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Research paper

Planar compliance realized with a hand composed of multiple 2-joint fingers

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ABSTRACT

In this paper, a geometric construction based means of realizing any specified planar compliance for an object held by a compliant hand is developed. It is shown that the elastic behavior of an object held by a multi-serial parallel mechanism (a multi-finger compliant hand) is more simply and equivalently modeled by a fully-parallel dual elastic mechanism. Synthesis procedures are developed for the realization of an arbitrary compliance with compliant hands using geometric constraints on the fully-parallel elastic dual. Kinematic topologies addressed are those associated with hands having 2 or 3 fingers for which each finger has 2 joints.

1. Introduction

In robotic manipulation, compliance is needed to regulate contact forces and to improve accuracy in constrained relative positioning. A general model for Cartesian compliance is a rigid-body supported by an elastic suspension. A compliant behavior is described by the relationship between the force and torque applied to the body and the resulting displacement of the body. For small displacements, the relationship is linear and can be characterized by a symmetric positive definite matrix, the compliance matrix **C**, or its inverse, the stiffness matrix **K**.

1.1. Related work

Prior work has addressed the realization of compliant behavior. Most early work focused on compliance synthesis with either a fully parallel mechanism or a fully serial mechanism [1–6]. In this work, however, little or no consideration for mechanism geometry was included in the synthesis processes.

In [7,8], stiffness synthesis methods with planar parallel mechanisms having specific constructions were developed. In [9–14], the analysis and synthesis of compliance associated with mechanisms composed of distributed elastic components were addressed.

In closely related previous work in *planar* compliance synthesis [15–18], geometry-based approaches to the design of fully parallel or fully serial mechanisms with m $(3 \leq m \leq 6)$ elastic components were developed. In each case, necessary and sufficient conditions on the mechanism geometry were presented for the realization of an arbitrary compliance. In [15], the concept of *dual elastic mechanisms* in parallel and serial construction for the realization of planar compliances was defined. It was shown that the space of compliant behaviors realized by a parallel mechanism is identical to that of its serial elastic dual. Thus, a parallel mechanism can be replaced by its dual elastic serial mechanism (and vice versa) in synthesis procedures for the realization of a compliance.

In more recent work, multi-serial parallel mechanisms have been considered. In [19], multiple serial elastic mechanisms rigidly connected to a single body were used to attain a desired planar compliant behavior for the body.

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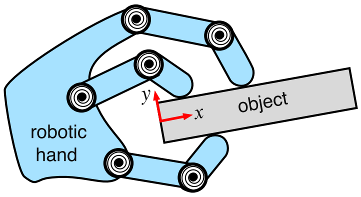

Fig. 1. Compliant robotic hand with multiple 2-joint fingers. Each finger has two joints with modulated elastic properties. The object is held by the fingers in point contact.

These previous means to achieve general elastic behavior with elastic mechanisms all assume that the compliance reference frame is attached to the mechanism distal body. In this paper, the compliance associated with a hand having modulated elastic properties is considered. Here, the compliance reference frame is attached to an object that is compliantly constrained at locations on its periphery. An example of this is illustrated in Fig. 1, which shows a robotic hand supporting an object with multiple fingertips contacting the object surface.

This approach yields a high degree of adjustability. With control of each finger, a hand can grasp and regrasp any object in a range of shapes. The ability to both change finger joint stiffness and change fingertip locations on the object surface allows a hand the ability to change the held object's compliance without influencing the pose of the object.

Many researchers have investigated the compliant behavior associated with a robotic hand [20–27]. Most work has focused on calculating the Cartesian compliance of a given grasp configuration/joint stiffness combination provided by a compliant hand or on determining the appropriate joint stiffnesses for a given grasp configuration/Cartesian compliance combination. The mathematical description of the elastic behavior associated with a hand involves calculating the compliance at each fingertip, inverting the matrix to obtain the stiffness matrix, then adding the stiffness matrices for each finger to obtain the object stiffness. Due to the complexity of this process, numerical algorithmic procedures [20,26] or numerical optimization procedures [22,25] have been used. These numerical approaches, however, neither consider the geometry restriction on the fingers nor guarantee that the desired compliance is actually attained.

This paper addresses the synthesis of an arbitrary planar compliance with a multi-finger hand. Since the approach developed is based on necessary and sufficient conditions on the hand geometry, the synthesis procedure ensures that the specified compliant behavior for the object is achieved.

1.2. Contribution of the paper

This paper addresses the realization of an arbitrary planar compliance with a robotic hand having multiple fingers. The main contributions of the paper are:

1. The concept of dual elastic mechanisms is extended to more general cases so that a fully parallel mechanism is used to equivalently model the elastic behavior of a multi-finger hand;
2. Geometric construction-based synthesis procedures are developed for compliance realization with a robotic hand with multiple fingers, each having 2 elastic joints.

The synthesis procedure developed in this paper is based on analytical and geometrical procedures and the use of necessary and sufficient (mathematical) conditions. Thus, the mechanism geometry is taken into account in the selection of each elastic component and the resulting hand is guaranteed to realize the desired compliant behavior (if its realization is possible). These results eliminate the need for numerical optimization in the realization of a desired grasp compliance.

1.3. Overview

This paper addresses means of providing a held object an arbitrary planar (3×3) compliance matrix with a compliant hand. In the hand topologies considered, each finger has 2 joints for which the passive compliance in each is selected or modulated.

The paper is outlined as follows. In Section 2, some technical background needed for compliance realization with a compliant hand is provided. In Section 3, the concept of dual elastic mechanisms is first reviewed, and a fully parallel elastic model of a hand with multiple 2-joint fingers in point contact with an object is developed. In Section 4, compliance realization with a 2-finger hand is addressed. A new geometry-based synthesis procedure is first developed for 4-spring parallel mechanisms. The new procedure accounts for the constraints associated with contact only on the body surface. Then, using these results and the concept of dual elastic mechanisms, a synthesis procedure for a 2-finger hand is presented. Similarly, in Section 5, a new 6-spring parallel mechanism synthesis procedure and a synthesis procedure for a 3-finger hand are developed. In Section 6, numerical examples are provided to demonstrate the synthesis procedures. Finally, a brief summary is presented in Section 7.

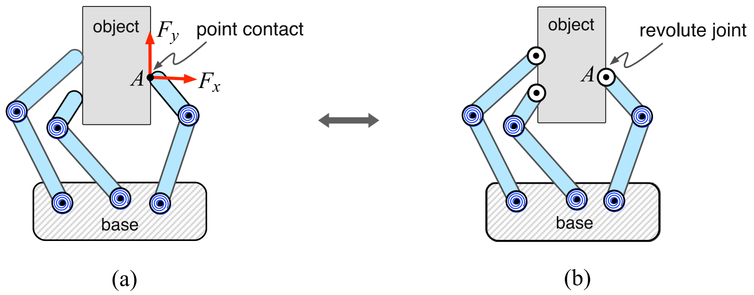

Fig. 2. A compliant grasp can be modeled as a system of multi-serial mechanisms connected in parallel. (a) At the contact point, only force (no torque) is transmitted to the body. (b) When a grasp is achieved, finger/object contact is equivalent to a free revolute joint at the contact point.

2. Technical background

In this section, the background needed for the realization of planar compliance with either a parallel or a serial mechanism is provided. First, a physical model of a compliant hand interacting with a held object is presented. Next, screw representation of a mechanism configuration is reviewed. Then, previously developed requirements on the distribution of elastic components relative to the elastic center are reviewed.

2.1. Physical model of a robotic compliant hand

In a hand, each finger/object combination can be viewed as a serial mechanism. A compliant grasp then can be modeled as a system of serial mechanisms connected in parallel to the held object. Unlike the fully serial mechanisms studied in previous work [18] (in which each joint has elastic behavior), the hard point contact between the fingertip and the held body is inelastic. As such, at the point of contact, only force (no torque) can be transmitted to the body as illustrated in Fig. 2a.

It can be seen that, with the above assumptions, each finger can provide only point compliance (yielding a 2×2 compliance matrix at the contact point) to the body regardless of the number of the compliant joints in the finger. When a grasp is achieved, since each fingertip maintains contact with the object and does not slip along the object surface, the finger most distal link can only rotate about the point of contact. As such, the connection between the fingertip and the contacted object is equivalent to a free revolute joint at the contact point as shown in Fig. 2b. Thus, when in hard point contact with the held object, each 2-joint finger illustrated can be viewed as a $3R$ serial mechanism with the last joint having no elastic behavior. Because all fingers contact the same body, they act in parallel yielding a multi-serial parallel mechanism. In this paper, the approach to grasp compliance realization is based on this model.

2.2. Stiffness realization with a parallel mechanism

A parallel elastic mechanism consists of a set of springs connecting two bodies. The geometry of each spring can be represented by a unit wrench $\mathbf{w}$ defined as the *spring wrench*. In Plücker ray coordinates, the planar spring wrench associated with a translational spring has the form:

$$\mathbf{w} = \begin{bmatrix} \hat{\mathbf{n}} \\ d \end{bmatrix}, \tag{1}$$

where the unit 2-vector $\hat{\mathbf{n}}$ indicates the direction of the wrench (spring axis) and where

$$d = (\mathbf{r}_P \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{k}}, \tag{2}$$

where $\mathbf{r}_P$ is the position vector from the origin to any point P along the spring axis, and $\hat{\mathbf{k}}$ is the unit vector orthogonal to the plane. A unit wrench $\mathbf{w}$ can be used to represent the axis of a spring, and geometrically, represent a line in the plane.

If a parallel mechanism consists of n springs $\mathbf{w}_i$ $(1, \ldots, n)$, then the Cartesian stiffness of the mechanism is [4]:

$$\mathbf{K} = k_1 \mathbf{w}_1 \mathbf{w}_1^T + k_2 \mathbf{w}_2 \mathbf{w}_2^T + \cdots + k_n \mathbf{w}_n \mathbf{w}_n^T, \tag{3}$$

where $k_i \geq 0$ is the spring rate associated with $\mathbf{w}_i$.

Thus, to achieve a specified stiffness $\mathbf{K}$ with an n-spring parallel mechanism, a set of n spring wrenches $\mathbf{w}_i$'s and the corresponding spring rates k_i's must be identified such that Eq. (3) is satisfied.

2.3. Compliance realization with a serial mechanism

A serial mechanism consists of links connected by a series of n joints. The location of a joint J can be represented by a unit twist $\mathbf{t}$ defined as the *joint twist*. In Plücker axis coordinates, the planar joint twist associated with a revolute joint has the form:

$$\mathbf{t} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix}, \tag{4}$$

where

$$\mathbf{u} = -\boldsymbol{\Omega}\mathbf{r}_J, \tag{5}$$

where $\mathbf{r}_J$ is the position vector of joint J relative to the coordinate frame, and where $\boldsymbol{\Omega} \in \mathbb{R}^{2\times 2}$ is the anti-symmetric matrix:

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{6}$$

Hence, for any given unit twist in Eq. (4), the corresponding instantaneous center (joint location) $\mathbf{r}_J$ is determined by Eq. (5) which can be equivalently expressed as:

$$\mathbf{r}_J = \boldsymbol{\Omega}\mathbf{u}. \tag{7}$$

If the location of joint J, $\mathbf{r}_J$, is specified, the joint twist can be calculated using Eqs. (4) and (5). Thus, a unit twist $\mathbf{t}$ can be used to represent the location of a joint, and geometrically, represent a point in the plane.

If a serial mechanism consists of n joints described by joint twists $\mathbf{t}_i$ $(1, \dots, n)$, then the Cartesian compliance of the mechanism is [4]:

$$\mathbf{C} = c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + \cdots + c_n\mathbf{t}_n\mathbf{t}_n^T, \tag{8}$$

where $c_i \geq 0$ is the joint compliance associated with $\mathbf{t}_i$.

Thus, to achieve a specified compliance $\mathbf{C}$ with an n-joint serial mechanism, a set of n joint twists $\mathbf{t}_i$'s and the corresponding joint compliances c_i's must be identified such that Eq. (8) is satisfied.

2.4. Distribution of elastic component locations

For a given planar stiffness (compliance) matrix, there is a unique point in the plane such that, if the coordinate frame is located at that point, the stiffness (compliance) matrix can be expressed in diagonal form. This point is called the center of stiffness (center of compliance). For the planar case, the center of stiffness and the center of compliance are coincident. When a compliance is realized with a parallel [19] or a serial [18] mechanism, the elastic components must surround the stiffness center, and the distance and orientation of the set of components relative to the stiffness center must satisfy additional conditions. Those for a parallel elastic mechanism are reviewed below.

If the frame used in describing the elastic behavior is located at the stiffness center and along the principal axes, the stiffness matrix has diagonal form:

$$\mathbf{K} = \mathrm{diag}[k_\chi, k_\eta, k_\tau], \tag{9}$$

where k_χ, k_η and k_τ are the principal stiffnesses and are uniquely determined by the behavior. A circle of radius ρ_k

$$\rho_k = \sqrt{\frac{k_\tau}{k_\chi + k_\eta}} \tag{10}$$

whose center is the stiffness center indicates the amount of torsional stiffness relative to translational stiffness.

Suppose that an n-spring parallel mechanism with spring wrenches $\mathbf{w}_i$ $(i = 1, 2, \dots, n)$ realizes the stiffness $\mathbf{K}$, and that $r_{\max}$ and $r_{\min}$ are the maximum and minimum distances from the spring wrenches to the stiffness center C_k, then as proved in [19],

$$r_{\min} \quad \leq \quad \sqrt{\frac{k_\tau}{k_\chi + k_\eta}} \quad \leq \quad r_{\max}. \tag{11}$$

Thus, to realize the elastic behavior, the spring axes can neither all intersect circle Γ_k of radius ρ_k centered at C_k, nor all not intersect circle Γ_k.

In addition, if we denote

$$\theta_\chi = \sin^{-1}\left(\sqrt{\frac{k_\chi}{k_\chi + k_\eta}}\right), \tag{12}$$

and denote l_θ^- and l_θ^+ as the 2 lines passing through the stiffness center C_k with angles $-\theta_\chi$ and θ_χ with respect to the principal χ-axis respectively, then l_θ^- and l_θ^+ separate the plane into 2 areas Λ_χ and Λ_η as illustrated in Fig. 3. As shown in [19], the perpendicular vectors $\mathbf{r}_i$ from the stiffness center to the spring wrenches $\mathbf{w}_i$ must be distributed in both areas and the spring wrenches must surround the stiffness center.

The geometry associated with each of these conditions is illustrated in Fig. 3. The spring spatial distribution conditions are summarized as:

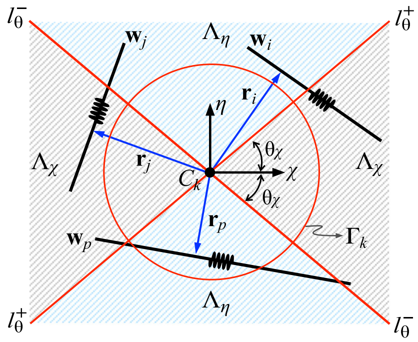

Fig. 3. Spring distribution conditions for a parallel mechanism. At least one spring axis intersects circle Γ_k and at least one spring axis does not intersect circle Γ_k. Vectors $\mathbf{r}_i$'s are located in both areas Λ_χ and Λ_η and span at least a half plane.

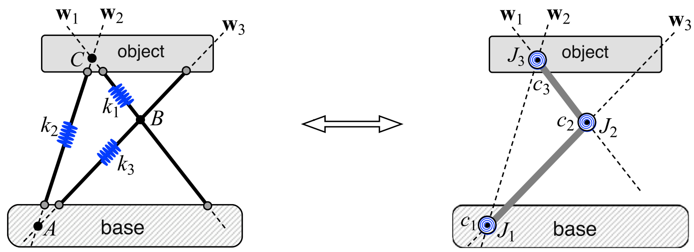

Fig. 4. Dual elastic mechanisms in parallel and serial construction. Triangle ABC formed by the three spring axes in the parallel mechanism is coincident with triangle $J_1J_2J_3$ formed by the three joints in the serial mechanism.

(i) At least one spring wrench intersects circle Γ_k and at least one spring does not intersect Γ_k;
(ii) The perpendicular vectors $\mathbf{r}_i$ from C_k to each spring wrench must be located in both areas Λ_χ and Λ_η bounded by lines l_θ^+ and l_θ^-, and all $\mathbf{r}_i$'s must span at least a half plane.

Note that conditions (i) and (ii) are *necessary* conditions for the realization of a planar compliant behavior with a parallel mechanism having any number of elastic components. To ensure a parallel mechanism realizes a stiffness, additional conditions are required for the mechanism with a specified number of elastic components [15–18].

3. Dual elastic mechanisms

The concept of dual elastic mechanisms was developed in [15] for parallel and serial planar mechanisms having 3 elastic components. It was shown that a pair of dual elastic mechanisms (one fully serial, the other fully parallel) have an identical space of realizable compliant behaviors. Thus, a serial mechanism of 3 elastic joints can be replaced by a parallel mechanism of 3 springs for the realization of any full-rank compliant behavior (and vice versa). This concept cannot be directly used for compliant hands since the planar compliance matrix associated with each finger is not full rank.

In this section, the concept of dual elastic mechanisms is reviewed in more detail. Then, the concept is extended to mechanisms having less than 3 elastic components, i.e., elastic mechanisms yielding rank deficient planar stiffness matrices.

3.1. Dual elastic mechanisms with three components

Suppose a parallel mechanism has three line spring wrenches $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$. The three lines of action form a triangle. Consider the three unit twists $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ centered at the triangle's three vertexes, i.e., $\mathbf{t}_i$ is centered at the intersection of $\mathbf{w}_j$ and $\mathbf{w}_p$. Then, $\mathbf{t}_i$ is reciprocal to both wrenches $\mathbf{w}_j$ and $\mathbf{w}_p$ ($\mathbf{t}_i^T\mathbf{w}_j = 0$, $\mathbf{t}_i^T\mathbf{w}_p = 0$).

Now consider the serial mechanism having joint twists $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$, respectively, for which the three line spring axes in the parallel mechanism are coincident with the triangle formed by the three revolute joints in the serial mechanism. Such a pair of parallel and serial mechanisms are defined as *dual elastic* mechanisms [15]. A pair of dual elastic mechanisms is illustrated in Fig. 4.

Given a pair of dual elastic mechanisms with spring wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and joint twists $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$, for a given elastic behavior described with stiffness matrix $\mathbf{K}$ or with compliance matrix $\mathbf{C} = \mathbf{K}^{-1}$, as shown in [15],

$$\mathbf{w}_i^T\mathbf{C}\mathbf{w}_j = 0 \quad \Longleftrightarrow \quad \mathbf{t}_i^T\mathbf{K}\mathbf{t}_j = 0, \quad i \neq j, \tag{13}$$

and an elastic behavior can be realized with one mechanism if and only if it can be realized with its dual elastic mechanism. If there are no bounds on the elastic properties in each elastic component, the realizable spaces of elastic behaviors for the two

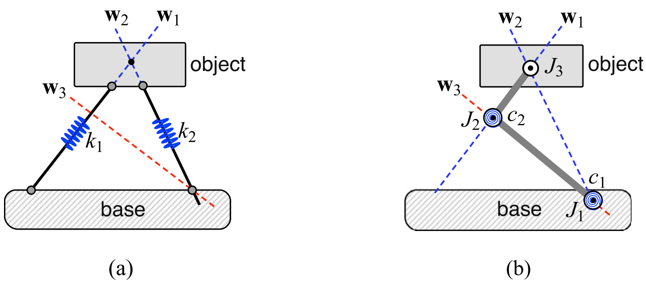

Fig. 5. Dual elastic mechanisms having two elastic components. (a) A 2-spring parallel mechanism. (b) A dual serial mechanism corresponding to the 2-spring system. The mechanism has three joints but only 2 are elastic. The free joint is located at the intersection of the two springs.

mechanisms are exactly the same. Also, it can be proved that, if k_i is the spring constant associated with spring wrench $\mathbf{w}_i$ in the parallel mechanism and c_i is the joint compliance associated with joint twist $\mathbf{t}_i$, then k_i and c_i satisfy:

$$c_i k_i = \frac{1}{(\mathbf{t}_i^T \mathbf{w}_i)^2}, \quad i = 1, 2, 3. \tag{14}$$

Note that

$$d_i = \mathbf{t}_i^T \mathbf{w}_i \tag{15}$$

indicates the distance of twist $\mathbf{t}_i$ from wrench $\mathbf{w}_i$.

The concept of dual elastic mechanisms can be extended to 2-spring and 1-spring systems as described below. This concept will be used in modeling 2-joint fingers in contact with an object for which both joints or only one joint has elastic properties.

3.2. Dual elastic mechanisms with two components

For a 2-spring parallel mechanism with spring wrenches $(\mathbf{w}_1, \mathbf{w}_2)$ and spring rates $k_1 > 0$ and $k_2 > 0$, consider the addition of an arbitrary spring $\mathbf{w}_3$ that intersects both $\mathbf{w}_1$ and $\mathbf{w}_2$ (Fig. 5a). A 3-joint elastic serial mechanism dual to the 3-spring system $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ can then be constructed. If the spring rate of $\mathbf{w}_3$ approaches zero, $k_3 \to 0$, the corresponding joint compliance in the dual elastic serial mechanism $c_3 \to +\infty$. Thus, if $k_3 = 0$, the corresponding joint J_3 in the dual serial mechanism is a free joint located at the intersection of springs $\mathbf{w}_1$ and $\mathbf{w}_2$. This joint does not provide elastic behavior but only provides constraint in the kinematic chain. Since $\mathbf{w}_3$ is arbitrary, the locations of the other two joints J_1 and J_2 can be anywhere on the axes of $\mathbf{w}_2$ and $\mathbf{w}_1$, respectively as shown in Fig. 5b. The joint compliances can be determined by:

$$c_i = \frac{1}{k_i (\mathbf{t}_i^T \mathbf{w}_i)^2} = \frac{1}{k_i d_i^2}, \quad i = 1, 2. \tag{16}$$

Note that the serial elastic mechanism dual to a 2-spring parallel mechanism is not unique. Below, we show that any 3-joint serial mechanism dual to a 2-spring parallel mechanism (as constructed in Fig. 5) provides the same elastic behavior.

Let $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$ be the 3 joint twists in the serial mechanism, and let c_1 and c_2 be the joint compliances at J_1 and J_2 calculated by Eq. (16). Suppose the joint compliance at J_3 has a large value: $c_3 \gg c_1$, $c_3 \gg c_2$. Then, the compliance associated with the serial mechanism is:

$$\mathbf{C} = c_1 \mathbf{t}_1 \mathbf{t}_1^T + c_2 \mathbf{t}_2 \mathbf{t}_2^T + c_3 \mathbf{t}_3 \mathbf{t}_3^T \tag{17}$$

which can be expressed as

$$\mathbf{C} = \mathbf{T}\, \mathrm{diag}[c_1, c_2, c_3]\, \mathbf{T}^T, \tag{18}$$

where $\mathbf{T} = [\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3] \in \mathbb{R}^{3\times 3}$ is the twist matrix. Taking the inverse of Eq. (18):

$$\mathbf{K} = \mathbf{T}^{-T}\, \mathrm{diag}\left[\frac{1}{c_1}, \frac{1}{c_2}, \frac{1}{c_3}\right] \mathbf{T}^{-1}. \tag{19}$$

Letting $c_3 \to +\infty$ and using Eq. (16),

$$\begin{aligned} \mathbf{K} &= \mathbf{T}^{-T}\, \mathrm{diag}\left[k_1 (\mathbf{t}_1^T \mathbf{w}_1)^2,\ k_2 (\mathbf{t}_2^T \mathbf{w}_2)^2,\ 0\right] \mathbf{T}^{-1} \\ &= \mathbf{T}^{-T}\, \mathrm{diag}\left[k_1 d_1^2,\ k_2 d_2^2,\ 0\right] \mathbf{T}^{-1}. \end{aligned} \tag{20}$$

If $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] \in \mathbb{R}^{3\times 3}$ is the wrench matrix of the parallel mechanism, then $\mathbf{D} = \mathrm{diag}\left[d_1,\ d_2,\ d_3\right]$ captures the reciprocal relations between the dual elastic mechanisms [15] given by

$$\mathbf{W}^T \mathbf{T} \ = \ \mathbf{T}^T \mathbf{W} = \mathrm{diag}\left[(\mathbf{t}_1^T \mathbf{w}_1),\ (\mathbf{t}_2^T \mathbf{w}_2),\ (\mathbf{t}_3^T \mathbf{w}_3)\right]$$

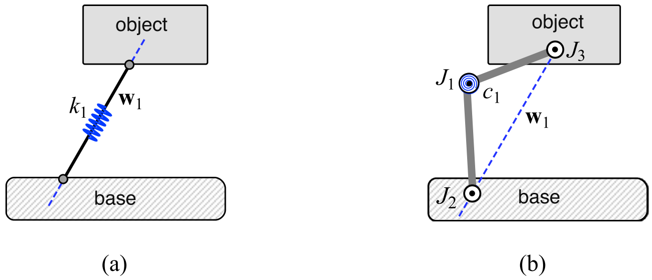

Fig. 6. Dual elastic mechanisms having one elastic component. (a) A line spring with spring wrench $\mathbf{w}_1$. (b) The dual serial elastic mechanism having 3 joints in which only one is elastic. Free joints J_2 and J_3 must be anywhere along the line of action of $\mathbf{w}_1$.

$$= \operatorname{diag}\left[d_1,\, d_2,\, d_3\right] \ = \mathbf{D}. \tag{21}$$

Thus,

$$\mathbf{T}^{-1} = \mathbf{D}^{-1}\mathbf{W}^T, \qquad \mathbf{T}^{-T} = \mathbf{W}\mathbf{D}^{-1}. \tag{22}$$

Substituting the relations of Eq. (22) into Eq. (20) yields:

$$\mathbf{K} = \mathbf{W}\operatorname{diag}\left[k_1,\, k_2,\, 0\right]\, \mathbf{W}^T = k_1\mathbf{w}_1\mathbf{w}_1^T + k_2\mathbf{w}_2\mathbf{w}_2^T, \tag{23}$$

which is in the form of Eq. (3).

Therefore, the 2-elastic joint, $3R$ serial mechanism provides the same elastic behavior as the 2-spring parallel mechanism. Note that a dual elastic serial mechanism is not unique for rank deficient stiffness $\mathbf{K}$. The compliant behavior provided by the dual serial mechanism is independent of the choice of the locations of the two joints J_1 and J_2 on the spring axes $\mathbf{w}_2$ and $\mathbf{w}_1$, respectively. From Eq. (16), it can be seen that the selections of J_1 and J_2 locations on the axes only impact the values of c_1 and c_2.

Note again that the serial mechanism in Fig. 5b has three revolute joints but only two are elastic. If the joint without elastic properties, J_3, is located on an object surface, the serial mechanism can be viewed as a finger for which the 2 finger joints are each along one spring axis. Therefore, a 2-spring parallel system can be converted to a 2-joint elastic finger in contact with an object (and vice versa).

3.3. Dual elastic mechanisms with one component

The concept of dual elastic mechanisms can also be extended to mechanisms having one elastic component. Because position control of a free joint in a finger is not possible, the observations (included below for completeness of the rank-deficient planar elastic mechanisms) are not subsequently used.

Consider a single spring with stiffness k_1 and spring wrench $\mathbf{w}_1$ as illustrated in Fig. 6a. The spring provides force only along the line of action of $\mathbf{w}_1$. Now consider a 3-joint serial mechanism $J_1J_2J_3$ as illustrated in Fig. 6b. In this mechanism, free joints J_2 and J_3 are located on the line of action of $\mathbf{w}_1$, and only joint J_1 is elastic. It can be seen that the serial mechanism provides an elastic force only along the $\mathbf{w}_1$ axis. If the value of joint compliance c_1 at J_1 is properly chosen, the serial mechanism acts exactly the same as the single spring $\mathbf{w}_1$.

In the mechanism of Fig. 6b, joints J_2 and J_3 can be anywhere along line $\mathbf{w}_1$ while the elastic joint can be located anywhere other than on line $\mathbf{w}_1$. It can be proved that when the location of the elastic joint J_1 is determined, the value of joint compliance is uniquely determined by

$$c_1 = \frac{1}{k_1(\mathbf{t}_1^T\mathbf{w}_1)^2} = \frac{1}{k_1 d_1^2}, \tag{24}$$

where d_1 is the distance of the twist $\mathbf{t}_1$ from the wrench $\mathbf{w}_1$ as defined in Eq. (15).

3.4. Application of dual elastic mechanisms

Since any 2-joint finger in contact with an object can be replaced by a 2-spring parallel mechanism using its elastic dual, the synthesis procedure used in realizing a compliance with a $2n$-spring ($n \geq 2$) parallel mechanism can be used to realize the same behavior with an n-finger compliant hand. If each finger has two elastic joints, the number of springs n in the corresponding parallel mechanism must be even and no less than 4.

Fig. 7 illustrates the relationship between a 4-spring parallel system and a 2-finger compliant hand elastic dual. In the 4-spring system $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$ shown in Fig. 7a, if the mechanism is separated into 2 groups: $(\mathbf{w}_1, \mathbf{w}_2)$ and $(\mathbf{w}_3, \mathbf{w}_4)$, then the mechanism can be replaced with a hand having 2 fingers as illustrated in Fig. 7b. The compliance in each joint can be calculated using Eq. (16). Note that, to perform the parallel system to finger conversion, 2 springs must intersect at the location where the fingertip contacts the object and one of these 2 springs must pass through the finger base (palm).

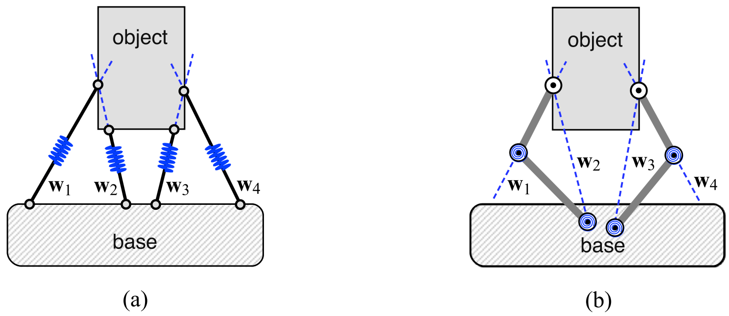

Fig. 7. A 4-spring parallel mechanism transformed into a compliant hand having two 2-joint fingers. (a) A 4-spring parallel mechanism with spring wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$. (b) A 2-finger compliant hand. Each finger has two elastic joints. The held object constitutes the third link in each finger serial mechanism.

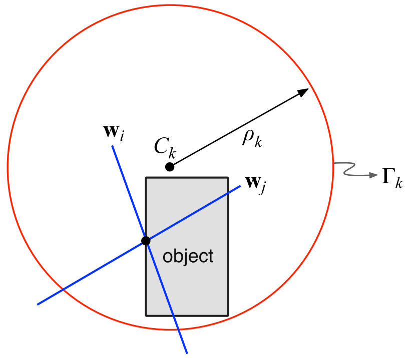

Fig. 8. Limitation of a compliant hand in realizing an arbitrary compliant behavior. If the held body is inside circle Γ_k determined by the principal stiffnesses of a desired compliant behavior, the compliance cannot be achieved by any compliant hand.

Also note that the transformation of a 4-spring parallel system into a 2-finger compliant hand is not unique. Different ways to separate the 4 springs yield a compliant hand having different finger configurations. For example, if the 4-spring mechanism is separated into 2 different groups: $(\mathbf{w}_1, \mathbf{w}_3)$ and $(\mathbf{w}_2, \mathbf{w}_4)$, then the compliant hand obtained using its dual elastic serial mechanism will have two fingers with different geometry.

The realization of an elastic behavior with a compliant hand can be achieved by first designing a parallel mechanism that realizes the compliance, then converting that mechanism into a set of multiple fingers. This geometric construction based approach does not require a matrix inversion. In Sections 4 and 5, the results of [16,18] will be modified to address elastic behavior synthesis using 2- and 3-finger planar hands.

3.5. Limits on grasp compliance

As stated earlier, each finger in a compliant hand provides point compliance to the held object at the fingertip in contact with the body. Since the contact point must be on the surface of the object, the space of realizable compliant behavior depends in part on the size and shape of the object. A significant amount of compliant behaviors cannot be achieved with a compliant hand for a given held object. For example, if a held body is completely inside circle Γ_k having radius ρ_k determined by the principal stiffnesses (Eq. (10)) of stiffness $\mathbf{K}$, this elastic behavior cannot be attained by *any* compliant hand regardless of the number of fingers in the hand and the number of joints in each finger. For this case, because all springs must pass through finger contact points, each spring wrench $\mathbf{w}_i$ in the parallel mechanism must intersect circle Γ_k as shown in Fig. 8, which violates the necessary condition (i) on the spring location distribution described in Section 2.4. Therefore, this evaluation should be the first step in the design of a compliant grasp.

4. Stiffness realization with a 2-finger hand

In this section, the realization of a held object's elastic behavior using a 2-finger hand in which each finger has 2 elastic joints is addressed. First, stiffness synthesis with a 4-spring parallel mechanism is presented. Next, the 4-spring parallel mechanism is converted to a hand with 2 fingers. Then, the limitations of 2-finger hands in the realization of an arbitrary stiffness are presented.

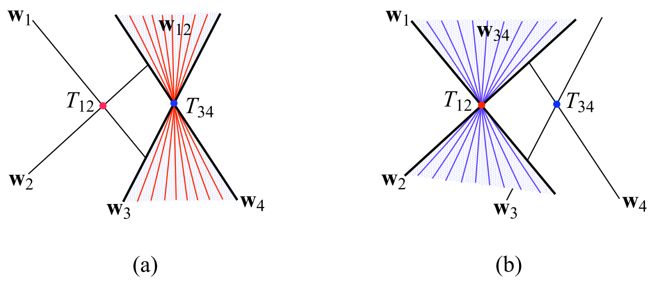

Fig. 9. Geometric implications of some of the necessary and sufficient conditions for a 4-spring parallel mechanism to realize an arbitrary stiffness. (a) Wrench $\mathbf{w}_{12}$ must pass through point T_{34} and lie in the shaded area bounded by the two axes of wrenches $\mathbf{w}_3$ and $\mathbf{w}_4$. (b) Wrench $\mathbf{w}_{34}$ must pass through point T_{12} and lie in the shaded area bounded by the two axes of wrenches $\mathbf{w}_1$ and $\mathbf{w}_2$.

4.1. Stiffness synthesis for a 4-spring mechanism

A geometric approach to achieve an arbitrary compliance with a 4-spring parallel mechanism was addressed in [16]. The procedure in [16] (summarized in Section 4.1.1 below), however, cannot be directly converted to a 2-finger hand used to grasp an object because each fingertip contacts only the held body's surface. As such, to convert the elastic behavior of a parallel mechanism to that of a compliant hand, pairs of wrenches must intersect at locations on the body surface. Thus, the synthesis procedure presented in [16] for general stiffness realization must be modified to satisfy this additional constraint. Below, the realization conditions on a 4-spring parallel mechanism are first reviewed, then a new synthesis procedure that considers each fingertip location requirement is developed.

4.1.1. Review of 4-spring realization conditions

The set of *necessary and sufficient* conditions on the geometry of any 4-spring parallel mechanism to realize a given compliance [16] are summarized below.

Consider a given stiffness $\mathbf{K}$ and a 4-spring parallel mechanism described by spring wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$. For any two wrenches $\mathbf{w}_i$ and $\mathbf{w}_j$, let T_{ij} be the intersection of the two wrench axes and $\mathbf{t}_{ij}$ be a twist having its instantaneous center at T_{ij}. Twist $\mathbf{t}_{ij}$ can be calculated by

$$\mathbf{t}_{ij} = \mathbf{w}_i \times \mathbf{w}_j. \tag{25}$$

Let $\mathbf{w}_{ij}$ be the wrench associated with $\mathbf{t}_{ij}$ obtained through the stiffness mapping:

$$\mathbf{w}_{ij} = \mathbf{K}\mathbf{t}_{ij}. \tag{26}$$

Then, equivalent to that presented in [16], we have:

Proposition 1. *A stiffness* $\mathbf{K}$ *can be realized with a 4-spring parallel mechanism having spring wrenches* $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ *and* $\mathbf{w}_4$ <u>*if and only if*</u> *the following two conditions are both satisfied:*

(a) Wrench $\mathbf{w}_{12}$ *passes through point* T_{34} *and wrench* $\mathbf{w}_{13}$ *passes through point* T_{24}*;*
(b) Wrench $\mathbf{w}_{12}$ *lies in the area bounded by the axes of* $\mathbf{w}_3$ *and* $\mathbf{w}_4$ *that does not contain* T_{12}*; and wrench* $\mathbf{w}_{34}$ *lies in the area bounded by the axes of* $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *that does not contain* T_{34}*.*

Note that Proposition 1(a) implies the satisfaction of 2 equality conditions:

$$\mathbf{t}_{34}^T\mathbf{K}\mathbf{t}_{12} = 0, \tag{27}$$

$$\mathbf{t}_{24}^T\mathbf{K}\mathbf{t}_{13} = 0, \tag{28}$$

which are also equivalent to two additional geometric conditions: 1) wrench $\mathbf{w}_{34}$ passes through point T_{12}, and 2) wrench $\mathbf{w}_{24}$ passes through point T_{13}. These 2 conditions ensure that the given $\mathbf{K}$ can be expressed in the form of Eq. (3) for $n = 4$. Proposition 1(b) implies 4 inequality conditions that can be alternatively expressed in terms of wrench $\mathbf{w}_{13}$ relative to wrenches $\mathbf{w}_2$ and $\mathbf{w}_4$, and in terms of wrench $\mathbf{w}_{24}$ relative to wrenches $\mathbf{w}_1$ and $\mathbf{w}_3$. These 4 conditions ensure that all k_is are positive. The results of Proposition 1 for equality condition Eq. (27) and the 4 inequality conditions are illustrated in Fig. 9.

Once the geometries of the 4 springs satisfying the conditions in Proposition 1 are determined, a passive realization of the given stiffness can be achieved by choosing each spring constant using:

$$k_1 = \frac{\mathbf{t}_{24}^T\mathbf{K}\mathbf{t}_{34}}{(\mathbf{t}_{24}^T\mathbf{w}_1)(\mathbf{t}_{34}^T\mathbf{w}_1)}, \tag{29}$$

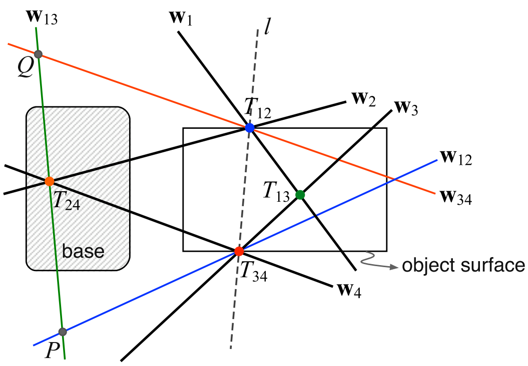

Fig. 10. Synthesis of a compliance with a 4-spring parallel mechanism based on geometry. The realization is achieved by selecting the spring axes of the four springs in the mechanism.

$$k_2 = \frac{\mathbf{t}_{14}^T \mathbf{K} \mathbf{t}_{34}}{(\mathbf{t}_{14}^T \mathbf{w}_2)(\mathbf{t}_{34}^T \mathbf{w}_2)}, \tag{30}$$

$$k_3 = \frac{\mathbf{t}_{12}^T \mathbf{K} \mathbf{t}_{14}}{(\mathbf{t}_{12}^T \mathbf{w}_3)(\mathbf{t}_{14}^T \mathbf{w}_3)}, \tag{31}$$

$$k_4 = \frac{\mathbf{t}_{13}^T \mathbf{K} \mathbf{t}_{12}}{(\mathbf{t}_{13}^T \mathbf{w}_4)(\mathbf{t}_{12}^T \mathbf{w}_4)}. \tag{32}$$

4.1.2. New 4-spring synthesis procedure

In this subsection, a new synthesis procedure is developed for the realization of an arbitrary compliance with a 4-spring parallel mechanism that can be converted to a 2-finger hand for grasping an object. Since the free joint (the fingertip in contact with the held body) of a serial mechanism is located at the intersection of the 2 springs in its parallel elastic dual, in the synthesis of a 4-spring parallel mechanism, 2 pairs of springs must intersect at two different points on the object surface. This requirement is enforced in the process described below. The geometry associated with the sequence of operations in the synthesis procedure is illustrated in Fig. 10.

1. Choose a point on the body surface. This point T_{12} will be the intersection of two spring axes $\mathbf{w}_1$ and $\mathbf{w}_2$ (and ultimately the contact point of a fingertip on the object). Then the unit twist $\mathbf{t}_{12}$ at T_{12} is obtained using Eqs. (4) and (5), and the wrench resulting from $\mathbf{t}_{12}$ is:
$$\mathbf{w}_{12} = \mathbf{K}\mathbf{t}_{12}. \tag{33}$$
The line of action of $\mathbf{w}_{12}$ is obtained.
2. Determine the point at which the other two spring wrenches $\mathbf{w}_3$ and $\mathbf{w}_4$ intersect. This point T_{34} is located at the intersection of line $\mathbf{w}_{12}$ and the body surface.
The unit twist $\mathbf{t}_{34}$ centered at T_{34} is calculated using Eq. (4). Then wrench $\mathbf{w}_{34}$ corresponding to $\mathbf{t}_{34}$ is obtained:
$$\mathbf{w}_{34} = \mathbf{K}\mathbf{t}_{34}. \tag{34}$$
By Proposition 1(a), line $\mathbf{w}_{34}$ will pass through T_{12}, i.e., satisfy Eq. (27).
The line passing through points T_{12} and T_{34} is denoted as l and is illustrated in Fig. 10.
3. Choose $\mathbf{w}_{13}$ along the hand base such that its intersections with $\mathbf{w}_{12}$ and $\mathbf{w}_{34}$ are both on the same side of l and for which twist $\mathbf{t}_{13} = \mathbf{C}\mathbf{w}_{13}$ is centered at the opposite side of line l. Line $\mathbf{w}_{13}$ intersects lines $\mathbf{w}_{12}$ and $\mathbf{w}_{34}$ at points P and Q respectively as shown in Fig. 10.
The appropriate placement of T_{13} can always be accomplished by following the steps described in [16]. The center of twist $\mathbf{t}_{13}$, T_{13} will be the intersection of spring axes $\mathbf{w}_1$ and $\mathbf{w}_3$.
4. Choose a point T_{24} along the wrench $\mathbf{w}_{13}$ axis between points P and Q such that the line passing through points (T_{12}, T_{24}) and the line passing through points (T_{24}, T_{34}) both intersect the hand base.
The unit twist at T_{24}, $\mathbf{t}_{24}$, can be calculated using Eq. (4) and the equality condition in Eq. (28) is satisfied. Point T_{24} will be the intersection of spring axes $\mathbf{w}_2$ and $\mathbf{w}_4$.
5. Determine the 4 spring axes. The four lines passing through points (T_{12}, T_{13}), (T_{12}, T_{24}), (T_{34}, T_{13}) and (T_{24}, T_{34}) as shown in Fig. 10 are identified as the four spring axes $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$ respectively for the parallel mechanism.
6. Calculate the value of stiffness k_i for each spring using Eqs. (29)–(32).

Using this synthesis process, the four spring axes selected will satisfy the realization conditions in Proposition 1 and the wrench intersection points T_{12} and T_{34} will be located on the surface of the object.

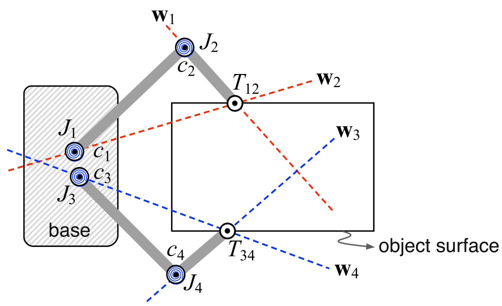

Fig. 11. A 2-finger hand obtained from the conversion of a 4-spring parallel mechanism. The intersections of two spring pairs T_{12} and T_{34} are located on the surface of the held body.

4.2. Conversion of 4-spring mechanism to 2-finger hand

Using the concept of dual elastic mechanisms, two 2-spring systems $(\mathbf{w}_1, \mathbf{w}_2)$ and $(\mathbf{w}_3, \mathbf{w}_4)$ can be converted to two serial mechanisms each having 2 elastic joints.

1. Choose the joint locations for each finger.
   (a) For spring pair $(\mathbf{w}_1, \mathbf{w}_2)$, choose a point J_1 on the hand base anywhere along the axis of $\mathbf{w}_2$, and choose a point J_2 anywhere along the axis of $\mathbf{w}_1$. The configuration of the first finger $J_1 J_2 T_{12}$ (as shown in Fig. 11) is determined.
   (b) For spring pair $(\mathbf{w}_3, \mathbf{w}_4)$, choose a point J_3 on the hand base anywhere along the axis of $\mathbf{w}_4$, and choose a point J_4 anywhere along the axis of $\mathbf{w}_3$. The configuration of the second finger $J_4 J_3 T_{34}$ (as shown in Fig. 11) is determined.
2. Determine the joint compliance for each elastic joint.
   (a) Calculate the joint twists $\mathbf{t}_i$ at joints J_i $(i = 1, 2, 3, 4)$ using Eq. (4).
   (b) Calculate the value of joint compliance at each finger joint using Eq. (16),

$$c_i = \frac{1}{k_i(\mathbf{t}_i^T \mathbf{w}_i)^2}, \qquad i = 1, 2, 3, 4. \tag{35}$$

With this final step, the configurations and the joint elastic properties of the 2 fingers are identified. The given stiffness is realized with the 2-finger hand.

4.3. Limitations of 2-finger hands

A 2-finger hand has the simplest construction needed to obtain force closure, but may not be able to achieve both a stable grasp of an object and a given compliance.

Although the locations of finger base joints J_1 and J_3 can be selected anywhere at the hand base along the axes of $\mathbf{w}_2$ and $\mathbf{w}_4$, the locations of these joints cannot, in general, be predetermined. This is because if these two joint locations are specified, the axes of $\mathbf{w}_2$ and $\mathbf{w}_4$ and their intersection T_{24} are determined, and a line ($\mathbf{w}_{13}$ in Fig. 10) passing through T_{24} may yield a twist $\mathbf{t}_{13}$ located on the same side of line l as T_{24}, which may violate one of the inequality conditions associated with Proposition 1.

In the synthesis procedure presented in Section 4.1.2, the first finger contact point with the body can be selected arbitrarily on the held body surface, while the second finger contact point is determined by the intersection of a line associated with the first contact point. Since the held body is finite, the line may not meet the body. If for all possible points on the body surface T_{12}, the corresponding line $\mathbf{w}_{12}$ does not intersect the body, then the given compliance cannot be attained for the body by *any* hand with 2 fingers.

The other main limitation of a 2-finger hand is that a stable grasp may not be obtained or maintained with the 2 fingers even if the configurations and joint elastic properties of each finger are identified. Although a fingertip in contact with an object can be modeled as a free revolute joint as described in Section 2.1, this model is valid only if the grasp (no-slip contact) is maintained. Since the contact kinematic constraint is unidirectional and unilateral, a fingertip may actually slide on the body surface, and the forces from the 2 fingers may result in body rotation and grasp failure. To prevent rotation, the value of friction and the relative position of the 2 contact points must satisfy a relatively complicated force closure condition. For Coulomb friction, a simple to evaluate necessary condition is illustrated in Fig. 12 for contact points A and B. Suppose that the friction cones at A and B are $c_{A\mu}$ and $c_{B\mu}$. A necessary condition for a stable grasp is that the line passing through A and B must be inside both cones as shown in Fig. 12a. The actual reaction force imposed on the body (within the friction cone) must also consider the compliance of each finger (which is beyond the scope of this paper). If line AB is outside one of the two cones (Fig. 12b), slipping occurs between the fingertip and the body surface, and a grasp cannot be achieved.

It can be seen that, in the realization of compliance with a 2-finger hand, the specified stiffness, the size and shape of the object, and the value of friction all play significant roles. For a given object, there is no guarantee that any 2-finger hand can achieve the desired compliance and a stable grasp. To realize more general planar compliant behavior with a compliant hand, more fingers (≥ 3) are needed.

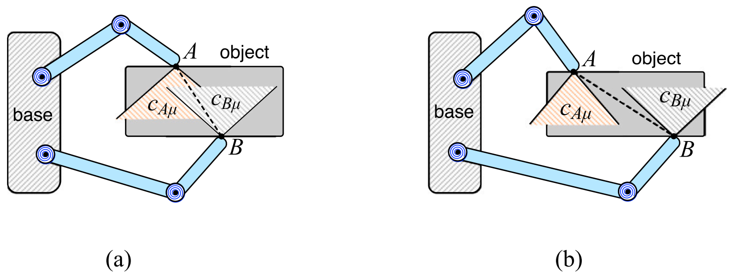

Fig. 12. Necessary condition to maintain a grasp. (a) Non-slip condition: line AB must be inside both friction cones c_{A_μ} and c_{B_μ} at the contact points A and B. (b) Sliding condition: line AB is outside one of the two friction cones.

5. Compliance realization with a 3-finger hand

In this section, the realization of an arbitrary compliance with a 3-finger hand is addressed. First, compliance synthesis with a 6-spring parallel mechanism in which 3 pairs of springs intersect at specified locations is presented. Then, the 6-spring parallel mechanism is converted to a 3-finger hand. A discussion on the 3-finger hand synthesis is then provided.

5.1. Stiffness synthesis for a 6-spring mechanism

In general, for any 6-spring parallel mechanism, if the 6 spring wrenches $\mathbf{w}_i$ $(i = 1, 2, \dots, 6)$ are not all tangent to any quadratic curve, an arbitrary stiffness matrix can always be expressed as [19]:

$$\mathbf{K} = k_1\mathbf{w}_1\mathbf{w}_1^T + k_2\mathbf{w}_2\mathbf{w}_2^T + \cdots + k_6\mathbf{w}_6\mathbf{w}_6^T, \tag{36}$$

where each coefficient k_i is uniquely determined by the following process.

A 3×3 symmetric matrix has 6 independent entries and can be represented by a 6-vector. Suppose a 3×3 symmetric matrix $\mathbf{Q}$ has entries q_{ij}, the associated 6-vector can be defined as:

$$\tilde{\mathbf{q}} = [q_{11}, q_{12}, q_{13}, q_{22}, q_{23}, q_{33}]^T. \tag{37}$$

With this representation, the stiffness matrix $\mathbf{K}$ and the rank-1 symmetric matrices $\mathbf{w}_i\mathbf{w}_i^T$ can be represented by 6-vectors $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{w}}_i$. If we denote

$$\tilde{\mathbf{W}} = [\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2, \dots, \tilde{\mathbf{w}}_6] \in \mathbb{R}^{6\times 6}, \qquad \mathbf{k} = [k_1, k_2, \dots, k_6]^T \in \mathbb{R}^6, \tag{38}$$

then, Eq. (36) can be written as:

$$\tilde{\mathbf{k}} = \tilde{\mathbf{W}}\mathbf{k}. \tag{39}$$

The spring stiffness vector $\mathbf{k}$ can be calculated by

$$\mathbf{k} = \tilde{\mathbf{W}}^{-1}\tilde{\mathbf{k}}. \tag{40}$$

Although for the generic case, an arbitrary stiffness matrix $\mathbf{K}$ can be uniquely expressed in the form of Eq. (36), the coefficients k_i may not all be positive. Thus, conditions to ensure that all k_i are positive are needed. Below, previously developed conditions [18,19] on 6-spring mechanisms that ensure each $k_i > 0$ are summarized, and then a modified procedure for the synthesis of a 6-spring parallel mechanism in which spring intersection locations are considered is developed (to facilitate the transformation to a 3-finger hand).

5.1.1. Review of 6-spring realization conditions

Consider a parallel mechanism described by 6 spring wrenches $\mathbf{w}_i$ $(i = 1, 2, \dots, 6)$. Suppose that $\mathbf{t}_{ij}$ $(i \neq j)$ is the unit twist centered at the intersection point T_{ij} of two wrench axes $\mathbf{w}_i$ and $\mathbf{w}_j$, then

$$\mathbf{t}_{ij}^T\mathbf{w}_i = 0, \qquad \mathbf{t}_{ij}^T\mathbf{w}_j = 0. \tag{41}$$

For any 5 non-concurrent spring wrenches, there is a unique quadratic curve that is tangent to the lines of action of the 5 spring wrenches. This quadratic curve is useful in selecting the location of the 6th spring wrench in the realization of a stiffness and can be determined by the procedure that follows.

Consider a spring wrench $\mathbf{w}_r$ in a 6-spring parallel mechanism, a 3×3 matrix $\mathbf{A}_r$ associated with the other 5 spring wenches can be calculated by:

$$\mathbf{A}_r = (\mathbf{t}_{im}^T\mathbf{w}_p)(\mathbf{t}_{jn}^T\mathbf{w}_p)\mathbf{t}_{ij}\mathbf{t}_{mn}^T - (\mathbf{t}_{ij}^T\mathbf{w}_p)(\mathbf{t}_{mn}^T\mathbf{w}_p)\mathbf{t}_{im}\mathbf{t}_{jn}^T, \tag{42}$$

where (i, j, m, n, p) is an arbitrary permutation of the set $\{1, 2, 3, 4, 5, 6\}$ excluding r. The symmetric matrix associated with $\mathbf{A}_r$ is:

$$\mathbf{G}_r = \mathbf{A}_r + \mathbf{A}_r^T. \tag{43}$$

The quadratic curve g_r tangent to the 5 spring axes in the xy-plane is determined by the equation:

$$g_r(x, y) = \mathbf{t}^T \mathbf{G}_r^{-1} \mathbf{t} = 0, \tag{44}$$

where $\mathbf{t}$ is a unit twist with instantaneous center at (x, y) that is defined by

$$\mathbf{t} = [y, -x, 1]^T. \tag{45}$$

As proved in [19], in order for Eq. (36) to have a solution, spring wrench $\mathbf{w}_r$ cannot be tangent to curve $g_r = 0$. For example, if the first 5 spring wrenches $\mathbf{w}_i$ $(i = 1, 2, \ldots, 5)$ are already selected, the 6th spring wrench $\mathbf{w}_6$ must either: (i) intersect quadratic curve $g_6 = 0$, or (ii) have no intersection with the curve. Also, it is proved [19] that the coefficient k_6 does not change its sign if and only if $\mathbf{w}_6$ maintains its intersection relationship with curve g_6. Note that the quadratic curve determined by 5 spring wrenches (Eq. (44)) is independent of the stiffness matrix $\mathbf{K}$.

Another quadratic curve that is important in selecting the spring locations is defined by the given stiffness matrix and any given 4 spring wrenches [18]. For a stiffness matrix $\mathbf{K}$ and any permutation (i, j, m, n) of $\{1, 2, 3, 4, 5, 6\}$, a 3×3 matrix $\mathbf{B}_{ijmn}$ is defined as:

$$\mathbf{B}_{ijmn} = (\mathbf{t}_{ij}^T \mathbf{K} \mathbf{t}_{mn})(\mathbf{t}_{im} \mathbf{t}_{jn}^T) - (\mathbf{t}_{im}^T \mathbf{K} \mathbf{t}_{jn})(\mathbf{t}_{ij} \mathbf{t}_{mn}^T). \tag{46}$$

The symmetric matrix associated with $\mathbf{B}_{ijnm}$ is:

$$\mathbf{H}_{ijmn} = \mathbf{B}_{ijmn} + \mathbf{B}_{ijmn}^T. \tag{47}$$

Then, the quadratic curve associated with stiffness $\mathbf{K}$ and 4 spring wrenches $(\mathbf{w}_i, \mathbf{w}_j, \mathbf{w}_m, \mathbf{w}_n)$ is:

$$h_{ijmn}(x, y) = \mathbf{t}^T \mathbf{H}_{ijmn}^{-1} \mathbf{t} = 0, \tag{48}$$

where $\mathbf{t}$ is the unit twist defined in Eq. (45).

With the 2 quadratic curves defined in Eqs. (44) and (48), the elastic behavior realization conditions on a 6-spring parallel mechanism can be expressed as:

Proposition 2. *For a given stiffness matrix* $\mathbf{K}$*, consider a 6-spring parallel mechanism described by spring wrenches* $\mathbf{w}_i$ $(i = 1, 2, \ldots, 6)$.

- *(a) When spring* $\mathbf{w}_r$ *varies in the plane while all the other five springs are constant, the corresponding* k_r *in Eq. (36) does not change sign if and only if* $\mathbf{w}_r$ *maintains its intersection property with the quadratic curve* g_r *that is tangent to the other 5 spring wrenches (determined by Eq. (44)).*
- *(b) For any 2 wrenches* $\mathbf{w}_i$ *and* $\mathbf{w}_j$*, the corresponding coefficients* k_i *and* k_j *in Eq. (36) have the same sign if and only if only one of the two intersects a single branch of the quadratic curve determined by* $\mathbf{K}$ *and the other four spring wrenches using Eq. (48).*
- *(c)* $\mathbf{K}$ *can be realized with a given mechanism if and only if for every combination of two springs, only one spring axis intersects a single branch of the quadratic curve determined by* $\mathbf{K}$ *and the other four springs using Eq. (48).*

The quadratic curves defined in Eqs. (44) and (48) will be used in the new synthesis procedure for a 6-spring parallel mechanism developed below for the realization of a given stiffness when springs must intersect at specific locations.

5.1.2. New 6-spring mechanism synthesis procedure

Since fingertip contact with the held body must be on the body surface, the intersections of 3 pairs of springs must be located on the surface of the held body. This requirement is enforced in the process described below. The geometry associated with each operation in the synthesis procedure is illustrated in Fig. 13.

1. For the given stiffness $\mathbf{K}$, calculate the location of the stiffness center C_k and the principal stiffnesses (k_χ, k_η, k_τ). Then obtain circle Γ_k and two lines l_θ^+ and l_θ^-.
   Since the realization conditions (Proposition 2) for 6-spring parallel mechanisms do not have equality constraints, the spring distribution conditions of Section 2.4 are useful in the selection of the locations of the springs.
2. Choose 2 points on the body surface. These points T_{12} and T_{34} will be where spring pairs $(\mathbf{w}_1, \mathbf{w}_2)$ and $(\mathbf{w}_3, \mathbf{w}_4)$ intersect. The corresponding unit twists $\mathbf{t}_{12}$ and $\mathbf{t}_{34}$ are obtained using Eq. (4).
3. Choose two lines passing through T_{12} that will be the axes of 2 springs $\mathbf{w}_1$ and $\mathbf{w}_2$. In selecting the 2 lines, one $(\mathbf{w}_2)$ must pass through the finger base. Similarly, choose two lines passing through T_{34}, which will be the axes of 2 springs $\mathbf{w}_3$ and $\mathbf{w}_4$ (with $\mathbf{w}_4$ passing through the finger base).
   When choosing these 4 lines, the spring distribution conditions described in Section 2.4 should be satisfied if possible.
   The unit twists $\mathbf{t}_{13}$, $\mathbf{t}_{14}$, $\mathbf{t}_{23}$ and $\mathbf{t}_{24}$ are calculated using Eq. (25) and the quadratic curve h_{1234} associated with stiffness $\mathbf{K}$ and the 4 wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$ is obtained using Eq. (48).
4. Judiciously choose a point T_{56} on the body surface. Choose 2 lines passing through T_{56} such that only one of the 2 lines intersects quadratic curve h_{1234} and at least one line intersects the hand base. These 2 lines are the axes of springs $\mathbf{w}_5$ and $\mathbf{w}_6$ (with $\mathbf{w}_6$ intersecting the hand base as shown in Fig. 13). By Proposition 2b, the corresponding coefficients k_5 and k_6 must have the same sign. Calculate k_5 using Eq. (40).

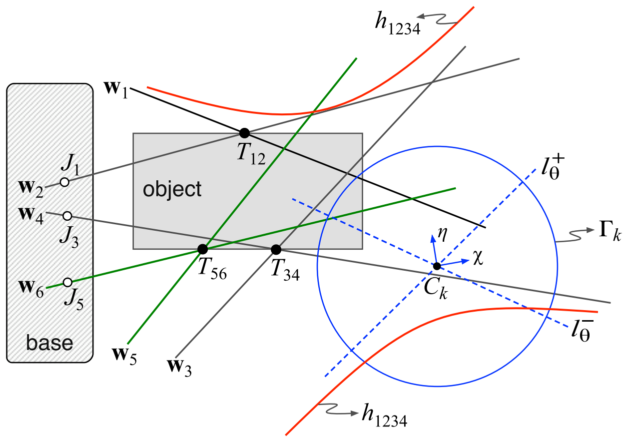

Fig. 13. Compliance synthesis with a 6-spring parallel mechanism. The intersections of three pairs of springs, T_{12}, T_{34} and T_{56}, must be on the surface of the held body.

(a) If $k_5 > 0$, then k_6 must be positive.
(b) If $k_5 < 0$, obtain the quadratic curve g_5 associated with the five spring wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_6)$ using Eq. (44). Then, rotate $\mathbf{w}_5$ about point T_{56} such that it changes the intersection relation with curve g_5, i.e., if $\mathbf{w}_5$ intersects g_5 (e.g., line l_5 shown in Fig. 14), then rotate it about T_{56} to a location that does not intersect the curve (e.g., line l_5' in Fig. 14). In this process, the intersection conditions for $\mathbf{w}_5$ and $\mathbf{w}_6$ relative to curve h_{1234} should be maintained. In satisfying these conditions, both k_5 and k_6 are positive.

5. Adjust the spring positions to ensure that each coefficient k_i is positive. For the obtained six spring wrenches, calculate the corresponding coefficients k_i using Eq. (40).

   (a) If one coefficient is negative (e.g., $k_1 < 0$), then obtain the quadratic curve $h_{2345} = 0$ determined by Eq. (48) using the 4 spring wrenches $(\mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5)$, then rotate $\mathbf{w}_6$ (or $\mathbf{w}_1$) about T_{56} (or T_{12}) such that Proposition 2b is satisfied for $\mathbf{w}_1$ and $\mathbf{w}_6$ (only one of $(\mathbf{w}_1, \mathbf{w}_6)$ intersects $h_{2345} = 0$). Thus, k_1 is positive.
   Alternatively, for $k_1 < 0$, obtain the quadratic curve $g_1 = 0$ determine by the other 5 spring wrenches using Eq. (42), then rotate $\mathbf{w}_1$ about T_{12} such that it changes the contact property with g_1. Thus, k_1 is positive.
   (b) If two coefficients are negative (e.g., k_1 and k_2), $\mathbf{w}_1$ or $\mathbf{w}_2$ must be located so that only one of them intersects curve h_{3456}. Use the process described in Step 4(b) for $\mathbf{w}_5$ and $\mathbf{w}_6$ to make both k_1 and k_2 positive (using instead g_1 or g_2 defined in Eq. (44)).

   Since $\mathbf{K}$ is positive definite, the number of negative coefficients in Eq. (36) is always 3 or less. As such, at most two adjustment iterations are needed.
6. Calculate the value of stiffness k_i for each joint using Eq. (40).

With the final step, the locations of 6 springs and their spring rates are determined.

5.2. Conversion of 6-spring mechanism to 3-finger hand

Using the concept of dual elastic mechanisms, 2-spring subsystems $(\mathbf{w}_1, \mathbf{w}_2)$, $(\mathbf{w}_3, \mathbf{w}_4)$ and $(\mathbf{w}_5, \mathbf{w}_6)$ are converted to serial mechanisms each having 2 elastic joints.

1. Choose the first joint location for each finger.
   For spring pair $(\mathbf{w}_1, \mathbf{w}_2)$, choose J_1 at the hand base along the axis of $\mathbf{w}_2$.
   Similarly, joints J_3 and J_5 at the hand base are chosen for the other 2 fingers along the axes of $\mathbf{w}_4$ and $\mathbf{w}_6$, respectively, as shown in Fig. 15.
2. Choose the second joint location for each finger.
   For spring pair $(\mathbf{w}_1, \mathbf{w}_2)$, choose a point J_2 anywhere along the axis of $\mathbf{w}_1$. The configuration of the first finger $J_1 J_2 T_{12}$ (as shown in Fig. 15) is determined.
   Similarly, joints J_4 and J_6 are chosen for the other 2 fingers along axes $\mathbf{w}_3$ and $\mathbf{w}_5$, respectively as shown in Fig. 15.
3. Determine the joint compliance for each elastic joint.
   (a) Calculate the joint twists $\mathbf{t}_i$ at joints J_i $(i = 1, 2, \ldots, 6)$ using Eq. (4).

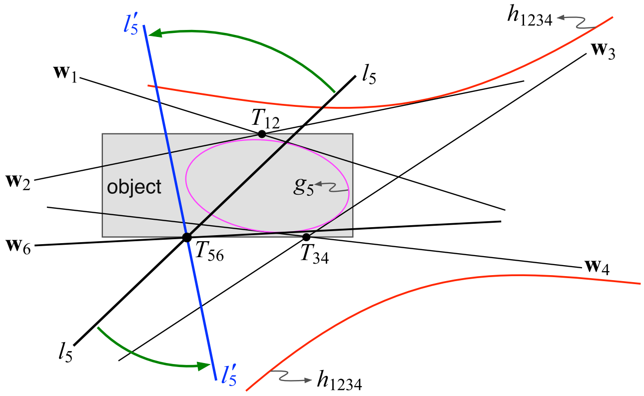

Fig. 14. Adjustment of a spring location to ensure positive spring coefficients. Spring $\mathbf{w}_5$ is rotated about point T_{56} based on its geometric relations to the two quadratic curves h_{1234} and g_5.

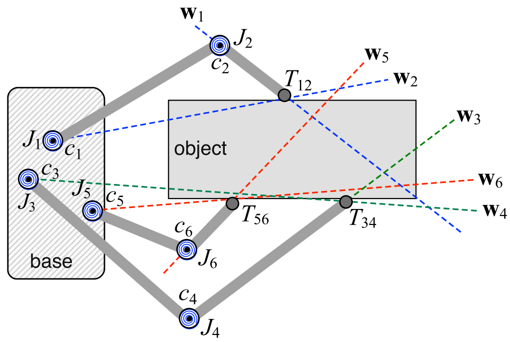

Fig. 15. Synthesis of a 3-finger compliant hand. The 3-finger hand is obtained from a 6-spring parallel mechanism that realizes the given compliance.

(b) Calculate the value of joint compliance at each finger joint using Eq. (16),

$$c_i = \frac{1}{k_i(\mathbf{t}_i^T \mathbf{w}_i)^2}, \qquad i = 1, 2, \ldots, 6. \tag{49}$$

With this final step, the configurations and the joint elastic properties of the 3 fingers are identified. The given stiffness is realized with the 3-finger hand.

5.3. Discussion on the 3-finger hand synthesis

Synthesis of a 3-finger compliant hand is based on the synthesis of a 6-spring parallel mechanism for the realization of a given stiffness. Since compliance realization with a 6-spring mechanism does not have equality constraints, a larger design space is available for selecting the spring locations. Although the intersection of 2 spring wrench pairs can be chosen arbitrarily on the surface of the held body, the selection should be judicious so that the necessary condition on the distribution of springs described in Section 2.4 is satisfied. For example, if in the selection of the first 4 springs, the condition cannot be satisfied, then in the selection of 5th and 6th spring locations (Step 4), the condition must be satisfied.

Since a finger contact only provides unidirectional kinematic constraint, the selections of fingertip contact point (the intersections of spring pairs) at the held body's surface should be judicious to achieve a stable grasp. For example, for the polygonal object illustrated in Fig. 16, if T_{12} and T_{34} are already selected, the intersection of springs $\mathbf{w}_5$ and $\mathbf{w}_6$ should be located on the opposite side between points A and B (Fig. 16a). If T_{34} is selected outside segment AB (Fig. 16b), a stable grasp may not be achieved.

Unlike compliance realization with a fully parallel or fully serial mechanism in which the locations of connection to the reference body are arbitrary, synthesis of a multi-finger hand depends on the shape of the held body because connections to the held object must be on its surface. As such, there is no guarantee that a given stiffness can be realized by a 3-finger hand on a given body. For example, in the synthesis process, the third fingertip (point T_{56}, the intersection of $\mathbf{w}_5$ and $\mathbf{w}_6$) cannot be selected inside the quadratic curve h_{1234}. Otherwise, both $\mathbf{w}_5$ and $\mathbf{w}_6$ will intersect the quadratic curve (Fig. 17), which violates Proposition 2b. If on the body surface, no point satisfies this condition, the given elastic behavior cannot be realized by the 3-finger hand.

In the 3-finger hand elastic behavior synthesis, the geometry of the hand cannot be completely specified. However, some geometric properties for each finger can be specified. In the synthesis process, the finger base joint locations can be predetermined

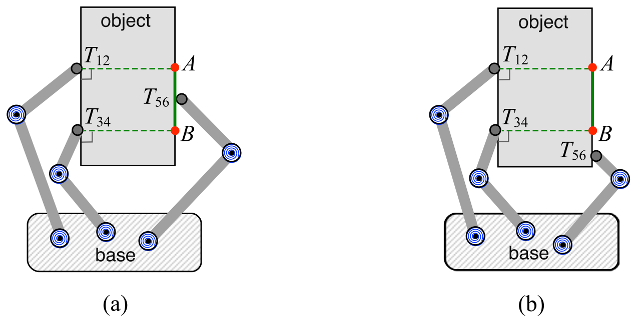

Fig. 16. Selection of fingertip contact in the synthesis procedure. (a) Fingertip T_{56} selected on the body between segment AB for stable grasp. (b) Fingertip T_{56} outside segment AB may yield rotation.

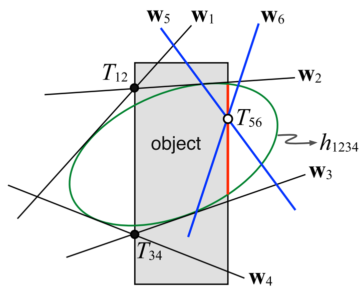

Fig. 17. Location restriction on the third fingertip. If point T_{56} is located inside the quadratic curve h_{1234}, the given compliance cannot be realized.

at the hand base only for the first 2 fingers (J_1 and J_3 in Fig. 15). Since the finger base joint for the third finger (J_5 in Fig. 15) must be located along $\mathbf{w}_6$ (which is selected based on the quadratic curve determined by the first 4 springs), its location cannot be independently specified. For each finger, since the second elastic joint can be located anywhere on the corresponding spring wrench axis, only one of the finger link lengths can be specified. To realize a compliance with a 3-finger hand having given geometry, more elastic joints in each finger are needed.

6. Examples

In this section, numerical examples are provided to illustrate the synthesis procedures. The compliance be to realized is obtained from an optimization of the compliance for a peg-in-hole assembly problem [28]. The dimensions of the rectangular peg and the hand base are illustrated in Fig. 18. In the coordinate frame shown, the desired compliance is:

$$\mathbf{C} = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 10 & -1.8 \\ 0 & -1.8 & 0.37 \end{bmatrix}. \tag{50}$$

The stiffness matrix is:

$$\mathbf{K} = \mathbf{C}^{-1} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 0.8043 & 3.9130 \\ 0 & 3.9130 & 21.7391 \end{bmatrix}.$$

Using the process described in [18], the stiffness center C_k is calculated to be at

$$\mathbf{r}_k = [4.8649,\ 0]^T.$$

The 3 principal stiffnesses are calculated to be:

$$[k_\chi, k_\eta, k_\tau] = [\,50,\ 0.8043,\ 2.7027\,].$$

The radius of circle Γ_k is calculated using Eq. (10) to be:

$$\rho_k = 0.2306.$$

Using Eq. (12), the 2 lines l_θ^+ and l_θ^- are determined. The compliance center, circle Γ_k and lines l_θ^+ and l_θ^- are illustrated in Fig. 18.

Below, the compliance is first synthesized for a 2-finger hand, then, for a 3-finger hand. In both synthesis processes, it is required that the contact point of each fingertip with the peg be located on the peg surface not to exceed half of the peg length so that the fingers do not interfere with the mating part during assembly.

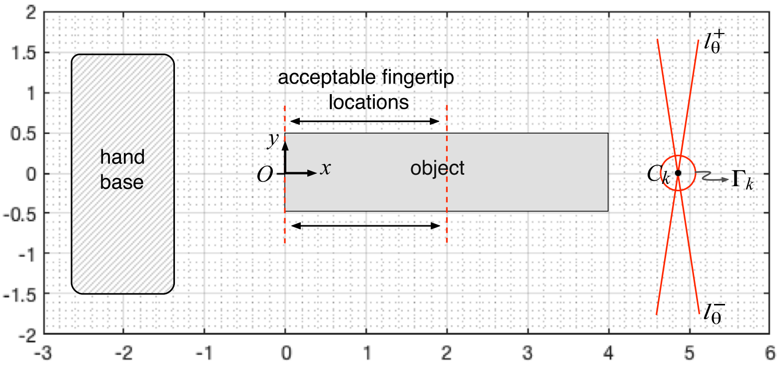

Fig. 18. Compliance realization for a given rectangular object and hand base.

6.1. Compliance synthesis for a 2-finger hand

Using the synthesis procedure presented in Section 4.1, a 4-spring parallel mechanism is obtained. Then, the 4-spring parallel mechanism is converted to a 2-finger compliant hand using the process described in Section 4.2.

6.1.1. Synthesis for a 4-spring parallel mechanism

A 4-spring parallel mechanism that realizes the given compliance can be obtained by the following steps. The geometry associated with each step is illustrated in Fig. 19a.

1. Choose a point T_{12} on the body with $x < 2$. This point will be the intersection of springs $\mathbf{w}_1$ and $\mathbf{w}_2$. Calculate the wrench associated with twist $\mathbf{t}_{12}$ centered at T_{12}:

   $$\mathbf{w}_{12} = \mathbf{K}\mathbf{t}_{12}.$$

   Bound the locations of T_{12} such that $\mathbf{w}_{12}$ intersects the opposite edge with $x < 2$. For the given compliance and all x in the range $0.3 \le x < 2$, $T_{12} = (x, 0.5)$ yields a wrench satisfying this condition. Thus, any point on the line segment ($0.3 \le x < 2$, $y = 0.5$) can be selected for T_{12}.
2. Choose point T_{34}, the intersection of spring $\mathbf{w}_3$ and $\mathbf{w}_4$. This point is determined by the intersection of wrench $\mathbf{w}_{12}$ with the opposite edge. Note that when T_{12} moves along the peg edge with $0.3 \le x < 2$, T_{34} changes along the opposite edge. Here, T_{12} is selected such that T_{34} is located at the opposite edge having the same horizontal coordinate x. This point is determined to be $T_{12} = (1.3748, 0.5)$ and the corresponding point on the opposite edge to be $T_{34} = (1.3748, -0.5)$.
3. Choose $\mathbf{w}_{13}$. Here, the selected line orientation is along the y-axis and passes through point $(-2.5, 0)$. Then,

   $$\mathbf{w}_{13} = [0,\ -1,\ 2.5]^T.$$

   Calculate the twist:

   $$\mathbf{t}_{13} = \mathbf{C}\mathbf{w}_{13} = [0,\ -14.5000,\ 2.7250]^T.$$

   The center of $\mathbf{t}_{13}$, T_{13}, is determined to be $(5.3211, 0)$.
4. Choose T_{24} on wrench $\mathbf{w}_{13}$ between points P and Q. This point is selected to be $(-2.5, 0)$.
5. Determine the 4 spring axes. Since $\mathbf{w}_1$ passes through points T_{12} and T_{13}, $\mathbf{w}_1$ can be determined by normalizing the wrench

   $$\hat{\mathbf{w}}_1 = \mathbf{t}_{12} \times \mathbf{t}_{13}.$$

   The other 3 spring wrenches can be determined similarly. The 4 unit spring wrenches obtained are:

   $$[\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4] = \begin{bmatrix} -0.9921 & -0.9918 & 0.9921 & -0.9918 \\ 0.1257 & -0.1280 & 0.1257 & 0.1280 \\ 0.6688 & 0.3199 & 0.6688 & -0.3199 \end{bmatrix}$$

   and are illustrated in Fig. 19a.
6. Calculate the spring stiffness for each spring. Using Eqs. (29)–(32), the 4 spring rates are:

   $$[k_1,\ k_2,\ k_3,\ k_4] = [23.9700,\ 1.4323,\ 23.9699,\ 1.4323].$$

With this step, a 4-spring parallel mechanism is obtained.

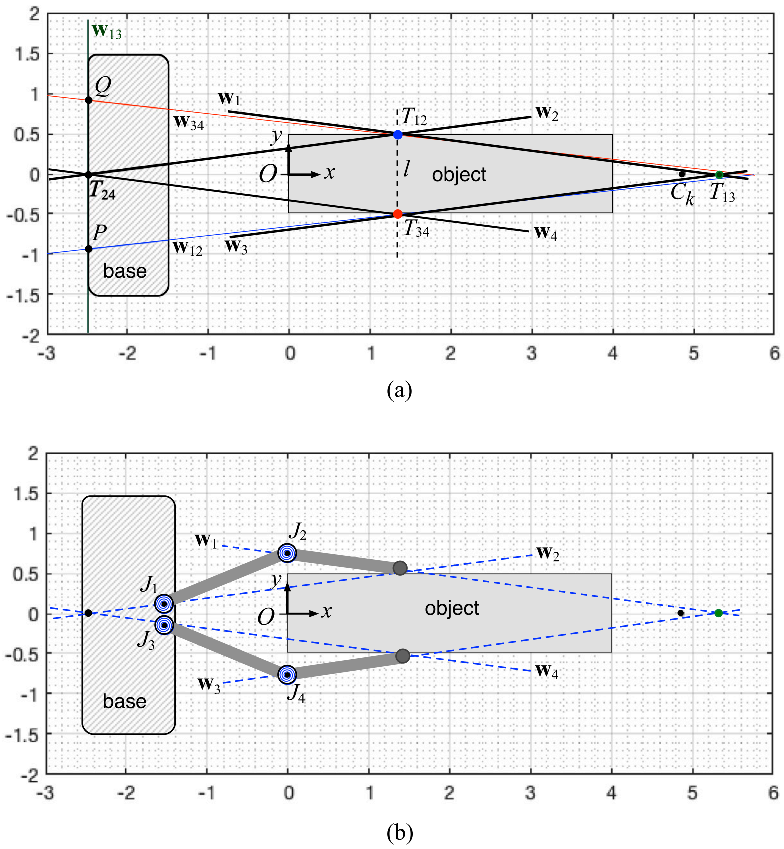

Fig. 19. Compliance realization for a 2-finger compliant hand. (a) A 4-spring parallel mechanism that realizes the given compliance. (b) A 2-finger hand converted from the 4-spring parallel mechanism using the concept of dual elastic mechanisms.

6.1.2. Conversion to a 2-finger hand

Using the concept of dual elastic mechanisms, the 4-spring parallel mechanism is converted to a 2-finger hand.

(1) For spring wrench pair $(\mathbf{w}_1, \mathbf{w}_2)$, choose a point J_2 along the axis of $\mathbf{w}_1$ and choose a point J_1 at the hand base along the axis of $\mathbf{w}_2$. These 2 points are the locations of the 2 elastic joints of the first finger.
Here, J_1 with $x_1 = -1.5$ and J_2 with $x_2 = 0$ are selected. Then, using the reciprocal condition ($\mathbf{t}_i^T \mathbf{w}_j = 0$), the y-components of the 2 points can be determined yielding

$$\mathbf{t}_1 = [0.1290,\ 1.5,\ 1]^T,$$
$$\mathbf{t}_2 = [0.6741,\ 0,\ 1]^T.$$

Therefore, the locations of J_1 and J_2 are $(-1.5, 0.1290)$ and $(0, 0.6741)$.

(2) For spring wrench pair $(\mathbf{w}_3, \mathbf{w}_4)$, joints J_3 and J_4 can be selected the same way as in Step 1. Here, these 2 joints are selected symmetrical about the x-axis as shown in Fig. 19b. The 2 joint locations are $(-1.5, -0.1290)$ and $(0, -0.6741)$. The corresponding joint twists are:

$$\mathbf{t}_3 = [-0.1290,\ 1.5,\ 1]^T,$$
$$\mathbf{t}_4 = [-0.6741,\ 0,\ 1]^T.$$

(3) Determine the elastic property for each compliant joint. Using Eq. (16), the joint compliance for joints J_i are obtained:

$$[c_1,\ c_2,\ c_3,\ c_4] = [0.0784,\ 5.7449,\ 0.0784,\ 5.7449].$$

With this step, a symmetric 2-finger hand that realizes the given compliance is obtained. The geometry of the 2-finger hand is illustrated in Fig. 19b.

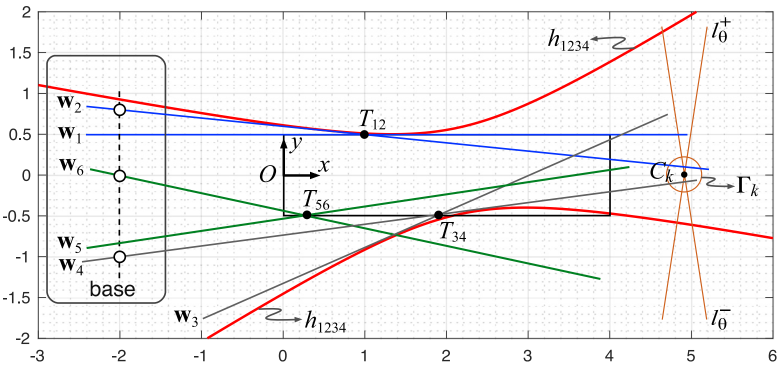

Fig. 20. Selection of spring wrenches in a 6-spring parallel mechanism. The spring distribution conditions should be considered for the first 4 springs. The selection of the last 2 springs is based on the quadratic curve determined by the first 4 springs.

6.1.3. Result verification

Since each finger can be viewed as a 3-joint serial mechanism with the fingertip being a free joint (having joint compliance $c = +\infty$), to numerically validate the obtained result, a large value is assigned to the 2 free joints at the fingertip T_{12} and T_{34}: $c_{12} = c_{34} = 10^6$.

The compliance matrix associated with the first finger is calculated using

$$\mathbf{C}_1 = c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + c_{12}\mathbf{t}_{12}\mathbf{t}_{12}^T,$$

and the corresponding stiffness matrix is calculated to be:

$$\mathbf{K}_1 = \mathbf{C}_1^{-1} = \begin{bmatrix} 25.0118 & -2.8072 & -16.3653 \\ -2.8072 & 0.4020 & 1.9563 \\ -16.3653 & 1.9562 & 10.8721 \end{bmatrix}.$$

The compliance matrix associated with the second finger is calculated using

$$\mathbf{C}_2 = c_3\mathbf{t}_3\mathbf{t}_3^T + c_4\mathbf{t}_4\mathbf{t}_4^T + c_{34}\mathbf{t}_{34}\mathbf{t}_{34}^T,$$

and the corresponding stiffness matrix is calculated to be:

$$\mathbf{K}_2 = \mathbf{C}_2^{-1} = \begin{bmatrix} 25.0118 & 2.8072 & 16.3653 \\ 2.8072 & 0.4020 & 1.9563 \\ 16.3653 & 1.9562 & 10.8721 \end{bmatrix}.$$

The stiffness of the 2-finger hand is:

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 = \begin{bmatrix} 50.0236 & 0 & 0 \\ 0 & 0.8040 & 3.9126 \\ 0 & 3.9126 & 21.7442 \end{bmatrix},$$

and the compliance of the hand is:

$$\mathbf{C} = \mathbf{K}^{-1} = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 10.0014 & -1.7996 \\ 0 & -1.7996 & 0.3698 \end{bmatrix}$$

which is very close to the desired compliance in Eq. (50). Thus, the realization of the elastic behavior is achieved.

6.2. Compliance synthesis for a 3-finger hand

The use of an additional finger allows constraints on joint base locations to be satisfied. The locations are selected to be $(-2, 0.8)$, $(-2, 0)$ and $(-2, -1)$. Using the synthesis procedure presented in Section 5.1, a 6-spring parallel mechanism is obtained. Then, the 6-spring parallel mechanism is converted to a 3-finger compliant hand as described in Section 5.2.

6.2.1. Synthesis for a 6-spring parallel mechanism

A 6-spring parallel mechanism that realizes the given compliance $\mathbf{C}$ can be obtained by the following steps.

(1) Choose the first fingertip-peg contact point T_{12} and the first spring pair $(\mathbf{w}_1, \mathbf{w}_2)$ intersecting at T_{12}. Here, T_{12} is chosen to be at $(1, 0.5)$. When selecting $\mathbf{w}_1$ and $\mathbf{w}_2$, the spring distribution conditions described in Section 2.4 should be considered. Here,

$\mathbf{w}_1$ is selected to be along the peg edge and $\mathbf{w}_2$ is selected to pass through the finger base joint J_1 at point $(-2, 0.8)$. With this selection, $\mathbf{w}_2$ intersects circle Γ_k as illustrated in Fig. 20. The 2 spring wrenches are:

$$\mathbf{w}_1 = [1,\ 0,\ -0.5]^T,$$
$$\mathbf{w}_2 = [-0.9950,\ 0.0995,\ 0.5970]^T.$$

(2) Choose the second fingertip contact point T_{34} and the second spring pair $(\mathbf{w}_3, \mathbf{w}_4)$ which intersect at T_{34}. Here, T_{34} is chosen to be at $(1.9, -0.5)$, $\mathbf{w}_3$ and $\mathbf{w}_4$ are selected to pass through points $(3, 0)$ and the finger base joint J_3 at $(-2, -1)$. The 2 spring wrenches are:

$$\mathbf{w}_3 = [0.9104,\ 0.4138,\ 1.2414]^T,$$
$$\mathbf{w}_4 = [0.9919,\ 0.1272,\ 0.7376]^T.$$

(3) Select the third fingertip contact point T_{56} and the third spring pair $(\mathbf{w}_5, \mathbf{w}_6)$ intersecting at T_{56}.
(i) First generate the quadratic curve h_{1234} determined by the first 4 spring wrenches ($\mathbf{w}_i$, $i = 1, 2, 3, 4$). Matrix $\mathbf{B}_{1234}$ associated with the 4 spring wrenches is calculated using Eq. (46) as:

$$\mathbf{B}_{1234} = (\mathbf{t}_{12}\mathbf{K}\mathbf{t}_{34})\mathbf{t}_{13}\mathbf{t}_{24}^T - (\mathbf{t}_{13}\mathbf{K}\mathbf{t}_{24})\mathbf{t}_{12}\mathbf{t}_{34}^T,$$

where $\mathbf{t}_{ij}$ is the twist at the intersection of $\mathbf{w}_i$ and $\mathbf{w}_j$ and can be calculated by Eq. (25). For the 4 wrenches selected in Steps 1 and 2, the symmetric matrix associated with $\mathbf{B}_{1234}$ calculated using Eq. (47) is:

$$\mathbf{H}_{1234} = \begin{bmatrix} 0.0032 & 0.0076 & -0.0008 \\ 0.0076 & -0.1010 & 0.0345 \\ -0.0008 & 0.0345 & -0.0160 \end{bmatrix}.$$

The inverse of $\mathbf{H}_{1234}$ is:

$$\mathbf{H}_{1234}^{-1} = \begin{bmatrix} 220.4964 & 48.4515 & 93.0662 \\ 48.4515 & -26.6373 & -59.7431 \\ 93.0662 & -59.7431 & -195.5482 \end{bmatrix}.$$

The quadratic curve h_{1234} calculated using Eq. (48) is:

$$h_{1234}(x, y) = \mathbf{t}^T\mathbf{H}_{1234}^{-1}\mathbf{t} = 0$$

where $\mathbf{t} = [y, -x, 1]^T$ is the unit twist centered at (x, y). As shown in Fig. 20, curve h_{1234} is a hyperbola having two branches.
(ii) For spring pair $(\mathbf{w}_5, \mathbf{w}_6)$, only one is allowed to intersect curve h_{1234}. Here, T_{56} is selected to be at $(0.3, -0.5)$, wrench $\mathbf{w}_5$ is selected to pass through point $(2, -0.25)$ without intersecting curve h_{1234}, and $\mathbf{w}_6$ is selected to pass through the finger base joint J_5 at $(-2, 0)$. As shown in Fig. 20, $\mathbf{w}_6$ intersects the curve. With this selection, k_5 and k_6 have the same sign. The selected 2 spring wrenches are:

$$\mathbf{w}_5 = [0.9894,\ 0.1455,\ 0.5383]^T,$$
$$\mathbf{w}_6 = [-0.9772,\ 0.2124,\ -0.4249]^T.$$

(4) Adjust the spring locations to achieve passive realization.
For the selected 6 spring wrenches, calculate the coefficients k_i using Eq. (40). The 6 spring rates are:

$$\mathbf{k} = [-1.8745,\ 29.0059,\ 0.5321,\ 17.5103,\ 4.6414,\ 0.9891]^T.$$

Since $k_1 < 0$, at least one spring location needs to be adjusted. Here consider the quadratic curve g_1 determined by the 5 spring wrenches $(\mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6)$. The matrix $\mathbf{A}_1$ associated with the 5 spring wrenches is calculated using Eq. (42):

$$\mathbf{A}_1 = (\mathbf{t}_{23}^T\mathbf{w}_4)(\mathbf{t}_{56}^T\mathbf{w}_4)\mathbf{t}_{25}\mathbf{t}_{36}^T - (\mathbf{t}_{25}^T\mathbf{w}_4)(\mathbf{t}_{36}^T\mathbf{w}_4)\mathbf{t}_{23}\mathbf{t}_{56}^T.$$

The symmetric matrix associated with $\mathbf{A}_1$ is calculated as:

$$\mathbf{G}_1 = \mathbf{A}_1 + \mathbf{A}_1^T = \begin{bmatrix} 0.0009 & 0.0110 & -0.0020 \\ 0.0110 & 0.0382 & -0.0214 \\ -0.0020 & -0.0214 & 0.0080 \end{bmatrix}.$$

The inverse of $\mathbf{G}_1$ is:

$$\mathbf{G}_1^{-1} = 10^3 \begin{bmatrix} 2.6586 & 0.7987 & 2.7914 \\ 0.7987 & 0.1876 & 0.6987 \\ 2.7914 & 0.6987 & 2.6813 \end{bmatrix}.$$

The quadratic curve g_1 is obtained by:

$$g_1(x, y) = \mathbf{t}^T\mathbf{G}_1^{-1}\mathbf{t} = 0$$

and is illustrated in Fig. 21. It can be seen that $\mathbf{w}_1$ intersects hyperbola g_1.

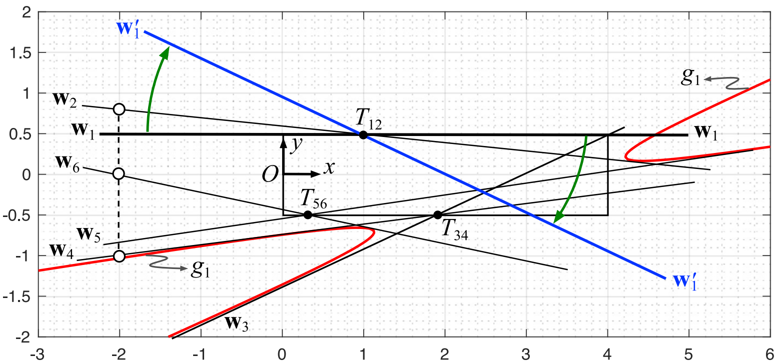

Fig. 21. Adjustment of selected spring locations. To make k_1 positive, rotate $\mathbf{w}_1$ about T_{12} to location $\mathbf{w}_1'$ so that it does not intersect quadratic curve g_1.

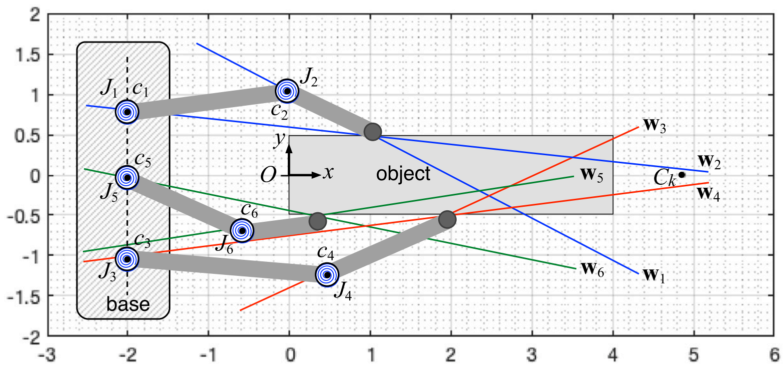

Fig. 22. Compliance synthesis for a 3-finger hand. The geometry of each finger is obtained from the corresponding spring pair in the 6-spring parallel mechanism that realizes the given behavior.

Wrench $\mathbf{w}_1$ is rotated about T_{12} to a location $\mathbf{w}_1'$ such that it has no intersection with curve g_1. The selected wrench $\mathbf{w}_1'$ passes through point $(2, 0)$ and is calculated to be:

$$\mathbf{w}_1' = [0.8944,\ -0.4472,\ -0.8944]^T.$$

With the adjusted $\mathbf{w}_1$, the 6 spring coefficients are calculated using Eq. (40) to be:

$$\mathbf{k} = [0.6191,\ 26.6801,\ 0.0336,\ 19.3915,\ 3.6346,\ 0.4455]^T.$$

The selected 6-spring parallel mechanism passively realizes the given compliance $\mathbf{C}$.

6.2.2. Conversion to a 3-finger hand

Using the concept of dual elastic mechanisms, the obtained 6-spring parallel mechanism is converted to a 3-finger hand.

For each spring pair, since the finger base joint is already determined, only one joint location needs to be selected along the corresponding spring wrench.

For spring pair $(\mathbf{w}_1, \mathbf{w}_2)$, the second elastic joint J_2 can be anywhere along the axis of $\mathbf{w}_1$. Here, the location of J_2 is selected to be (0, 1).

For spring pair $(\mathbf{w}_3, \mathbf{w}_4)$, the location of elastic joint J_4 is selected to have $x = 0.5$. Then the joint location of J_4 is determined to be $(0.5, -1.1363)$.

For spring pair $(\mathbf{w}_5, \mathbf{w}_6)$, the location of elastic joint J_6 is selected to have $x = -0.5$. The joint location of J_6 is determined to be $(-0.5, -0.6176)$.

The selections of the elastic joint locations are illustrated in Fig. 22. The joint twists for the 6 elastic joints are:

$$\mathbf{T} = \begin{bmatrix} 0.8 & 1 & -1 & -1.1363 & 0 & -0.6176 \\ 2 & 0 & 2 & -0.5 & 2 & 0.5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The joint compliance for each elastic joint is calculated using Eq. (16):

$$\mathbf{c} = [1.4021,\ 0.2366,\ 22.1935,\ 0.2512,\ 0.4000,\ 27.6614].$$

Thus, the hand configuration and elastic properties that realizes the given compliance **C** are identified.

Using the process presented in Section 6.1.3 for 2-finger hand case, the obtained result are numerically validated. Similar to the 2-finger hand, a large value of compliance is assigned to the 3 free joints T_{12}, T_{34} and T_{56}: $c_{12} = c_{34} = c_{56} = 10^6$. The stiffness of the 3-finger hand calculated is:

$$\mathbf{K} = \begin{bmatrix} 50.0012 & 0 & 0.0006 \\ 0 & 0.8043 & 3.9130 \\ 0.0006 & 3.9130 & 21.7394 \end{bmatrix},$$

and the compliance calculated is:

$$\mathbf{C} = \begin{bmatrix} 0.0200 & 0 & 0 \\ 0 & 10.0015 & -1.8002 \\ 0 & -1.8002 & 0.3700 \end{bmatrix}$$

which confirms that the desired elastic behavior is achieved by the 3-finger hand.

7. Summary

In this paper, the realization of any specified planar compliance for an object held by a compliant hand having multiple fingers is addressed. The concept of dual elastic mechanisms is extended to non-full rank stiffnesses and is then used for the realization of compliance with 2- and 3-finger compliant hands in which each finger has 2 elastic joints. Synthesis procedures for both 2-finger and 3-finger hands are developed. The limitations on achievable compliant behaviors for hands having 2, 3 and an arbitrary number of fingers are discussed. Since the synthesis procedures are based on the necessary and sufficient conditions, the resulting hand is guaranteed to realize any specified compliance if its realization is possible. The theory presented in this paper can be used in the design of general multi-serial mechanism compliance and of compliant hands for robotics applications.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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