# Value of Information Analysis in Non-stationary Stochastic Decision Environments: A Reliability-assisted POMDP Approach

Chaolin Song[1,2], Chi Zhang[1], Abdollah Shafieezadeh[1]*, Rucheng Xiao[2]

[1]Risk Assessment and Management of Structural and Infrastructure Systems (RAMSIS) Lab, Department of Civil, Environmental, and Geodetic Engineering, The Ohio State University, Columbus, OH, 43210, United States

[2]Department of Bridge Engineering, Tongji University, Shanghai, 200092, China

## ABSTRACT

Optimal management of systems over their service life as they face a multitude of uncertainties remains a significant challenge. While additional information can reduce uncertainties, collecting new information incurs cost and may include observation error. Value of Information (VoI) analysis facilitates quantitative assessment of the expected net benefits of collecting new information. Moreover, partially observable Markov decision processes (POMDPs) can be integrated within VoI analysis to efficiently capture the sequential decision-making environments for systems. The assumption of stationary environment in existing POMDP frameworks may not be valid, however, in many applications such as deterioration processes which are often non-stationary. To address this gap, this paper presents a new approach called VoI-R-POMDP. A new POMDP framework is proposed to accurately describe non-stationary processes using multiple integrated transition models. New strategies based on reliability concepts are developed to accurately and efficiently determine the parameters of the proposed POMDP model based on prior information. A new formulation of the observation function based on Bayes' theorem is also derived. The proposed framework is applied to a corroding beam example. Results indicate that VoI-R-POMDP can accurately and efficiently describe the deterioration process and thus provide accurate VoI estimates for non-stationary systems.

**Key words**: *Value of information, Reliability methods, Partially observable Markov decision processes, Bayes' theorem, Non-stationary environments*

**Nomenclature**

| | |
|---|---|
| $a^0$ | An inspection action |
| $a^1$ | A maintenance action |
| $a_N^1$ | The do-nothing maintenance action |
| $a_R^1$ | The replacement maintenance action |
| $a(t=n)$ | The action to be executed at time step $n$ for a POMDP model |
| $A$ | The set of actions in POMDPs |
| $A_{i,j}$ | The probability of providing "Signal of $E_{z,i}$" when the system is in the state $s_j$. |
| $\boldsymbol{b}$ | The belief state |
| $\boldsymbol{b_0}$ | The initial belief state |
| $\boldsymbol{b}_{t=n}$ | The belief state of a POMDP model at time step $n$ |
| $C_{prior}$ | Expected cost of the optimal decision without additional information |
| $C_{pre-post}$ | Expected cost of the optimal decision with additional information |
| $d$ | The number of defined damage levels |
| $\boldsymbol{b}_{t=n,R}$ | The probabilities of being different states determined by reliability methods |
| $E_i$ | A discrete event describing the state $s_i$ of POMDPs |
| $E_{z,i}$ | A discrete event describing the observation outcome $z_i$ of POMDPs |
| $\mathrm{E}(C\|\boldsymbol{Y}=\boldsymbol{y})$ | Conditional expected cost with the optimal decision given the observation result $\boldsymbol{Y} = \boldsymbol{y}$ |
| $f_i$ | A limit state function that describes the event $E_i$ |
| $h(\cdot)$ | A function that models the monitored quantity |
| $k$ | The number of inspection measurements |
| $m$ | A choice of management strategy from the set |
| $M$ | The set of management strategy |
| $\boldsymbol{O}$ | The observation matrices in POMDPs |
| $O(z,a,s)$ | The probability of providing observation outcome $z$ when the system state is $s$ and the selected action is $a$ |
| $E_{z,i}^I(t=n)$ | The event indicated by the inspection outcome at time step $n$ |
| $r(t=n)$ | The received reward of a POMDP model at time step $n$ |
| $\boldsymbol{R}$ | The reward matrices in PODMPs |
| $R(s,a)$ | The reward for the case where the system state is $s$ and action is $a$ |
| $s^0$ | The state of a POMDP model at inspection sub-steps |
| $s^1$ | The state of a POMDP model at maintenance sub-steps |
| $s(t=n)$ | The state of a POMDP model at time step $n$ |
| $s_i$ | The $i^{\mathrm{th}}$ state in the set $S$ |
| $S$ | The set of states in POMDPs |
| $\boldsymbol{T}$ | The transition matrices in POMDPs |
| $\boldsymbol{T}_a$ | The transition matrix with action $a$ |
| $T(s,a,s')$ | The transition probability from state $s$ to $s'$ with action $a$ being performed |
| $V_n^*$ | The maximum expected reward calculated by a POMDP model considering $n$ steps |
| VoI | Value of information |
| $\mathrm{VoI}_{\mathrm{f,R}}$ | Value of an information flow determined by the proposed POMDP framework |
| $\mathrm{VoI}_{\mathrm{c,R}}$ | Value of a current inspection action determined by the proposed POMDP framework |
| $\boldsymbol{x}$ | A realization of the random vector $\boldsymbol{X}$ |
| $\boldsymbol{X}$ | The random vector representing the system uncertainty |
| $\boldsymbol{y}$ | A realization of the inspection outcomes $\boldsymbol{Y}$ |
| $\boldsymbol{Y}$ | The vector of inspection outcomes |
| $z$ | The observation outcome |
| $z(t=n)$ | The observation outcome received at time step $n$ for a POMDP model |
| $Z$ | The set of observation outcomes in POMDPs |

| | |
|---|---|
| $\mathbf{\Theta}$ | $\mathbf{\Theta} = \{T, O, R, \gamma\}$ is the set of parameters in a POMDP model |
| $\tau_l$ | $\tau_l = T\left(s_{l-1,i}, a_N, s_{l.i}\right)$ is the transition probability from $s_{l-1,i}$ to $s_{l.i}$ with "do nothing" action |
| $\boldsymbol{\epsilon}$ | The measurement error of an inspection action |
| $\Gamma_{i,j}$ | The probability of receiving "Signal of $E_{z,i}$" when the system is currently in state $s_j$ |
| $\gamma$ | The discounting factor in POMDPs |
| $\boldsymbol{\mathcal{M}}_{l,l}$ | The submatrix of the transition matrix $\boldsymbol{T}_{a_N^1}$ |
| $\boldsymbol{\mathcal{T}}_{l,v}$ | The submatrix of the transition matrix $\boldsymbol{T}_{a_R^1}$ |

## 1. Introduction

Aging deterioration and shock-type hazards threaten the functionality of systems (e.g., machines, structures or infrastructure systems [1, 2]) over their service life, thus requiring multiple maintenance treatments to manage the risks. The presence of uncertainties in loads, system conditions, the environment and the subsequent effects on systems poses a substantial challenge for developing optimal maintenance policies especially when the budget is limited [3]. As a result, management of uncertainties is necessary and crucial for decision making; collecting new information from inspections and structural health monitoring (SHM) systems can help with improved characterization of involved uncertainties. Nevertheless, the cost savings by this improvement are not guaranteed to be significant, as the collected information also suffers from measurement errors and incurs cost that may surpass monetary benefits gained by the information. To address this challenge, the concept of Value of Information (VoI) has been proposed to quantitatively measure the expected benefit (or in a more common sense, utility increment) from future information. Quantifying the VoI has been an important step towards maximizing the potential benefits from inspections or continuous SHM systems.

The mathematical framework of VoI was first proposed in [4, 5]. Based on Bayes' theorem, if a decision maker knows the inspection outcome, the prior belief will be updated, and her choice of action can potentially change. The increase in utility is defined as the conditional value of information (CVI), and the expected value of CVI is named as the value of information. In the last few decades, the concept of VoI has been adopted and applied in various scientific and engineering fields. For example, Pozzi and Der Kiureghian [6] outlined the framework of assessing the VoI for long-term SHM and introduced a Monte Carlo approach to quantify the values. Zonta et al. [7] proposed assessing the value of a monitoring system for maintenance decision making of a pedestrian bridge using the VoI concept. Straub [8] investigated the potential of using reliability methods to reduce the computational burden of VoI analysis. Zitrou et al. [9] performed a VoI-based sensitivity analysis to identify the model parameters that have significant impact on a maintenance optimization problem. Long et al. [10] investigated the relation between VoI and parameters including the number of sensors, sensor locations, measurement noise, and the Type-I error for the indication of the system states. Iannacone et al. [11] proposed a formulation to calculate the VoI considering both deterioration processed and shock occurrences. Bjørnsen et al. [12] proposed a semi-quantitative process considering the strength of background knowledge, on which the probabilistic model of the system is established, in the VoI assessment and decision-making. Zou et al. [13] developed a holistic decision making-based approach to enable assessing the VoI for a sequence of future tests.

As VoI explicitly analyzes the utility gained from inspections in terms of reduction in the expected loss, some studies have used VoI as a metric to optimize the way of collecting information (e.g., inspection plans or sensor placement). Straub and Faber [14, 15] developed an approach to determine component inspection plans under system effects. Zhang et al. [16] proposed a framework integrating the VoI and risk-based inspection planning and applied the approach to an example concerning fatigue degradation of steel structures to illustrate the optimization of the monitoring variables, period and quality of inspections. Fauriat and Zio [17] proposed an aperiodic sequential inspection policy where the next time of action is determined by maximizing the VoI based on the condition of the system. Neves [18] developed a dynamic decision-making framework based on decision trees and VoI analysis to determine the optimal SHM and maintenance decisions in bridges. Farhan et al. [19] studied the optimal inspection time and maintenance strategy for a welded joint in an offshore wind turbine support structure based on VoI. Malings and Pozzi [20] adopted a greedy optimization technique to maximize the VoI metric and determine the optimal plan iteratively. Malings and Pozzi [21] further compared the performance of the VoI metric-based greedy optimization with genetic algorithm and a conditional entropy-based heuristic approach for the arrangement of sensor placement. The same authors investigated VoI-based dynamic adjustment of sensor placements and scheduling as new information updates the knowledge about the system [22].

Despite the above advances, determining the optimal actions with the prior or pre-posterior information can still be computationally very complex due to the increasing number of decision times. This task is even more challenging when considering the availability of future inspections, of which results are dynamic and correlated with the later optimal decisions. If the problem is simplified that one single decision is to be made based only on the current information, both the VoI and Service Life Cost (SLC) can be overestimated. To address this limitation, more advanced techniques are required to efficiently determine the optimal management policy and SLC.

Partially observable Markov decision processes (POMDPs) provide a sound mathematical framework for life cycle analysis [23-26]. Compared with Markov decision processes (MDPs) [27] where the state is fully observed with absolute certainty at each time step, POMDP assumes that the true state is not fully revealed and allows updating the belief about the state with arriving observations. Thus, POMDP offers a significantly more realistic framework for the analysis and management of the service life of systems. Exact solutions of POMDP need to consider all possible actions and observation results, which can incur a high computational burden. An efficient strategy for solving large PODMP problems is the point-based algorithm [28-38], which samples a set of points as a representation of the belief space. Note that in structure and infrastructure management, a common approach to model degradation processes is Markov-based where the probability of states depends only on the previous state of the system and therefore independent of the history prior to the last state [39, 40]. Relative to general probabilistic models that may not necessarily require this assumption, Markov-based models can substantially reduce the complexity of modeling systems in uncertain environments, while providing a capable mathematical framework for sequential decision making through Markov decision processes. In engineering applications, POMDP has been successfully adopted for robot navigation [41, 42], search for victims [43], and management of systems (e.g., corroded structures [44-47], machines [26, 48] and transmission networks [49]). Memarzadeh and Pozzi [50] proposed utilizing two POMDP models for VoI analysis, and the two models are established with the only difference that one model has an additional information flow (or an additional inspection action under the information flow) and the other does not. The difference between the minimum SLCs obtained from these two models is named the value of information flow (or the value of the inspection action). A stochastic allocation model and a fee-based allocation model are specially designed to describe the information flow with the information being collected at a given probability and at a given cost at each time step, respectively. Except for the inspections that provide the information flow, it is also assumed that there are ordinary scheduled observations that only reveal if the system has failed or not. To include both the fee-based inspections and ordinary observations, each time step in the POMDP is split into an inspection sub-step, where the inspection can be executed to help with the later maintenance decisions, and a maintenance sub-step, where the ordinary observation occurs, revealing whether the system has failed or not. Li and Pozzi [39] additionally investigated the relation between VoI and multiple features of the monitoring system, such as the measurement accuracy and repair cost, based on the stochastic allocation model assumption. However, existing POMDP frameworks adopt stationary transition functions, and therefore cannot well describe a time-dependent deterioration process. Moreover, accurately determining a POMDP model based on the prior knowledge remains a challenge. For instance, even with known measurement errors, it is still challenging to derive the observation function in POMDPs, as the knowledge of the system uncertainties and monitoring process is not rigorously connected with the POMDP model. Thus, gaps remain regarding accurate modeling of uncertainties of aging systems in non-stationary environments with the POMDP model.

To address these gaps, this paper proposes a new approach called VoI-R-POMDP to facilitate the POMDP-based VoI analysis. The contributions of the paper can be summarized as follows. First, a new POMDP framework is proposed that incorporates multiple models to describe different deterioration rates with the ability for the belief state to transition among these models with aging. Thus, a time-dependent deterioration process can be more accurately described by the proposed framework relative to existing POMDP models, where the stationary deterioration assumption can result in considerable errors. Second, new methods are developed for accurate and efficient POMDP model definition. Here, the transition functions are estimated using Maximum Likelihood Estimation (MLE) based on probabilities that are derived from reliability methods as benchmark. A new formulation based on Bayes' theorem is introduced to derive accurate observation functions. The proposed approach is applied to a corroding beam example to calculate value of both current inspection action and information flow. The probabilities of failure from the proposed POMDP model and existing POMDP models are compared in terms of accuracy. The relations among VoI, the replacement cost, the inspection accuracy and costs are investigated.

The rest of the paper is organized as follows. Section 2 provides an overview of the fundamental theory of VoI and the POMDP framework. Section 3 presents the proposed VoI-R-POMDP approach. Section 4 includes a detailed numerical example to demonstrate the performance of the proposed method. Concluding remarks are presented in Section 5.

## 2. Preliminaries

This section provides an overview of VoI concept and the POMDP framework. Readers are referred to [8, 39, 50, 51] for more details.

### 2.1 Value of Information

The VoI analysis quantifies the expected utility improvement of a system by acquiring additional information. A system can denote a machine, a structure or an infrastructure system, which is providing a service and is managed to retain the functionality over time. As introduced in [8], VoI can be generally formulated as:

$$\mathrm{VoI} = C_{prior} - C_{pre-post} \tag{1}$$

where $C_{prior}$ and $C_{pre-post}$ are the expected costs of the optimal decisions with and without the additional information, respectively.

The high complexity of optimal decision making for management of structures and infrastructure systems stems in part from the multitude of endogenous or exogenous uncertainties (e.g., various environmental or load conditions, workmanship, human error and occurrence of future events [52]) that they face. Following the classical reliability framework, the random vector $\boldsymbol{X}$ is used here to represent the uncertainty associated with the phenomena of interest. For instance, when the event of interest is the failure of a transmission tower under extreme wind loads, the random vector $\boldsymbol{X}$ will include the parameters that have considerable influence on the performance of the tower under wind loads including, among others, the modulus of elasticity, yield stress of the main legs and other elements of the tower, and the uncertainty in the wind-induced loading [53]. Thus, the relation between the failure or survival of the tower and a realization of $\boldsymbol{X}$ can be defined using a function known as the limit state function or performance function in reliability analysis. In this environment, the decision maker can choose a maintenance strategy $m$ from a set of available strategies $M$ at a given time during the decision horizon of the system. Then, $C_{prior}$ can be typically calculated by:

$$C_{prior} = \min_{m \in M} \mathbb{E}_{\boldsymbol{X}}[c(m, \boldsymbol{x})] = \min_{m \in M} \int_{\boldsymbol{X}} c(m, \boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{2}$$

where $c(m, \boldsymbol{x})$ is the cost corresponding to a given strategy $m$ and a realization $\boldsymbol{x}$ of the random vector. $\mathrm{E}_{\boldsymbol{X}}$ denotes the expectation with respect to $\boldsymbol{X}$, and $f_{\boldsymbol{X}}(\cdot)$ is the joint probability density function (PDF) of the random vector $\boldsymbol{X}$.

The information provided by the event of inspection can affect the optimal decision. For instance, measuring the deflections of a bridge in a load test can reduce the uncertainties on the modulus of elasticity and the cross-section areas of the horizontal and diagonal bars of a truss bridge [54]. Different measured defections in the load test yield different beliefs of the current state of the bridge (e.g., an unusually large defection can indicate a high failure probability of the bridge in the future), thus resulting in different maintenance actions. Let $\boldsymbol{Y}$ denote the vector of the inspection or monitoring outcome. With a realization of the inspection outcome $\boldsymbol{y}$, the conditional expected cost with the optimal decision can be formulated as:

$$\mathbb{E}_{\boldsymbol{X}|\boldsymbol{Y}}(C|\boldsymbol{Y} = \boldsymbol{y}) = \min_{m \in M} \int_{\boldsymbol{X}} c(m, \boldsymbol{x}) f_{\boldsymbol{X}|\boldsymbol{Y}}(\boldsymbol{x}|\boldsymbol{y}) d\boldsymbol{x} \tag{3}$$

where $f_{\boldsymbol{X}|\boldsymbol{Y}}(\cdot)$ is the conditional joint PDF of the random vector $\boldsymbol{X}$ given the inspection outcome $\boldsymbol{Y} = \boldsymbol{y}$. Considering all possible outcomes of these measurement quantities, $C_{pre-post}$ can be expressed as follows:

$$C_{pre-post} = \mathbb{E}_{\boldsymbol{Y}}[\mathbb{E}_{\boldsymbol{X}|\boldsymbol{Y}}(C|\boldsymbol{Y}=\boldsymbol{y})] = \int_{\boldsymbol{Y}} \left[\min_{m\in M} \int_{\boldsymbol{X}} c(m,\boldsymbol{x}) f_{\boldsymbol{X}|\boldsymbol{Y}}(\boldsymbol{x}|\boldsymbol{y}) d\boldsymbol{x}\right] f_{\boldsymbol{Y}}(\boldsymbol{y}) d\boldsymbol{y} \quad (4)$$

where $\mathrm{E}_{\boldsymbol{Y}}$ denotes the expectation with respect to $\boldsymbol{Y}$ and $f_{\boldsymbol{Y}}(\cdot)$ is the joint PDF of the inspection outcome $\boldsymbol{Y}$.

Thus, VoI quantifies the difference between the minimum expected SLC with prior and pre-posterior information. The random vector $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$ formulates the probabilistic model representing the uncertainties of a system. The vector $\boldsymbol{Y}$ formulates the inspection outcomes of the system, and the limit state function formulates the performance of the system. At least one source of uncertainties needs to be considered via statistical approaches in the computation of VoI. As in realistic applications, the agent typically needs to consider uncertainties from multiple sources (e.g., loads, system conditions and the environment). Quantifying the VoI has been an important step for evaluating the optimality of a SHM system or maximizing the potential benefits from SHM.

## 2.2 POMDP

As introduced, the VoI has been successfully implemented in many applications. However, a significant computational challenge remains regarding the determination of the optimal management strategy ($m$ in Eq. (2) and Eq. (4)). Numerous actions can be taken with increasing number of decision times, thus, evaluating all the possibilities can become computationally intractable. POMDP-based VoI analysis, which offers a powerful tool for life cycle analysis, is therefore proposed to remedy this challenge. This section briefly introduces the basic framework of the POMDP.

A POMDP can be considered as a tuple composed of seven components ($S, A, Z, \boldsymbol{T}, \boldsymbol{O}, \boldsymbol{R}, \gamma$), where $S$, $A$ and $Z$ are the discrete set of states, actions, and observation outcomes, respectively; $\boldsymbol{T}$, $\boldsymbol{O}$ and $\boldsymbol{R}$ are the matrices that define the transition function, observation function and reward function, respectively; and $\gamma$ is the discounting factor between 0 and 1. $T(s, a, s')$ denotes the transition probability from state $s$ to state $s'$ with action $a$ being performed at this time step. $O(z, a, s)$ denotes the probability of providing observation outcome $z$ when the system state is $s$ and the selected action is $a$. $R(s, a)$ represents the reward for the case where the system state is $s$ and action is $a$. Thus, to fully describe the transition, observation and reward function, the size of matrices $\boldsymbol{T}$, $\boldsymbol{O}$ and $\boldsymbol{R}$ should be $|S| \times |S| \times |A|$, $|S| \times |Z| \times |A|$ and $|S| \times |A|$, respectively. Moreover, $\boldsymbol{b}$ denotes the belief state and contains the probabilities of being in different states, and $\boldsymbol{b}_0$ denotes the initial belief. In the reinforcement learning literature, the decision maker can also be referred as the agent, while the system is referred to as the environment.

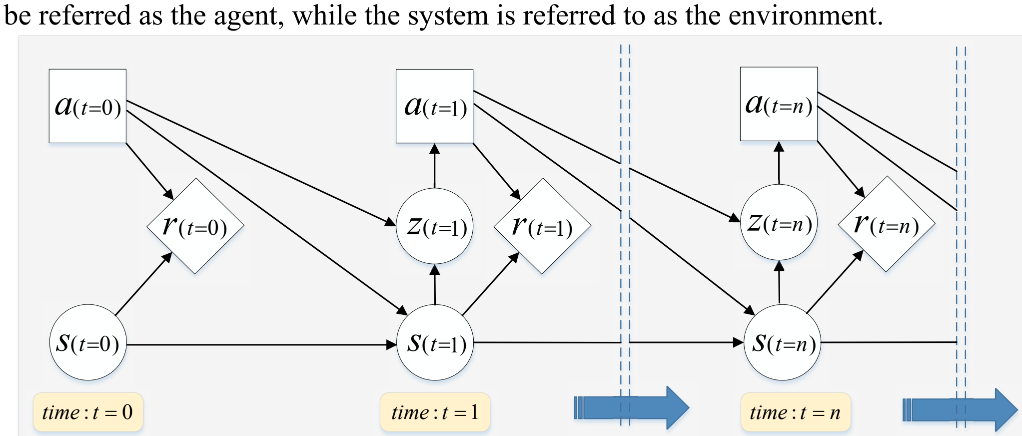



**Fig. 1.** Illustration of the POMDP

The general POMDP framework is illustrated in Fig. 1. Following the typical illustration of Bayesian networks, circles here represent random variables, squares represent decision variables and diamonds represent utility variables. The system starts from an initial state $s(t{=}0)$ that belongs to the finite discrete set $S$. The agent can choose one action from the action set $A$. The state subsequently changes to $s(t{=}1)$ with the chosen action $a(t{=}0)$, and this transition probability has been defined by the transition function. The agent can also receive an observation that provides information about the current state, and the conditional probability of receiving the observation result $z(t{=}1)$ is defined by the observation function. Subsequently, the agent chooses an action, and the state may change again. This process can continue to an infinite-horizon where at each time step, the agent receives a reward based on the corresponding state and the action.

A key feature of POMDPs is that they allow modeling the uncertainty about the system state at each time step. The belief state is based on both the initial belief and the observation outcome. Starting from the belief $\boldsymbol{b}$, conducting action $a$ and receiving an observation $z$, the updated belief $\boldsymbol{b}'$ can be formulated as follows with Bayes' theorem:

$$\boldsymbol{b}'(s') = \frac{O(z, a, s') \sum_{s \in S} \boldsymbol{b}(s) T(s, a, s')}{P_z} \tag{5}$$

where $P_z$ is the probability of receiving the observation outcome $z$. $P_z$ can be calculated by:

$$P_z = \sum_{s \in S} \boldsymbol{b}(s) \sum_{s' \in S} T(s, a, s') O(z, a, s') \tag{6}$$

POMDPs can be solved using dynamic programming or iteration techniques. The maximum expected reward considering $n$ steps, namely $V_n^*(\boldsymbol{b})$, can be constructed based on the maximum expected reward with $n-1$ steps, namely $V_{n-1}^*(\boldsymbol{b})$. This relationship is formulated by the well-known Bellman equation as follows:

$$V_n^*(\boldsymbol{b}, \boldsymbol{\Theta}) = \max_a \left\{ \sum_{s \in S} \boldsymbol{b}(s) R(s, a) + \gamma \sum_{z \in Z} P_z V_{n-1}^*(\boldsymbol{b}', \boldsymbol{\Theta}) \right\} \tag{7}$$

where $\boldsymbol{b}$ is the input of the belief; $\boldsymbol{b}'$ and $P_z$ have been formulated by Eq. (7) and (8), respectively; and $\boldsymbol{\Theta} = \{\boldsymbol{T}, \boldsymbol{O}, \boldsymbol{R}, \gamma\}$ is the set of parameters in a POMDP model.

It has been proven that $V_n^*$ is a piecewise linear and convex function, which can be presented by a number of linear functions [30]. However, as the number of steps grows, the complexity of $V_n^*$ can increase substantially to a degree that exact computation is no longer viable. Several random-pointed methods that can approximate $V_n^*$ have been proposed [30]. With these efficient solvers, POMDPs can be used to determine the optimal maintenance and inspection policies and obtain the maximum expected reward (or minimizing the expected cost) considering infinite steps $V_\infty^*$.

## 3. The proposed approach

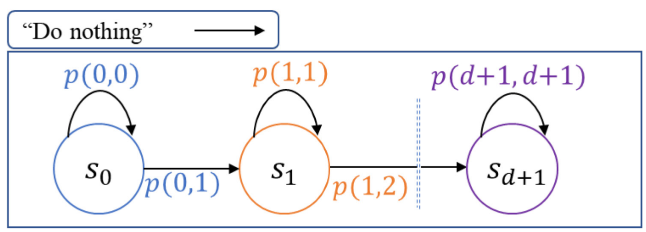
As POMDPs can efficiently determine optimal policies considering available inspection and maintenance actions, POMDPs can be used to obtain the minimum SLCs with and without pre-posterior information. Therefore, the VoI can be determined as the difference between the two SLCs. However, several gaps remain including, among others, the non-stationary description of the environment, and accurate and efficient determination of the POMDP model based on prior information. For instance, when the rate of deterioration increases with aging, a constant transition function (or matrix) cannot accurately describe the physical phenomenon. Even if the physical or mathematical function to model the inspection action and the corresponding measurement errors are known, determining the observation function in the POMDP remains a challenge. These gaps limit the application of POMDP for determining the VoI in some engineering problems.

To address these gaps, this paper proposes a new approach, called VoI-R-POMDP, which is able to estimate the value of both current inspection and information flow. The rest of this section presents the proposed VOI-R-POMDP approach followed by the methods to determine the proposed POMDP model.

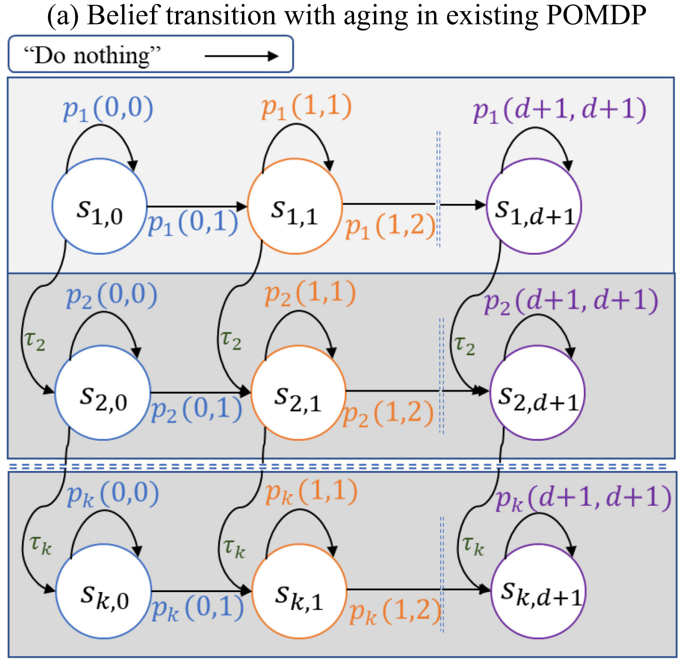
### 3.1 VOI-R-POMDP

A classical POMDP model typically assumes that the environment can be described by a stationary transition function. Let $S = [s_0, s_1, \dots, s_{d+1}]$ denote the set of states in the original POMDP model. $s_0$ represents the intact state, $s_1$, …, $s_d$ are damaged states and $s_{d+1}$ is the failure state (the damage condition of $s_i$ is more severe than $s_j$ if $i > j$). Fig. 2 (a) shows the deterioration process modeled by a classical POMDP model. The system can transition from state $s_i$ to a more severe state $s_{i+1}$ with probability $p(i, i+1)$, or the system can remain in the same state with probability $p(i, i)$. These probabilities remain constant in the POMDP model. Thus, a time-dependent or non-stationary deterioration process may not be accurately described; and a time-dependent transition model is needed. However, the naïve strategy of simply appending a state describing the time onto the set of system states is not applicable, as the property of time is significantly different from that of the system states. In managing structures and infrastructure systems, the system state typically cannot be fully known by the agent and needs to be updated with the observation, while the time state is perfectly known and cannot be changed. An alternative is to duplicate the states into discretized slices of time layers [55]. However, the number of states will increase significantly, as well as the number of actions, resulting in a high computational burden.

To address this gap, this paper proposes a new approach to efficiently model the non-stationary transition. Let us consider a system with the deterioration rate increasing with aging. It is assumed that a transition function can well describe the initial deterioration process, while another transition function can well model the most severe deterioration phase. The true deterioration process during the service life typically acts between these two transition functions. Considering $k$ deterioration models, where each deterioration model can be regarded as a POMDP model with constant transition probabilities from one state to the others, the set of states is expanded to $[S_1, \dots, S_k]$, where $S_l$ ($l \in [1, \dots, k]$) is $[s_{l,0}, s_{l,1}, \dots, s_{l,d+1}]$ with the first subscript denoting the deterioration model and the second subscript the physical state of the system. For instance, $s_{1,0}$ and $s_{2,0}$ represent the intact state in different deterioration models; thus, the model will include different transition probabilities to capture the transition from $s_{1,0}$ and $s_{2,0}$.




(a) Belief transition with aging in existing POMDP




(b) Belief transition with aging in proposed non-stationary POMDP

**Fig. 2.** Illustration of the proposed POMDP framework

Subsequently, in order to model the time-dependent transition effect, the belief transitions from $S_n$ to $S_{n+1}$ with "do nothing" action ( a "do nothing" action indicates the increase of the age here). Fig. 2 (b) illustrates the deterioration process modeled by the proposed POMDP framework. Each state $s_{l,i}$ can enter a more severe state $s_{l,i+1}$ with the transition probability $p_l(i, i+1)$ and remain in the current state with the transition probability $p_l(i,i)$. Different from existing POMDP models, in the proposed approach, the state $s_{l,i}$ can also enter a physically identical state $s_{l+1,i}$ that has a higher deterioration rate with the transition probability of $\tau_{l+1}$ $(l \in [1, \dots, k-1], i \in [0, \dots, d])$. Thus, with aging, the belief gradually transitions from $S_1$ to $S_k$ which models a more severe deterioration process, capturing the time-dependent transitions. Note that a maintenance action (e.g., "replacement") can reduce the rate of deterioration; therefore, the transition function can be set to transition to an early belief (e.g., the initial one). As Fig. 3 shows, with the replacement action denoted as $a_R^1$, the transition probabilities starting from an arbitrary state $s_{l,j}$ to another state $s_{v,i}$ $(l, v \in [1, \dots, k],\ i, j \in [0, \dots, d+1])$ can be represented by $T[s_{l,j}, a_R^1, s_{v,i}] = p(s_{l,j}, s_{v,i})$. In order to transition to the initial belief in the proposed POMDP model, $p(s_{l,j}, s_{v,i})$ can be defined as $\boldsymbol{b}_0(s_{v,i})$, which is often zero when $v > l$. Thus, the belief state becomes the initial belief after this replacement action is performed.

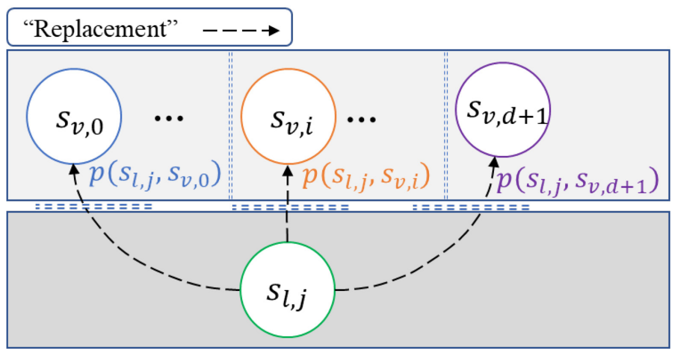
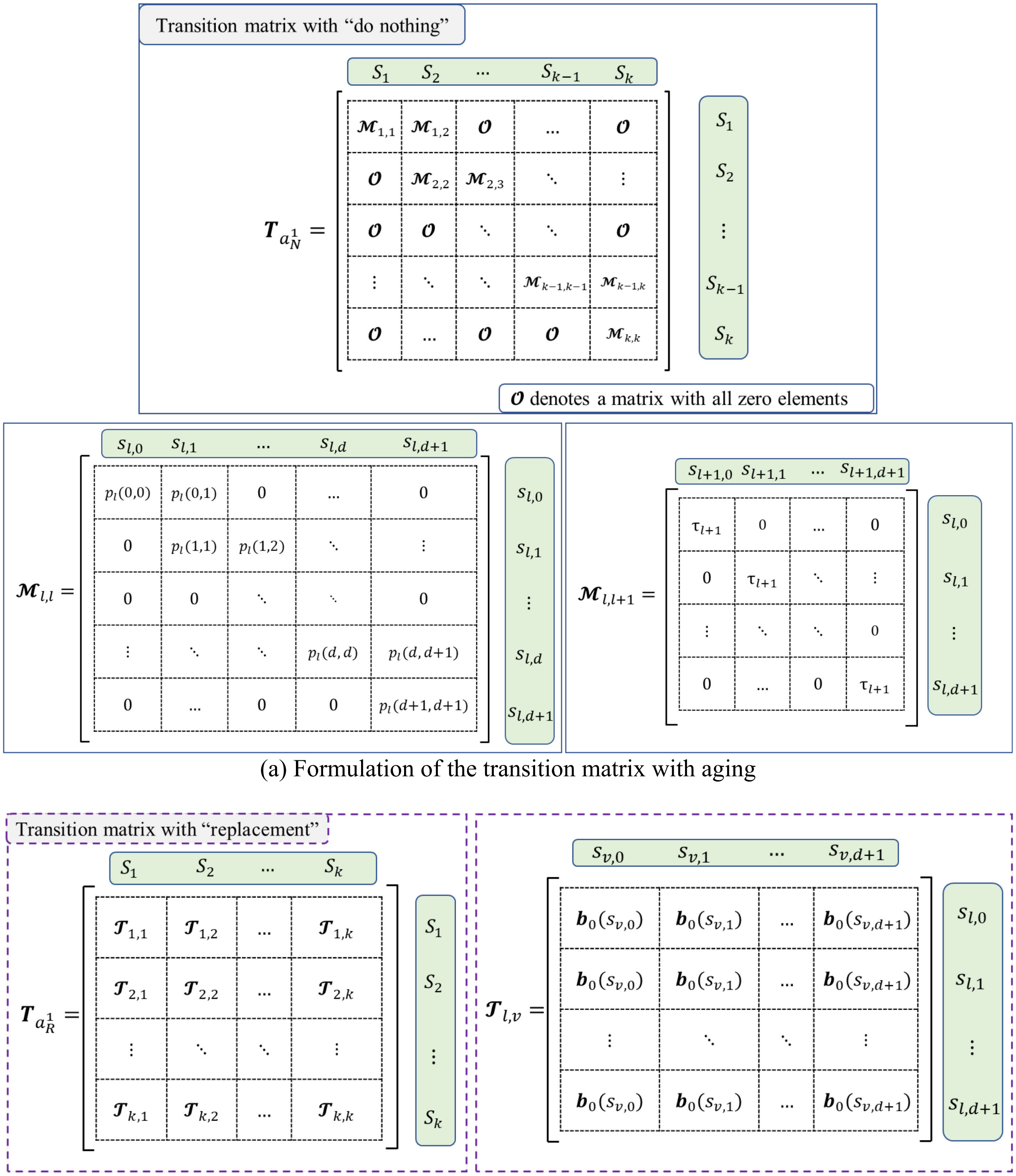



**Fig. 3**. Belief transition with replacement in the proposed POMDP

The formulation of the transition matrices with "do nothing" and "replacement" maintenance actions in the proposed POMDP framework is shown in Fig. 4. Each element in the matrix represents the transition probability from the corresponding row state to the corresponding column state. Note that as multiple deterioration models are considered in the proposed framework, $\boldsymbol{\mathcal{M}}_{l,l}$ $(l \in [1, \dots, k])$ and $\boldsymbol{\mathcal{M}}_{l,l+1}$ $(l \in [1, \dots, k-1])$ are submatrices of the transition matrix $\boldsymbol{T}_{a_N^1}$, and $\boldsymbol{\mathcal{T}}_{l,v}$ $(l, v \in [1, \dots, k])$ are submatrices of the transition matrix $\boldsymbol{T}_{a_R^1}$. Thus, the sum of probabilities in the same row of $\boldsymbol{\mathcal{M}}_{l,l}$ $(l \in [1, \dots, k])$, $\boldsymbol{\mathcal{M}}_{l,l+1}$ $(l \in [1, \dots, k-1])$ or $\boldsymbol{\mathcal{T}}_{l,v}$ $(l, v \in [1, \dots, k])$ is not necessarily identical to one, while the sum of probabilities in the same row of $\boldsymbol{T}_{a_N^1}$ or $\boldsymbol{T}_{a_R^1}$ is one. The formulation of the transition matrix in Fig. 4 (a) is consistent with Fig. 2 (b). With aging, each state can enter a more severe state or remain in the current state or enter a physically identical state that has a higher deterioration rate. The formulation in Fig. 4 (b) is consistent with Fig. 3. It is assumed that, with replacement, the belief state becomes the initial belief. Thus, the transition probability to an arbitrary state equals the initial belief of this state.




(a) Formulation of the transition matrix with aging

(b) Formulation of the transition matrix with replacement

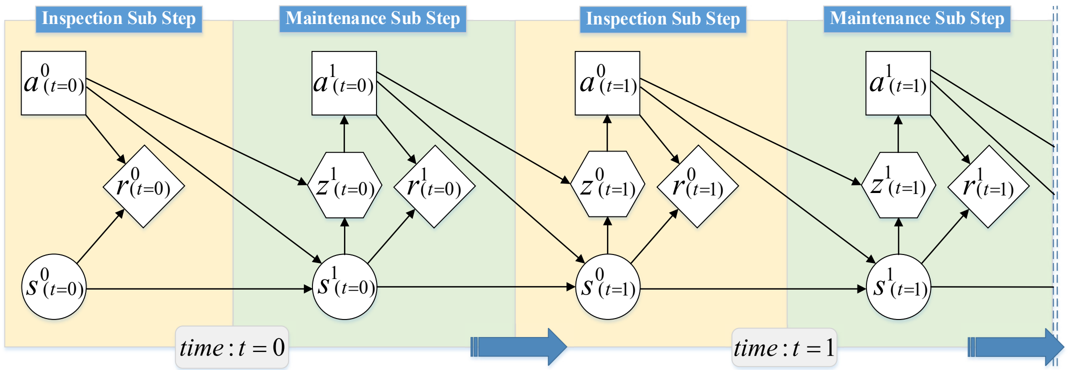
**Fig. 4**. Illustration of the transition matrices in the proposed POMDP framework

Let $\boldsymbol{\Theta}_{\mathrm{R}}$ denote the parameters of the proposed POMDP model and $\boldsymbol{b}_{\mathbf{0}}$ the belief for the initial state. To split the effects from the inspection actions and maintenance actions, the two-sub-step adjustment proposed by Memarzadeh and Pozzi [50] is adopted here. Let $V_I^*(\boldsymbol{b}_{\mathbf{0}}, \boldsymbol{\Theta}_{\mathrm{R}})$ denote the minimum cost of the POMDP model starting from an inspection sub-step as shown in Fig. 5 (a), and $V_M^*(\boldsymbol{b}_{\mathbf{0}}, \boldsymbol{\Theta}_{\mathrm{R}})$ denote the minimum cost of the POMDP model starting from a maintenance sub-step as shown in Fig. 5 (b). Note that compared to $V_M^*(\boldsymbol{b}_{\mathbf{0}}, \boldsymbol{\Theta}_{\mathrm{R}})$, $V_I^*(\boldsymbol{b}_{\mathbf{0}}, \boldsymbol{\Theta}_{\mathrm{R}})$ considers the availability of an inspection action at the beginning of the inspection sub-step. Thus, the difference between $V_I^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_\mathrm{R})$ and $\gamma^+ V_M^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_\mathrm{R})$ can be defined as the value of this current inspection action ($\gamma^+$ is the discounting factor in the two-sub-step POMDP model):

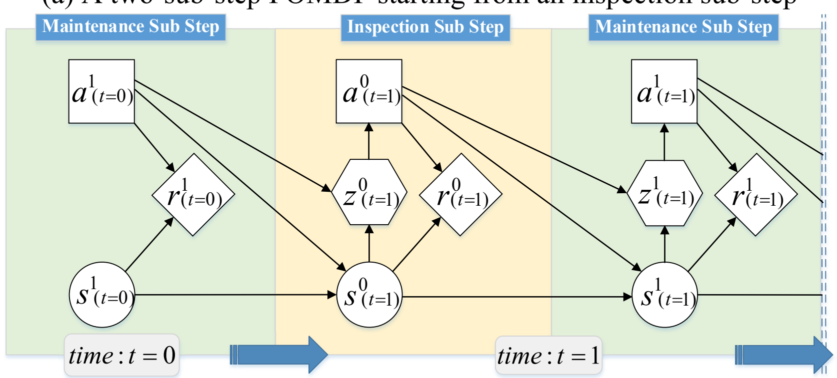
$$\mathrm{VoI}_{c,R} = V_I^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_\mathrm{R}) - \gamma^+ V_M^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_\mathrm{R}) \tag{8}$$

Moreover, let $\boldsymbol{\Theta}_0$ denote the parameters of the original POMDP model without access to inspection actions, while the other parameters, e.g., the transition function, remain the same as $\boldsymbol{\Theta}_\mathrm{R}$. Let $V_\infty^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_0)$ denote the minimum service life cost without any inspection actions. The difference between $V_I^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_\mathrm{R})$ and $V_\infty^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_0)$ represents the value of this information flow as follows:

$$\mathrm{VoI}_{f,R} = V_I^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_\mathrm{R}) - V_\infty^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_0) \tag{9}$$




(a) A two-sub-step POMDP starting from an inspection sub-step




(b) A two-sub-step POMDP starting from a maintenance sub-step

**Fig. 5.** Illustration of the two-sub-step POMDP

The proposed POMDP framework can reliably model time-dependent or non-stationary environments compared with the existing models. Nevertheless, the determination of the number of deterioration models, and the corresponding transition functions and observation function based on the prior knowledge is challenging. The rest of this section focuses on methods that are developed in this research for efficient and accurate definition of the parameters of the proposed POMDP model. As these approaches are built on reliability concepts to obtain accurate time-dependent belief of the system state and to facilitate the establishment of the proposed POMDP model, the VoI analysis method is called VoI-R-POMDP. Note that as the POMDP model can determine the optimal policy of managing the system, the action of collecting additional information is not performed when the expected cost is larger than the expected utility improvement. Thus, though the POMDP-based VoI analysis determines the net VoI result, the result cannot be negative.

### 3.2 Definition of the transition function

In reliability analysis, a performance function is typically used to describe the performance of a system. The failure event occurs when the system performance cannot meet a prescribed requirement.

Inspired by the approach in [8], we propose defining the states based on the performance of the system. For instance, if a structural component under fatigue deterioration is to be investigated, the failure of the component is determined by the crack depth. $d$ damage levels are considered besides the failure and intact states of the system. The system states can be defined as: the intact state $s_0$, damaged state $s_1$, …, $s_d$, and the failure state $s_{d+1}$. Mutually exclusive and collectively exhaustive events $E_i$ ($i = 0, \dots, d+1$) can be correspondingly defined to indicate that the system is in state $s_i$. These events can be described by the same performance function with different criteria (e.g., the failure event occurs when the crack depth is larger than 50 mm; the intact event happens when the crack depth is less than 10 mm). Let $f(\boldsymbol{x})$ denote the performance function and $\mathbb{c}_i$ and $\mathbb{c}_{i+1}$ denote the lower bound and the upper bound of the performance, which defines the event $E_i$. The relation between the states and the performance function can be formulated as:

$$\begin{cases} s_i = 1 \leftrightarrow f(\boldsymbol{x}) \in [\, \mathbb{c}_i, \mathbb{c}_{i+1}) \\ s_i = 0 \leftrightarrow f(\boldsymbol{x}) \notin [\, \mathbb{c}_i, \mathbb{c}_{i+1}) \end{cases} \tag{10}$$

Thus, the system is in state $s_i$ when $f_i(\boldsymbol{x}) \leq 0$, indicating that the event $E_i$ has occurred. The belief of system state at time step $n$ can be formulated as $\{\Pr(E_0, \boldsymbol{X}, t \in [n-1, n],), \Pr(E_1, \boldsymbol{X}, t \in [n-1, n]), \dots, \Pr(E_{d+1}, \boldsymbol{X}, t \in [n-1, n])\}$, which represents the probabilities of these events in a specific time period. These probabilities can be efficiently estimated with time-dependent reliability methods [56-58] and the initial belief $\boldsymbol{b}_0$ can be correspondingly determined. Moreover, whether the environment can be described by a stationary transition function can also be observed based on these probabilities. As the relationship between the rate of failure and aging deviates further from a linear model, the number of the deterioration models can be adaptively increased until the non-stationary environment can be accurately described.

In the following, the determination of the transition function is discussed. With $k$ deterioration models, the set of system states in the proposed POMDP approach can be defined as $[S_1, \dots, S_k]$, where $S_l$ ($l \in [1, \dots, k]$) is $[s_{l,0}, s_{l,1}, \dots, s_{l,d+1}]$. When the maintenance action is "do nothing" (denoted as $a_N^1$), the system in damaged state $s_{l,i}$ deteriorates and enter state $s_{l,i+1}$, or remains in the current state $s_{l,i}$, or enters the state $s_{l+1,i}$ which is physically identical but with a higher deterioration rate. The relation between the belief state at time $n$ and that at time $n+1$ can be expressed as:

$$\begin{aligned} \boldsymbol{b}_{t=n+1}\left(s_{l,i+1}\right) = \boldsymbol{b}_{t=n}\left(s_{l,i}\right) \times p_l(i, i+1) + \boldsymbol{b}_{t=n}\left(s_{l,i+1}\right) \times p_l(i+1, i+1) \\ + \boldsymbol{b}_{t=n}\left(s_{l-1,i+1}\right) \times \tau_l \end{aligned} \tag{11}$$

Considering these physically identical states, the belief that the system is in state $s_i$ at time step $n$ is $\sum_{l=1}^{k} \boldsymbol{b}_{t=n}\left(s_{l,i}\right)$. From another point of view, the probabilities of being in states $s_i$ can be estimated with reliability methods, denoted as $\boldsymbol{b}_{t=n,R}(s_i)$. Considering the results provided by the reliability analysis as benchmarks, the errors from the transition functions can be measured. The optimal transition probabilities can be determined by the maximum likelihood estimation (MLE) as follows:

$$[\boldsymbol{\tau}, \boldsymbol{p}] = \underset{(\boldsymbol{\tau}, \boldsymbol{p})}{\operatorname{argmin}}\, L(\boldsymbol{b}_{t=1,R}, \dots, \boldsymbol{b}_{t=n,R} | \boldsymbol{b}_0, \boldsymbol{\tau}, \boldsymbol{p}) \tag{12}$$

where $\boldsymbol{p} = [\boldsymbol{p}_1, \dots, \boldsymbol{p}_k]^{\mathrm{T}}$, $\boldsymbol{p}_l = \begin{bmatrix} p_l(0,0) & p_l(1,1) & \cdots & p_l(d,d) \\ p_l(0,1) & p_l(1,2) & \cdots & p_l(d,d+1) \end{bmatrix}$ ($l \in [1, \dots, k]$), and $\boldsymbol{\tau} = [\tau_2, \dots, \tau_k]$ (with $\boldsymbol{\tau}$ and $\boldsymbol{p}$, the transition matrix of Fig. 4 (a) can be determined). Note that the dimensions of $\boldsymbol{p}_l$, $\boldsymbol{p}$ and $\boldsymbol{\tau}$ are $2 \times |d+1|$, $|2k| \times |d+1|$ and $1 \times |k-1|$, respectively. $\boldsymbol{b}_0$ is the initial belief state, $L$ is the likelihood function. Here, it is assumed that relative difference between $\boldsymbol{b}_t$ and $\boldsymbol{b}_{t,R}$ follows a standard normal distribution.

When the maintenance action is not "do nothing" (e.g., "minor repair", "major repair" or "replacement"), the definition of the transition function depends on the belief about the effect of the maintenance actions. For instance, when "replacement" $a_R^1$ is performed, the system can be considered as a new system, thus the transition function can be set such that the belief becomes the initial one.

### 3.3 Definition of the observation function

Inspection actions can provide additional information to better characterize the uncertainty and support decisions. In realistic applications, the inspection action can provide either equality-type or inequality-type outcomes. However, the POMDP model can only utilize discrete signals to update the belief state. To fully leverage the collected the information, this paper generally defines that $q$ discrete inspection outcomes $z_i$ $(i = 1, \ldots, q)$ are provided by the inspection action, and each outcome $z_i$ indicates the occurrence of the event $E_{z,i}$. For instance, for a structural component under fatigue deterioration, the "Alarm" inspection outcome can be described as the event that the crack depth has become larger than 40 mm. The model can more accurately describe the equality-type inspection with more discrete outcomes $z_i$.

Let $\Gamma_{i,j}$ represent the probability of receiving "Signal of $E_{z,i}$" when the system is currently in state $s_j$; thus, $\Gamma_{i,j}$ can be formulated as:

$$\Gamma_{i,j} = \Pr\left[E_{z,i}^{I}(t{=}n) | a_I^0, E_j(t{=}n)\right] \tag{13}$$

where $a_I^0$ denotes the inspection action, $E_{z,i}^{I}(t{=}n)$ denotes the event indicated by the inspection at time step $n$, and $E_j(t{=}n)$ is the event that actually happens at time step $n$. Next in the process is the determination of $\Gamma_{i,j}$. Given an inspection $\boldsymbol{y}$ at time step $n$, the conditional probability that the event $E_{z,i}$ happens at this time step can be expressed as:

$$\Pr\left[E_{z,i}(t{=}n) | \boldsymbol{y}(t{=}n)\right] = \int_{\boldsymbol{X}} I\left[(\boldsymbol{x}, t{=}n) \in \Omega_{E_{z,i}}\right] \cdot f_{\boldsymbol{X}|\boldsymbol{Y}}[\boldsymbol{x}|\boldsymbol{y}(t{=}n)] d\boldsymbol{x} \tag{14}$$

Following Bayes' theorem, $f_{\boldsymbol{X}|\boldsymbol{Y}}[\boldsymbol{x}|\boldsymbol{y}(t{=}n)]$ can be expressed as:

$$f_{\boldsymbol{X}|\boldsymbol{Y}}[\boldsymbol{x}|\boldsymbol{y}(t{=}n)] = \frac{L(\boldsymbol{x}, t{=}n)\, f(\boldsymbol{x})}{\int_{\boldsymbol{X}} L(\boldsymbol{x}, t{=}n)\, f(\boldsymbol{x})\, d\boldsymbol{x}} \tag{15}$$

where $L(\cdot)$ is the likelihood function. Considering the involvement of time, the likelihood function can be recast as [8]:

$$L(\boldsymbol{x}, t{=}n) = \prod_{i=1}^{k} f_{\epsilon_i}\left[y_{i,t=n} - h_i(\boldsymbol{x}, t{=}n)\right] \tag{16}$$

where $\boldsymbol{y}_{t=n} = \left[y_{1,t=n}, .., y_{k,t=n}\right]$ are the measurements at time step $n$, $h_i(\cdot)$ is the function that models the corresponding monitored quantity, and $\boldsymbol{\epsilon} = [\epsilon_1, .., \epsilon_k]$ is the vector of the measurement errors. With Eq. (15) and (16), $\Pr\left[E_{z,i}(t{=}n)|\boldsymbol{y}(t{=}n)\right]$ can be reformulated as:

$$\Pr\left[E_{z,i}(t{=}n) | \boldsymbol{y}(t{=}n)\right] = \frac{\int_{\boldsymbol{X}} I\left[(\boldsymbol{x}, t{=}n) \in \Omega_{E_{z,i}}\right] \cdot L(\boldsymbol{x}, t{=}n)\, f(\boldsymbol{x})\, d\boldsymbol{x}}{\int_{\boldsymbol{X}} L(\boldsymbol{x}, t{=}n)\, f(\boldsymbol{x})\, d\boldsymbol{x}} \tag{17}$$

The accuracy of the inspection action determines $h(\boldsymbol{X})$ and the PDF of the observation error $f_{\boldsymbol{\epsilon}}$. Considering that the inspection action is $a_I^0$, the condition probability $\Pr\left[E_{z,i}(t{=}n)|a_I^0, \boldsymbol{y}(t{=}n)\right]$ is:

$$\Pr\left[E_{z,i}(t{=}n) | \boldsymbol{y}(t{=}n), a_I^0\right] = \frac{\int_{\boldsymbol{X}} I\left[(\boldsymbol{x}, t{=}n) \in \Omega_{E_{z,i}}\right] \cdot L(\boldsymbol{x}, t{=}n, a_I^0)\, f(\boldsymbol{x})\, d\boldsymbol{x}}{\int_{\boldsymbol{X}} L(\boldsymbol{x}, t{=}n, a_I^0)\, f(\boldsymbol{x})\, d\boldsymbol{x}} \tag{18}$$

where $L(\boldsymbol{x}, t{=}n, a_I^0)$ is the likelihood function with the $h(\cdot)$ and $f_{\boldsymbol{\epsilon}}$ being determined based on the inspection action $a_I^0$.

The joint PDF of $\boldsymbol{y}$ may not be directly obtained because a complex mathematical or physical function is required to model this monitoring process. Alternatively, we can first determine the joint distribution of $\boldsymbol{X}$ and then sample from the conditional distribution $f_{\boldsymbol{Y}|\boldsymbol{X}}$. Thus, considering all possible observations, the probability that $E_{z,i}$ happens based on all possible observation outcomes is:

$$\Pr\left[E_{z,i}^{I}(t{=}n) | a_I^0\right] = \int_{\boldsymbol{X}} f_{\boldsymbol{X}}(\boldsymbol{x}) \int_{\boldsymbol{Y}} \Pr\left[E_{z,i}(t{=}n) | \boldsymbol{y}(t{=}n), a_I^0\right] \cdot f_{\boldsymbol{Y}|\boldsymbol{X}}\left[\boldsymbol{y}(t{=}n) | \boldsymbol{x}, a_I^0\right] d\boldsymbol{y} d\boldsymbol{x} \tag{19}$$

Subsequently, the probability that event $E_{z,i}$ is indicated by the observation but event $E_j$ has truly occurred at the inspection time step $n$ can be derived as:

$$\begin{aligned}&\Pr\left[E_{z,i}^{I}(t{=}n) \cap E_j(t{=}n) | a_I^0\right] \\ &= \int_{\boldsymbol{X}} I[(\boldsymbol{x}, t{=}n) \in \Omega_{E_j}] \cdot f_{\boldsymbol{X}}(\boldsymbol{x}) \int_{\boldsymbol{Y}} \Pr\left[E_{z,i}(t{=}n) | \boldsymbol{y}(t{=}n), a_I^0\right] \cdot f_{\boldsymbol{Y}|\boldsymbol{X}}[\boldsymbol{y}(t{=}n) | \boldsymbol{x}, a_I^0]\, d\boldsymbol{y} d\boldsymbol{x}\end{aligned} \tag{20}$$

With Bayes' theorem, $\Gamma_{i,j}$ can be expressed as:

$$\begin{aligned}\Gamma_{i,j} &= \frac{\Pr\left[E_{z,i}^{I}(t{=}n) \cap E_j(t{=}n) | a_I^0\right]}{\Pr\left[E_j(t{=}n) | a_I^0\right]} \\ &= \frac{\int_{\boldsymbol{X}} I[(\boldsymbol{x}, t{=}n) \in \Omega_{E_j}] \cdot f_{\boldsymbol{X}}(\boldsymbol{x}) \int_{\boldsymbol{Y}} \Pr\left[E_{z,i}(t{=}n) | \boldsymbol{y}(t{=}n), a_I^0\right] \cdot f_{\boldsymbol{Y}|\boldsymbol{X}}[\boldsymbol{y}(t{=}n) | \boldsymbol{x}, a_I^0]\, d\boldsymbol{y} d\boldsymbol{x}}{\int_{\boldsymbol{X}} I\left[(\boldsymbol{x}, t{=}n) \in \Omega_{E_j}\right] \cdot f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}}\end{aligned} \tag{21}$$

Note that $\Gamma_{i,j}$ is identical to $\Pr\left(z_i | a_I^0, s_j\right)$ with $z_{si}$ denoting the "Signal of $E_{z,i}$" outcome. After determining all $\Gamma_{i,j}$ considering the given inspection action, the set of observation probabilities or the observation function will be established for the proposed POMDP model. MCS is a basic approach for determining $\Gamma_{i,j}$. The same sampling strategy as in [8] (first, sample from $\boldsymbol{X}$ and then sample multiple $\boldsymbol{Y}$ based on each $\boldsymbol{x}$) is used in this paper. More advanced techniques, such as IS and surrogate models, can be potentially used here.

## 4. Illustrative application

### 4.1 Engineering model

In order to further illustrate the proposed approach, the VoI analysis is investigated for a corroding simply supported beam [59]. As shown in Fig. 6, the span of the beam is $L$ = 5 m and cross-section of the beam is rectangular. It is assumed that the section uniformly corrodes, and the relation between the section and time can be formulated as:

$$h(t) = h_0 - 2wt \tag{22}$$

$$b(t) = b_0 - 2wt \tag{23}$$

where $h_0$ and $b_0$ are the initial depth and width of the cross section, respectively and $w$ is the time-dependent corrosion rate that is assumed to be $\theta\sqrt{t}$ m/year, where $\theta$ is also a random variable listed in Table 1.

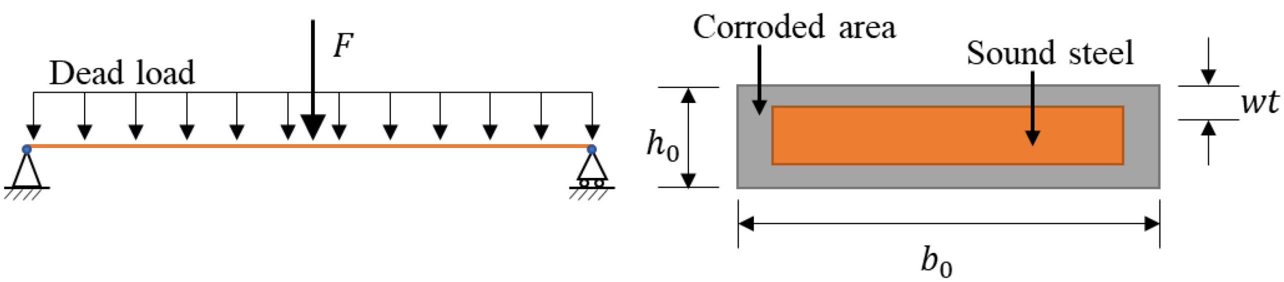



**Fig. 6.** The corroding beam and the cross section

The beam is subjected to both the gravity load (the force density is 78500 N/m$^3$) and a concentrated load $F$. Due to the functionality constraint, it is assumed that the external load is applied in the weak axis. The details of the simply supported beam and the probabilistic information are listed in Table 1.

**Table 1** Parameters of the crack growth model

| Variable | Distribution | Mean | Standard deviation |
|---|---|---|---|

| | | | |
|---|---|---|---|
| $u_y$ | Deterministic | $2.1\times10^{8}$ (Pa) | - |
| $h_0$ | Lognormal | 0.045 (m) | $4\times10^{-3}$ |
| $b_0$ | Lognormal | 0.20 (m) | 0.01 |
| $\theta$ | Lognormal | $2.5\times10^{-5}$ (m/year) | $2.5\times10^{-6}$ |
| $F$ | Deterministic | 8000 (N) | - |

Let $u_y$ denote the yield strength of the steel. The structure fails when the stress of the beam exceeds $u_y$. The performance function is formulated as:

$$g(\boldsymbol{x}, t) = \frac{(b_0 - 2wt)(h_0 - 2wt)^2 u_y}{4} - (\frac{FL}{4} + \frac{78500 b_0 h_0 L^2}{8}) \tag{24}$$

where $\boldsymbol{X} = [h_0, b_0, \theta]^T$ is a vector of the random variables that describe the uncertainties for the initial size of structure and the corrosion rate, $\boldsymbol{x}$ is a realization of $\boldsymbol{X}$, $t$ is the time parameter, $w$ is identical to $\theta\sqrt{t}$ and $F$ is the external load.

The information collection action in this example includes measuring the width and depth of the sound area using inspection. The likelihood function describing the inspection outcome at time step $n$ can be thus formulated as:

$$L(\boldsymbol{x}) = f_{\epsilon_1}\left[y_{h,t=n} - h(\boldsymbol{x}, t=n)\right] \times f_{\epsilon_2}\left[y_{b,t=n} - b(\boldsymbol{x}, t=n)\right] \tag{25}$$

where $y_{h,t=n}$ and $y_{b,t=n}$ are the measured width and depth at time step $n$, respectively, $h(\boldsymbol{x}, t=n)$, which is defined by Eq. (22), represents the depth of the sound steel at time step $n$, and $b(\boldsymbol{x}, t=n)$, which is defined by Eq. (23), represents the width of the sound steel at time step $n$. In this example, $f_{\epsilon 1}$ and $f_{\epsilon 2}$ are both defined as a zero-mean normal PDF with the standard deviation of $\sigma_{\epsilon 1} = \sigma_{\epsilon 2} = 1$ mm.

It is assumed that an inspection action and a maintenance action can be taken every two years. Note the exact choice of the optimal maintenance action at each time step depends on the current belief of the state of the beam, which changes with aging and is updated with different inspection outcomes in the POMDP model. The available choices of actions in this example are listed as follows:

1) Inspection actions: "do inspection", denoted as $a_I^0$, and "do nothing", denoted as $a_N^0$.
2) Maintenance actions: "replacement", denoted as $a_R^1$, and "do nothing", denoted as $a_N^1$.

The goal is to estimate $\mathrm{VoI}_{c,R}$ for performing an inspection action at the initial time step and $\mathrm{VoI}_{f,R}$ for performing the inspection actions in the service life of the beam. The cost of failure $c_F$ is the sum of the costs of replacing the beam, the incurred user cost, and the potential injury and casualty. Assuming that the failure cost is proportional to the cost of the bridge in [7], $c_F$ is estimated as \$$1.16\times10^{5}$. The cost of replacing the beam $c_R$ and the cost of inspection $c_I$ are estimated as \$$2\times10^{4}$ and \$200, respectively. The magnitude of the applied cost model is consistent with [8, 54]. In engineering applications, more detailed cost models can be applied based on the specific application.

## 4.2 The reliability assisted POMDP model

The proposed POMDP model is determined following the proposed approaches in Section 3. Three states including $s_0$, $s_1$, and the failure state $s_2$ are considered here based on the maximum stress under the dead load and external load as shown in Table 2.

**Table 2** Definition of states

| State condition | Maximum stress | Cost |
|---|---|---|
| $s_0$ | $<0.9\sigma_u$ | 0 |
| $s_1$ | $\geq 0.9\sigma_u$ & $< \sigma_u$ | 0 |
| $s_2$ | $\geq \sigma_u$ | $1.16\times10^{6}$ |

The time-dependent probabilities of the states of the beam can be efficiently determined by reliability methods. MCS is adopted in this example as the benchmark. Approximation methods, such as

First-Order Reliability Methods (FORM) that transform the non-standard normal variables to standard normal variables using transformation techniques (e.g., Rosenblatt transformation [52]) and approximate the failure probability based on the most probable failure point, can also be applied here. Table 3 lists the probabilities of being in these states at different time steps.

**Table 3** The probabilities of being in different system states

| State condition | Probability at the 1st time step | Probability at the 5th time step | Probability at the 10th time step | Probability at the 20th time step |
|---|---|---|---|---|
| $s_0$ | $9.9535\times10^{-1}$ | $9.8231\times10^{-1}$ | $8.8974\times10^{-1}$ | $3.6244\times10^{-1}$ |
| $s_1$ | $4.5137\times10^{-3}$ | $1.6047\times10^{-2}$ | $9.0260\times10^{-2}$ | $3.6889\times10^{-1}$ |
| $s_2$ | $1.3208\times10^{-4}$ | $1.6391\times10^{-3}$ | $2.0002\times10^{-2}$ | $2.6867\times10^{-1}$ |

Four transition models are used to describe the time-dependent deterioration process in the proposed POMDP framework. The details of defining the transition function have been explained in Section 3. The initial belief is determined as $\boldsymbol{b}_0 = [0.9962, 0.0038, 0]$. $\boldsymbol{\tau}$ and $\boldsymbol{p}$ in Eq. (12) are determined as follows:

$$\boldsymbol{\tau} = [0.1683, 0.1714, 0.1484]$$

$$\boldsymbol{p}_1 = \begin{bmatrix} 0.8305 & 0.7975 \\ 0.0012 & 0.0342 \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} 0.8229 & 0.7799 \\ 0.0057 & 0.0487 \end{bmatrix}, \boldsymbol{p}_3 = \begin{bmatrix} 0.8286 & 0.8466 \\ 0.0230 & 0.0050 \end{bmatrix}, \boldsymbol{p}_4 = \begin{bmatrix} 0.8313 & 0.8333 \\ 0.1687 & 0.1667 \end{bmatrix}$$

The formulation of the transition matrix has been illustrated by Fig. 4. To validate the accuracy of the proposed framework, the probability of failure with MCS, the probability of failure determined by the transition function in the proposed POMDP model and that in existing POMDP models are compared in Fig. 7. POMDP model-1 and POMPD model-2 are determined to fit the deterioration process at the beginning of the service life and at the end of service life, respectively. Considering the results from MCS as the benchmark, the probability of failure increases nonlinearly with aging. Thus, POMDP model-1 and POMPD model-2, which are based on the stationary transition assumption, have considerable errors in probability estimations. In comparison, the proposed POMDP model accurately describes the deterioration process over the entire service life of the beam.

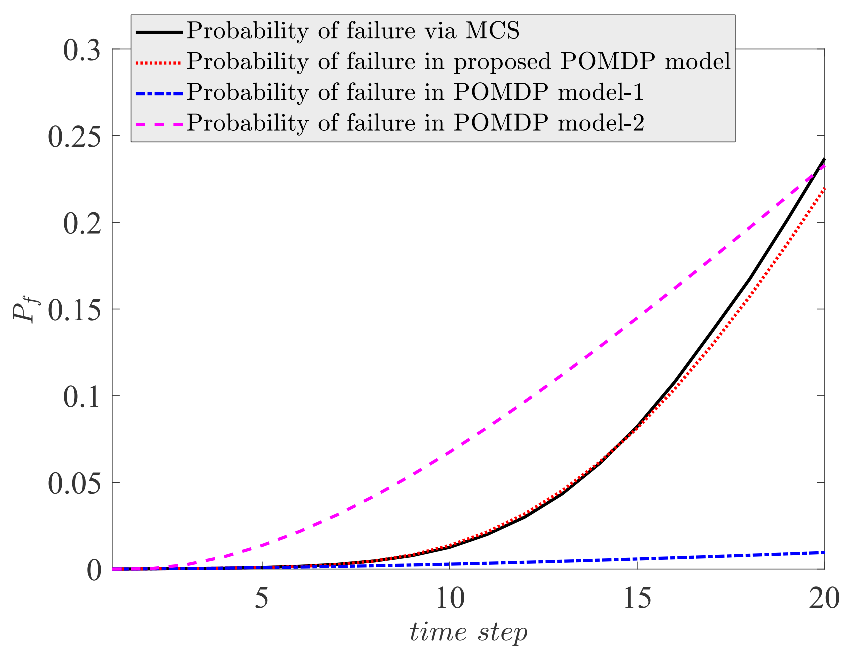
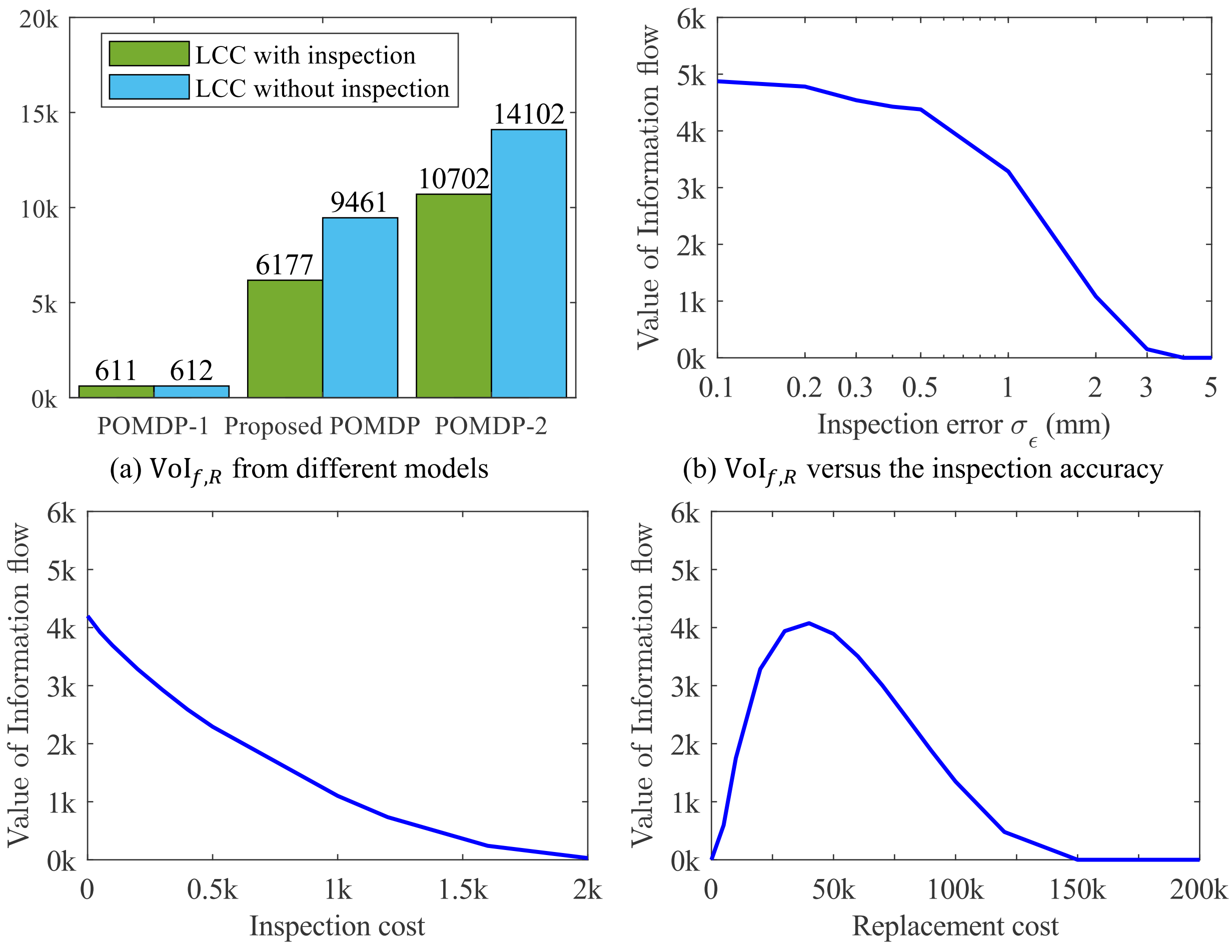



**Fig. 7.** The comparison among the failure probabilities from different POMDP models

For the sake of illustration, two signals are considered as the inspection outcomes in the POMDP model: signal of $s_0$, called $z_{s0}$, indicating that the beam is in state $s_0$, and signal of $s_1$, called $z_{s1}$, indicating that the beam is in state $s_1$. The observation probability can be determined as explained in Section 3.3. $O(z_{s0}, a_I^0, s_0)$ is 0.9982 and $O(z_{s1}, a_I^0, s_1)$ is 0.5355.

### 4.3 The value of information

After establishing the proposed POMDP model, value of the current inspection action $\text{VoI}_{c,R}$ can be determined based on Eq. (8). The minimum service life cost with pre-posterior information $V_I^*(\boldsymbol{b_0}, \boldsymbol{\Theta})$ is -$6,177. The minimum service life cost without additional inspection action at the initial time step is $V_M^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_{\text{R}})$ =-$6,122. Thus, the value of the current information is $\text{VoI}_{c,R}$ = $0. On the other hand, value of information flow can be similarly determined based on Eq. (9). The minimum service life cost without inspection action is $V_\infty^*(\boldsymbol{b_0}, \boldsymbol{\Theta}_0)$ =-$9,237. Thus, the value of information flow is $\text{VoI}_{f,R}$ = $3,430.




(a) $\text{VoI}_{f,R}$ from different models

(b) $\text{VoI}_{f,R}$ versus the inspection accuracy

(c) $\text{VoI}_{f,R}$ versus the inspection cost

(d) $\text{VoI}_{f,R}$ versus the replacement cost

**Fig. 8.** Parametric investigation on the value of information flow

The probability of failure is relatively low at the beginning time step, thus the inspection may not be necessary and the value of the current information $\text{VoI}_{c,R}$ is 0. As the corrosion rate of the beam is assumed to be time-dependent, the probability of failure increases at a higher rate with aging. Collecting additional information through inspection is more essential when the failure probability is larger. Thus, in this non-stationary environment, the flow of information in the service life can bring considerable benefit

($VoI_{f,R}$=\$3,285). To further investigate the influence of POMDP parameters on VoI, $\mathrm{VoI}_{f,R}$ is investigated parametrically as Fig. 8 shows. In Fig. 8 (a), it is observed that the minimum service life costs with and without inspection actions and $\mathrm{VoI}_{f,R}$ from the proposed POMDP model are between the corresponding values from POMDP model-1 and POMDP model-2. This observation is consistent with Fig. 7, where the transition function of POMDP model-1 significantly underestimates the probability of failure, while the transition function of POMDP model-2 overestimates the probability of failure in the aging process. The existing POMDP models with stationary deterioration processes can introduce large errors in estimating the service life cost and VoI in a non-stationary environment. The significance of the proposed approach is thus highlighted. Fig. 8 (b) shows the relation between the $\mathrm{VoI}_{f,R}$ and the error of the inspection action when error of inspection increases from 0.1 mm to 5 mm. As illustrated in the figure, $\mathrm{VoI}_{f,R}$ decreases slowly with the increase of the error. However, after the error reaches the threshold of 0.5 mm, the rate of reduction increases until $\mathrm{VoI}_{f,R}$ comes close to zero. The trend in the figure is based on the assumption that the inspection cost remains the same with different inspection errors. In reality, the inspection cost is typically inversely proportional to the inspection error, thus, the selection of the inspection accuracy should be investigated to reach the optimality of $\mathrm{VoI}_{f,R}$. The relation between $\mathrm{VoI}_{f,R}$ and the cost of the inspection action is shown in Fig. 8 (c). The value of information decreases monotonically with the increase of the inspection cost. When the inspection cost is around \$2,000, $\mathrm{VoI}_{f,R}$ drops to almost zero. Fig. 8 (d) illustrates the relation between $\mathrm{VoI}_{f,R}$ and the cost of the replacement action. It is noted that when the replacement cost increases from \$0 to around $\$5\times10^4$, the value of information flow shows a significant increase from \$0 to more than \$4,000. Subsequently, when the replacement cost increases further, the value of information drops until it reaches zero. When replacement is considerably cheap (even cheaper than the inspection), replacement can be applied without collecting information in advance. On the other hand, when replacement is extremely expensive (even more expensive than the failure cost), the decision maker prefers to "do nothing" and thus, the collected information cannot yield a better maintenance decision.

## 5. Conclusion

This paper presented a new approach for value of information (VoI) analysis called VoI-R-POMDP for systems with time-dependent deterioration processes and introduced methods for accurate determination of the POMDP model. A novel POMDP framework, which establishes multiple transition models describing different deterioration rates, is proposed to accurately model the non-stationary deterioration process in realistic applications. Accurate and efficient strategies are also proposed for the POMDP model definition. The transition function is estimated using the maximum likelihood estimation considering the probabilities that are derived from reliability methods. A new formulation of the observation function based on Bayes' theorem is proposed to accurately describe the inspection process in the POMDP model. The VoI-R-POMDP approach is investigated for a corroding beam example to demonstrate its performance. The proposed approach can provide considerably more accurate estimates of values of both current inspection and information flow in the case where the deterioration process is non-stationary. The paper is concluded by discussing limitations and possible extensions. First, when the limit state function or the function that models monitored quantities become expensive-to-evaluate, advanced sampling techniques [60, 61] or surrogate models [62, 63] can be applied to reduce the computational costs. Moreover, the decomposition approach in [50] can be extended for VoI analysis of a system with multiple components.

## CRediT authorship contribution statement

**Chaolin Song**: Conceptualization, Methodology, Formal Analysis, Writing - original draft. **Chi Zhang**: Methodology, Validation, Writing - review & editing. **Abdollah Shafieezadeh**: Methodology, Validation, Writing - review & editing, Supervision, **Rucheng Xiao**: Conceptualization, Supervision.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Acknowledgments**

This research was partly funded by the U.S. National Science Foundation (NSF) through award CMMI-2000156; the Lichtenstein endowment at The Ohio State University; and the China Scholarship Council. Opinions and findings presented are those of the authors and do not necessarily reflect the views of the sponsors.

## Reference

[1] O. Avci, O. Abdeljaber, S. Kiranyaz, M. Hussein, M. Gabbouj, D.J.J.a.p.a. Inman, A Review of Vibration-Based Damage Detection in Civil Structures: From Traditional Methods to Machine Learning and Deep Learning Applications, (2020).

[2] C. Duan, C.J.I.A.T.o.M. Deng, Prognostics of Health Measures for Machines with Aging and Dynamic Cumulative Damage, (2020).

[3] H. Mo, G. Sansavini, M. Xie, Performance-based maintenance of gas turbines for reliable control of degraded power systems, Mechanical Systems and Signal Processing, 103 (2018) 398-412.

[4] H. Raiffa, R. Schlaifer, Applied statistical decision theory, (1961).

[5] R.A. Howard, Information value theory, IEEE Transactions on systems science and cybernetics, 2 (1966) 22-26.

[6] M. Pozzi, A. Der Kiureghian, Assessing the value of information for long-term structural health monitoring, Health monitoring of structural and biological systems 2011, International Society for Optics and Photonics, 2011, pp. 79842W.

[7] D. Zonta, B. Glisic, S. Adriaenssens, Value of information: impact of monitoring on decision-making, Structural Control and Health Monitoring, 21 (2014) 1043-1056.

[8] D. Straub, Value of information analysis with structural reliability methods, Structural Safety, 49 (2014) 75-85.

[9] A. Zitrou, T. Bedford, A. Daneshkhah, Robustness of maintenance decisions: Uncertainty modelling and value of information, Reliability Engineering & System Safety, 120 (2013) 60-71.

[10] L. Long, M. Döhler, S. Thöns, Determination of structural and damage detection system influencing parameters on the value of information, Structural Health Monitoring, (2020) 1475921719900918.

[11] L. Iannacone, P. Francesco Giordano, P. Gardoni, M. Pina Limongelli, Quantifying the value of information from inspecting and monitoring engineering systems subject to gradual and shock deterioration, Structural Health Monitoring, (2021) 1475921720981869.

[12] K. Bjørnsen, J.T. Selvik, T. Aven, A semi-quantitative assessment process for improved use of the expected value of information measure in safety management, Reliability Engineering & System Safety, 188 (2019) 494-502.

[13] G. Zou, M.H. Faber, A. González, K. Banisoleiman, Computing the value of information from periodic testing in holistic decision making under uncertainty, Reliability Engineering & System Safety, 206 (2021) 107242.

[14] D. Straub, M.H. Faber, System effects in generic risk-based inspection planning, J. Offshore Mech. Arct. Eng., 126 (2004) 265-271.

[15] D. Straub, M.H. Faber, Risk based inspection planning for structural systems, Structural safety, 27 (2005) 335-355.

[16] W.-H. Zhang, J. Qin, D.-G. Lu, S. Thöns, M.H. Faber, VoI-informed decision-making for SHM system arrangement, Structural Health Monitoring, (2020) 1475921720962736.

[17] W. Fauriat, E. Zio, Optimization of an aperiodic sequential inspection and condition-based maintenance policy driven by value of information, Reliability Engineering & System Safety, 204 (2020) 107133.

[18] A.C. Neves, J. Leander, I. González, R. Karoumi, An approach to decision-making analysis for implementation of structural health monitoring in bridges, Structural Control and Health Monitoring, 26 (2019) e2352.

[19] M. Farhan, R. Schneider, S. Thöns, Predictive information and maintenance optimization based on decision theory: a case study considering a welded joint in an offshore wind turbine support structure, Structural Health Monitoring, (2021) 1475921720981833.

[20] C. Malings, M. Pozzi, Conditional entropy and value of information metrics for optimal sensing in infrastructure systems, Structural Safety, 60 (2016) 77-90.

[21] C. Malings, M. Pozzi, Submodularity issues in value-of-information-based sensor placement, Reliability Engineering & System Safety, 183 (2019) 93-103.

[22] C. Malings, M. Pozzi, Value-of-information in spatio-temporal systems: Sensor placement and scheduling, Reliability Engineering & System Safety, 172 (2018) 45-57.

[23] K.J. Astrom, Optimal control of Markov processes with incomplete state information, Journal of mathematical analysis and applications, 10 (1965) 174-205.

[24] R.D. Smallwood, E.J. Sondik, The optimal control of partially observable Markov processes over a finite horizon, Operations research, 21 (1973) 1071-1088.
[25] C. Duan, V. Makis, C. Deng, An integrated framework for health measures prediction and optimal maintenance policy for mechanical systems using a proportional hazards model, Mechanical Systems and Signal Processing, 111 (2018) 285-302.
[26] H. Jin, F. Han, Y. Sang, An optimal maintenance strategy for multi-state deterioration systems based on a semi-Markov decision process coupled with simulation technique, Mechanical Systems and Signal Processing, 139 (2020) 106570.
[27] R. Bellman, Dynamic programming, Science, 153 (1966) 34-37.
[28] A.R. Cassandra, M.L. Littman, N.L. Zhang, Incremental pruning: A simple, fast, exact method for partially observable Markov decision processes, arXiv preprint arXiv:1302.1525, (2013).
[29] H. Kurniawati, D. Hsu, W.S. Lee, Sarsop: Efficient point-based pomdp planning by approximating optimally reachable belief spaces, Robotics: Science and systems, Zurich, Switzerland., 2008.
[30] M.L. Littman, A tutorial on partially observable Markov decision processes, Journal of Mathematical Psychology, 53 (2009) 119-125.
[31] N. Meuleau, K.-E. Kim, L.P. Kaelbling, A.R. Cassandra, Solving POMDPs by searching the space of finite policies, arXiv preprint arXiv:1301.6720, (2013).
[32] S.C. Ong, S.W. Png, D. Hsu, W.S. Lee, POMDPs for robotic tasks with mixed observability, Robotics: Science and systems, 2009, pp. 4.
[33] P. Poupart, Exploiting structure to efficiently solve large scale partially observable Markov decision processes, Citeseer, 2005.
[34] T. Smith, R. Simmons, Focused real-time dynamic programming for MDPs: Squeezing more out of a heuristic, AAAI, 2006, pp. 1227-1232.
[35] T. Smith, R. Simmons, Heuristic search value iteration for POMDPs, arXiv preprint arXiv:1207.4166, (2012).
[36] T. Smith, R. Simmons, Point-based POMDP algorithms: Improved analysis and implementation, arXiv preprint arXiv:1207.1412, (2012).
[37] M.T. Spaan, F.A. Oliehoek, The MultiAgent Decision Process toolbox: software for decision-theoretic planning in multiagent systems, Proc. of the AAMAS Workshop on Multi-Agent Sequential Decision Making in Uncertain Domains (MSDM), 2008, pp. 107-121.
[38] M.T. Spaan, N. Vlassis, Perseus: Randomized point-based value iteration for POMDPs, Journal of artificial intelligence research, 24 (2005) 195-220.
[39] S. Li, M. Pozzi, What makes long-term monitoring convenient? A parametric analysis of value of information in infrastructure maintenance, Structural Control and Health Monitoring, 26 (2019) e2329.
[40] W.-H. Zhang, D.-G. Lu, J. Qin, S. Thöns, M.H. Faber, Value of information analysis in civil and infrastructure engineering: a review, Journal of Infrastructure Preservation and Resilience, 2 (2021) 1-21.
[41] M.T. Spaan, N. Spaan, A point-based POMDP algorithm for robot planning, IEEE International Conference on Robotics and Automation, 2004. Proceedings. ICRA'04. 2004, IEEE, 2004, pp. 2399-2404.
[42] A. Foka, P. Trahanias, Real-time hierarchical POMDPs for autonomous robot navigation, Robotics and Autonomous Systems, 55 (2007) 561-571.
[43] R.Z.B. Bravo, A. Leiras, F.L.J.P. Cyrino Oliveira, O. Management, The Use of UAV s in Humanitarian Relief: An Application of POMDP-Based Methodology for Finding Victims, 28 (2019) 421-440.
[44] H. Ellis, M. Jiang, R.B. Corotis, Inspection, maintenance, and repair with partial observability, Journal of Infrastructure Systems, 1 (1995) 92-99.
[45] E. Fereshtehnejad, A. Shafieezadeh, A randomized point-based value iteration POMDP enhanced with a counting process technique for optimal management of multi-state multi-element systems, Structural Safety, 65 (2017) 113-125.
[46] K. Papakonstantinou, M. Shinozuka, Optimum inspection and maintenance policies for corroded structures using partially observable Markov decision processes and stochastic, physically based models, Probabilistic Engineering Mechanics, 37 (2014) 93-108.
[47] K.G. Papakonstantinou, M. Shinozuka, Planning structural inspection and maintenance policies via dynamic programming and Markov processes. Part II: POMDP implementation, Reliability Engineering & System Safety, 130 (2014) 214-224.
[48] C. Bunks, D. McCarthy, T. Al-Ani, Condition-based maintenance of machines using hidden Markov models, Mechanical Systems and Signal Processing, 14 (2000) 597-612.
[49] M. Compare, P. Baraldi, P. Marelli, E. Zio, Partially observable Markov decision processes for optimal operations of gas transmission networks, Reliability Engineering & System Safety, 199 (2020) 106893.

[50] M. Memarzadeh, M. Pozzi, Value of information in sequential decision making: Component inspection, permanent monitoring and system-level scheduling, Reliability Engineering & System Safety, 154 (2016) 137-151.
[51] M. Pozzi, A. Der Kiureghian, Assessing the value of alternative bridge health monitoring systems, 6th International Conference on Bridge Maintenance, Safety and Management, IABMAS, 2012.
[52] R.E. Melchers, A.T. Beck, Structural reliability analysis and prediction, John wiley & sons, 2018.
[53] Y. Mohammadi Darestani, A. Shafieezadeh, K. Cha, Effect of modelling complexities on extreme wind hazard performance of steel lattice transmission towers, Structure and Infrastructure Engineering, 16 (2020) 898-915.
[54] C. Zhang, Z. Wang, A. Shafieezadeh, Value of Information Analysis via Active Learning and Knowledge Sharing in Error-Controlled Adaptive Kriging, IEEE Access, 8 (2020) 51021-51034.
[55] L. Liu, G.S. Sukhatme, A solution to time-varying Markov decision processes, IEEE Robotics and Automation Letters, 3 (2018) 1631-1638.
[56] S. Chakraborty, S. Tesfamariam, Subset simulation based approach for space-time-dependent system reliability analysis of corroding pipelines, Structural Safety, 90 (2021) 102073.
[57] X. Han, D.Y. Yang, D.M. Frangopol, Time-variant reliability analysis of steel plates in marine environments considering pit nucleation and propagation, Probabilistic Engineering Mechanics, 57 (2019) 32-42.
[58] D.-C. Feng, S.-C. Xie, Y. Li, L. Jin, Time-dependent reliability-based redundancy assessment of deteriorated RC structures against progressive collapse considering corrosion effect, Structural Safety, 89 (2021) 102061.
[59] C. Jiang, D. Wang, H. Qiu, L. Gao, L. Chen, Z. Yang, An active failure-pursuing Kriging modeling method for time-dependent reliability analysis, Mechanical Systems and Signal Processing, 129 (2019) 112-129.
[60] I. Papaioannou, S. Geyer, D. Straub, Improved cross entropy-based importance sampling with a flexible mixture model, Reliability Engineering & System Safety, 191 (2019) 106564.
[61] W. Du, Y. Luo, Y. Wang, Time-variant reliability analysis using the parallel subset simulation, Reliability Engineering & System Safety, 182 (2019) 250-257.
[62] Z. Wang, A. Shafieezadeh, REAK: Reliability analysis through Error rate-based Adaptive Kriging, Reliability Engineering & System Safety, 182 (2019) 33-45.
[63] C. Zhang, Z. Wang, A. Shafieezadeh, Error quantification and control for adaptive kriging-based reliability updating with equality information, Reliability Engineering & System Safety, 207 (2021) 107323.