$

Here by symmetry, we can assume $\varepsilon_j \leq \varepsilon_i$. Since Remark 5.8 and Lemma A.5,

$$\int_X L_{2,\rho}^{m_0}(V_i) V_j \rho^{m_0} d\mu_g = O(\delta) \varepsilon_{i,j}.$$

For the other term, one can apply Remark 5.8 and Lemma A.7 to get

$$\begin{aligned} & -\kappa_\gamma \oint_M \lim_{\rho \rightarrow 0} \rho^{m_0} \partial_\rho (V_i) V_j d\sigma_h \\ &= (1 + O(\delta)) \oint_M \chi_i w_i^{(n+2\gamma)/(n-2\gamma)} v_j d\sigma_h \\ &= (1 + O(\delta)) \varepsilon_{i,j} - (1 + O(\delta)) \oint_M (v_i^{(n+2\gamma)/(n-2\gamma)} - \chi_i w_i^{(n+2\gamma)/(n-2\gamma)}) v_j d\sigma_h \\ &= (1 + O(\delta)) \varepsilon_{i,j} + O(\varepsilon_i^{2\gamma} \delta^{-2\gamma}) \varepsilon_{i,j}. \end{aligned}$$

Combing the above two estimates, one gets (i) when $\gamma \in (0, 1)$.

Suppose $\gamma \in (1, \min\{2, \frac{n}{2}\})$. It follows from (2-11) that

$$\begin{aligned} (5-8) \quad e_{i,j} &= \kappa_\gamma \int_X L_{4,\rho}^{m_1}(V_i) V_j \rho^{m_1} d\mu_g + \kappa_\gamma \oint_M \lim_{\rho \rightarrow 0} \rho^{m_1} \partial_\rho \Delta_\rho^{m_1}(V_i) V_j d\sigma_h \\ &= I_1 + I_2. \end{aligned}$$

Claim 1. $I_1 = O(\delta) \varepsilon_{i,j}$.

Proof. It follows from [Case and Chang 2016, Theorem 3.1] that $L_{4,\rho}^{m_1}$ has the decomposition

$$L_{4,\rho}^{m_1} = L_{2,\rho}^{m_1+2} \circ L_{2,\rho}^{m_0} = L_{2,\rho}^{m_1+2} \circ L_{2,\rho}^{m_0},$$

where by definition one has

$$L_{2,\rho}^{m_1+2} = L_{2,\rho}^{m_1} - 2\rho^{-1} \partial_\rho + J[g].$$

Since

$$\begin{aligned} L_{2,\rho_a}^{m_0}(U_{a,\varepsilon,\delta}) &= \chi_\delta L_{2,\rho_a}^{m_0} W_\varepsilon + (1 - \chi_\delta) L_{2,\rho_a}^{m_0} G_a^\gamma \\ &\quad + 2\langle \nabla \chi_\delta, \nabla (W - \varepsilon^{(n-2\gamma)/2} G_a^\gamma) \rangle_{g_a} + \Delta_{\rho_a}^{m_0} \chi_\delta (W_\varepsilon - \varepsilon^{(n-2\gamma)/2} G_a^\gamma), \end{aligned}$$

using the estimates in (2-14) and Lemma A.1, we arrive at the following estimates in $B_+^N(0, 2\delta)$ which are

$$\begin{aligned}
L_{4,\rho_a}^{m_1}(U_{a,\varepsilon,\delta}) &= L_{2,\rho_a}^{m_1+2}(\chi_\delta L_{2,\rho_a}^{m_0} W_\varepsilon + (1-\chi_\delta) L_{2,\rho_a}^{m_0} G_a^\gamma) + O(\varepsilon^{(n-2\gamma)/2} \delta^{2\gamma-n-3} \mathbf{1}_{\{\delta \leq |x| \leq 2\delta\}}) \\
&= \chi_\delta L_{2,\rho_a}^{m_1+2}(L_{2,\rho_a}^{m_0} W_\varepsilon) + 2\langle \nabla \chi_\delta, \nabla(L_{2,\rho_a}^{m_0}(W_\varepsilon - G_a^\gamma)) \rangle_{g_a} \\
&\quad + (\Delta_{\rho_a}^{m_1+2} \chi_i) L_{2,\rho_a}^{m_0}(W_\varepsilon - G_a^\gamma) + O(\varepsilon^{(n-2\gamma)/2} \delta^{2\gamma-n-3} \mathbf{1}_{\{\delta \leq |x| \leq 2\delta\}}) \\
&= \chi_\delta L_{2,\rho_a}^{m_1+2}(L_{2,\rho_a}^{m_0} W_\varepsilon) + O(\varepsilon^{(n-2\gamma)/2} \delta^{2\gamma-n-3} \mathbf{1}_{\{\delta \leq |x| \leq 2\delta\}}).
\end{aligned}$$

By (2-2), we have

$$\begin{aligned}
L_{4,\rho}^{m_1}(V_i) &= (\rho_{a_i}/\rho)^{(n+8-2\gamma)/2} L_{4,\rho_{a_i}}^{m_1}(U_i) \\
&= (\rho_{a_i}/\rho)^{(n+8-2\gamma)/2} [\chi_\delta L_{2,\rho_{a_i}}^{m_1+2}(L_{2,\rho_{a_i}}^{m_0} W_i) + O(\varepsilon^{(n-2\gamma)/2} \delta^{2\gamma-n-3} \mathbf{1}_{\{\delta \leq d_{g_{a_i}}(x, a_i) \leq 2\delta\}})] \\
&= \chi_\delta L_{2,\rho}^{m_1+2}(L_{2,\rho}^{m_0} \tilde{V}_i) + O(\varepsilon^{(n-2\gamma)/2} \delta^{2\gamma-n-3} \mathbf{1}_{\{\delta/2 \leq d_g(x, a_i) \leq 4\delta\}}).
\end{aligned}$$

Here $\tilde{V}_i = (\rho_{a_i}/\rho)^{(n-2\gamma)/2} W_i$. Then by Lemma A.6

$$\begin{aligned}
I_1 &= \kappa_\gamma \int_X \chi_\delta L_{2,\rho}^{m_1+2}(L_{2,\rho}^{m_0} \tilde{V}_i) V_j \rho^{m_1} d\mu_g \\
&\quad + O\left(\int_X \varepsilon_i^{(n-2\gamma)/2} \delta^{2\gamma-n-3} \mathbf{1}_{\{\delta/2 \leq d_g(x, a_i) \leq 4\delta\}} V_j \rho^{m_1} d\mu_g\right) \\
&= \kappa_\gamma \int_X \chi_\delta L_{2,\rho}^{m_1+2}(L_{2,\rho}^{m_0} \tilde{V}_i) V_j \rho^{m_1} d\mu_g + O(\delta) \varepsilon_{i,j} \\
&= \kappa_\gamma \int_X \chi_\delta L_{2,\rho}^{m_1}(L_{2,\rho}^{m_0} \tilde{V}_i) V_j \rho^{m_1} d\mu_g \\
&\quad - 2\kappa_\gamma \int_X \chi_\delta \rho^{-1} \partial_\rho(L_{2,\rho}^{m_0} \tilde{V}_i) V_j \rho^{m_1} d\mu_g + O(\delta) \varepsilon_{i,j} \\
&= \kappa_\gamma \int_X \chi_\delta L_{2,\rho}^{m_1}(L_{2,\rho}^{m_0} \tilde{V}_i) V_j \rho^{m_1} d\mu_g + O(\delta) \varepsilon_{i,j}.
\end{aligned}$$

It follows from integration by parts that

$$\begin{aligned}
&\int_X \chi_\delta L_{2,\rho}^{m_1}(L_{2,\rho}^{m_0} \tilde{V}_i) V_j \rho^{m_1} d\mu_g - \int_X L_{2,\rho}^{m_0}(\tilde{V}_i) L_{2,\rho}^{m_1}(\chi_\delta V_j) \rho^{m_1} d\mu_g \\
&= - \oint_M \lim_{\rho \rightarrow 0} \rho^{m_1} \partial_\rho(L_{2,\rho}^{m_0} \tilde{V}_i) V_j d\sigma_h + \oint_M \lim_{\rho \rightarrow 0} \rho^{m_1} \partial_\rho(\chi_\delta V_j) L_{2,\rho}^{m_0} \tilde{V}_i d\sigma_h \\
&= 0.
\end{aligned}$$

Then

$$I_1 = \kappa_\gamma \int_X (L_{2,\rho}^{m_0} \tilde{V}_i) L_{2,\rho}^{m_1} (\chi_\delta V_j) \rho^{m_1} d\mu_g + O(\delta) \varepsilon_{i,j} = O(\delta) \varepsilon_{i,j}. \quad \square$$

To deal with I_2 in (5-8), we have

$$\begin{aligned} \lim_{\rho \rightarrow 0} \rho^{m_1} \partial_\rho \Delta_\rho^{m_1} (V_i) &= \phi_{a_i}^{(n+2\gamma)/(n-2\gamma)} \lim_{\rho_{a_i} \rightarrow 0} \rho_{a_i}^{m_1} \partial_{\rho_{a_i}} \Delta_{\rho_{a_i}}^{m_1} (W_i) \\ &= \kappa_\gamma^{-1} \phi_{a_i}^{(n+2\gamma)/(n-2\gamma)} \chi_\delta w_i^{(n+2\gamma)/(n-2\gamma)}. \end{aligned}$$

Hence

$$\begin{aligned} I_2 &= \oint_M \phi_{a_i}^{(n+2\gamma)/(n-2\gamma)} \chi_\delta w_i^{(n+2\gamma)/(n-2\gamma)} v_j d\sigma_h \\ &= (1 + O(\delta)) \oint_M \chi_\delta w_i^{(n+2\gamma)/(n-2\gamma)} v_j d\sigma_h \\ &= (1 + O(\delta)) \varepsilon_{i,j} + O(\varepsilon_i^{2\gamma} \delta^{-2\gamma}) \varepsilon_{i,j}. \end{aligned}$$

Inserting the estimates of I_1 and I_2 into (5-8), we get the desired result. $\square$

6. Algebraic topological argument

In this section, we will outline the algebraic topological argument by [Bahri and Coron 1988]. We omit some standard proofs. Readers are encouraged to find them in [Mayer and Ndiaye 2017a].

To introduce the *neighborhood of potential critical points at infinity* of $\mathcal{E}_h^\gamma$, we first choose some $v_0 > 1$ and $v_0 \approx 1$, and some $\mu_0 > 0$ and $\mu_0 \approx 0$. With the later quantities fixed, for $p \in \mathbb{N}^*$, and $0 < \mu \leq \mu_0$, we define $V(p, \mu)$ the (p, μ) -neighborhood of potential critical points at infinity of $\mathcal{E}_h^\gamma$ by the following formula

$$\begin{aligned} V(p, \mu) &:= \{u \in W_+^{\gamma,2}(M) : \exists a_1, \dots, a_p \in M, \alpha_1, \dots, \alpha_p > 0, \\ &\quad 0 < \varepsilon_1, \dots, \varepsilon_p \leq \mu, \|u - \sum_{i=1}^p \alpha_i v_{a_i, \varepsilon_i, \delta}\| \leq \varepsilon, \\ &\quad \frac{\alpha_i}{\alpha_j} \leq v_0 \text{ and } \varepsilon_{i,j} \leq \mu, i \neq j = 1, \dots, p\}, \end{aligned}$$

where $\|\cdot\|$ denotes the standard $W^{\gamma,2}$ -norm.

Next, we introduce the sublevels of our Euler–Lagrange functional corresponding to the quantized values due to the involved bubbling phenomena. They are the sets L_p ($p \in \mathbb{N}$) defined as follows

$$L_p := \{u \in W_+^{\gamma,2}(M) : \mathcal{E}_h^\gamma[u] \leq (p+1)^{2\gamma/n} \mathcal{J}_{\mathbb{S}^n}^\gamma\} \quad \text{for } p \geq 1,$$

and

$$L_0 := \emptyset.$$

As in classical calculus of variations and classical critical points theory where Ekeland variational principle and deformation lemma play *dual role* in producing Palais–Smale sequences, here also for the Ekeland variational principle in the calculus of variations at infinity underlying the Aubin–Schoen’s minimizing technique, we have the following deformation lemma which plays the corresponding role in the critical point theory at infinity behind the barycenter technique that we are going to use. It follows from the profile decomposition (Lemma 5.1) and same arguments as in others applications of the algebraic topological argument of Bahri and Coron [1988].

Lemma 6.1 (deformation lemma). *Assuming that $\mathcal{E}_h^\gamma$ has no critical points, then for every $p \in \mathbb{N}^*$, there exists $0 < \mu_p < \mu_0$ such that, for every $0 < \mu \leq \mu_p$, it holds that (L_p, L_{p-1}) retracts by deformation onto $(L_{p-1} \cup A_p, L_{p-1})$ with $V(p, \tilde{\mu}) \subset A_p \subset V(p, \mu)$ where $0 < \tilde{\mu} < \frac{\mu}{4}$ is a very small positive real number which depends on μ .*

On the other hand, since we are in the global case, and no variant of the positive mass theorem is known to hold, then clearly there is no variational barrier available. However, as the mass there is another global invariant of the variational problem which is the interaction. Using the later information we will establish a *multiple* variational barrier estimate (see Proposition 6.3) which will play *dual role* in the application of the algebraic topological argument for existence.

Now we present some topological properties of the space of formal barycenter of M , that we need for our barycenter technique for existence. To do that we recall that for $p \in \mathbb{N}^*$ the set of formal barycenters of M of order p is defined as

$$B_p(M) = \left\{ \sum_{i=1}^p \alpha_i \delta_{a_i} : a_i \in M, \alpha_i \geq 0, i = 1, \dots, p, \sum_{i=1}^p \alpha_i = 1 \right\}, \quad B_0(M) = \emptyset,$$

where δ_a for $a \in M$ is the Dirac measure at a . Moreover we have the existence of $\mathbb{Z}_2$ orientation classes

$$(6-1) \quad w_p \in H_{(n+1)p-1}(B_p(M), B_{p-1}(M))$$

and that the cap product acts as follows

$$(6-2) \quad H^l(M^p/\sigma_p) \times H_k(B_p(M), B_{p-1}(M)) \xrightarrow{\cap} H_{k-l}(B_p(M), B_{p-1}(M)).$$

On the other hand, since M is a closed n -dimensional manifold, we have

$$\text{an orientation class } 0 \neq O_M^* \in H^n(M),$$

and there is a natural way to see $O_M^* \in H^n(M)$ as a nontrivial element of $H^n(M^p/\sigma_p)$; see [Mayer and Ndiaye 2017a, pages 532–533], namely

$$(6-3) \quad O_M^* \simeq O_p^* \quad \text{with } 0 \neq O_p^* \in H^n((M^p)/\sigma_p).$$

Recalling (6-2), and identifying O_M^* and O_p^* via (6-3), we have the following well-known formula.

Lemma 6.2. *There holds*

$$\begin{aligned} H^n((M^p)/\sigma_p) \times H_{(n+1)p-1}(B_p(M), B_{p-1}(M)) \\ \xrightarrow{\quad} H_{(n+1)p-(n+1)}(B_p(M), B_{p-1}(M)) \\ \xrightarrow{\quad \partial \quad} H_{(n+1)p-n-2}(B_{p-1}(M), B_{p-2}(M)), \end{aligned}$$

and

$$\omega_{p-1} = \partial(O_M^* \frown w_p).$$

Next we define for $p \in \mathbb{N}^*$ and $\varepsilon > 0$

$$f_p(\varepsilon) : B_p(M) \rightarrow W_+^{\gamma,2}(M) : \sigma = \sum_{i=1}^p \alpha_i \delta_{a_i} \in B_p(M) \rightarrow f_p(\varepsilon)(\sigma) = \sum_{i=1}^p \alpha_i v_{a_i, \varepsilon, \delta}.$$

Using the $f_p(\varepsilon)$, we express the multiple variational barrier in the following proposition:

Proposition 6.3. *There exists $v_0 > 1$ such that for every $p \in \mathbb{N}^*$, $p \geq 2$ and every $0 < \mu \leq \mu_0$, there exists $\varepsilon_p := \varepsilon_p(\mu)$ such that for every $0 < \varepsilon \leq \varepsilon_p$ and for every $\sigma = \sum_{i=1}^p \alpha_i \delta_{a_i} \in B_p(M)$, we have:*

- (1) *If $\sum_{i \neq j} \varepsilon_{i,j} > \mu$ or there exist $i_0 \neq j_0$ such that $\frac{\alpha_{i_0}}{\alpha_{j_0}} > v_0$, then*

$$\mathcal{E}_h^\gamma[f_p(\varepsilon)(\sigma)] \leq p^{2\gamma/n} \mathcal{Y}_{\mathbb{S}^n}^\gamma.$$

- (2) *If $\sum_{i \neq j} \varepsilon_{i,j} \leq \mu$ and for every $i \neq j$ we have $\frac{\alpha_i}{\alpha_j} \leq v_0$, then*

$$\mathcal{E}_h^\gamma[f_p(\varepsilon)(\sigma)] \leq p^{2\gamma/n} \mathcal{Y}_{\mathbb{S}^n}^\gamma (1 + C_6 \varepsilon^{(n-2\gamma)/2} - C_7(p-1) \varepsilon^{(n-2\gamma)/2}),$$

where $C_6, C_7 > 0$ depend on n, γ, g, δ .

Proof. Notice that in the definition of $f_p(\varepsilon)$ we are taking all ε_i the same. The proof is the same as the one of Proposition 3.1 in [Mayer and Ndiaye 2017a] using Propositions 2.2, 2.3 and Lemmas 5.5, 5.6, 5.9 and Propositions 3.1, 4.3, 4.4. $\square$

Now we start transporting the topology of the manifold M into the sublevels of the Euler–Lagrange functional $\mathcal{E}_h^\gamma$ by bubbling via $v_{a_i, \varepsilon, \delta}$. But before that, we first recall the definition of the selection map defined inside the neighborhood of potential critical points at infinity. For every $p \in \mathbb{N}^*$, there exists $0 < \mu_p \leq \mu_0$ such that for every $0 < \mu \leq \mu_p$ there holds:

- (6-4) $\forall u \in V(p, \mu)$ the minimization problem, $\min_{B_\mu^p} \|u - \sum_{i=1}^p \alpha_i v_{a_i, \varepsilon_i, \delta}\|$ has a solution, which is unique up to permutations,

where B_μ^p is defined as

$$B_\mu^p := \{(\bar{\alpha}, A, \bar{\lambda}) : \varepsilon_i \leq \mu, i = 1, \dots, p, \frac{\alpha_i}{\alpha_j} \leq v_0 \text{ and } \varepsilon_{i,j} \leq \mu, i \neq j = 1, \dots, p\}$$

where $(\bar{\alpha}, A, \bar{\lambda}) \in \mathbb{R}_+^p \times M^p \times (0, +\infty)^p$ and v_0 is as in Proposition 6.3. Furthermore we define the selection map via

$$s_p : V(p, \mu) \rightarrow (M)^p / \sigma_p : u \rightarrow s_p(u) = A \quad \text{and} \quad A \text{ is given by (6-4).}$$

Recalling (6-1) we have:

Lemma 6.4. *Assuming that $\mathcal{E}_h^\gamma$ has no critical points and $0 < \mu \leq \mu_1$, then up to taking μ_1 smaller and ε_1 smaller too, we have that for every $0 < \varepsilon \leq \varepsilon_1$, there holds*

$$f_1(\varepsilon) : (B_1(M), B_0(M)) \rightarrow (L_1, L_0)$$

is well defined and satisfies

$$(f_1(\varepsilon))_*(w_1) \neq 0 \quad \text{in} \quad H_n(L_1, L_0).$$

Proof. The proof follows from the same arguments as the ones used in the proof of Lemma 4.2 in [Mayer and Ndiaye 2017a] by using the selection map s_1 , Lemma 6.1 and Proposition 3.1, 4.3, 4.4. $\square$

Next we use the previous lemma and *pile up masses* by bubbling via $v_{a,\varepsilon,\delta}$ in a recursive way. Still recalling (6-1) we have:

Lemma 6.5. *Assuming that $\mathcal{E}_h^\gamma$ has no critical points and $0 < \mu \leq \mu_{p+1}$, then up to taking μ_{p+1} smaller, and ε_p and ε_{p+1} smaller too, we have that for every $0 < \varepsilon \leq \min\{\varepsilon_p, \varepsilon_{p+1}\}$, there holds*

$$f_{p+1}(\varepsilon) : (B_{p+1}(M), B_p(M)) \rightarrow (L_{p+1}, L_p)$$

and

$$f_p(\varepsilon) : (B_p(M), B_{p-1}(M)) \rightarrow (L_p, L_{p-1})$$

are well defined and satisfy

$$(f_p(\varepsilon))_*(w_p) \neq 0 \quad \text{in} \quad H_{np-1}(L_p, L_{p-1})$$

implies

$$(f_{p+1}(\varepsilon))_*(w_{p+1}) \neq 0 \quad \text{in} \quad H_{n(p+1)-1}(L_{p+1}, L_p).$$

Proof. The proof follows from the same arguments as the ones used in the proof of Lemma 4.3 in [Mayer and Ndiaye 2017a], by using the selection map s_p , Lemma 6.1 and Proposition 6.3. $\square$

Finally we use the strength of Proposition 6.3 — namely point (ii) — to give a criterion ensuring that the recursive process of *piling up* masses via Lemma 6.5 will lead to a topological contradiction after a very large number of steps.

Lemma 6.6. *Setting*

$$p^* := \left[1 + \frac{\mathcal{C}_6}{\mathcal{C}_7} \right] + 1,$$

we have that $\forall 0 < \varepsilon \leq \varepsilon_{p^}$ there holds*

$$f_{p^*}(\varepsilon)[B_{p^*}(M)] \subset L_{p^*-1}.$$

Proof. The proof is a direct application of Proposition 6.3 . □

Proof of Theorems 1.1 and 1.2. It follows by a contradiction argument from Lemmas 6.4–6.6. □

7. Case (I-2): low dimension in AH

In this section, we want to show that our method could also apply to some asymptotically hyperbolic case. Suppose (X^{n+1}, g_+) is an asymptotically hyperbolic manifold with conformal infinity $(M^n, [h])$. Assume also ρ is the geodesic defining function of a representative metric h . Furthermore we require

$$(7-1) \quad R[g_+] + n(n+1) = o(\rho) \text{ as } \rho \rightarrow 0 \text{ uniformly on } M.$$

Then it follows from [Kim et al. 2018, Lemma 2.3] that the mean curvature $H = 0$. According to [Kim et al. 2018, Lemma 2.2 and 2.4], for any point $a \in M$, there exist $h_a \in [h]$ (write h_a as h for short) and the geodesic defining function ρ_a near M such that the metric $g = \rho_a^2 g_+$ has the following expansion

$$(7-2) \quad \begin{aligned} g^{ij}(x) &= \delta_{ij} + 2\pi_{ij}x_N + \frac{1}{3}R_{ikjl}[h]x_kx_l + g_{,Nk}^{ij}x_Nx_k \\ &\quad + (3\pi_{ik}\pi_{kj} + R_{iNjN}[g])x_N^2 + O(|x|^3) \\ \sqrt{|g|}(x) &= 1 - \frac{1}{6}\text{Ric}[h]_{ij}x_ix_j - \left(\frac{1}{2}\|\pi\|^2 + \text{Ric}[g]_{NN}\right)x_N^2 + O(|x|^3) \\ &\quad \text{in } B_+^N(0, \delta). \end{aligned}$$

In terms of Fermi coordinates around a . Here π is the second fundamental form of $(M, h) \subset (\bar{X}, g)$. Every tensor in the expansion is computed at $a = 0$.

As in (3-2), we define

$$U_{a,\varepsilon,\delta}(x) = \chi_\delta W_\varepsilon(\Psi_a(x)) + (1 - \chi_\delta(\Psi_a(x)))\varepsilon^{(n-2\gamma)/2}G_a^\gamma(x)$$

for $C_0\varepsilon < \delta \leq \delta_0 \leq 1$. We shall consider the case $n < 2 + 2\gamma$ and $\gamma \in (0, 1)$, which is a global case, notice this implies $n = 3$ and $\gamma \in (\frac{1}{2}, 1)$.

Proposition 7.1. *Suppose that $n < 2 + 2\gamma$ and $\gamma \in (0, 1)$. If (7-1) holds and δ_0 small enough and C_0 large enough, then there exists a constant $\mathcal{C}_8 > 0$ such that*

$$(7-3) \quad \mathcal{E}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \bar{\mathcal{E}}_h^\gamma[U_{a,\varepsilon,\delta}] \leq \mathcal{Y}_{\mathbb{S}^n}^\gamma + \varepsilon^{n-2\gamma}\mathcal{C}_8(n, \gamma, g, \delta) + o(\varepsilon^{n-2\gamma}).$$

Proof. The proof is similar to the one of Proposition 3.1. The energy inequality of (2-6) in [Case 2017] goes through verbatim in AH setting for $\gamma \in (0, 1)$. One just needs to use the expansion of the metric in (7-2) instead of (3-1). $\square$

Once the above proposition is established, then we have the corresponding self-action estimates in Lemma 5.5. Although (5-7) will be changed to $O(W_\varepsilon)$, the interaction estimates Lemma 5.9 still holds in this case. Therefore, one can also run the critical points at infinity approach.

Appendix: Some estimates

In this appendix, we will provide some details for the estimates used in the previous sections.

Lemma A.1. *Suppose $n > 2\gamma$. $W_\varepsilon = W_{\varepsilon,0}$ is defined in (2-17). Denote $|x| = |\bar{x}|^2 + x_N^2$ on $\mathbb{R}_+^N$, then:*

- (1) $W_\varepsilon(\bar{x}, x_N) = O(\varepsilon^{(n-2\gamma)/2}(\varepsilon^2 + |x|^2)^{-(n-2\gamma)/2})$.
- (2) $\partial_N W_\varepsilon(\bar{x}, x_N) = O(\varepsilon^{(n-2\gamma)/2} x_N^{2\gamma-1}(\varepsilon^2 + |x|^2)^{-n/2})$.
- (3) $\nabla_{\bar{x}} W_\varepsilon(\bar{x}, x_N) = O(\varepsilon^{(n-2\gamma)/2}(\varepsilon^2 + |x|^2)^{-(n-2\gamma+1)/2})$.
- (4) $\nabla_{\bar{x}}^2 W_\varepsilon(\bar{x}, x_N) = O(\varepsilon^{(n-2\gamma)/2}(\varepsilon^2 + |x|^2)^{-(n-2\gamma+2)/2})$.
- (5) $\partial_N \nabla_{\bar{x}}^2 W_\varepsilon(\bar{x}, x_N) = O(\varepsilon^{(n-2\gamma)/2} x_N^{2\gamma-1}(\varepsilon^2 + |x|^2)^{-(n+2)/2})$, for $\gamma > 1$.

Proof. These estimates follow from [Mayer and Ndiaye 2017b, Corollary 5.2]. One of crucial observation in [Mayer and Ndiaye 2017b, (47)] is that $W_{\varepsilon,\sigma}$ in (2-17) can be interpreted as the interaction of standard bubbles on $\mathbb{R}^n$. $\square$

Let us use the notation

$$W = W_1(|\bar{x}|, x_N) \quad \text{and} \quad r = |\bar{x}|.$$

We have the following list of formulae. Here we borrow the notations $\mathcal{F}_i$ from [Kim et al. 2018, Lemma B.6].

Lemma A.2. *If $n > 2\gamma + 4$, then*

$$\begin{aligned} \mathcal{A}_1 &= \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} r \partial_N W \partial_r W dx = \frac{1}{4} \left[\frac{n}{2} \mathcal{F}_2 + \left(\frac{n}{2} - 1 \right) \mathcal{F}_3 + \mathcal{F}_7 \right], \\ \mathcal{A}_2 &= \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r \partial_{NN}^2 W \partial_r W dx = -(5-2\gamma) \mathcal{A}_1 + \frac{n}{2} (\mathcal{F}_2 - \mathcal{F}_3), \\ \mathcal{A}_3 &= \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} r^2 \partial_N W \partial_{rr}^2 W dx = -(n+1) \mathcal{A}_1 - \mathcal{F}_9. \end{aligned}$$

Proof. Integration by parts gives

$$\begin{aligned}
\mathcal{A}_2 &= \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r \partial_{NN}^2 W \partial_r W \, dx \\
&= -(5-2\gamma) \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} r \partial_N W \partial_r W \, dx - \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r \partial_N W \partial_{rN}^2 W \, dx \\
&= -(5-2\gamma) \mathcal{A}_1 - \frac{1}{2} \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r \partial_r |\partial_N W|^2 \, dx \\
&= -(5-2\gamma) \mathcal{A}_1 + \frac{n}{2} \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} |\partial_N W|^2 \, dx \\
&= -(5-2\gamma) \mathcal{A}_1 + \frac{n}{2} (\mathcal{F}_2 - \mathcal{F}_3).
\end{aligned}$$

Using (2-23), one obtains

$$\begin{aligned}
-(1-2\gamma) \mathcal{A}_1 &= \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r \Delta W \partial_r W \, dx \\
&= \mathcal{F}_7 + (n-1) \mathcal{F}_3 + \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r \partial_{NN}^2 W \partial_r W \, dx \\
&= \mathcal{F}_7 + (n-1) \mathcal{F}_3 - (5-2\gamma) \mathcal{A}_1 + \frac{n}{2} (\mathcal{F}_2 - \mathcal{F}_3).
\end{aligned}$$

One can combine the above two equalities to get $\mathcal{A}_1$ and $\mathcal{A}_2$. Similarly

$$\begin{aligned}
\mathcal{A}_3 &= \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} r^2 \partial_N W \partial_{rr}^2 W \, dx \\
&= -(n+1) \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} r \partial_N W \partial_r W \, dx - \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} r^2 \partial_r W \partial_{rN}^2 W \, dx \\
&= -(n+1) \mathcal{A}_1 - \mathcal{F}_9.
\end{aligned}$$

□

Lemma A.3. Suppose $n > 2\gamma + 4$ and $\gamma \in (1, \min\{2, \frac{n}{2}\})$, then $\mathcal{C}_4$ defined in (4-19) is positive.

Proof. Inserting the expressions of $\mathcal{A}_1$ and $\mathcal{A}_3$ into the previous lemma into (4-19) gives

$$\begin{aligned}
n\mathcal{C}_4 &= -\frac{n(n-2\gamma)}{2} \mathcal{F}_1 - \frac{n}{2} \mathcal{F}_2 - \left(\frac{n}{2} - 1\right) \mathcal{F}_3 + \frac{n-2\gamma}{2} \mathcal{F}_5 \\
&\quad - \frac{n^2 - n + 4}{(n-1)(n+2)} \mathcal{F}_6 - \mathcal{F}_7 - \frac{6}{n+2} \mathcal{F}_9 \\
&= I_1 + I_2 + \frac{n^2 - n + 4}{(n-1)(n+2)} \mathcal{F}_6
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= -\frac{n}{2}\mathcal{F}_2 - \left(\frac{n}{2} - 1\right)\mathcal{F}_3 - \mathcal{F}_7 - \frac{6}{n+2}\mathcal{F}_9 \\
&= -\frac{2(2-\gamma)(12\gamma(\gamma+2) + 5n^2 - 8(\gamma+2)n)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)}A_3B_2, \\
I_2 &= -\frac{n(n-2\gamma)}{2}\mathcal{F}_1 + \frac{n-2\gamma}{2}\mathcal{F}_5 \\
&= \frac{n(n-2\gamma)(-4\gamma^2 + 3n^2 - 18n + 28)}{2(\gamma+1)(n-4)(n-4-2\gamma)(n-4+2\gamma)}A_3B_2.
\end{aligned}$$

Here we were using the expression of $\mathcal{F}_i$ in [Kim et al. 2018, Lemma B.6]. Now it is not hard to show $I_1 + I_2 > 0$ for $n > 4 + 2\gamma$ and $\gamma \in (1, \min\{2, \frac{n}{2}\})$. Consequently, $\mathcal{C}_4 > 0$. $\square$

Lemma A.4. *Suppose that $0 < 2\varepsilon \leq \delta \leq 1$, $n = 2\gamma + 4 = 7$ and W is defined in (4-20), then*

$$\begin{aligned}
\int_{B_+^N(0, \delta/\varepsilon)} W^2 dx &= \frac{5\pi}{32}\alpha_{7,3/2}^2 |\mathbb{S}^6| \log\left(\frac{\delta}{\varepsilon}\right) + O(1), \\
\int_{B_+^N(0, \delta/\varepsilon)} r^2 (\partial_N W)^2 dx &= \frac{7\pi}{32}\alpha_{7,3/2}^2 |\mathbb{S}^6| \log\left(\frac{\delta}{\varepsilon}\right) + O(1), \\
\int_{B_+^N(0, \delta/\varepsilon)} r^2 (\partial_r W)^2 dx &= \frac{63\pi}{32}\alpha_{7,3/2}^2 |\mathbb{S}^6| \log\left(\frac{\delta}{\varepsilon}\right) + O(1), \\
\int_{B_+^N(0, \delta/\varepsilon)} x_N r \partial_N W \partial_r W dx &= \frac{7\pi}{32}\alpha_{7,3/2}^2 |\mathbb{S}^6| \log\left(\frac{\delta}{\varepsilon}\right) + O(1), \\
\int_{B_+^N(0, \delta/\varepsilon)} x_N r^2 \partial_N W (\partial_{rr}^2 W - r^{-1} \partial_r W) dx &= -\frac{63\pi}{64}\alpha_{7,3/2}^2 |\mathbb{S}^6| \log\left(\frac{\delta}{\varepsilon}\right) + O(1),
\end{aligned}$$

where $\alpha_{7,3/2}$ is defined in (2-26) and $|\mathbb{S}^6|$ is the volume of a 6 dimensional sphere.

Proof. We show how to get the second estimate, the others follow from this similarly:

$$\begin{aligned}
\partial_r W &= -4\alpha_{7,3/2} \frac{|\bar{x}|(x_N^2 + 8x_N + 1 + |\bar{x}|^2)}{[(1+x_N)^2 + |\bar{x}|^2]^4}, \\
\partial_N W &= -4\alpha_{7,3/2} \frac{x_N(x_N^2 + 8x_N + 7 + |\bar{x}|^2)}{[(1+x_N)^2 + |\bar{x}|^2]^4}, \\
\partial_{rr}^2 W - r^{-1} \partial_r W &= 24\alpha_{7,3/2} \frac{|\bar{x}|^2(x_N^2 + 10x_N + 1 + |\bar{x}|^2)}{[(1+x_N)^2 + |\bar{x}|^2]^5}.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_{B_+^N(0, \delta/\varepsilon)} r^2 (\partial_N W)^2 dx \\
&= 16\alpha_{7,3/2}^2 \int_{\mathbb{R}_+^N \cap \{x_N \leq \delta/\varepsilon\}} \frac{r^2 x_N^2 (x_N^2 + 8x_N + 7 + |\bar{x}|^2)^2}{[(1+x_N)^2 + |\bar{x}|^2]^8} d\bar{x} dx_N + O(1) \\
&= 16\alpha_{7,3/2}^2 \int_0^{\delta/\varepsilon} \int_{\mathbb{R}^n} \frac{x_N^2}{(1+x_N)^3} \frac{s^2 \left(\frac{x_N+7}{x_N+1} + s^2\right)^2}{(1+s^2)^8} s^6 ds dx_N + O(1) \\
&= 16\alpha_{7,3/2}^2 \int_0^{\delta/\varepsilon} \int_{\mathbb{R}^n} \frac{x_N^2}{(1+x_N)^3} \frac{s^8}{(1+s^2)^6} ds dx_N + O(1) \\
&= \frac{7\pi}{32} \alpha_{7,3/2}^2 \log\left(\frac{\delta}{\varepsilon}\right) + O(1). \quad \square
\end{aligned}$$

Suppose χ_δ is defined in (2-28) and $W_{\varepsilon,\sigma}$ is defined in (2-17). Let $\Psi_a : \mathcal{O}(a) \rightarrow B_+^N(0, 2\delta)$ be the Fermi coordinate map. Let us use the short notation $V_i = V_{a_i, \varepsilon_i, \delta}$ in (5-1), $\chi_i = \chi_\delta(\Psi_{a_i})$, $W_i = W_{\varepsilon_i}(\Psi_{a_i})$.

Lemma A.5. *Suppose $\gamma \in (0, 1)$ and $C_0\varepsilon_j \leq C_0\varepsilon_i \leq \delta < \delta_0$ is small enough, then*

- (1) $\int_X \rho^{m_0} \chi_i W_i V_j d\mu_g \leq C\delta^2 \varepsilon_{i,j}$,
- (2) $\int_X \rho^{m_0} \varepsilon_i^{(n-2\gamma)/2} \delta^{2\gamma-n-1} \mathbf{1}_{\{1/2\delta \leq d_{g_{a_i}}(x, a_i) \leq 4\delta\}} V_j d\mu_g \leq C\delta \varepsilon_{i,j}$,

where $\varepsilon_{i,j}$ is defined in (5-3).

Proof. We use the techniques in [Brendle 2005, Lemma B.4].

(1) Assume that δ_0 is small enough such that the support of χ_i is contained in $\{x \in X : d_g(x, a_i) \leq 4\delta\}$. Denote

$$\mathcal{A} = \{x \in X : 2d_g(a_j, x) \leq \varepsilon_i + d_g(a_i, a_j)\} \cap \{d_g(x, a_i) \leq 4\delta\},$$

$$\mathcal{A}^c = \{x \in X : 2d_g(a_j, x) > \varepsilon_i + d_g(a_i, a_j)\} \cap \{d_g(x, a_i) \leq 4\delta\}.$$

Then it follows from Lemma A.1 that

$$\begin{aligned}
& \int_X \rho^{m_0} \chi_i W_i V_j d\mu_g \\
&= \int_{\mathcal{A} \cup \mathcal{A}^c} \rho^{m_0} \chi_i W_i V_j d\mu_g \\
&\leq C \left(\int_{\mathcal{A}} + \int_{\mathcal{A}^c} \right) \rho^{m_0} \left(\frac{\varepsilon_i}{\varepsilon_i^2 + d_g(x, a_i)^2} \right)^{(n-2\gamma)/2} \left(\frac{\varepsilon_j}{\varepsilon_j^2 + d_g(x, a_j)^2} \right)^{(n-2\gamma)/2} d\mu_g \\
&= I_1 + I_2.
\end{aligned}$$

For I_2 , we have

$$\begin{aligned} I_2 &\leq C \int_{\{d_g(x, a_i) \leq 4\delta\}} \rho^{m_0} \left(\frac{\varepsilon_i}{\varepsilon_i^2 + d_g(x, a_i)^2} \right)^{(n-2\gamma)/2} \left(\frac{\varepsilon_j}{\varepsilon_i^2 + d_g(a_i, a_j)^2} \right)^{(n-2\gamma)/2} d\mu_g \\ &\leq C \delta^2 \frac{\varepsilon_i^{(n-2\gamma)/2} \varepsilon_j^{(n-2\gamma)/2}}{(\varepsilon_i^2 + d_g(a_i, a_j)^2)^{(n-2\gamma)/2}} \\ &\leq C \delta^2 \varepsilon_{i,j}, \end{aligned}$$

where in the last inequality we used $\varepsilon_j \leq \varepsilon_i$. To deal with I_1 , notice that on $\mathcal{A}$, one has

$$\varepsilon_i + d_g(x, a_i) \geq \varepsilon_i + d_g(a_i, a_j) - d_g(a_j, x) \geq \frac{1}{2}(\varepsilon_i + d_g(a_i, a_j)).$$

Consequently $d_g(a_i, a_j) \leq \delta + 2d_g(x, a_i) \leq 9\delta$ and $\mathcal{A} \subset \{d_g(x, a_j) \leq 5\delta\}$. Then

$$\begin{aligned} I_1 &\leq C \int_{\{d_g(x, a_j) \leq 5\delta\}} \left(\frac{\varepsilon_i}{\varepsilon_i^2 + d_g(a_i, a_j)^2} \right)^{(n-2\gamma)/2} \left(\frac{\varepsilon_j}{\varepsilon_j^2 + d_g(x, a_j)^2} \right)^{(n-2\gamma)/2} d\mu_g \\ &\leq C \delta^2 \frac{\varepsilon_i^{(n-2\gamma)/2} \varepsilon_j^{(n-2\gamma)/2}}{(\varepsilon_i^2 + d_g(a_i, a_j)^2)^{(n-2\gamma)/2}} \leq C \delta^2 \varepsilon_{i,j}. \end{aligned}$$

Combining the estimates of I_1 and I_2 , we can prove (1).

(2) Taking δ_0 small enough such that

$$\begin{aligned} \int_X \rho^{m_0} \varepsilon_i^{(n-2\gamma)/2} \delta^{2\gamma-n-1} \mathbf{1}_{\{1/2\delta \leq d_{g_{a_i}}(x, a_i) \leq 4\delta\}} V_j d\mu_g \\ \leq C \frac{1}{\delta} \int_{\{1/2\delta \leq d_g(x, a_i) \leq 8\delta\}} \rho^{m_0} \left(\frac{\varepsilon_i}{\varepsilon_i^2 + d_g(x, a_i)^2} \right)^{(n-2\gamma)/2} \\ \times \left(\frac{\varepsilon_j}{\varepsilon_j^2 + d_g(x, a_j)^2} \right)^{(n-2\gamma)/2} d\mu_g. \end{aligned}$$

One can use the proof of (1) without significant change to conclude (2). $\square$

Similarly we have:

Lemma A.6. Suppose that $\gamma \in (1, \min\{2, \frac{n}{2}\})$, and $C_0 \varepsilon_j \leq C_0 \varepsilon_i \leq \delta < \delta_0$ small enough, then:

- (1) $\int_X \rho^{m_1} \chi_i W_i V_j d\mu_g \leq C \delta^4 \varepsilon_{i,j}.$
- (2) $\int_X \rho^{m_1} \varepsilon_i^{(n-2\gamma)/2} \delta^{2\gamma-n-3} \mathbf{1}_{\{1/2\delta \leq d_{g_{a_i}}(x, a_i) \leq 4\delta\}} V_j d\mu_g \leq C \delta \varepsilon_{i,j}.$

Now let us prove some interaction estimates on the boundary.

Lemma A.7. Suppose that $\gamma \in (0, 1) \cup (1, \min\{2, \frac{n}{2}\})$, and $C_0 \varepsilon_j \leq C_0 \varepsilon_i \leq \delta \leq \delta_0$, $v_i = v_{a_i, \varepsilon_i, \delta}$ is defined in (5-1). Then

$$\oint_M |v_i^{(n+2\gamma)/n-2\gamma} - \chi_i w_i^{(n+2\gamma)/n-2\gamma}| v_j d\sigma_h \leq C \frac{\varepsilon_i^{2\gamma}}{\delta^{2\gamma}} \varepsilon_{i,j}.$$

here w_i and v_i are defined in (2-17) and (5-1).

Proof. Since M and $\bar{X}$ are smooth compact manifolds, the metric $d_g(x, a)$ and $d_h(x, a)$ for $x, a \in M$ are comparable. Notice that by Lemma A.1, (2-8) and (2-14)

$$|v_i^{(n+2\gamma)/(n-2\gamma)} - \chi_i w_i^{(n+2\gamma)/(n-2\gamma)}| \leq C(1 - \chi_i) \left(\frac{\varepsilon_i}{\varepsilon_i^2 + d_h(x, a_i)^2} \right)^{(n+2\gamma)/2}.$$

Define

$$\mathcal{A} = \{x \in M : 2d_h(a_j, x) \leq \varepsilon_i + d_h(a_i, a_j)\} \cap \left\{d_h(x, a_i) \geq \frac{\delta}{2}\right\},$$

$$\mathcal{A}^c = \{x \in M : 2d_h(a_j, x) > \varepsilon_i + d_h(a_i, a_j)\} \cap \left\{d_h(x, a_i) \geq \frac{\delta}{2}\right\}.$$

Then

$$\begin{aligned} & \oint_M (v_i^{(n+2\gamma)/(n-2\gamma)} - \chi_i w_i^{(n+2\gamma)/(n-2\gamma)}) v_j d\sigma_h \\ & \leq C \int_{\mathcal{A} \cup \mathcal{A}^c} \left(\frac{\varepsilon_i}{\varepsilon_i^2 + d_h(x, a_i)^2} \right)^{(n+2\gamma)/2} \left(\frac{\varepsilon_j}{\varepsilon_j^2 + d_h(x, a_j)^2} \right)^{(n-2\gamma)/2} d\sigma_h \\ & \leq C \int_{\mathcal{A}} \frac{\varepsilon_i^{(n+2\gamma)/2}}{\delta^{2\gamma} (\varepsilon_i^2 + d_h(a_i, a_j)^2)^{n/2}} \left(\frac{\varepsilon_j}{\varepsilon_j^2 + d_h(x, a_j)^2} \right)^{(n-2\gamma)/2} d\sigma_h \\ & \quad + C \int_{\{d_h(x, a_i) > \delta/2\}} \left(\frac{\varepsilon_i}{\varepsilon_i^2 + d_h(x, a_i)^2} \right)^{(n+2\gamma)/2} \left(\frac{\varepsilon_j}{\varepsilon_j^2 + d_h(a_i, a_j)^2} \right)^{(n-2\gamma)/2} d\sigma_h \\ & \leq C \frac{\varepsilon_i^{2\gamma}}{\delta^{2\gamma}} \frac{\varepsilon_i^{(n-2\gamma)/2} \varepsilon_j^{(n-2\gamma)/2}}{(\varepsilon_i^2 + d_h(a_i, a_j)^2)^{(n-2\gamma)/2}} \\ & \leq C \frac{\varepsilon_i^{2\gamma}}{\delta^{2\gamma}} \varepsilon_{i,j}. \end{aligned}$$

In the last inequality we used $\varepsilon_j \leq \varepsilon_i$. □

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