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ASYMPTOTICS OF THE POISSON KERNEL AND GREEN'S FUNCTIONS OF THE FRACTIONAL CONFORMAL LAPLACIAN

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Abstract. We study the asymptotics of the Poisson kernel and Green's functions of the fractional conformal Laplacian for conformal infinities of asymptotically hyperbolic manifolds. We derive sharp expansions of the Poisson kernel and Green's functions of the conformal Laplacian near their singularities. Our expansions of the Green's functions answer the first part of the conjecture of Kim-Musso-Wei[21] in the case of locally flat conformal infinities of Poincare-Einstein manifolds and together with the Poisson kernel asymptotic is used also in our paper [25] to show solvability of the fractional Yamabe problem in that case. Our asymptotics of the Green's functions on the general case of conformal infinities of asymptotically hyperbolic space is used also in [29] to show solvability of the fractional Yamabe problem for conformal infinities of dimension 3 and fractional parameter in $(\frac{1}{2}, 1)$ corresponding to a global case left by previous works.

1. **Introduction.** In the last decades there has been a lot of study about fractional order operators in Analysis and Geometric Analysis as well. In both fields, the recurrent themes are existence, regularity and sharp estimates, see [3], [4], [5], [6], [7], [8], [11], [10], [30], [17], [18], [14], [15]). In this paper we are interested in the issue of existence, regularity and sharp estimates in the context of Conformal Geometry. Precisely, we study the issue of existence, regularity and sharp asymptotics of the Poisson and Green's functions of the fractional conformal Laplacian on conformal infinities of asymptotically hyperbolic manifolds.

To introduce the fractional conformal Laplacian, we first recall some definitions in the theory of asymptotically hyperbolic metrics. Given $X = X^{n+1}$ a smooth

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manifold with boundary $M = M^n$ and $n \geq 2$ we say that ϱ is a defining function of the boundary M in X, if

$$\varrho > 0 \quad \text{in} \quad X, \quad \varrho = 0 \quad \text{on} \quad M \quad \text{and} \quad d\varrho \neq 0 \quad \text{on} \quad M.$$

A Riemannian metric g^+ on X is said to be conformally compact, if for some defining function ϱ, the Riemannian metric

$$g := \varrho^2 g^+ \tag{1}$$

extends to $\overline{X} := X \cup M$ so that $(\overline{X},\, g)$ is a compact Riemannian manifold with boundary M and interior X. Clearly this induces a conformal class of Riemannian metrics

$$[h] = [g|_{TM}]$$

on M, where TM denotes the tangent bundle of M, when the defining functions ϱ vary and the resulting conformal manifold $(M, [h])$ is called conformal infinity of $(X,\, g^+)$. Moreover a Riemannian metric g^+ in X is said to be asymptotically hyperbolic, if it is conformally compact and its sectional curvature tends to -1 as one approaches the conformal infinity of $(X,\, g^+)$, which is equivalent to

$$|d\varrho|_{\bar{g}} = 1$$

on M, see [26], and in such a case $(X,\, g^+)$ is called an asymptotically hyperbolic manifold. Furthermore a Riemannian metric g^+ on X is said to be conformally compact Einstein or Poincaré-Einstein (PE), if it is asymptotically hyperbolic and satisfies the Einstein equation

$$Ric_{g^+} = -ng^+,$$

where Ric_{g^+} denotes the Ricci tensor of $(X,\, g^+)$.

On one hand for every asymptotically hyperbolic manifold $(X,\, g^+)$ and every choice of the representative h of its conformal infinity $(M, [h])$, there exists a geodesic defining function y of M in X such that in a tubular neighborhood of M in X, the Riemannian metric g^+ takes the following normal form

$$g^+ = \frac{dy^2 + h_y}{y^2}, \tag{2}$$

where h_y is a family of Riemannian metrics on M satisfying $h_0 = h$ and y is the unique such a one in a tubular neighborhood of M. Furthermore we say that the conformal infinity $(M,\, [\hat{h}])$ of an asymptotically hyperbolic manifold $(X,\, g^+)$ is locally flat, if h is locally conformally flat, and clearly this is independent of the representative h of $[h]$. Moreover we say that $(M, [h])$ is umbilic, if (M, h) is umbilic in $(X,\, g)$ where g is given by (1) and y is the unique geodesic defining function given by (2), and this is clearly independent of the representative h of $[h]$, as easily seen from the uniqueness of the normal form (2) or Lemma 2.3 in [15]. Similarly we say that $(M, [h])$ is minimal if $H_g = 0$ with H_g denoting the mean curvature of $(M,\, h)$ in $(\overline{X},\, g)$ with respect to the inward direction, and this is again clearly independent of the representative of h of $[h]$, as easily seen from Lemma 2.3 in [15]. Finally we say that $(M, [h])$ is totally geodesic, if $(M, [h])$ is umbilic and minimal.

Remark 1. We remark that in the conformally compact Einstein case, h_y as in (2) has an asymptotic expansion which contains only even powers of y, at least up to order n, see [8]. In particular the conformal infinity $(M, [h])$ of any Poincaré-Einstein manifold (X, g^+) is totally geodesic.

Remark 2. As every 2-dimensional Riemannian manifold is locally conformally flat, we will say locally flat conformal infinity of a Poincaré-Einstein manifold to mean just the conformal infinity of a Poincaré-Einstein manifold when the dimension is either 2 or which is further locally flat if the dimension is bigger than 2.

On the other hand, for any asymptotically hyperbolic manifold (X, g^+) with conformal infinity $(M, [h])$, Graham-Zworsky[17] have attached a family of scattering operators $S(s)$ which is a meromorphic family of pseudo-differential operators on M defined on $\mathbb{C}$, by considering Dirichlet-to-Neumann operators for the scattering problem for $(X,\ g^+)$ and a meromorphic continuation argument. Indeed it follows from [17] and [28] that for every $f \in C^\infty(M)$, and for every $s \in \mathbb{C}$ such that $Re(s) > \frac{n}{2}$ and $s(n-s)$ is not an L^2-eigenvalue of $-\Delta_{g^+}$, the following generalized eigenvalue problem

$$-\Delta_{g^+}u - s(n-s)u = 0 \quad \text{in} \quad X \tag{3}$$

has a solution of the form

$$u = Fy^{n-s} + Gy^s, \quad F,\ G \in C^\infty(\overline{X}), \quad F|_{y=0} = f,$$

where y is given by (2) and for those values of s the scattering operator $S(s)$ on M is defined as

$$S(s)f = G|_M. \tag{4}$$

Furthermore using a meromorphic continuation argument, Graham-Zworsky[17] extend $S(s)$ defined by (4) to a meromorphic family of pseudo-differential operators on M defined on all $\mathbb{C}$ and still denoted by $S(s)$ with only a discrete set of poles including the trivial ones $s = \frac{n}{2}, \frac{n}{2}+1, \cdots,$ which are simple poles of finite rank, and possibly some others corresponding to the L^2-eigenvalues of $-\Delta_{g^+}$. Using the regular part of the scattering operators $S(s)$, to any $\gamma = s - \frac{n}{2} \in (0,1)$ such that

$$\left(\frac{n}{2}\right)^2 - \gamma^2 < \lambda_1(-\Delta_{g^+})$$

with $\lambda_1(-\Delta_{g^+})$ denoting the first eigenvalue of $-\Delta_{g_+}$, Chang-Gonzalez[8] have attached the following fractional order pseudo-differential operators referred to as fractional conformal Laplacians or fractional Paneitz operators

$$P^\gamma[g^+,\ h] := -d_\gamma S\left(\frac{n}{2}+\gamma\right), \tag{5}$$

where d_γ is a positive constant depending only on γ and chosen such that the principal symbol of $P^\gamma[g^+, h]$ is exactly the same as the one of the fractional Laplacian $(-\Delta_h)^\gamma$, when

$$X = \mathbb{R}^{n+1}_+,\ M = \mathbb{R}^n,\ h = g_{\mathbb{R}^n} \quad \text{and} \quad g^+ = g_{\mathbb{H}^{n+1}}.$$

When there is no possible confusion with the metric g^+, we just use the simple notation

$$P^\gamma_h := P^\gamma[g^+,\ h].$$

Similarly to the other well studied conformally covariant differential operators, Chang-Gonzalez[8] associate to each P^γ_h the curvature quantity

$$Q^\gamma_h := P^\gamma_h(1).$$

The Q^γ_h are referred to as fractional scalar curvatures, fractional Q-curvatures or simply Q^γ-curvatures. Of particular importance to conformal geometry is the conformal covariance property verified by P^γ_h

$$P^\gamma_{h_u}(v) = v^{-\frac{n+2\gamma}{n-2\gamma}} P^\gamma_h(uv) \quad \text{for} \quad h_v = v^{\frac{4}{n-2\gamma}} \quad \text{and} \quad 0 < v \in C^\infty(M). \tag{6}$$

The fractional Yamabe problem is the problem of finding conformal metrics of with constant Q^γ-curvature. As in the classical Yamabe problem, see [31], its study deeply depends on the existence, regularity and sharp asymptotic of the Green's function of P^γ_h.

In this paper, we show existence, regularity and sharp asymptotics of the Poisson kernel K_g and Green's functions Γ_g under weighted Neumann boundary conditions of the Chang-Gonzalez[8] extension problem associated to P^γ_h and the Green's function G_h of P^γ_h. Indeed recalling (12), we prove:

Theorem 1.1. *Let $(X,\ g^+)$ be an asymptotically hyperbolic manifold with conformal infinity $(M,[h])$ of dimension $n\geq 2$. If*

$$\frac{1}{2}\neq\gamma\in(0,1)\quad and\quad \lambda_1(-\Delta_{g^+})>s(n-s)\quad for\ s=\frac{n}{2}+\gamma,$$

then the Poison kernel K_g and the Green's functions Γ_g and G_h respectively for

$$\begin{cases}D_gU=0 & in\quad X\\ U=f & on\quad ,\end{cases}$$

$$\begin{cases}D_gU=0 & in\quad X\\ -d^*_\gamma\lim_{y\to 0}y^{1-2\gamma}\partial_yU=f & on\quad M\end{cases}$$

and

$$\left\{P^\gamma_h u=f\quad on\quad M\right.$$

exist and we may expand in g-normal Fermi-coordinates around $\xi\in M$

$$\begin{aligned}K_g(z,\xi)\ &\in\eta_\xi(z)\left(p_{n,\gamma}\tfrac{y^{2\gamma}}{|z|^{n+2\gamma}}+\textstyle\sum_{l=-n-2\gamma}^{2m+5-2\gamma}y^{2\gamma}H_{1+l}(z)\right)+y^{2\gamma}C^{2m,\alpha}(X)\\ \Gamma_g(z,\xi)\ &\in\eta_\xi(z)\left(\tfrac{g_{n,\gamma}}{|z|^{n-2\gamma}}+\textstyle\sum_{l=-n}^{2m+3}H_{1+2\gamma+l}(z)\right)+C^{2m,\alpha}(X)\\ G_h(x,\xi)\ &\in\eta_\xi(x)\left(\tfrac{g_{n,\gamma}}{|x|^{n-2\gamma}}+\textstyle\sum_{l=-n}^{2m+3}H_{1+2\gamma+l}(x)\right)+C^{2m,\alpha}(M)\end{aligned}$$

with $H_l\in C^\infty(\mathbb{R}^{n+1}_+\setminus\{0\})$ being homogeneous of order l, η_ξ is a cut-off function as in (23), $p_{n,\gamma}$ is as in (9), and $g_{n,\gamma}$ is as in (41), provided $H_g=0$.

In the case of locally flat conformal infinities of Poincare-Einstein manifolds, we have:

Theorem 1.2. *Let $(X,\ g^+)$ be a Poincaré-Einstein manifold with conformal infinity $(M,[h])$ of dimension $n=2$ or $n\geq 3$ and $(M,[h])$ is locally flat. If*

$$\frac{1}{2}\neq\gamma\in(0,1)\quad and\quad \lambda_1(-\Delta_{g^+})>s(n-s)\quad for\ s=\frac{n}{2}+\gamma,$$

then the Poisson kernel K_g and the Green's functions Γ_g and G_h respectively for

$$\begin{cases}D_gU=0 & in\quad X\\ U=f & on\quad M\end{cases}$$

$$\begin{cases}D_gU=0 & in\quad X\\ -d^*_\gamma\lim_{y\to 0}y^{1-2\gamma}\partial_yU=f & on\quad M\end{cases}$$

and

$$\left\{P^\gamma_h u=f\quad on\quad M\right.$$

are respectively of class $y^{2\gamma}C^{2,\alpha}$ *and* $C^{2,\alpha}$ *away from the singularity and admit for every* $a \in M$ *locally in* g_a*-normal Fermi-coordinates an expansion around* a

$$\begin{aligned}
K_a(z) &\in p_{n,\gamma}\tfrac{y^{2\gamma}}{|z|^{n+2\gamma}} + y^{2\gamma}H_{-2\gamma}(z) + y^{2\gamma}H_{1-2\gamma}(z) + y^{2\gamma}H_{2-2\gamma}(z) + y^{2\gamma}C^{2,\alpha}(X)\\
\Gamma_a(z) &\in \tfrac{g_{n,\gamma}}{|z|^{n-2\gamma}} + H_{2\gamma}(z) + H_{1+2\gamma}(z) + C^{2,\alpha}(X)\\
G_a(x) &\in \tfrac{g_{n,\gamma}}{|x|^{n-2\gamma}} + H_{2\gamma}(x) + H_{1+2\gamma}(x) + C^{2,\alpha}(M),
\end{aligned}$$

where g_a *is as in* (46), $K_a = K_{g_a}(\cdot, a)$, $\Gamma_a = \Gamma_{g_a}(\cdot, a)$ *and* $G_a = G_{h_a}(\cdot, a)$ *and* $H_k \in C^\infty(\overline{\mathbb{R}^n_+} \setminus \{0\})$ *are homogeneous of degree* k.

To prove Theorem 1.1 and Theorem 1.2, we use the method of Lee-Parker[22] of killing deficits successively. However difficulties arise due the the rigidity involved in the problem (see (2)) and the lack of classical regularity theory. To overcome the rigidity issue, we work with the space of homogeneous functions rather than the one of polynomials as done in [22]. To handle the regularity issue, we show some higher order regularity results for the Dirichlet problem and the weighted Neumann boundary problem of the Chang-Gonzalez[8] extension problem for P^γ_h which are of independent interest, see Proposition 2 and Proposition 3. We point out that even if the estimates in Proposition 2 and Proposition 3 are weak, they are enough for our purpose and in turn get improved by the estimates of the Poisson kernel and Green's function in Theorem 1.1 and Theorem 1.2 that they imply. On the other hand, we would like to emphasize that the expansion of Γ_a in Theorem 1.2 answers the first part of the Conjecture of Kim-Musso-Wei[21] about the asymptotics of Γ_a and gives the definition of the fractional mass, see our work [25], Definition 4.3 and Lemma 4.1.

The structure of the paper is as follows: In Section 2 we fix some notations. In Section 3 we develop a non-homogeneous extension of some aspects of the works of Chang-Gonzalez[8] and Graham-Zworsky[17]. It is divided in two subsections. In the first one, namely Subsection 3.1, we develop a non-homogeneous scattering theory, define the associated non-homogeneous fractional operator and its relation to a non-homogeneous uniformly degenerate boundary value problem. In Subsection 3.2 we discuss the conformal property of the non-homogeneous fractional operator. We point out that Section 3 even being of independent interest contains estimates which are used in Section 5 and in [25], and a regularity result that we use in [25]. Section 4 is concerned with the study of the Poisson kernel K_g and the Green's function Γ_g under weighted Neumann boundary conditions of the Chang-Gonzalez extension problem of P^γ_h, and the Green's function G_h of P^γ_h all in the general case of asymptotically hyperbolic manifolds with minimal conformal infinity. In Section 5 we sharpen the results obtained in Section 4 in the particular case of a locally flat conformal infinity of a Poincaré-Einstein manifold.

2. **Notations and preliminaries.** In this section we fix some notations. First of all let $X = X^{n+1}$ be a manifold of dimension $n+1$ with boundary $M = M^n$ and closure $\overline{X}$ with $n \geq 2$.

In the following, for any Riemannian metric $\bar{h}$ defined on M, $a \in M$ and $r > 0$, we use the notation $B^{\bar{h}}_r(a)$ to denote the geodesic ball with respect to $\bar{h}$ of radius r and center a. We also denote by $d_{\bar{h}}(x, y)$ the geodesic distance with respect to $\bar{h}$ between two points x and y of M. $inj_{\bar{h}}(M)$ stands for the injectivity radius of $(M, \bar{h})$. $dV_{\bar{h}}$ denotes the Riemannian measure associated to the metric $\bar{h}$ on M.

For $a \in M$ we use the notation $\exp_{\bar h}^{a}$ to denote the exponential map with respect to $\bar h$ on M.

Similarly for any Riemannian metric $\bar g$ defined on $\overline{X}$, $a \in M$ and $r > 0$ we use the notation $B_r^{\bar g, +}(a)$ to denote the geodesic half ball with respect to $\bar g$ of radius r and center a. We also denote by $d_{\bar g}(x, y)$ the geodesic distance with respect to $\bar g$ between two points $x \in M$ and $y \in \overline{X}$. $inj_{\bar g}(\overline{X})$ stands for the injectivity radius of $(\overline{X}, \bar g)$. $dV_{\bar g}$ denotes the Riemannian measure associated to the metric $\bar g$ on $\overline{X}$. For $a \in M^n$ we use the notation $\exp_a^{\bar g, +}$ to denote the exponential map with respect to $\bar g$ on $\overline{X}$.

$\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{N}^*$ the set of positive integers and for $k \in \mathbb{N}^*$, $\mathbb{R}^k$ stands for the standard k-dimensional Euclidean space, $\mathbb{R}^k_+$ the open positive half-space of $\mathbb{R}^k$, and $\bar{\mathbb{R}}^k_+$ its closure in $\mathbb{R}^k$. For simplicity we use the notation $\mathbb{R}_+ := \mathbb{R}^1_+$, and $\bar{\mathbb{R}}_+ := \bar{\mathbb{R}}^1_+$. For $r > 0$ we denote respectively

$$B_r^{\mathbb{R}^k}(0) \quad \text{and} \quad B_r^{\mathbb{R}^k_+}(0) = B_r^{\mathbb{R}^k}(0) \cap \mathbb{R}^k_+ \simeq]0, r[\times B_r^{\mathbb{R}^{k-1}}(0)$$

the open and open upper half ball of $\mathbb{R}^k$ of center 0 and radius r, and set $B_r = B_r^{\mathbb{R}^n}$ and $B_r^+ = B_r^{\mathbb{R}^{n+1}_+}$. For $k \in \mathbb{N}^*$, we set

$$S^k = \partial B_1^{\mathbb{R}^{k+1}}(0) \ \text{ and } \ S^k_+ = S^k \cap \mathbb{R}^{k+1}_+.$$

We also denote by $\nabla^{\perp}_{S^n_+}$ the normal part of the standard gradient $\nabla_{S^n_+}$ on S^n_+.

For $p \in \mathbb{N}^*$, let M^p denotes the Cartesian product of p copies of M. We define $(M^2)^* := M^2 \setminus Diag(M^2)$, where $Diag(M^2) = \{(a, a) : a \in M\}$ is the diagonal of M.

For $1 \leq p \leq \infty$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $\beta \in]0, 1[$ and $\bar h$ a Riemannian metric defined on M,

$$L^p(M, \bar h), \ W^{s,p}(M, \bar h), \ C^k(M, \bar h) \quad \text{and} \quad C^{k,\beta}(M, \bar h)$$

stand respectively for the p-Lebesgue, (s, p)-Sobolev space, k-continuously differentiable space and k-continuously differentiable space of Hölder exponent β, all on M and with respect to $\bar h$, if the definition required a metric structure. Similarly for $1 \leq p \leq \infty$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $\beta \in]0, 1[$ and $\bar g$ a Riemannian metric defined on $\overline{X}$,

$$L^p_f(\overline{X}, \bar g), \ W^{s,p}_f(\overline{X}, \bar g), \ C^k(\overline{X}, \bar g) \text{ and } C^{k,\beta}(\overline{X}, \bar g)$$

stand respectively for the weighted p-Lebesgue, (s, p)-Sobolev space, continuously differentiable space of order k and k-continuously differentiable space of Hölder exponent β, all on $\overline{X}$, and as above with respect to $\bar g$ and a measurable function $f > 0$ on X , if required. For precise definitions and properties see [1], [9], [13], [12] and [32]. $C^\infty_0(X)$ means element in $C^\infty(X)$ vanishing on M to infinite order.

For $\epsilon > 0$ and small $o_\epsilon(1)$ means quantities which tend to 0 as ϵ tends to 0. $O(1)$ stands for quantities which are bounded. For $x \in \mathbb{R}$ we use the notation $O(x)$ and $o_\epsilon(x)$ to mean respectively $|x|O(1)$ and $|x|o_\epsilon(1)$. Large positive constants are usually denoted by C and the value of C is allowed to vary from formula to formula and also within the same line. Similarly small positive constants are denoted by c and their values may vary from formula to formula and also within the same line.

We define

$$d^*_\gamma = \frac{d_\gamma}{2\gamma}, \tag{7}$$

cf. (5). Furthermore, we set

$$c_{n,3}^{\gamma} = \int_{\mathbb{R}^n} \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\gamma}{2}} dx, \tag{8}$$

and

$$p_{n,\gamma} = \frac{1}{c_{n,3}^{\gamma}} \tag{9}$$

Let (X, g^+) be an asymptotically hyperbolic manifold of dimension $n+1$ with $n \geq 2$ and minimal conformal infinity $(M, [h])$. Then, because of (2) and minimality of the conformal infinity, we can consider a geodesic defining function y splitting the metric

$$g = y^2 g^+, \;\; g = dy^2 + h_y \text{ near } M \quad \text{and} \quad h = h_y \lfloor_M$$

in such a way, that $H_g = 0$. Moreover using the existence of conformal normal coordinates, cf. [19], there exists for every $a \in M$ a conformal factor

$$0 < u_a \in C^\infty(M) \quad \text{satisfying} \quad \frac{1}{C} \leq u_a \leq C, \;\; u_a(a) = 1 \quad \text{and} \quad \nabla u_a(a) = 0, \tag{10}$$

inducing a conformal normal coordinate system close to a on M, in particular in normal coordinates with respect to

$$h_a = u_a^{\frac{4}{n-2\gamma}} h$$

we have for some small $\epsilon > 0$

$$h_a = \delta + O(|x|^2), \;\; \det h_a \equiv 1 \quad \text{on} \quad B_\epsilon^{h_a}(a).$$

As clarified in Subsection 3.2 the conformal factor u_a then naturally extends onto X via

$$u_a = (\frac{y_a}{y})^{\frac{n-2\gamma}{2}},$$

where y_a close to the boundary M is the unique geodesic defining function, for which

$$g_a = y_a^2 g^+, \;\; g_a = dy_a^2 + h_{a,y_a} \text{ near } M \quad \text{with} \quad h_a = h_{a,y_a} \lfloor_M$$

and there still holds $H_{g_a} = 0$. Consequently

$$g_a = \delta + O(y + |x|^2) \quad \text{and} \quad \det g_a = 1 + O(y^2) \text{ in } B_\epsilon^{g_a,+}(a).$$

3. **Non-homogeneous scattering theory.** In this section we extend some PDE aspects of the works of Chang-Gonzalez[8] and Graham-Zworsky[17] to a non-homogeneous setting and in the general framework of asymptotically hyperbolic manifolds. It is of independent interest, but in it we derive estimates that are used in Section 5 and [25], and an existence and regularity result used in [25] to construct barrier solutions in order to compare different types of bubbles via maximum principle. We divide this section in two subsections.

3.1. **Scattering operators and uniformly degenerate equations.** In this subsection we extend some parts of the works of Chang-Gonzalez[8] and Graham-Zworski[17] to a non-homogeneous setting in the context of asymptotically hyperbolic manifolds. First of all let (X, g^+) be an asymptotically hyperbolic manifold with conformal infinity $(M, [h])$ and y the unique geodesic defining function associated to h given by (2). Then we have the normal form

$$y^2 g^+ = g = dy^2 + h_y \quad \text{near} \quad M$$

with $y > 0$ in X, $y = 0$ on M and $|dy|_g = 1$ near M. Furthermore let

$$\Box_{g^+} = -\Delta_{g^+} - s(n-s),$$

where by definition

$$s = \frac{n}{2} + \gamma, \quad \gamma \in (0,1), \quad \gamma \neq \frac{1}{2} \quad \text{and} \quad s(n-s) \in (0, \frac{n^2}{4}).$$

According to Mazzeo and Melrose [26], [27], [28]

$$\sigma(-\Delta_{g^+}) = \sigma_{pp}(-\Delta_{g^+}) \cup [\frac{n^2}{4}, \infty), \ \sigma_{pp}(-\Delta_{g^+}) \subset (0, \frac{n^2}{4}),$$

where $\sigma(-\Delta_{g^+})$ and $\sigma_{pp}(-\Delta_{g^+})$ are respectively the spectrum and the pure point spectrum of L^2-eigenvalues of $-\Delta_{g^+}$. Using the work of Graham-Zworski[17], see equation (3.9) therein, we may solve

$$\begin{cases} \Box_{g^+} u = f & \text{in} \quad X \\ y^{s-n} u = \underline{v} & \text{on} \quad M \end{cases}$$

for $s(n-s) \notin \sigma_{pp}(-\Delta_{g^+})$ and $f \in y^{n-s+1} C^\infty(\overline{X}) + y^{s+1} C^\infty(\overline{X})$ in the form

$$\begin{cases} u = y^{n-s} A + y^s B & \text{in} \quad X \\ A, \ B \in C^\infty(\overline{X}), \quad A = \underline{v} & \text{on} \quad M. \end{cases}$$

As in the case $f = 0$, which corresponds to the generalized eigenvalue problem of Graham-Zworsky[17], this gives rise to a Dirichlet-to-Neumann map $S_f(s)$ via

$$\underline{v} = A\lfloor_M \longrightarrow B\lfloor_M = \overline{v},$$

which we refer to as non-homogeneous scattering operator and denote it by $S_f(s)$. Clearly $S_0(s) = S(s)$ and $S_f(s)$ is invertible, since the standard scattering operator $S_0(s)$ is invertible, cf. equation (1.2) in [20]. We define the non-homogeneous fractional operators by

$$P^\gamma_{f,h} = -d_\gamma S_f(s),$$

where d_γ is as in (5). Following [15] we find by conformal covariance of the conformal Laplacian that

$$\Box_{g^+} u = f \overset{U = y^{s-n} u}{\Longleftrightarrow} D_g U = y^{-s-1} f, \tag{11}$$

where

$$D_g U = -div_g(y^{1-2\gamma} \nabla_g U) + E_g U \tag{12}$$

and with $L_g = -\Delta_g + \frac{R_g}{c_n}$ denoting the conformal Laplacian on (X, g)

$$E_g := y^{\frac{1-2\gamma}{2}} L_g y^{\frac{1-2\gamma}{2}} - (\frac{R_{g^+}}{c_n} + s(n-s)) y^{(1-2\gamma)-2}, \quad c_n = \frac{4n}{n-1}. \tag{13}$$

Thus we find for $\phi, \psi \in C^\infty(\overline{X})$, that

$$\begin{cases} \Box_{g^+} u = y^{n-s+1}\phi + y^{s+1}\psi & \text{in} \quad X \\ y^{s-n}u = \underline{v} & \text{on} \quad M \end{cases} \quad \xLeftrightarrow{U=y^{s-n}u} \quad \begin{cases} D_g U = y^{-2\gamma}\phi + \psi & \text{in} \quad X \\ U = \underline{v} & \text{on} \quad M \end{cases}.$$

Note, that such a solution U is of the form

$$U = A + By^{2\gamma} = \sum A_i y^i + \sum B_i y^{i+2\gamma} + U_0$$

for some $U_0 \in C_0^\infty(X)$ and has principal terms

$$\begin{cases} \underline{v} + \overline{v}y^{2\gamma} & \text{for} \quad \gamma < \frac{1}{2} \\ \underline{v} + A_1 y + \overline{v}y^{2\gamma} & \text{for} \quad \gamma > \frac{1}{2}. \end{cases}$$

As for the case $\gamma > \frac{1}{2}$, expanding the boundary metric h_y, we find

$$h_y = h_0 + h_1 y + O(y^2) \quad \text{with} \quad h_1 = 2\Pi_g$$

and Π_g denoting the second fundamental form of (M, h) in $(\overline{X}, g)$. Still according to [17] we may solve

$$\begin{cases} \Box_{g^+} u = y^{n-s+2}\phi + y^{s+1}\psi & \text{in} \quad X \\ y^{s-n}u = \underline{v} \ \text{on} \ M \end{cases}$$

for $\phi, \psi \in C^\infty(\overline{X})$ in the form

$$\begin{cases} u = y^{n-s}A + y^s B & \text{in} \quad X \\ A,\ B \in C^\infty(\overline{X}), \quad A = \underline{v} & \text{on} \quad M \end{cases}$$

with asymptotic

$$A = \sum A_i y^i, \quad A_0 = \underline{v}, \quad A_1 = 0$$

at a point, where $H_g = 0$, i.e. the mean curvature vanishes. Thus for $\gamma > \frac{1}{2}$

$$\begin{cases} \Box_{g^+} u = y^{n-s+2}\phi + y^{s+1}\psi & \text{in} \quad X \\ y^{s-n}u = \underline{v} & \text{on} \quad M \end{cases} \quad \xLeftrightarrow{U=y^{s-n}u} \quad \begin{cases} D_g U = y^{1-2\gamma}\phi + \psi & \text{in} \quad X \\ U = \underline{v} & \text{on} \quad M \end{cases}$$

with principal terms

$$U = \underline{v} + \overline{v}y^{2\gamma} + o(y^{2\gamma})$$

at a point with $H_g = 0$ - just like in the case $\gamma < \frac{1}{2}$ - and there holds $\overline{v} = \frac{1}{2\gamma}\lim_{y\to 0} y^{1-2\gamma}\partial_y U$.

We summarize the latter discussion in the following proposition.

Proposition 1. *Let (X, g^+) be a $(n+1)$-dimensional asymptotically hyperbolic manifold with conformal infinity $(M, [h])$ of dimension $n \geq 2$ being minimal in case $\gamma \in (\frac{1}{2}, 1)$ and y the unique geodesic defining function associated to h given by (2). Assuming that*

$$s = \frac{n}{2} + \gamma, \quad \gamma \in (0,1), \quad \gamma \neq \frac{1}{2}, \quad s(n-s) \notin \sigma_{pp}(\Delta_{g^+})$$

and $f \in y^{n-s+2}C^\infty(\overline{X}) + y^{s+1}C^\infty(\overline{X})$, then for every $\underline{v} \in C^\infty(M)$

$$P^\gamma_{f,h}(\underline{v}) = -d^*_\gamma \lim_{y\to 0} y^{1-2\gamma}\partial_y U^f,$$

where U^f *is the unique solution to*

$$\begin{cases} D_g U = y^{-s-1} f & \text{in } X \\ U = \underline{v} & \text{on } M \end{cases}$$

and d^*_γ *is as in* (7). *Moreover* U^f *satisfies*

$$U^f = A + y^{2\gamma} B, \quad A,\ B \in C^\infty(\overline{X})$$

and A *and* B *satisfy the asymptotics*

$$\begin{cases} A = \sum A_i y^i, & A_i \in C^\infty(M), \quad A_0 = \underline{v} \ \text{ and } \ A_1 = 0 \\ B = \sum B_i y^i, & B_i \in C^\infty(M) \quad \text{ and } \quad - d_\gamma B_0 = -d_\gamma \overline{v} = P^\gamma_{f,h}(\underline{v}), \end{cases}$$

where d_γ *is as in* (5), *hence* $U^f = \underline{v} + \overline{v} y^{2\gamma} + o(y^{2\gamma})$.

3.2. **Conformal property of the non-homogeneous scattering operator.** In this subsection we study the conformal property of the non-homogeneous scattering operator $P^\gamma_{h,f}$ of the previous subsection. To this end we first consider as background data (X, g^+) with conformal infinity $(M, [h])$ with $n \geq 2$ and y the associated unique geodesic definition function such that

$$g = y^2 g^+, \quad g = dy^2 + h_y \quad \text{close to } M \quad \text{and} \quad h = g\lfloor_M$$

as in (2). From (13) it is easy to see, that in g-normal Fermi coordinates (y, x)

$$E_g = \frac{n-2\gamma}{2} \frac{\partial_y \sqrt{g}}{\sqrt{g}} y^{-2\gamma} \quad \text{close to} \quad M. \tag{14}$$

We assume further that $(M, [h])$ is minimal and $\Box_{g^+}$ is positive, i.e.

$$H_g = 0 \quad \text{and} \quad \lambda_1(-\Delta_{g^+}) > s(n-s).$$

Then $\partial_y \sqrt{g} = 0$ on M^{n+1} and we may assume

$$\partial_y \sqrt{g} \in y C^\infty(\overline{X}) \tag{15}$$

whence D_g is well defined on

$$W^{1,2}_{y^{1-2\gamma}} = W^{1,2}_{y^{1-2\gamma}}(X, g) = \overline{C^\infty(X)}^{\|\cdot\|_{W^{1,2}_{y^{1-2\gamma}}(X,g)}}$$

with

$$\|u\|^2_{W^{1,2}_{y^{1-2\gamma}}(X,g)} = \int_X y^{1-2\gamma}(|du|^2_g + u^2) dV_g$$

and becomes positive under Dirichlet condition, cf. (11), so

$$\partial_y \sqrt{g} \in y C^\infty(\overline{X}) \quad \text{and} \quad \langle \cdot, \cdot \rangle_{D_g} \simeq \langle \cdot, \cdot \rangle_{W^{1,2}_{y^{1-2\gamma}}}.$$

Let us consider now a conformal metric $\tilde{h} = \varphi^{\frac{4}{n-2\gamma}} h$ on M. We then find a unique geodesic defining function $\tilde{y} > 0$, precisely unique in a tubular neighborhood of M, such that

$$\tilde{g} = d\tilde{y}^2 + \tilde{h}_y \quad \text{close to} \quad M, \quad \tilde{y}^{-2}\tilde{g} = g^+ = y^{-2} g \quad \text{and} \quad \tilde{h} = \varphi^{\frac{4}{n-2}} h = (\frac{\tilde{y}}{y})^2 h \quad \text{on} \quad M.$$

So we may naturally extend $\varphi = (\frac{\tilde{y}}{y})^{\frac{n-2\gamma}{2}}$ onto X and by the conformal relation

$$\tilde{g} = (\frac{\tilde{y}}{y})^2 g = \varphi^{\frac{4}{n-2\gamma}} g,$$

we still have $\langle\cdot,\cdot\rangle_{D_{\tilde g}} \simeq \langle\cdot,\cdot\rangle_{W^{1,2}_{\tilde y^{1-2\gamma}}}$. Putting $\tilde y = \alpha y$, the equation

$$|dy|_g^2 = 1 = |d\tilde y|^2_{\tilde g} = 1 + 2\frac{y}{\alpha}\langle d\alpha, dy\rangle_g + (\frac{y}{\alpha})^2|d\alpha|^2_g$$

for the geodesic defining functions implies $\partial_y\alpha = -\frac{1}{2}\frac{y}{\alpha}|d\alpha|^2_g$. Since $\tilde g = \alpha^2 g$ by definition, we firstly find $H_g = 0 \Longrightarrow H_{\tilde g} = 0$, i.e. minimality is preserved as already observed by Gonzalez-Qing[15], and secondly $\tilde y = \alpha_0 y + O(y^3)$. Thus on the one hand side the properties

$$\partial_{\tilde y}\sqrt{\tilde g} \in \tilde y C^\infty \text{ and } \langle\cdot,\cdot\rangle_{D_{\tilde g}} \simeq \langle\cdot,\cdot\rangle_{W^{1,2}_{\tilde y^{1-2\gamma}}}$$

are preserved under a conformal change of the metric on the boundary. Moreover we obtain a conformal transformation for the extension operators $D_{\tilde g}$ and D_g subjected to Dirichlet and weighted Neumann boundary conditions. Put $\tilde u = (\frac{y}{\tilde y})^{n-s}u$. As for the Dirichlet case, (11) directly shows

$$\begin{cases} D_g u = f & \text{in} \quad X \\ u = v & \text{on} \quad M \end{cases} \Longleftrightarrow \begin{cases} D_{\tilde g}\tilde u = (\frac{y}{\tilde y})^{s+1} f & \text{in} \quad X \\ \tilde u = (\frac{y}{\tilde y})^{n-s} v & \text{on} \quad M. \end{cases}$$

Moreover there holds

$$\lim_{y\to 0} y^{1-2\gamma}\partial_y u = v \Longleftrightarrow \lim_{\tilde y \to 0}\tilde y^{1-2\gamma}\partial_{\tilde y}\tilde u = (\frac{y}{\tilde y})^{n-s+2\gamma} v,$$

since $\tilde y = \alpha_0 y + O(y^3)$, whence for the weighted Neumann case we obtain

$$\begin{cases} D_g u = f & \text{in} \quad X \\ \lim_{y\to 0} y^{1-2\gamma}\partial_y u = v & \text{on} \quad M \end{cases} \Longleftrightarrow \begin{cases} D_{\tilde g}\tilde u = (\frac{y}{\tilde y})^{s+1} f & \text{in} \quad X \\ \lim_{\tilde y\to 0}\tilde y^{1-2\gamma}\partial_{\tilde y}\tilde u = (\frac{y}{\tilde y})^{n-s+2\gamma} v & \text{on} \quad M. \end{cases}$$

We may rephrase this via $\varphi = (\frac{\tilde y}{y})^{\frac{n-2\gamma}{2}} = (\frac{\tilde y}{y})^{n-s}$ as

$$\begin{cases} D_g(\varphi u) = \varphi^{\frac{s+1}{n-s}} f & \text{in} \quad X \\ \varphi u = \varphi v & \text{on} \quad M \end{cases} \Longleftrightarrow \begin{cases} D_{\tilde g} u = f & \text{in } X \\ u = v & \text{on } M \end{cases}$$

and

$$\begin{cases} D_g(\varphi u) = \varphi^{\frac{s+1}{n-s}} f & \text{in} \quad X \\ \lim_{y\to 0} y^{1-2\gamma}\partial_y(\varphi u) = \varphi^{\frac{n+2\gamma}{n-2\gamma}} v & \text{on} \quad M^n \end{cases} \Longleftrightarrow \begin{cases} D_{\tilde g}\tilde u = f & \text{in} \quad X \\ \lim_{\tilde y\to 0}\tilde y^{1-2\gamma}\partial_{\tilde y} u = v & \text{on} \quad M. \end{cases}$$

Noticing $\frac{s+1}{n-s} = \frac{n+2+2\gamma}{n-2\gamma}$ we thus have shown

$$\begin{array}{ccc} P^\gamma_{f,\tilde h}(\underline{v}) = \overline{v} & \Longleftrightarrow & \begin{cases} D_{\tilde g} u = f & \text{in} \quad X \\ u = \underline{v} & \text{on} \quad M \\ -d^*_\gamma \lim_{\tilde y\to 0}\tilde y^{1-2\gamma}\partial_{\tilde y} u = \overline{v} & \text{on} \quad M \end{cases} \\ \Updownarrow & & \Updownarrow \\ P^\gamma_{\varphi^{\frac{s+1}{n-s}} f, h}(\varphi\underline{v}) = \varphi^{\frac{n+2\gamma}{n-2\gamma}}\overline{v} & \Longleftrightarrow & \begin{cases} D_g(\varphi u) = \varphi^{\frac{n+2+2\gamma}{n-2\gamma}} f \text{ in } X \\ \varphi u = \varphi\underline{v} \text{ on } M \\ -d^*_\gamma \lim_{y\to 0} y^{1-2\gamma}\partial_y(\varphi u) = \varphi^{\frac{n+2\gamma}{n-2\gamma}}\overline{v}, \end{cases} \end{array}$$

where the last equation on the right hand side holds on M. Therefore the non-homogeneous fractional operator verifies the conformal property

$$P^\gamma_{f,\tilde h}(\underline{v}) = \varphi^{-\frac{n+2\gamma}{n-2\gamma}} P^\gamma_{\varphi^{\frac{s+1}{n-s}} f, h}(\varphi\underline{v}) \text{ for } \tilde h = \varphi^{\frac{4}{n-2\gamma}} h$$

or equivalently

$$P^{\gamma}_{\tilde f,\tilde h}(\underline{v})=\varphi^{-\frac{n+2\gamma}{n-2\gamma}}P^{\gamma}_{f,h}(\varphi\underline{v})\ \text{ for }\ \tilde h=\varphi^{\frac{4}{n-2\gamma}}h\ \text{ and }\ \tilde f=\varphi^{\frac{-s-1}{n-s}}f,$$

hence extending the conformal property of the homogeneous fractional operator to the non-homogeneous setting. We remark that

$$P^{\gamma}_{h}=P^{\gamma}_{0,h}.$$

4. **Fundamental solutions in the asymptotically hyperbolic case.** In this section, keeping the notations of the previous one, for an asymptotically hyperbolic manifold (X,g^{+}) with conformal infinity $(M,[h])$, we study the existence and asymptotic behavior of the Poisson kernel $K_g:=K^{\gamma}_g$ of D_g, the Green's functions $\Gamma_g:=\Gamma^{\gamma}_g$ of D_g under weighted normal boundary condition and $G_h:=G^{\gamma}_h$ of the fractional conformal Laplacian P^{γ}_h, i.e.

$$\begin{cases}D_gK_g(\cdot,\xi)=0 \quad \text{in} \quad X \quad \text{and for all} \quad \xi\in M\\ \lim_{y\to 0}K_g(y,x,\xi)=\delta_{\xi}(x) \quad \text{and for all} \quad x,\ \xi\in M\end{cases}$$

and

$$\begin{cases}D_g\Gamma_g(\cdot,\xi)=0 \ \text{ in } X \quad \text{and for all} \quad \xi\in M\\ -d^*_{\gamma}\lim_{y\to 0}y^{1-2\gamma}\partial_y\Gamma_g(y,x,\xi)=\delta_{\xi}(x) \quad \text{and for all} \quad x,\xi\in M,\end{cases}$$

where d^*_{γ} is given by (7), and $P^{\gamma}_hG^{\gamma}_h(x,\xi)=\delta_{\xi}(x),\ \ x\in M$. So by definition

$$K_g:(\overline{X}\times M)\setminus Diag(M)\longrightarrow\mathbb{R}_+$$

is the Green's function to the extension problem

$$\begin{cases}D_gU=0 \ \text{ in } \quad X\\ U=\underline{v} \ \text{ on } \ M,\end{cases}$$

while

$$\Gamma_g:(\overline{X}\times M)\setminus Diag(M)\longrightarrow\mathbb{R}$$

is the Green's function to the dual problem

$$\begin{cases}D_gU=0 \quad \text{in} \quad X\\ -d^*_{\gamma}\lim_{y\to 0}y^{1-2\gamma}\partial_yU=\overline{v} \quad \text{on} \quad M\end{cases}$$

and

$$G_h:(M\times M)\setminus Diag(M)\longrightarrow\mathbb{R}.$$

is the Green's function of the nonlocal problem $P^{\gamma}_h\underline{v}=\overline{v}$ on M. They are linked via

$$\Gamma_g(z,\xi)=\int_M K_g(z,x)G_h(x,\xi)dV_h(x),\qquad z\in\overline{X},\quad \xi\in M. \tag{16}$$

4.1. **Study of the Poisson kernel for D_g .** In this subsection we study the Poisson kernel K_g focusing on the existence issue and its asymptotics. We follow the method of Lee-Parker[22] of killing deficits successively. However, due to the rigidity property involved in the problem, see the normal form (2), we have to work close to the boundary in Fermi coordinates rather than normal ones. To compensate this we are forced to pass from the space of polynomials used in [22] to the space of homogeneous functions. We start with recalling some related facts in the case of the standard Euclidean space $\mathbb{R}^{n+1}_+$. According to [4] on $\mathbb{R}^{n+1}_+$

$$K(y,x,\xi)=K^{\gamma}(y,x,\xi)=p_{n,\gamma}\frac{y^{2\gamma}}{(y^2+|x-\xi|^2)^{\frac{n+2\gamma}{2}}},\ \ (y,x)\in\overline{\mathbb{R}}^{n+1}_+,\ \ \xi\in\mathbb{R}^n \tag{17}$$

where $p_{n,\gamma}$ is as in (9), is the Poisson kernel of the operator

$$D=-div(y^{1-2\gamma}\nabla(\,\cdot\,)),$$

namely the Green's function of the extension problem

$$\begin{cases}Du=0 & \text{in} \quad \mathbb{R}^{n+1}_+\\ u=f & \text{on} \quad \mathbb{R}^n,\end{cases}$$

i.e.

$$\begin{cases}DK(y,x,\xi)=0 & \text{in} \quad \mathbb{R}^{n+1}_+,(y,x)\in\mathbb{R}^{n+1}_+,\xi\in\mathbb{R}^n\\ K(y,x,\xi)\to\delta_\xi(x) & \text{for} \quad (y,x)\in\mathbb{R}^{n+1}_+,\xi\in\mathbb{R}^n,\ y\to 0.\end{cases} \tag{18}$$

We will construct the Poisson kernel for D_g, cf. (12), namely the Green's function of the analogous extension problem

$$\begin{cases}D_gu=0 & \text{in} \ \ X\\ u=f & \text{on} \quad M,\end{cases}$$

i.e. K_g solves for $z\in X$ and $\xi\in M$

$$\begin{cases}D_gK_g(z,\xi)=0 & \text{in} \quad X\\ K(z,\xi)\to\delta_\xi(x) & \text{for} \quad y\to 0,\end{cases}$$

where $z=(y,x)\in X$ for z close to M. To that end we identify

$$\xi\in M\cap U\subset U\cap X \text{ with } 0\in B^{\mathbb{R}^{n+1}}_\epsilon(0)\cap\mathbb{R}^n\subset B^{\mathbb{R}^{n+1}}_\epsilon(0)\cap\overline{\mathbb{R}}^{n+1}_+$$

for some open neighborhood U of ξ in X and small $\epsilon>0$, and write $K(z)=K(z,0)$. We then have

$$D_gK=-\frac{\partial_p}{\sqrt{g}}(\sqrt{g}g^{p,q}y^{1-2\gamma}\partial_qK)+E_gK=f\in yH_{-n-2\gamma-1}C^{\infty} \tag{19}$$

on $B^{\mathbb{R}^{n+1}}_\epsilon(0)\cap\mathbb{R}^{n+1}_+$ due (15), which relies on minimality $H_g=0$, where by definition

$$H_l=\{\varphi\in C^{\infty}(\overline{\mathbb{R}^{n+1}_+}\setminus\{0\})\mid\varphi \text{ is homogeneous of degree } l\}. \tag{20}$$

The next lemma allows us to solve homogeneous deficits homogeneously.

Lemma 4.1. *For $\frac{1}{2}\neq\gamma\in(0,1)$ and $f_l\in yH_{l-1}$, $l\in\mathbb{N}-n-2\gamma$ there exists $K_{1+2\gamma+l}\in y^{2\gamma}H_{l+1}$ such, that*

$$DK_{1+2\gamma+l}=f_l.$$

Proof. First of all the Stone-Weierstraß Theorem implies

$$\langle Q_l^k(y,x) = y^{2\gamma+2k}P_l(x) \mid k,l \in \mathbb{N} \text{ and } P_l \in \Pi_l\rangle \underset{\text{dense}}{\subset} y^{2\gamma}C^0(\overline{B}_1(0)\cap \mathbb{R}^{n+1}_+)$$

with $\Pi_l = \{\varphi \in C^\infty(\mathbb{R}^n) \mid \varphi \text{ is homogeneous of degree } l\}$ and an easy induction argument shows, that we have a unique representation

$$Q_l^k = \sum |z|^{2i} A_{2k+l-2i}$$

with D-harmonics of the form $A_m(y,x) = \sum y^{2\gamma+2l}P_{m-2l}(x)$, $DA_m = 0$. Since

$$y^{2\gamma}C^0(S^n_+) \underset{\text{dense}}{\subset} L^2_{y^{1-2\gamma}}(S^n_+),$$

we thus obtain a D-harmonic basis $E = \{e_k^i\}$ for $L^2_{y^{1-2\gamma}}(S^n_+)$ with

$$De_k^i = 0, \;\; k = deg(e_k^i), \;\; k \in \mathbb{N}+2\gamma \text{ and } \;\; i \in \{1,\ldots,d_k\},$$

where d_k denotes the dimension of the space of D-harmonics of degree k. We may assume, that e_k^i, e_k^j for $i \neq j$ are orthogonal with respect to the scalar product on $L^2_{y^{1-2\gamma}}(S^n_+)$. Moreover on S^n_+ we have

$$\begin{aligned}
0 = &- De_k^i = \partial_y(y^{1-2\gamma}\partial_y e_k^i) + y^{1-2\gamma}\Delta_x e_k^i = \nabla y^{1-2\gamma}\nabla e_k^i + y^{1-2\gamma}\Delta e_k^i \\
= &\nabla y^{1-2\gamma}\nabla e_k^i + y^{1-2\gamma}\frac{\Delta_{S^n}}{r^2}e_k^i + y^{1-2\gamma}[\partial_r^2 + \frac{n\partial_r}{r}]e_k^i \\
= &\nabla^\perp_{S^n_+} y^{1-2\gamma}\nabla^\perp_{S^n_+} e_k^i + div_{S^n_+}(y^{1-2\gamma}\nabla_{S^n_+} e_k^i) + k(k+n-1)y^{1-2\gamma}e_k^i,
\end{aligned}$$

whence due to

$$\begin{aligned}
\nabla^\perp_{S^n_+} y^{1-2\gamma}\nabla^\perp_{S^n_+} e_k^i = &\langle \nabla y^{1-2\gamma}, \nu_{S^n_+}\rangle\langle \nu_{S^n_+}, \nabla e_k^i\rangle = (1-2\gamma)y^{-2\gamma}\langle e_{n+1}, \nu_{S^n_+}\rangle r\partial_r e_k^i \\
= &(1-2\gamma)ky^{1-2\gamma}e_k^i
\end{aligned}$$

there holds for $D_{S^n_+} = -div_{S^n_+}(y^{1-2\gamma}\nabla_{S^n_+}\cdot)$

$$D_{S^n_+}e_k^i = k(k+n-2\gamma)y^{1-2\gamma}e_k^i.$$

Therefore $E = \{e_k^i\}$ is an orthogonal basis of $y^{2\gamma-1}D_{S^n_+}$-eigenfunctions with eigenvalues

$$\lambda_k = k(k+n-2\gamma).$$

By the same argument solving

$$\begin{cases} Du = f \in L^2_{y^{2\gamma-1}}(\mathbb{R}^{n+1}_+) & \text{in } \;\; \mathbb{R}^{n+1}_+ \\ u = 0 \quad \text{on} \;\; \mathbb{R}^n \end{cases} \tag{21}$$

with homogeneous f, u of degree λ, $\lambda+1+2\gamma$ is equivalent to solving

$$\begin{cases} D_{S^n_+}u = f + (\lambda+1+2\gamma)(\lambda+n+1)y^{1-2\gamma}u & \text{in } \;\; S^n_+ \\ u = 0 \quad \text{on} \;\; \partial S^n_+ = S^{n-1} \end{cases}$$

and thus, writing $u = \sum a_{i,k}e_k^i$, $y^{2\gamma-1}f = \sum b_{j,l}e_l^j$, also equivalent to solving

$$\sum a_{i,k}(k(k+n-2\gamma) - (\lambda+1+2\gamma)(\lambda+n+1))e_k^i = \sum b_{j,l}e_l^j$$

and the latter system is always solvable in case

$$k(k+n-2\gamma) - (\lambda+1+2\gamma)(\lambda+n+1) \neq 0 \quad \text{for all} \quad k,\; n,\; \lambda \in \mathbb{N}. \tag{22}$$

This observation allows us to prove the lemma, by whose assumptions

$$deg(f_l) = \lambda = m - n - 2\gamma, \;\; m \in \mathbb{N}.$$

And we know

$$deg(e_k^i) = k = m' + 2\gamma, \quad m' \in \mathbb{N}.$$

Plugging these values into (22), solvability of (21) is a consequence of

$$(m' + 2\gamma)(m' + n) - (m - n + 1)(m + 1 - 2\gamma) \neq 0 \quad \text{for all} \quad m', \quad n, m \in \mathbb{N}$$

and this holds true for $\frac{1}{2} \neq \gamma \in (0,1)$. Thus we have proven solvability of

$$\begin{cases} DK_{1+2\gamma+l} = f_l & \text{in} \quad \mathbb{R}^{n+1}_+ \\ K_{1+2\gamma+l} = 0 & \text{on} \quad \mathbb{R}^n \setminus \{0\} \end{cases}$$

with $K_{1+2\gamma+l}$ being homogeneous of degree $1 + 2\gamma + l$. We are left with showing $K_{1+2\gamma+l} \in y^{2\gamma} H_{l+1}$. But this follows easily from Proposition 2 below. □

Now recalling (19) we may use Lemma 4.1 to solve (18) successively, since

$$D_g K_{1+2\gamma+l} = f_l + (D_g - D)K_{1+2\gamma+l} \in f_l + yH_l C^\infty$$

due to (15) and $K_{1+2\gamma+l} \in y^{2\gamma} H_{l+1}$. With a suitable cut-off function

$$\eta_\xi : \overline{X} \longrightarrow \mathbb{R}^+, \quad supp(\eta_\xi) = B_\epsilon^+(\xi) = B_\epsilon^{g,+}(\xi) \;\; \text{for} \quad M \ni \xi \sim 0 \in \mathbb{R}^n \text{ and } \epsilon > 0 \quad \text{small} \tag{23}$$

and for the meaning of $B_\epsilon^{g,+}(\xi)$ see Section 2, we then find

$$K_g = \eta_\xi (K + \sum_{l=-n-2\gamma}^{m+2-2\gamma} K_{1+2\gamma+l}) + \kappa_m$$

for $m \in \mathbb{N}$ and a weak solution

$$\begin{cases} D_g \kappa_m = -D_g \left(\eta_\xi (K + \sum_{l=-n-2\gamma}^{m+2-2\gamma} K_{1+2\gamma+l}) \right) = h_m & \text{in} \quad X \\ \kappa_m = 0 & \text{on} \quad M \end{cases}$$

with $h_m \in yC^{m,\alpha}$.

The following weak regularity statement will be sufficient for our purpose.

Proposition 2. *Let $h \in yC^{2k+3,\alpha}(X)$ and $u \in W^{1,2}_{y^{1-2\gamma}}(X)$ be a weak solution of*

$$\begin{cases} D_g u = h & in \quad X \\ u = 0 & on \quad M. \end{cases}$$

Then u is of class $y^{2\gamma} C^{2k,\beta}(X)$, provided $H_g = 0$.

Putting these facts together before giving the proof of Proposition 2, we have the existence of K_g and can describe its asymptotic.

Corollary 1. *Let $\frac{1}{2} \neq \gamma \in (0,1)$. Then K_g exists and we may expand in g-normal Fermi-coordinates around $\xi \in M$*

$$K_g(z,\xi) \in \eta_\xi(z) \left(p_{n,\gamma} \frac{y^{2\gamma}}{|z|^{n+2\gamma}} + \sum_{l=-n-2\gamma}^{2m+5-2\gamma} y^{2\gamma} H_{1+l}(z) \right) + y^{2\gamma} C^{2m,\alpha}(X)$$

with $H_l \in C^\infty(\mathbb{R}^{n+1}_+ \setminus \{0\})$ being homogeneous of order l and $p_{n,\gamma}$ is as in (9), provided $H_g = 0$.

Proof of Proposition 2. We use the Moser iteration argument. First let $p, q = 1, \dots, n+1$ and $i, j = 1, \dots, n$ such, that $g_{n+1,i} = g_{y,i} = 0$. The statement clearly holds by standard local regularity away from the boundary, since D_g is strongly elliptic there. Now fixing a point $\xi \in M$ and a cut-off function

$$\eta \in C_0^\infty(B_{r_2}^+(0), \mathbb{R}_+),\ \eta \equiv 1 \text{ on } B_{r_1}^+(0) \text{ for } 0 < r_1 < r_2 \ll 1, \text{ where } \xi \sim 0 \in \mathbb{R}^n,$$

we pass to g-normal Fermi-coordinates around ξ and estimate for some $\lambda \geq 2$ and $\alpha \in \mathbb{N}^n$

$$\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}|\nabla_z(|\partial_x^\alpha u|^{\frac{\lambda}{2}}\eta)|^2 \leq 2\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}|\nabla_z|\partial_x^\alpha u|^{\frac{\lambda}{2}}|^2\eta^2 + 2\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}|\partial_x^\alpha u|^\lambda|\nabla_z\eta|^2 \tag{24}$$

and

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}|\nabla_z|\partial_x^\alpha u|^{\frac{\lambda}{2}}|^2\eta^2 = & \frac{\lambda^2}{4}\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}\nabla_z\partial_x^\alpha u\nabla_z\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2 \\
= & \frac{\lambda^2}{4(\lambda-1)}\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}\nabla_z\partial_x^\alpha u\nabla_z(\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2) \\
& - \frac{\lambda^2}{2(\lambda-1)}\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}\nabla_z\partial_x^\alpha u\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\nabla_z\eta\eta \\
\leq & \frac{\lambda^2}{4(\lambda-1)}\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}\nabla_z\partial_x^\alpha u\nabla_z(\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2) \\
& + \frac{\lambda^2}{8}\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}|\nabla_z\partial_x^\alpha u|^2|\partial_x^\alpha u|^{\lambda-2}\eta^2 + \frac{\lambda^2}{2(\lambda-1)^2}\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}|\partial_x^\alpha u|^\lambda|\nabla_z\eta|^2.
\end{aligned}$$

Absorbing the second summand above this implies

$$\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}|\nabla_z(|\partial_x^\alpha u|^{\frac{\lambda}{2}})|^2\eta^2 \leq \frac{\lambda^2}{2(\lambda-1)}\int_{\mathbb{R}_+^{n+1}} D(\partial_x^\alpha u)\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2 + \mathcal{I} \tag{25}$$

with

$$\mathcal{I} = \frac{\lambda^2}{(\lambda-1)^2}\int_{\mathbb{R}_+^{n+1}} y^{1-2\gamma}|\partial_x^\alpha u|^\lambda|\nabla_z\eta|^2$$

Due to $D(\partial_x^\alpha u) = \partial_x^\alpha Du$, and the structure of the metric

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1}} D(\partial_x^\alpha u)\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2 = & \int_{\mathbb{R}_+^{n+1}} \partial_x^\alpha(D_g u)\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2 \\
& - \int_{\mathbb{R}_+^{n+1}} \partial_x^\alpha((D_g - D)u)\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2
\end{aligned} \tag{26}$$

$$= \int_{\mathbb{R}^{n+1}_+} \partial_x^\alpha [h + \frac{\partial_p\sqrt{g}}{\sqrt{g}} y^{1-2\gamma} g^{p,q}\partial_q u + y^{1-2\gamma}\partial_i((g^{i,j}-\delta^{i,j})\partial_j u) - \frac{n-2\gamma}{2}\frac{\partial_y\sqrt{g}}{\sqrt{g}} y^{-2\gamma} u]\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2 = I_1+\ldots+I_4.$$

We may assume $|\nabla_z^k \eta| \le \frac{C}{\epsilon^k}$ for $k=0,1,2$, where $\epsilon = r_2 - r_1$. Then

$$|I_1| \le C\int_{\mathbb{R}^{n+1}_+} |\nabla_x^{|\alpha|} h||\nabla_x^{|\alpha|} u|^{\lambda-1}\eta^2. \tag{27}$$

Using integrations by parts and (15)

$$\begin{aligned}
|I_2| \le & |\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}\partial_q\partial_x^\alpha(\frac{\partial_p\sqrt{g}}{\sqrt{g}} g^{p,q} u)\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2| \\
&+ |\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}\partial_x^\alpha(\partial_q(\frac{\partial_p\sqrt{g}}{\sqrt{g}} g^{p,q})u)\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2| \\
\le & |\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}\partial_y\partial_x^\alpha(\frac{\partial_y\sqrt{g}}{\sqrt{g}} u)\partial_x^\alpha u|\partial_x^\alpha u|^{\lambda-2}\eta^2| \\
&+ \frac{C_{|\alpha|}}{\lambda}\sum_{m\le|\alpha|}\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\nabla_x^m u||\partial_x^\alpha u|^{\frac{\lambda-2}{2}}|\nabla_x|\partial_x^\alpha u|^{\frac{\lambda}{2}}|\eta^2 \\
&+ C_{|\alpha|}\sum_{m\le|\alpha|}\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\nabla_x^m u||\partial_x^\alpha u|^{\lambda-1}[|\nabla_x\eta|\eta+\eta^2] \\
\le & \frac{C_{|\alpha|}}{\lambda}\sum_{m\le|\alpha|}\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\nabla_x^m u|^{\frac{\lambda}{2}}|\nabla_z|\partial_x^\alpha u|^{\frac{\lambda}{2}}|\eta^2 \\
&+ C_{|\alpha|}\sum_{m\le|\alpha|}\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\nabla_x^m u|^{\lambda}[|\nabla_z\eta|\eta+\eta^2].
\end{aligned} \tag{28}$$

Using integration by parts and recalling $i,j=1,\ldots,n$

$$\begin{aligned}
|I_3| \le & \frac{C}{\lambda}\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\partial_x^\alpha((g^{i,j}-\delta^{i,j})\partial_j u)||\partial_x^\alpha u|^{\frac{\lambda-2}{2}}|\partial_i|\partial_x^\alpha u|^{\frac{\lambda}{2}}|\eta^2 \\
&+ C\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\partial_x^\alpha((g^{i,j}-\delta^{i,j})\partial_j u)||\partial_x^\alpha u|^{\lambda-1}|\partial_i\eta|\eta \\
\le & \frac{C}{\lambda^2}\sup_{B^+_{r_2}}|g^{i,j}-\delta^{i,j}|\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\partial_i|\partial_x^\alpha u|^{\frac{\lambda}{2}}||\partial_j|\partial_x^\alpha u|^{\frac{\lambda}{2}}|\eta^2 \\
&+ \frac{C_{|\alpha|}}{\lambda}\sum_{m\le|\alpha}\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\nabla_x^m u||\partial_x^\alpha u|^{\frac{\lambda-2}{2}}|\nabla_x|\partial_x^\alpha u|^{\frac{\lambda}{2}}|\eta^2
\end{aligned} \tag{29}$$

$$+\frac{C}{\lambda}\sup_{B^+_{r_2}}|g^{i,j}-\delta^{i,j}|\int_{\mathbb{R}^{n+1}_+}y^{1-2\gamma}|\partial_j|\partial^\alpha_x u|^{\frac{\lambda}{2}}||\partial^\alpha_x u|^{\frac{\lambda}{2}}|\partial_i\eta|\eta$$

$$+C_{|\alpha|}\sum_{m\le|\alpha}\int_{\mathbb{R}^{n+1}_+}y^{1-2\gamma}|\nabla^m_x u|^\lambda|\nabla_x\eta|\eta.$$

Using (15)

$$|I_4|\le C_{|\alpha|}\sum_{m\le|\alpha|}\int_{\mathbb{R}^{n+1}_+}y^{1-2\gamma}|\nabla^m_x u|^\lambda\eta^2 \tag{30}$$

Applying Hölder's and Young's inequality to (27)-(30) we obtain

$$\sum_{i=1}^4|I_i|\le\frac{C\sup_{B^+_{r_2}}|g-\delta|}{\lambda^2}\int_{\mathbb{R}^{n+1}_+}y^{1-2\gamma}|\nabla_z|\partial^\alpha_x u|^{\frac{\lambda}{2}}|^2\eta^2+\frac{C_{|\alpha|}}{\epsilon^2}\sum_{k\le|\alpha|}\|\nabla^k_x u\|^\lambda_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}$$
$$+C\|y^{2\gamma-1}\nabla^{|\alpha|}_x h\|_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}\|\nabla^{|\alpha|}_x u\|^{\lambda-1}_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}.$$

We may assume $C\sup_{B^+_{r_2}}|g-\delta|<\frac{1}{2}$, whence in view of (25) and (4.1)

$$\int_{\mathbb{R}^{n+1}_+}y^{1-2\gamma}|\nabla_z|\partial^\alpha_x u|^{\frac{\lambda}{2}}|^2\eta^2\le\frac{C_{|\alpha|}\lambda}{\epsilon^2}\sum_{k\le|\alpha|}\|\nabla^k_x u\|^\lambda_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}$$
$$+C\lambda\|y^{2\gamma-1}\nabla^{|\alpha|}_x h\|_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}\|\nabla^{|\alpha|}_x u\|^{\lambda-1}_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})},$$

so (24) implies

$$\int_{\mathbb{R}^{n+1}_+}y^{1-2\gamma}|\nabla_z(|\partial^\alpha_x u|^{\frac{\lambda}{2}}\eta)|^2\le\frac{C_{|\alpha|}\lambda}{\epsilon^2}\sum_{k\le|\alpha|}\|\nabla^k_x u\|^\lambda_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}$$
$$+C\lambda\|y^{2\gamma-1}\nabla^{|\alpha|}_x h\|_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}\|\nabla^{|\alpha|}_x u\|^{\lambda-1}_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}. \tag{31}$$

The weighted Sobolev inequality of Fabes-Kenig-Seraponi [10] Theorem 1.2 with $\kappa=\frac{n+1}{n}$ then shows

$$r_2^{-\frac{n+2\gamma}{n+1}}\|\partial^\alpha_x u\|^\lambda_{L^{\kappa\lambda}_{y^{1-2\gamma}}(B^+_{r_1})}\le\frac{C_{|\alpha|}\lambda}{\epsilon^2}\sum_{k\le|\alpha|}\|\nabla^k_x u\|^\lambda_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}$$
$$+C\lambda\|y^{2\gamma-1}\nabla^{|\alpha|}_x h\|_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}\|\nabla^{|\alpha|}_x u\|^{\lambda-1}_{L^\lambda_{y^{1-2\gamma}}(B^+_{r_2})}.$$

By rescaling we may assume for some $0<\epsilon_0\ll 1$, that

$$\|u\|_{L^2_{y^{1-2\gamma}}}+\sum_{k=0}^{|\alpha|}\|y^{2\gamma-1}\nabla^k_x h\|_{L^\infty_{y^{1-2\gamma}}(B^+_{(2+|\alpha|)\epsilon_0})}=1, \tag{32}$$

and putting $\lambda_i=2(\frac{n+1}{n})^i$ and $\rho_i=\epsilon_0(1+\frac{1}{2^i})$ we obtain

$$\|\nabla^{|\alpha|}_x u\|_{L^{\lambda_{i+1}}_{y^{1-2\gamma}}(B^+_{\rho_{i+1}})}\le\sqrt[\lambda_i]{C_{|\alpha|,\epsilon_0}\lambda_i 2^{2i}}\times$$
$$\sup_{m\le|\alpha|}[\|\nabla^m_x u\|_{L^{\lambda_i}_{y^{1-2\gamma}}(B^+_{r_2})}+\|\nabla^m_x u\|^{\frac{1}{2}}_{L^{\lambda_i}_{y^{1-2\gamma}}(B^+_{r_2})}],$$

where we have used $\frac{1}{2} \leq \frac{\lambda_i - 1}{\lambda_i} < 1$. Iterating this inequality then shows

$$\|\nabla_x^{|\alpha|} u\|_{L^\infty_{y^{1-2\gamma}}(B^+_{\epsilon_0})} \leq C_{\alpha,\epsilon_0}(1 + \sup_{m \leq |\alpha|} \|\nabla_x^m u\|_{L^2_{y^{1-2\gamma}}(B^+_{2\epsilon_0})}) \leq C_{\alpha,\epsilon_0},$$

where the last inequality follows from iterating (31) with $\lambda = 2$ and (32). Rescaling back we conclude

$$\sum_{k=0}^m \|\nabla_x^k u\|_{L^\infty_{y^{1-2\gamma}}(B^+_{\epsilon_0})} \leq C_{m,\epsilon_0}[\|u\|_{L^2_{y^{1-2\gamma}}} + \sum_{k=0}^m \|y^{2\gamma-1}\nabla_x^k h\|_{L^\infty_{y^{1-2\gamma}}}]. \tag{33}$$

Note, that $D(\partial_x^\alpha u) = \partial_x^\alpha h + \partial_x^\alpha((D - D_g)u)$, where

$$\begin{aligned}
&\partial_x^\alpha((D-D_g)u) \\
&\quad = \partial_x^\alpha[\frac{\partial_p\sqrt{g}}{\sqrt{g}} y^{1-2\gamma} g^{p,q}\partial_q u + \partial_i(y^{1-2\gamma}(g^{i,j} - \delta^{i,j})\partial_j u) - \frac{n-2\gamma}{2}\frac{\partial_y\sqrt{g}}{\sqrt{g}} y^{-2\gamma} u] \\
&\quad = \partial_q\partial_x^\alpha(\frac{\partial_p\sqrt{g}}{\sqrt{g}} y^{1-2\gamma} g^{p,q} u) - \partial_x^\alpha(\partial_q(\frac{\partial_p\sqrt{g}}{\sqrt{g}} y^{1-2\gamma} g^{p,q})u) \\
&\qquad + \partial_i\partial_x^\alpha(y^{1-2\gamma}(g^{i,j} - \delta^{i,j})\partial_j u) - \frac{n-2\gamma}{2}\partial_x^\alpha(\frac{\partial_y\sqrt{g}}{\sqrt{g}} y^{-2\gamma} u).
\end{aligned} \tag{34}$$

In particular, since $-\partial_p(y^{1-2\gamma} g^{p,q}\partial_q v) = Dv - \partial_i(y^{1-2\gamma}(g^{i,j} - \delta^{i,j})v_j)$ we may write

$$\partial_p(y^{1-2\gamma} g^{p,q}\partial_q\partial_x^\alpha u) = \partial_x^\alpha h + h^\alpha + \sum \partial_p h_p^\alpha,$$

where h^α, h_p^α depend only on $x-$derivatives of u of order up to $|\alpha|$, and due to (15), (33), there holds

$$\sum_{|\alpha|=0}^m \|\frac{h^\alpha}{y^{1-2\gamma}}, \frac{h_p^\alpha}{y^{1-2\gamma}}\|_{L^\infty_{y^{1-2\gamma}}(B^+_{\epsilon_0})} \leq C_{m,\epsilon_0}[\|u\|_{L^2_{y^{1-2\gamma}}} + \sum_{k=0}^m \|y^{2\gamma-1}\nabla_x^k h\|_{L^\infty_{y^{1-2\gamma}}}],$$

for all $m \in \mathbb{N}$. Then Zamboni[33] Theorem 5.2 shows Hölder regularity, i.e. for all $m \in \mathbb{N}$

$$\sum_{k=0}^m \|\nabla_x^k u\|_{C^{0,\alpha}(B^+_{\frac{\epsilon_0}{2}})} \leq C_{m,\epsilon_0}[\|u\|_{L^2_{y^{1-2\gamma}}} + \sum_{k=0}^m \|y^{2\gamma-1}\nabla_x^k h\|_{L^\infty_{y^{1-2\gamma}}}]. \tag{35}$$

This allows us to integrate the equation directly. Indeed from (34) we have

$$D(\partial_x^\alpha u) = \partial_x^\alpha h + \partial_x^\alpha((D - D_g)u) = \partial_x^\alpha h + \partial_y(y^{2-2\gamma} f_1^\alpha) + y^{1-2\gamma} f_2^\alpha,$$

where by definition $f_1^\alpha = \partial_x^\alpha(\frac{\partial_y\sqrt{g}}{y\sqrt{g}}u)$ and

$$\begin{aligned}
f_2^\alpha = &\partial_i\partial_x^\alpha(\frac{\partial_j\sqrt{g}}{\sqrt{g}} g^{i,j} u) - \partial_x^\alpha(y^{2\gamma-1}\partial_q(\frac{\partial_p\sqrt{g}}{\sqrt{g}} y^{1-2\gamma} g^{p,q})u) \\
&+ \partial_i\partial_x^\alpha((g^{i,j} - \delta^{i,j})\partial_j u) - \frac{n-2\gamma}{2}\partial_x^\alpha(\frac{\partial_y\sqrt{g}}{y\sqrt{g}} u).
\end{aligned}$$

This implies

$$-\partial_y(y^{1-2\gamma}\partial_y\partial_x^\alpha u) = \partial_y(y^{2-2\gamma} f_1^\alpha) + y^{1-2\gamma}(f_2^\alpha + \Delta_x\partial_x^\alpha u + y^{2\gamma-1}\partial_\alpha^x h)$$

and we obtain

$$\partial_x^\alpha u(y,x) = y^{2\gamma}\bar{u}_0^\alpha(x) - \int_0^y \sigma\tilde{f}_1^\alpha(\sigma,x)d\sigma - \int_0^y \sigma^{2\gamma-1}\int_0^\sigma \tau^{1-2\gamma}\tilde{f}_2^\alpha(\tau,x)d\tau d\sigma, \tag{36}$$

where by definition we may write with smooth coefficients $f_{i,\beta}$

$$\tilde{f}_1^\alpha = \sum_{|\beta|\leq|\alpha|} f_{1,\beta}\partial_x^\beta u \quad\text{and}\quad \tilde{f}_2^\alpha = \frac{\partial_x^\alpha h}{y^{1-2\gamma}} + \sum_{|\beta|\leq|\alpha|+2} f_{2,\beta}\partial_x^\beta u. \tag{37}$$

Let $h\in yC^{l,\lambda'}$. Then (35) shows

$$\forall|\alpha|\leq l \,:\, \nabla_x^{|\alpha|}u\in C^{0,\lambda},$$

whence $\forall\,|\alpha|\leq l-2 \,:\, \tilde{f}_i^\alpha\in C^{0,\lambda}$ due to (37). In particular (36) implies

$$\partial_x^\alpha u(y,x) = y^{2\gamma}\bar{u}_0^\alpha(x) + o(y^{2\gamma}),$$

so $\bar{u}_0^\alpha\in C^{l+2,\lambda}$ anyway by interior regularity. We define

$$\bar{u}^\alpha = y^{-2\gamma}\partial_x^\alpha u,\quad \bar{f}_1^\alpha = y^{-2\gamma}\tilde{f}_1^\alpha,\quad \bar{f}_2^\alpha = y^{-2\gamma}\tilde{f}_2^\alpha. \tag{38}$$

We then find from (36), that

$$\begin{aligned}\bar{u}^\alpha(y,x) =&\bar{u}_0^\alpha(x) - y^{-2\gamma}\int_0^y \sigma^{1+2\gamma}\bar{f}_1^\alpha(\sigma,x)d\sigma - y^{-2\gamma}\int_0^y\sigma^{2\gamma-1}\int_0^\sigma \tau\bar{f}_2^\alpha(\tau,x)d\tau d\sigma\\ =&\bar{u}_0^\alpha(x)+\bar{u}_1^\alpha(y,x)+\bar{u}_2^\alpha(y,x),\end{aligned} \tag{39}$$

where according to (37), (38) we may write with smooth coefficients $f_{i,\beta}$

$$\bar{f}_1^\alpha = \sum_{|\beta|\leq|\alpha|} f_{1,\beta}\bar{u}^\beta \quad\text{and}\quad \bar{f}_2^\alpha = \frac{\partial_x^\alpha h}{y} + \sum_{|\beta|\leq|\alpha|+2} f_{2,\beta}\bar{u}^\beta. \tag{40}$$

Then (36) and $\forall\,|\alpha|\leq l-2 \,:\, \tilde{f}_i^\alpha\in C^{0,\lambda}$ already show

$$\forall|\alpha|\leq l-2 \,:\, \bar{u}^\alpha\in C^{0,\lambda}$$

and we may assume $\forall|\alpha|\leq l-2-2m \,:\, \partial_y^{2m}\bar{u}^\alpha\in C^{0,\lambda}$ inductively, whence according to (40)

$$\forall|\alpha|\leq l-2-2(m+1) \,:\, \partial_y^{2m}\bar{f}_i^\alpha\in C^{0,\lambda}.$$

Then (39) implies via Taylor expansion

$$\forall|\alpha|\leq l-2-2(m+1) \,:\, \partial_y^{2m+2}\bar{u}_i^\alpha,\ \partial_y^{2m+2}\bar{u}^\alpha\in C^{0,\alpha}.$$

Thus we have proven $\forall\,|\alpha|\leq l-2-2m \,:\, \partial_y^{2m}\partial_x^\alpha u\in C^{0,\lambda}$ for some $\lambda>0$. However, since there are only even powers in the y-derivative, we only find $u\in C^{l-3,\lambda}$ for $l\in 2\mathbb{N}$. The proof is thereby complete. ☐

4.2. **Green's function for D_g under weighted Neumann boundary condition.** In this subsection we study the Green's function Γ_g. As in the previous one we consider the existence and asymptotics issue. To do that we use the method of Lee-Parker[22] and have the same difficulties to overcome as in the previous subsection. We first note that on $\mathbb{R}^{n+1}_+$

$$\Gamma(y,x,\xi)=\Gamma^\gamma(y,x,\xi)=\frac{g_{n,\gamma}}{(y^2+|x-\xi|^2)^{\frac{n-2\gamma}{2}}},\quad (y,x)\in\overline{\mathbb{R}}^{n+1}_+,\quad \xi\in\mathbb{R}^n \tag{41}$$

for some $g_{n,\gamma}>0$ is the Green's function to the dual problem

$$\begin{cases} Du=0 \quad\text{in}\quad \mathbb{R}^{n+1}_+\\ -d_\gamma^*\lim_{y\to 0}y^{1-2\gamma}\partial_y u(y,\cdot)=f \quad\text{on}\quad \mathbb{R}^n,\end{cases}$$

i.e.

$$\begin{cases} D\Gamma(,\xi) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \ \xi \in \mathbb{R}^n \\ -d^*_\gamma \lim_{y\to 0} y^{1-2\gamma}\Gamma(y,x,\xi) = \delta_\xi(x), \quad x, \ \xi \in \mathbb{R}^n. \end{cases}$$

We will construct the Green's function Γ_g for the analogous problem

$$\begin{cases} D_g u = 0 \quad \text{in} \quad X \\ -d^*_\gamma \lim_{y\to 0} y^{1-2\gamma}\partial_y u(y,\cdot) = f \quad \text{on} \quad M \end{cases}$$

for $D_g = -div_g(y^{1-2\gamma}\nabla_g(\,\cdot\,)) + E_g$, i.e. for $z \in X$ and $\xi \in M$

$$\begin{cases} D_g\Gamma_g(\cdot,\xi) = 0 \quad \text{in} \quad X \\ -d^*_\gamma \lim_{y\to 0} y^{1-2\gamma}\Gamma_g(z,\xi) = \delta_\xi(x), \end{cases} \tag{42}$$

where $z = (y,x) \in X$ in g-normal Fermi-coordinates close to M. To that end we identify

$$\xi \in M \cap U \subset U \cap X \quad \text{with} \quad 0 \in B^{\mathbb{R}^{n+1}}_\epsilon(0) \cap \mathbb{R}^n \subset B^{\mathbb{R}^{n+1}}_\epsilon(0) \cap \overline{\mathbb{R}}^{n+1}_+$$

as in the previous subsection, and write $\Gamma(z) = \Gamma(z,0)$. On $B^{\mathbb{R}^{n+1}}_\epsilon(0) \cap \mathbb{R}^{n+1}_+$ we then have

$$D_g\Gamma = -\frac{\partial_p}{\sqrt{g}}(\sqrt{g}g^{p,q}y^{1-2\gamma}\partial_q\Gamma) + E_g\Gamma = f \in y^{1-2\gamma}H_{-n+2\gamma-1}C^\infty. \tag{43}$$

Again we may solve homogeneous deficits homogeneously.

Lemma 4.2. *For $\frac{1}{2} \neq \gamma \in (0,1)$ and $f_l \in y^{1-2\gamma}H_{l+2\gamma-1}$, $l \in \mathbb{N} - n$ there exists $\Gamma_{1+2\gamma+l} \in H_{1+2\gamma+l}$ such, that*

$$D\Gamma_{1+2\gamma+l} = f_l \quad \textit{in} \ \mathbb{R}^{n+1} \quad \textit{and} \quad \lim_{y\to 0} y^{1-2\gamma}\partial_y\Gamma_{1+2\gamma+l} = 0 \ \textit{on} \ \mathbb{R}^n \setminus \{0\}.$$

Proof. This time we use

$$\langle Q^k_l(y,x) = y^{2k}P_l(x) \mid \ k, \ l \in \mathbb{N} \quad \text{and} \quad P_l \in \Pi_l\rangle \underset{\text{dense}}{\subset} C^0(\overline{B}^{\mathbb{R}^{n+1}}_1(0) \cap \mathbb{R}^{n+1}_+),$$

to obtain a orthogonal basis $E = \{e^i_k\}$ for $L^2_{y^{1-2\gamma}}(S^n_+)$ consisting of D-harmonics of the form

$$e^i_k = A_m\lfloor_{S^n_+}, \quad A_m(y,x) = \sum y^{2l}P_{k-2l}(x), \quad DA_m = 0$$

and we have $D_{S^n_+}e^i_k = k(k+n-2\gamma)y^{1-2\gamma}e^i_k$. Then for homogeneous f, u of degree λ, $\lambda + 1 + 2\gamma$ solving

$$\begin{cases} Du = f \in L^2_{y^{2\gamma-1}}(\mathbb{R}^{n+1}_+) \quad \text{in} \quad \mathbb{R}^{n+1}_+ \\ \lim_{y\to 0} y^{1-2\gamma}\partial_y u = 0 \quad \text{on} \quad \mathbb{R}^n \end{cases}$$

is, when writing $u = \sum a_{i,k}e^i_k$, $y^{2\gamma-1}f = \sum b_{j,l}e^j_l$, equivalent to solving

$$\sum a_{i,k}(k(k+n-2\gamma) - (\lambda+1+2\gamma)(\lambda+n+1))e^i_k = \sum b_{j,l}e^j_l$$

and the latter system is always solvable in case

$$k(k+n-2\gamma) - (\lambda+1+2\gamma)(\lambda+n+1) \neq 0 \ \text{ for all } \ k, \ n, \ \lambda \in \mathbb{N}. \tag{44}$$

As for proving the lemma there holds

$$deg(f_l) = \lambda = m - n \quad \text{and} \quad deg(e^i_k) = k = m' \quad \text{for some} \quad m, \ m' \in \mathbb{N}$$

and plugging this into (44) we verify for $\frac{1}{2} \neq \gamma \in (0,1)$

$$m'(m'+n-2\gamma)-(m-n+1+2\gamma)(m+1)\neq 0 \quad \text{for all} \quad n\,,m,\; m'\in\mathbb{N}.$$

This shows homogeneous solvability, whereas regularity of the solution follows from Proposition 3. ☐

Analogously to the case of the Poisson kernel we may solve (42) successively using Lemma 4.2 and obtain

$$\Gamma_g=\eta_\xi(\Gamma+\sum_{l=-n}^{m}\Gamma_{1+2\gamma+l})+\gamma_m$$

for $m\geq 0$, where η_ξ is as in (23) and a weak solution

$$\begin{cases} D_g\gamma_m=-D_g\left(\eta_\xi(\Gamma+\sum_{l=-n}^{m}\Gamma_{1+2\gamma+l})\right)=y^{1-2\gamma}h_m & \text{in} \quad X\\ \lim_{y\to 0}y^{1-2\gamma}\partial_y\gamma_m=0 & \text{on} \quad M\end{cases}$$

with $h_m\in C^{m,\alpha}$. As in the previous subsection a weak regularity statement is sufficient for our purpose.

Proposition 3. *Let $h\in y^{1-2\gamma}C^{2k+3,\alpha}(X)$ and $u\in W^{1,2}_{y^{1-2\gamma}}(X)$ be a weak solution of*

$$\begin{cases} D_gu=h & \text{in} \quad X\\ \lim_{y\to 0}y^{1-2\gamma}\partial_yu=0 & \text{on} \quad M.\end{cases}$$

Then u is of class $C^{2k,\beta}(X)$, provided $H_g=0$.

As in the previous subsection, putting these facts together before presenting the proof of Proposition 3, we have the existence of Γ_g and can describe its asymptotics.

Corollary 2. *Let $\frac{1}{2}\neq\gamma\in(0,1)$. Then Γ_g exists and we may expand in g-normal Fermi-coordinates around $\xi\in M$*

$$\Gamma_g(z,\xi)\;\in\eta_\xi(z)\left(\frac{g_{n,\gamma}}{|z|^{n-2\gamma}}+\sum_{l=-n}^{2m+3}H_{1+2\gamma+l}(z)\right)+C^{2m,\alpha}(X)$$

with $H_l\in C^\infty(\mathbb{R}^{n+1}_+\setminus\{0\})$ being homogeneous of order l and $g_{n,\gamma}$ is as in (41), provided $H_g=0$.

Proof of Proposition 3. As in the previous subsection we use the Moser iteration argument. Indeed by exactly the same arguments as the ones used when proving Proposition 2 we recover Hölder regularity (35) and integrating the equation directly we find the analogue of (36), namely

$$\begin{aligned}\partial_x^\alpha u(y,x)=&u_0^\alpha(x)-\int_0^y\sigma\tilde{f}_1^\alpha(\sigma,x)d\sigma-\int_0^y\sigma^{2\gamma-1}\int_0^\sigma\tau^{1-2\gamma}\tilde{f}_2^\alpha(\tau,x)d\tau d\sigma\\ =&u_0^\alpha(x)+u_1^\alpha(y,x)+u_2^\alpha(y,x),\end{aligned}\tag{45}$$

where $\tilde{f}_1,\ \tilde{f}_2$ are given by (37). Let $h\in y^{1-2\gamma}C^{l,\lambda'}$. Then (35) and (37) show

$$\forall|\alpha|\leq l-2\;:\;\tilde{f}_i^\alpha\in C^{0,\lambda}.$$

In particular (45) implies

$$\partial_x^\alpha u(y,x)=u_0^\alpha(x)+O(y),$$

so $u_0^\alpha\in C^{l+2,\lambda}$ anyway by interior regularity and we may assume inductively

$$\forall|\alpha|\leq l-2-2m\;:\;\partial_y^{2m}\partial_x^\alpha u\in C^{0,\lambda},$$

whence according to (37)

$$\forall|\alpha| \leq l-2-2(m+1) \ : \ \partial_y^{2m} \tilde{f}_i^\alpha \in C^{0,\lambda}.$$

Then (45) implies via Taylor expansion

$$\forall|\alpha| \leq l-2-2(m+1) \ : \ \partial_y^{2m+2} u_i^\alpha, \ \partial_y^{2m+2} \partial_x^\alpha u \in C^{0,\lambda}$$

Thus we have proven $\forall| \alpha| \leq l-2-2m \ : \ \partial_y^{2m}\partial_x^\alpha u \in C^{0,\lambda}$ for some $\lambda > 0$. However, since there are only even powers in the y-derivative, we only find $u \in C^{l-3,\lambda}$ for $l \in 2\mathbb{N}$. The proof is thereby complete. □

4.3. **Green's function for the fractional conformal Laplacian.** In this short subsection we study the Green's function G_h^γ of P_h^γ. We derive its existence and asymptotics as a consequence of the results of the previous subsections and formula (16).

Corollary 3. *Let $\frac{1}{2} \neq \gamma \in (0,1)$. Then G_h exists and we may expand in h-normal-coordinates around $\xi \in M$*

$$G_h(x,\xi) \ \in \eta_\xi(x) \left(\frac{g_{n,\gamma}}{|x|^{n-2\gamma}} + \sum_{l=-n}^{2m+3} H_{1+2\gamma+l}(x) \right) + C^{2m,\alpha}(M)$$

with $H_l \in C^\infty(\mathbb{R}^n \setminus \{0\})$ being homogeneous of order l, provided $H_g = 0$.

To end this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. It follows directly from Corollary 1, Corollary 2, and Corollary 3. □

5. **Locally flat conformal infinities of PE-manifolds.** In this section we sharpen the results of Section 4 in the case of Poincaré-Einstein manifold (X, g^+) with locally flat conformal infinity $(M, [h])$.

5.1. **Fermi-coordinates in this particular case.** By our assumptions we have a geodesic defining function y splitting the metric

$$g = y^2 g^+, \ g = dy^2 + h_y \quad \text{near} \quad M \quad \text{and} \quad h = h_y \lfloor_M$$

and for every $a \in M$ a conformal factor as in (10), whose conformal metric $h_a = u_a^{\frac{4}{n-2\gamma}} h$ close to a admits an Euclidean coordinate system, $h_a = \delta$ on $B_\epsilon^{h_a}(a)$. As clarified in subsection 3.2 and recalling Remark 1, this gives rise to a geodesic defining function y_a, for which

$$g_a = y_a^2 g^+, \ \ g_a = dy_a^2 + h_{a,y_a} \ \text{ near } \ M \ \text{ with } \ h_a = h_{a,y_a}\lfloor_M \ \text{ and } \ \delta = h_a\lfloor_{B_\epsilon^{h_a}(a)}, \tag{46}$$

the boundary $(M, [h_a])$ is totally geodesic and the extension operator D_{g_a} is positive. As observed by Kim-Musso-Wei[21] in the case $n \geq 3$, cf. Lemma 43 in [21], and for $n = 2$ due to Remark 1 and the existence of isothermal coordinates we have

$$g_a = \delta + O(y_a^n) \quad \text{on} \quad B_\epsilon^{g_a,+}(a) \tag{47}$$

in g_a-normal Fermi-coordinates around $a$for some small $\epsilon > 0$. Therefore the previous results on the fundamental solutions in the case of an asymptotically hyperbolic manifold with minimal conformal infinity of Section 4 are applicable. We collect them in the following subsection.

5.2. **Fundamental solutions in this particular case.** In this subsection we sharpen the results of Section 4 in the case of a Poincaré-Einstein manifold (X, g^+) with locally flat conformal infinity $(M, [h])$.

To do that let us first recall that $K_a = K_{g_a}(\cdot, a)$, $\Gamma_a = \Gamma_{g_a}(\cdot, a)$ and $G_a = G_{h_a}(\cdot, a)$. From (47) we then find

$$D_{g_a} K_a \in y H_{-2\gamma-2} C^\infty, \quad D_{g_a}\Gamma_a \in y^{1-2\gamma} H_{2\gamma-2} C^\infty$$

for the lowest order deficits in (19) and (43). Then in view of Lemmas 4.1, 4.2 the corresponding expansions given by Corollaries 1, 2, 3 are

$$K_a(z) \in \eta_\xi(z)\left(p_{n,\gamma}\frac{y^{2\gamma}}{|z|^{n+2\gamma}} + \sum_{l=0}^{2m+6} y^{2\gamma} H_{l-2\gamma}(z)\right) + y^{2\gamma} C^{2m,\alpha}(X)$$

$$\Gamma_a(z) \in \eta_\xi(z)\left(\frac{g_{n,\gamma}}{|z|^{n-2\gamma}} + \sum_{l=0}^{2m+4} H_{l+2\gamma}(z)\right) + C^{2m,\alpha}(X)$$

$$G_a(x) \in \eta_\xi(x)\left(\frac{g_{n,\gamma}}{|x|^{n-2\gamma}} + \sum_{l=0}^{2m+4} H_{l+2\gamma}(x)\right) + C^{2m,\alpha}(M).$$

Recalling (20) there holds $y^{2\gamma}H_{l-2\gamma} \subset C^{m,\alpha}$ for $l > m$ and $H_{l+2\gamma} \subset C^{m,\alpha}$ for $l \geq m$. We have therefore proven the following result.

Corollary 4. *Let (X, g^+) be a Poincaré-Einstein manifold with conformal infinity $(M, [h])$ of dimension $n = 2$ or $n \geq 3$ and $(M, [h])$ is locally flat. If*

$$\frac{1}{2} \neq \gamma \in (0,1) \quad \textit{and} \quad \lambda_1(-\Delta_{g^+}) > s(n-s) \quad \textit{for } s = \frac{n}{2} + \gamma,$$

then the Poison kernel K_g and the Green's functions Γ_g and G_h respectively for

$$\begin{cases} D_g U = 0 & \textit{in} \quad X \\ U = f & \textit{on} \quad M \end{cases}$$

$$\begin{cases} D_g U = 0 & \textit{in} \quad X \\ -d_\gamma^* \lim_{y\to 0} y^{1-2\gamma}\partial_y U = f & \textit{on} \quad M \end{cases}$$

and

$$\begin{cases} P_h^\gamma u = f & \textit{on} \quad M \end{cases}$$

are respectively of class $y^{2\gamma}C^{2,\alpha}$ and $C^{2,\alpha}$ away from the singularity and admit for every $a \in M$ locally in g_a-normal Fermi-coordinates an expansion around a

$$K_a(z) \in p_{n,\gamma}\frac{y^{2\gamma}}{|z|^{n+2\gamma}} + y^{2\gamma}H_{-2\gamma}(z) + y^{2\gamma}H_{1-2\gamma}(z) + y^{2\gamma}H_{2-2\gamma}(z) + y^{2\gamma}C^{2,\alpha}(X)$$

$$\Gamma_a(z) \in \frac{g_{n,\gamma}}{|z|^{n-2\gamma}} + H_{2\gamma}(z) + H_{1+2\gamma}(z) + C^{2,\alpha}(X)$$

$$G_a(x) \in \frac{g_{n,\gamma}}{|x|^{n-2\gamma}} + H_{2\gamma}(x) + H_{1+2\gamma}(x) + C^{2,\alpha}(M),$$

where g_a is as in (46) and $H_k \in C^\infty(\overline{\mathbb{R}^n_+} \setminus \{0\})$ are homogeneous of degree k.

Finally, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. It is exactly the statement of Corollary 4. $\square$

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