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Compliance Realization With Planar Serial Mechanisms Having Fixed Link Lengths

In this article, the synthesis of any specified planar compliance with a serial elastic mechanism having previously determined link lengths is addressed. For a general n-joint serial mechanism, easily assessed necessary conditions on joint locations for the realization of a given compliance are identified. Geometric construction-based synthesis procedures for five-joint and six-joint serial mechanisms having kinematically redundant fixed link lengths are developed. By using these procedures, a given serial manipulator can achieve a large set of different compliant behaviors by using variable stiffness actuation and by adjusting the mechanism configuration. [DOI: 10.1115/1.4053819]

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1 Introduction

To regulate contact forces and ensure accurate relative positioning, passive compliance is needed in constrained robotic manipulation. A general model for compliance is a rigid body supported by an elastic suspension. A compliant behavior is characterized by the relationship between a force (wrench) applied to the body and the resulting displacement (twist) of the body. If small displacements are considered, the wrench–twist relationship can be represented by a symmetric positive definite matrix, the compliance matrix **C**, or the stiffness matrix **K**, the inverse of **C**.

In practice, an elastic suspension can be achieved by elastic components connected in parallel or in series. Realization of a given compliance involves identifying the geometric and elastic properties of each component such that the desired compliance is attained. This article focuses on serial mechanisms with revolute joints, each having some type of passive compliance. The previous work in this area addressed the problem of finding any mechanism (one with unspecified geometry) to realize a selected compliance. Here, we address the issue of assessing whether a given mechanism is capable of realizing a selected compliance and, if so, how it must be configured to do so. A serial manipulator having fixed link lengths can achieve a large set of different compliant behaviors by adjusting the joint compliance (e.g., using a cobot with variable stiffness actuation [1]) and by adjusting the mechanism configuration (using kinematic redundancy).

1.1 Related Work.

Many researchers investigated general compliant behaviors. In the *analysis* of spatial compliance, screw theory and Jacobian analysis [2–9], and Lie groups [10,11] have been widely used. In the recent work on the *synthesis* of compliance, mechanisms are designed to realize any specified compliance. Most previous synthesis approaches were based on an algebraic rank-1 decomposition of the stiffness/compliance matrix [12–15]. In Refs. [16,17], some geometric considerations on the mechanism were included in the synthesis procedures. In Ref. [18], a completely geometry-based approach to the realization of an arbitrary *spatial* stiffness was presented.

In Refs. [19,20], the synthesis of planar stiffness with parallel mechanisms having specific topologies was presented. In Refs. [21–26], compliant behaviors associated with mechanisms composed of distributed elastic components were investigated.

In closely related work in the realization of *planar* compliances [27–30], geometry-based approaches were developed for the design of fully parallel or fully serial mechanisms having n ($3 \le n \le 6$) elastic components. Necessary and sufficient conditions on the elastic component *locations* of corresponding mechanisms of a given topology were identified for the realization of any specified planar compliant behavior. The link lengths in these mechanisms were not considered in the synthesis procedures [27–30].

In Ref. [31], conditions required to achieve a special isotropic compliance in a 2D Euclidean space with a serial mechanism with specified link lengths was presented.

1.2 Contribution of the Paper.

Previously developed necessary and sufficient conditions [27–30] on mechanism geometry for the realization of a given compliance must be satisfied for any n-component ($3 \le n \le 6$) mechanism. These conditions are the foundation for the development of general planar compliance synthesis procedures. The main limitations of this prior work are as follows:

(1) Each n-joint serial mechanism had no constraints imposed on its link lengths. Thus, the serial mechanism obtained from the synthesis procedure to realize one selected compliance is very unlikely to be able to realize a different compliance.
(2) The issue of whether or how a specified compliance can be realized by a *given* mechanism was not addressed.

These restrictions limit the use of the existing theories in practical application and are the motivation of this work. When link lengths are considered, the distance between two adjacent joints J_i and J_{i+1} is constant. In selecting a configuration of an n-joint serial mechanism for the realization of a compliance, $(n-1)$ nonlinear constraints on the n-joint locations must be satisfied. The main contributions of this article are as follows:

(1) Identification of a set of necessary conditions on a general n-joint serial mechanism. These conditions provide greater insight into the distribution of joint locations of a serial mechanism in the realization of a compliance.
(2) Development of new synthesis procedures that take into account the known link lengths of any specific serial manipulator. By using these procedures, a large and continuous, but constrained, space of compliances can be realized with a

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single mechanism by identifying its configuration and joint compliances.

1.3 Overview. This article addresses the passive realization of an arbitrary planar (3×3) compliance with a serial compliant mechanism having fixed link lengths and variable stiffness actuators.

This article is outlined as follows. In Sec. 2, screw representation of planar mechanism configuration is first reviewed. A set of *necessary* conditions on the geometry of a general n-joint serial mechanism for the realization of a given compliance is then identified. Necessary and sufficient conditions for the realization of a compliance with $5R$ and $6R$ mechanisms with prescribed link lengths are presented in Secs. 3 and 4, respectively. Geometry-based synthesis procedures for these mechanisms to realize a given compliance are developed. In Sec. 5, a numerical example is provided to demonstrate the synthesis procedures for both $5R$ and $6R$ mechanisms. Finally, a brief discussion and summary are presented in Secs. 6 and 7.

2 Technical Background

In this section, the technical background needed for planar compliance realization with an n-joint serial mechanism is presented. First, the use of screw representation to describe mechanism configuration is reviewed. Next, a requirement on the compliance center location expressed in terms of mechanism joint locations is derived, and a requirement on the distribution of joint locations relative to the compliance center is identified. Then, screw representation of link length constraints and the associated geometric restrictions are presented.

2.1 Elastic Behavior Realized With a Serial Mechanism. First, screw representations of a point on a plane and a line on a plane are reviewed. The realization of a planar compliance at a mechanism configuration represented by a set of screws is then summarized.

2.1.1 Screw Representation of Points/Lines in a Plane. It is known that in a plane, a point can be represented by a unit twist $\mathbf{t}$ and a line can be represented by a unit wrench $\mathbf{w}$ [32]. In Plücker axis coordinates, a planar unit twist $\mathbf{t}$ has the following form:

$$\mathbf{t} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} \tag{1}$$

where $\mathbf{u} = \mathbf{r} \times \hat{\mathbf{k}}$, $\mathbf{r}$ is the position vector of the point (instantaneous center of the twist) with respect to the coordinate frame, and $\hat{\mathbf{k}}$ is the unit vector perpendicular to the plane. Thus, for any unit twist $\mathbf{t}$, the location of its instantaneous center is expressed as follows:

$$\mathbf{r} = \mathbf{\Omega u} \tag{2}$$

where $\mathbf{\Omega}$ is the 2×2 antisymmetric matrix defined as follows:

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

Thus, $\mathbf{t}$ is uniquely described by the location $\mathbf{r}$ of a point J as shown in Fig. 1(*a*).

In Plücker ray coordinates, a unit wrench $\mathbf{w}$ has the following form:

$$\mathbf{w} = \begin{bmatrix} \mathbf{n} \\ d \end{bmatrix} \tag{4}$$

where $\mathbf{n}$ is a unit 2-vector indicating the direction of the wrench and where

$$d = (\mathbf{r}_p \times \mathbf{n}) \cdot \hat{\mathbf{k}} \tag{5}$$

where $\mathbf{r}_p$ is the position vector from the origin to any point on the wrench axis. The axis of $\mathbf{w}$ is uniquely defined as the line l having direction $\mathbf{n}$ with perpendicular distance d to the origin (as shown in Fig. 1(*b*)). Thus, any line in the plane can be represented by a unit wrench.

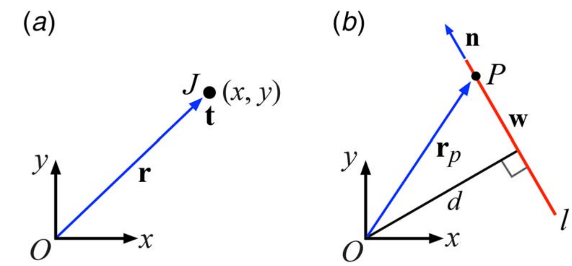

Fig. 1 Screw representation of a point and a line. (*a*) A point *J* is uniquely represented by a unit twist t in Eq. (1). The point location r identifies the instantaneous center of the twist. (*b*) A line *l* is uniquely represented by a unit wrench w in Eq. (4). The line is the line-of-action (axis) of the wrench.

If a line l represented by wrench $\mathbf{w}$ passes through a point J represented by twist $\mathbf{t}$, then the two screws must be reciprocal:

$$\mathbf{t}^T\mathbf{w} = 0 \tag{6}$$

These properties will be used in the synthesis of compliance with a serial mechanism.

2.1.2 Compliance Realization With a Serial Mechanism. Consider a serial mechanism with n revolute joints J_i ($i = 1, 2, \ldots, n$). If each joint location J_i is described by joint twist $\mathbf{t}_i$, the mechanism Cartesian compliance $\mathbf{C}$ is expressed as follows [15]:

$$\mathbf{C} = c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + \cdots + c_n\mathbf{t}_n\mathbf{t}_n^T \tag{7}$$

where $c_i \geq 0$ is the joint compliance at joint J_i, $i = 1, 2, \ldots, n$. Thus, to passively realize a compliance $\mathbf{C}$ with an n-joint serial mechanism, a set of n-joint twists $\mathbf{t}_i$ and corresponding joint compliances c_i that satisfy Eq. (7) need to be identified. For a specified joint twist, the location of the associated joint is determined by Eq. (2).

An n-joint planar serial mechanism with fixed link lengths has n degrees-of-freedom. If each joint has modulated passive compliance c_i, there are n additional independent variables. If the end-effector position and orientation with respect to the base joint are specified, the total number of independent variables associated with a compliant mechanism is $(2n-3)$. Since a planar 3×3 passive compliance matrix is symmetric, it has six independent parameters. To realize an arbitrary compliance at an end-effector pose relative to the robot base, the number of joints, n, must satisfy

$$2n - 3 \geq 6 \implies n \geq 4.5$$

Since n is an integer,

$$n \geq 5$$

Therefore, to achieve an arbitrary compliant behavior with a serial mechanism having fixed link lengths, a given position of the base, and a given position and orientation of the end-effector, the mechanism must have at least five joints.

2.2 Compliance Center Relative Position. For every planar fully elastic behavior, there is a unique point at which the compliance matrix can be diagonalized. This point is called the center of compliance. For the planar case, the center of compliance and the center of stiffness are coincident. A general planar compliance matrix has the following form:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{v} \\ \mathbf{v}^T & c_{22} \end{bmatrix} \tag{8}$$

where $\mathbf{C}_{11} \in \mathbb{R}^{2\times2}$, $\mathbf{v} \in \mathbb{R}^2$ and $c_{22} > 0$. The location $\mathbf{r}_c$ of the compliance center C_c is determined by

$$\mathbf{r}_c = \frac{\boldsymbol{\Omega}\mathbf{v}}{c_{22}} \tag{9}$$

where the 2×2 matrix $\boldsymbol{\Omega}$ is defined in Eq. (3).

The relationship between the location of the compliance center and the configuration of a mechanism capable of realizing the behavior is presented in Ref. [33] for the general *spatial* case. For the planar case, the relationship can be expressed in a simpler form.

Proposition 1. Suppose a planar compliance $\mathbf{C}$ is realized at a particular configuration of an n-joint mechanism. If $\mathbf{r}_i$ is the position vector of each joint in an arbitrary coordinate frame, and c_i is the corresponding joint compliance, then the location of the center of compliance is expressed as follows:

$$\mathbf{r}_c = \frac{c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \cdots + c_n\mathbf{r}_n}{c_1 + c_2 + \cdots + c_n} \tag{10}$$

Thus, the center of compliance is the joint compliance c_i weighted average of the joint locations $\mathbf{r}_i$. It can be seen that the location of the compliance center in Eq. (10) takes the same form as the location of the mass center for particle masses, which indicates the analogy between the two types of centers. Therefore, the compliance center must be within the convex hull formed by the n-joint locations.

2.3 Joint Location Distribution Conditions.

The condition that the compliance center must be inside the area determined by the joint locations is only a *necessary* condition to realize the behavior. Most compliant behaviors cannot be achieved by a serial mechanism even if the compliance center is located within the corresponding area associated with the mechanism geometry. Necessary and sufficient conditions for mechanisms having 3, 4, 5, and 6 joints are identified in Refs. [27–30].

Below, an additional set of easily assessed necessary conditions on the distribution of elastic components is identified.

At the center of compliance, a compliance matrix can be expressed in diagonal form in the principal frame by screw transformation:

$$\mathbf{C} = \mathrm{diag}\left[\lambda_x, \lambda_y, \lambda_\tau\right] \tag{11}$$

where λ_x and λ_y are the two translational principal compliances and λ_τ is the rotational principal compliance.

For a given $\mathbf{C}$ realized by an n-joint mechanism in which d_i is the distance of joint J_i from the compliance center, denote:

$$d_{\min} = \min\{d_1, d_2, \ldots, d_n\} \tag{12}$$

$$d_{\max} = \max\{d_1, d_2, \ldots, d_n\} \tag{13}$$

Suppose that, in the principal frame, each joint J_i has coordinates (x_i, y_i) and

$$d^x_{\min} = \min\{|x_1|, \ldots, |x_n|\}, \quad d^x_{\max} = \max\{|x_1|, \ldots, |x_n|\} \tag{14}$$

$$d^y_{\min} = \min\{|y_1|, \ldots, |y_n|\}, \quad d^y_{\max} = \max\{|y_1|, \ldots, |y_n|\} \tag{15}$$

Then, $d^x_{\min}$ and $d^x_{\max}$ indicate the minimum and maximum distances from joints to the principal y-axis, and $d^y_{\min}$ and $d^y_{\max}$ indicate the minimum and maximum distances from the joints to the principal x-axis. The distribution of joint locations relative to the compliance center must satisfy geometric constraints determined by the three principal compliances.

Proposition 2. *Suppose a compliance* $\mathbf{C}$ *with principal compliances* $(\lambda_x, \lambda_y, \lambda_\tau)$ *is realized by an n-joint serial mechanism. Then,*

(i) *the distances of the joints to the principal axes must satisfy*:

$$d^x_{\min} \le \sqrt{\frac{\lambda_y}{\lambda_\tau}} \le d^x_{\max} \tag{16}$$

$$d^y_{\min} \le \sqrt{\frac{\lambda_x}{\lambda_\tau}} \le d^y_{\max} \tag{17}$$

(ii) *the distances of the joints to the compliance center C_c must satisfy:*

$$d_{\min} \le \sqrt{\frac{\lambda_x + \lambda_y}{\lambda_\tau}} \le d_{\max} \tag{18}$$

Proof. Consider the principal frame at the center of compliance in which the $\mathbf{C}$ matrix is in the diagonal form of Eq. (11). Suppose $\mathbf{r}_i = [x_i, y_i]^T$ is the position vector of joint J_i, then using Eq. (1), the corresponding joint twist $\mathbf{t}_i$ is expressed as follows:

$$\mathbf{t}_i = [y_i, -x_i, 1]^T \tag{19}$$

The distance from the frame origin (the compliance center) to joint J_i is expressed as follows:

$$d_i = \sqrt{x_i^2 + y_i^2} \tag{20}$$

If $\mathbf{C}$ is realized by a mechanism at the configuration described by joint twists $(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n)$, then

$$\mathbf{C} = \sum_{i=1}^{n} c_i \mathbf{t}_i \mathbf{t}_i^T = \sum_{i=1}^{n} c_i \begin{bmatrix} y_i \\ -x_i \\ 1 \end{bmatrix} [y_i, -x_i, 1] \tag{21}$$

where each $c_i > 0$ is the joint compliance of J_i.

Thus,

$$\lambda_x = c_1 y_1^2 + c_2 y_2^2 + \cdots + c_n y_n^2 \tag{22}$$

$$\lambda_y = c_1 x_1^2 + c_2 x_2^2 + \cdots + c_n x_n^2 \tag{23}$$

$$\lambda_\tau = c_1 + c_2 + \cdots + c_n \tag{24}$$

From Eqs. (23) and (24),

$$\lambda_y \le (c_1 + c_2 + + \cdots + c_n)(d^x_{\max})^2 = \lambda_\tau (d^x_{\max})^2$$

Hence,

$$\frac{\lambda_y}{\lambda_\tau} \le (d^x_{\max})^2 \quad \Longleftrightarrow \quad \sqrt{\frac{\lambda_y}{\lambda_\tau}} \le d^x_{\max} \tag{25}$$

Similarly,

$$d^x_{\min} \le \sqrt{\frac{\lambda_y}{\lambda_\tau}} \tag{26}$$

Thus, inequality (16) is proved. By using the same reasoning, inequality (17) is proved.

To prove inequality (18), adding Eqs. (22) and (23) yields

$$\begin{aligned} \lambda_x + \lambda_y &= c_1(x_1^2 + y_1^2) + c_2(x_2^2 + y_2^2) + \cdots + c_n(x_n^2 + y_n^2) \\ &= c_1 d_1^2 + c_2 d_2^2 + \cdots + c_n d_n^2 \\ &\le (c_1 + c_2 + \cdots + c_n) d_{\max}^2 \\ &= \lambda_\tau d_{\max}^2 \end{aligned}$$

Similarly,

$$\lambda_\tau d_{\min}^2 \le \lambda_x + \lambda_y$$

Thus,

$$\lambda_\tau d_{\min}^2 \le \lambda_x + \lambda_y \le \lambda_\tau d_{\max}^2 \tag{27}$$

which leads to inequality (18). ∎

The inequalities in Proposition 2 have geometric significance. The two equations

$$x = \sqrt{\frac{\lambda_y}{\lambda_\tau}} \quad \text{and} \quad x = -\sqrt{\frac{\lambda_y}{\lambda_\tau}} \tag{28}$$

define two lines l_x^+ and l_x^- parallel to and symmetric about the principal y-axis as illustrated in Fig. 2. The two equations

$$y = \sqrt{\frac{\lambda_x}{\lambda_\tau}} \quad \text{and} \quad y = -\sqrt{\frac{\lambda_x}{\lambda_\tau}} \tag{29}$$

define two lines l_y^+ and l_y^- parallel to and symmetric about the principal x-axis (Fig. 2). Proposition 2(i) states that to realize a given compliance with a configuration of a serial mechanism, the joint locations cannot be either all inside or all outside area Λ_x between l_x^- and l_x^+; i.e., at least two joint locations must be separated by only one line of l_x^- and l_x^+. The same statement holds for area Λ_y between the other two lines l_y^- and l_y^+.

Inequality (18) imposes restriction on the distance from the joint locations to the compliance center. To realize a given compliance with a serial mechanism, the joint locations must surround the compliance center C_c and cannot be either all inside or all outside circle Γ_c centered at C_c having radius:

$$r_\lambda = \sqrt{\frac{\lambda_x + \lambda_y}{\lambda_\tau}} \tag{30}$$

as illustrated in Fig. 2. Therefore, to realize a specified compliance with a serial mechanism, the space reachable by the mechanism joints must include the compliance center C_c and the joint locations cannot be enclosed by circle Γ_c.

2.4 Implications of Joint Location Restrictions.

Proposition 1 requires that the mechanism joints surround the compliance center. Proposition 2 places requirement on how the joints surround the compliance center. Although there is some overlap in conditions (i) and (ii) of Proposition 2, the two sets of inequalities are independent.

For an n-joint mechanism with each link length l_i, the boundary of the space reachable by the last (most distal) joint J_n is a circle Γ_w of radius r_w centered at the base joint J_1. Propositions 1 and 2 also impose restrictions on the distance d_c between the mechanism base J_1 and compliance center C_c, and on the radius r_w of Γ_w. Since the joints must surround the compliance center and circle Γ_w cannot be contained by circle Γ_c, the following conditions must be satisfied:

$$r_\lambda - r_w < d_c < r_w \tag{31}$$

which also requires

$$r_w > \frac{1}{2} r_\lambda = \frac{1}{2}\sqrt{\frac{\lambda_x + \lambda_y}{\lambda_\tau}} \tag{32}$$

If any condition in inequalities (31)–(32) is not satisfied, the given compliance cannot be realized by the mechanism.

Note that the conditions in Propositions 1 and 2 and their implications in Eqs. (31)–(32) are only necessary conditions to achieve a given compliance. To ensure passive realization of a compliance, additional conditions are needed [27–30].

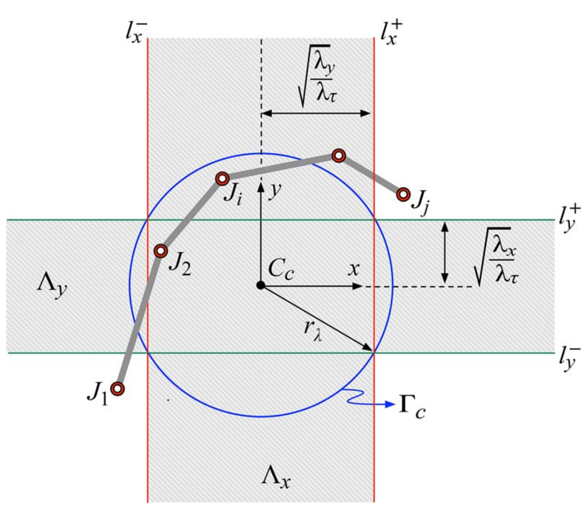

Fig. 2 Joint location restrictions relative to the compliance center and principal axes. At least one joint must be located inside circle Γ_c and at least one joint must be outside Γ_c. At least one joint must be located within area Λ_x between lines l_x^- and l_x^+; and at least one joint must be located outside Λ_x. At least one joint must be located inside the area Λ_y between lines l_y^- and l_y^+, and at least one joint must be located outside Λ_y.

2.5 Screw Representation of Link Length Constraints.

If the location of joint J_i is specified, then the locus of possible joint locations of J_{i+1} is a circle Γ_i of radius l_i centered at J_i. Suppose joint J_i is located at a given position (x_i, y_i), then joint J_{i+1} must be located at a point (x, y), which satisfies:

$$(x - x_i)^2 + (y - y_i)^2 = l_i^2 \tag{33}$$

Using screw representation, point (x, y) is associated with a unit twist $\mathbf{t}$ given by

$$\mathbf{t} = [y, -x, 1]^T \tag{34}$$

As such, Eq. (33) can be written in a homogeneous form as follows:

$$\mathbf{t}^T(\mathbf{T}_i^T \mathbf{E}_i \mathbf{T}_i)\mathbf{t} = 0 \tag{35}$$

where

$$\mathbf{E}_i = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^T & -l_i^2 \end{bmatrix}, \quad \mathbf{T}_i = \begin{bmatrix} 1 & 0 & -y_i \\ 0 & 1 & x_i \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathbf{I}$ is the 2×2 identity matrix.

Suppose $\mathbf{t}$ is a unit twist located on circle Γ_i of Eq. (35) and suppose $\mathbf{w}$ is the corresponding wrench $\mathbf{w} = \mathbf{K}\mathbf{t}$. Then,

$$\mathbf{t} = \mathbf{C}\mathbf{w}$$

Substituting the aforementioned equation into Eq. (35) yields

$$\mathbf{w}^T(\mathbf{C}\mathbf{T}_i^T \mathbf{E}_i \mathbf{T}_i \mathbf{C})\mathbf{w} = 0 \tag{36}$$

Let

$$\mathbf{G}_i = \mathbf{C}(\mathbf{T}_i^T \mathbf{E}_i \mathbf{T}_i)\mathbf{C} \tag{37}$$

then $\mathbf{G}_i$ is a 3×3 symmetric matrix that relates acceptable joint locations to acceptable line locations. The collection of all wrenches corresponding to the twists on circle Γ_i as mapped through $\mathbf{K}$ is expressed as follows:

$$\mathbb{W}_i = \{\mathbf{w} : \ \mathbf{w}^T \mathbf{G}_i \mathbf{w} = 0\} \tag{38}$$

Consider a different mapping from wrenches (lines) to twists (points) defined by:

$$\mathbf{t} = \mathbf{G}_i \mathbf{w} \tag{39}$$

Then,

$$\mathbf{w}^T \mathbf{G}_i \mathbf{w} = 0 \quad \Longleftrightarrow \quad \mathbf{t}^T \mathbf{G}_i^{-1} \mathbf{t} = 0 \tag{40}$$

If we denote

$$\mathbb{T}_i = \{\mathbf{t} : \ \mathbf{t}^T \mathbf{G}_i^{-1} \mathbf{t} = 0\} \tag{41}$$

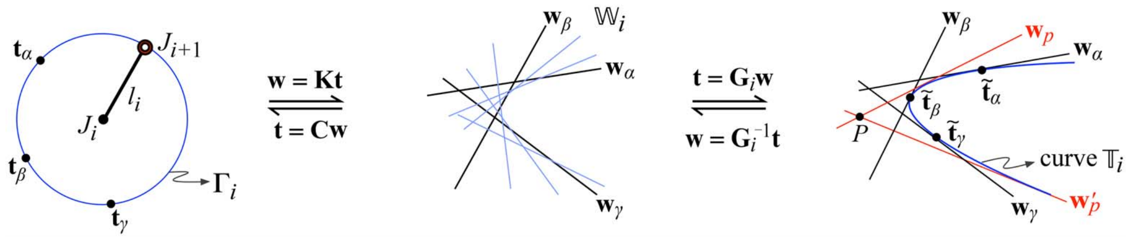

Fig. 3 Twists t on circle Γ_i and the corresponding wrenches $\mathbf{w}\in\mathbb{T}_i$ through the stiffness mapping. Wrench w must be tangent to the quadratic curve $\mathbb{T}_i$. For point P not enclosed by $\mathbb{T}_i$, there are two wrenches $\mathbf{w}_p$ and $\mathbf{w}'_p$ with axes passing through P and tangent to $\mathbb{T}_i$. The twist t corresponding to any wrench tangent to T_i must be located on circle Γ_i.

then Eq. (39) defines a 1-to-1 mapping from $\mathbb{W}_i$ to $\mathbb{T}_i$. It can be proved that the set $\mathbb{T}_i$ defined in Eq. (41) is a quadratic curve in the plane. All wrenches $\mathbf{w}\in\mathbb{W}_i$ corresponding to the twists on circle Γ_i (through the stiffness mapping $\mathbf{w}=\mathbf{K}\mathbf{t}$) must be tangent to curve $\mathbb{T}_i$ (as illustrated in Fig. 3). Conversely, if a wrench $\mathbf{w}$ is tangent to the quadratic curve $\mathbb{T}_i$, then the twist $\mathbf{t}=\mathbf{C}\mathbf{w}$ must be located on circle Γ_i. For any given point P in the plane not enclosed by curve $\mathbb{T}_i$, there are two wrenches with axes passing through P and tangent to $\mathbb{T}_i$. The two twists corresponding to the two wrenches obtained by the compliance mapping must both be located on circle Γ_i. This property will be used in the synthesis procedure presented below for a serial mechanism with fixed link lengths. The use of lines rather than points allows the placement of two points (joint locations) to be considered simultaneously.

3 Compliance Realization With a 5*R* Mechanism

In this section, the realization of an arbitrary compliance with a five-joint serial mechanism having specified link lengths is addressed. Since each link length is fixed, the distance between two adjacent joints J_i and J_{i+1} is constrained, i.e., $\|J_iJ_{i+1}\|=l_i$. To impose this constraint, a new set of realization conditions is identified first. Then, a geometry-based synthesis procedure for the realization of compliance with a 5*R* serial mechanism with specified link lengths is developed.

3.1 Realization Condition. Consider a 5*R* serial mechanism having specified link lengths. A given compliance $\mathbf{C}$ can be passively realized with the mechanism at a configuration if and only if $\mathbf{C}$ can be expressed as follows:

$$\mathbf{C}=c_1\mathbf{t}_1\mathbf{t}_1^T+c_2\mathbf{t}_2\mathbf{t}_2^T+\cdots+c_5\mathbf{t}_5\mathbf{t}_5^T \tag{42}$$

with $c_i\geq 0$. A set of necessary and sufficient conditions on the mechanism configuration for the realization of $\mathbf{C}$ without considering the link length restrictions was presented in Ref. [29]. Below, a different set of conditions is presented.

As proved in Ref. [29], to realize a given compliance $\mathbf{C}$ with a 5*R* mechanism, any joint J_s in the mechanism must be located on a quadratic curve determined by $\mathbf{C}$ and the locations of the other four joints (J_i, J_j, J_p, J_q). This curve is characterized by a 3×3 symmetric matrix $\mathbf{A}_{ijpq}$ constructed below.

Consider a 3×3 matrix $\mathbf{H}_{ijpq}$ defined as follows:

$$\mathbf{H}_{ijpq}=(\mathbf{w}_{ij}^T\mathbf{C}\mathbf{w}_{pq})(\mathbf{w}_{ip}\mathbf{w}_{jq}^T)-(\mathbf{w}_{ip}^T\mathbf{C}\mathbf{w}_{jq})(\mathbf{w}_{ij}\mathbf{w}_{pq}^T) \tag{43}$$

where $\mathbf{w}_{ij}$ is the unit wrench passing through joints J_i and J_j. The symmetric matrix associated with $\mathbf{H}_{ijpq}$ is expressed as follows:

$$\mathbf{A}_{ijpq}=\mathbf{H}_{ijpq}+\mathbf{H}_{ijpq}^T \tag{44}$$

Consider an arbitrary unit twist $\mathbf{t}$ located at (x, y) expressed in the form of Eq. (34). The equation

$$f_{ijpq}(x,y)=\mathbf{t}^T\mathbf{A}_{ijpq}\mathbf{t}=0 \tag{45}$$

defines a quadratic (conic) curve in the xy-plane.

It is proved [30] that the curve defined in Eq. (45) passes through the four joints (J_i, J_j, J_p, J_q) and that the compliance matrix $\mathbf{C}$ can be expressed in the form of Eq. (42), if and only if the one remaining joint is located on the curve. However, this condition alone does not ensure a *passive* realization of the compliance, since Eq. (45) does not require that the coefficients c_i in Eq. (42) are all nonnegative. A set of necessary and sufficient conditions for passive realization of a compliance with a 5*R* serial mechanism is described in this section.

Suppose that five-joint twists $\mathbf{t}_i$ satisfy Eq. (42) for a selected set of c_is. Consider the unit wrench $\mathbf{w}_{ij}$ passing through two joints J_i and J_j, and consider the corresponding twist $\mathbf{t}_{ij}$ defined by

$$\mathbf{t}_{ij}=\mathbf{C}\mathbf{w}_{ij} \tag{46}$$

Equation (46) can be viewed as a mapping from a line represented by $\mathbf{w}_{ij}$ into a point represented by $\mathbf{t}_{ij}$ through the compliance. As proved in Ref. [29], to ensure that all coefficients c_i in Eq. (42) are nonnegative, $\mathbf{t}_{ij}$ must be located inside the triangle formed by the other three joints J_p, J_q, and J_s. For example, if $\mathbf{t}_{12}$ is located within the triangle formed by joints J_3, J_4, and J_5 as shown in Fig. 4, then the coefficients c_3, c_4, and c_5 in Eq. (42) must be positive. If the equivalent condition also holds for twist $\mathbf{t}_{34}$ (or $\mathbf{t}_{45}$, $\mathbf{t}_{35}$), then all five coefficients c_i in Eq. (42) must be positive. Thus, we have:

PROPOSITION 3. *A 5J serial mechanism realizes a given compliance* $\mathbf{C}$ *at a configuration in which the joint twists are* ($\mathbf{t}_1$, $\mathbf{t}_2$, …, $\mathbf{t}_5$) *if and only if*

(i) *each joint is located on the quadratic curve of Eq. (45) determined by four of the five joints, and*
(ii) *for any permutation* (i, j, p, q, s) *from* $\{1, 2, 3, 4, 5\}$, *twist* $\mathbf{t}_{ij}$ *is located within triangle* $J_pJ_qJ_s$ *and twist* $\mathbf{t}_{pq}$ *is located within triangle* $J_iJ_jJ_s$.

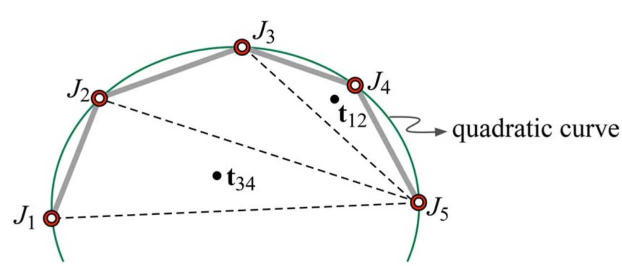

Fig. 4 Realization condition on mechanism configuration. One joint must be located on the quadratic curve determined by the other four joint locations; twists $\mathbf{t}_{12}$ and $\mathbf{t}_{34}$ are located within triangles $J_3J_4J_5$ and $J_1J_2J_5$, respectively.

Note that as shown in Ref. [29], when the five-joint locations are identified, the joint compliance of joint J_s can be uniquely determined with (i, j, p, q) being any permutation of {1, 2, 3, 4, 5} excluding s, i.e., $(i, j, p, q) = \{1, 2, 3, 4, 5\} \setminus s$:

$$c_s = \frac{\mathbf{w}_{ij}^T \mathbf{C} \mathbf{w}_{pq}}{(\mathbf{w}_{ij}^T \mathbf{t}_s)(\mathbf{w}_{pq}^T \mathbf{t}_s)} \tag{47}$$

3.2 Construction-Based Synthesis Procedure.

The synthesis of a compliance with a given mechanism (one having specified link lengths) is primarily based on the conditions presented in Proposition 3 with additional guidance provided by Propositions 1 and 2. The synthesis procedure identifies a configuration of a 5*R* mechanism by determining the location of each joint. The joint compliance $c_i \geq 0$ at each joint is also determined in the procedure.

As stated in Sec. 2.1.2, $n = 5$ is the minimum number of joints in a serial mechanism needed to achieve an arbitrary compliance if the link lengths and the locations of the base joint J_1 and the endpoint joint J_n are specified. In the geometry-based synthesis process, for the system to have sufficient degrees-of-freedom, only one joint location can be specified (e.g., for $n = 5$, either only base location J_1 or only distal joint location J_5) to reliably obtain the specified compliance.

For a given compliance matrix $\mathbf{C}$, first calculate (1) the location of the compliance center C_c; (2) the three principal compliances λ_x, λ_y, and λ_τ; and (3) the directions of the principal axes. By using these values, the circle Γ_c defined in Eq. (30), and the four lines parallel to the principal axes defined in Eqs. (28)–(29) are constructed to provide guidance in the selection of joint locations.

In the synthesis procedure described here, two twists $\mathbf{t}_{12}$ and $\mathbf{t}_{34}$ must be located in triangles $J_3J_4J_5$ and $J_1J_2J_5$, respectively. The locations of these twists are selected first to satisfy condition (ii) of Proposition 3 before determining the locations of J_2 and J_3 in the subsequent steps.

A more detailed description of the synthesis procedure is presented below. The geometry corresponding to each step in the procedure is illustrated in Figs. 5(*a*)–5(*d*).

(1) Identify the location of one joint, typically the base joint J_1, arbitrarily.

(2) Choose the location of J_2. Since the location of J_1 (with joint twist $\mathbf{t}_1$) is specified, the locus of J_2 locations is a circle of radius l_1, Γ_1. The collection of lines passing through J_1 and J_2 is a pencil $\mathbb{P}_{12}$ of lines at J_1. If we denote the collection of all twists obtained by the compliance mapping:

$$\mathbb{T}_{12} = \{\mathbf{t} = \mathbf{C}\mathbf{w} : \ \mathbf{w} \in \mathbb{P}_{12}\}$$

then the centers of all twists in $\mathbb{T}_{12}$ form a straight line represented by wrench $\mathbf{w}_1 = \mathbf{K}\mathbf{t}_1$, which is the locus of twist $\mathbf{t}_{12}$ locations. Since twist $\mathbf{t}_{12}$ must be in the triangle formed by J_3, J_4, and J_5, this line must intersect circle Γ_w defined in Sec. 2.4 for the compliance to be realized by the mechanism. Judiciously select point $\mathbf{t}_{12}$ on the line such that conditions in Propositions 2 and 3 are easier to satisfy. The line associated with wrench $\mathbf{w}_{12} = \mathbf{K}\mathbf{t}_{12}$ will pass through J_1 with a slope determined by $\mathbf{t}_{12}$. The intersection of line $\mathbf{w}_{12}$ and circle Γ_1 determines the location of J_2 as shown in Fig. 5(*a*).

(3) Select the location of twist $\mathbf{t}_{34}$ such that it lies within the triangle formed by the locations of J_1, J_2, and J_5. Since the location of J_5 is not yet determined, the location of $\mathbf{t}_{34}$ is selected before selecting J_3 and J_4 separately so that the triangle condition will be satisfied for virtually all possible locations of J_5. The location of $\mathbf{t}_{34}$ is selected based on the selected two joints (J_1, J_2) and quadratic curve $\mathbb{T}_2$ associated with circle Γ_2:

$$\mathbf{t}^T \mathbf{G}_2^{-1} \mathbf{t} = 0 \tag{48}$$

where $\mathbf{G}_2$ is the 3×3 matrix defined in Eq. (37).

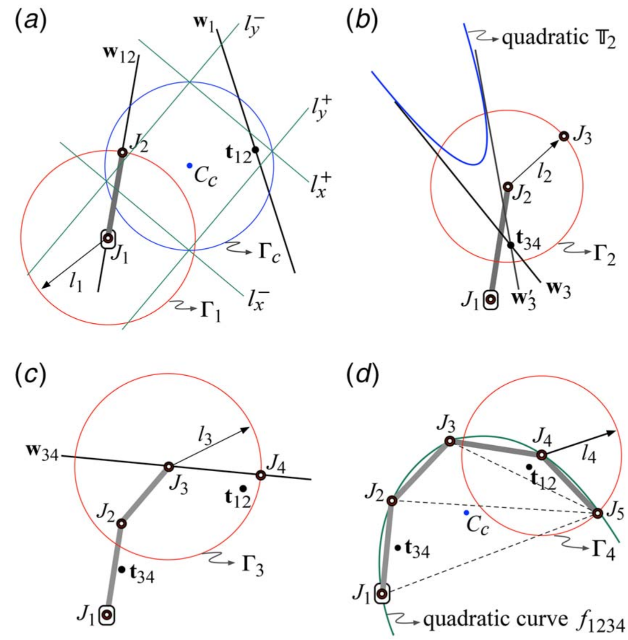

Fig. 5 Synthesis of a compliance with a 5*R* serial mechanism: (*a*) select the location of joint J_2 on circle Γ_1, (*b*) select the location of joint J_3 so that it is on circle Γ_2, (c) select the location of joint J_4 on circle Γ_3, and (*d*) locate joint J_5 at the intersection of circle Γ_4 and curve f_{1234}

Here, $\mathbf{t}_{34}$ is selected to be close to line segment $\overline{J_1J_2}$ and not enclosed by $\mathbb{T}_2$.

(4) Select the location of J_3. The locus of J_3 locations is a circle Γ_2 of radius l_2 centered at J_2 (x_2, y_2).

Consider the wrench $\mathbf{w}_3$ that passes through $\mathbf{t}_{34}$ and is tangent to the curve defined in Eq. (48). Mathematically, $\mathbf{w}_3$ satisfies the following two equations:

$$\mathbf{t}_{34}^T \mathbf{w}_3 = 0 \tag{49}$$

$$\mathbf{w}_3^T \mathbf{G}_2 \mathbf{w}_3 = 0 \tag{50}$$

Solving these two equations yields two lines (or unit wrenches). Choose one $\mathbf{w}_3$ from the two solutions, then the location of J_3 is determined by twist:

$$\mathbf{t}_3 = \mathbf{C}\mathbf{w}_3 \tag{51}$$

Since $\mathbf{w}_3$ is tangent to curve $\mathbb{T}_2$, by the results obtained in Sec. 2.5, joint twist $\mathbf{t}_3$, and therefore joint J_3, must be located on circle Γ_2 (Fig. 5(*b*)).

(5) Select the location of J_4. The locus of J_4 is a circle Γ_3 of radius l_3 centered at J_3. Determine the line defined by:

$$\mathbf{w}_{34} = \mathbf{K}\mathbf{t}_{34} \tag{52}$$

It can be proved that if $\mathbf{t}_{34}$ is close to line $\mathbf{w}_{12}$, twist $\mathbf{t}_{12}$ is close to line $\mathbf{w}_{34}$.

Since $\mathbf{w}_3$ passes through point $\mathbf{t}_{34}$ as selected in step 4, $\mathbf{w}_{34}$ must pass through J_3. Line $\mathbf{w}_{34}$ and circle Γ_3 intersect at two points. Select J_4 to be the one closer to point $\mathbf{t}_{12}$ (Fig. 5(*c*)) to ensure that $\mathbf{t}_{12}$ is inside triangle $J_3J_4J_5$.

(6) Select the location of J_5. The locus of J_5 locations is a circle Γ_4 of radius l_4 centered at J_4. A quadratic curve f_{1234} passing through the four joints (J_1, J_2, J_3, J_4) is determined using Eq. (45). This curve intersects circle Γ_4 at two points. Select J_5 so that $\mathbf{t}_{12}$ is inside triangle $J_3J_4J_5$ and $\mathbf{t}_{34}$ is inside triangle $J_1J_2J_5$ (Fig. 5(*d*)).

(7) Determine the joint compliances. The five-joint compliances at the joint locations are each calculated using Eq. (47).

The process described earlier enforces link length constraints. For the selected five joints, the conditions in Proposition 3 are satisfied, which guarantees that each joint compliance calculated in step 6 is positive. Therefore, the compliance is passively achieved by the mechanism in the selected configuration.

4 Compliance Realization With a 6*R* Mechanism

In this section, the synthesis of a planar compliance with a 6*R* mechanism having given link lengths is addressed. As the number of joints increases, the mechanism degrees-of-freedom are increased. As such, more constraints can be considered in the synthesis process. First, new compliance realization conditions on a general 6*R* serial mechanism are presented. Then, a synthesis procedure for the realization of compliance with a given 6*R* mechanism with a set of constraints is developed.

4.1 Realization Condition. Consider a 6*R* serial mechanism with each joint J_i represented by joint twist $\mathbf{t}_i$ ($i = 1, 2, \ldots, 6$). Any given compliance $\mathbf{C}$ can be expressed in the following form:

$$\mathbf{C} = c_1 \mathbf{t}_1 \mathbf{t}_1^T + c_2 \mathbf{t}_2 \mathbf{t}_2^T + \cdots + c_6 \mathbf{t}_6 \mathbf{t}_6^T \tag{53}$$

For any given configuration, the coefficients c_is in Eq. (53) can be uniquely determined using the following procedure.

Since $\mathbf{C}$ is symmetric, Eq. (53) can be expressed in the vector form as follows:

$$\tilde{\mathbf{c}} = c_1 \tilde{\mathbf{t}}_1 + c_2 \tilde{\mathbf{t}}_2 + \cdots + c_6 \tilde{\mathbf{t}}_6 \tag{54}$$

where $\tilde{\mathbf{c}} = [c_{11}, c_{12}, c_{13}, c_{22}, c_{23}, c_{33}]^T$, and $\tilde{\mathbf{t}}_i$ is the six-vector representation of $\mathbf{t}_i \mathbf{t}_i^T$. If we denote:

$$\mathbf{c} = [c_1, c_2, \ldots, c_6]^T \in \mathbb{R}^6, \quad \tilde{\mathbf{T}} = \left[\tilde{\mathbf{t}}_1, \tilde{\mathbf{t}}_2, \ldots, \tilde{\mathbf{t}}_6\right] \in \mathbb{R}^{6\times 6}$$

then, Eq. (54) is expressed as follows:

$$\tilde{\mathbf{c}} = \tilde{\mathbf{T}} \mathbf{c} \tag{55}$$

The joint compliance variables are obtained using:

$$\mathbf{c} = \tilde{\mathbf{T}}^{-1} \tilde{\mathbf{c}} \tag{56}$$

Thus, for any given compliance $\mathbf{C}$, the coefficients c_i are uniquely determined by Eq. (56) for a given mechanism configuration.

Note that the coefficients c_is from Eq. (56) may be positive or negative. A necessary and sufficient condition for the passive realization of a given compliance, however, is that each c_i in Eq. (56) is nonnegative:

$$\tilde{\mathbf{T}}^{-1} \tilde{\mathbf{c}} \geq 0 \tag{57}$$

The six inequalities in Eq. (57) impose constraints on the mechanism configuration. However, due to the matrix inverse operation of $\tilde{\mathbf{T}}$, the geometric significance of these inequalities is not evident. Thus, the conditions in Eq. (57) cannot be used directly in a geometric construction-based synthesis procedure to achieve a given compliance.

In Ref. [30], it was shown that two joint compliances c_i and c_j have the same sign if and only if the two joints are separated by the quadratic curve of Eq. (45) determined by the other four joints, and that all six c_i are positive if and only if every two joints are separated by the quadratic curve of Eq. (45) determined by the other four joints. Here, a property on any two joint locations and their corresponding joint compliances is identified.

Consider a set of six joints J_is for the realization of a given compliance $\mathbf{C}$. If joint J_5 is located on the quadratic curve f_{1234} determined by four other joints (J_1, J_2, J_3, J_4), then $c_6 = 0$ in Eq. (53) regardless of the location of J_6. Now consider varying the location of J_5 while keeping all other joint locations unchanged (Fig. 6).

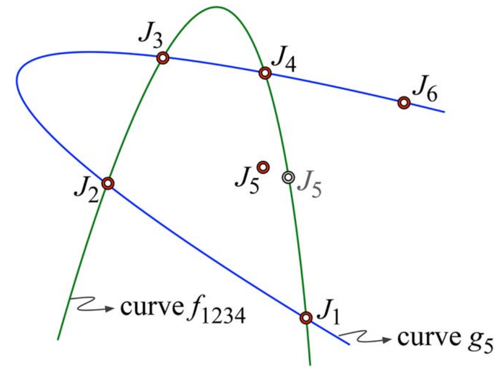

Fig. 6 Synthesis of a compliance with six elastic joints. If J_5 and J_6 are separated by curve f_{1234}, c_5 and c_6 must have the same sign. If J_5 moves without crossing curve g_5, c_5 maintains the same sign.

As shown in Ref. [30], the value of c_5 can be calculated as follows:

$$c_5 = \frac{(\mathbf{w}_{13}^T \mathbf{t}_6)(\mathbf{w}_{24}^T \mathbf{t}_6)(\mathbf{w}_{12}^T \mathbf{C} \mathbf{w}_{34}) - (\mathbf{w}_{12}^T \mathbf{t}_6)(\mathbf{w}_{34}^T \mathbf{t}_6)(\mathbf{w}_{13}^T \mathbf{C} \mathbf{w}_{24})}{D_{56}} \tag{58}$$

where $\mathbf{w}_{ij}$ is the unit wrench passing through J_i and J_j, and where the denominator D_{56} is expressed as follows:

$$\begin{aligned} D_{56} = & (\mathbf{w}_{12}^T \mathbf{t}_5)(\mathbf{w}_{34}^T \mathbf{t}_5)(\mathbf{w}_{13}^T \mathbf{t}_6)(\mathbf{w}_{24}^T \mathbf{t}_6) \\ & - (\mathbf{w}_{12}^T \mathbf{t}_6)(\mathbf{w}_{34}^T \mathbf{t}_6)(\mathbf{w}_{13}^T \mathbf{t}_5)(\mathbf{w}_{24}^T \mathbf{t}_5) \end{aligned} \tag{59}$$

Since all joint locations except J_5 are unchanged, the numerator of c_5 in Eq. (58) does not change. A sign change of c_5 depends only on the denominator D_{56} in Eq. (59). Let $\mathbf{t} = [y, -x, 1]^T$ and consider the function $g_5(x, y)$ defined by

$$\begin{aligned} g_5(x, y) = & (\mathbf{w}_{12}^T \mathbf{t})(\mathbf{w}_{34}^T \mathbf{t})(\mathbf{w}_{13}^T \mathbf{t}_6)(\mathbf{w}_{24}^T \mathbf{t}_6) \\ & - (\mathbf{w}_{12}^T \mathbf{t}_6)(\mathbf{w}_{34}^T \mathbf{t}_6)(\mathbf{w}_{13}^T \mathbf{t})(\mathbf{w}_{24}^T \mathbf{t}) \end{aligned} \tag{60}$$

Then,

$$g_5(x, y) = 0 \tag{61}$$

defines a quadratic curve in the plane. It can be seen that this curve passes through the five joints (J_1, J_2, J_3, J_4, J_6), and thus, it is uniquely determined by the locations of these five joints.

If c_5 changes its sign, joint J_5 must cross curve $g_5(x, y) = 0$. Thus, if J_5 moves without crossing curve $g_5(x, y)$ such that J_5 and J_6 are separated by curve f_{1234}, then c_5 and c_6 are either both positive or both negative. If J_5 crosses curve $g_5(x, y)$ with J_5 and J_6 being separated by curve f_{1234}, then both c_5 and c_6 change their sign. Note that this property is also true for any two joints (J_i, J_j) and their corresponding joint compliances (c_i, c_j) and will be used for the synthesis procedure for 6*R* mechanisms having fixed link lengths.

4.2 Construction-Based Synthesis Procedures. In this section, synthesis procedures used to realize a given compliance with a 6*R* serial mechanism are presented. In the process, the first joint J_1 (base joint) and the most distal joint J_6 (connected to the end-effector) are specified. First, a procedure that uses five elastic joints in a 6*R* mechanism is presented, Then, a procedure for which all six joints are elastic is presented.

4.2.1 Compliance Synthesis With Five Elastic Joints. This synthesis procedure identifies the locations of the six joints and the corresponding joint compliances (with one joint compliance equal to zero). This procedure is based on the 5*R* synthesis procedure presented in Sec. 3.2. The geometry corresponding to each step in the procedure is illustrated in Fig. 7.

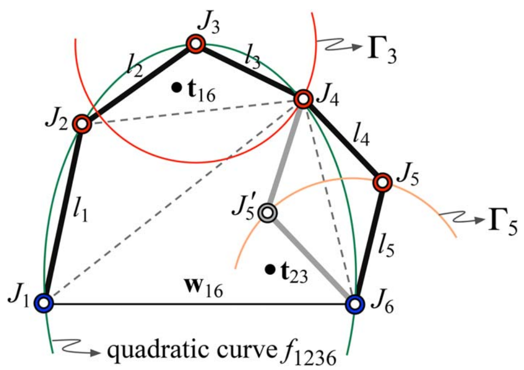

Fig. 7 Synthesis of a compliance with a 6*R* serial mechanism. The locations of J_1 and J_6 are specified.

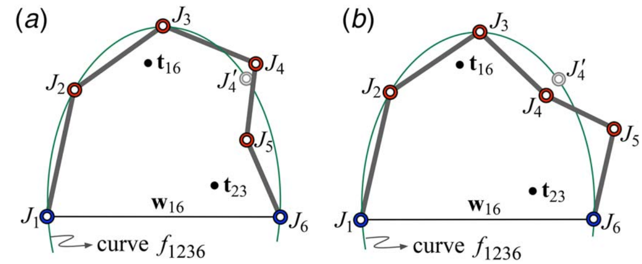

Fig. 8 Synthesis of a compliance with a 6*R* serial mechanism. There are two configurations (*a*) and (*b*) at which joints J_4 and J_5 are separated by curve f_{1236}. The joint compliances c_4 and c_5 have the same sign at these two configurations.

(1) Select the locations of two joints arbitrarily, typically, the base joint J_1 and the distal joint J_6.
(2) Calculate the twist $\mathbf{t}_{16}$ associated with wrench $\mathbf{w}_{16}$. Since J_1 and J_6 are specified, the wrench $\mathbf{w}_{16}$ passing through J_1 and J_6 is determined and

$$\mathbf{t}_{16} = \mathbf{C}\mathbf{w}_{16} \tag{62}$$

The location of $\mathbf{t}_{16}$ is obtained using Eq. (2).
(3) Select J_2. Follow step 2 in the procedure for 5*R* mechanisms presented in Sec. 3.2.
(4) Select J_3. Follow steps 3 and 4 in the procedure for 5*R* mechanisms presented in Sec. 3.2.
(5) Select J_4. Determine the quadratic curve f_{1236} associated with the four selected joints (J_1, J_2, J_3, J_6) using Eq. (45) and determine circle Γ_3 of radius l_3 centered at J_3. Joint J_4 is at the intersection of circle Γ_3 and curve f_{1236}. In selecting the location of J_4, the distance between J_4 and J_6 must be less than (l_4+l_5) and $\mathbf{t}_{23}$ must be inside triangle $J_4J_6J_1$.
(6) Determine the location of J_5. Since J_4 and J_6 are specified, J_5 is determined by the intersection of the two circles Γ_4 and Γ_5 centered at J_4 and J_6 with radius l_4 and l_5, respectively. There are two possible locations for J_5 as shown in Fig. 7. Choose either one.
(7) Determine the joint compliances. Since the compliance is effectively realized by five elastic joints (J_1, J_2, J_3, J_4, J_6) in the 6*R* mechanism, $c_5=0$. The other five-joint compliances can be calculated using either Eq. (47) or (56).

In this procedure, although only five joints are used to provide joint compliance, six joints are needed for the kinematic mobility necessary to satisfy the geometric constraints (specified locations of J_1 and J_6).

4.2.2 Compliance Synthesis With Six Elastic Joints. Synthesis of a compliance with all six nonzero compliance joints is outlined as follows:

(1) Identify a configuration of the 6*R* mechanism that realizes the given compliance with five joints as described in the procedure of Sec. 4.2.1. Suppose that in the realization, the given compliance is realized with five elastic joints (J_1, J_2, J_3, J_4, J_6) as illustrated in Fig. 7. The procedure ensures that all five-joint compliances c_1, c_2, c_3, c_4, and c_6 are positive.
(2) Move J_4 away from the quadratic curve f_{1236} such that J_4 and J_5 are separated by f_{1236}. Since joints J_3 and J_6 are selected, the mechanism is equivalent to a four-bar mechanism with J_3 and J_6 fixed. By rotating link J_3J_4 (or J_6J_5) about J_3 (or J_6), J_4 is moved away from the curve. Two example configurations are shown in Fig. 8. Since joints J_4 and J_5 are separated by curve f_{1236} for each configuration, the joint compliances c_4 and c_5 at these two joints must have the same sign. Since the two configurations are only slightly varied from a configuration at which $c_4>0$, both c_4 and c_5 are positive at one of these two configurations.
(3) Calculate the joint compliances using Eq. (56) at the two configurations selected in step 2, and choose the one that has all positive joint compliances.

With the final step, the configuration of the mechanism and all joint compliances are determined and the compliance is passively achieved with the 6*R* mechanism.

5 Example

In this section, a numerical example is provided to illustrate the synthesis procedures. In a global frame, the compliance matrix to be realized is expressed as follows:

$$\mathbf{C} = \begin{bmatrix} 12.50\,\text{m/N} & -13.54\,\text{m/N} & 3.41\,\text{N}^{-1} \\ -13.54\,\text{m/N} & 15.91\,\text{m/N} & -3.74\,\text{N}^{-1} \\ 3.41\,\text{N}^{-1} & -3.74\,\text{N}^{-1} & 0.95\,(\text{N}\cdot\text{m})^{-1} \end{bmatrix}$$

By using Eq. (9), the compliance center is calculated to be

$$\mathbf{r}_c = [3.9368,\ 3.5895]^T$$

The two principal axes and the corresponding two translational principal compliances are expressed as follows:

$$[\mathbf{e}_1, \mathbf{e}_2] = \begin{bmatrix} -0.9926 & 0.1218 \\ -0.1218 & -0.9926 \end{bmatrix}, \quad [\lambda_x, \lambda_y] = [0.2457,\ 1.2004]$$

and the rotational principal compliance is $\lambda_\tau=0.95$. The circle Γ_c and the two pairs of lines in the principal frame at the compliance center C_c are calculated as follows:

$$\Gamma_c:\ r_\lambda = 1.2338, \quad l_x^{\pm}:\ x' = \pm 1.1241, \quad l_y^{\pm}:\ y' = \pm 0.5086$$

and are illustrated in Fig. 9.

In the following, the synthesis of $\mathbf{C}$ with a 5*R* serial mechanism having given link lengths is first performed. Then, the synthesis of $\mathbf{C}$ with a 6*R* serial mechanism is presented.

5.1 5*R* Mechanism Synthesis. Consider a 5*R* serial mechanism in which each link has the same length:

$$l_1 = l_2 = l_3 = l_4 = 1\,\text{m}$$

The space reachable by the last joint J_5 is a circle Γ_w of radius r_w centered at J_1. The radius r_w is expressed as follows:

$$r_w = \sum l_i = 4\,\text{m}$$

(1) Select the location of J_1. To satisfy inequality (31), the distance between joint J_1 and the compliance center C_c must be less than 4 m. Here, J_1 is located at position (3, 3)

(inside the rectangle enclosed by the four lines $l_x^{\pm}$ and $l_y^{\pm}$ as illustrated in Fig. 9), which satisfies the two lower bound inequalities of (16) and (17). The joint twist of J_1 is expressed as follows:

$$\mathbf{t}_1 = [3, -3, 1]^T$$

(2) Select the location of J_2. Based on the selected location of J_1 and Proposition 2, position (3, 4) is selected to be the location of J_2, which is separated from J_1 by l_y^+. The joint twist of J_2 is expressed as follows:

$$\mathbf{t}_2 = [4, -3, 1]^T$$

The wrench passing through J_1 and J_2 is expressed as follows:

$$\mathbf{w}_{12} = [0, 1, 3]^T$$

The twist associated with $\mathbf{w}_{12}$ is expressed as follows:

$$\mathbf{t}_{12} = \mathbf{C}\mathbf{w}_{12} = [-3.31, 4.69, -0.89]^T$$

which is located at (5.2697, 3.7191).

(3) Select the center of twist $\mathbf{t}_{34}$. By using Eq. (37), the matrix $\mathbf{G}_2$ is expressed as follows:

$$\mathbf{G}_2 = \begin{bmatrix} 0.6276 & -4.3893 & 0.1510 \\ -4.3893 & 10.0249 & -1.1749 \\ 0.1510 & -1.1749 & 0.0417 \end{bmatrix}$$

The associated quadratic curve $\mathbb{T}_2$ is expressed as follows:

$$\mathbb{T}_2: \ \mathbf{t}^T \mathbf{G}_2^{-1} \mathbf{t} = 0$$

and is illustrated in Fig. 9.

Choose a point (not enclosed by curve $\mathbb{T}_2$) close to line segment $\overline{J_1 J_2}$. This point is selected to be (3.2, 3.2), which is the center of $\mathbf{t}_{34}$. Then, twist $\mathbf{t}_{34}$ is expressed as follows:

$$\mathbf{t}_{34} = [3.2, -3.2, 1]^T$$

(4) Select the location of J_3. Consider a line that passes through the center of $\mathbf{t}_{34}$ (3.2, 3.2) and is tangent to curve $\mathbb{T}_2$. There are two lines represented by wrenches $\mathbf{w}_3$ and $\mathbf{w}'_3$ satisfying these conditions as shown in Fig. 9. Here, unit wrench $\mathbf{w}_3$ is selected, using Eqs. (49)–(50):

$$\mathbf{w}_3 = [0.9945, 0.1046, -2.8476]^T$$

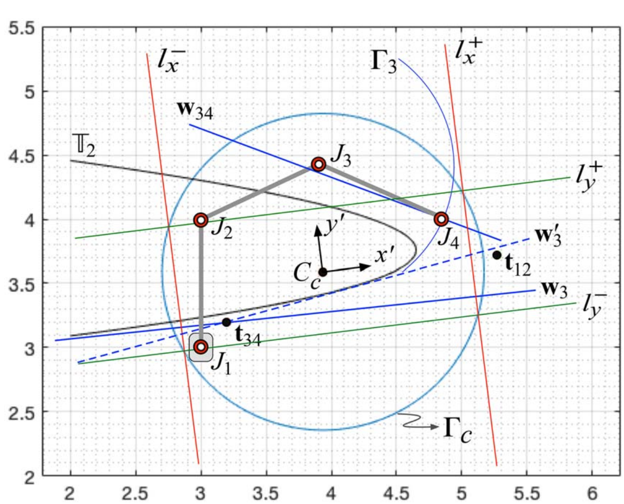

Fig. 9 Compliance synthesis process. Selection of joint locations for joints J_1, J_2, J_3, and J_4.

By the results presented in Sec. 3.2, the twist associated with $\mathbf{w}_3$ through the compliance mapping must be located on circle Γ_2. This twist is calculated to be:

$$\hat{\mathbf{t}}_3 = \mathbf{C}\mathbf{w}_3 = [1.3043, -1.1508, 0.2947]^T$$

The center of twist $\hat{\mathbf{t}}_3$, (3.9050, 4.4255), is the location of J_3. The (unit) joint twist of J_3 is expressed as follows:

$$\mathbf{t}_3 = [4.4255, -3.9050, 1]^T$$

(5) Select the location of J_4. Joint J_4 must be on circle Γ_3 with radius $l_3 = 1$ centered at joint J_3.

The wrench associated with twist $\mathbf{t}_{34}$ is calculated to be

$$\mathbf{w}_{34} = \mathbf{K}\mathbf{t}_{34} = [-1.2780, 0.4969, 7.5962]^T$$

The intersection of line $\mathbf{w}_{34}$ and circle Γ_3 will be the location of J_4. If (x, y) are the coordinates of J_4, then the corresponding joint twist is $\mathbf{t}_4 = [y, -x, 1]^T$. The location of J_4 is obtained by solving the following equations:

$$(x - x_3)^2 + (y - y_3)^2 = l_3 = 1$$

$$\mathbf{t}_4^T \mathbf{w}_{34} = 0$$

For the two sets of solutions to the equations, select the one that better causes the set of joints to surround the compliance center. Here, the solution $(x, y) = (4.8370, 4.0632)$ is selected to be the location of J_4. The joint twist of J_4 is expressed as follows:

$$\mathbf{t}_4 = [4.0632, -4.8370, 1]^T$$

(6) Determine the location of J_5. For the selected four joints (J_1, J_2, J_3, J_4), a quadratic curve passing through these four joints is obtained using Eq. (45):

$$f_{1234}: \ \mathbf{t}^T \mathbf{A}_{1234} \ \mathbf{t} = 0$$

where twist $\mathbf{t}$ is defined in Eq. (34) and the symmetric matrix $\mathbf{A}_{1234}$ calculated from Eqs. (43)–(44) is expressed as follows:

$$\mathbf{A}_{1234} = \begin{bmatrix} 0.1429 & 0.1184 & 0.8554 \\ 0.1184 & -0.1892 & -1.2253 \\ 0.8554 & -1.2253 & -7.3636 \end{bmatrix}$$

The intersection of the quadratic curve and circle Γ_4 occurs outside of Γ_c at (5.6552, 3.4882), which is the location of J_5. Thus, all five-joint locations are identified and shown in Fig. 10. The joint twist of J_5 is expressed as follows:

$$\mathbf{t}_5 = [3.4882, -5.6552, 1]^T$$

Thus, all five-joint locations are identified and shown in Fig. 10.

Since all conditions in Proposition 3 are satisfied for the selected five-joint locations, the given compliance is passively realized by the serial mechanism.

(7) Determine the joint compliances c_i. By using Eq. (47):

$$\mathbf{c} = [0.3352, 0.1643, 0.1344, 0.0867, 0.2294] \ (\mathrm{N \cdot m})^{-1}$$

With this final step, the mechanism configuration and joint compliances are identified. By using the obtained results for joint compliances identified earlier and the joint twists:

$$[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5] = \begin{bmatrix} 3 & 4 & 4.4255 & 4.0632 & 3.4882 \\ -3 & -3 & -3.9050 & -4.8370 & -5.6552 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

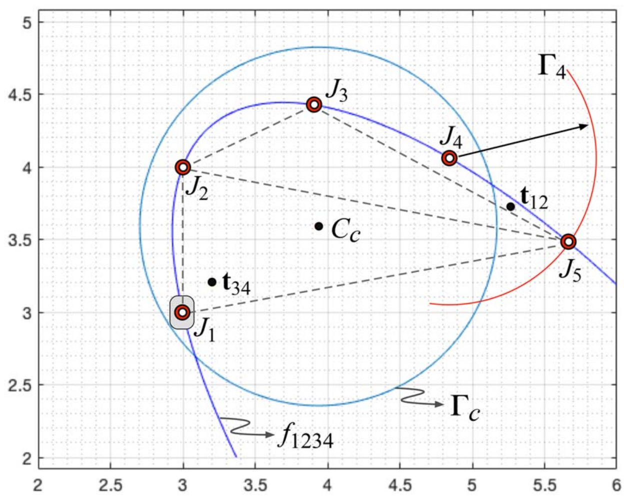

Fig. 10 Selection of the location of joint J_5. The joint must be located on quadratic curve f_{1234} determined by the other four joints.

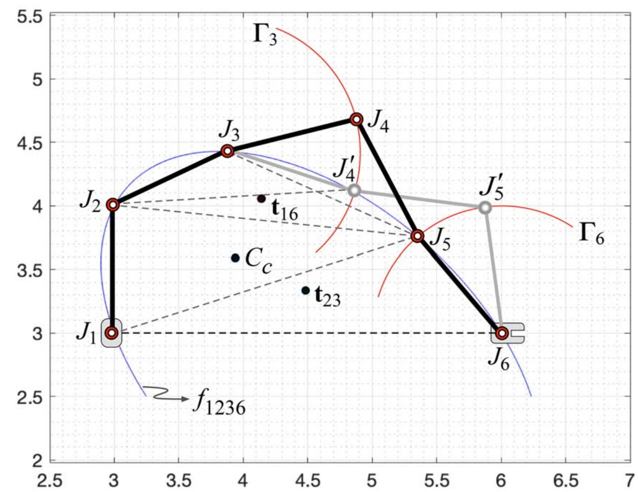

Fig. 11 Synthesis of C with a 6R serial mechanism with the base joint location J_1 and last joint location J_6 specified. The synthesis procedure yields a mechanism for which only five of the six joints are elastic.

The calculated compliance is expressed as follows:

$$\sum_{i=1}^{5} c_i \mathbf{t}_i \mathbf{t}_i^T = \begin{bmatrix} 12.50 & -13.54 & 3.41 \\ -13.54 & 15.91 & -3.74 \\ 3.41 & -3.74 & 0.95 \end{bmatrix}$$

which verifies the realization.

5.2 6*R* Mechanism Synthesis. If a 6*R* serial mechanism is considered, due to the increase in the degrees-of-freedom, the locations of joints J_1 and J_6 can be specified arbitrarily based on the conditions of Proposition 2. As stated in Sec. 4.2.1, a given compliance can be realized by a 6*R* mechanism with only five effective elastic joints. In the following, the synthesis of **C** with five joints is performed. Then, using the procedure described in Sec. 4.2.2, the synthesis of **C** with all six joints having nonzero compliance is performed.

5.2.1 Realization With Five Elastic Joints. Select joints J_1 and J_6 to be located at positions (3,3) and (3,6), respectively. The locations of J_2 and J_3 can be selected using the process of 5*R* mechanism synthesis described in Sec. 4.2.1. Here, J_2 and J_3 locations are selected to be the same as that in Sec. 5.1: J_2 is located at (3, 4), and J_3 is located at (3.9050, 4.4255). Since joint J_6 is specified, for five-joint synthesis, we only need to select one joint, either J_4 or J_5, to be located on the quadratic curve f_{1236} determined by the four joint locations (J_1, J_2, J_3, J_6).

By using Eq. (45), curve f_{1236} is calculated and illustrated in Fig. 11. The wrench associated with the line passing through J_1 and J_6 is determined as follows:

$$\mathbf{w}_{16} = [1, 0, -3]^T$$

The twist associated with $\mathbf{w}_{16}$ is calculated to be:

$$\mathbf{t}_{16} = \mathbf{C}\mathbf{w}_{16} = [2.2700, -2.3200, 0.56]^T$$

which is located at (4.1429, 4.0536) as shown in Fig. 11.

Similarly, unit wrench $\mathbf{w}_{23}$ and the corresponding twist $\mathbf{t}_{23}$ are calculated to be

$$\mathbf{w}_{23} = [0.9050, 0.4255, -2.3435]^T$$

$$\mathbf{t}_{23} = [-2.4401, 3.2807, -0.7316]^T$$

Twist $\mathbf{t}_{23}$ is located at (4.4840, 3.3351) and is illustrated in Fig. 11.

First, consider the case that J_4 is on curve f_{1236}. The location of J_4 is the intersection of circle Γ_3 and curve f_{1236}, which is calculated to be J'_4 (4.8546, 4.1120). Since $\mathbf{t}_{16}$ is outside triangle $J_2J_3J'_4$, the realization condition (ii) of Proposition 3 is not satisfied, and the given compliance cannot be passively realized at this configuration.

Now consider the case that J_5 is located on curve f_{1236}. The location of J_5 can be determined by the intersection of circle Γ_5 and curve f_{1236}, which is point (5.3498, 3.7598). Since (J_1, J_2, J_3, J_5, J_6) are all located on curve f_{1236}, the location of J_4 is irrelevant ($c_4 = 0$). From Fig. 11, it can be seen that $\mathbf{t}_{16}$ is located inside triangle $J_2J_3J_5$ and $\mathbf{t}_{23}$ is located inside triangle $J_1J_5J_6$. Thus, all conditions in Proposition 3 are satisfied, and the given compliance can be passively realized at this configuration.

The five elastic joint twists selected are given as follows:

$$[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_5, \mathbf{t}_6] = \begin{bmatrix} 3 & 4 & 4.4255 & 3.7598 & 3 \\ -3 & -3 & -3.9050 & -5.3498 & -6 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

By using Eq. (47), the five-joint compliances are calculated as follows:

$$[c_1, c_2, c_3, c_5, c_6] = [0.3313, 0.1744, 0.1292, 0.2651, 0.05]$$

with units $(\mathrm{N \cdot m})^{-1}$. It is readily verified that:

$$\sum c_i \mathbf{t}_i \mathbf{t}_i^T = \begin{bmatrix} 12.50 & -13.54 & 3.41 \\ -13.54 & 15.91 & -3.74 \\ 3.41 & -3.74 & 0.95 \end{bmatrix}$$

5.2.2 Realization With Six Joints. Consider a configuration close to that obtained in Sec. 5.2.1. When all six joints have nonzero passive compliances, J_4 and J_5 must be separated by curve f_{1236}. There are infinitely many possible locations of J_4 and J_5 on the opposite sides of curve f_{1236} that satisfy link length restrictions. Figure 12 illustrates two cases (J_4, J_5) and (J'_4, J'_5) in which the two joints are separated by curve f_{1236}.

First, consider the case that J_5 is outside of the curve. Choose the location of J_5 at (5.6, 3.9156) on Γ_5, then the location of J_4 can be obtained by solving the equations:

$$(x - 5.6)^2 + (y - 3.9156)^2 = l_4^2 = 1$$

$$(x - 3.9050)^2 + (y - 4.4255)^2 = l_3^2 = 1$$

which yields (4.6185, 3.7249) as the location of J_4.

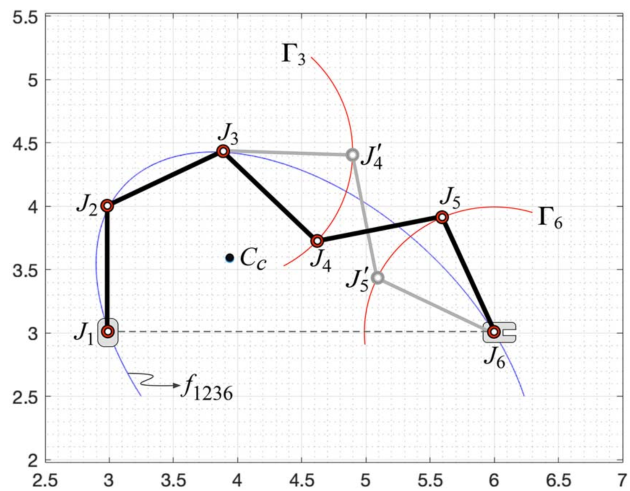

Fig. 12 Synthesis of C with a 6*R* serial mechanism with the base joint location J_1 and last joint location J_6 specified. In the synthesis, all six joints are elastic. Note that joints J_4 and J_5 must be separated by curve f_{1236}.

For the selected six joints (J_1, J_2, J_3, J_4, J_5, J_6) shown in Fig. 12, the joint twist matrix is expressed as follows:

$$[\mathbf{T}] = \begin{bmatrix} 3 & 4 & 4.4255 & 3.7249 & 3.9165 & 3 \\ -3 & -3 & -3.9050 & -4.6185 & -5.6000 & -6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

By using Eq. (56), the six-joint compliances are obtained:

$$\mathbf{c} = [0.3214, 0.1758, 0.1104, 0.1297, 0.1450, 0.0678]$$

with units $(\mathrm{N}\cdot\mathrm{m})^{-1}$.

The obtained results can be verified using Eq. (53):

$$\sum_{i=1}^{6} c_i \mathbf{t}_i \mathbf{t}_i^T = \begin{bmatrix} 12.50 & -13.54 & 3.41 \\ -13.54 & 15.91 & -3.74 \\ 3.41 & -3.74 & 0.95 \end{bmatrix}$$

which confirms that passive realization of the given compliance is achieved.

If the location of the fifth joint is chosen to be inside the curve, for example, at J_5' (5.1, 3.4359) as shown in Fig. 12, the location of the fourth joint is determined to be at J_4' (4.9049, 4.4167). At this configuration (J_1, J_2, J_3, J_4', J_5', J_6), the values of the joint compliances are calculated to be:

$$\mathbf{c} = [0.2833, 0.2905, -0.0775, 0.2097, 0.1902, 0.0537]$$

Since c_3 is negative, the given compliance $\mathbf{C}$ cannot be passively achieved at this configuration.

If a different compliance is desired at the same end-effector location, the joints J_1 and J_6 can be selected at the same locations for the mechanism as specified in Sec. 5.2.1. Following the steps as described earlier, the synthesis procedure will yield a different mechanism configuration and a different set of joint compliances that realize the desired compliant behavior.

6 Discussion

It is known that any planar compliance can be realized with an n-joint ($n \geq 3$) serial mechanism if there are no constraints on the link lengths. To increase the space of realizable compliant behaviors when link lengths are specified, more joints are needed. For a mechanism having the number of joints $n \leq 5$, increasing n by 1 increases the dimension of realizable space by 1. When $n \geq 6$, increasing the number of joints does not significantly enlarge the realizable space.

If link lengths are specified, a minimum of five joints are needed to realize an arbitrary compliance (within a space bounded by inequalities) at given base and end-effector positions. Thus, for a given compliance, a given 5*R* or given 6*R* serial mechanism can be configured to achieve the behavior.

In each step of the processes presented in Secs. 3 and 4, the selection of joint locations is not unique. Since the procedure is geometric construction based, graphics tools can be used to select a better mechanism configuration in the realization of compliance.

In the second step of the procedure presented in Sec. 3, it is suggested that point $\mathbf{t}_{34}$ be selected close to the line segment formed by the two selected joints J_1 and J_2. It can be proved that, if $\mathbf{t}_{34}$ is on segment $\overline{J_1J_2}$, then point $\mathbf{t}_{12}$ must be on the line passing through J_3 and J_4 selected by the procedure. Thus, if $\mathbf{t}_{34}$ is close to $\overline{J_1J_2}$, $\mathbf{t}_{12}$ is close to the line passing through J_3 and J_4. However, if $\mathbf{t}_{34}$ is selected on $\overline{J_1J_2}$, the matrix $\mathbf{A}_{1234}$ in Eq. (44) will be rank-deficient, and the associated quadratic curve f_{1234} in Eq. (45) will be degenerate. For this case, the conditions of Proposition 3 cannot be applied. As such, $\mathbf{t}_{34}$ should be selected close to but not on segment $\overline{J_1J_2}$.

7 Summary

In this article, the realization of planar compliance with a serial mechanism having specified link lengths is addressed. Insight into the joint distribution of a general serial mechanism in the realization of a given compliance is provided. Synthesis procedures to achieve a compliance with a serial mechanism having fixed link lengths and either five or six elastic joints are developed. The theories presented in this article enable one to assess the ability of a given compliant serial mechanism to realize any given compliance and to select a configuration of a given mechanism that achieves a realizable elastic behavior. Since the developed synthesis procedures are completely geometry based, computer graphics tools can be used in the process to obtain a mechanism configuration that attains the desired compliance.

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Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

The authors attest that all data for this study are included in the paper. Data provided by a third party listed in Acknowledgment.

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