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Planar Compliance Realization With Two 3-Joint Serial Mechanisms

In this paper, the realization of any specified planar compliance with two 3R serial elastic mechanisms is addressed. Using the concept of dual elastic mechanisms, it is shown that the realization of a compliant behavior with two serial mechanisms connected in parallel is equivalent to its realization with a 6-spring fully parallel mechanism. Since the spring axes of a 6-spring parallel mechanism indicate the geometry of a dual 3R serial mechanism, a new synthesis procedure for the realization of a stiffness matrix with a 6-spring parallel mechanism is first developed. Then, this result is extended to a geometric construction-based synthesis procedure for two 3-joint serial mechanisms. [DOI: 10.1115/1.4053284]

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1 Introduction

In robotic manipulation, some form of compliance is needed to provide accurate relative positioning for constrained motion and to avoid excessive contact forces when interacting with other objects. A general model of compliance is a rigid body suspended by an elastic system. In robotic applications, compliance can be attained by the robot itself and/or by an elastic device at the robot end-effector. An elastic behavior is characterized by the relationship between a wrench **w** (force and torque) applied to the body and the resultant twist **t** (translation and rotation) of the body. If small displacement from an equilibrium is considered, the elastic behavior can be described by a linear mapping:

$$\mathbf{w} = \mathbf{K}\mathbf{t} \quad \text{or} \quad \mathbf{t} = \mathbf{C}\mathbf{w} \tag{1}$$

where the stiffness **K** and compliance **C** (the inverse of **K**) are symmetric positive semi-definite (PSD) matrices.

An elastic suspension can be obtained by a general network of passive elastic components. The elastic behavior of the network is determined by (1) the way components are connected (in parallel or in series), (2) the geometric configuration of components, and (3) the elastic properties (joint or spring stiffness) of each component. Each of these needs to be identified to achieve a desired Cartesian compliant behavior at the reference body. The two networks with the simplest topology are as follows: (1) a fully *parallel* mechanism having springs independently connected to the reference body (Fig. 1(*a*)), and (2) a fully *serial* mechanism having compliances at each joint ultimately connected to the reference body (Fig. 1(*b*)).

A significant amount of prior work has addressed compliance realization with serial mechanisms having kinematic redundancy. Unlike a redundant fully parallel mechanism, a redundant serial mechanism can change its configuration without end-point motion. Therefore, a larger space of compliances can be attained not only by adjusting the joint compliances, but also by varying the configuration of this type of mechanism. Also, a redundant serial mechanism can continuously provide a desired compliance when the suspended reference body (the end-effector) is moving in space. However, a fully serial mechanism has the following significant limitations relative to fully parallel mechanisms.

(1) The suspended weight of the mechanism is larger, especially when the mechanism has more joints and longer links. The suspending joint forces may result in relatively large deformation and thus influence the accuracy of the manipulator;
(2) The set of realizable elastic behaviors is more restricted. A relatively large set of compliant behaviors cannot be achieved even if each joint compliance has infinite range [1–4].

The set of realizable compliant behaviors using fully serial mechanisms is restricted in that a desired planar compliant behavior can be realized by a serial mechanism only if the center of compliance (the location where the compliance matrix is diagonal) is inside the convex hull formed by the mechanism joint locations [1–4]. If the center of a desired compliance is beyond the mechanism workspace, the compliance cannot be realized regardless of the values of joint compliances. This restriction prevents serial mechanisms from providing the appropriate compliance in many tasks, since most manipulations require that the compliance center be projected away from the mechanism (e.g., Ref. [5]).

The two limitations of fully serial elastic mechanisms do not exist for multi-serial parallel mechanisms. As such, two or more serial mechanisms connected in parallel can be used to support and move an object as well as attain a much larger set of compliant behaviors. Using two serial mechanisms, the forces and moments at each joint due to the weight are reduced, enabling the dexterous manipulation of larger and heavier objects, and the space of realizable compliant behaviors is dramatically increased.

As stated previously, a planar compliance (or stiffness) center is the location at which the compliance (or stiffness) matrix can be expressed in diagonal form, i.e., the location at which translational and rotational aspects are completely decoupled. Figure 2 illustrates

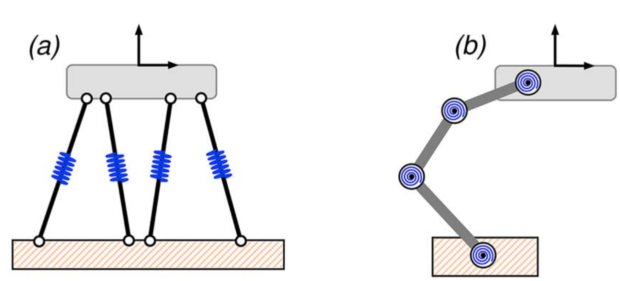

Fig. 1 Compliant mechanisms having simple topology: (*a*) fully parallel mechanism and (*b*) fully serial mechanism

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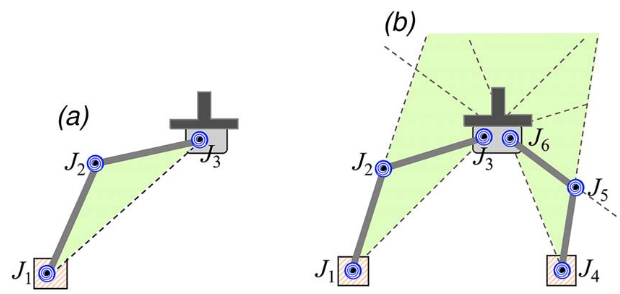

Fig. 2 Space of compliance centers associated with two types of elastic suspensions: (*a*) a simple fully serial mechanism with elastic joints. The compliance center locus is the triangle $J_1J_2J_3$ formed by the three joints and (*b*) two fully serial mechanisms with elastic joints. The compliance center locus is the union of triangles formed by all combinations of lines along the sides of triangles $J_1J_2J_3$ and $J_4J_5J_6$.

the space of realizable compliance centers associated with a 3*R* serial mechanism (Fig. 2(*a*)) and the space of realizable compliance centers associated with two 3*R* serial mechanisms connected in parallel (Fig. 2(*b*)). For the fully serial mechanism shown in Fig. 2(*a*), the locus of the compliance centers achievable by the mechanism is the triangle $J_1J_2J_3$ [1]. The mechanism cannot realize a compliance with a center located above the end-effector. When two serial mechanisms are used, the space of the realizable centers is not just the union of the center spaces associated with each serial mechanism, i.e., the two triangles $J_1J_2J_3$ and $J_4J_5J_6$. It is the union of the triangles formed by all combinations of the straight lines along the sides of the two triangles [2,3] as shown in Fig. 2(*b*). It can be seen that the locus of compliance center locations achievable by the multi-serial parallel mechanism extends far beyond the workspaces of the two serial mechanisms.

1.1 Related Work.

In the analysis of general compliant behavior, screw theory [6–9] and Lie groups [10,11] have been used. In recent work on the realization of compliance, the design of mechanisms to achieve a specified elastic behavior has been addressed. Most early approaches were based on a rank-1 decomposition of the compliance/stiffness matrix [12–18]. In some work on the synthesis of compliance [16–18], geometric constraints on the mechanisms were considered in the procedures. In Ref. [19], a fully geometric construction-based approach to the realization of *spatial* compliance was developed.

In Refs. [20,21], approaches to attain an isotropic compliance in a Euclidean space with a fully serial mechanism were presented. In Refs. [22,23], compliance analysis and synthesis for flexure mechanisms were addressed. In Refs. [24,25], stiffness synthesis for parallel mechanisms was presented.

In closely related work in the realization of *planar* compliance [1–4], geometry-based approaches were used in the design of mechanisms (both fully serial and fully parallel) having three to six elastic components. Necessary and sufficient conditions on the corresponding mechanism geometry were identified for the realization of an arbitrary compliance. It was shown that, as the number of the elastic components n increases, the *dimension* of the space of realizable compliances is similarly increased for $n \leq 6$.

In almost all prior work, mechanisms considered for the realization of compliance were either fully parallel or fully serial. Very little work has addressed the compliance synthesis with *multiple* serial mechanisms. When compliance realization of multi-serial parallel mechanisms has been addressed [26], optimization has been used to select mechanism geometry and elastic behavior that is unlikely to attain the desired Cartesian compliance.

1.2 Partial-Geometric Approach.

For two serial mechanisms that are independently connected to a suspended body, if $\mathbf{C}_i$ is the compliance matrix of each serial mechanism, then the Cartesian stiffness $\mathbf{K}$ of the system is

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 \tag{2}$$

where $\mathbf{K}_i = \mathbf{C}_i^{-1}$ is the stiffness matrix associated with each serial mechanism. Thus, if a given stiffness $\mathbf{K}$ is algebraically decomposed arbitrarily into two rank-3 matrices as in Eq. (2), then, each $\mathbf{K}_i$ can be realized with a 3*R* serial mechanism using a systematic geometric construction process [1]. As such, any stiffness $\mathbf{K}$ can be achieved with limited geometric considerations by two serial mechanisms by varying the mechanism geometry and its inherent passive compliant behavior in each component.

Although a decomposition of $\mathbf{K}$ into the form of Eq. (2) yields two serial mechanisms that realize the behavior, the algebraic decomposition process of Eq. (2) does not take into account the geometric properties of the mechanisms. It is known that for any given symmetric PSD matrix, there are infinitely many algebraic decompositions of the matrix in the form of Eq. (2). The geometry of each serial mechanism would largely depend on how the stiffness matrix is decomposed. A purely algebraic decomposition of a stiffness matrix may result in mechanisms that are geometrically awkward or violate task-imposed geometric constraints. Therefore, an approach that takes into account the geometry of components in each step of the synthesis process is needed.

1.2.1 Full-Geometric Approach.

In this paper, a new approach to realize a given compliance with a multi-serial parallel mechanism is developed. The mechanism considered consists of two 3-joint serial mechanisms connected in parallel. The approach allows one to select the geometry of each component.

The main contributions of the paper are as follows:

(1) Identification of the elastic duality between a manipulator consisting of two 3-joint compliant arms and a fully parallel mechanism consisting of 6 springs.
(2) Development of a geometry-based procedure for the realization of an arbitrary planar compliance with a two-arm manipulator.

Additional contributions include the following: (1) identification of necessary conditions on spring geometry for the realization of an arbitrary elastic behavior with any number of springs connected in parallel; (2) identification of a new set of necessary and sufficient conditions for the realization of any elastic behavior with six springs connected in parallel.

1.3 Overview.

In this paper, a geometry-based synthesis procedure for the realization of any given planar compliance with two 3*R* serial mechanisms connected in parallel is developed. The synthesis is based on the concept of *dual elastic mechanisms* [1].

The paper is outlined as follows. In Sec. 2, screw representation of planar mechanism configurations and the concept of dual elastic mechanisms are reviewed. In Sec. 3, necessary conditions on spring geometry of any parallel mechanism for the realization of a specified elastic behavior and a new set of necessary and sufficient conditions for the realization of any specified compliance with a six-spring parallel mechanism are identified. In Sec. 4, a geometry-based synthesis procedure for a two-arm mechanism to realize a given compliance is developed. In Sec. 5, a numerical example is provided to demonstrate the synthesis procedure. Finally, a discussion and summary are presented in Sec. 6.

2 Technical Background

In this section, the background needed for planar compliance realization with a multi-serial parallel mechanism is provided. First, screw representation of a planar mechanism configuration is reviewed. Next, the duality relationship between a planar parallel elastic mechanism and a planar serial elastic mechanism is presented.

2.1 Stiffness Realization With Parallel Connections. Consider a planar parallel mechanism consisting of a set of n line springs connected to a single reference body. The geometry of each spring can be represented by a unit wrench $\mathbf{w}_i$ defined as a *spring wrench*. In Plücker ray coordinates, the planar spring wrench associated with a line spring has the form:

$$\mathbf{w} = \begin{bmatrix} \mathbf{n} \\ d \end{bmatrix} \tag{3}$$

where the unit vector $\mathbf{n}$ indicates the direction of the wrench (spring axis) and where

$$d = (\mathbf{r} \times \mathbf{n}) \cdot \tilde{\mathbf{k}} \tag{4}$$

where $\mathbf{r}$ is the position vector from the origin to any point on the line and $\tilde{\mathbf{k}}$ is the unit vector orthogonal to the plane.

It can be seen that the line-of-action (axis) of wrench $\mathbf{w}$ uniquely defines a line in the plane. Conversely, any line in the plane is represented by a unique unit wrench $\mathbf{w}$ in the form of Eq. (3) such that the line is the axis of $\mathbf{w}$. Thus, a unit wrench $\mathbf{w}$ can be used to represent the axis of a spring, and geometrically, represent a line in the plane.

If a stiffness matrix $\mathbf{K}$ is realized with a fully parallel mechanism having n springs $\mathbf{w}_i$ (1, …, n), then [15]

$$\mathbf{K} = k_1\mathbf{w}_1\mathbf{w}_1^T + k_2\mathbf{w}_2\mathbf{w}_2^T + \cdots + k_n\mathbf{w}_n\mathbf{w}_n^T \tag{5}$$

where $k_i \geq 0$ is the stiffness of spring $\mathbf{w}_i$.

Conversely, if a stiffness matrix $\mathbf{K}$ can expressed in the form of Eq. (5) with each $k_i \geq 0$, then the stiffness is realized with a parallel mechanism having its geometry described by spring wrenches $\mathbf{w}_i$. Thus, to realize a specified stiffness $\mathbf{K}$ with a parallel mechanism, a set of spring wrenches $\mathbf{w}_i$'s and the corresponding spring rates k_i must be identified such that Eq. (5) is satisfied.

2.2 Compliance Realization With Serial Connections. Consider a planar serial mechanism consisting of n revolute joints. The location of a joint can be represented by a unit twist $\mathbf{t}_i$ defined as a *joint twist*. In Plücker axis coordinates, the planar motion joint twist associated with a joint has the form:

$$\mathbf{t} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} \tag{6}$$

where $\mathbf{u} = \mathbf{r} \times \tilde{\mathbf{k}}$, $\tilde{\mathbf{k}}$ is the unit vector orthogonal to the plane and $\mathbf{r}$ is the position vector of the revolute joint with respect to the coordinate frame:

$$\mathbf{r} = \mathbf{\Omega}\mathbf{u} \tag{7}$$

where $\mathbf{\Omega}$ is the 2×2 skew-symmetric matrix defined as follows:

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

Thus, for any given unit twist in Eq. (6), the corresponding instantaneous center location $\mathbf{r}$ is determined by Eq. (7).

Since $\mathbf{\Omega}^{-1} = \mathbf{\Omega}^T = -\mathbf{\Omega}$, Eq. (7) can be expressed as

$$\mathbf{u} = -\mathbf{\Omega}\mathbf{r} \tag{9}$$

Thus, the location of a point uniquely describes a unit twist.

If a compliance matrix $\mathbf{C}$ is realized with a fully serial mechanism having n joints described by joint twists $\mathbf{t}_i$, (1, …, n), then [15]

$$\mathbf{C} = c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + \cdots + c_n\mathbf{t}_n\mathbf{t}_n^T \tag{10}$$

where $c_i \geq 0$ is the joint compliance at joint $\mathbf{t}_i$.

Conversely, if a compliance matrix $\mathbf{C}$ can be expressed in the form of Eq. (10) with each $c_i \geq 0$, then the compliance is realized at the serial mechanism configuration described by joint twists $\mathbf{t}_i$. Thus, to realize a specified compliance $\mathbf{C}$ with a serial mechanism, the mechanism configuration $\mathbf{t}_i$'s and the corresponding joint compliances c_i's must be identified such that Eq. (10) is satisfied.

2.3 Reciprocal Screws. A wrench $\mathbf{w}$ expressed in Plücker ray coordinates and a twist $\mathbf{t}$ expressed in Plücker axis coordinates are reciprocal if and only if

$$\mathbf{t}^T\mathbf{w} = 0 \tag{11}$$

Geometrically, the reciprocal condition Eq. (11) for the planar case indicates that the line defined by wrench $\mathbf{w}$ passes through the point defined by twist $\mathbf{t}$.

If a line passes through two points J_i and J_j that are represented by unit twists $\mathbf{t}_i$ and $\mathbf{t}_j$ respectively, the wrench $\mathbf{w}_{ij}$ associated with the line must satisfy:

$$\mathbf{t}_i^T\mathbf{w}_{ij} = 0, \quad \mathbf{t}_j^T\mathbf{w}_{ij} = 0 \tag{12}$$

Thus, for planar screws, wrench $\mathbf{w}_{ij}$ can be uniquely determined by the cross product operation:

$$\mathbf{w}_{ij} = \mathbf{t}_i \times \mathbf{t}_j \tag{13}$$

Similarly, if a point is located at the intersection of two lines that are represented by wrenches $\mathbf{w}_i$ and $\mathbf{w}_j$, respectively, the twist $\mathbf{t}_{ij}$ associated with the point must satisfy:

$$\mathbf{t}_{ij}^T\mathbf{w}_i = 0, \quad \mathbf{t}_{ij}^T\mathbf{w}_j = 0 \tag{14}$$

The planar twist $\mathbf{t}_{ij}$ can be determined by

$$\mathbf{t}_{ij} = \mathbf{w}_i \times \mathbf{w}_j \tag{15}$$

For planar screws, a unit twist reciprocal to two non-parallel wrenches is unique. These properties will be used in the description of realization conditions in planar parallel, serial, and multi-serial parallel mechanisms. The reciprocal relation between spring wrenches in a parallel mechanism and joint twists in a serial mechanism is the foundation for the concept of *dual elastic mechanisms* [1] as reviewed below.

2.4 Planar Dual Elastic Mechanisms. Suppose a three-spring planar parallel mechanism has spring wrenches $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. The lines of action of $\mathbf{w}_i$ form a triangle. Consider the three unit twists ($\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$) centered at the triangle vertices, i.e., $\mathbf{t}_i$ is centered at the intersection of $\mathbf{w}_j$ and $\mathbf{w}_p$. Then, $\mathbf{t}_i$ is reciprocal to wrenches ($\mathbf{w}_j$, $\mathbf{w}_p$).

Also consider a planar serial mechanism having these same three joint twists $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$. The triangle formed by the three line spring axes in the parallel mechanism is coincident with the triangle formed by the vertices identified by the three revolute joints in the serial mechanism. A pair of parallel and serial elastic mechanisms satisfying these conditions are defined as *dual elastic* mechanisms [1].

Figure 3 illustrates a pair of dual elastic mechanisms for the generic case. The triangle $A_1A_2A_3$ in the three-spring parallel mechanism of Fig. 3(*a*) is coincident with triangle $J_1J_2J_3$ in the three-joint serial mechanism of Fig. 3(*b*). Each spring wrench $\mathbf{w}_i$ in the parallel mechanism is reciprocal to two joint twists ($\mathbf{t}_j$, $\mathbf{t}_p$) in the serial mechanism.

Given a pair of dual elastic mechanisms with spring wrenches ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$) and joint twists ($\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$), for a given elastic behavior described in stiffness matrix $\mathbf{K}$ or compliance matrix $\mathbf{C} = \mathbf{K}^{-1}$, it is proved [1] that

$$\mathbf{t}_i^T\mathbf{K}\mathbf{t}_j = 0 \Longleftrightarrow \mathbf{w}_i^T\mathbf{C}\mathbf{w}_j = 0, \quad \forall i \neq j \tag{16}$$

and that an elastic behavior can be realized with one mechanism if and only if it can be realized with its dual elastic mechanism. When their elastic properties have infinite variability, the space of realizable elastic behaviors for each mechanism is exactly the same.

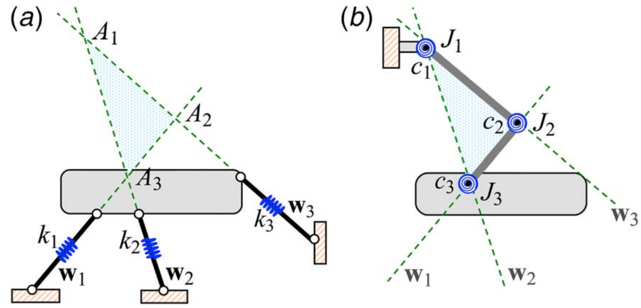

Fig. 3 Planar dual elastic mechanisms in parallel and serial construction: (*a*) a 3-spring parallel mechanism and (*b*) dual elastic 3*R* serial mechanism. The triangle formed by the three spring axes in the parallel mechanism is coincident with the triangle formed by the three joints in the serial mechanism. The two mechanisms have the exact same space of realizable compliant behaviors when k_i in (*a*) and c_i in (*b*) have infinite variability.

Also, it can be proved that, if k_i is the spring constant associated with spring wrench $\mathbf{w}_i$ in the parallel mechanism and c_i is the joint compliance associated with joint twist $\mathbf{t}_i$, then k_i and c_i satisfy:

$$k_i c_i = \frac{1}{(\mathbf{w}_i^T \mathbf{t}_i)^2}, \quad i = 1, 2, 3 \tag{17}$$

2.5 Application of Dual Elastic Mechanisms. Since any three-joint serial elastic mechanism can be replaced with its dual parallel elastic mechanism (and vice versa) without changing the realizability of compliant behaviors, a multi-serial manipulator system can be replaced by a fully parallel mechanism. Therefore, two 3-joint serial manipulators can be replaced by a six-spring parallel manipulator and the existing synthesis procedures for compliance realization with a six-spring parallel mechanism can be modified to achieve the desired elastic behavior with a two-arm manipulator. This process does not require a matrix inverse (as in Eq. (2)) for which the geometric significance of spring wrenches or joint twists is lost.

The transformation of two 3*R* serial mechanisms to a six-spring parallel mechanism is unique; whereas, the reverse transformation is not.

Figure 4 illustrates the transformation of a six-spring parallel mechanism into two 3*R* serial mechanisms. The six springs are separated into two groups ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$) and ($\mathbf{w}_4$, $\mathbf{w}_5$, $\mathbf{w}_6$). The three-spring system ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$) is converted into its dual three-joint serial mechanism $J_1J_2J_3$, and the three-spring system ($\mathbf{w}_4$, $\mathbf{w}_5$, $\mathbf{w}_6$) is converted into its dual three-joint serial mechanism $J_4J_5J_6$. Since the locations of joints in each serial mechanism are at the intersection of the corresponding two spring axes, the twist centered at joint J_p can be determined using Eq. (15),

$$\tilde{\mathbf{t}}_p = \mathbf{w}_i \times \mathbf{w}_j \tag{18}$$

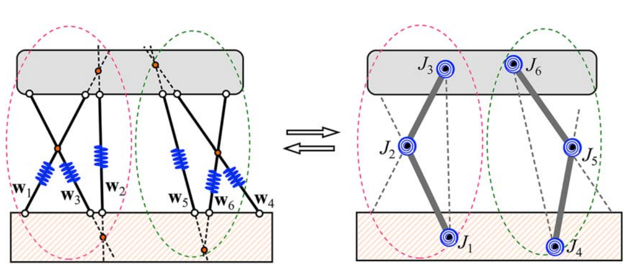

Fig. 4 A six-spring parallel mechanism transformed into a two-serial parallel mechanism using dual elastic mechanisms. Each set of three springs is replaced by a three-joint serial mechanism having three elastic joints (and vice versa).

By normalizing $\tilde{\mathbf{t}}_p$, the corresponding joint twist (unit twist), $\mathbf{t}_p$ is obtained. The joint compliances can be calculated using Eq. (17),

$$c_i = \frac{1}{k_i(\mathbf{w}_i^T \mathbf{t}_i)^2}, \quad i = 1, 2, \ldots, 6 \tag{19}$$

Because the separation of the six springs into two 3-spring systems is not unique, if two different three-spring systems ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_4$) and ($\mathbf{w}_3$, $\mathbf{w}_5$, $\mathbf{w}_6$) are instead considered, two different serial mechanisms will be obtained.

3 Stiffness Realization With a Parallel Mechanism

In this section, stiffness realization with a parallel mechanism is addressed. First, new relationships between mechanism geometry and the location of the compliance center are presented. These relationships yield a set of necessary conditions on spring positions and orientations relative to the stiffness center that must be satisfied in order to achieve a given compliance. Then, a new set of necessary and sufficient conditions on spring configurations for the realization of stiffness with a six-spring parallel mechanism is identified.

3.1 Mechanism Geometry and Stiffness Center. In the realization of a stiffness with a parallel mechanism, the springs must surround the center of stiffness. Below, we show that an additional requirement on the distribution of springs relative to the center of stiffness must be satisfied.

Consider a planar parallel mechanism with n springs. The axis of each spring can be represented by a unit wrench $\mathbf{w}_i$ having the form of Eq. (3):

$$\mathbf{w}_i = \begin{bmatrix} \mathbf{n}_i \\ d_i \end{bmatrix} \tag{20}$$

where $d_i = (\mathbf{r}_i \times \mathbf{n}_i) \cdot \tilde{\mathbf{k}}$ indicates the distance from the coordinate frame to the spring axis, and $\mathbf{r}_i$ is the perpendicular position vector from the coordinate frame to the spring axis. Since both $\mathbf{n}_i$ and $\tilde{\mathbf{k}}$ are unit vectors,

$$\|\mathbf{r}_i\| = |d_i| \tag{21}$$

If each spring wrench $\mathbf{w}_i$ and corresponding spring stiffness k_i are given, the Cartesian stiffness $\mathbf{K}$ of the mechanism is as follows:

$$\mathbf{K} = k_1\mathbf{w}_1\mathbf{w}_1^T + k_2\mathbf{w}_2\mathbf{w}_2^T + \cdots + k_n\mathbf{w}_n\mathbf{w}_n^T \tag{22}$$

If the coordinate frame origin is located at the center of stiffness, then each vector $\mathbf{r}_i$ in Eq. (21) is the perpendicular position vector from the stiffness center to the axis of wrench $\mathbf{w}_i$. As proved in Ref. [27],

$$k_1\mathbf{r}_1 + k_2\mathbf{r}_2 + \cdots + k_n\mathbf{r}_n = 0 \tag{23}$$

Thus, the center of stiffness is the center of the n spring axes weighted by the corresponding spring rate values k_i. Additional requirements on the distribution of spring locations are identified below.

At the stiffness center, a coordinate frame can be oriented such that the stiffness matrix has diagonal form:

$$\mathbf{K} = \mathrm{diag}(k_x, k_y, k_\tau) \tag{24}$$

where k_x and k_y are the two translational principal stiffnesses and k_τ is the rotational principal stiffness. If we denote

$$r_{\min} = \min\{\|\mathbf{r}_1\|, \ldots, \|\mathbf{r}_n\|\} \tag{25}$$

$$r_{\max} = \max\{\|\mathbf{r}_1\|, \ldots, \|\mathbf{r}_n\|\} \tag{26}$$

then, we have:

Proposition 1. *Suppose a stiffness $\boldsymbol{K}$ is realized by an n-spring parallel mechanism. If r_{min} and r_{max} are the minimum and* *maximum distances from the stiffness center to the spring axes, then*

$$r_{\min} \le \sqrt{\frac{k_\tau}{k_x + k_y}} \le r_{\max} \tag{27}$$

where k_x, k_y, and k_τ are the principal stiffnesses of $\mathbf{K}$.

Proof. Proof. Consider the coordinate frame located at the stiffness center for which the stiffness matrix is diagonal:

$$\mathbf{K} = \mathrm{diag}(k_x, k_y, k_\tau) \tag{28}$$

Each spring wrench described in this frame is

$$\mathbf{w}_i = \begin{bmatrix} \mathbf{n}_i \\ d_i \end{bmatrix} \tag{29}$$

Then,

$$\mathbf{K} = \sum_{i=1}^{n} k_i \mathbf{w}_i \mathbf{w}_i^T = \sum_{i=1}^{n} k_i \begin{bmatrix} \mathbf{n}_i \\ d_i \end{bmatrix} [\mathbf{n}_i^T, d_i]$$
$$= \begin{bmatrix} \sum_{i=1}^n k_i \mathbf{n}_i \mathbf{n}_i^T & \sum_{i=1}^n k_i d_i \mathbf{n}_i \\ \sum_{i=1}^n k_i d_i \mathbf{n}_i^T & \sum_{i=1}^n k_i d_i^2 \end{bmatrix}$$

Since $\mathbf{n}_i$ is a unit vector, $\mathrm{trace}(\mathbf{n}_i \mathbf{n}_i^T) = 1$,

$$k_x + k_y = \mathrm{trace}\left(\sum_{i=1}^{n} k_i \mathbf{n}_i \mathbf{n}_i^T\right) = k_1 + k_2 + \cdots + k_n$$

The value of the (3, 3) entry of $\mathbf{K}$ is

$$k_\tau = k_1 d_1^2 + k_2 d_2^2 + \cdots + k_n d_n^2 \tag{30}$$

Since $d_i^2 = \|\mathbf{r}_i\|^2$,

$$(k_1 + \cdots + k_n) r_{\min}^2 \le k_\tau \le (k_1 + \cdots + k_n) r_{\max}^2$$

which is equivalent to

$$r_{\min}^2 \le \frac{k_\tau}{k_x + k_y} \le r_{\max}^2$$

and leads to

$$r_{\min} \le \sqrt{\frac{k_\tau}{k_x + k_y}} \le r_{\max}$$

■

Thus, to realize a given stiffness with a parallel mechanism, the spring axes in the mechanism must enclose the stiffness center C_k, and some, but not all, spring axes must intersect circle Γ_k of radius

$$r_k = \sqrt{\frac{k_\tau}{k_x + k_y}} \tag{31}$$

centered at the stiffness center C_k (Fig. 5).

For a given elastic behavior, the principal stiffnesses not only restrict the distances between the springs and the stiffness center but also the directions of spring axes. Suppose $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors along the two principal axes. In the principal frame at the stiffness center, let θ_i be the angle between the (perpendicular) position vector $\mathbf{r}_i$ and the principal axis $\mathbf{e}_1$, as shown in Fig. 6(*a*). Then, the direction of $\mathbf{r}_i$ is

$$\mathbf{u}_i = [\cos\theta_i,\ \sin\theta_i]^T \tag{32}$$

Since $\mathbf{r}_i$ is perpendicular to wrench $\mathbf{w}_i$, the unit direction two-vector $\mathbf{n}_i$ in Eq. (29) can be expressed as follows:

$$\mathbf{n}_i = [-\sin\theta_i,\ \cos\theta_i]^T \tag{33}$$

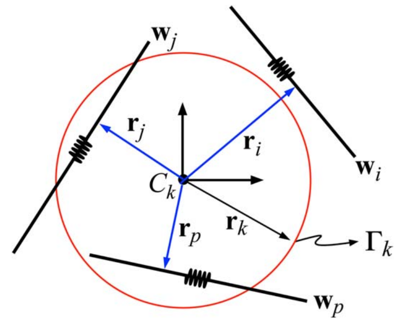

Fig. 5 Spring axes in a parallel mechanism. At least one spring axis must intersect circle Γ_k and at least one spring axis must not intersect Γ_k.

Thus,

$$\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} = \sum k_i \mathbf{n}_i \mathbf{n}_i^T = \sum k_i \begin{bmatrix} -\sin\theta_i \\ \cos\theta_i \end{bmatrix} [-\sin\theta_i,\ \cos\theta_i]$$

which leads to

$$k_x = \sum k_i \sin^2\theta_i \tag{34}$$

Let

$$\sin\theta_{\min} = \min\{|\sin\theta_i|,\ \ i = 1, 2, \ldots, n\} \tag{35}$$

$$\sin\theta_{\max} = \max\{|\sin\theta_i|,\ \ i = 1, 2, \ldots, n\} \tag{36}$$

then,

$$k_x = \sum k_i \sin^2\theta_i \le \left(\sum k_i\right) \sin^2\theta_{\max} = (k_x + k_y)\sin^2\theta_{\max} \tag{37}$$

Therefore,

$$\sin\theta_{\max} \ge \sqrt{\frac{k_x}{k_x + k_y}} \tag{38}$$

Similarly,

$$\sin\theta_{\min} \le \sqrt{\frac{k_x}{k_x + k_y}} \tag{39}$$

Let

$$\theta_x = \sin^{-1}\left(\sqrt{\frac{k_x}{k_x + k_y}}\right) \tag{40}$$

and denote l_θ^- and l_θ^+ to be the two lines passing through stiffness center C_k and having angles $-\theta_x$ and θ_x with respect to the *x*-axis respectively, then, the two lines separate the plane into two areas Λ_x and Λ_y as illustrated in Fig. 6(*b*). The perpendicular vectors $\mathbf{r}_i$ from the stiffness center to the springs cannot be either all in area Λ_x or all in area Λ_y.

Also, the perpendicular position vectors $\mathbf{r}_i$s cannot all be within a half plane defined by a line passing through the compliance center, i.e, the space positively spanned by $\mathbf{r}_i$s must be contain at least a half plane. This can be proved by Eq. (23). In fact, if all $\mathbf{r}_i$s are inside a half plane, then Eq. (23) cannot hold for coefficients $k_i > 0$.

In summary, we have:

Proposition 2. *Suppose a stiffness* $\boldsymbol{K}$ *is realized by an* n-*spring parallel mechanism and* $\boldsymbol{r}_i$ *is the perpendicular vector from the stiffness center to the spring axis of* $\boldsymbol{w}_i$. *Then, the set of* $\boldsymbol{r}_i$ *vectors must*

(i) Be located in both areas Λ_x and Λ_y bounded by lines l_θ^+ and l_θ^- as illustrated in Fig. 6(*b*);

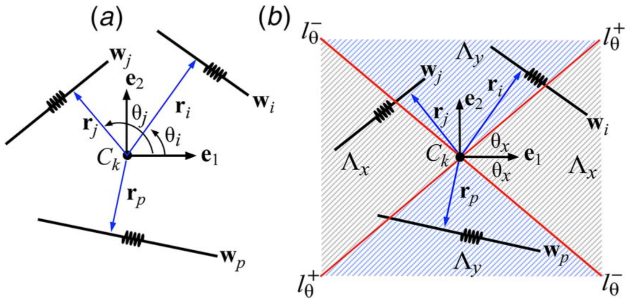

Fig. 6 The restriction on the directions of springs related to the principal stiffnesses: (*a*) angle θ_i between position vector $\mathbf{r}_i$ and the principal axis and (*b*) areas Λ_x and Λ_y defined by lines l_θ^- and l_θ^+. At least one $\mathbf{r}_i$ must be in Λ_x and at least one $\mathbf{r}_i$ must be in Λ_y.

(ii) Span a space of at least a half plane.

Figure 7 illustrates four cases in which the given compliance cannot be achieved by the illustrated parallel mechanism regardless of the values of spring rates.

In the case of Fig. 7(*a*), all spring axes intersect circle Γ_k determined by Eq. (31); therefore, Propositions 1 is not satisfied. The spring locations in the parallel mechanism will not yield sufficient moment about the stiffness center required by the stiffness matrix. Thus, the behavior cannot be achieved by the mechanism regardless the values of the spring constants.

In the cases of Figs. 7(*b*) and 7(*c*), Proposition 1 is satisfied in both cases. In the case of Fig. 7(*b*), all perpendicular position vectors of springs are located in the area Λ_y, therefore, condition (i) of Proposition 2 is not satisfied. The springs in this case will yield excessive force in the *x*-direction relative to the *y*-direction regardless of the spring rate values. Similarly, in the case of Fig. 7(*c*), all spring position vectors are located in the area Λ_x. The springs in this case will yield excessive force in the *y*-direction relative to the *x*-direction regardless of the spring rate values.

In the case of Fig. 7(*d*), Proposition 1 and condition (i) of Proposition 2 are both satisfied, however, the spring position vectors ($\mathbf{r}_i$, $\mathbf{r}_j$, $\mathbf{r}_p$) do not span more than a half plane, therefore, condition (ii) of Proposition 2 is not satisfied. Thus the compliant behavior cannot be realized by the mechanism.

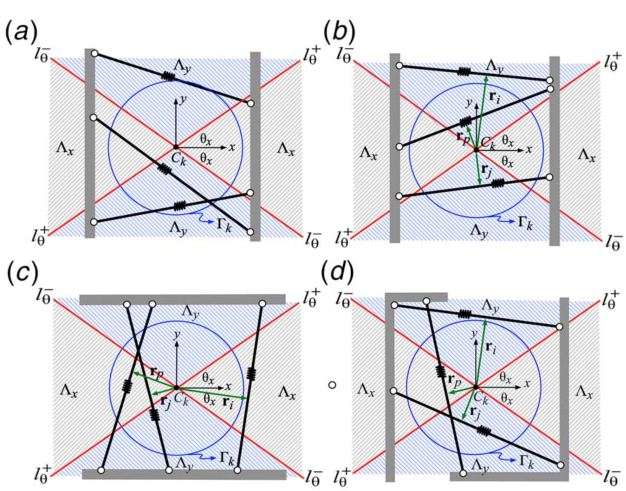

Fig. 7 Cases in which a given stiffness cannot be achieved by the parallel mechanisms: (*a*) all spring axes intersect circle Γ_k, (*b*) all position vectors $\mathbf{r}_i$ from the stiffness center to the spring axes are located in area Λ_y, (*c*) all vectors $\mathbf{r}_i$ are located in area Λ_x, and (*d*) the set of $\mathbf{r}_i$s do not span more than a half plane

Note that the conditions in Propositions 1 and 2 are only *necessary* conditions on the distribution of spring locations of a parallel mechanism. To realize a given elastic behavior, additional conditions are required [1–4].

3.2 Realization Conditions for Six-Spring Mechanisms.

In previous work [4], sets of realization conditions for six-spring parallel mechanisms to achieve an arbitrary compliance were presented. Since the realization of a stiffness matrix with six parallel springs is essential for the synthesis of two 3-joint serial mechanisms, a more physically intuitive set of conditions for the realization of a stiffness with a six-spring parallel mechanism is developed below.

If a given stiffness matrix $\mathbf{K}$ is realized by a six-spring parallel mechanism with spring wrenches $\mathbf{w}_i$ and spring rates k_i, then,

$$\mathbf{K} = k_1\mathbf{w}_1\mathbf{w}_1^T + k_2\mathbf{w}_2\mathbf{w}_2^T + \cdots + k_6\mathbf{w}_6\mathbf{w}_6^T \tag{41}$$

where each $k_i \geq 0$.

For any given mechanism geometry, the coefficients k_is in Eq. (41) can be uniquely determined using the following procedure.

For an arbitrary 3×3 symmetric matrix $\mathbf{A}$ having entries a_{ij}, denote $\hat{\mathbf{a}}$ as the six-vector of the six independent entries of $\mathbf{A}$:

$$\hat{\mathbf{a}} = [a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}]^T \tag{42}$$

With this representation, symmetric matrices $\mathbf{K}$ and $\mathbf{w}_i\mathbf{w}_i^T$ can be described as six-vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{w}}_i$. Denote

$$\mathbf{k} = [k_1, k_2, \ldots, k_6]^T, \quad \hat{\mathbf{W}} = [\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \ldots, \hat{\mathbf{w}}_6] \in \mathbb{R}^{6\times 6} \tag{43}$$

Then, Eq. (41) can be equivalently expressed as follows:

$$\hat{\mathbf{k}} = \hat{\mathbf{W}}\mathbf{k} \tag{44}$$

For the generic case in which $\hat{\mathbf{W}}$ is full rank, the spring rate vector $\mathbf{k}$ can be determined by

$$\mathbf{k} = \hat{\mathbf{W}}^{-1}\hat{\mathbf{k}} \tag{45}$$

Thus, for any given symmetric matrix $\mathbf{K}$ and any set of six spring wrenches $\mathbf{w}_i$, $\mathbf{K}$ can be expressed in the form of Eq. (41) and the coefficients k_i are uniquely determined by Eq. (45) if $\hat{\mathbf{W}}$ is full rank.

The coefficients k_is from Eq. (45), however, are not guaranteed to be non-negative, a requirement for passive realization. Thus, $\mathbf{K}$ can be passively realized by the six springs if and only if

$$\hat{\mathbf{W}}^{-1}\hat{\mathbf{k}} \geq \mathbf{0} \tag{46}$$

Condition (46) imposes six inequalities on the six spring wrenches $\mathbf{w}_i$s. These inequalities cannot be directly used in the synthesis of a mechanism for the realization of a given stiffness because the physical meaning of these inequalities is lost in the matrix inverse operation.

In the synthesis procedure presented in Sec. 4, a set of six spring wrenches is first selected based on two geometric considerations that ensure a physically realizable solution is obtained. One ensures that $\hat{\mathbf{W}}$ is full rank, and the other ensures that the stiffness coefficients calculated by Eq. (45) are positive. After the wrench locations are selected, the corresponding spring rates are determined using Eq. (45).

3.2.1 *Solution Existence and Uniqueness.*

Equation (45) indicates that when the six spring wrenches $\mathbf{w}_i$ are determined and the corresponding set of six-vectors $\hat{\mathbf{w}}_i$ are linearly independent, each coefficient k_i can be uniquely determined for any given $\mathbf{K}$. If, however, the six 6-vectors $\hat{\mathbf{w}}_i s$ are not linearly independent, $\hat{\mathbf{W}}$ is not full rank, and an arbitrary stiffness matrix $\mathbf{K}$ cannot be expressed in the form of Eq. (41).

The linear independence of the set of six-vectors $\hat{\mathbf{w}}_i$ ($i = 1, 2, \ldots, 6$) is equivalent to the linear independence of the six rank-1 matrices $\mathbf{w}_i\mathbf{w}_i^T$ in $\mathbb{R}^{3\times 3}$. Below, a geometric necessary and sufficient condition for the set of $\mathbf{w}_i\mathbf{w}_i^T s$ to be linearly *dependent* is identified.

If the $\mathbf{w}_i\mathbf{w}_i^T s$ are dependent, then one matrix $\mathbf{w}_i\mathbf{w}_i^T$ can be expressed as a linear combination of the other five matrices. Here, we suppose, without loss of generality, that

$$\mathbf{w}_6\mathbf{w}_6^T = \alpha_1\mathbf{w}_1\mathbf{w}_1^T + \alpha_2\mathbf{w}_2\mathbf{w}_2^T + \cdots + \alpha_5\mathbf{w}_5\mathbf{w}_5^T \tag{47}$$

Consider the intersection of two spring wrenches $\mathbf{w}_i$ and $\mathbf{w}_j$ represented by twist $\mathbf{t}_{ij}$. Then,

$$\mathbf{t}_{ij}^T\mathbf{w}_i = 0, \quad \mathbf{t}_{ij}^T\mathbf{w}_j = 0 \tag{48}$$

Multiplying Eq. (47) with $\mathbf{t}_{12}^T$ from the left and $\mathbf{t}_{34}$ from the right yields

$$(\mathbf{t}_{12}^T\mathbf{w}_6)(\mathbf{w}_6^T\mathbf{t}_{34}) = \alpha_5(\mathbf{t}_{12}^T\mathbf{w}_5)(\mathbf{w}_5^T\mathbf{t}_{34}) \tag{49}$$

Similarly,

$$(\mathbf{t}_{13}^T\mathbf{w}_6)(\mathbf{w}_6^T\mathbf{t}_{24}) = \alpha_5(\mathbf{t}_{13}^T\mathbf{w}_5)(\mathbf{w}_5^T\mathbf{t}_{24}) \tag{50}$$

Equating α_5 in Eqs. (49) and (50) yields

$$(\mathbf{t}_{12}^T\mathbf{w}_5)(\mathbf{t}_{34}^T\mathbf{w}_5)(\mathbf{t}_{13}^T\mathbf{w}_6)(\mathbf{t}_{24}^T\mathbf{w}_6) = (\mathbf{t}_{12}^T\mathbf{w}_6)(\mathbf{t}_{34}^T\mathbf{w}_6)(\mathbf{t}_{13}^T\mathbf{w}_5)(\mathbf{t}_{24}^T\mathbf{w}_5) \tag{51}$$

Conversely, it can be proved that if Eq. (51) is satisfied, $\mathbf{w}_i\mathbf{w}_i^T s$ must be linearly dependent (proved in the Appendix). Thus, Condition (51) is the necessary and sufficient condition for the linear dependence of the six rank-1 matrices $\mathbf{w}_i\mathbf{w}_i^T s$. The necessary and sufficient condition for Eq. (44) to be solvable then is that Eq. (51) is not satisfied.

To obtain the geometric significance of condition (51), consider the mapping from the wrench space $\mathbb{W}$ into the real field $\mathbb{R}$ defined as follows:

$$\begin{aligned} F_6(\mathbf{w}) = &(\mathbf{t}_{12}^T\mathbf{w})(\mathbf{t}_{34}^T\mathbf{w})(\mathbf{t}_{13}^T\mathbf{w}_5)(\mathbf{t}_{24}^T\mathbf{w}_5) \\ &- (\mathbf{t}_{12}^T\mathbf{w}_5)(\mathbf{t}_{34}^T\mathbf{w}_5)(\mathbf{t}_{13}^T\mathbf{w})(\mathbf{t}_{24}^T\mathbf{w}) \end{aligned} \tag{52}$$

where $\mathbf{w} \in \mathbb{W}$ is an arbitrary wrench. For any given five wrenches $\mathbf{w}_i$ $(i = 1, 2, \ldots, 5)$, Eq. (52) defines a function $F_6(\mathbf{w}) : \mathbb{W} \to \mathbb{R}$.

Let $\mathbf{B}_6$ be the 3×3 matrix defined as

$$\mathbf{B}_6 = (\mathbf{t}_{13}^T\mathbf{w}_5)(\mathbf{t}_{24}^T\mathbf{w}_5)\mathbf{t}_{12}\mathbf{t}_{34}^T - (\mathbf{t}_{12}^T\mathbf{w}_5)(\mathbf{t}_{34}^T\mathbf{w}_5)\mathbf{t}_{13}\mathbf{t}_{24}^T \tag{53}$$

and the corresponding symmetric matrix is

$$\mathbf{M}_6 = \mathbf{B}_6 + \mathbf{B}_6^T \tag{54}$$

Then, the function defined in Eq. (52) can be expressed in quadratic form:

$$F_6(\mathbf{w}) = \mathbf{w}^T\mathbf{M}_6\mathbf{w} \tag{55}$$

It can be seen that

$$F_6(\mathbf{w}_i) = 0, \quad \text{for } i = 1, 2, 3, 4, 5 \tag{56}$$

and that $\mathbf{w}_6\mathbf{w}_6^T$ can be expressed as a linear combination of the other five matrices $\mathbf{w}_1\mathbf{w}_1^T, \mathbf{w}_2\mathbf{w}_2^T, \ldots, \mathbf{w}_5\mathbf{w}_5^T$ if and only if

$$F_6(\mathbf{w}_6) = 0 \tag{57}$$

Consider the mapping from twist space $\mathbb{T}$ to wrench space $\mathbb{W}$ defined as follows:

$$\mathbf{w} = \mathbf{M}_6^{-1}\mathbf{t} \tag{58}$$

Then,

$$\mathbf{w}^T\mathbf{M}_6\mathbf{w} = 0 \Longleftrightarrow \mathbf{t}^T\mathbf{M}_6^{-1}\mathbf{t} = 0 \tag{59}$$

If an arbitrary unit twist $\mathbf{t}$ defined in the xy-plane is

$$\mathbf{t} = [y, -x, 1]^T \tag{60}$$

then,

$$f_6(x, y) = \mathbf{t}^T\mathbf{M}_6^{-1}\mathbf{t} = 0 \tag{61}$$

defines a quadratic curve in the xy-plane. Then, $\mathbf{w}_6$ satisfies Eq. (57) if and only if it is tangent to the quadratic curve $f_6 = 0$ defined by Eq. (61). Equation (44) is unsolvable if and only if the sixth spring wrench $\mathbf{w}_6$ is tangent to the quadratic curve determined by the other five spring wrenches. Thus, when the first five spring wrenches are determined, the sixth spring must be selected such that it either crosses the quadratic curve $f_6 = 0$ or does not meet the curve.

Since a quadratic curve tangent to five given non-concurrent lines is unique and the spring wrenches can be numbered arbitrarily, the six rank-1 matrices $\mathbf{w}_i\mathbf{w}_i^T$ $(i = 1, 2, \ldots, 6)$ are linearly dependent if and only if the six spring axes are all tangent to a single quadratic curve. Therefore, Eq. (44) is solvable for an arbitrary stiffness if and only if the six spring axes are not all tangent to *any* quadratic curve (Fig. 8).

In a six-spring parallel mechanism, for any spring wrench $\mathbf{w}_s$, the quadratic curve f_s tangent to the other five spring wrenches can be determined as follows:

$$\mathbf{B}_s = (\mathbf{t}_{ip}^T\mathbf{w}_r)(\mathbf{t}_{jq}^T\mathbf{w}_r)\mathbf{t}_{ij}\mathbf{t}_{pq}^T - (\mathbf{t}_{ij}^T\mathbf{w}_r)(\mathbf{t}_{pq}^T\mathbf{w}_r)\mathbf{t}_{ip}\mathbf{t}_{jq}^T \tag{62}$$

where (i, j, p, q, r) is an arbitrary permutation of the set $\{1, 2, 3, 4, 5, 6\}$ excluding s. The corresponding symmetric matrix is

$$\mathbf{M}_s = \mathbf{B}_s + \mathbf{B}_s^T \tag{63}$$

The quadratic curve is determined by the equation:

$$f_s = \mathbf{t}^T\mathbf{M}_s^{-1}\mathbf{t} = 0 \tag{64}$$

where $\mathbf{t}$ is the unit twist defined in Eq. (60).

Note that the tangent condition for six springs only ensures the existence and uniqueness of the solution to Eq. (44); it does not ensure that each k_i obtained by Eq. (45) is non-negative. For passive realization, additional conditions to ensure a non-negative solution $k_i \geq 0$ are needed.

3.2.2 The Coefficient Signs and Spring Geometry. As shown in Ref. [4], for any PSD matrix $\mathbf{K}$ and a set of six springs $\{\mathbf{w}_i, i = 1, 2, \ldots, 6\}$, if $\mathbf{K}$ is expressed in the form of Eq. (41), the number of coefficients k_is that are negative cannot exceed 3. Thus, if all six k_is have the same sign, they must all be positive. Below, the relation between the sign of k_i and the spring geometry is identified.

For a given stiffness $\mathbf{K}$ and a six-spring parallel mechanism, any four spring wrenches $(\mathbf{w}_i, \mathbf{w}_j, \mathbf{w}_p, \mathbf{w}_q)$ define a quadratic curve as follows.

Consider the 3×3 matrix $\mathbf{G}_{ijpq}$ defined as follows:

$$\mathbf{G}_{ijpq} = (\mathbf{t}_{ij}^T\mathbf{K}\mathbf{t}_{pq})(\mathbf{t}_{ip}\mathbf{t}_{jq}^T) - (\mathbf{t}_{ip}^T\mathbf{K}\mathbf{t}_{jq})(\mathbf{t}_{ij}\mathbf{t}_{pq}^T) \tag{65}$$

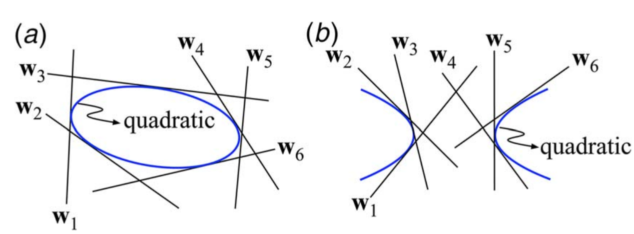

Fig. 8 The six rank-1 matrices $\mathbf{w}_i\mathbf{w}_i^T$ ($i = 1, 2, \ldots, 6$) are linearly dependent if and only if the six spring axes are all tangent to a quadratic curve: (*a*) a quadratic curve with one branch and (*b*) a quadratic curve with two branches

where $\mathbf{t}_{ij}$ is the unit twist centered at the intersection of wrenches $\mathbf{w}_i$ and $\mathbf{w}_j$. The symmetric matrix associated with $\mathbf{G}_{ijpq}$ is follows:

$$\mathbf{H}_{ijpq} = \mathbf{G}_{ijpq} + \mathbf{G}_{ijpq}^T \tag{66}$$

For an arbitrary unit twist $\mathbf{t}$ defined in Eq. (60), the equation

$$h_{ijpq}(x, y) = \mathbf{t}^T \mathbf{H}_{ijpq}^{-1} \mathbf{t} = 0 \tag{67}$$

defines a quadratic curve in the plane. This curve depends on the given stiffness matrix $\mathbf{K}$ and, as proved in Ref [4], is tangent to the four spring wrenches ($\mathbf{w}_i$, $\mathbf{w}_j$, $\mathbf{w}_p$, $\mathbf{w}_q$).

Note the quadratic curve h_{ijpq} may have a single branch (ellipse or parabola) or two branches (hyperbola). As proved in Ref. [4], two coefficients k_r and k_s have the same sign if and only if, of the two wrenches $\mathbf{w}_r$ and $\mathbf{w}_s$, only one intersects a single branch of the quadratic curve h_{ijpq} determined by the other four spring wrenches defined in Eq. (67).

Consider the selection of the sixth spring location when the other five have already been selected. In order to obtain the coefficient k_6 associated with $\mathbf{w}_6$, multiplying Eq. (41) by $\mathbf{t}_{12}^T$ and $\mathbf{t}_{34}$ (from the left and right, respectively) yields

$$\mathbf{t}_{12}^T \mathbf{K} \mathbf{w}_{34} = (\mathbf{t}_{12}^T \mathbf{w}_5)(\mathbf{w}_5^T \mathbf{t}_{34}) k_5 + (\mathbf{t}_{12}^T \mathbf{w}_6)(\mathbf{w}_6^T \mathbf{t}_{34}) k_6 \tag{68}$$

Similarly,

$$\mathbf{t}_{13}^T \mathbf{K} \mathbf{t}_{24} = (\mathbf{t}_{13}^T \mathbf{w}_5)(\mathbf{w}_5^T \mathbf{t}_{24}) k_5 + (\mathbf{t}_{13}^T \mathbf{w}_6)(\mathbf{w}_6^T \mathbf{t}_{24}) k_6 \tag{69}$$

Solving Eqs. (68)–(69) for k_6 yields

$$k_6 = \frac{(\mathbf{t}_{13}^T \mathbf{w}_5)(\mathbf{t}_{24}^T \mathbf{w}_5)(\mathbf{t}_{12}^T \mathbf{K} \mathbf{t}_{34}) - (\mathbf{t}_{12}^T \mathbf{w}_5)(\mathbf{t}_{34}^T \mathbf{w}_5)(\mathbf{t}_{13}^T \mathbf{K} \mathbf{t}_{24})}{F_6(\mathbf{w}_6)} \tag{70}$$

where the denominator $F_6(\mathbf{w})$ is the function defined in Eq. (52).

Since the five spring wrenches ($\mathbf{w}_1$, $\mathbf{w}_2$, …, $\mathbf{w}_5$) are already selected, the numerator of Eq. (70) is constant and the sign of k_6 only depends on the denominator $F_6(\mathbf{w}_6)$. To ensure the existence of a solution, $\mathbf{w}_6$ must either (i) intersect the quadratic curve $f_6 = 0$ or (ii) have no intersection with the curve. The coefficient k_6 changes sign if and only if $\mathbf{w}_6$ changes its intersection case (i) to case (ii) or vice versa.

Figure 9 illustrates the relations between the signs of k_is and the curves h_{1234} and f_6. Coefficients k_5 and k_6 have the same sign if and only if either only $\mathbf{w}_5$ or only $\mathbf{w}_6$ intersects a single branch of h_{1234} (Fig. 9(*a*)). If the first five wrenches are selected, k_6 changes sign if and only if $\mathbf{w}_6$ moves from a location where it intersects curve f_6 to a location $\mathbf{w}_6'$ where it does not intersect the curve or vice versa (Fig. 9(*b*)).

In summary, for a stiffness matrix $\mathbf{K}$ and a six-spring parallel mechanism with spring wrenches $\mathbf{w}_i$ ($i = 1, 2, …, 6$), if $\mathbf{K}$ is expressed in the form of Eq. (41), then,

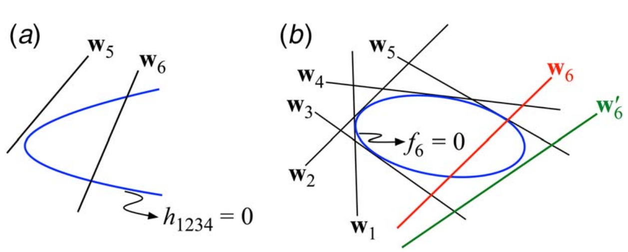

Fig. 9 The sign of k_i and the quadratic curves: (*a*) k_5 and k_6 have the same sign if and only if either only $\mathbf{w}_5$ or only $\mathbf{w}_6$ intersects curve h_{1234} determined by the other four springs and (*b*) if the first five spring are selected, k_6 changes sign if and only if $\mathbf{w}_6$ varies from a location of intersecting curve f_6 to a location $\mathbf{w}_6'$ not intersecting the curve or vice versa

(1) Any two coefficients k_r and k_s have the same sign if and only if either only $\mathbf{w}_r$ or only $\mathbf{w}_s$ intersects the curve of Eq. (67) determined by the other 4 wrenches;
(2) When $\mathbf{w}_s$ varies in the plane while all the other five springs are constant, the corresponding k_s does not change its sign if and only if $\mathbf{w}_s$ maintains its intersection property with the quadratic curve f_s of Eq. (64) tangent to the other five spring wrenches.

3.2.3 Realization Conditions. The linear independence condition on the distribution of springs described in Sec. 3.2.1 ensures the solution existence and uniqueness. The separation conditions on any two spring axes described in Sec. 3.2.2 ensure the solution is positive valued. In summary, we have:

Proposition 3. *Consider a parallel mechanism described by six spring wrenches $\mathbf{w}_i$. A stiffness matrix $\mathbf{K}$ can be realized by the mechanism if and only if the following two conditions are satisfied*:

(i) All six spring axes are not tangent to any quadratic curve;
(ii) For any combination of two springs, only one spring axis crosses a single branch of the quadratic curve of Eq. (67) determined by the other four springs.

Note that, in a six-spring parallel mechanism, there are 15 combinations of two springs ($\mathbf{w}_i$, $\mathbf{w}_j$). In a synthesis procedure, however, it is not necessary to check all combinations. If the combinations of one spring with each of the others ensure that these k_is have the same sign, then the six spring constants must be all positive (since $\mathbf{K}$ is positive definite). For example, if each of the five springs paired with the sixth spring (e.g., ($\mathbf{w}_i$, $\mathbf{w}_6$), $i = 1, 2, 3, 4, 5$) satisfies Condition (ii) of Proposition 3, then all k_i must be positive.

4 Stiffness Synthesis for Two 3-Joint Serial Mechanisms

In this section, a synthesis procedure for the realization of an arbitrary stiffness matrix $\mathbf{K}$ with two 3-joint serial mechanisms is developed. First, a new geometry-based procedure for the synthesis of a six-spring parallel mechanism is presented. Then, using the concept of dual elastic mechanisms, the obtained six-spring parallel mechanism is converted into two 3-joint serial mechanisms.

4.1 Synthesis With a Six-Spring Parallel Mechanism. In Ref. [4], a synthesis procedure for six-spring parallel mechanisms was presented. In the process of Ref. [4], several *sufficient* conditions were used to ensure that the spring rates k_i are positive. Here, a new six-spring mechanism synthesis procedure that uses only the necessary and sufficient conditions of Proposition 3 is presented.

A planar stiffness matrix $\mathbf{K}$ can be partitioned as follows:

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & k_{33} \end{bmatrix}$$

for which the location of the center of stiffness is given by

$$\mathbf{r}_c = -\mathbf{\Omega} \mathbf{A}^{-1} \mathbf{b} \tag{71}$$

where $\mathbf{\Omega}$ is the 2×2 matrix defined in Eq. (8). The unit twist $\mathbf{t}_c$ associated with the stiffness center C_k is calculated using Eqs. (6) and (9).

The steps for spring selections are outlined below and illustrated in Fig. 10.

(1) Select four spring axes ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$) with the guidance provided below and obtain the quadratic curve associated with these four springs.
  (i) Since the stiffness center must be surrounded by the spring axes, wrench locations relative to the stiffness center should be considered. The four spring locations should be selected such that: (1) the conditions of

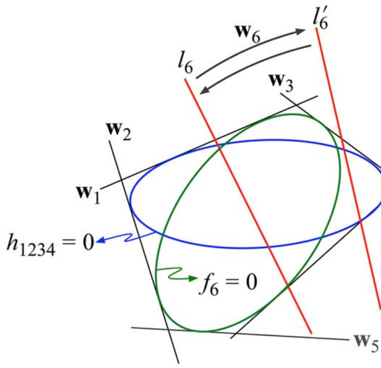

Fig. 10 Stiffness synthesis with a six-spring parallel mechanism. Selection of $\mathbf{w}_5$ and $\mathbf{w}_6$: $\mathbf{w}_6$ intersects curve h_{1234} while $\mathbf{w}_5$ does not.

Propositions 1 and 2 are easily satisfied and (2) knowledge that the intersection of all six wrenches will eventually correspond to the joint locations of two serial mechanisms.

(ii) For the selected four spring wrenches $\mathbf{w}_i$ ($i=1, 2, 3, 4$), the corresponding quadratic curve $h_{1234}=0$ is obtained using Eq. (67).

(2) Select $\mathbf{w}_5$ and $\mathbf{w}_6$ such that k_5 and k_6 are positive.

(i) Choose $\mathbf{w}_5$ such that it does not intersect curve $h_{1234}=0$; choose $\mathbf{w}_6$ such that it only intersects a single branch of the curve. These selections ensure that coefficients k_5 and k_6 have the same sign. Evaluate the value of k_6 using Eq. (70). If $k_6>0$, then both k_5 and k_6 are positive. Go to Step 3.

(ii) If $k_6<0$, obtain the quadratic curve f_6 associated with the five spring wrenches ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$) using Eq. (61). Then, move $\mathbf{w}_6$ such that it changes the intersection relation with curve f_6. For example, if $\mathbf{w}_6$ intersects f_6 at location l_6 as shown in Fig. 10, then move it to a location that does not meet the curve such as location l_6'. In this process, the geometric condition for $\mathbf{w}_5$ and $\mathbf{w}_6$ with respect to curve h_{1234} described in (i) should be maintained. In satisfying these conditions, both k_5 and k_6 are positive.

(3) Adjust the spring locations to ensure that all coefficients k_is are positive. For the obtained six spring wrenches, calculate the six coefficients k_i using Eq. (45).

(i) If one coefficient is negative, obtain the quadratic curve determined by four of the other five spring wrenches and then move a corresponding spring wrench based on the curve. For example, if $k_1<0$, obtain curve $h_{2345}=0$ determined by ($\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$); then move $\mathbf{w}_6$ such that condition (ii) of Proposition 3 is satisfied. As such, $k_1>0$.

(ii) If two coefficients are negative, then use the process described in Step 2(ii) for $\mathbf{w}_5$ and $\mathbf{w}_6$ to make the two coefficients positive. For example, if k_1 and k_2 are negative, then move either $\mathbf{w}_1$ or $\mathbf{w}_2$ relative to f_1 or f_2 defined in Eq. (64) to change its intersection property with the curve. In this process, the intersection property of $\mathbf{w}_1$ and $\mathbf{w}_2$ with curve h_{3456} should be maintained. As such, both k_1 and k_2 are positive.

Since the number of negative coefficients does not exceed 3, at most two adjustment iterations are needed.

With the final step, the six spring wrenches $\mathbf{w}_i$ and the corresponding stiffnesses k_i are identified.

4.2 Conversion to Two 3-Joint Serial Mechanisms.

Using the concept of dual elastic mechanisms, the six-spring parallel mechanism obtained in Sec. 4.1 is converted into two serial mechanisms.

(1) Separate the six springs into two groups ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$) and ($\mathbf{w}_4$, $\mathbf{w}_5$, $\mathbf{w}_6$).

The separation is not unique. Different groupings will yield different serial mechanisms. One can separate the springs based on the desired serial mechanism geometry and constraints.

(2) Obtain the dual serial mechanism for each group.

(i) For ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$), the three joint locations (J_1, J_2, J_3) are determined by the twists:

$$\tilde{\mathbf{t}}_1=\mathbf{w}_2\times\mathbf{w}_3,\quad \tilde{\mathbf{t}}_2=\mathbf{w}_1\times\mathbf{w}_3,\quad \tilde{\mathbf{t}}_3=\mathbf{w}_1\times\mathbf{w}_2$$

The corresponding joint compliances are calculated using Eq. (19).

(ii) Similarly, for ($\mathbf{w}_4$, $\mathbf{w}_5$, $\mathbf{w}_6$), the joint locations and corresponding joint compliances are obtained.

With this final step, two serial mechanisms that realize the given stiffness $\mathbf{K}$ are obtained.

5 Example

In this section, a numerical example is provided to illustrate the synthesis procedures. In a global frame, the compliance matrix to be realized is

$$\mathbf{K}=\begin{bmatrix} 3 & -2 & -8\\ -2 & 6 & 11\\ -8 & 11 & 30\end{bmatrix}$$

In the realization of $\mathbf{K}$, it is desired that the obtained two 3-joint serial mechanisms are below the center of the stiffness.

Using Eq. (71), the stiffness center is calculated to be

$$\mathbf{r}_c=[1.2143,\ 1.8571]^T$$

The unit twist at the center is

$$\mathbf{t}_c=[1.8571,\ -1.2143,\ 1]^T$$

The two principal axes and the corresponding two translational principal stiffnesses are as follows:

$$[\mathbf{e}_1,\ \mathbf{e}_2]=\begin{bmatrix} 0.8944 & -0.4472\\ 0.4472 & 0.8944\end{bmatrix},\quad [k_x,\ k_y]=[2,\ 7]$$

and the rotational principal stiffness is $k_\tau=1.7857$. Using Eqs. (31) and (40), the radii of circle Γ_k and angle θ_x are calculated to be

$$\Gamma_k: r_k=0.4725,\quad \theta_x=28.1255\ \text{deg}$$

The stiffness center C_k, the circle Γ_k, the principal axes ($\mathbf{e}_1$, $\mathbf{e}_2$), and the two lines l_θ^- and l_θ^+ for this stiffness matrix are illustrated in Fig. 11.

Following the procedure described in Sec. 4, a six-spring parallel mechanism that realizes the given behavior is first obtained. Then, the parallel mechanism is converted into two 3-joint serial mechanisms.

5.1 Synthesis for a Six-Spring Parallel Mechanism.

With the steps described in Sec. 4.1, a six-spring parallel mechanism is synthesized.

(1) *Select four spring axes ($\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$, $\boldsymbol{w}_4$).*

The spring axis selections are based on the conditions of Propositions 1 and 2 and illustrated in Fig. 10. Here, $\mathbf{w}_1$ and $\mathbf{w}_3$ are selected not to meet circle Γ_k having angles −45 deg and 60 deg with the x-axis, respectively; $\mathbf{w}_2$ and $\mathbf{w}_4$ are selected to meet circle Γ_k having angles 75 deg and −45 deg with the

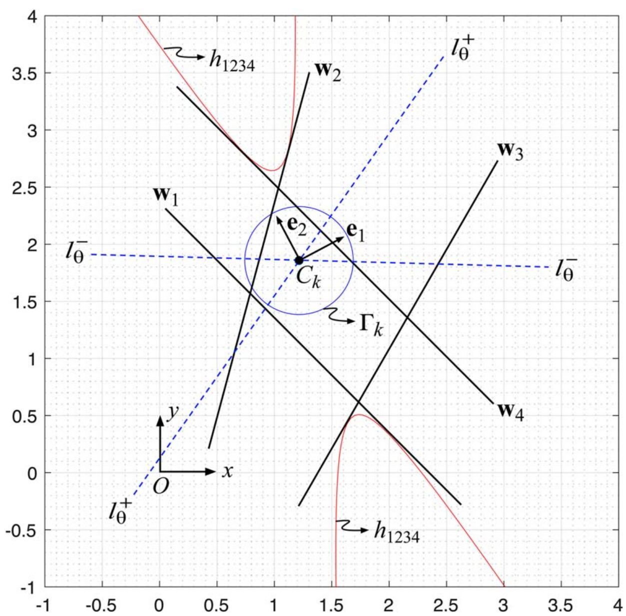

Fig. 11 Selection of the first four spring axes. The four spring axes are selected to satisfy the conditions of Propositions 1 and 2. The quadratic curve h_{1234} tangent to the 4 springs is a hyperbola.

x-axis, respectively. The four spring wrenches are as follows:

$$[\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4] = \begin{bmatrix} 0.7071 & 0.2588 & 0.5000 & 0.7071 \\ -0.7071 & 0.9659 & 0.8660 & -0.7071 \\ -1.6500 & 0.3500 & 1.2000 & -2.5000 \end{bmatrix}$$

Using Eq. (67), the quadratic curve $h_{1234}=0$ associated with the four spring wrenches is obtained:

$$19.1154x^2 + 13.5743xy + 0.5215y^2 - 73.3346x - 20.0868y + 67.7967 = 0$$

which corresponds to the hyperbola illustrated in Fig. 11.

(2) *Select the remaining two spring axes $\mathbf{w}_5$ and $\mathbf{w}_6$.*
To ensure the corresponding spring constants k_5 and k_6 have the same sign, only one of the two spring axes must intersect one branch of h_{1234}. Here, $\mathbf{w}_5$ is selected to be

$$\mathbf{w}_5 = [0.8660, 0.5, \ -0.3]^T$$

which does not intersect curve h_{1234} as shown in Fig. 12. The axis of the sixth spring is first selected to be vertical and pass through $x=2.5$, i.e.,

$$\mathbf{w}_6' = [0, 1, 2.5]^T$$

which intersects curve h_{1234}. For the currently selected spring wrenches ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$, $\mathbf{w}_6'$), using Eq. (67), the spring constant k_6 is calculated to be

$$k_6 = -1.0418$$

which indicates that both k_5 and k_6 are negative.
To change the sign of k_5 and k_6, the quadratic curve f_6 tangent to the five spring wrenches ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$) is the hyperbola obtained using Eq. (61) and is illustrated in Fig. 12. It can be seen that $\mathbf{w}_6'$ intersects f_6. Move $\mathbf{w}_6'$ to a new location $\mathbf{w}_6''$ such that it does not meet curve f_6, then the corresponding coefficients k_5 and k_6 are both positive. Here, $\mathbf{w}_6''$ is selected to be

$$\mathbf{w}_6'' = [0, 1, 2]^T$$

(3) *Make adjustment to ensure all positive coefficients k_is.*
For the six spring wrenches ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$, $\mathbf{w}_6''$), calculate the coefficients k_i using Eq. (45),

$$\mathbf{k} = [2.2606, 2.2732, \ -0.5905, 2.8649, 0.5773, 1.6149]^T$$

Since only k_3 is negative, the location of any one spring relative to $\mathbf{w}_3$ needs to be adjusted. Here, consider the pair of ($\mathbf{w}_3$, $\mathbf{w}_6''$) and the quadratic curve h_{1245} associated with ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_4$, $\mathbf{w}_5$) obtained using Eq. (67). As shown in Fig. 12, h_{1245} is an ellipse tangent to the four wrench axes and not intersected by $\mathbf{w}_3$ or $\mathbf{w}_6''$. To ensure that k_3 and k_6 are both positive, either $\mathbf{w}_3$ or $\mathbf{w}_6''$ must be moved to intersect the ellipse. Here, $\mathbf{w}_6''$ is translated further to the right to location $\mathbf{w}_6$ to cross the ellipse. The spring wrench $\mathbf{w}_6$ selected is

$$\mathbf{w}_6 = [0, 1, 1.8]^T$$

With the six selected spring wrenches ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$, $\mathbf{w}_6$), the corresponding spring constants calculated using Eq. (45) are as follows:

$$\mathbf{k} = [2.1395, 2.2776, 0.1293, 3.2507, 0.1605, 1.0428]^T$$

Thus, a six-spring parallel mechanism that passively realizes the given $\mathbf{K}$ is obtained.

5.2 Conversion to Two 3-Joint Serial Mechanisms.

With the steps described in Sec. 4.2, the obtained six-spring parallel mechanism is transformed into two 3-joint serial mechanisms.

(1) *Separate the six springs into two groups.*
Since it is desired that each joint in the serial mechanisms be below the stiffness center (below line l_c as illustrated in Fig. 13), the two groups separated are ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_5$) and ($\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_6$).

(2) *Obtain the dual elastic serial mechanism for each three-spring group.*

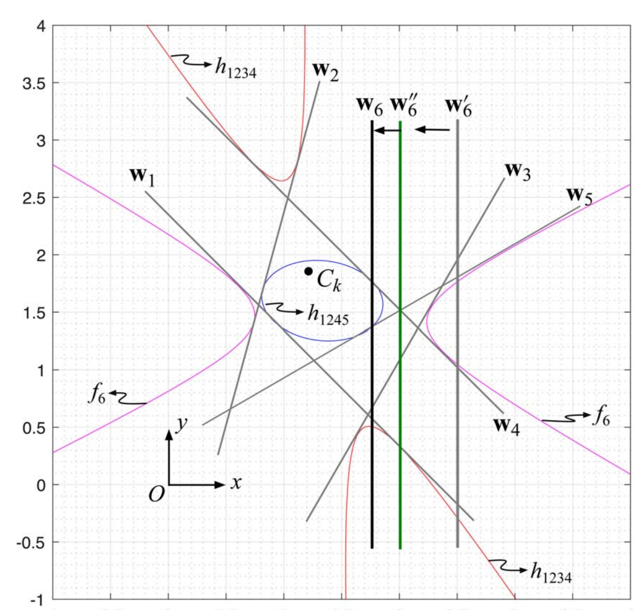

Fig. 12 Selection of the fifth and sixth springs. The sixth spring is first moved from $\mathbf{w}_6''$ to $\mathbf{w}_6$ (changing the intersection property with curve f_6) to make k_6 positive. Then, it is moved for $\mathbf{w}_6$ to intersect curve h_{1245} to make $k_3$positive.

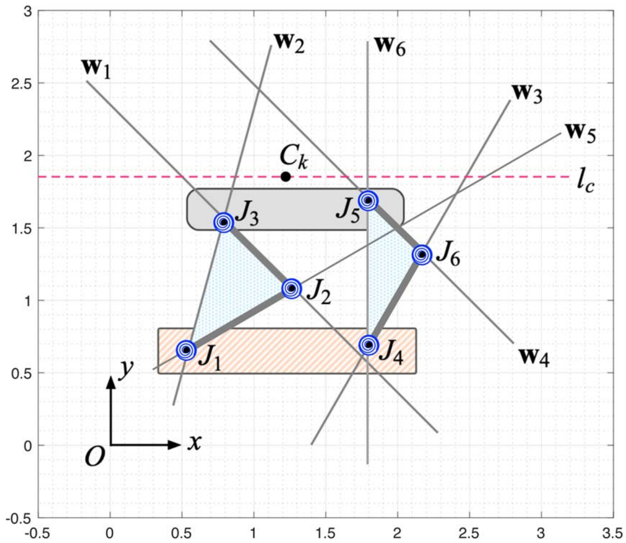

Fig. 13 Conversion of the six-spring parallel mechanism into two 3-joint serial mechanisms

For the three-spring system ($\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_5$), the three joints of the dual serial mechanism are located at the three vertexes of the triangle formed by the three spring axes as shown in Fig. 13. The joint twists ($\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$) at the joints J_1, J_2 and J_3 are calculated using Eq. (15) to be

$$[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3] = \begin{bmatrix} 0.6573 & 1.0738 & 1.5548 \\ -0.5385 & -1.2598 & -0.7789 \\ 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

and the corresponding joint compliances calculated using Eq. (19) are

$$c_1 = 0.3025, \quad c_2 = 0.1523, \quad c_3 = 2.6890$$

Similarly, for the three-spring system ($\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_6$), the joint twists ($\mathbf{t}_4$, $\mathbf{t}_5$, $\mathbf{t}_6$) at the joints J_4, J_5 and J_6 are calculated to be

$$[\mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6] = \begin{bmatrix} 0.7176 & 1.7356 & 1.3630 \\ -1.8000 & -1.8000 & -2.1726 \\ 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

The corresponding joint compliances are as follows:

$$c_4 = 0.5937, \quad c_5 = 29.8472, \quad c_6 = 6.9068$$

With this final step, the configurations (illustrated in Fig. 13) and the joint compliances of the two 3-joint serial mechanism are identified.

To validate the result, the Cartesian compliances associated with the two serial mechanisms are calculated. For mechanism $J_1J_2J_3$, the Cartesian compliance is calculated to be

$$\mathbf{C}_1 = c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + c_1\mathbf{t}_1\mathbf{t}_1^T = \begin{bmatrix} 36.6695 & -19.4520 & 24.2798 \\ -19.4520 & 10.9779 & -13.2290 \\ 24.2798 & -13.2290 & 16.4249 \end{bmatrix}$$

For mechanism $J_4J_5J_6$, the Cartesian compliance is calculated to be

$$\mathbf{C}_2 = c_4\mathbf{t}_4\mathbf{t}_4^T + c_5\mathbf{t}_5\mathbf{t}_5^T + c_6\mathbf{t}_6\mathbf{t}_6^T = \begin{bmatrix} 103.0418 & -114.4624 & 61.6415 \\ -114.4624 & 131.2301 & -69.7994 \\ 61.6415 & -69.7994 & 37.3477 \end{bmatrix}$$

Since the two serial mechanisms are independently connected to the body, the overall stiffness of the system is

$$\mathbf{K} = \mathbf{C}_1^{-1} + \mathbf{C}_2^{-1} = \begin{bmatrix} 3 & -2 & -8 \\ -2 & 6 & 11 \\ -8 & 11 & 30 \end{bmatrix}$$

which confirms that the given stiffness is realized by the two serial mechanisms.

6 Discussion and Summary

In this section, the new six-spring mechanism synthesis procedure is compared to that previously obtained [4] and a brief summary is presented.

6.1 Discussion. The synthesis of a multi-serial parallel mechanism presented in this paper is based on the concept of dual elastic mechanisms and the new procedure developed for six-spring parallel mechanisms. Compared to the previous work [4] in which sufficient conditions are used, the synthesis procedure developed in this paper uses only necessary and sufficient conditions. Thus, the space of spring candidates for the realization of a given elastic behavior is significantly enlarged.

The realization conditions are represented by the wrench locations relative to quadratic curves f_s defined by Eq. (64) and h_{ijpq} defined by Eq. (67). The shapes of these quadratic curves are determined by the eigenvalues of the 2×2 leading block in $\mathbf{M}_s^{-1}$ of Eq. (64) or in $\mathbf{H}_{ijpq}^{-1}$ of Eq. (67). If the two eigenvalues are non-zero with the same sign, the curve is an ellipse; if the two eigenvalues are non-zero with opposite signs, the curve is a hyperbola; if one eigenvalue is zero, the curve is a parabola. It can be seen that in the generic case, the quadratic curves used in the synthesis are either ellipses or hyperbolas.

Unlike the previous process, the synthesis procedure presented in this paper may require iteration that involves adjusting the spring locations by evaluating the spring coefficients and the corresponding quadratic curves. If one spring location is changed, all spring coefficients k_i will be changed. Since the sign change of a spring coefficient depends on the change of intersection relations of the corresponding wrench with respect to a quadratic curve, when the location change of a spring is not large, the intersection relations of other springs with respect to other curves is likely maintained.

In the synthesis process, the first four spring locations are arbitrary. Thus, some mechanism geometric constraints can be considered and enforced in the process. For example, if one joint location in each serial mechanism is specified, then, those two points can be selected as the intersections of two springs in the selection of the first four springs. For example, the specified points might be on the base of each serial mechanism, or both on the reference body.

6.2 Summary. In this paper, the realization of any planar compliance with a type of multi-serial parallel mechanism (two 3-joint serial mechanisms connected in parallel) is addressed. It is shown that the realization of a compliant behavior with this type of mechanism is equivalent to its realization with a 6-spring fully parallel mechanism. For any given elastic behavior, conditions on the distribution of springs in a parallel mechanism relative to the center of stiffness are identified. A new synthesis procedure that uses only *necessary and sufficient* conditions for the realization is developed. The obtained six-spring parallel mechanism is transformed into two 3-joint serial mechanisms. The theories presented in this paper enable one to achieve any specified elastic behavior with a multi-serial parallel mechanism with some control over the mechanism geometry.

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Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

The authors attest that all data for this study are included in the paper. Data provided by a third party listed in Acknowledgment. No data, models, or codes were generated or used for this paper.

Appendix

Consider the function defined by Eq. (52):

$$F_6(\mathbf{w}) = (\mathbf{t}_{12}^T\mathbf{w})(\mathbf{t}_{34}^T\mathbf{w})(\mathbf{t}_{13}^T\mathbf{w}_5)(\mathbf{t}_{24}^T\mathbf{w}_5) - (\mathbf{t}_{12}^T\mathbf{w}_5)(\mathbf{t}_{34}^T\mathbf{w}_5)(\mathbf{t}_{13}^T\mathbf{w})(\mathbf{t}_{24}^T\mathbf{w}) \tag{A1}$$

Then, Eq. (51) is equivalent to $F_6(\mathbf{w}_6) = 0$. We prove that, if $F_6(\mathbf{w}_6) = 0$, then, the determinant of the 6×6 matrix $\hat{\mathbf{W}}$ defined in Eq. (43) must be zero, i.e.,

$$\det(\hat{\mathbf{W}}) = 0 \tag{A2}$$

If $\det(\hat{\mathbf{W}}) \neq 0$, the inverse of $\hat{\mathbf{W}}$ can be expressed as

$$\hat{\mathbf{W}}^{-1} = \frac{\operatorname{adj}(\hat{\mathbf{W}})}{\det(\hat{\mathbf{W}})} \tag{A3}$$

where $\operatorname{adj}(\hat{\mathbf{W}})$ is the adjoint matrix of $\hat{\mathbf{W}}$. Thus, Eq. (45) can be expressed as

$$\mathbf{k} = \frac{\operatorname{adj}(\hat{\mathbf{W}})\hat{\mathbf{k}}}{\det(\hat{\mathbf{W}})} \tag{A4}$$

If we denote the sixth component of $\operatorname{adj}(\hat{\mathbf{W}})\hat{\mathbf{k}}$ as k_6*, then

$$k_6 = \frac{k_6^*}{\det(\hat{\mathbf{W}})} \tag{A5}$$

Comparing Eq. (5) with Eq. (70),

$$\det(\hat{\mathbf{W}}) \neq 0 \Longleftrightarrow F_6(\mathbf{w}_6) \neq 0 \tag{A6}$$

Thus, if $F_6(\mathbf{w}_6) = 0$, $\det(\hat{\mathbf{W}})$ must be zero and the column vectors of $\hat{\mathbf{W}}$ must be linearly dependent, which implies that $\mathbf{w}_i\mathbf{w}_i^T$ must be linearly dependent. Therefore, Eq. (51) is also a sufficient condition for the six rank-1 matrices s to be linearly dependent.

References

[1] Huang, S., and Schimmels, J. M., 2017, "Geometric Construction-Based Realization of Planar Elastic Behaviors With Parallel and Serial Manipulators," ASME. J. Mech. Rob., **9**(5), p. 051006.
[2] Huang, S., and Schimmels, J. M., 2018, "Geometric Approach to the Realization of Planar Elastic Behaviors With Mechanisms Having Four Elastic Components," ASME. J. Mech. Rob., **10**(4), p. 041004.
[3] Huang, S., and Schimmels, J. M., 2019, "Geometry Based Synthesis of Planar Compliances With Redundant Mechanisms Having Five Compliant Components," Mech. Mach. Theory, **134**, pp. 645–666.
[4] Huang, S., and Schimmels, J. M., 2020, "Synthesis of Planar Compliances With Mechanisms Having Six Compliant Components: Geometric Approach," ASME. J. Mech. Rob., **12**(3), p. 031013.
[5] Whitney, D. E., and Nevins, J. L., 1979, "What Is the Remote Center Compliance (RCC) and What Can It Do," Proceedings of the 9th International Symposium and Exposition on Industrial Robots, Washington, DC, pp. 135–152.
[6] Ball, R. S., 1900, *A Treatise on the Theory of Screws*, Cambridge University Press, London, UK.
[7] Dimentberg, F. M., 1965, *The Screw Calculus and its Applications in Mechanics.* Foreign Technology Division, Wright-Patterson Air Force Base, Dayton, Ohio. Document No. FTD-HT-23-1632-67.
[8] Griffis, M., and Duffy, J., 1991, "Kinestatic Control: A Novel Theory for Simultaneously Regulating Force and Displacement," ASME. J. Mech. Des., **113**(4), pp. 508–515.
[9] Patterson, T., and Lipkin, H., 1993, "Structure of Robot Compliance," ASME. J. Mech. Des., **115**(3), pp. 576–580.
[10] Loncaric, J., 1987, "Normal Forms of Stiffness and Compliance Matrices," IEEE J. Rob. Autom., **3**(6), pp. 567–572.
[11] Zefran, M., and Kumar, V., 2002, "A Geometrical Approach to the Study of the Cartesian Stiffness Matrix," ASME. J. Mech. Des., **124**(1), pp. 30–38.
[12] Huang, S., and Schimmels, J. M., 1998, "The Bounds and Realization of Spatial Stiffnesses Achieved With Simple Springs Connected in Parallel," IEEE. Trans. Rob. Autom., **14**(3), pp. 466–475.
[13] Roberts, R. G., 1999, "Minimal Realization of a Spatial Stiffness Matrix with Simple Springs Connected in Parallel," IEEE. Trans. Rob. Autom., **15**(5), pp. 953–958.
[14] Ciblak, N., and Lipkin, H., 1999, "Synthesis of Cartesian Stiffness for Robotic Applications," Proceedings of the IEEE International Conference on Robotics and Automation, Detroit, MI, pp. 2147–2152.
[15] Huang, S., and Schimmels, J. M., 2002, "The Duality in Spatial Stiffness and Compliance As Realized in Parallel and Serial Elastic Mechanisms," ASME J. Dyn. Syst., Meas. Control, **124**(1), pp. 76–84.
[16] Choi, K., Jiang, S., and Li, Z., 2002, "Spatial Stiffness Realization With Parallel Springs Using Geometric Parameters," IEEE. Trans. Rob. Autom., **18**(3), pp. 264–284.
[17] Hong, M. B., and Choi, Y. J., 2009, "Screw System Approach to Physical Realization of Stiffness Matrix with Arbitrary Rank," ASME. J. Mech. Rob., **1**(2), p. 021007.
[18] Huang, S., and Schimmels, J. M., 2011, "Realization of An Arbitrary Planar Stiffness With a Simple Symmetric Parallel Mechanism," ASME. J. Mech. Rob., **3**(4), p. 041006.
[19] Huang, S., and Schimmels, J. M., 2018, "Geometric Construction-Based Realization of Spatial Elastic Behaviors in Parallel and Serial Manipulators," IEEE Trans. Rob., **34**(3), pp. 764–780.
[20] Verotti, M., and Belfiore, N. P., 2016, "Isotropic Compliance in E(3): Feasibility and Workspace Mapping," ASME. J. Mech. Rob., **8**(6), p. 061005.
[21] Verotti, M., Masarati, P., Morandini, M., and Belfiore, N., 2016, "Isotropic Compliance in the Special Euclidean Group SE(3)," Mech. Mach. Theory, **98**, pp. 263–281.
[22] Su, H.-J., Dorozhin, D., and Vance, J., 2009, "A Screw Theory Approach for the Conceptual Design of Flexible Joints for Compliant Mechanisms," ASME. J. Mech. Rob., **1**(4), p. 041009.
[23] Yu, J., Li, S., Su, H.-J., and Culpepper, M. L., 2011, "Screw Theory Based Methodology for the Deterministic Type Synthesis of Flexure Mechanisms," ASME. J. Mech. Rob., **3**(3), p. 031008.
[24] Simaan, N., and Shoham, M., 2003, "Stiffness Synthesis of a Variable Geometry Six-Degrees-of-Freedom Double Planar Parallel Robot," Int. J. Rob. Res., **22**(9), pp. 757–775.
[25] Wen, K., Shin, C.-B., Seo, T.-W., and Lee, J.-W., 2016, "Stiffness Synthesis of 3-DOF Planar 3RPR Parallel Mechanisms," Robotica, **34**(12), pp. 2776–2787.
[26] Cutkosky, M. R., and Kao, I., 1989, "Computing and Controlling the Compliance of a Robotic Hand," IEEE. Trans. Rob. Autom., **5**(2), pp. 617–622.
[27] Huang, S., and Schimmels, J. M., 2022, "The Relationship Between Mechanism Geometry and the Centers of Stiffness and Compliance," Mech. Mach. Theory, **167**, p. 104565.