Topology and its Applications 316 (2022) 108101

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Normalizers of chains of discrete p-toral subgroups in compact Lie groups

Eva Belmont [a], Natàlia Castellana [b], Jelena Grbić [c], Kathryn Lesh [d,*], Michelle Strumila [e]

[a] *Department of Mathematics, University of California San Diego, La Jolla, CA, USA*
[b] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, and Centre de Recerca Matemàtica, Barcelona, Spain*
[c] *School of Mathematical Sciences, University of Southampton, Southampton, UK*
[d] *Department of Mathematics, Union College, Schenectady NY, USA*
[e] *School of Mathematics and Statistics, Melbourne University, Parkville VIC, Australia*

ARTICLE INFO

Article history:
Received 15 April 2020
Received in revised form 16 December 2021
Available online 7 April 2022

MSC:
primary 55R35
secondary 20D20, 20E25, 22E40

Keywords:
Classifying spaces
Compact Lie groups
Fusion
p-completion

ABSTRACT

In this paper we study the normalizer decomposition of a compact Lie group G using the information of the fusion system $\mathcal{F}$ of G on a maximal discrete p-toral subgroup. We prove that there is an injective map from the set of conjugacy classes of chains of $\mathcal{F}$-centric, $\mathcal{F}$-radical discrete p-toral subgroups to the set of conjugacy classes of chains of p-centric, p-stubborn continuous p-toral subgroups. The map is a bijection when $\pi_0(G)$ is a finite p-group. We also prove that the classifying space of the normalizer of a chain of discrete p-toral subgroups of G is mod p equivalent to the classifying space of the normalizer of the corresponding chain of p-toral subgroups.
© 2022 Elsevier B.V. All rights reserved.

1. Introduction

In [8], Dwyer formalized and unified three homology decompositions for the p-completed classifying space of a finite group G based on a collection of p-subgroups: the centralizer decomposition, the subgroup decomposition, and the normalizer decomposition. The first two had been studied in [11] and [12] for compact Lie groups, and the normalizer decomposition was new in this context. Dwyer showed that for a given collection of subgroups of a finite group G, either all three decompositions give the correct homotopy type for $BG^{\wedge}_{p}$ or none of them do. Such decompositions in the setting of Lie groups have since been studied

* Corresponding author.
E-mail addresses: ebelmont@ucsd.edu (E. Belmont), natalia@mat.uab.cat (N. Castellana), j.grbic@soton.ac.uk (J. Grbić), leshk@union.edu (K. Lesh), mstrumila@student.unimelb.edu.au (M. Strumila).

https://doi.org/10.1016/j.topol.2022.108101
0166-8641/© 2022 Elsevier B.V. All rights reserved.

by other authors, for example [6,14,19,20]. In particular, in [14] Libman gives a normalizer decomposition and then unifies the three homology decompositions for Lie groups, as Dwyer did in the finite group case.

Recently, the homotopy theory of p-local compact groups [3, Defn 4.2] has provided a new, more general framework for dealing with the homotopy type of p-completed classifying spaces of compact Lie groups, in addition to other examples coming from finite loop spaces ([4]). One works with discrete p-toral groups (Definition 2.1) instead of p-toral groups. The formal structure of a p-local compact group consists of a triple $(S, \mathcal{F}, \mathcal{L})$ where $\mathcal{F}$ is a saturated fusion system over the discrete p-toral group S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. But in view of [15, Thm B] a p-local compact group is equivalent to just a pair $(S, \mathcal{F})$, namely a saturated fusion system over a discrete p-toral group.

When the p-local compact group arises from a compact Lie group G, it encodes the essential p-local information needed to uniquely determine the homotopy type of $BG^\wedge_p$ (see [3], [5], [18], [15]). A great advantage of studying Lie groups via this theory is that it reduces the study of a topological group to the study of a collection of *discrete* subgroups. There are also other interesting examples of p-local compact groups. For example, one can construct a p-local compact group capturing the homotopy type of a p-compact group (an $\mathbb{F}_p$-finite loop space together with a chosen p-complete delooping, see [7]). Other examples of p-local compact groups are given in [4] and [9].

To state the form of a normalizer decomposition more precisely, consider a collection $\mathcal{C}$ of closed subgroups of a Lie group G. Define $\overline{s}d(\mathcal{C})$ to be the poset of G-conjugacy classes of chains of proper inclusions in $\mathcal{C}$, say $\mathbf{H}_* := (\mathbf{H}_0 \subset \cdots \subset \mathbf{H}_k)$. One can construct a functor $\delta\colon \overline{s}d(\mathcal{C}) \to \mathrm{Top}$, and a natural transformation from δ to the constant functor with value BG, to induce a map

$$\left(\operatorname*{hocolim}_{\overline{s}d(\mathcal{C})} \delta\right)^\wedge_p \to BG^\wedge_p \tag{1.1}$$

such that $\delta(\mathbf{H}_*) \simeq BN_G(\mathbf{H}_0 \subset \cdots \subset \mathbf{H}_k) := B\big(\bigcap_i N_G(\mathbf{H}_i)\big)$. The following statement collects results of [14, Thm C, D], [12, Thm 1.4], and [3, Lemma 9.7] that establish collections for which the normalizer decomposition correctly computes the p-completed homotopy type of BG.

Theorem 1.2. *Let G be a compact Lie group and let $\mathcal{C}$ be either (i) the collection of nontrivial p-radical p-toral subgroups or (ii) the collection of p-stubborn p-toral subgroups or (iii) the collection of p-centric p-toral subgroups of G (see Definition 4.1). Then (1.1) is an equivalence.*

Our program's goal, taken up in a forthcoming work [1], is a computable setup that generalizes the normalizer decomposition (1.1) from compact Lie groups to p-local compact groups. The formalism is a straightforward generalization of the earlier work of Libman [13] giving a normalizer decomposition for p-local *finite* groups. In a result similar to Theorem 1.2, [1] will also show that if the p-local compact group corresponds to the fusion system $\mathcal{F}$, then the full subcategory of $\mathcal{F}$ consisting of $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups (Definition 4.1) is sufficient to determine the homotopy type of the p-completed classifying space. This result is in the literature for finite groups ([10, Thm 1.5]) and p-local finite groups ([2, Thm 3.5]), but not for p-local compact groups.

When it comes to actual computations, the analysis of the spaces coming into our normalizer decompositions for p-local compact groups can be delicate. This paper is largely in service of understanding the spaces in the decompositions that we obtain in certain examples. In particular, we need to understand what happens in the case of a p-local compact group that arises from a compact Lie group, because we want to compare the decomposition we obtain in [1] with the earlier one of Libman for the corresponding Lie group [14].

We turn to a description of the contents of this paper and how they fit into our program. Let G be a compact Lie group, and let $\mathbf{S} \subseteq G$ be a maximal p-toral subgroup of G with maximal discrete p-toral

subgroup $S \subseteq \mathbf{S}$. The corresponding fusion system $\mathcal{F}$ associated to G is the category whose objects are the discrete p-toral subgroups of S, and whose morphisms are given by homomorphisms induced by conjugation by elements of G. The goal of this paper is to establish that the left side of (1.1), which is described in terms of chains of continuous p-toral groups and the action of G, can instead be described in terms of $\mathcal{F}$, i.e. in terms of chains of discrete p-toral groups of G and morphisms in $\mathcal{F}$. There are two issues: the indexing category, and the values of the functor δ.

Our first theorem addresses the indexing category, by relating conjugacy classes of chains of discrete p-toral subgroups of a compact Lie group G to conjugacy classes of chains of continuous p-toral subgroups of G. The following theorem establishes that the desired classes of chains can all be found by considering the p-stubborn p-toral subgroups of G, which are classified in [17] for classical groups. (See Definition 2.5 for p-discretization.)

Theorem 4.3. *Let $\mathbf{S}$ be a maximal p-toral subgroup of a compact Lie group G, with p-discretization $S \subseteq \mathbf{S}$. The closure map $P \mapsto \overline{P}$ defines an injective map*

$$\begin{array}{c} \{P_0 \subseteq \ldots \subseteq P_k \subseteq S \,|\, \text{all } P_i \text{ are } \mathcal{F}\text{-centric and } \mathcal{F}\text{-radical}\}\,/G \\ \Big\downarrow \\ \{\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k \subseteq \mathbf{S} \,|\, \text{all } \mathbf{P}_i \text{ are } p\text{-toral, } p\text{-centric, and } p\text{-stubborn }\}\,/G. \end{array}$$

The map is a one-to-one correspondence if $\pi_0 G$ is a p-group.

Our second theorem deals with the values of the functor δ in (1.1). In particular, we relate the mod p homotopy type of the classifying spaces of normalizers of chains of discrete p-toral subgroups to those of chains of continuous p-toral subgroups. Since our decomposition for p-local compact groups will involve the former, this theorem will relate *(i)* the decomposition given by our p-local compact group methods applied to the case of a compact Lie group and *(ii)* the decomposition for a compact Lie group that is obtained by [14].

Theorem 5.1. *Let $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k$ be a chain of p-toral subgroups of a compact Lie group G, and let $P_0 \subseteq \ldots \subseteq P_k$ be a chain of discrete p-toral subgroups such that each P_i is a p-discretization of $\mathbf{P}_i$. Then*

$$N_G\left(P_0 \subseteq \ldots \subseteq P_k\right) \longrightarrow N_G\left(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k\right)$$

induces a mod p equivalence of classifying spaces.

The proof introduces the outer automorphism group of a chain $H_0 \subseteq \cdots \subseteq H_k$ (Definition 5.2), which turns out to be a finite group and plays an important role in the argument. (See Proposition 5.3, Lemma 5.6, and diagram (5.14).)

In summary, this paper provides the technical results necessary to compare two normalizer decompositions for classifying spaces of compact Lie groups: the one obtained by applying [1] to a p-local compact group arising from a Lie group G, and the earlier one due to Libman [14], obtained by techniques using G-actions. The two decompositions are related by taking closures of discrete p-toral subgroups, which brings up surprisingly subtle issues. Hence we develop some useful tools for studying the relationship between discrete p-toral groups and their closures, as well as the relationship between the classifying spaces of their respective normalizers.

Notation. Throughout the paper, G denotes a compact Lie group. Our convention for conjugation is that $c_g(x) = g^{-1}xg$. We generally use a boldface font to denote a topological group, as opposed to a discrete

group, with the exception of the ambient Lie group G itself. For example, we use $\mathbf{P}$ for a p-toral group, and P for a discrete p-toral group.

Organization. Section 2 includes background material on p-toral and discrete p-toral subgroups of a compact Lie group. Section 3 presents a key technical result on p-discretization of pairs, which allows us to understand how chains of discrete p-toral subgroups conjugate inside their closures. Some of the results of this section already appear in [3], but we present some simplified proofs. Section 4 contains the proof of Theorem 4.3. Lastly, in Section 5 we introduce the group of outer automorphisms of a chain and we prove Theorem 5.1.

Acknowledgments. This paper is the first part of the authors' Women in Topology III project. A second part of that project will appear in a separate article [1]. We thank the organizers of the Women in Topology III workshop, as well as the Hausdorff Research Institute for Mathematics, where the workshop was held. The Women in Topology III workshop was supported by NSF grant DMS-1901795, the AWM ADVANCE grant NSF HRD-1500481, and Foundation Compositio Mathematica. The second author was partially supported by Spanish State Research Agency through the FEDER-MEC grant MTM2016-80439-P, and the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). Finally, the authors gratefully acknowledge exchanges with Bill Dwyer, who suggested the approach to p-discretizations used in Section 3, and also thank the anonymous referee for a careful reading.

2. p-toral and discrete p-toral subgroups of Lie groups

In this section, we give background material on p-toral and discrete p-toral subgroups of a compact Lie group G. First, the definitions.

Definition 2.1.

(1) A group is *p-toral of rank r* if it is an extension of a torus of rank r by a finite p-group.
(2) A *discrete p-torus of rank r* is a group isomorphic to a product $(\mathbb{Z}/p^\infty)^r$.
(3) A *discrete p-toral group of rank r* is an extension of a discrete p-torus of rank r by a finite p-group.

Remark. When we want to emphasize the difference between a p-toral group and a discrete p-toral group, we will sometimes refer to the former as a *continuous p-toral group*.

The p-toral subgroups of a compact Lie group G play a key role in the analysis of the mod p homology of the classifying space of G, analogous to the role played by the p-subgroups in the case of a finite group. However, there is a key difference between the finite and topological contexts: subgroups of finite p-groups are finite p-groups, but subgroups of p-toral groups need not be p-toral. For example, S^1 is a p-toral group, but it has finite subgroups of order prime to p (certainly not p-toral) as well as the subgroup $\mathbb{Z}/p^\infty \subset S^1$ (also not p-toral, for a different reason). By contrast, a subgroup of a discrete p-toral group is necessarily another discrete p-toral group. This feature of discrete p-toral subgroups of a compact Lie group G gives them an advantage over continuous p-toral subgroups as tools to approximate G.

A compact Lie group G admits both maximal continuous p-toral subgroups and maximal discrete p-toral subgroups, both of which have properties analogous to those of the Sylow p-subgroups of a finite group.

Proposition 2.2 *([3, Prop. 9.3]). Let G be a compact Lie group.*

(1) *Every p-toral subgroup (respectively, discrete p-toral subgroup) of G is contained in a maximal one.*
(2) *All maximal p-toral subgroups (respectively, discrete p-toral subgroups) are conjugate in G.*

Unfortunately, discrete p-toral subgroups will not be good approximations to continuous p-toral groups when their group-theoretic properties interact badly with their embeddings. In particular, a discrete p-toral subgroup can have smaller rank than its closure, and this can occur even when the groups are just tori. For example, $\mathbb{Z}/p^\infty$ can be embedded via a homomorphism as a dense subgroup of $S^1 \times S^1$. To set apart good approximations from bad ones, we have the following definition.

Definition 2.3 *([3, Defn. 9.1]).* A discrete p-toral subgroup $P \subseteq G$ is *snugly embedded* if P is a maximal discrete p-toral subgroup of $\overline{P}$.

Lemma 2.4 *([3, Prop. 9.2]). If $P \subseteq G$ is a snugly embedded discrete p-toral group, then $P \hookrightarrow \overline{P}$ induces a homotopy equivalence $(BP)^\wedge_p \simeq \left(B\overline{P}\right)^\wedge_p$.*

Not all discrete p-toral subgroups of a compact Lie group G are snugly embedded. However, since any p-toral group possesses maximal discrete p-toral subgroups by Proposition 2.2, a p-toral group can always be approximated by a snugly embedded discrete p-toral group. A more compact terminology will be helpful.

Definition 2.5. Let $\mathbf{P}$ be a p-toral subgroup of G, and let $P \subseteq \mathbf{P}$ be a snugly embedded discrete p-toral subgroup with $\overline{P} = \mathbf{P}$. We say that P is a *p-discretization* of $\mathbf{P}$.

In particular, a p-discretization of $\mathbf{P}$ is characterized by being a maximal discrete p-toral subgroup of $\mathbf{P}$.

Example 2.6. A torus has only one p-discretization, namely the subgroup consisting of all p-torsion elements. Similarly, Proposition 2.2(2) establishes that any abelian p-toral group has a unique p-discretization.

In general, however, a p-toral group $\mathbf{P}$ that has multiple components has many p-discretizations. If $P \subseteq \mathbf{P}$ is one such, then the others are all conjugate to P in $\mathbf{P}$ by Proposition 2.2(2). The stabilizer of P is $N_{\mathbf{P}}(P)$, so the approximations are parametrized by $\mathbf{P}/N_{\mathbf{P}}(P)$. (See also Remark 3.7.)

The simplest nontrivial example with more than one discretization is the 2-toral group $\mathbf{P} = O(2) \cong S^1 \rtimes \{\pm 1\}$, where -1 is represented by reflection over the y-axis. An obvious 2-discretization is given by the subgroup $P = \mathbb{Z}/2^\infty \rtimes \{\pm 1\}$. A direct matrix calculation shows that $N_{\mathbf{P}}(P) = P$, so in fact the 2-discretizations are parametrized by $\mathbf{P}/P \cong S^1/(\mathbb{Z}/2^\infty)$. The other parametrizations are given by

$$P' = (\mathbb{Z}/2^\infty \times \{1\}) \sqcup (\xi \cdot \mathbb{Z}/2^\infty \times \{-1\}) \tag{2.7}$$

where ξ is any fixed element of S^1. And indeed, the proof of [3, 9.3] establishes that, in general, the different p-discretizations of a p-toral group $\mathbf{P}$ can be obtained by conjugation by an element of the torus of $\mathbf{P}$.

We close this section by observing that, as in (2.7), any p-discretization must start with the unique p-discretization of the torus.

Lemma 2.8. *If $\mathbf{P}$ is a p-toral group with maximal torus $\mathbf{T}$, and T_p denotes the p-torsion elements of $\mathbf{T}$, then any p-discretization P of $\mathbf{P}$ must contain T_p.*

Proof. By Proposition 2.2, T_p can be expanded to a p-discretization P' of $\mathbf{P}$, and P' is conjugate to P in $\mathbf{P}$. However, T_p is a normal subgroup of $\mathbf{P}$ and hence is stabilized by the conjugation. Thus $T_p \subseteq P$ as well. □

3. p-discretizations of pairs

Let $\mathbf{P}$ be a p-toral group. Proposition 2.2 tells us that all p-discretizations of $\mathbf{P}$ are conjugate in $\mathbf{P}$. It also tells us that if $\mathbf{Q} \subseteq \mathbf{P}$ is a p-toral subgroup and Q is a p-discretization of $\mathbf{Q}$, then Q can be expanded to

a p-discretization P of $\mathbf{P}$. However, it is likely that there is more than one way to expand Q; that is, there can be different pairs of discrete p-toral subgroups (Q, P_1) and (Q, P_2) that are p-discretizations for the pair $(\mathbf{Q}, \mathbf{P})$. The main goal of this section is to establish the following proposition, which can be thought of as a uniqueness statement about p-discretizations of pairs. The point is that P_1 and P_2 are conjugate in $\mathbf{P}$ by an element that fixes Q. Hence the pair (Q, P_1) is conjugate in $\mathbf{P}$ to the pair (Q, P_2).

Proposition 3.1. *If P_1 and P_2 are p-discretizations of $\mathbf{P}$, and $Q \subseteq P_1 \cap P_2$, then there exists $\mathbf{y} \in C_{\mathbf{P}}(Q)$ such that $c_{\mathbf{y}}(P_1) = P_2$.*

Our approach is based on a non-canonical (and non-topological) splitting of p-toral groups, for which we use the following standard homological lemma.

Lemma 3.2. *Let K be a finite group, and let*

$$0 \longrightarrow I \longrightarrow X \longrightarrow V \longrightarrow 0$$

be a short exact sequence of $\mathbb{Z}[K]$-modules. If V is uniquely $|K|$-divisible and I is an injective $\mathbb{Z}$-module, then there is a splitting $X \cong I \times V$ as $\mathbb{Z}[K]$-modules.

Proof. Because I is an injective abelian group, there is a retraction of abelian groups $r\colon X \to I$, which in turn defines a section $s\colon V \to X$ of abelian groups. However, $\mathrm{Hom}_{\mathbb{Z}}(V, X)$ is uniquely $|K|$-divisible (because V is), so we can define a new section $\tilde{s}\colon V \to X$ by averaging over the elements of K,

$$\tilde{s} := \frac{1}{|K|} \sum_{y \in K} y^{-1} s y.$$

Then $\tilde{s}$ is a section of $X \to V$ as $\mathbb{Z}[K]$-modules, which establishes the lemma. □

We use the following notation for the splitting result below. Let $\mathbf{T} = \mathbb{R}^r/\mathbb{Z}^r$ be a rank r torus, whose subgroup of torsion elements is denoted $T_{\mathbb{Q}} := \mathbb{Q}^r/\mathbb{Z}^r$. The quotient of $\mathbf{T}$ by the torsion elements is denoted $T_\infty := \mathbf{T}/T_{\mathbb{Q}}$. If p is a prime, then the p-torsion subgroup of $\mathbf{T}$ is denoted $T_p := (\mathbb{Z}/p^\infty)^r$, and we write $T_{p'}$ for the subgroup of $\mathbf{T}$ consisting of torsion elements of order prime to p, i.e. the product of all the subgroups $(\mathbb{Z}/q^\infty)^r$ over primes $q \neq p$.

Lemma 3.3. *Let $\mathbf{P}$ be a p-toral group with p-discretization $P \subseteq \mathbf{P}$. There exists a (non-canonical, discontinuous) group homomorphism $\mathbf{P} \to P$ that splits the inclusion $P \hookrightarrow \mathbf{P}$. Any such splitting has the property that if P' is another p-discretization of $\mathbf{P}$, then $P' \to \mathbf{P} \to P$ is an isomorphism.*

Proof. Let $\mathbf{T}$ be the maximal torus of $\mathbf{P}$ and let $K = \pi_0\mathbf{P} = \mathbf{P}/\mathbf{T}$, a finite p-group. By Lemma 3.2 applied to $\mathbf{T}$ (considered as a discrete group), there is a split short exact sequence of $\mathbb{Z}[K]$-modules

$$0 \longrightarrow T_p \longrightarrow \mathbf{T} \longrightarrow T_{p'} \times T_\infty \longrightarrow 0.$$

Note that by Lemma 2.8, $T_p \subseteq P$. Further, because $T_{p'} \times T_\infty$ is split from $\mathbf{T}$ as a $\mathbb{Z}[K]$-module, we know $T_{p'} \times T_\infty$ is normal in $\mathbf{P}$ and we can define the quotient $\tilde{P} := \mathbf{P}/(T_{p'} \times T_\infty)$, which is a discrete p-toral group. (We note here that we have completely discarded the topology on the torus. The key tool resulting from Lemma 3.3 is Lemma 3.5, and the topology is not needed there.)

Consider the commutative ladder of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_p & \longrightarrow & P & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \underbrace{T_p \times T_{p'} \times T_\infty}_{\mathbf{T}} & \longrightarrow & \mathbf{P} & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & T_p & \longrightarrow & \tilde{P} & \longrightarrow & K & \longrightarrow & 0. \end{array} \tag{3.4}$$

By construction, the compositions of the two maps in the first and third columns are identity maps on T_p and K, respectively. Hence the composite $f := q \circ i \colon P \to \tilde{P}$ is an isomorphism. Then $f^{-1} \circ q \colon \mathbf{P} \to P$ is the required group homomorphism splitting the inclusion $P \subseteq \mathbf{P}$.

If $P' \subset \mathbf{P}$ is another p-discretization of $\mathbf{P}$, then P' also contains T_p (Lemma 2.8). Therefore we can substitute P' for P in (3.4) and the composite $P' \to \mathbf{P} \to \tilde{P}$ will still be an isomorphism. Hence composing with the isomorphism $f^{-1} \colon \tilde{P} \to P$ finishes the proof. □

Using the splitting, we are able to show a sense in which a p-discretization $P \subseteq \mathbf{P}$ is able to capture conjugation information present in $\mathbf{P}$. The statement below is a slight generalization of [3, Lemma 9.4(a)] and a couple of statements in its proof.

Lemma 3.5. *Let $\mathbf{P}$ be a p-toral subgroup with p-discretization P. Let Q_1 and Q_2 be subgroups of P, and suppose that a group homomorphism $f \colon Q_1 \to Q_2$ is induced by conjugation in $\mathbf{P}$. Then f can be induced by conjugation in P.*

Proof. Suppose that $f \colon Q_1 \to Q_2$ is given by conjugation by $\mathbf{y} \in \mathbf{P}$. Let $r \colon \mathbf{P} \to P$ be the retraction provided by Lemma 3.3, and consider the commutative diagram

$$\begin{array}{ccccccc} Q_1 & \hookrightarrow & P & \hookrightarrow & \mathbf{P} & \xrightarrow[r]{} & P \\ {\scriptstyle f = c_{\mathbf{y}}}\downarrow & & & & \downarrow{\scriptstyle c_{\mathbf{y}}} & & \downarrow{\scriptstyle c_{r(\mathbf{y})}} \\ Q_2 & \hookrightarrow & P & \hookrightarrow & \mathbf{P} & \xrightarrow[r]{} & P. \end{array}$$

Although we cannot fill in the rectangle, because conjugation by $\mathbf{y}$ may not take P to P, we do know that (by assumption) that Q_1 and Q_2 are contained in P. Since $P \hookrightarrow \mathbf{P} \xrightarrow{r} P$ is the identity map of P, the compositions across the top and bottom rows corestrict to the identity maps on Q_1 and Q_2, respectively. Therefore conjugation by $\mathbf{y} \in \mathbf{P}$ and $r(\mathbf{y}) \in P$ induce the same map $f \colon Q_1 \to Q_2$. □

We now have all the tools we need to establish Proposition 3.1.

Proof of Proposition 3.1. Since P_1 and P_2 are both p-discretizations of $\mathbf{P}$, they are conjugate in $\mathbf{P}$. Choose $\mathbf{y} \in \mathbf{P}$ such that $c_{\mathbf{y}}(P_1) = P_2$. It is possible that $c_{\mathbf{y}}$ does not stabilize Q, so let $Q' = c_{\mathbf{y}}(Q)$. Then Q and Q' are both subgroups of P_2, and $c_{\mathbf{y}} \colon Q \to Q'$. By Lemma 3.5, there exists $x \in P_2$ such that $c_x = c_{\mathbf{y}} \colon Q \to Q'$. Define $\mathbf{y}' = \mathbf{y} \cdot x^{-1}$. Then $\mathbf{y}'$ still conjugates P_1 to P_2, but $\mathbf{y}'$ centralizes Q. □

In the remainder of this section, we give two applications of Proposition 3.1. First, we prove that the outer automorphism group in G of a p-discretization is the same as that of its closure. This is proved in [3, Lemma 9.4] using a different point of view. Second, we prove that for the purpose of understanding the

mod p homology of classifying spaces, centralizers and normalizers can be computed either in a discrete p-toral group or a continuous p-toral group. These two results are the base cases for inductions to establish the corresponding results for chains of subgroups in Section 5.

Lemma 3.6. *Let P be a p-discretization of a p-toral subgroup $\mathbf{P}$ of G. Then $\mathrm{Out}_G(P) \cong \mathrm{Out}_G(\mathbf{P})$.*

Proof. We want to prove that the natural map

$$\mathrm{Out}_G(P) := \frac{N_G(P)}{C_G(P)\cdot P} \longrightarrow \frac{N_G(\mathbf{P})}{C_G(\mathbf{P})\cdot \mathbf{P}} =: \mathrm{Out}_G(\mathbf{P})$$

is an isomorphism. To show that it is an epimorphism, suppose that $\mathbf{n} \in N_G(\mathbf{P})$. Because $c_{\mathbf{n}}(P)$ and P are both discretizations of $\mathbf{P}$, there exists $\mathbf{x} \in \mathbf{P}$ such that $c_{\mathbf{x}}(c_{\mathbf{n}}P) = P$. Therefore $\mathbf{n}\cdot\mathbf{x} \in N_G(P)$, and it represents the same class as $\mathbf{n}$ in $\mathrm{Out}_G(\mathbf{P})$.

To show injectivity, first suppose that $n \in N_G(P) \cap \mathbf{P}$. We would like to show that n is already in $C_G(P)\cdot P$. However, Lemma 3.5 tells us that the automorphism of P induced by n can be induced by some $y \in P$. Hence $n\cdot y^{-1} = c \in C_G(P)$ and $n = c\cdot y \in C_G(P)\cdot P$, as required.

To finish, suppose that $n \in N_G(P)\cap[C_G(\mathbf{P})\cdot\mathbf{P}]$, say $n = \mathbf{c}\cdot\mathbf{x}$ with $\mathbf{c} \in C_G(\mathbf{P}) = C_G(P)$ and $\mathbf{x} \in \mathbf{P}$. Then $\mathbf{x} = n\cdot\mathbf{c}^{-1}$ normalizes P. The previous argument shows that $\mathbf{x} \in C_G(P)\cdot P$, and hence $n = \mathbf{c}\cdot\mathbf{x} \in C_G(P)\cdot P$, as required. □

Remark 3.7. Observe that $N_{\mathbf{P}}(P) = N_G(P)\cap\mathbf{P}$, and the proof of Lemma 3.6 establishes that $N_G(P)\cap\mathbf{P} = (P\cdot C_G(P))\cap\mathbf{P}$. Since $C_G(P)\cap\mathbf{P} = C_G(\mathbf{P})\cap\mathbf{P} = Z(\mathbf{P})$, we have actually proved that if P is a p-discretization of $\mathbf{P}$, then $N_{\mathbf{P}}(P) = P\cdot Z(\mathbf{P})$. This gives a refinement to the discussion of Example 2.6: p-discretizations of $\mathbf{P}$ are parametrized by $\mathbf{P}/N_{\mathbf{P}}(P) = \mathbf{P}/(P\cdot Z(\mathbf{P}))$. We recover the result that if $\mathbf{P}$ is abelian ($Z(\mathbf{P}) = \mathbf{P}$), then the p-discretization is unique. Indeed, if the torus is central in $\mathbf{P}$ then there is a unique p-discretization of $\mathbf{P}$, and otherwise there are infinitely many.

For our final result of this section, note that if $\mathbf{Q} \subseteq \mathbf{P}$ is an inclusion of p-toral subgroups, then both $N_{\mathbf{P}}(\mathbf{Q})$ and $C_{\mathbf{P}}(\mathbf{Q})$ (and hence $Z(\mathbf{Q})$) are p-toral ([12, Lemma A.3]).

Proposition 3.8. *Let $Q \subseteq P$ be p-discretizations of p-toral groups $\mathbf{Q} \subseteq \mathbf{P}$. Then $C_P(Q)$ is a p-discretization of $C_{\mathbf{P}}(\mathbf{Q})$, and $N_P(Q)$ is a p-discretization of $N_{\mathbf{P}}(\mathbf{Q})$.*

Proof. Let D be a p-discretization of $C_{\mathbf{P}}(\mathbf{Q})$; we note that $Z(Q)$ is necessarily contained in D, since $Z(\mathbf{Q})$ has only one p-discretization. Since D commutes with Q, their product $D\cdot Q$ is a discrete p-toral subgroup of $N_{\mathbf{P}}(\mathbf{Q})$. We can expand $D\cdot Q$ to a p-discretization N of $N_{\mathbf{P}}(\mathbf{Q})$, and then enlarge N to a p-discretization P' of $\mathbf{P}$. So we have compatible p-discretizations

$$\begin{array}{ccccccc} D & \longrightarrow & D\cdot Q & \longrightarrow & N & \longrightarrow & P' \\ \downarrow & & & & \downarrow & & \downarrow \\ C_{\mathbf{P}}(\mathbf{Q}) & & \longrightarrow & & N_{\mathbf{P}}(\mathbf{Q}) & \longrightarrow & \mathbf{P}. \end{array}$$

By construction, $Q \subseteq N\cap\mathbf{Q}$, and since Q is a maximal discrete p-toral subgroup of $\mathbf{Q}$, we know that $Q = N\cap\mathbf{Q}$. Therefore N normalizes Q (because N normalizes both itself and $\mathbf{Q}$), so $N \subseteq N_{P'}(Q) \subseteq N_{\mathbf{P}}(\mathbf{Q})$. But N is a maximal discrete p-toral subgroup of $N_{\mathbf{P}}(\mathbf{Q})$, so in fact $N = N_{P'}(Q)$. Similarly, we have $D \subseteq C_{P'}(Q) \subseteq C_{\mathbf{P}}(\mathbf{Q})$ and maximality gives us $D = C_{P'}(Q)$.

However, we have another p-discretization of $\mathbf{P}$, namely P. Notice that Q is contained both P (by assumption) and P' (by construction), so Proposition 3.1 says that there exists $\mathbf{y} \in C_{\mathbf{P}}(Q)$ with $c_{\mathbf{y}}(P') = P$. We obtain the two commutative diagrams

$$\begin{array}{ccc} C_{P'}(Q) & \xrightarrow{c_{\mathbf{y}}} & C_P(Q) \\ \downarrow & & \downarrow \\ C_{\mathbf{P}}(\mathbf{Q}) & \xrightarrow[c_{\mathbf{y}}]{} & C_{\mathbf{P}}(\mathbf{Q}) \end{array} \qquad \begin{array}{ccc} N_{P'}(Q) & \xrightarrow{c_{\mathbf{y}}} & N_P(Q) \\ \downarrow & & \downarrow \\ N_{\mathbf{P}}(\mathbf{Q}) & \xrightarrow[c_{\mathbf{y}}]{} & N_{\mathbf{P}}(\mathbf{Q}). \end{array}$$

The left vertical arrows are p-discretizations by construction, therefore the right vertical arrows are p-discretizations as well. □

Corollary 3.9. *If P is a p-discretization of $\mathbf{P}$, then $Z(P)$ is a p-discretization of $Z(\mathbf{P})$.*

Proof. Apply Proposition 3.8 with $P = Q$. □

4. Chains of p-centric, p-stubborn subgroups of G

In [3], Broto, Levi and Oliver construct a saturated fusion system associated to a compact Lie group G, denoted $\mathcal{F}_S(G)$, where S is a maximal discrete p-toral subgroup of G. It is a category whose objects are the subgroups of S and whose morphisms are homomorphisms induced by conjugation in G. The purpose of this section is to compare the collection of $\mathcal{F}_S(G)$-centric, $\mathcal{F}_S(G)$-radical subgroups (Definition 4.1) with the analogous collection of continuous p-toral subgroups, namely the p-centric, p-stubborn subgroups. In our forthcoming normalizer decomposition for p-local compact groups [1], the indexing category will be conjugacy classes of chains of $\mathcal{F}_S(G)$-centric, $\mathcal{F}_S(G)$-radical subgroups of S. In this section, we show that when $\pi_0 G$ is a p-group, the set of such conjugacy classes is in one-to-one correspondence with conjugacy classes of chains of p-centric, p-stubborn subgroups of G (Theorem 4.3). Further, even when $\pi_0 G$ is not a p-group, there is still an injection from the first set to the second.

First we need some definitions, taken from the definitions for a fusion system (see [3, Def. 2.6 and pp 380] and [12, Def. 1.3]). For streamlined notation, we suppress both G and the maximal discrete p-toral subgroup S.

Definition 4.1. Fix a compact Lie group G and a maximal discrete p-toral subgroup $S \subseteq G$.

(1) For discrete p-toral groups
   (a) A subgroup $P \subseteq S$ is *$\mathcal{F}$-centric* if whenever $Q \subseteq S$ is G-conjugate to P, we have $C_S(Q) = Z(P)$. (In particular, $C_S(P) = Z(P)$.)
   (b) A subgroup $P \subseteq S$ is *$\mathcal{F}$-radical* if $\mathrm{Out}_G(P) := N_G(P)/[C_G(P) \cdot P]$ has no nontrivial normal subgroups.

(2) For continuous p-toral groups
   (a) A p-toral subgroup $\mathbf{P} \subseteq G$ is *p-centric in G* if $Z(\mathbf{P})$ is a maximal p-toral subgroup of $C_G(\mathbf{P})$.
   (b) A p-toral subgroup $\mathbf{P} \subseteq G$ is *p-stubborn in G* if $N_G(\mathbf{P})/\mathbf{P}$ is finite and has no nontrivial normal p-subgroups.

Note that although the concepts of $\mathcal{F}$-centric and $\mathcal{F}$-radical depend on both S and G, we omit them from the notation because S and G are always clear from context, and the omission gives a slimmer notation. We also observe that the properties of being p-centric and p-stubborn are closed under G-conjugation since $c_g(C_G(\mathbf{P})) = C_G(c_g(\mathbf{P}))$ and $c_g(N_G(\mathbf{P})) = N_G(c_g(\mathbf{P}))$ for any $g \in G$.

Remark 4.2. A p-toral subgroup $\mathbf{P} \subseteq G$ is *p-radical* if $N_G(\mathbf{P})/\mathbf{P}$ has no nontrivial normal p-toral subgroups (no assumption that $N_G(\mathbf{P})/\mathbf{P}$ is finite). For finite groups, the collection of p-radical subgroups was introduced by Bouc. In the compact Lie case, the p-radical subgroups were featured in [14].

However, when $\mathbf{P} \subseteq G$ is p-centric, then p-radical and p-stubborn become equivalent, because $\mathrm{Out}_G(\mathbf{P})$ is finite ([3, Lemma 9.4]) and the short exact sequence

$$0 \to C_G(\mathbf{P})/Z(\mathbf{P}) \to N_G(\mathbf{P})/\mathbf{P} \to \mathrm{Out}_G(\mathbf{P}) \to 0$$

has $C_G(\mathbf{P})/Z(\mathbf{P})$ finite of order prime to p.

With the necessary vocabulary in hand, we are able to state the main theorem for this section.

Theorem 4.3. *Let $\mathbf{S}$ be a maximal p-toral subgroup of a compact Lie group G, with p-discretization $S \subseteq \mathbf{S}$. The closure map $P \mapsto \overline{P}$ defines an injective map*

$$\begin{array}{c} \{P_0 \subseteq \ldots \subseteq P_k \subseteq S \,|\, \text{all } P_i \text{ are } \mathcal{F}\text{-centric and } \mathcal{F}\text{-radical}\}\,/G \\ \downarrow \\ \{\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k \subseteq \mathbf{S} \,|\, \text{all } \mathbf{P}_i \text{ are } p\text{-toral, } p\text{-centric, and } p\text{-stubborn}\}\,/G. \end{array}$$

The map is a one-to-one correspondence if $\pi_0 G$ is a p-group.

The first task is to show that the map of Theorem 4.3 actually exists. That is, we need to establish that the closure of a discrete p-toral subgroup of S that is $\mathcal{F}$-centric and $\mathcal{F}$-radical is p-centric and p-stubborn. (It is certainly p-toral.) To use the results of Section 3, we need to know that the discrete p-toral groups we are dealing with are snugly embedded. The following lemma can be found in [3, Corollary 3.5, Lemma 9.9], but we give an elementary proof here that does not use the bullet construction.

Lemma 4.4. *Let G be a compact Lie group with maximal discrete p-toral subgroup S, and let P be a subgroup of S. If P is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then P is snugly embedded.*

Proof. Let $\mathbf{P} = \overline{P}$, and expand P to a p-discretization Q' of $\mathbf{P}$. To prove that P is snugly embedded, we would like to prove that $Q' = P$. Expand Q' further to a p-discretization S' of $\mathbf{S}$, so that we have $P \subseteq Q' \subseteq S' \subseteq \mathbf{S}$. Using Proposition 3.1 with $P \subseteq S \cap S'$, choose $\mathbf{y} \in C_{\mathbf{S}}(P)$ such that $c_{\mathbf{y}}(S') = S$, and let $Q = c_{\mathbf{y}}(Q')$. Now we have $P \subseteq Q \subseteq S \subseteq \mathbf{S}$ and Q is a p-discretization of $\mathbf{P}$, and our goal has become to prove $Q = P$.

Consider the homomorphism

$$N_Q(P)/Z(P) \to N_G(P)/C_G(P). \tag{4.5}$$

Because P is $\mathcal{F}$-centric by assumption, $C_S(P) = Z(P)$. Therefore the centralizer of P in $Q \subseteq S$ is $Z(P)$ as well, and (4.5) is a monomorphism. We assert that the image is a normal subgroup of $N_G(P)/C_G(P)$. To prove this, we must take an element $g \in N_G(P)$ and prove that we can adjust g by an element of $x \in C_G(P)$ so that $g \cdot x$ normalizes $N_Q(P)$. Given that $g \cdot x$ would certainly normalize P, it is sufficient to construct x so that $g \cdot x$ normalizes Q.

Let $Q'' = c_g(Q)$. Because $g \in N_G(P) \subseteq N_G(\mathbf{P})$, the groups Q'' and Q are both p-discretizations of $\mathbf{P}$, and we have $P \subseteq Q \cap Q''$. Proposition 3.1 gives us an element $x \in C_{\mathbf{P}}(P)$ such that $c_x(Q'') = Q$. Therefore $g \cdot x$ normalizes both Q and P and we conclude that (4.5) is the inclusion of a normal subgroup. Taking the quotient on both sides by P, we find that $N_Q(P)/P$ is a normal subgroup of $\mathrm{Out}_G(P)$.

However, we have assumed that P is $\mathcal{F}$-radical, meaning that $\mathrm{Out}_G(P)$ has no nontrivial p-subgroups. Therefore $N_Q(P)/P$ must be the trivial group, that is, $N_Q(P) = P$. Because P and Q are discrete p-toral groups, $N_Q(P) = P$ implies that the inclusion of P into Q cannot be proper [3, Lemma 1.8], so $P = Q$. Hence P is a maximal discrete p-toral subgroup of $\mathbf{P}$, as required. □

Now that we know that $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of G must be snug, we can use the results of Section 3. Our next proposition shows that the map of Theorem 4.3 can be defined. That is, we show that the closure of an $\mathcal{F}$-centric and $\mathcal{F}$-radical discrete p-toral subgroup is in fact p-centric and p-stubborn. (See also the argument given in [3, Prop 8.4 and Lemma 9.6] for p-centricity.)

Proposition 4.6. *If P is a p-discretization of $\mathbf{P}$, and P is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then $\mathbf{P}$ is p-centric and p-stubborn.*

Proof. To show that $\mathbf{P}$ is p-centric, we must show that $Z(\mathbf{P})$ is a maximal p-toral subgroup of $C_G(\mathbf{P})$. To see this, suppose that $\mathbf{H} \subseteq C_G(\mathbf{P})$ is a maximal p-toral subgroup. Then $\mathbf{H}$ necessarily contains $Z(\mathbf{P})$, because all choices for $\mathbf{H}$ are conjugate in $C_G(\mathbf{P})$ and $Z(\mathbf{P}) \triangleleft C_G(\mathbf{P})$. We construct a p-discretization of $\mathbf{H}$ by expanding $Z(P) \subseteq \mathbf{H}$ to a p-discretization H of $\mathbf{H}$. Then we further expand the discrete p-toral subgroup $H \cdot P$ to a maximal discrete p-toral subgroup S' of G. (Note that S' does not have to have the same closure as S.)

All maximal discrete p-toral subgroups of G are conjugate, so there exists $g \in G$ with $c_g(S') = S$. Let $Q = c_g(P) \subseteq S$ and $J = c_g(H) \subseteq S$ so we have the following picture:

$$\begin{array}{ccccccccc} Z(P) & \longrightarrow & H & \longrightarrow & P \cdot H & \longrightarrow & S' & \longrightarrow & G \\ \downarrow{\scriptstyle c_g} & & \downarrow{\scriptstyle c_g} & & \downarrow{\scriptstyle c_g} & & \downarrow{\scriptstyle c_g} & & \downarrow{\scriptstyle c_g} \\ Z(Q) & \longrightarrow & J & \longrightarrow & Q \cdot J & \longrightarrow & S & \longrightarrow & G. \end{array}$$

By construction, $\mathbf{H} = C_{\mathbf{S}'}(\mathbf{P}) = C_{\mathbf{S}'}(P)$, and further, H is a p-discretization of $\mathbf{H}$ and $H \subseteq S'$. This means $H = C_{S'}(P)$, and so $J = C_S(Q)$.

But we know that $Q \subseteq S$ is G-conjugate to P, and by definition of $\mathcal{F}$-centric, that implies $Z(Q) = C_S(Q) = J$. Hence $Z(P) = H$ as well. Lastly, because P is $\mathcal{F}$-centric and $\mathcal{F}$-radical, P is snug by Lemma 4.4, and therefore $Z(P)$ is a p-discretization of $Z(\mathbf{P})$ by Corollary 3.9. Since H is a p-discretization of $\mathbf{H}$ by construction, and $H = Z(P)$, we find that $\mathbf{H} = Z(\mathbf{P})$, as required to prove that $\mathbf{P}$ is p-centric.

To show that $\mathbf{P}$ is p-stubborn, we must show that $N_G\mathbf{P}/\mathbf{P}$ is finite and contains no nontrivial normal p-subgroups. Consider the short exact sequence

$$1 \longrightarrow C_G(\mathbf{P})/Z(\mathbf{P}) \longrightarrow N_G(\mathbf{P})/\mathbf{P} \longrightarrow \underbrace{N_G(\mathbf{P})/\left[C_G(\mathbf{P}) \cdot \mathbf{P}\right]}_{\mathrm{Out}_G(\mathbf{P})} \longrightarrow 1. \tag{4.7}$$

The right-hand term is $\mathrm{Out}_G(\mathbf{P})$, which by Lemma 3.6 is isomorphic to $\mathrm{Out}_G(P)$. The definition of $\mathcal{F}$-radical tells us that $\mathrm{Out}_G(P)$ contains no nontrivial normal p-subgroups, and hence the same is true of $\mathrm{Out}_G(\mathbf{P})$.

Turning our attention to the left-hand term in (4.7), we have proved that $Z(\mathbf{P})$ is a maximal p-toral subgroup of $C_G(\mathbf{P})$. The quotient $C_G(\mathbf{P})/Z(\mathbf{P})$ is a compact Lie group, but cannot contain an S^1 by maximality of $Z(\mathbf{P})$. Hence $C_G(\mathbf{P})/Z(\mathbf{P})$ is finite. Again by maximality of $Z(\mathbf{P})$, we know $C_G(\mathbf{P})/Z(\mathbf{P})$ has no p-torsion elements, so it has order prime to p. As a consequence, the image of a nontrivial normal p-subgroup of $N_G(\mathbf{P})/\mathbf{P}$ would be a nontrivial p-subgroup of $\mathrm{Out}_G(\mathbf{P}) = \mathrm{Out}_G(P)$, a contradiction of the assumption that P is $\mathcal{F}$-radical.

It remains to establish that $N_G(\mathbf{P})/\mathbf{P}$ is finite, for which is it sufficient to know that the right-hand term, $\mathrm{Out}_G(\mathbf{P}) \cong \mathrm{Out}_G(P)$, is finite. If $\mathrm{Out}_G(P)$ is not finite, then it has a nontrivial torus, and therefore

an infinite torsion subgroup. However, any torsion subgroup of $\mathrm{Out}_G(P)$ is finite [3, Prop. 1.5]. Therefore $\mathrm{Out}_G(\mathbf{P})$ is finite, and hence $N_G(\mathbf{P})/\mathbf{P}$ is finite with no nontrivial normal p-subgroups, meaning that $\mathbf{P}$ is p-stubborn, as required. □

So far, our progress toward proving Theorem 4.3 is to establish that the statement makes sense: the function actually exists! Next we establish that the function is injective.

Lemma 4.8. *Let $\mathbf{S} \subseteq G$ be a maximal p-toral subgroup of the Lie group G, and fix a p-discretization S of $\mathbf{S}$. Suppose that $P_0 \subseteq \ldots \subseteq P_k$ and $Q_0 \subseteq \ldots \subseteq Q_k$ are two chains of snug discrete p-toral subgroups of S that both have $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k$ as their closure. Then $P_0 \subseteq \ldots \subseteq P_k$ and $Q_0 \subseteq \ldots \subseteq Q_k$ are conjugate in $\mathbf{P}_k$ (and hence necessarily in G).*

Proof. We induct on k. The base case $k = 0$ is true because P_0 and Q_0 are both p-discretizations of $\mathbf{P}_0$, and hence are conjugate in $\mathbf{P}_0$. Now suppose that $P_0 \subseteq \ldots \subseteq P_{k-1}$ and $Q_0 \subseteq \ldots \subseteq Q_{k-1}$ are conjugate by $\mathbf{x} \in \mathbf{P}_{k-1}$. Then $c_{\mathbf{x}}(P_k)$ and Q_k are p-discretizations of $\mathbf{P}_k$, and $Q_{k-1} \subseteq c_{\mathbf{x}}(P_k) \cap Q_k$. By Proposition 3.1, there exists $\mathbf{y} \in C_{\mathbf{P}_k}(Q_{k-1})$ such that $c_{\mathbf{y}}\left(c_{\mathbf{x}}(P_k)\right) = Q_k$. Hence $\mathbf{x} \cdot \mathbf{y} \in \mathbf{P}_k$ and conjugates $P_0 \subseteq \ldots \subseteq P_k$ to $Q_0 \subseteq \ldots \subseteq Q_k$. □

To address the extent to which the function in Theorem 4.3 is surjective, we first need to check whether it is always possible to obtain a discretization of a given chain in $\mathbf{S}$ within the chosen p-discretization S.

Lemma 4.9. *Let $\mathbf{S} \subseteq G$ be a maximal p-toral subgroup of the Lie group G, and let $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k$ be a chain of p-toral subgroups of $\mathbf{S}$. Fix a p-discretization $S \subseteq \mathbf{S}$. Then there exists an $\mathbf{S}$-conjugate of $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k$ that has a chain of p-discretizations $P_0 \subseteq \ldots \subseteq P_k$ contained in S.*

Proof. Choose a p-discretization P_0 of $\mathbf{P}_0$, and expand it one group at a time to a chain of p-discretizations $P_0 \subseteq \ldots \subseteq P_k \subseteq S'$ of $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k \subset \mathbf{S}$. Since S and S' are both p-discretizations of $\mathbf{S}$, there exists $\mathbf{s} \in \mathbf{S}$ such that $c_{\mathbf{s}}(S') = S$. Then $c_{\mathbf{s}}\left(P_0 \subseteq \ldots \subseteq P_k\right)$ is a p-discretization inside S of $c_{\mathbf{s}}\left(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k\right)$. □

Proof of Theorem 4.3. Most of the proof is in the preceding results. We have proved that the function exists (Proposition 4.6) and that it is injective (Lemma 4.8). Further, Lemma 4.9 lays the groundwork for an epimorphism statement, since a chain in the target has a simultaneous p-discretization in the chosen S. To finish the proof of the epimorphism statement, we must show that

(1) if $\mathbf{P}$ is p-stubborn and p-centric, with p-discretization $P \subseteq S$, then P is $\mathcal{F}$-centric, and
(2) if in addition $\pi_0 G$ is a p-group, then P is also $\mathcal{F}$-radical.

Suppose that $\mathbf{P}$ is p-stubborn and p-centric. Then $C_G(\mathbf{P})/Z(\mathbf{P}) \subseteq N_G(\mathbf{P})/\mathbf{P}$ is finite, and has order prime to p because $\mathbf{P}$ is p-centric. Mapping from $\mathbf{S}$ to G gives a monomorphism $C_{\mathbf{S}}(\mathbf{P})/Z(\mathbf{P}) \hookrightarrow C_G(\mathbf{P})/Z(\mathbf{P})$ from a p-toral group ([12, A.3]) to a finite group of order prime to p, and therefore the map is null. We conclude that $Z(\mathbf{P}) = C_{\mathbf{S}}(\mathbf{P})$. But by Proposition 3.8 and Corollary 3.9, we know that the groups $Z(P) \subseteq C_S(P)$ are p-discretizations of $Z(\mathbf{P}) = C_{\mathbf{S}}(\mathbf{P})$, respectively. Maximality implies that $Z(P) = C_S(P)$, that is, the group P satisfies the required condition to be $\mathcal{F}$-centric.

We must also check that if $Q \subseteq S$ is G-conjugate to P, then $C_S(Q) = Z(Q)$. The subgroup Q is snug, because P is. Let $\mathbf{Q}$ be the closure of Q, and observe that $\mathbf{Q}$ is G-conjugate to $\mathbf{P}$, by the same element that takes Q to P. Further, $\mathbf{Q}$ is p-stubborn and p-centric, because $\mathbf{P}$ is, and those properties are preserved by conjugation in G. Since $Q \subseteq \mathbf{Q}$ is a p-discretization, the argument of the previous paragraph shows that Q is $\mathcal{F}$-centric as well.

We must still show that P is $\mathcal{F}$-radical when we know that $\pi_0 G$ is a p-group, that is, we must show that $\mathrm{Out}_G(P)$ has no nontrivial normal p-subgroups. By Lemma 3.6, $\mathrm{Out}_G(P) \cong \mathrm{Out}_G(\mathbf{P})$, so we can use the short exact sequence of (4.7). The key ingredient is that if $\pi_0 G$ is a p-group, then $C_G(\mathbf{P})$ is p-toral [12, A.5], and hence $C_G(\mathbf{P})/\mathbf{P} \subseteq N_G(\mathbf{P})/\mathbf{P}$ is also p-toral. However, $\mathbf{P}$ is p-stubborn by assumption, so $N_G(\mathbf{P})/\mathbf{P}$ is finite, and $C_G(\mathbf{P})/\mathbf{P}$ is therefore a finite p-group. If $\mathrm{Out}_G(\mathbf{P})$ had a nontrivial normal p-subgroup, then its inverse image in $N_G(\mathbf{P})/\mathbf{P}$ would be a normal p-subgroup, in contradiction of the assumption that $\mathbf{P}$ is p-stubborn. □

5. Normalizers

In Section 3 we studied relative discretizations and proved that centralizers and normalizers of p-toral groups inside other p-toral groups are compatible with p-discretizations (Proposition 3.8). In this section, we leverage the results of Section 3 to prove that if $P_0 \subseteq \ldots \subseteq P_k$ is a chain of p-discretizations of $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k$, then the corresponding map of G-normalizers induces a mod p homology isomorphism on classifying spaces.

Theorem 5.1. *Let $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k$ be a chain of p-toral subgroups of a compact Lie group G, and let $P_0 \subseteq \ldots \subseteq P_k$ be a chain of discrete p-toral subgroups such that each P_i is a p-discretization of $\mathbf{P}_i$. Then*

$$N_G(P_0 \subseteq \ldots \subseteq P_k) \longrightarrow N_G(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)$$

induces a mod p equivalence of classifying spaces.

Our forthcoming work on the normalizer decomposition of a p-local compact group will use Theorem 5.1 to establish that, when applied to a compact Lie group, our decomposition recovers a version of the theorem of Libman [14] using p-toral subgroups that are both p-centric and p-stubborn.

Our strategy to prove Theorem 5.1 is to study the "outer automorphism" group of a chain separately from the "inner automorphisms" of the chain.

Definition 5.2. Let H be a group, and let $P_0 \subseteq \ldots \subseteq P_k$ be a chain of subgroups of H. We define $\mathrm{Out}_H(P_0 \subseteq \ldots \subseteq P_k)$ as the quotient

$$\frac{N_H(P_0 \subseteq \ldots \subseteq P_k)}{C_H(P_k) \cdot N_{P_k}(P_0 \subseteq \ldots \subseteq P_k)}.$$

Note that the definition makes no restriction on the subgroups in the chain. In particular, in the next proposition, we establish the relationship between the outer automorphism group of a chain of continuous p-toral subgroups and that of a p-discretization of the chain.

Proposition 5.3. *Let $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k$ be a chain of p-toral subgroups of G, and let $P_0 \subseteq \ldots \subseteq P_k$ be a chain of p-discretizations of the p-toral chain. Then inclusion of normalizers induces an isomorphism*

$$\underbrace{\mathrm{Out}_G(P_0 \subseteq \ldots \subseteq P_k)}_{\frac{N_G(P_0 \subseteq \ldots \subseteq P_k)}{C_G(P_k) \cdot N_{P_k}(P_0 \subseteq \ldots \subseteq P_k)}} \longrightarrow \underbrace{\mathrm{Out}_G(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)}_{\frac{N_G(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)}{C_G(\mathbf{P}_k) \cdot N_{\mathbf{P}_k}(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)}}. \tag{5.4}$$

Proof. To show that the map is an epimorphism, we induct on k. The case $k = 0$ is Lemma 3.6. For the inductive hypothesis, assume that we have $g \in N_G(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)$, and that g stabilizes $P_0 \subseteq \ldots \subseteq P_{k-1}$. We need to adjust g to stabilize P_k as well.

Suppose that $c_g(P_k) = P'_k$. Then $P_{k-1} \subseteq P_k \cap P'_k$, and by Proposition 3.1 we can find $\mathbf{y} \in C_{\mathbf{P}_k}(P_{k-1})$ that conjugates P'_k to P_k. Now we have an element $g \cdot \mathbf{y}$ that stabilizes $P_0 \subseteq \ldots \subseteq P_k$. Since $\mathbf{y}$ is in $N_{\mathbf{P}_k}(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)$, we conclude that we have an epimorphism

$$N_G(P_0 \subseteq \ldots \subseteq P_k) \xrightarrow{\text{epi}} \frac{N_G(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)}{N_{\mathbf{P}_k}(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)},$$

and therefore (5.4) is also surjective.

To establish injectivity, consider $n \in N_G(P_0 \subseteq \ldots \subseteq P_k)$ such that $[n]$ is in the kernel of (5.4). We can write $n = c \cdot \mathbf{x}$ where $c \in C_G(\mathbf{P}_k) = C_G(P_k)$ and $\mathbf{x} \in N_{\mathbf{P}_k}(\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k)$. Then in fact $\mathbf{x}$ stabilizes $P_0 \subseteq \ldots \subseteq P_k$, since both n and c do so. By Lemma 3.5, the automorphism of P_k induced by $\mathbf{x}$ can be induced by some element $y \in P_k$, and then $\mathbf{x} \cdot y^{-1} \in C_{\mathbf{P}_k}(P_k)$. Further, since $\mathbf{x}$ and $\mathbf{x} \cdot y^{-1}$ both stabilize $P_0 \subseteq \ldots \subseteq P_k$, so does y. We have expressed $n = c \cdot \mathbf{x} = (c \cdot \mathbf{x} \cdot y^{-1}) \cdot y$ as an element in the denominator of the left side of (5.4), which completes the proof. □

Corollary 5.5. *Let P be a p-discretization of a p-toral group $\mathbf{P}$, and let $Q_0 \subseteq \ldots \subseteq Q_k$ be a chain of snug subgroups of P. Then inclusions of normalizers induce isomorphisms*

$$\begin{array}{ccc} \mathrm{Out}_P(Q_0 \subseteq \ldots \subseteq Q_k) & \longrightarrow & \mathrm{Out}_{\mathbf{P}}(\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_k) \\ & \searrow \quad \nearrow_{\cong \text{ Prop. 5.3}} & \\ & \mathrm{Out}_{\mathbf{P}}(Q_0 \subseteq \ldots \subseteq Q_k). & \end{array}$$

Proof. The left diagonal map is clearly a monomorphism. To see that it is also an epimorphism, note that any automorphism of Q_k induced by conjugation in $\mathbf{P}$ can also be induced by conjugation in P (Lemma 3.5). □

Outer automorphism groups of chains turn out to be finite, generalizing the result of [3, Prop. 9.4(b)] for a single group.

Lemma 5.6. *Let G be a compact Lie group and let $P_0 \subseteq \ldots \subseteq P_k$ be a chain of snug discrete p-toral subgroups. Then $\mathrm{Out}_G(P_0 \subseteq \ldots \subseteq P_k)$ is finite.*

Proof. We induct on k. The base case is given by [3, Prop. 9.4(b)]. Now suppose that $\mathrm{Out}_G(P_0 \subseteq \ldots \subseteq P_{k-1})$ is finite, and consider the homomorphism induced by deleting the smallest element of the chain:

$$\frac{N_G(P_0 \subseteq \ldots \subseteq P_k)}{C_G(P_k) \cdot N_{P_k}(P_0 \subseteq \ldots \subseteq P_k)} \longrightarrow \frac{N_G(P_1 \subseteq \ldots \subseteq P_k)}{C_G(P_k) \cdot N_{P_k}(P_1 \subseteq \ldots \subseteq P_k)}. \tag{5.7}$$

The target is finite by the inductive hypothesis, so we need only show that the map is injective. The homomorphism of the numerators is certainly injective. Now suppose that

$$n \in N_G(P_0 \subseteq \ldots \subseteq P_k) \ \cap \ [C_G(P_k) \cdot N_{P_k}(P_1 \subseteq \ldots \subseteq P_k)].$$

Then $n = c \cdot x$ for some $c \in C_G(P_k)$ and $x \in N_{P_k}(P_1 \subseteq \ldots \subseteq P_k)$. The element x also normalizes P_0, since n and c do. Hence $n = c \cdot x \in C_G(P_k) \cdot N_{P_k}(P_0 \subseteq \ldots \subseteq P_k)$, as desired. □

We turn now to the comparison of the normalizers and centralizers of continuous and discrete p-toral subgroups of G. The starting point is Proposition 3.8, which tells us that if $Q \subseteq P$ are p-discretizations for p-toral groups $\mathbf{Q} \subseteq \mathbf{P}$, then $C_P(Q) \to C_{\mathbf{Q}}(\mathbf{P})$ and $N_P(Q) \to N_{\mathbf{P}}(\mathbf{Q})$ are p-discretizations as well (and therefore induce mod p homology equivalences on classifying spaces by Lemma 2.4).

To prove Theorem 5.1, we need to relate normalizers of p-toral subgroups of an ambient Lie group G (which could itself be p-toral) with normalizers of their p-discretizations. We begin with a special case.

Proposition 5.8. *Let P be a p-discretization of a p-toral group $\mathbf{P}$, and let $Q_0 \subseteq \ldots \subseteq Q_k$ be a chain of snugly embedded discrete p-toral subgroups of P, with closures $\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_k$. Then the inclusion*

$$N_P\left(Q_0 \subseteq \ldots \subseteq Q_k\right) \longrightarrow N_{\mathbf{P}}\left(\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_k\right)$$

induces a mod p homology isomorphism on classifying spaces.

Proof. We induct on the length of the chain. The case $k = 0$ is provided by Proposition 3.8.

For the inductive hypothesis, assume that for any p-toral group $\mathbf{B}$ and p-discretization $B \subseteq \mathbf{B}$, the inclusion

$$N_B\left(Q_0 \subseteq \ldots \subseteq Q_{k-1}\right) \longrightarrow N_{\mathbf{B}}\left(\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_{k-1}\right) \tag{5.9}$$

induces a mod p homology isomorphism of classifying spaces. Consider the relationship of the desired statement for k to the corresponding outer automorphism groups:

$$\begin{array}{ccc} N_P\left(Q_0 \subseteq \ldots \subseteq Q_k\right) & \twoheadrightarrow & \mathrm{Out}_P\left(Q_0 \subseteq \ldots \subseteq Q_k\right) \\ \downarrow & & \downarrow \cong \text{ by Corollary 5.5} \\ N_{\mathbf{P}}\left(\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_k\right) & \twoheadrightarrow & \mathrm{Out}_{\mathbf{P}}\left(\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_k\right). \end{array} \tag{5.10}$$

The horizontal maps are epimorphisms, and the map of kernels is given by

$$C_P(Q_k) \cdot N_{Q_k}\left(Q_0 \subseteq \ldots \subseteq Q_k\right) \longrightarrow C_{\mathbf{P}}(\mathbf{Q}_k) \cdot N_{\mathbf{Q}_k}\left(\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_k\right). \tag{5.11}$$

To streamline notation, let $Q_* := (Q_0 \subseteq \ldots \subseteq Q_k)$ and $\mathbf{Q}_* := (\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_k)$. Then (5.10) induces a commutative ladder of classifying spaces, where the rows are fibrations:

$$\begin{array}{ccccc} B\left(C_P(Q_k) \cdot N_{Q_k}\left(Q_*\right)\right) & \longrightarrow & B\left(N_P\left(Q_*\right)\right) & \longrightarrow & B\,\mathrm{Out}_P\left(Q_*\right) \\ \downarrow & & \downarrow & & \downarrow \cong \\ B\left(C_{\mathbf{P}}(\mathbf{Q}_k) \cdot N_{\mathbf{Q}_k}\left(\mathbf{Q}_*\right)\right) & \longrightarrow & B\left(N_{\mathbf{P}}\left(\mathbf{Q}_*\right)\right) & \longrightarrow & B\,\mathrm{Out}_{\mathbf{P}}\left(\mathbf{Q}_*\right). \end{array} \tag{5.12}$$

Since the base spaces are the same, it is sufficient to prove that the map between fibers is a mod p homology isomorphism; a Serre spectral sequence argument then establishes that the middle map is also an isomorphism on mod p homology.

To understand the map in (5.11), we need to understand the map on each factor, and also on their intersection. The groups $C_P(Q_k)$ and $N_{Q_k}(Q_*)$ are commuting subgroups of P, and their intersection is $Z(Q_k)$. We have a central extension

$$0 \longrightarrow Z(Q_k) \longrightarrow C_P(Q_k) \times N_{Q_k}\left(Q_*\right) \longrightarrow C_P(Q_k) \cdot N_{Q_k}\left(Q_*\right) \longrightarrow 0$$

and the analogous one involving $\mathbf{Q}_*$ and $\mathbf{P}$. The fibrations induced by these short exact sequences are principal, and we have the following commutative diagram of horizontal fibrations:

$$\begin{array}{ccccc} BC_P(Q_k) \times BN_{Q_k}(Q_*) & \longrightarrow & B\left(C_P(Q_k) \cdot N_{Q_k}(Q_*)\right) & \longrightarrow & B^2Z(Q_k) \\ \downarrow & & \downarrow & & \downarrow \\ BC_{\mathbf{P}}(\mathbf{Q}_k) \times BN_{\mathbf{Q}_k}(\mathbf{Q}_*) & \longrightarrow & B\left(C_{\mathbf{P}}(\mathbf{Q}_k) \cdot N_{\mathbf{Q}_k}(\mathbf{Q}_*)\right) & \longrightarrow & B^2Z(\mathbf{Q}_k). \end{array} \tag{5.13}$$

First consider the fibers. The map on the first factor is a mod p homology isomorphism by Proposition 3.8. For the second factor, we can apply the inductive hypothesis, because $N_{Q_k}(Q_*)$ is actually $N_{Q_k}(Q_0 \subseteq \ldots \subseteq Q_{k-1})$ (the normalizer of a shorter chain); likewise $N_{\mathbf{Q}_k}(\mathbf{Q}_*)$ is $N_{\mathbf{Q}_k}(\mathbf{Q}_0 \subseteq \ldots \subseteq \mathbf{Q}_{k-1})$. Therefore $BN_{Q_k}(Q_*) \to BN_{\mathbf{Q}_k}(\mathbf{Q}_*)$ is a mod p homology isomorphism.

Turning to the base, we know that $BZ(Q_k) \to BZ(\mathbf{Q}_k)$ induces a mod p homology isomorphism by Corollary 3.9. The Rothenberg-Steenrod spectral sequence [16, Corollary 7.29] then shows that $B^2Z(Q_k) \to B^2Z(\mathbf{Q}_k)$ likewise induces an isomorphism on mod p homology.

We apply the Serre spectral sequence to (5.13). The base spaces are simply connected. The maps between the bases and the fibers are mod p homology isomorphisms. Hence the map of total spaces is a mod p homology isomorphism as well. Feeding this result back into (5.12) finishes the proof. □

Finally, we arrive at the proof of this section's main result.

Proof of Theorem 5.1. Let P_* denote the chain $P_0 \subseteq \ldots \subseteq P_k$, and similarly let $\mathbf{P}_*$ denote the chain $\mathbf{P}_0 \subseteq \ldots \subseteq \mathbf{P}_k$. We compare the normalizers via the ladder of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_G(P_k) \cdot N_{P_k}(P_*) & \longrightarrow & N_G(P_*) & \longrightarrow & \mathrm{Out}_G(P_*) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \text{ by Proposition 5.3} & & \\ 0 & \longrightarrow & C_G(\mathbf{P}_k) \cdot N_{\mathbf{P}_k}(\mathbf{P}_*) & \longrightarrow & N_G(\mathbf{P}_*) & \longrightarrow & \mathrm{Out}_G(\mathbf{P}_*) & \longrightarrow & 0. \end{array} \tag{5.14}$$

We are in the exact same situation as in the proof of Proposition 5.8. The inclusion $N_{P_k}(P_*) \hookrightarrow N_{\mathbf{P}_k}(\mathbf{P}_*)$ induces an isomorphism on mod p homology of classifying spaces by Proposition 5.8. The centralizers $C_G(P_k)$ and $C_G(\mathbf{P}_k)$ are equal. And lastly, $C_G(P_k) \cap N_{P_k}(P_*) = Z(P_k)$ and $C_G(\mathbf{P}_k) \cap N_{\mathbf{P}_k}(\mathbf{P}_*) = Z(\mathbf{P}_k)$, and $Z(P_k) \to Z(\mathbf{P}_k)$ induces an isomorphism on mod p homology of classifying spaces by Corollary 3.9. □

References

[1] E. Belmont, N. Castellana, J. Grbić, K. Lesh, M. Strumila, Normalizer decompositions of p-local compact groups, in preparation.

[2] Carles Broto, Natàlia Castellana, Jesper Grodal, Ran Levi, Bob Oliver, Subgroup families controlling p-local finite groups, Proc. Lond. Math. Soc. (3) 91 (2) (2005) 325–354, MR 2167090.

[3] Carles Broto, Ran Levi, Bob Oliver, Discrete models for the p-local homotopy theory of compact Lie groups and p-compact groups, Geom. Topol. 11 (2007) 315–427, MR 2302494.

[4] Carles Broto, Ran Levi, Bob Oliver, An algebraic model for finite loop spaces, Algebraic Geom. Topol. 14 (5) (2014) 2915–2981, MR 3276851.

[5] Andrew Chermak, Fusion systems and localities, Acta Math. 211 (1) (2013) 47–139, MR 3118305.

[6] Natàlia Castellana, Ran Levi, Dietrich Notbohm, Homology decompositions for p-compact groups, Adv. Math. 216 (2) (2007) 491–534, MR 2351369.

[7] W.G. Dwyer, C.W. Wilkerson, Homotopy fixed-point methods for Lie groups and finite loop spaces, Ann. Math. (2) 139 (2) (1994) 395–442, MR 1274096.

[8] W.G. Dwyer, Homology decompositions for classifying spaces of finite groups, Topology 36 (4) (1997) 783–804, MR 1432421.

[9] Alex González, Toni Lozano, Albert Ruiz, Some new examples of simple p-local compact groups, Publ. Mat. 63 (2) (2019) 445–489, MR 3980932.

[10] Jesper Grodal, Higher limits via subgroup complexes, Ann. Math. (2) 155 (2) (2002) 405–457, MR 1906592.

[11] Stefan Jackowski, James McClure, Homotopy decomposition of classifying spaces via elementary abelian subgroups, Topology 31 (1) (1992) 113–132, MR 1153240.

[12] Stefan Jackowski, James McClure, Bob Oliver, Homotopy classification of self-maps of BG via G-actions I, II, Ann. Math. (2) 135 (1) (1992) 183–270, MR 1147962 (93e:55019a).
[13] Assaf Libman, The normaliser decomposition for p-local finite groups, Algebraic Geom. Topol. 6 (2006) 1267–1288, MR 2253446.
[14] Assaf Libman, Orbit spaces, Quillen's theorem A and Minami's formula for compact Lie groups, Fundam. Math. 213 (2) (2011) 115–167, MR 2800583 (2012f:55010).
[15] Ran Levi, Assaf Libman, Existence and uniqueness of classifying spaces for fusion systems over discrete p-toral groups, J. Lond. Math. Soc. (2) 91 (1) (2015) 47–70, MR 3338608.
[16] John McCleary, A User's Guide to Spectral Sequences, second ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001, MR 1793722.
[17] Bob Oliver, p-stubborn subgroups of classical compact Lie groups, J. Pure Appl. Algebra 92 (1) (1994) 55–78, MR 1259669 (94k:57055).
[18] Bob Oliver, Existence and uniqueness of linking systems: Chermak's proof via obstruction theory, Acta Math. 211 (1) (2013) 141–175, MR 3118306.
[19] Jolanta Słomińska, Homotopy decompositions of orbit spaces and the Webb conjecture, Fundam. Math. 169 (2) (2001) 105–137, MR 1852376.
[20] Alexei Strounine, Homology decompositions for classifying spaces of compact Lie groups, Trans. Am. Math. Soc. 352 (6) (2000) 2643–2657, MR 1637102.