Topology and its Applications 316 (2022) 108107

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

An infinity operad of normalized cacti

Luciana Basualdo Bonatto [a], Safia Chettih [b], Abigail Linton [c], Sophie Raynor [d], Marcy Robertson [e,*], Nathalie Wahl [f]

[a] *Mathematical Institute, University of Oxford, Oxford, UK*
[b] *Department of Mathematics, Southwestern University, Georgetown, TX, USA*
[c] *School of Mathematical Sciences, University of Southampton, UK*
[d] *Department of Mathematics and Statistics, Macquarie University, NSW, Australia*
[e] *School of Mathematics and Statistics, The University of Melbourne, Melbourne, Victoria, Australia*
[f] *Department of Mathematical Sciences, University of Copenhagen, Copenhagen, Denmark*

ARTICLE INFO

Article history:
Received 3 July 2020
Received in revised form 27 October 2020
Available online 9 April 2022

MSC:
18
55P
55P48

Keywords:
Operads
Infinity operads
Cactus operad

ABSTRACT

We endow the normalized cacti with the structure of an ∞-operad by showing that its existing composition laws are associative up to all higher homotopies. The higher homotopies are encoded by a new topological operad of bracketed trees which we relate both to an enrichment of the dendroidal category Ω and to the Boardman-Vogt W-construction on the operad of operads.

Contents

* Corresponding author.
E-mail addresses: luciana.bonatto@maths.ox.ac.uk (L. Basualdo Bonatto), chettihs@southwestern.edu (S. Chettih), a.linton@soton.ac.uk (A. Linton), sophie.raynor@mq.edu.au (S. Raynor), marcy.robertson@unimelb.edu.au (M. Robertson), wahl@math.ku.dk (N. Wahl).

https://doi.org/10.1016/j.topol.2022.108107
0166-8641/

1. Introduction

Gluing surfaces along their boundary allows us to define composition laws that have been used to define cobordism categories, as well as operads and props associated to surfaces. These have played an important role in recent years, for example in constructing topological field theory or computing the homology of the moduli space of Riemann surfaces. Of particular interest is the cobordism category whose morphism spaces are moduli spaces of Riemann surfaces. It has long been known that such moduli spaces admit a graph model: they have the homotopy type of spaces of metric fat graphs [7,21,34]. The composition of moduli spaces induced by the gluing of surfaces was modeled using graphs in [17, Construction 3.29]. Though the resulting composition is associative on the associated cellular chain complex, it is not associative on the space level, and, at present, it is not known how to make it associative, or even coherently homotopy associative [17, Remark 3.31]. In genus 0, this graph model of the cobordism category includes normalized cacti (e.g. [41, Remark 2.8]), whose composition was also known not to be associative [24, Remark 2.3.19]. The goal of our paper is to show that the composition of normalized cacti is associative up to all higher homotopies, and hence normalized cacti form an ∞-operad in the way detailed below. We expect that the technique presented here can be extended to likewise show that the composition in the graph model of the cobordism category is also associative up to all higher homotopies.

A cactus is a treelike configuration of circles (Fig. 1). The cactus operad, originally introduced by Voronov [39, Section 2.7], and its spineless version, introduced by Kaufmann [24, Section 2.3], are models for the framed and unframed little disc operads respectively [24, Section 3.2.1]. Operadic composition is by insertion: identifying the outside contour of one cactus with the lobe of another cactus and scaling the inserted cactus appropriately. Here we work with the spineless version for simplicity.

A cactus is *normalized* if each circle in the cactus has circumference of length one. The space of all normalized cacti with k-lobes is denoted by $\mathcal{C}\mathrm{act}^1(k)$ and these spaces assemble into the symmetric sequence $\mathcal{C}\mathrm{act}^1 = \{\mathcal{C}\mathrm{act}^1(k)\}_{k\geq 0}$, with each $\mathcal{C}\mathrm{act}^1(k) \subset \mathcal{C}\mathrm{act}(k)$ a homotopy equivalent subspace, for $\mathcal{C}\mathrm{act}(k)$ the space of all cacti with k lobes. (See [24, Section 2.3].) Composition of normalized cacti is defined by insertion as for the cactus operad, but instead of scaling the inserted cactus to the size of the lobe it is inserted in, one scales the lobe to the size of the inserted cactus. Surprisingly, as illustrated in Fig. 16, this new composition is not associative ([24, Remark 2.3.19]). So, normalized cacti do *not* form an operad. This non-associative composition is, however, the one relevant to the graph model of the cobordism category, as we explain in Remark 5.1.

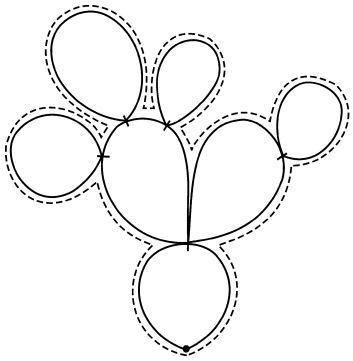


Fig. 1. Spineless cactus with 7 lobes, with its outside the dotted line.

Operads can be described as algebras over the operad of operads $\mathcal{O}$, an operad whose elements can be represented by certain trees (Definition 2.8). In Section 3, we define a bracketing of a tree and use it to construct a new topological operad $B\mathcal{O}$ (Definition 3.9) whose algebras are homotopy associative versions of operads: Any $B\mathcal{O}$–algebra has an underlying symmetric sequence and a preferred composition, but the composition is only associative up to coherent homotopy. The operad $B\mathcal{O}$ is the realization of an operad whose operations lie in the poset of bracketings of the trees in $\mathcal{O}$.

Given a composition on a symmetric sequence, an action of the operad $B\mathcal{O}$ gives a hands-on way to keep track of the homotopies required to show that it is coherently homotopy associative. We illustrate how to construct a $B\mathcal{O}$–algebra in practice by showing:

Theorem A *(Theorem 5.12). The symmetric sequence* $\{\mathcal{C}\mathrm{act}^1(k)\}_{k\geq 0}$ *of normalized cacti, together with the* $\mathcal{C}\mathrm{act}^1$ *composition described above, extends to a* $B\mathcal{O}$*–algebra structure.*

In this paper, we show that this hands-on notion of an operad up to homotopy is related to more well-known notions of ∞-operads. There exist several models of ∞-operads in the literature: lax operads, obtained by resolving the operad of operads [8], dendroidal sets or spaces satisfying an inner Kan or Segal condition [11, Proposition 6.3, Theorem 8.15] and [11, Definition 8.1], Lurie's ∞-operads [22, Section 2.5], [10, Corollary 1.2] and Barwick's complete Segal operads [10, Theorem 1.1]. These models are all Quillen equivalent, and also Quillen equivalent to topological operads themselves [12, Theorem 8.15]. We compare our construction with two of these notions of ∞-operads. In Section 4, we show that any $B\mathcal{O}$–algebra defines a dendroidal space satisfying a weak Segal condition, and in Section 6, we show that any $B\mathcal{O}$-algebra is also a lax operad in the sense of [8].

To associate a dendroidal space to a $B\mathcal{O}$-algebra, we first construct a topological enrichment $\widetilde{\Omega}_0$ of the dendroidal category Ω. The objects of $\widetilde{\Omega}_0$ are trees, as for Ω, but morphisms are the realization of certain posets of bracketings in trees, defined in a similar fashion to the operad $B\mathcal{O}$. Diagrams over this thickened dendroidal category $\widetilde{\Omega}_0$ are types of homotopy coherent dendroidal spaces. By defining a nerve functor that takes a $B\mathcal{O}$-algebra to the category of reduced $\widetilde{\Omega}_0$-diagrams that satisfy a strict Segal condition, we prove the following:

Theorem B *(Theorem 4.8). There is an isomorphism of categories between* $B\mathcal{O}$*-algebras and the category of reduced* $\widetilde{\Omega}_0$*-diagrams that satisfy a strict Segal condition.*

In Proposition 4.10, we show that a $\widetilde{\Omega}_0$–diagram can be rectified to an equivalent Ω–diagram, i.e. an actual dendroidal space. Moreover, we show that this rectified dendroidal space satisfies a weak version of the Segal condition if, and only if, the original $\widetilde{\Omega}_0$–diagram satisfies such a condition. We describe this rectification explicitly in the case of normalized cacti in Corollary 5.13.

The other notion of ∞-operads we consider are a more classical notion of an operad up to homotopy obtained by resolving the operad of operads $\mathcal{O}$ via the Boardman-Vogt W-construction. A $W\mathcal{O}$–algebra is an operad up to homotopy, where the symmetric group action, the unit and associativity relation are all assumed to hold only up to coherent homotopy. For this reason, $W\mathcal{O}$-algebras are called lax operads in the Ph.D. thesis [8]. Algebras over the operad $W\mathcal{O}$ are a model for ∞-operads: there exists a zig-zag of Quillen equivalences between the category of $W\mathcal{O}$-algebras and reduced dendroidal Segal spaces. (For example this can be seen by combining Theorem 4.1 of [1] with either Theorem 1.1 of [3] or Theorem 8.15 of [12].) However, the operad $W\mathcal{O}$ is not easy to work with directly. Indeed, its elements are trees (from the W–construction) whose vertices are themselves decorated by trees (from the operad $\mathcal{O}$), where the first trees compose by grafting and the second trees compose by vertex substitution. In Section 6, we show that the operad $B\mathcal{O}$ is actually isomorphic to a quotient $W_0\mathcal{O}$ of $W\mathcal{O}$:

Theorem C. *[Theorem 6.4] There exists an isomorphism of topological operads $W_0\mathcal{O} \cong B\mathcal{O}$, for $W_0\mathcal{O}$ an explicit quotient of the operad $W\mathcal{O}$. In particular, any $B\mathcal{O}$–algebra is also a $W\mathcal{O}$-algebra.*

A $W_0\mathcal{O}$–algebra (or equivalently $B\mathcal{O}$–algebra) is a homotopy operad where the composition is still only homotopy associative, but where the symmetric group action and unit are strict. We show in Corollary 6.5 that every $B\mathcal{O}$-algebra is homotopy equivalent to a strict operad.

The relationship between the operad $B\mathcal{O} \cong W_0\mathcal{O}$ and the dendroidal category Ω is detailed in Theorem B, in which we show a "bracketed version" of the equivalence between O–algebras and appropriate Ω–diagrams. As the operad $W\mathcal{O}$ is a more complete resolution of $\mathcal{O}$, it is natural to ask if an even larger resolution of the dendroidal category Ω extends this relationship. For a category $\mathcal{K}$, there exists a resolution similar to the W–construction, namely the "explosion" $\widetilde{\mathcal{K}}$ of the category, as studied by Segal [37, Appendix B] and Leitch [25]. This "explosion" has the property that $\widetilde{\mathcal{K}}$–diagrams are coherently homotopy $\mathcal{K}$–diagrams. Applying this construction to the category Ω, one could expect that $W\mathcal{O}$–algebras are related to $\widetilde{\Omega}$–diagrams in the same way that $B\mathcal{O} = W_0\mathcal{O}$–algebras are related to $\widetilde{\Omega}_0$–diagrams. We show in Theorem A.6 that this does not quite hold, proving instead that there is an embedding of the category of $W\mathcal{O}$-algebras as a full subcategory of the category of $\widetilde{\Omega}$–diagrams satisfying a strict Segal condition.

By combining Theorem A and Theorem B or Theorem C, normalized cacti are a rare example of an ∞-operad that does not arise via the application of a nerve construction to a known (discrete or topological) operad. Indeed, to our knowledge, only a few such examples exist in the literature; see the weak operad of configuration spaces [20, Corollary 5], the configuration categories of [6] or examples that arise as a result of completion as in [4, Proposition 5.1].

The input of the construction of our explicit infinity operad structure on normalized cacti is a pre-given composition that we show to be associative up to coherent homotopy by using the operad $B\mathcal{O} = W\mathcal{O}/\sim$. The homotopies are constructed using the contractible space of basepoint preserving monotone reparametrizations of the circle (see the proof of Theorem 5.12). To extend the results to the cobordism category of graphs described above, one would need to replace $\mathcal{O}$ by the operad $P\mathcal{O}$, whose algebras are all symmetric properads [42, Section 14.1.2], define a resolution "$BP\mathcal{O}$", as the appropriate quotient of the W-construction applied to $P\mathcal{O}$. Our expectation is that these same reparametrizations of the circle will likewise provide all the necessary homotopies to provide an infinity composition in the graph model of the cobordism category.

Acknowledgments

This work was done as part of the Women in Topology Workshop in August 2019, supported by the Hausdorff Research Institute for Mathematics, NSF grant DMS 1901795, the AWM ADVANCE grant NSF-HRD-1500481, and Foundation Compositio Mathematica (ref. 386). Additional work by L.B.B. and M.R.

was carried out while in residence at MSRI in 2020. L.B.B. was supported by CNPq (201780/2017-8). S.R. acknowledges the support of the Centre of Australian Category Theory and Australian Research Council grants DP160101519 and FT160100393. N.W. was supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151) and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 772960).

In addition, we would like to thank Philip Hackney, Gijs Heuts, Muriel Livernet, Claudia Scheimbauer and Bruno Vallette for helpful conversations, suggested references and comments, and the referee for helpful comments and suggestions.

2. Preliminaries on operads

A *symmetric sequence* in a symmetric monoidal category $\mathcal{S}$ is a collection $\mathcal{P} = \{\mathcal{P}(k)\}_{k\geq 0}$ of objects in $\mathcal{S}$ in which each $\mathcal{P}(k)$ comes equipped with an action of the symmetric group Σ_k. In this paper, our symmetric monoidal category $\mathcal{S}$ will either be the discrete category of sets, the category of simplicial sets, or the category of topological spaces with their standard Cartesian products.

An *operad* in $\mathcal{S}$ is a symmetric sequence $\mathcal{P} = \{\mathcal{P}(k)\}_{k\geq 0}$ together with a distinguished element $\iota \in \mathcal{P}(1)$, called the *unit*, and a collection of composition maps

$$\circ_i : \mathcal{P}(k) \times \mathcal{P}(j) \longrightarrow \mathcal{P}(k+j-1),$$

$1 \leq i \leq k$, which are associative, unital, and equivariant. For more complete details see, for example, [27, Definition 11]. Given an operad $\mathcal{P}$, a symmetric sequence $\mathcal{Q} = \{\mathcal{Q}(k) \subseteq \mathcal{P}(k)\}_{k\geq 0}$ is a *suboperad* of $\mathcal{P}$ if the restriction of the composition maps in $\mathcal{P}$ induce an operad structure on $\mathcal{Q}$. A *morphism of operads* $f : \mathcal{P} \to \mathcal{Q}$ is a family of equivariant maps

$$\{f(k) : \mathcal{P}(k) \to \mathcal{Q}(k)\}_{k\geq 0}$$

that are compatible with composition and units.

Remark 2.1. It is equivalent to work with individual compositions

$$\circ_i \colon \mathcal{P}(k) \times \mathcal{P}(j_i) \to \mathcal{P}(k + j_i - 1)$$

or with all $\circ_i$-compositions simultaneously. In the latter case, the simultaneous compositions are denoted by a map

$$\gamma_{\mathcal{P}} \colon \mathcal{P}(k) \times \mathcal{P}(j_1) \times \ldots \times \mathcal{P}(j_k) \to \mathcal{P}(\Sigma_{i=1}^{k} j_i).$$

(e.g.: [27, Proposition 13]).

More generally, we will use *colored* operads. For any non-empty set $\mathfrak{C}$, a $\mathfrak{C}$-*colored symmetric sequence* is a family of objects $\mathcal{P} := \{\mathcal{P}(c; c_1, \ldots, c_k)\}_{k\geq 0}$ in $\mathcal{S}$, where $(c; c_1, \ldots, c_k)$ ranges over every list of colors in $\mathfrak{C}$ together with a map $\sigma^\star : \mathcal{P}(c; c_1, \ldots, c_k) \to \mathcal{P}(c; c_{\sigma(1)}, \ldots, c_{\sigma(k)})$ for each $\sigma \in \Sigma_k$. A $\mathfrak{C}$-*colored operad* is a $\mathfrak{C}$-colored symmetric sequence $\mathcal{P}$ together with a family of partial composition maps

$$\circ_i \colon \mathcal{P}(c; c_1, \ldots, c_k) \times \mathcal{P}(d; d_1, \ldots, d_j) \to \mathcal{P}(c; c_1, \ldots, c_{i-1}, d_1, \ldots, d_j, c_{i+1} \ldots, c_k)$$

defined only when $c_i = d$, together with an element $\iota_c \in \mathcal{P}(c; c)$ for each $c \in \mathfrak{C}$, which satisfies unit, equivariance and associativity conditions. For more details see, for example, [1, Definition 1.1]. When the

color set is $\mathfrak{C} = \{*\}$, a $\mathfrak{C}$–colored operad is a one-colored operad. In this paper we will refer to both operads and colored operads as "operads", only mentioning the color set when necessary.

An *algebra* over a ($\mathfrak{C}$–colored) operad $\mathcal{P}$ is a collection of objects $\{X(c)\}_{c\in\mathfrak{C}}$ in $\mathcal{S}$ together with evaluation maps

$$\alpha\colon \mathcal{P}(c; c_1, \dots, c_k) \times X(c_1) \times \cdots \times X(c_k) \longrightarrow X(c)$$

satisfying appropriate associativity, unit and equivariance conditions, see e.g. [1, Definition 1.2]. The category of $\mathcal{P}$-algebras in $\mathcal{S}$ is denoted $\mathcal{P}-\mathrm{Alg}_{\mathcal{S}}$.

Our main example of a colored operad will be the $\mathbb{N}$-colored operad $\mathcal{O}$, whose algebras are the (non-colored) operads, see Definition 2.8. In Section 5, we will also make use of the following operad:

Example 2.2. Let X be a fixed space in $\mathcal{S}$. The *coendomorphism operad* of X, $\mathrm{CoEnd}(X)$, has an underlying symmetric sequence with arity k spaces

$$\mathrm{CoEnd}(n)(X) := \mathrm{Map}(X, X^{\times k}).$$

The symmetric groups act by permuting the factors of $f = (f_1, \dots, f_k) \in \mathrm{CoEnd}(k)$. If $f = (f_1, \dots, f_k) \in \mathrm{CoEnd}(k)(X)$ and $g = (g_1, \dots, g_j) \in \mathrm{CoEnd}(j)(X)$ the partial compositions

$$\circ_i : \mathrm{CoEnd}(k)(X) \times \mathrm{CoEnd}(j)(X) \longrightarrow \mathrm{CoEnd}(k + j - 1)(X)$$

are given by

$$f \circ_i g = (f_1\,, \dots\,, f_{i-1}\,, g_1 \circ f_i\,, \dots\,, g_j \circ f_i\,, f_{i+1}\,, \dots\,, f_k).$$

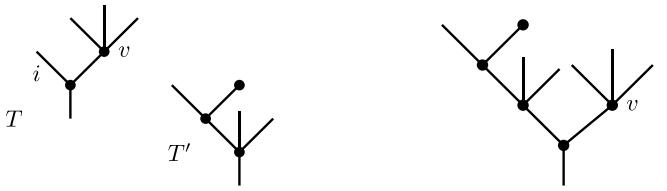
2.1. Trees

Throughout this paper, we use trees to model operad compositions and as the basis of our main constructions. A graph G is a tuple $(V(G), H(G), s, i)$ where $V(G)$ is a set of vertices, $H(G)$ a set of half-edges, $s : H(G) \to V(G)$ is the source map and $i : H(G) \to H(G)$ is an involution. Orbits of the involution i are called *edges* of G and the set of edges is denoted by $E(G)$. An edge represented by a pair $\{h, i(h)\}$ with $i(h) \neq h$ is called an *internal edge*, and the set of internal edges is denoted $iE(G)$. Edges corresponding to orbits of fixed points of the involution are *external*.

A *tree* is a simply connected graph. All our trees will be *rooted*, i.e. they come with a distinguished "outgoing" external edge called the *root*. All other external edges are "incoming" and called *leaves*. The set of leaves is denoted $L(T)$. The *arity* of T is the number of leaves $|L(T)|$. The root of the tree is denoted $R(T)$.

Note that a rooted tree can be canonically made into a directed graph by setting all the edges to point towards the root. Then note that the set of edges incident to a vertex always has a unique *outgoing edge*, the one closest to the root, and all other edges are *incoming edges*. The number of incoming edges of a vertex v is called the *arity* of the vertex and denoted by $|v|$, with $|v| \geq 0$ any natural number.

We allow the special tree $\eta = |$, with no vertices and a single edge. The trees with a single vertex and n leaves are called *n-corollas* and denoted C_n. A rooted tree S is a *subtree* of T if $V(S) \subseteq V(T)$ preserving the arity, $H(S) \subseteq H(T)$, and the structure maps for S are restrictions of the structure maps for T, defining $i(h) = h$ in S if $i(h) = h'$ in T with $h' \notin H(S)$. A *planar tree* is a rooted tree together with a preferred isotopy class of embedding into the plane. Note that this is equivalent to defining a planar tree as a tree with a canonical ordering of the incoming edges at each vertex.



(a) Example trees T and T' (b) The grafting $T \circ_i T'$

Fig. 2. Grafting of trees.

We use planar trees to model operad compositions via an operation called grafting. Given trees T and T', of arity n and m respectively, and a leaf $i \in L(T)$, the *grafting* of T' onto T along the leaf i is defined to be the tree $T \circ_i T'$ obtained by attaching the root of T' to the leaf i of T so that they form a new internal edge in the grafted tree (Fig. 2). Grafting of trees is also used to model the free operad generated by a symmetric sequence, as we will explain now. To avoid confusion later, when we will have to decorate vertices of trees by other trees, we will use blackboard fonts for the trees in the free operad construction (and later the associated W-construction in Section 6.1), as we will soon apply this construction to a symmetric sequence of trees, which will give (blackboard) trees of (plain) trees.

Definition 2.3. Let $\mathcal{P} = \{\mathcal{P}(c; c_1, \dots, c_k)\}_{c_i, c \in \mathfrak{C}}$ be a $\mathfrak{C}$-colored symmetric sequence in $\mathcal{S}$. A planar tree $\mathbb{T}$ is $\mathfrak{C}$-colored if it is equipped with a map $f : E(\mathbb{T}) \to \mathfrak{C}$, we refer to $f(e)$ as the color of the edge e. A $\mathfrak{C}$-colored planar tree $\mathbb{T}$ is *decorated* by $\mathcal{P}$ if each vertex $v \in V(\mathbb{T})$ is labeled by an operation in $p_v \in \mathcal{P}(out(v); in(v))$, where $out(v)$ is the color of the outgoing edge of v, and $in(v)$ is the list of colors of the incoming edges of v, ordered by the planar structure. The *free operad* $F(\mathcal{P})$ on $\mathcal{P}$ is the $\mathfrak{C}$-colored operad whose k-ary operations are the $\mathfrak{C}$-colored, $\mathcal{P}$ decorated, planar trees $\mathbb{T}$ of arity k with leaves labeled by a bijection $\lambda : \{1, \dots, k\} \to L(\mathbb{T})$.

Explicitly, for each $c, c_1, \dots, c_k \in \mathfrak{C}$,

$$F(\mathcal{P})(c; c_1, \dots, c_k) := \Big(\coprod_{(\mathbb{T}, f, \lambda)} \prod_{v \in V(\mathbb{T})} \mathcal{P}(out(v); in(v)) \Big) / \sim,$$

where $(\mathbb{T}, f, \lambda)$ runs over all isomorphism classes of leaf-labeled $\mathfrak{C}$-colored planar trees with k leaves such that $f(\lambda(i)) = c_i$, $f(R(\mathbb{T})) = c$, and where the equivalence relation is generated by the following:

(∗) two labeled trees $(\mathbb{T}, f, \lambda, (p_v)_{v \in V(\mathbb{T})})$ and $(\mathbb{T}', f', \lambda', (p'_w)_{w \in V(\mathbb{T}')})$ are equivalent if there exists a non-planar isomorphism $\alpha : \mathbb{T} \to \mathbb{T}'$ such that $f \circ \alpha = f'$, $\alpha \circ \lambda = \lambda'$, and $\sigma_v(\alpha) p_v = p_{\alpha(v)}$, for $\sigma_v(\alpha)$ the permutation on $in(v)$ induced by α.

The symmetric group acts on $F(\mathcal{P})$ by permuting the labels of the leaves, acting on λ, and composition in $F(\mathcal{P})$ is given by grafting of trees, with $\circ_i$ grafting at the leaf $\lambda(i)$. For full details see, for example, the construction under Corollary 3.3 [1].

We now employ the free operad construction to define a class of free operads $\Omega(T)$ generated by a planar tree T. This will play a fundamental role in the definition of the dendroidal category (Section 2.2), which describes a model for ∞-operads.

Example 2.4. A planar tree T generates a free colored operad $\Omega(T)$ as follows. The set of colors of $\Omega(T)$ is the set of edges $\mathfrak{C} = E(T)$. We define a discrete $E(T)$-colored symmetric sequence $X(T)$ by

$$X(T)(e; e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \begin{cases} \{\sigma v\} & \text{if } (e; e_1, \dots, e_n) = (\text{out}(v); \text{in}(v)), \text{ for } v \in V(T), \\ \emptyset & \text{otherwise,} \end{cases}$$

with its built in free symmetric group action. Then $\Omega(T) := FX(T)$ is the free operad on the collection $X(T)$. Explicitly, $\Omega(T)$ is an $E(T)$-colored operad with

$$\Omega(T)(e; e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \begin{cases} \sigma S & \text{if } (e; e_1, \dots, e_n) = (R(S); L(S)),\ S \subset T, \\ \emptyset & \text{otherwise.} \end{cases}$$

for $S \subset T$ a subtree of T. Composition, as in the free operad, is given by grafting of subtrees. For further details, see Section 2.2 and just above Definition 2.3.1 in [31].

2.2. The dendroidal category Ω

The model we use for ∞-operads is that of dendroidal Segal spaces that satisfy the *weak Segal condition*. Dendroidal spaces are diagrams of the dendroidal category.

The dendroidal category Ω is the full subcategory of colored operads whose objects are the free operads $\Omega(T)$ generated by trees (as in Example 2.4). In other words, objects of Ω are planar isomorphism classes of planar rooted trees and morphisms in Ω are defined to be operad maps

$$\text{Hom}_\Omega(S, T) = \text{Hom}_{\mathsf{Op}}(\Omega(S), \Omega(T)).$$

Morphisms in Ω can be described as a composition of four types of elementary morphisms: isomorphisms, degeneracies, inner and outer face maps. In terms of trees, *isomorphisms* are non-planar tree isomorphisms, *inner face maps* are of the form $\partial_e : T/e \to T$, where T/e is the tree obtained from T by contracting an inner edge $e \in iE(T)$. If v is a vertex of T with only one inner edge attached to it then T/v is the tree obtained from T by chopping off the vertex v and the inclusion $\partial_v : T/v \to T$ is an *outer face map*. A *degeneracy* is a map $s_v : T/v \to T$ where T/v is obtained from T by deleting a vertex v, with $|v| = 1$, in T.

In the opposite category Ω^{op}, outer face maps correspond to restriction to certain allowed subtrees, while inner face maps correspond to edge collapses. For more details and plenty of examples see [32,31].

Remark 2.5. Our definition of Ω differs slightly from the usual definition in that we have chosen our objects to be planar trees. Technically, what we have described here is the equivalent category Ω' from [31, 2.3.2].

Definition 2.6. A *dendroidal space* X is an Ω-diagram $X : \Omega^{op} \to \mathcal{S}$, where $\mathcal{S}$ is either the category of simplicial sets or topological spaces.

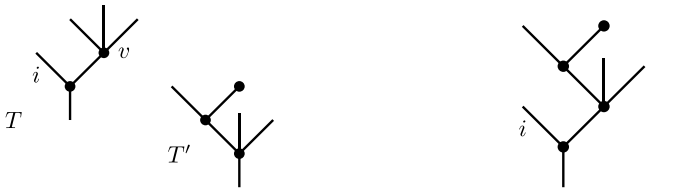
The evaluation of X at a tree T is denoted $X(T)$. A dendroidal space is called *reduced* if $X(\eta) = *$, where $\eta = |$. We will write $\mathcal{S}^{\Omega^{op}}$ for the category of dendroidal spaces.

For any vertex v in a tree $T \in \Omega$, we have an associated outer face map in Ω

$$C_v \longrightarrow T$$

taking the unique vertex of the corolla to $v \in V(T)$, where C_v is the corolla with $|v|$ leaves. Likewise, for any internal edge between vertices u and v in T, there is a commuting diagram in Ω

$$\begin{array}{ccc} \eta & \longrightarrow & C_u \\ \downarrow & & \downarrow \\ C_v & \longrightarrow & T. \end{array}$$



(a) Example trees T and T' (b) The substitution $T \bullet_v T'$

Fig. 3. Tree substitution (Compare with grafting in Fig. 2).

Let $\mathrm{Sk}_1(T)$ be the category whose objects are the edges and vertices of T, thought of copies of η and corollas C_v, and whose morphisms are associated to edge inclusions in T, as in the top left corner of the above diagram.

For a dendroidal space X, the *Segal map* is the unique map from $X(T)$ to the limit $\lim_{\mathrm{Sk}_1(T)^{op}} X$ induced by the corolla inclusions. When $X(\eta) = *$, this limit becomes a product over the value of X at the corollas, and the Segal map becomes the map

$$\chi\colon X(T) \longrightarrow \prod_{v \in V(T)} X(C_v)$$

with components the restriction to the value of X at each corolla.

The category of Ω-diagrams admits two Quillen model category structures: the Reedy model structure and the projective model structure which are Quillen equivalent (see e.g. [5, Remark 2.5]). Throughout, we take the *projective model structure* in which a morphism of Ω-diagrams is a weak equivalence or fibration if it is entrywise a weak equivalence or fibration.

Definition 2.7. A dendroidal space $X \in \mathcal{S}^{\Omega^{op}}$ satisfies a *strict Segal condition* if the Segal map is an isomorphism for each $\eta \neq T \in \Omega$.

2.3. The operad of operads

One of the main constructions in this paper is the operad $B\mathcal{O}$. This operad builds on an $\mathbb{N}$-colored operad $\mathcal{O}$ called the *operad of operads*, whose algebras are one-colored operads.

Let T be a planar tree. For a vertex $v \in V(T)$ with arity $|v| = m$ and a planar tree T' with m leaves, the *substitution* $T \bullet_v T'$ is obtained by removing the vertex v from T and identifying the incoming and outgoing edges of v with the leaves and root of T', respectively. An example is shown in Fig. 3.

A *labeled* planar tree is a triple (T, σ, τ), consisting of a planar tree T equipped with bijections $\sigma : |V(T)| \to V(T)$ and $\tau : |L(T)| \to L(T)$. Two such triples (T, σ, τ) and (T', σ', τ') are isomorphic if there is a planar tree isomorphism $T \to T'$ that respects the labeling σ, τ. We represent a labeled planar tree (T, σ, τ) by writing above each leaf $\ell \in L(T)$ the number $\tau^{-1}(\ell)$, and writing by each vertex $v \in V(T)$ the number $\sigma^{-1}(v)$, as depicted in Fig. 4.

We also define a tree substitution that is *compatible with the labellings of the leaves*. Let (T, σ, τ) and (T', σ', τ') be two planar labeled trees with $|V(T)| = k$, $|V(T')| = l$ and $|L(T')| = |\sigma(i)| = m_i$. The map τ' encodes a permutation in the symmetric group with m_i elements. We obtain a new planar tree $(\tau'_{\sigma(i)})T$ by applying the permutation τ' on the m_i incoming edges of the vertex $\sigma(i) \in V(T)$. We then define

$$T \bullet_{\sigma(i),\tau'} T' = (\tau'_{\sigma i})T \bullet_{\sigma(i)} T'. \tag{2.1}$$

In particular, $V(T \bullet_{\sigma(i),\tau'} T') = \{V(T) - \sigma(i)\} \amalg V(T')$. The labeling on the vertices of $T \bullet_{\sigma(i),\tau'} T'$ is given by the map $\sigma \circ_i \sigma'$, which is the induced bijection $\{1, \ldots, k + l - 1\} \to V(T \bullet_{\sigma(i),\tau'} T')$

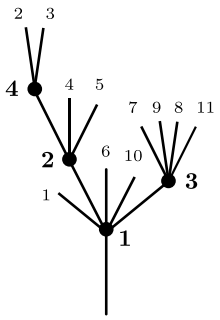


Fig. 4. Example of a labeled planar tree in $\mathcal{O}(11; 5, 3, 4, 2)$.

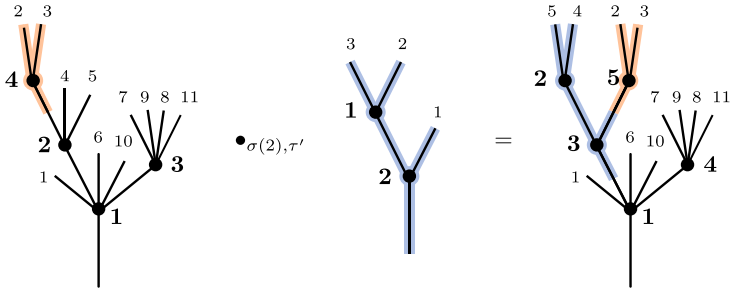


Fig. 5. Example of composition in $\mathcal{O}$ where τ' is different to the planar order of $in(\sigma(2))$.

$$j \mapsto \begin{cases} \sigma(j) & 1 \leq j < i \\ \sigma'(j-i+1) & i \leq j \leq i+l \\ \sigma(j-l+1) & i+l < j \leq k+l-1. \end{cases}$$

An example is shown in Fig. 5. In the case where the order induced by τ' on the m_i incoming edges of $\sigma(i)$ is the same as the order induced by the planar structure, then $T \bullet_{\sigma(i),\tau'} T' = T \bullet_{\sigma(i)} T'$.

Definition 2.8. The *operad of operads* $\mathcal{O}$ is the $\mathbb{N}$–colored operad, for which

$$\mathcal{O}(n; m_1, \ldots, m_k)$$

is the discrete space whose elements are isomorphism classes of *labeled* planar rooted trees (T, σ, τ) where T is a planar tree with k vertices and n leaves, with bijections $\sigma\colon |V(T)| \to V(T)$, $\tau\colon |L(T)| \to L(T)$, such that the vertex $\sigma(i)$ has arity m_i for each $1 \leq i \leq k$. The composition operation

$$\begin{aligned} \mathcal{O}(n; m_1, \ldots, m_k) \times \mathcal{O}(m_i; b_1, \ldots, b_l) &\xrightarrow{\circ_i} \mathcal{O}(n; m_1, \ldots, b_1, \ldots, b_l, \ldots, m_k) \\ ((T, \sigma, \tau), (T', \sigma', \tau')) &\longmapsto (T, \sigma, \tau) \circ_i (T', \sigma', \tau') \end{aligned}$$

is induced by tree substitution that is compatible with the labeling as in (2.1), where

$$(T, \sigma, \tau) \circ_i (T', \sigma', \tau') = (T \bullet_{\sigma(i),\tau'} T', \sigma \circ_i \sigma', \tau).$$

The unit for this composition, for the color n, is the element of $\mathcal{O}(n; n)$ represented by the corolla C_n equipped with the canonical left-right labeling. The symmetric group Σ_k acts on $(T, \sigma, \tau) \in \mathcal{O}(n; m_1, \ldots, m_k)$ by precomposition on the labeling σ of the vertices $V(T)$.

We further observe that, for each $m, n \in \mathbb{N}$,

$$\mathcal{O}(m;n) \cong \begin{cases} \Sigma_n & \text{for } m = n, \\ \emptyset & \text{when } m \neq n. \end{cases}$$

The isomorphism $\mathcal{O}(n;n) \cong \Sigma_n$ corresponds to labeling the leaves of a corolla C_n in all possible ways. The unique arity 0 operation in $\mathcal{O}$ is represented by the special tree $\eta \in \mathcal{O}(1;\emptyset)$. An $\mathcal{O}$-algebra, $\mathcal{P}$, is precisely a one-colored operad. That is to say, $\mathcal{P}$ has an underlying $\mathbb{N}$-graded object $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ in $\mathcal{S}$. Moreover, $\mathcal{P}$ admits actions $\mathcal{O}(n;n) \times \mathcal{P}(n) \to \mathcal{P}(n)$ for all n and thus $\mathcal{P}$ has an underlying symmetric sequence. By definition, we have

$$\mathcal{O}(n; m_1, \ldots, m_k) \times \mathcal{P}(m_1) \times \ldots \times \mathcal{P}(m_k) \subset F\mathcal{P}(n),$$

where $F\mathcal{P}$ is the free operad on the symmetric sequence $\mathcal{P}$, and

$$F\mathcal{P}(n) = \coprod_{k\in\mathbb{N}} \left(\coprod_{(m_1,\ldots,m_k)\in\mathbb{N}^k} \mathcal{O}(n; m_1, \ldots, m_k) \times \mathcal{P}(m_1) \times \ldots \times \mathcal{P}(m_k) \right)_{\Sigma_k}$$

so the action maps

$$\alpha : \mathcal{O}(n; m_1, \ldots, m_k) \times \mathcal{P}(m_1) \times \ldots \times \mathcal{P}(m_k) \to \mathcal{P}(n)$$

induce maps $F\mathcal{P}(n) \to \mathcal{P}(n)$ for all n, and by the algebra axioms, this is precisely the data of a symmetric operad in $\mathcal{S}$ (See [1, Example 1.5.6]). Note that, in particular, the $\circ_i$-compositions of an operad $\mathcal{P}$ are governed by the trees with one internal edge in $\mathcal{O}(n; m_1, m_2)$, where $n = m_1 + m_2 - 1$.

2.4. The relationship between operads and dendroidal spaces

Dendroidal spaces with $X(\eta) = *$ that satisfy a *strict* Segal condition are closely related to one-colored operads. Explicitly, every operad $\mathcal{P}$ can be viewed as a dendroidal space via the dendroidal *nerve* construction that defines a functor

$$N^d(\mathcal{P})(T) = \mathrm{Hom}_{\mathsf{Op}}(\Omega(T), \mathcal{P})$$

as T ranges over Ω. The nerve of the free operad $\Omega(T)$ is just the representable dendroidal space $\Omega[T] := \mathrm{Hom}_\Omega(-, T)$. A dendroidal space X is the *nerve* of an operad if, and only if, the Segal map of Definition 2.7 is an isomorphism for all T [12, Lemma 6.4; Proposition 6.5]. To put this altogether, there is an isomorphism of categories

$$\mathcal{O}\text{–Alg}_{\mathcal{S}} \cong (\mathcal{S}^{\Omega^{op}})_{strict}$$

where $\mathcal{O}$ is the colored operad whose algebras are one-colored operads (Definition 2.8, below) and $(\mathcal{S}^{\Omega^{op}})_{strict}$ denotes the category of reduced dendroidal spaces satisfying the strict Segal condition. We will prove similar statements for "thickened" versions of Ω in Theorem 4.8 and Theorem A.6.

3. The operad of brackets $\mathcal{BO}$

In this section we introduce a new topological operad called the *operad of bracketed trees*. In short, the operad $B\mathcal{O}$ captures a weak notion of an operad in the sense that a $B\mathcal{O}$-algebra is a symmetric sequence

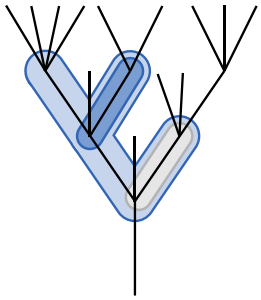


Fig. 6. Example of a tree bracketing with 3 nested subtrees.

with $\circ_i$-operations that are only associative up to higher homotopy. The construction of the operad $B\mathcal{O}$ allows one to check with relative ease whether a symmetric sequence with compositions assembles into an ∞-operad. In Theorem 5.12, we use this to show that normalized cacti admit such a structure. Moreover, we expect that this construction provides a general method that one can use to construct other examples of ∞-operads.

One could instead use the classical Boardman-Vogt W-construction on the operad $\mathcal{O}$ to obtain an operad $W\mathcal{O}$ whose algebras are homotopy operads (*lax operads* in the language of [8]). It is known to experts that bracketings in trees are related to this operad $W\mathcal{O}$, but the precise details are difficult to find in the literature. (However, see [33, Section 2.3], in particular Theorem 4, together with Remark 3.8 below, for an algebraic version of this in the case of non-symmetric operads.) In Section 6 we will show that $B\mathcal{O}$ identifies with a quotient of the operad $W\mathcal{O}$. Bracketings in trees have also appeared elsewhere, see e.g. [13,14], and the parenthesizations of [38, 2.6].

3.1. Bracketings of trees

We define in this section the poset of bracketings of a tree, starting with the definition of a bracketing:

Definition 3.1. A tree is called *large* if it has at least two vertices (or equivalently, at least one internal edge). A set $\{S_j\}_{j\in J}$ of subtrees of a tree T is *nested* if, for any $i, j \in J$, the set of common vertices $V(S_i) \cap V(S_j)$ is either $V(S_i)$, $V(S_j)$ or empty. A *bracketing* B of a tree T is a (possibly empty) collection $B = \{S_j\}_{j\in J}$ of nested large proper subtrees of T.

Recall from Section 2.1 that a subtree of T is a tree S whose vertices are a subset of the vertices of T, and whose half-edges are all the half-edges in T attached to such vertices. Therefore, a subtree is completely determined by its vertices. With this in mind, we will represent bracketings as in Fig. 6.

Definition 3.2. Bracketings of a tree T form a poset of bracketings $\mathcal{B}(T)$ with the relation $B' \leq B$ if $B' \subseteq B$.

We denote the geometric realization of the nerve of the poset $\mathcal{B}(T)$ by $|\mathcal{B}(T)|$. A point in

$$|\mathcal{B}(T)| = \coprod_{r\geq 0} N_r\mathcal{B}(T) \times \Delta^r / \sim$$

is a pair (B, t) with $B = B_0 \subset \cdots \subset B_r$ a sequence of bracketings and $t \in \Delta^r$. Such a pair (B, t) can be interpreted as a *weighted bracketing* with underlying set of brackets $B_r = \cup_{i=0}^r B_i$ and weights given by

$$t = (1, t_1, \ldots, t_r) \in \Delta^r = \{1 = t_0 \geq t_1 \geq \cdots \geq t_r \geq 0\}$$

where we assign the weight $t_0 = 1$ to all brackets in B_0, and for each $1 \leq i \leq r$, the weight t_i to all brackets in $B_i \backslash B_{i-1}$. In particular, a weighted bracketing with all brackets having weight 1 corresponds to a vertex

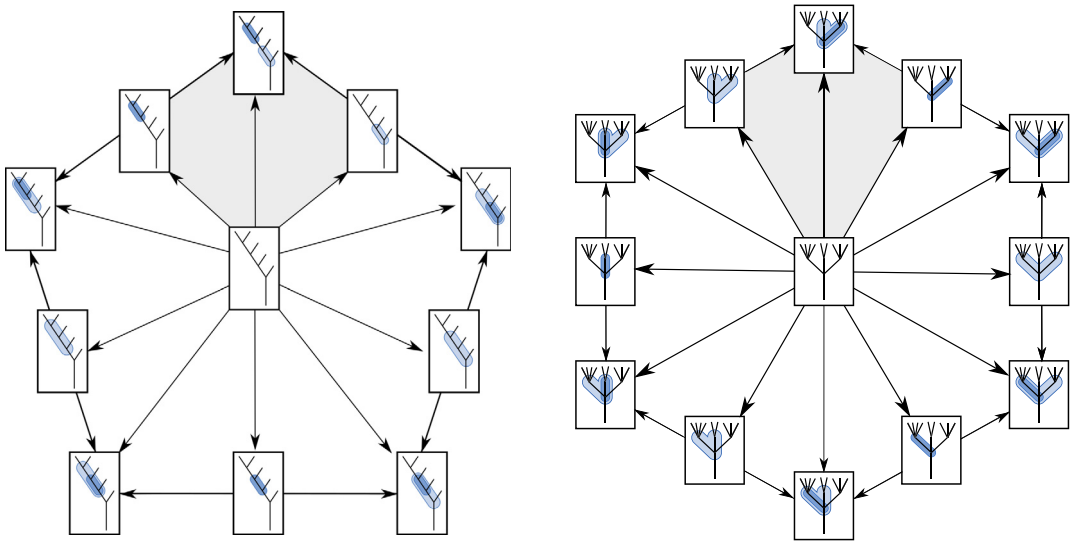


Fig. 7. Geometric realization of the poset $\mathcal{B}(T)$ of Examples 3.4 and 3.5.

$B = B_0$ in the nerve of the poset. Also, the equivalence relation on the realization implies that a bracket of weight 0 can be discarded. (See also Section 6 and in particular the proof of Lemma 6.8 where this point of view is used to relate $B\mathcal{O}$ to the operad $W\mathcal{O}$.)

Example 3.3. If $T = C_n$ then T does not admit any large subtree, therefore $\mathcal{B}(T) = \{\emptyset\}$ only has the empty (or trivial) bracketing.

Example 3.4. Let T be the tree  , then the space $|\mathcal{B}(T)|$ is depicted in Fig. 7 (left). Note that the initial object in the poset is the empty bracket, in the center of the pentagon (the 4th associahedron).

More generally, let T_n be a tree with n vertices such that no vertex is connected to more than two inner edges. For such trees, the set of vertices can always be given a total ordering, for instance by constructing a list starting with a vertex v connected to only one internal edge, and defining the next element of the list to be the vertex sharing an edge with v that has not yet been listed. Then a bracket of T_n can be immediately identified with a meaningful placement of parentheses on a word with n letters where the word is represented by the ordered set of vertices. Therefore, $|\mathcal{B}(T_n)|$ can always be identified with the n-th associahedron (see also Remark 3.8 for another approach to this statement).

Example 3.5. Consider a tree T with three inner edges all meeting at a single vertex. Note that the poset of bracketings depends only on the relative positions of the vertices (or analogously, the inner edges) of the tree T, and is independent of the number of leaves at each vertex. Therefore, the realization poset of bracketings of T is the hexagon (two-dimensional permutahedron) depicted in the Fig. 7 (right), using as an example the tree $T =$  .

Example 3.6. Fig. 8 depicts the realization of the poset of bracketings of a tree T with four inner edges meeting at a single vertex. Note that by fixing a large subtree S of T, the realization of the subposet of bracketings of T containing S will correspond to a subspace of the boundary of $|\mathcal{B}(T)|$. Each boundary face of top dimension is then associated to a subtree S of T, and two such faces S_1, S_2 share a subface if $\{S_1, S_2\}$ is nested. In this case, $|\mathcal{B}(T)|$ is the three-dimensional permutahedron.

Lemma 3.7. *For any tree T, the space $|\mathcal{B}(T)|$, is contractible.*

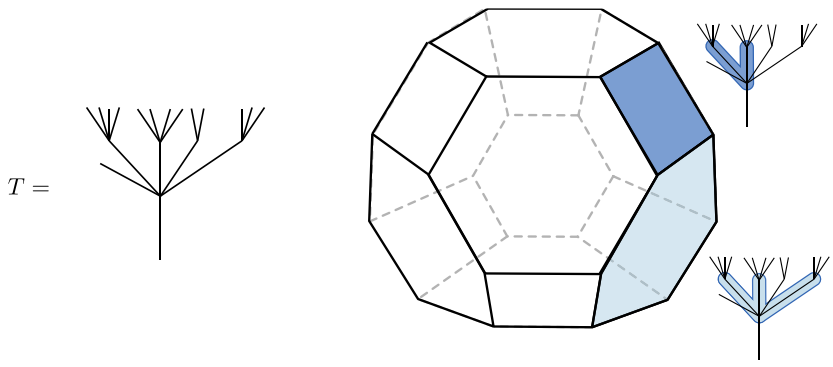


Fig. 8. Tree satisfying the conditions of Example 3.6 together with the geometric realization of its poset of bracketings.

Proof. The contractibility of the space $|\mathcal{B}(T)|$ follows directly from the fact that the poset $\mathcal{B}(T)$ has a minimal element, namely the empty bracketing. □

Remark 3.8. The spaces $|\mathcal{B}(T)|$ are related to the family of nestohedra that were first described by Feichtner and Sturmfels [18] and Postnikov [35]. Examples of nestohedra include simplices, permutahedra, Stasheff's associahedra and more generally Carr and Devadoss' graph associahedra [9]. Hypergraph polytopes are an interpretation of nestohedra by Došen and Petrić [15], with the advantage that they have a convenient tree notation (called *constructs*) to label the faces of the polytope and that encode the face inclusions. The space $|\mathcal{B}(T)|$ can be identified with a hypergraph polytope as follows. Using the definition and notation by Obradović [33, Section 2.2.1], the edge-graph $\mathbf{H}_T$ of T is the graph whose vertices correspond to the inner edges of T, and two vertices of $\mathbf{H}_T$ share an edge if the corresponding inner edges of T have a common vertex. A subgraph $\mathbf{S}$ of $\mathbf{H}_T$ uniquely defines a subforest $\langle \mathbf{S} \rangle$ of T whose internal edges correspond precisely with the vertices of $\mathbf{S}$, and each tree in this forest is necessarily large because it has an inner edge (see [33, Section 2.2.1, Lemma 3]). Then we have an order reversing bijection b between the hypergraph polytope of the edge-graph $\mathbf{H}_T$ and $\mathcal{B}(T)$ that can be recursively defined as follows: we take the construct $V(\mathbf{H}_T)$ given by the vertex set of $\mathbf{H}_T$ to the empty bracketing, and if $\mathbf{H}_T \setminus Y$ for $Y \subset V(\mathbf{H}_T)$ decomposes into n connected components $\mathbf{H}_{T_1}, \ldots, \mathbf{H}_{T_n}$, then for constructs C_i of $\mathbf{H}_{T_i}$ we take the construct $Y\{C_1, \ldots, C_n\}$ to the bracketing $\{\langle \mathbf{H}_{T_1} \rangle, \ldots, \langle \mathbf{H}_{T_n} \rangle, b(C_1), \ldots, b(C_n)\}$. The definition of constructs guarantees that these sets are nested and therefore defines a bracketing, and it is simple to check that this is an order reversing bijection.

3.2. An operad of bracketings

We'll use the bracketings $\mathcal{B}(T)$ to construct a topological operad. Let the collection

$$B\mathcal{O}(n; m_1, \ldots, m_k) = \coprod_{(T,\sigma,\tau) \in \mathcal{O}(n; m_1, \ldots, m_k)} |\mathcal{B}(T)|$$

define the $\mathbb{N}$-colored symmetric sequence $B\mathcal{O}$. So, elements of $B\mathcal{O}(n; m_1, \ldots, m_k)$ are tuples (T, σ, τ, B, t) where (T, σ, τ) is an element of $\mathcal{O}(n; m_1, \ldots, m_k)$ (Definition 2.8) and (B, t) is a weighted bracketing of T (i.e. a point in $|\mathcal{B}(T)|$).

To define operadic composition in $B\mathcal{O}$, we use the composition of trees in $\mathcal{O}$ and induce a bracketing of the resulting tree. Let (T, σ, τ, B) and (T', σ', τ', B') be labeled trees with bracketings. The composition in $\mathcal{O}$ (Definition 2.8) is given by the substitution of T' into the vertex $\sigma(i) \in V(T)$,

$$(T, \sigma, \tau) \circ_i (T', \sigma', \tau') = (T \bullet_{\sigma(i),\tau'} T', \sigma \circ_i \sigma', \tau).$$

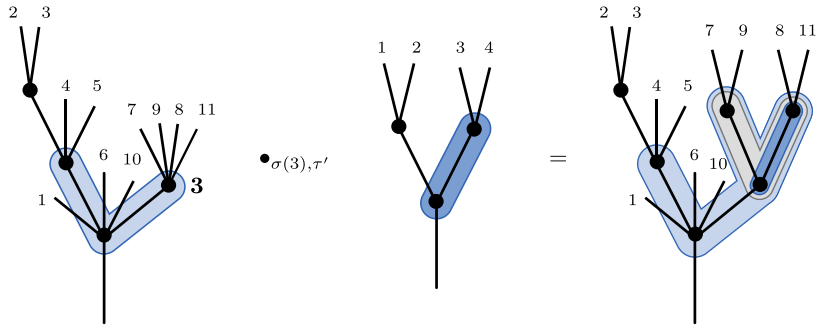


Fig. 9. Example of composition in $B\mathcal{O}$ with labeling of the vertices omitted for simplicity.

Since T' is canonically a subtree of $T \bullet_{\sigma(i),\tau'} T'$, the bracketing B' on T' defines a nested collection of subtrees of $T \bullet_{\sigma(i),\tau'} T'$. We also construct a nested collection of subtrees $\tilde{B} = \{\tilde{S}_j\}_{j\in J}$ on $T \bullet_{\sigma(i),\tau'} T'$ that is induced by the bracketing $B = \{S_j\}_{j\in J}$ on T. If $T' \neq \eta$, then $\tilde{B} \cong B$ is given by

$$\tilde{S}_j = \begin{cases} S_j & \text{if } \sigma(i) \notin V(S_j), \\ S_j \bullet_{\sigma(i),\tau'} T' & \text{if } \sigma(i) \in V(S_j). \end{cases} \tag{3.1}$$

If $T' = \eta$, then $\tilde{B} = \{\tilde{S}_j\}_{j\in J}$ is defined in the same way, unless $\sigma(i) \in V(S_j)$ and S_j has two vertices, in which case $S_j \bullet_{\sigma(i),\tau'} \eta$ is a corolla and is discarded as it is not large. That is, we replace J with another indexing set $J' \subset J$, which is the subset of indices j such that S_j is large.

We define a bracketing of the tree $T \bullet_{\sigma(i),\tau'} T'$ by

$$B'' = \begin{cases} \tilde{B} \cup B' \cup \{T'\} & \text{if } T' \text{ is large} \\ \tilde{B} & \text{else.} \end{cases} \tag{3.2}$$

See Fig. 9. This defines a composition of bracketings of trees. This composition is associative as follows. Suppose $S_j \subset T$ is a bracket with only two vertices v and w, and T' is a tree with at least two vertices. If we first compose η in v and then T' in w, the bracket S_j is discarded during the first composition, and then replaced by a new bracket T'. Reversing the order of these two compositions yields the same result because first composing T' in w will create a new bracket T', and S_j will not be discarded, but composing further η in v will equate S_j and T'. Otherwise, the associativity of the composition follows from the associativity on the composition in $\mathcal{O}$.

The composition also respects inclusions and thus is a poset map

$$\mathcal{B}(T) \times \mathcal{B}(T') \longrightarrow \mathcal{B}(T \bullet_{\sigma(i),\tau'} T'). \tag{3.3}$$

The realization of the poset map (3.3) induces a map between the geometric realizations of the nerve of the posets.

Also recall that the unary elements of $\mathcal{O}$, i.e. the elements of $\mathcal{O}(n;n)$ for some n, are given by labeled corollas. Since there are no non-trivial bracketings of corollas, unary elements of $B\mathcal{O}$ have the form $(C_n, \sigma, *, \emptyset, \emptyset) \in B\mathcal{O}(n;n)$ with $\sigma \in \Sigma_n$. In particular, the n-colored identity for the composition $\circ$ in $B\mathcal{O}$ is given by $(C_n, id_n, *, \emptyset, \emptyset) \in B\mathcal{O}(n;n)$. Therefore $B\mathcal{O}$ is an operad.

Definition 3.9. The *operad of bracketed trees* $B\mathcal{O}$ is the $\mathbb{N}$-colored topological operad with underlying symmetric sequence

$$B\mathcal{O}(n; m_1, \dots, m_k) = \coprod_{(T,\sigma,\tau)\in\mathcal{O}(n;m_1,\dots,m_k)} |\mathcal{B}(T)|$$

and composition given by combining the composition in $\mathcal{O}$ with the map (3.3) described above.

Remark 3.10.

(1) Note that there is an isomorphism of operads $\mathcal{O} \cong \pi_0 B\mathcal{O}$ and, by Lemma 3.7, the projection $B\mathcal{O} \to \mathcal{O}$ is a homotopy equivalence of operads.
(2) The topological operad $B\mathcal{O}$ is the realization of an operad in posets. Indeed, the space $B\mathcal{O}(n; m_1, \dots, m_k)$ is the realization of the poset of elements (T, σ, τ) of $\mathcal{O}(n; m_1, \dots, m_k)$ together with a bracketing of T, where two elements are comparable only if they have the same underlying element of $\mathcal{O}$. Likewise, the operad structure is defined as the realization of a map on the level of posets.

3.3. B$\mathcal{O}$-algebras

A *B$\mathcal{O}$-algebra* is an operad whose $\circ_i$-compositions are associative up to all higher homotopies. In particular, a $B\mathcal{O}$-algebra $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ has an underlying *symmetric sequence*. To see this, we note that the labeling of the leaves of a corolla $(C_n, \tau, *, \emptyset, \emptyset) \in B\mathcal{O}(n; n)$ identifies with elements of the symmetric group and we have isomorphisms

$$B\mathcal{O}(n; n) \cong \mathcal{O}(n; n) \cong \Sigma_n.$$

The action

$$B\mathcal{O}(n; n) \times \mathcal{P}(n) \longrightarrow \mathcal{P}(n)$$

makes $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ into a symmetric sequence.

$B\mathcal{O}$-algebras also have a notion of operadic $\circ_i$ *composition*. To see this, recall that such compositions are encoded in the operad $\mathcal{O}$ by the trees with exactly two vertices, one attached to the ith incoming edge of the other. As such trees admit no large, proper subtrees, they admit no non-trivial bracketing and we have isomorphisms for any $n, m \geq 0$

$$B\mathcal{O}(m + n - 1; m, n)|_{V(T)\leq 2} \cong \mathcal{O}(m + n - 1; m, n)|_{V(T)\leq 2}$$

between the components of the tuples $(T, \sigma, \tau, \emptyset, 0)$ (resp. (T, σ, τ)) with T having at most two vertices. It follows then that $\mathcal{P}$ is equipped with operadic $\circ_i$-compositions.

A $B\mathcal{O}$-algebra is not in general an operad, however. The brackets that arise in trees with more than two vertices capture the different choices one has in iterated compositions of $\circ_i$ operations. More explicitly, if $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ is a $B\mathcal{O}$–algebra, then for any collection of elements $x_i \in \mathcal{P}(m_i)$ that decorate the vertices of a tree $(T, \sigma, \tau) \in \mathcal{O}(n; m_1, \dots, m_k)$, we have a chosen composition of those elements, namely the one determined by $(T, \sigma, \tau, \emptyset, \emptyset) \in B\mathcal{O}(n; m_1, \dots, m_k)$. This "unbracketed" tree sits in the middle of a polytope of all possible elements (T, σ, τ, B, s) for any bracketing B, as in Fig. 7. The corners of this polytope correspond to the possible maximal bracketings of T (the maximal elements of $\mathcal{B}(T)$). Just like the corners of the Stasheff polytopes give all the possible ways to bracket a k–fold multiplication, these maximal bracketings correspond precisely to the possible ways to bracket the composition of $\circ_i$ operations, which are those defined using trees with exactly two vertices. The polytopes arising from the posets of bracketing in trees can be thought of as an operadic analogue of the Stasheff polytopes.

Remark 3.11. In [24, Definition 1.1.1], a *quasi-operad* is a symmetric sequence $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ together with operadic $\circ_i$-compositions and no further structure. In this way, a $B\mathcal{O}$-algebra is an extension of a quasi-operad. The operad $B\mathcal{O}$ is closely related to the W-construction of $\mathcal{O}$, whose algebras go under the

name *lax operads*, see Section 6, where we show that $B\mathcal{O}$-algebras can be described as strictly symmetric lax operads.

4. Thickening the category Ω

We have seen that operads are $\mathcal{O}$-algebras. Also recall from Section 2.4 that operads can be described as strict Segal dendroidal spaces. The dendroidal category Ω is defined as a full subcategory of the colored operads generated by trees. To obtain a similar description of $B\mathcal{O}$–algebras as certain "homotopy dendroidal Segal spaces," we construct a topological category $\widetilde{\Omega}_0$ that is a category with the same objects as Ω but its spaces of morphisms are built using posets similar to the posets used to define $B\mathcal{O}$. Theorem 4.8 establishes that this category, $\widetilde{\Omega}_0$, has the desired property that reduced Segal $\widetilde{\Omega}_0^{op}$–spaces are precisely $B\mathcal{O}$-algebras. In Section 4.3, we then show how rectification of diagrams can be used to produce an actual Segal dendroidal space from such a homotopy version of a dendroidal space.

Given any category $\mathcal{K}$ with a discrete set of objects, Leitch [25] constructed a new category $\widetilde{\mathcal{K}}$ with the property that $\widetilde{\mathcal{K}}$–diagrams are homotopy coherent $\mathcal{K}$–diagrams. A similar enrichment (the *explosion category*) was also used by Segal [37, Appendix B] to relate his Γ–space approach to infinite loop spaces to the operadic approach of Boardman-Vogt and May. Because $\widetilde{\Omega}_0$–diagrams are homotopy coherent Ω_0–diagrams, one can expect that the category $\widetilde{\Omega}_0$ is related to this construction of Leitch applied to Ω. In Section A, we construct an equivalence between these two categories, and show that strict Segal $\widetilde{\Omega}$–spaces are closely related to $W\mathcal{O}$–algebras.

4.1. Bracketing Ω and the category $\widetilde{\Omega}_0$

Recall from Section 2.2 that the objects of Ω are planar isomorphism classes of planar rooted trees. Morphisms in Ω are compositions of inner and outer face maps, degeneracies and isomorphisms of trees. Inner face maps $\partial_e : T/e \to T$ create inner edges and correspond to operadic composition, while outer face maps are subtree inclusions and are associated to projection maps. A degeneracy creates a vertex that is adjacent to exactly two edges. The category $\widetilde{\Omega}_0$ is a version of Ω with the same set of objects, but with the realization of a poset of bracketings over each composition of inner face maps.

We define the morphism spaces of $\widetilde{\Omega}_0$ as follows. Let $g : S \to T$ be a morphism in Ω. For each vertex $v \in V(S)$, let $C_v \subset S$ denote the corolla of the vertex v that is, $C_v = i_v(C_{|in(v)|})$ where $i_v : C_{|in(v)|} \to S$ is the composition of outer faces in Ω sending the vertex of the corolla $C_{|in(v)|}$ to v. Since g is alternatively considered as a map of operads between $\Omega(S)$ and $\Omega(T)$, the image in S of C_v under g is a subtree in T, which we denote

$$g(C_v) \subset T.$$

Note that the trees $g(C_v)$ are precisely the subtrees of T that correspond to expansion of vertices into subtrees, going from S to T, or collapsed by $g^{op} : T \to S$ in the opposite category Ω^{op}. These subtrees correspond to the part of g made out of inner face maps.

For a vertex $v \in V(S)$, let B_v^g be a bracketing of $g(C_v)$ as defined in Definition 3.1. We define a poset $\mathcal{L}_g$ whose objects are tuples $(B_v^g)_{v \in V(S)}$ of bracketings of the trees $g(C_v)$. The poset relation is componentwise inclusion. Taking the realization of these posets, for each morphism g we associate the space

$$L_g := \prod_{v \in V(S)} |\mathcal{B}(g(C_v))|$$

where $\mathcal{B}(g(C_v))$ is the poset of bracketings of the tree $g(C_v)$ as defined in Definition 3.2. Note also that $|\mathcal{B}(g(C_v))| = *$ if $g(C_v)$ admits only the trivial bracketing.

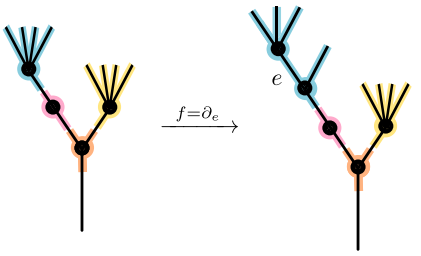
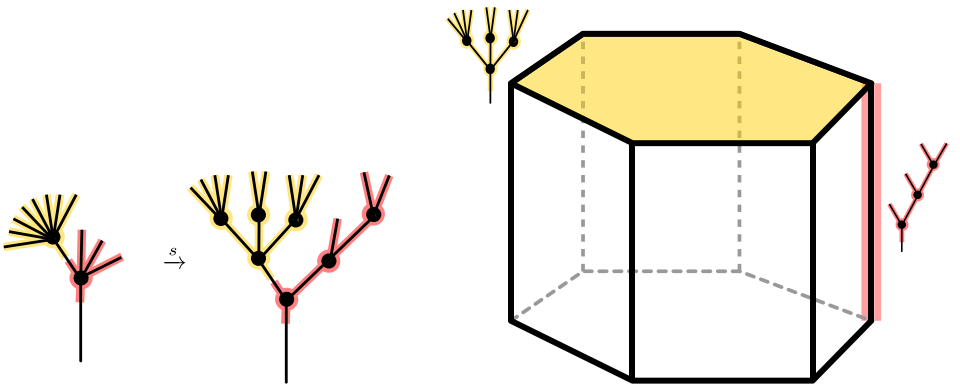


Fig. 10. Example of map f in Ω and the subtrees $f(C_v)$.



(a) Map s in Ω and the subtrees $s(C_v)$. (b) Space L_s for the map s of Figure 11(a)

Fig. 11. A map s in Ω and its corresponding space L_s.

Example 4.1. Consider the morphism $f \in \mathrm{Hom}_\Omega(R,S)$ of Fig. 10. Since the image of each corolla under f only admits a trivial bracketing,

$$L_f = \left|\mathcal{B}\left(\begin{smallmatrix}\diagup\diagdown\\ \diagup\diagdown\end{smallmatrix}\right)\right| \times \left|\mathcal{B}\left(\begin{smallmatrix}\diagup\diagdown\\ |\end{smallmatrix}\right)\right| \times \left|\mathcal{B}\left(\begin{smallmatrix}\diagup\diagdown\\ \diagup\end{smallmatrix}\right)\right| \times \left|\mathcal{B}\left(\begin{smallmatrix}\diagup\diagdown\\ \diagup\diagdown\end{smallmatrix}\right)\right| = *.$$

Example 4.2. Let s be the morphism in Fig. 11(a). By Example 3.4, if $s(C_v)$ has 3 vertices such that no vertex is connected to more than two inner edges, then $|\mathcal{B}(s(C_v))|$ is the 3rd associahedron, which is an interval. As in Example 3.5, when $s(C_v)$ is a tree whose three internal edges meet at a single vertex, the realization poset $|\mathcal{B}(s(C_v))|$ corresponds to a hexagon. Thus L_s is identified with the hexagonal prism of Fig. 11(b).

The space of morphisms between any two objects in $\widetilde{\Omega}_0$ is

$$\mathrm{Hom}_{\widetilde{\Omega}_0}(S,T) = \coprod_{g\in\mathrm{Hom}_\Omega(S,T)} L_g.$$

It remains to define composition in $\widetilde{\Omega}_0$. To do this, we first define a map of posets

$$\mathcal{L}_g \times \mathcal{L}_f \longrightarrow \mathcal{L}_{g\circ f} \tag{4.1}$$

for any two morphisms $f : R \to S$ and $g : S \to T$ in Ω, then we take the realization of this composition map to get a composition of spaces L_g. Let $(B_v^g)_{v\in V(S)} \in \mathcal{L}_g$ and $(B_w^f)_{w\in V(R)} \in \mathcal{L}_f$ be two collections of bracketings. So for each $v \in V(S)$, B_v^g is a bracketing of $g(C_v) \subset T$ and for each $w \in V(R)$, B_w^f is

a bracketing of the tree $f(C_w) \subset S$. To define the image of (4.1), we construct a bracketing of the tree $(g \circ f)(C_w)$ from the bracketings of f and g.

Fix a vertex $w \in V(R)$. For each $v \in f(C_w) \subset S$, there is the subtree $g(C_v) \subset (g \circ f)(C_w)$, as well as a bracketing B^g_v of $g(C_v)$. Also, for each bracket in $S_i \in B^f_w$, the image $g(S_i)$ is a subtree of $(g \circ f)(C_w)$. Therefore we have the following collections of subtrees in $(g \circ f)(C_w)$:

$$\begin{aligned}
\tilde{B}^g_{f(w)} &= \bigcup_{v \in f(C_w)} B^g_v \;=\; \{S_j : \; S_j \in B^g_v \text{ and } v \in f(C_w)\} \\
\tilde{B}^{g\circ f}_w &= \; \{g(C_v) : \; v \in f(C_w) \text{ and } g(C_v) \subsetneq (g \circ f)(C_w) \text{ is large}\} \\
\tilde{B}^f_w &= \; \{g(S_i) : \; S_i \in B^f_w \text{ and } g(S_y) \subsetneq (g \circ f)(C_w) \text{ is large}\}.
\end{aligned}$$

All of these are collections of proper large subtrees of $(g \circ f)(C_w)$. We set the bracketing $\tilde{B}_w$ of $(g \circ f)(C_w)$ to be the union

$$\tilde{B}_w := \tilde{B}^g_{f(w)} \cup \tilde{B}^{g\circ f}_w \cup \tilde{B}^f_w.$$

To see that $\tilde{B}_w$ is a bracketing of $(g \circ f)(C_w)$, it remains to verify that this collection is comprised of nested subtrees. First, each $B^g_v \subset \tilde{B}^g_{f(w)}$ is a bracketing of $g(C_v) \subset (g \circ f)(C_w)$, so it is nested. Moreover, the subtrees $g(C_v)$ are all disjoint and each tree of $\tilde{B}^g_{f(w)}$ is contained in a tree of $\tilde{B}^{g\circ f}_w$, so the union $\tilde{B}^g_{f(w)} \cup \tilde{B}^{g\circ f}_w$ is nested too. The $\tilde{B}^{g\circ f}_w \cup \tilde{B}^f_w$ is also nested, since each bracket $g(C_v)$ in the first set is included in each $g(S_y)$ of the second set whenever $v \in S_y$ and otherwise is disjoint from it. Hence $\tilde{B}^g_{f(w)} \cup \tilde{B}^f_w$ is also nested, and thus $\tilde{B}_w$ is nested.

Define the composition $(B^f_v)_{v\in V(S)} \circ (B^g_w)_{w\in V(R)}$ to be the collection

$$(\tilde{B}_w)_{w\in V(R)} \in \mathcal{L}_{g\circ f}.$$

Associativity of this composition is analogous to the associativity of the $B\mathcal{O}$ composition in Section 3.2. In most cases, the composition is associative because vertex substitution is associative. In a composition with a degeneracy, a vertex is removed and so a bracket may be discarded if it is no longer large. Any discarded bracket is recreated in a subsequent composition if it should not have been discarded in the total composition.

Furthermore, this composition definition respects componentwise inclusion and thus defines the poset map (4.1). The realization of this poset map induces a map

$$L_g \times L_f \;\longrightarrow\; L_{g\circ f}. \tag{4.2}$$

This defines a composition on the morphism spaces of $\tilde{\Omega}_0$.

Example 4.3. Let $f\colon R \to S$ and $g\colon S \to T$ be the morphisms in Fig. 12. Then R is a corolla $C_w = C_9$, and $f(C_w) \subset S$ is the proper subtree of S whose vertices are v_1, v_2, v_3. The images $g(C_{v_1}), g(C_{v_3}), g(C_{v_4}) \subset T$ are the corollas $C_{u_1}, C_{u_5}, C_{u_6}$ respectively and $g(C_{v_2})$ is the subtree with vertices u_2, u_3, u_4. The only images of corollas that admit a non-trivial bracketing are $f(C_w)$ and $g(C_{v_2})$. If the bracketing of $f(C_w)$ consists of the bracket B_1 in Fig. 13(a) and the bracketing of $g(C_{v_2})$ consists of B_2 in Fig. 13(b), then

$$\tilde{B}^g_{f(w)} = \{B_2\}, \quad \tilde{B}^{g\circ f}_w = \{g(C_{v_2})\}, \quad \tilde{B}^f_w = \{g(B_1)\}.$$

The bracketing $\tilde{B}_w \in \mathcal{B}((g \circ f)(C_w))$ is illustrated in Fig. 13(c).

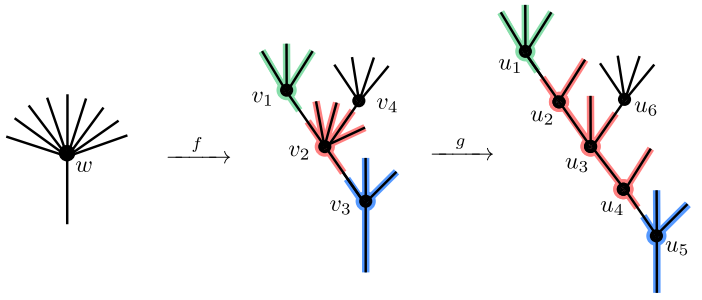
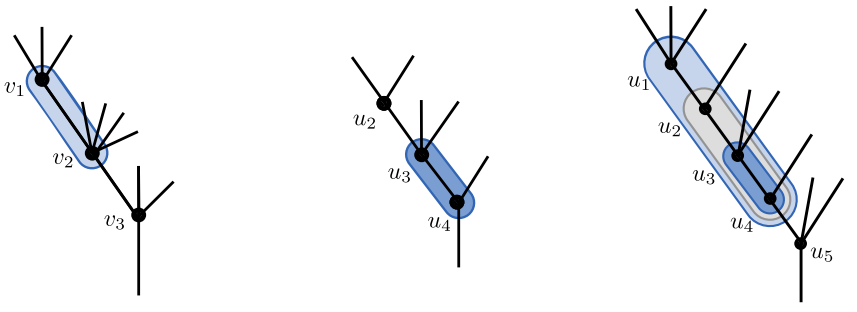


Fig. 12. Morphisms f and g in Ω.



(a) Bracketing B_1 of $f(C_w)$. (b) Bracketing B_2 of $g(C_{v_2})$. (c) Bracketing $\tilde{B}_w$ of $g \circ f(C_w)$.

Fig. 13. Example of a bracketing induced by f and g in Fig. 12.

By Example 3.4, if T_n is a tree with n vertices such that no vertex is connected to more than two inner edges, then $|\mathcal{B}(T_n)|$ is the nth associahedron. The 3rd associahedron is an interval. Thus,

$$L_f = |\mathcal{B}(f(C_w))| = \left|\mathcal{B}\left(\begin{smallmatrix}\curlyvee\\|\end{smallmatrix}\right)\right| \cong [0,1]$$
$$L_g = |\mathcal{B}(g(C_{v_1}))| \times \left|\mathcal{B}\left(\begin{smallmatrix}\curlyvee\\|\end{smallmatrix}\right)\right| \times |\mathcal{B}(g(C_{v_3}))| \cong [0,1].$$

Again by Example 3.4 and since $(g \circ f)(C_w)$ is a tree on five vertices, $L_{g\circ f} = |\mathcal{B}((g \circ f)(C_w))|$ is the 5th associahedron, which is a three dimensional polytope called an enneahedron.

Definition 4.4. The category $\widetilde{\Omega}_0$ has the same objects as Ω. Morphism spaces in $\widetilde{\Omega}_0$ are

$$\mathrm{Hom}_{\widetilde{\Omega}_0}(S,T) = \coprod_{g\in \mathrm{Hom}_\Omega(S,T)} L_g = \coprod_{g\in \mathrm{Hom}_\Omega(S,T)} \prod_{v\in V(S)} |\mathcal{B}(g(C_v))|$$

with composition (4.2) as described above.

Example 4.5. Suppose T_n is a planar tree with $(n+1)$ leaves and n vertices, each of which is connected to at most two inner edges. Let the inner edges of T_n be named $e_1, \ldots, e_{n-1}$. Morphisms $g \in \mathrm{Hom}_\Omega(C_{n+1}, T_n)$ are compositions of inner face maps $\partial_{e_1}, \ldots, \partial_{e_{n-1}}$ but since the order of the composition does not affect the total composition, there is only one such morphism g. Hence $\mathrm{Hom}_{\widetilde{\Omega}_0}(C_{n+1}, T_n) = L_g = |\mathcal{B}(T_n)|$. Thus $\mathrm{Hom}_{\widetilde{\Omega}_0}(C_{n+1}, T_n)$ is the nth associahedron by Example 3.4; the center point of the polytope is defined by the empty bracket, which is the initial object in the poset $\mathcal{B}(T_n)$.

Lemma 3.7 tells us that each bracketing space $|\mathcal{B}(g(C_v))|$ is contractible, which implies that each L_g is contractible. Let $p : \widetilde{\Omega}_0 \to \Omega$ be the functor that is the identity on objects and projects each morphism space L_g to g. By considering Ω as a discrete topological category, we have the following proposition.

Proposition 4.6. *The functor $p : \widetilde{\Omega}_0 \to \Omega$ induces a homotopy equivalence on morphism spaces.* □

This proposition will allow us to associate an actual dendroidal space to any homotopy dendroidal space in Section 4.3.

4.2. Homotopy dendroidal spaces

In Section 2.2, we defined a Segal condition for dendroidal spaces $X : \Omega^{op} \to \mathcal{S}$ using the Segal map

$$\chi\colon X(T) \longrightarrow \lim_{\mathrm{Sk}_1(T)^{op}} X.$$

We recall that the category $\mathrm{Sk}_1(T)$ has the vertices and edges of T as objects, with morphisms given by edge inclusions $\iota_e : \eta \to C_v$ into the corollas of adjacent vertices. The Segal map χ is the unique map to the limit induced by the edge and corolla inclusions

$$\iota_e : \eta \to T \quad \text{and} \quad \iota_v : C_{|v|} \to T.$$

Note that the spaces L_{ι_e} and L_{ι_v} in $\widetilde{\Omega}_0^{op}$ which lie above the morphisms ι_e and ι_v are always just a single point, so the Segal map exists unchanged for functors $X : \widetilde{\Omega}_0^{op} \to \mathcal{S}$. This allows us to make the following definition:

Definition 4.7. A *homotopy dendroidal space* X is a diagram $X : \widetilde{\Omega}_0^{op} \to \mathcal{S}$. A homotopy dendroidal space is *reduced* if $X(\eta) = *$ and satisfies the *strict* Segal condition if the Segal map is an *isomorphism* for each $\eta \neq T \in \Omega$.

Recall from Section 2.4 that one-colored operads are identified with reduced dendroidal Segal spaces via the dendroidal nerve

$$N^d : \mathcal{O}\text{-Alg} \to \mathcal{S}^{\Omega^{op}}.$$

The following theorem is a version of this nerve theorem for homotopy dendroidal spaces. We construct a functor

$$\Phi : B\mathcal{O}\text{-Alg}_{\mathcal{S}} \longrightarrow \mathcal{S}^{\widetilde{\Omega}_0^{op}}$$

and show that a reduced homotopy dendroidal space $X \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$ is strictly Segal if, and only if, $X \cong \Phi(\mathcal{P})$ for some $B\mathcal{O}$-algebra $\mathcal{P}$.

Write $(\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}$ for the full subcategory of $\widetilde{\Omega}_0$-diagrams whose objects are reduced homotopy dendroidal spaces satisfying the strict Segal condition. Then we have the following result:

Theorem 4.8. *There exists an isomorphism of categories*

$$\Phi : B\mathcal{O}\text{-Alg}_{\mathcal{S}} \xrightarrow{\cong} (\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}.$$

Proof. Given a $B\mathcal{O}$–algebra $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ with structure maps

$$\alpha_{\mathcal{P}} : B\mathcal{O}(n; m_1, \ldots, m_k) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_k) \longrightarrow \mathcal{P}(n)$$

we will define

$$\Phi(\mathcal{P}) = \Phi(\mathcal{P}, \alpha_{\mathcal{P}})\colon \widetilde{\Omega}_0^{op} \longrightarrow \mathcal{S}$$

as follows. We set $\Phi(\mathcal{P})(\eta) = *$. On objects $T \neq \eta$ of $\widetilde{\Omega}_0$, we set

$$\Phi(\mathcal{P})(T) = \prod_{w \in V(T)} \mathcal{P}(|w|).$$

Given a morphism $g : S \to T$ in Ω, we need to define maps

$$\Phi(\mathcal{P})(g) : L_g \times \prod_{w \in V(T)} \mathcal{P}(|w|) \longrightarrow \prod_{v \in V(S)} \mathcal{P}(|v|).$$

We proceed one vertex v at a time. As $L_g = \prod_{v \in V(S)} |\mathcal{B}(g(C_v))|$, at each $v \in V(S)$ we have projection maps

$$\pi_v : L_g \times \prod_{w \in V(T)} \mathcal{P}(|w|) \longrightarrow |\mathcal{B}(g(C_v))| \times \prod_{w \in V(g(C_v))} \mathcal{P}(|w|).$$

An application of the structure map $\alpha_{\mathcal{P}}$ defines a map

$$\alpha_v : |\mathcal{B}(g(C_v))| \times \prod_{w \in V(g(C_v))} \mathcal{P}(|w|) \longrightarrow \mathcal{P}(|v|). \qquad (*)$$

Indeed, an element of $|\mathcal{B}(g(C_v))|$ is a weighted bracketing (B, t) of the subtree $g(C_v) \subset T$. Because T is a planar tree, $g(C_v)$ inherits a planar structure. We consider $g(C_v)$ as an element of $\mathcal{O}$ by picking an ordering σ of its vertices $\{w_1, \ldots, w_k\}$, and labeling its leaves via the map τ ordering them according to its planar structure. This way $((g(C_v), \sigma, \tau), B, t)$ is an element of $B\mathcal{O}(|v|; |w_1|, \ldots, |w_k|)$. To define the map $(*)$, we first order the factors $\mathcal{P}(|w|)$ for $w \in V(g(C_v))$, in accordance with our chosen σ, and then apply $\alpha_{\mathcal{P}}$ noting that our choice of ordering does not affect the result by the equivariance of $\alpha_{\mathcal{P}}$. Finally we act on the resulting element of $\mathcal{P}(|v|)$ by the permutation induced by g that identifies the inputs of v with the leaves of $g(C_v)$, comparing the labeling τ from the planar structure of T to the planar ordering of $in(v)$ (which comes from the planar structure of S). We now set

$$\Phi(\mathcal{P})(g) := (\alpha_v \circ \pi_v)_{v \in V(S)}.$$

The fact that $\Phi(\mathcal{P})$ commutes with composition follows from the fact that composition in $\widetilde{\Omega}_0$ is defined exactly as the operadic composition of $B\mathcal{O}$ by taking the union of the brackets from the first morphism which remain large after applying the second morphisms, the brackets from the second morphism, and new "middle brackets", the images of the middle corollas, if they are large. It follows then that $\Phi(\mathcal{P}) : \widetilde{\Omega}_0 \to \mathcal{S}$ is a functor. Since $\Phi(\mathcal{P})(\eta) = *$, the Segal map is the map

$$\Phi(\mathcal{P})(T) \longrightarrow \prod_{v \in V(T)} \Phi(\mathcal{P})(C_v)$$

induced by the inclusions of the corollas. It is an isomorphism by definition of $\Phi(\mathcal{P})$.

The data required in the definition of the homotopy dendroidal space $\Phi(\mathcal{P})$ is the underlying symmetric sequence $\mathcal{P} = \{\mathcal{P}(m)\}$, the $B\mathcal{O}$-algebra structure maps $\alpha_{\mathcal{P}}$ and the projection maps π_v, all of which are natural under maps of $B\mathcal{O}$-algebras. Thus, the assignment $\mathcal{P} \mapsto \Phi(\mathcal{P})$ defines a functor

$$\Phi : B\mathcal{O}\text{−Alg}_{\mathcal{S}} \longrightarrow (\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}.$$

It remains to show that the functor Φ is an isomorphism of categories. Given two $B\mathcal{O}$-algebras $\mathcal{P}$ and $\mathcal{Q}$ with $\Phi(\mathcal{P}) = \Phi(\mathcal{Q})$, the underlying symmetric sequences $\{\mathcal{P}(n)\}_{n \geq 0}$ and $\{\mathcal{Q}(n)\}_{n \geq 0}$ are necessarily equal,

being the value at the corollas C_n and the corolla isomorphisms $\mathrm{Hom}_{\widetilde{\Omega}_0}(C_n, C_n) \cong \mathrm{Hom}_{\Omega}(C_n, C_n) \cong \Sigma_n$. Moreover, the structure maps $\alpha_{\mathcal{P}}$ and $\alpha_{\mathcal{Q}}$ likewise must agree as they agree with the evaluation of $\Phi(\mathcal{P}) = \Phi(\mathcal{Q})$ at corresponding morphisms in $\widetilde{\Omega}_0^{op}$. It follows that Φ is injective.

On the other hand, given any $X \in (\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}$, we can construct a $B\mathcal{O}$-algebra $\mathcal{P}_X$ by setting $\mathcal{P}_X(n) = X(C_n)$ with a symmetric group action induced by the image under X of the isomorphisms of C_n in $\widetilde{\Omega}_0$. The $B\mathcal{O}$-algebra structure maps of $\mathcal{P}_X$ are defined using the above identification of the spaces $B\mathcal{O}(n; m_1, \ldots, m_k)$ with morphism spaces in $\widetilde{\Omega}_0$. The fact that X is a functor will then give that $\mathcal{P}_X$ is a $B\mathcal{O}$–algebra. Thus the functor Φ is surjective. □

4.3. Rectifying homotopy dendroidal spaces

We have just seen that $B\mathcal{O}$-algebras correspond to homotopy dendroidal spaces satisfying the strict Segal condition. In this section we will show how to produce, from a $B\mathcal{O}$-algebra, an actual dendroidal space satisfying a weak version of the Segal condition.

A commutative diagram in a topological category $\mathcal{S}$ is a functor from a discrete category $\mathcal{K}$ to $\mathcal{S}$. A homotopy commutative diagram can be similarly described as a functor from a topological category $\widetilde{\mathcal{K}}$ to $\mathcal{S}$, with the homotopies encoded as paths in the spaces of morphisms. In this language, a homotopy commutative diagram $X : \widetilde{\mathcal{K}} \to \mathcal{S}$ can be *rectified*, or *strictified*, to a functor $X' : \mathcal{K} \to \mathcal{S}$ precisely when there is an equivalence $p : \widetilde{\mathcal{K}} \to \mathcal{K}$. We briefly recall this rectification of diagrams, which was used by Segal in [37], and treated in great generality by Dwyer and Kan [16]; see also [40, Sec 2] for a detailed account of what we will use here. Our examples will be $\mathcal{K} = \Omega$ with $\widetilde{\mathcal{K}} = \widetilde{\Omega}_0$.

Let $p : \widetilde{\mathcal{K}} \to \mathcal{K}$ be a functor between categories enriched over topological spaces. There is an induced functor

$$p^* : \mathcal{S}^{\mathcal{K}} \longrightarrow \mathcal{S}^{\widetilde{\mathcal{K}}}$$

defined by precomposition with p. The homotopy left Kan extension defines also a functor

$$p_! : \mathcal{S}^{\widetilde{\mathcal{K}}} \longrightarrow \mathcal{S}^{\mathcal{K}}$$

that can be explicitly given as follows: given a diagram $Y \in \mathcal{S}^{\widetilde{\mathcal{K}}}$, its evaluation at an object d of $\mathcal{K}$ is the realization of a simplicial space with space of k–simplices

$$(p_!Y(d))_k = \coprod_{c_0,\ldots,c_k \in ob(\widetilde{\mathcal{K}})} Y(c_0) \times \mathrm{Hom}_{\widetilde{\mathcal{K}}}(c_0, c_1) \times \cdots \times \mathrm{Hom}_{\widetilde{\mathcal{K}}}(c_{k-1}, c_k) \times \mathrm{Hom}_{\mathcal{K}}(c_k, d).$$

Lemma 4.9. *[40, Proposition 2.1] Let $p\colon \widetilde{\mathcal{K}} \to \mathcal{K}$ be a functor inducing a homotopy equivalence of morphism spaces, and let $Y\colon \widetilde{\mathcal{K}} \to \mathcal{S}$ be a diagram, with $p_!Y\colon \mathcal{K} \to \mathcal{S}$ its rectification as defined above. Then there exists a zig-zag of natural transformations $p^*p_!Y \leftarrow \overline{p^*p_!Y} \to Y$, which induces a homotopy equivalence on objects: $p^*p_!Y(d) \simeq \overline{p^*p_!Y}(d) \simeq Y(d)$.*

In the statement, $\overline{p^*p_!Y}$ is an explicit functor from $\widetilde{\mathcal{K}}$ to $\mathcal{S}$ associated to Y given by a two-sided bar construction (details in the proof of [40, Proposition 2.1]).

Proposition 4.6 states that the functor $p : \widetilde{\Omega}_0 \to \Omega$ induces a homotopy equivalence of morphism spaces. Below, we apply Lemma 4.9 to describe weak Segal dendroidal spaces that arise as the rectification of a homotopy dendroidal Segal space $Y \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$ in the following sense:

Consider the functors

$$\mathcal{S}^{\widetilde{\Omega}_0^{op}} \underset{p^*}{\overset{p_!}{\rightleftarrows}} \mathcal{S}^{\Omega^{op}}.$$

For any small category $\mathcal{K}$, the category of diagrams $\mathcal{S}^{\mathcal{K}^{op}}$ admits a projective model structure in which weak equivalences and fibrations are defined entrywise [23, 11.6.1]. In particular, when $\mathcal{S}$ is the category of topological spaces, every object is fibrant in this model structure. An application of [26, Proposition A.3.3.7] implies that $p_!$ and p^* are the left and right Quillen functor of a Quillen equivalence.

Proposition 4.10. *Let $\mathcal{S}$ be the category of topological spaces, and let $Y \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$ be a homotopy dendroidal space. Then the rectification $p_!Y \in \mathcal{S}^{\Omega^{op}}$ is a dendroidal space such that $p_!Y(\eta) \simeq *$, and such that the weak Segal map*

$$\tilde{\chi} : p_!Y(T) \xrightarrow{\simeq} \operatorname{holim}_{Sk_1(T)^{op}} p_!Y(C_v)$$

is a homotopy equivalence for each $\eta \neq T \in \Omega$ if, and only if, the same two properties hold for the homotopy dendroidal space Y.

Proof. Note first that, because p is the identity on objects, we have $p^*X(T) = X(T)$ for any $X : \Omega^{op} \to \mathcal{S}$ and any $T \in Obj(\Omega) \equiv Obj(\widetilde{\Omega}_0)$. It follows that $p_!Y(\eta) \simeq *$ if and only if $Y(\eta) \simeq *$ as $p_!Y(\eta) = p^*p_!Y(\eta) \simeq Y(\eta) \simeq *$.

We are left to show that the weak Segal map $\tilde{\chi}$ is a weak equivalence for every $T \neq \eta$ for $p_!Y : \Omega^{op} \to \mathcal{S}$ if and only if it is the case for the original functor $Y : \widetilde{\Omega}_0^{op} \to \mathcal{S}$. Recall that the Segal map χ and its weak version $\tilde{\chi}$ for both Ω^{op}–spaces and $\widetilde{\Omega}_0^{op}$–spaces are induced by corolla and edge inclusions ι_v and ι_e in $\widetilde{\Omega}_0$ and Ω, respectively. As p takes each map ι_v and ι_e in $\widetilde{\Omega}_0$, to the corresponding map in Ω, and p is the identity on objects, for any $X : \Omega^{op} \to \mathcal{S}$, we have

$$\begin{array}{ccc} X(T) & \xrightarrow{\tilde{\chi}} & \operatorname{holim}_{\mathrm{Sk}_1(T)^{op}} X(C_v) \\ \| & & \| \\ p^*X(T) & \xrightarrow{\tilde{\chi}} & \operatorname{holim}_{\mathrm{Sk}_1(T)^{op}} p^*X(C_v) \end{array}$$

in which the two horizontal maps describe the exact same map in $\mathcal{S}$.

The natural equivalences of functors $p^*p_!Y \leftarrow \overline{p^*p_!Y} \to Y$ of Lemma 4.9 give us the vertical homotopy equivalences in the following commuting diagram in $\mathcal{S}$

$$\begin{array}{ccc} p^*p_!Y(T) & \xrightarrow{\tilde{\chi}} & \operatorname{holim}_{\mathrm{Sk}_1(T)^{op}} p^*p_!Y(C_v) \\ \simeq\uparrow & & \uparrow\simeq \\ \overline{p^*p_!Y}(T) & \xrightarrow{\tilde{\chi}} & \operatorname{holim}_{\mathrm{Sk}_1(T)^{op}} \overline{p^*p_!Y}(C_v) \\ \simeq\downarrow & & \downarrow\simeq \\ Y(T) & \xrightarrow{\tilde{\chi}} & \operatorname{holim}_{\mathrm{Sk}_1(T)^{op}} Y(C_v), \end{array}$$

which gives the result using the previous remark in the case $X = p_!Y$. □

Remark 4.11. We used in the proof of Proposition 4.10 that every topological $\widetilde{\Omega}_0$ diagram is projectively fibrant to obtain that the homotopy limits preserve homotopy equivalences.

As the dendroidal category is a generalized Reedy category [2, Example 1.6], there is a Reedy model structure on the category of reduced dendroidal spaces. We do not currently have an enriched generalized Reedy model structure on $\widetilde{\Omega}_0$-spaces. An advantage of the Reedy model structure is that homotopy limits of fibrant objects are weakly equivalent to the limit. One expects that, using such Reedy model structures, one could prove a version of Proposition 4.10 with the actual Segal map χ replacing the weak Segal map $\tilde{\chi}$.

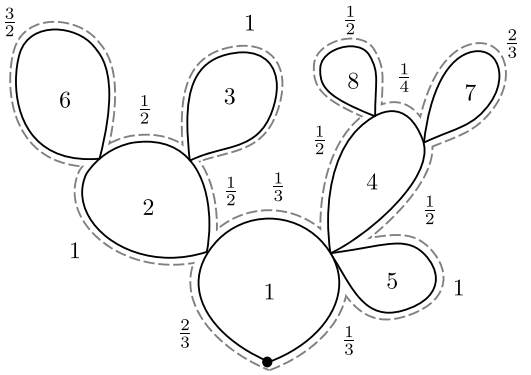


Fig. 14. Cactus with 8 lobes, its outside circle indicated by the dotted line.

5. Normalized Cacti as an infinity operad

The first goal of this section is to define an operad MS^+ and show that, despite not being an operad itself, normalized cacti and their composition can be described as elements and compositions inside MS^+. In Section 5.2, we will use MS^+ to show that normalized cacti and the normalized composition extends to define a $B\mathcal{O}$-algebra structure. Using the results of Sections 3 and 4, this implies that we have an explicit construction of an ∞-operad with underlying sequence the spaces $\mathcal{C}\mathrm{act}^1(n)$.

A cactus is a configuration of circles of various lengths attached to each other in a treelike fashion. In the original definition by Voronov [39, Section 2.7], there is a global basepoint associated to the "outside circle" of the cactus, as well as a basepoint for each circle (or *lobe*). A *spineless cactus* is a variant introduced by Kaufmann [24, Section 2.3], where the basepoint of each lobe is its closest point to the global basepoint along the outside circle. See Fig. 14 for an example. The space of all spineless cacti with k lobes is denoted $\mathcal{C}\mathrm{act}(k)$. The symmetric group acts on this space by permuting the labels of the lobes. The symmetric sequence $\mathcal{C}\mathrm{act} = \{\mathcal{C}\mathrm{act}(k)\}_{k\geq 0}$ is given a composition

$$\circ_i \colon \mathcal{C}\mathrm{act}(k) \times \mathcal{C}\mathrm{act}(j) \to \mathcal{C}\mathrm{act}(k+j-1)$$

that is defined by inserting the second cactus into the ith lobe of the first cactus and aligning its global basepoint with the basepoint of the ith lobe. The insertion is done by rescaling the second cactus so that its total length is equal to the length of the ith lobe of the first cactus, then identifying the outside circle of the second cactus with the ith lobe of the first cactus. This composition makes $\mathcal{C}\mathrm{act}$ into an operad, which is equivalent to the little 2-discs operad [24, Section 3.2.1]. A rigorous definition of this composition requires close attention to subtleties and we refer to [24, Section 2] for precise definitions.

The space of *normalized cacti* $\mathcal{C}\mathrm{act}^1(k) \subset \mathcal{C}\mathrm{act}(k)$ is the subspace of spineless cacti whose lobes all have length equal to 1 ([24, Definition 2.3.1]). They form a symmetric sequence $\mathcal{C}\mathrm{act}^1 = \{\mathcal{C}\mathrm{act}^1(k)\}_{k\geq 0}$. Composition of normalized cacti

$$\circ_i \colon \mathcal{C}\mathrm{act}^1(k) \times \mathcal{C}\mathrm{act}^1(j) \to \mathcal{C}\mathrm{act}^1(k+j-1), \tag{5.1}$$

is defined by reparameterizing the ith lobe of a cactus $x \in \mathcal{C}\mathrm{act}^1(k)$ to have length j, then identifying this lobe with the outer circle of the second cactus $y \in \mathcal{C}\mathrm{act}^1(j)$ and aligning their basepoints. In contrast to $\mathcal{C}\mathrm{act}$, the ith lobe of the first cactus is scaled instead of scaling the second cactus to the length of the ith lobe. See Fig. 15 for an example. This composition is not associative [24, Remark 2.3.19], as illustrated in Fig. 16. Thus $\mathcal{C}\mathrm{act}^1$ is *not* an operad.

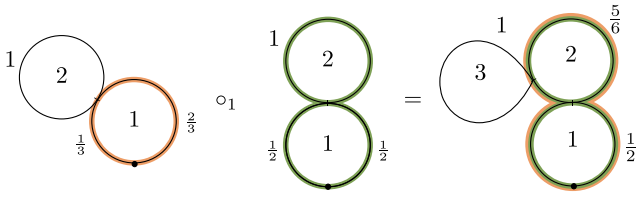


Fig. 15. A composition of normalized cacti.

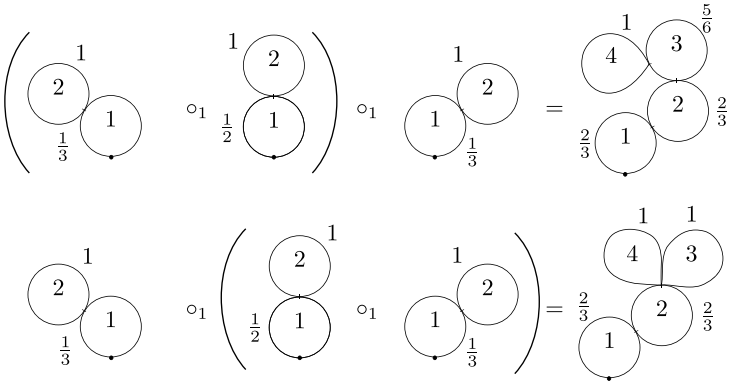


Fig. 16. Non-associativity in $\mathcal{C}\mathrm{act}^1$.

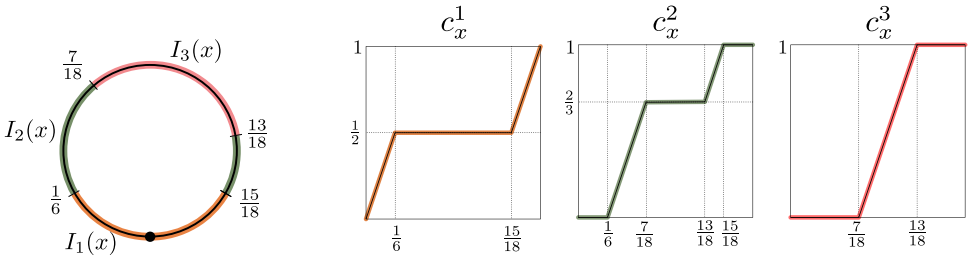
Remark 5.1 *(Composition in the graph cobordism category).* This composition of normalized cacti is highly relevant to the graph model of the cobordism category of Riemann surfaces mentioned in the introduction of the paper. To model the gluing of cobordisms, we use graphs to represent surfaces with potentially many incoming and outgoing boundary components. Normalized cacti are a simple case of this model, representing surfaces of genus zero with potentially many inputs but always just one output. Two surfaces are glued by attaching the incoming boundaries of the first surface to the outgoing boundaries of the second. According to [19] (see also [17, Theorem A]), we may assume that all incoming boundaries of a surface are disjoint embedded circles in the corresponding graph (like the lobes of the cactus, if they were pulled apart a little bit). Since these boundary circles are disjoint in the graph, they can be scaled independently to each match the length of an outgoing boundary in the graph of the second surface, just like scaling the ith lobe of the first cactus in $\mathcal{C}\mathrm{act}^1$ composition. There is no obvious way to define a "$\mathcal{C}$act-like" composition for such more general graphs, because the outgoing circles of the second surface cannot be assumed to be disjoint, and hence cannot be scaled independently to the appropriate length. (See [17, Section 3.3] for more details about this gluing of fat graphs.)

5.1. An operad MS^+ that contains $\mathcal{C}\mathrm{act}^1$

In their proof of the Deligne conjecture, McClure and Smith [29,30] introduced an operad MS equivalent the little 2-discs operad.[1] Later, Salvatore [36, Section 4] used similar methods to show directly that the operad MS is equivalent to the non-normalized cactus operad $\mathcal{C}\mathrm{act}$. Here we will define a variant of MS called MS^+, and, following [36], start by showing that it is an operad by proving that it embeds in $\mathrm{CoEnd}(S^1)$. We then show that normalized cacti are a subspace of the underlying symmetric sequence of MS^+ and that their composition can be written in terms of compositions in MS^+.

The space of operations $MS^+(k)$ is built from a space $\mathcal{F}(k)$, which we will show is homeomorphic to $\mathcal{C}\mathrm{act}^1(k)$. In fact, we can think of an element of $\mathcal{F}(k)$ as the outer circle of a cactus.

[1] The operad MS is denoted $\mathcal{C}'$ in [29, Section 5].



(a) $x \in \mathcal{F}(3)$. (b) Maps c_x^i.

Fig. 17. Element of $x \in \mathcal{F}(3)$ and associated projections.

Definition 5.2. [36, Definition 4.1] Let $S^1 = [0,1]/0\sim 1$ be the circle of circumference equal to 1. Define $\mathcal{F}(k)$ as the space of partitions $x = (I_1(x), \ldots, I_k(x))$ of S^1 into closed 1-manifolds $I_j(x) \subset S^1$, each of which have total length $\frac{1}{k}$, with pairwise disjoint interiors, and such that

(∗) there does not exist a cyclically ordered 4-tuple $(z_1; z_2; z_3; z_4) \in S^1$ with $z_1, z_3 \in \mathring{I}_j(x)$ and $z_2, z_4 \in \mathring{I}_i(x)$, for $j \neq i$.

For an example, see Fig. 17(a). The topology of $\mathcal{F}(k)$ is induced by the metric measuring the size of the overlap between partitions: for $x, y \in \mathcal{F}(k)$, $d(x,y) = 1 - \sum_{j=1}^{k} \ell(I_j(x) \cap I_j(y))$ for ℓ the length function on submanifolds of S^1.

The symmetric group Σ_k acts on $\mathcal{F}(k)$ by reindexing the labels of the 1-manifolds.

Definition 5.3. Given an element $x \in \mathcal{F}(k)$, we associate to each $I_j(x)$ a projection map $c_x^j : S^1 \to S^1$ that takes the quotient of S^1 under the identification of all the points in the same path component of $\overline{S^1 \setminus \mathring{I}_j}$ and then scales this circle by a factor of k. See Fig. 17(b) for an example. The *cactus map* $c_x \colon S^1 \to (S^1)^k$ is the collection of maps $c_x := (c_x^1, \ldots, c_x^k)$. Then there is a map

$$\begin{aligned} c : \mathcal{F}(k) &\longrightarrow Map(S^1, (S^1)^k) \\ x &\longmapsto c_x = (c_x^1, \ldots, c_x^k) \colon S^1 \to (S^1)^k. \end{aligned}$$

For any $x \in \mathcal{F}(k)$, we also use x to denote the configuration of circles in the image of the cactus map $c_x \colon S^1 \to (S^1)^k$. Condition (∗) in Definition 5.2 guarantees that this configuration is treelike, as it forces the submanifolds $I_j(x)$ to be nested. The global basepoint of x is the image of the basepoint of S^1 and a planar structure is induced by the orientation of the source S^1 (see [36, Definition 4.2]). Since each part of a partition $x \in \mathcal{F}(k)$ has equal length, x is a normalized cactus as shown in Fig. 18. This is the sketch of the proof for the next lemma.

Lemma 5.4. *[36, Section 4] For each $k \geq 1$, the space $\mathcal{F}(k)$ is homeomorphic to $\mathcal{C}\mathrm{act}^1(k)$.*

Recall the coendomorphism operad $\mathrm{CoEnd}(S^1)$ from Example 2.2, whose underlying symmetric sequence is a collection of $\mathrm{CoEnd}(k)(S^1) := \mathrm{Map}(S^1, (S^1)^k)$. We use the map

$$c : \mathcal{C}\mathrm{act}^1(k) \cong \mathcal{F}(k) \hookrightarrow \mathrm{Map}(S^1, (S^1)^k) = \mathrm{CoEnd}(S^1)(k)$$

to define an embedding of symmetric sequences.

Lemma 5.5. *The map $c : \mathcal{F}(k) \to Map(S^1, (S^1)^k)$ is a topological embedding.*

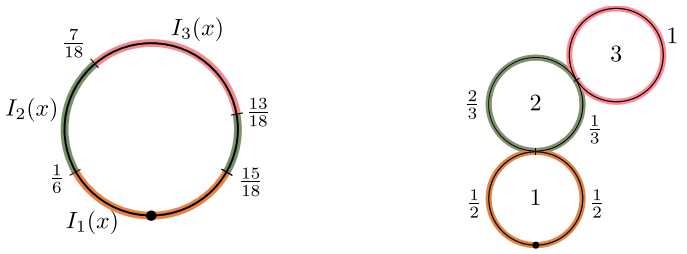


Fig. 18. An element x of $\mathcal{F}(3)$ and the corresponding normalized cactus c_x.

Proof. We first check injectivity. Given a map $c_x = (c_x^1, \ldots, c_x^k)$ in the image of c, we can completely determine $x \in \mathcal{F}(k)$. We know that each c_x^j is a "step-map" with linear of slope k over its non-constant parts, by the definition of c. (See Fig. 17(b).) Then $I_j(x)$ is precisely the subset of points of S^1 where the derivative $(c_x^j)'$ equals k. Continuity of c follows from the fact that the topology in the mapping space can be defined using the convergence metric, using likewise the metric on S^1. □

This embedding of symmetric sequences does not extend to an embedding of operads. As already mentioned, $\mathcal{C}\mathrm{act}^1$ is not an operad and one can check that the image of c is not a suboperad of $\mathrm{CoEnd}(S^1)$. Indeed, if we compose two elements in $\mathrm{CoEnd}(S^1)$ that came from elements of $\mathcal{F}$, their composition will not be in the image of any $\mathcal{F}(k)$ because all elements in the image of $\mathcal{F}(k)$ are piecewise linear graphs of slope 0 or k, and this property is not preserved by the composition in $\mathrm{CoEnd}(S^1)$.

Here we define the symmetric sequence $MS^+ = \{MS^+(k)\}_{k\geq 0}$, which is built from $\mathcal{F}(k)$ and a collection $Mon^+(I, \partial I)$ of scaling maps on the interval I. It has the important property that $\mathcal{C}\mathrm{act}^1(k) \subset MS^+(k)$ for each $k \geq 0$.

Definition 5.6 *(MS^+ as a symmetric sequence).* For each $k \geq 0$, we define the space $MS^+(k)$ as

$$MS^+(0) = *$$
$$MS^+(k) = \mathcal{F}(k) \times Mon^+(I, \partial I)$$

where $Mon^+(I, \partial I)$ is the space of strictly monotone self-maps of I that restrict to the identity on ∂I. We consider $Mon^+(I, \partial I)$ as a subspace of the space of self-maps of $S^1 = I/\partial I$. For each k, there is an action of the symmetric group Σ_k on $MS^+(k)$ by the reindexing of the labels of the 1-manifolds in $\mathcal{F}(k)$.

Remark 5.7. The operad MS that appears in [29,30,36] has an underlying symmetric sequence obtained by replacing $Mon^+(I, \partial I)$ by the larger space $Mon(I, \partial I)$ of weakly monotone maps. The inclusion $MS^+ \hookrightarrow MS$ is a homotopy equivalence as both $Mon(I, \partial I)$ and $Mon^+(I, \partial I)$ are contractible (in fact, they are both convex).

In order to show that MS^+ is an operad, we start by showing that each space of operations $MS^+(k)$ embeds in $\mathrm{CoEnd}(S^1)(k)$. We also check that the operad composition of $\mathrm{CoEnd}(S^1)$ preserves the image of MS^+, and hence is a suitable composition for MS^+, thus making MS^+ a suboperad of $\mathrm{CoEnd}(S^1)$.

Proposition 5.8. *There is a topological embedding $\phi : MS^+(k) \to \mathrm{CoEnd}(S^1)(k)$ that sends $(x, f) \in MS^+(k)$ to the composite*

$$S^1 \xrightarrow{f} S^1 \xrightarrow{c_x} (S^1)^k$$

where c_x is the cactus map as in Definition 5.3.

A version of Proposition 5.8 is stated for the operad MS in [36, Section 4]. As we rely heavily on this result we give more complete details here.

Proof. The fact that ϕ is continuous follows from Lemma 5.5, so we are left to check that ϕ is injective. Let $x \in \mathcal{F}(k)$. Recall that the map $c_x = (c_x^1, \dots, c_x^k)\colon S^1 \to (S^1)^k$ is a collection of "step-maps" of linear of slope k over its non-constant parts. Each map $c_x^j\colon S^1 \to S^1$ identifies points in the same path component of $\overline{S^1 \setminus \mathring{I}_j(x)}$ and linearly takes $I_j(x)$ (of length $1/k$) to a circle of circumference 1. So, these maps satisfy that

$$\frac{1}{k}\sum_{j=1}^{k} c_x^j = \mathrm{Id}_{S^1}.$$

In particular, this means that if $c_x \circ f = c_y \circ g$, then

$$f = (\frac{1}{k}\sum_{j=1}^{k} c_x^j) \circ f = \frac{1}{k}\sum_{j=1}^{k}(c_x^j \circ f) = \frac{1}{k}\sum_{j=1}^{k}(c_x^j \circ g) = (\frac{1}{k}\sum_{j=1}^{k} c_y^j) \circ g = g.$$

Moreover, as f, g are strictly monotone and hence invertible, for each $j = 1, \dots, k$,

$$c_x^j = (c_x^j \circ f) \circ f^{-1} = (c_y^j \circ g) \circ f^{-1} = (c_y^j \circ g) \circ g^{-1} = c_y^j.$$

This shows that $c_x = c_y$ and therefore the map is injective. □

Proposition 5.8 shows that MS^+ is a symmetric subsequence of CoEnd and this next lemma shows that the operad structure maps of CoEnd preserve this structure.

Lemma 5.9. *The operad structure maps of* CoEnd *preserve the symmetric subsequence* MS^+.

Proof. It suffices to consider the composition operations $\circ_i$ in CoEnd as defined in Example 2.2. Given (x, f) and (y, g) in MS^+, we need to check that the composition

$$S^1 \xrightarrow{f} S^1 \xrightarrow{c_x} (S^1)^k \xrightarrow{1\times g\times 1} (S^1)^k \xrightarrow{1\times c_y\times 1} (S^1)^{j+k-1} \tag{5.2}$$

is in the image of MS^+, where $1 \times g \times 1$ denotes the map where g acts only on the ith circle. For this, we will show two things:

(i) $(1 \times g \times 1) \circ c_x = c_{\tilde{x}} \circ \tilde{g}$, for some $\tilde{g} \in Mon^+(I, \partial I)$ and $\tilde{x} \in \mathcal{F}(k)$,
(ii) $(1 \times c_y \times 1) \circ c_x = c_z \circ h_{x,y}$ for some $h_{x,y} \in Mon^+(I, \partial I)$ and $z \in \mathcal{F}(j + k - 1)$.

For statement (i), the map $(1 \times g \times 1)$ acts only on the ith circle, so in the composition with c_x it only affects points in $I_i(x)$. Recall that we identify S^1 with $I/\partial I$. Suppose $I_i(x) = J_1 \sqcup \cdots \sqcup J_r$ with each J_s a subinterval of $[0, 1]$ and $I_i(x)$ of total length $\frac{1}{k}$. We obtain $\tilde{x} \in \mathcal{F}(k)$ from x by replacing each subinterval J_s by an interval $\tilde{J}_s$ of length $\frac{1}{k}\ell(g(c_x^i(J_s))))$ and shifting each path component of $[0, 1] \setminus \mathring{I}_i(x)$ accordingly. Then $I_i(\tilde{x}) = \tilde{J}_1 \sqcup \cdots \sqcup \tilde{J}_r$. This makes sense as, by construction, the total length of $I_i(\tilde{x})$ is again $\frac{1}{k}$. The map $\tilde{g}$ is defined as the canonical identification of x with $\tilde{x}$ that maps $I_i(x)$ to $I_i(\tilde{x})$ for all $i \in \{1, \dots, k\}$. See Fig. 19 for an example.

For statement (ii), we consider a composition $(1 \times c_y \times 1) \circ c_x : S^1 \to (S^1)^{j+k-1}$ with c_y on the ith position. Such a composition maps the rth partition $I_r(x)$, for $r \neq i$, to the rth (if $r < i$) or $(r + k - 1)$st (if $r > i$) component in the target by a slope k map, while $I_i(x)$ is mapped by slope jk maps to the remaining

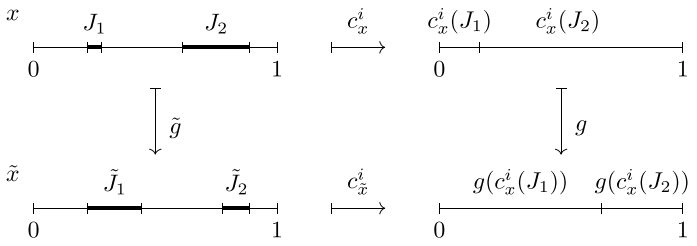


Fig. 19. An example of the commutative diagram $g \circ c_x^i = c_{\tilde{x}}^i \circ \tilde{g}$.

components. Let $h_{x,y} : S^1 \to S^1$ be the rescaling map that scales each $I_r(x)$ by a factor $\frac{k}{j+k-1}$ for $r \neq i$, and $I_i(x)$ by a factor $\frac{jk}{j+k-1}$. Then the image under $h_{x,y}$ of each $I_r(x)$ will be of size $\frac{1}{j+k-1}$ for $r \neq i$, while $I_i(x)$ will have image of total size $\frac{j}{j+k-1}$. Note that this gives a well-defined map in $Mon^+(I, \partial I)$ as the sum of the length of the images $h_{x,y}(I_r(x))$ is $(k-1)\frac{1}{j+k-1} + \frac{j}{j+k-1} = 1$. Subdividing the image under $h_{x,y}$ of $I_i(x)$ into j parts as prescribed by y, together with the images of the other $I_r(x)$'s, then defines $z \in \mathcal{F}(j+k-1)$. The relation $(1 \times c_y \times 1) \circ c_x = c_z \circ h_{x,y}$ holds by construction.

By putting (i) and (ii) together, the composition in (5.2) is given by $(h_{\tilde{x},y} \circ \tilde{g} \circ f, z) \in MS^+$. □

Therefore we have shown that MS^+ is a suboperad of $\mathrm{CoEnd}(S^1)$ via the embedding ϕ in Proposition 5.8.

Definition 5.10 *(MS^+ as an operad).* The symmetric sequence $MS^+ = \{MS^+(k)\}_{k\in\mathbb{N}}$ becomes an operad with composition

$$(x, f) \bullet_i (y, g) := \phi^{-1}(\phi(x, f) \circ_i \phi(y, g)) \tag{5.3}$$

where $\circ_i$ is the composition in $\mathrm{CoEnd}(S^1)$ defined in (5.2), and the pre-image exists as a consequence of Lemma 5.9.

We will often use scaling maps in $Mon^+(I, \partial I)$ to encode the scaling of lobes in the composition of normalized cacti. Given a partition $x = (I_1(x), \ldots, I_k(x)) \in \mathcal{F}(k) \cong \mathcal{C}\mathrm{act}^1(k)$, and natural numbers $m_1, \ldots, m_k \geq 0$, we let

$$g = g(x; m_1, \ldots, m_k) : S^1 \longrightarrow S^1 \tag{5.4}$$

be the element of $Mon^+(I, \partial I)$ that scales $I_j(x)$ by the factor $\frac{km_j}{m_1+\cdots+m_k}$, $1 \leq j \leq k$. Each $I_j(x)$ has total length $\frac{1}{k}$, so the image of $I_j(x)$ will have length $\frac{m_j}{m_1+\cdots+m_k}$ for each $1 \leq j \leq k$. (See Fig. 20 for an example.) Note that $g(x; 1, \ldots, 1) = \mathrm{id}$ is just the identity map on S^1.

We will now show that the $\circ_i$-compositions of normalized cacti from (5.1),

$$\circ_i : \mathcal{C}\mathrm{act}^1(k) \times \mathcal{C}\mathrm{act}^1(j) \longrightarrow \mathcal{C}\mathrm{act}^1(k + j - 1),$$

and, more generally, the $\mathcal{C}\mathrm{act}^1$–composition maps

$$\gamma_{\mathcal{C}\mathrm{act}^1} : \mathcal{C}\mathrm{act}^1(k) \times \mathcal{C}\mathrm{act}^1(m_1) \times \cdots \times \mathcal{C}\mathrm{act}^1(m_k) \longrightarrow \mathcal{C}\mathrm{act}^1(\Sigma_{i=1}^k m_i),$$

are restrictions of the corresponding compositions of appropriately chosen elements of MS^+.

For a collection of cacti $x \in \mathcal{C}\mathrm{act}^1(k)$ and $y_j \in \mathcal{C}\mathrm{act}^1(m_j)$, $1 \leq j \leq k$, the quasi-operad composition $\gamma_{\mathcal{C}\mathrm{act}^1}(x; y_1, \ldots, y_k)$ scales each lobe of x so that the ith lobe now has length m_i, and then inserts (without any further scaling) each y_i in place of the ith scaled lobe.

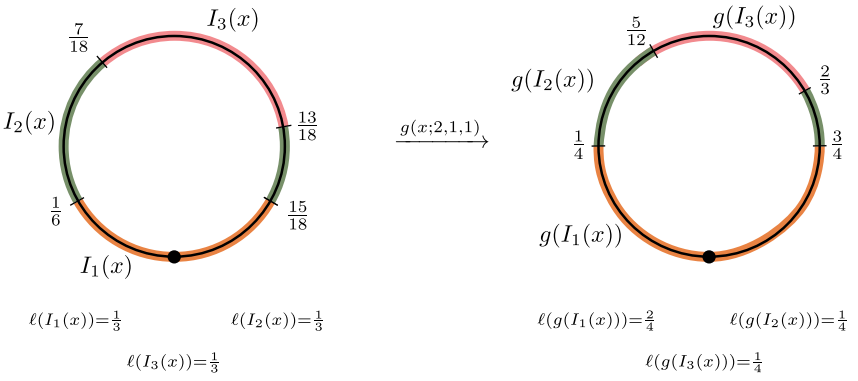


Fig. 20. Map $g(x;2,1,1)$, for x from Fig. 18.

Under the homeomorphism $\mathcal{C}\mathrm{act}^1(k) \cong \mathcal{F}(k)$ in Lemma 5.4, a normalized cactus $x \in \mathcal{C}\mathrm{act}^1(k)$ precisely corresponds to a partition $x \in \mathcal{F}(k)$ of $[0,1]$ into k submanifolds $I_j(x)$ of equal lengths $\frac{1}{k}$, satisfying the conditions of Definition 5.2. Since the ith lobe of x corresponds to the submanifold $I_i(x)$, there is a scaling by $\frac{1}{k}$ in the identification $\mathcal{C}\mathrm{act}^1(k)$ to take a lobe of length 1 to $I_i(x)$. In the next lemma we will use $x \in \mathcal{F}(k)$ and $y_j \in \mathcal{F}(m_j)$ for $1 \le j \le k$ to represent a sequence of cacti in $\mathcal{C}\mathrm{act}^1(k)$ and $\mathcal{C}\mathrm{act}^1(m_j)$ respectively. We will still denote the composition by $\circ_i$ or $\gamma_{\mathcal{C}\mathrm{act}^1}$.

Lemma 5.11. *Let γ_{MS^+} and $\gamma_{\mathcal{C}\mathrm{act}^1}$ denote the (quasi-)operad compositions in MS^+ and $\mathcal{C}\mathrm{act}^1$, respectively. Then for $x \in \mathcal{F}(k)$ and $y_j \in \mathcal{F}(m_j)$, with $1 \le j \le k$, we have*

$$\gamma_{MS^+}((x, g^{-1}(x; m_1, \dots, m_k)); (y_1, id), \dots, (y_k, id)) = (\gamma_{\mathcal{C}\mathrm{act}^1}(x; y_1, \dots, y_k), id)$$

in $MS^+(\sum m_j)$. In particular,

$$(x, g^{-1}(x; 1, \dots, m_i, \dots, 1)) \bullet_i (y_i, \mathrm{id}) = (x \circ_i y_i, \mathrm{id})$$

where $\bullet_i$ denotes the composition (5.3) of MS^+ and $\circ_i$ represents the composition (5.1) of $\mathcal{C}\mathrm{act}^1$.

Proof. Let $\mathcal{F}_{(m_1,\dots,m_k)}(k)$ denote the scaled version of $\mathcal{F}(k)$ where the ith partition, I_i, now has length m_i instead of $\frac{1}{k}$. In particular, we have that $\mathcal{F}(k) = \mathcal{F}_{(\frac{1}{k},\dots,\frac{1}{k})}(k)$ and $\mathcal{C}\mathrm{act}^1(k) = \mathcal{F}_{(1,\dots,1)}(k)$. Thus the homeomorphism $\mathcal{C}\mathrm{act}^1(k) \cong \mathcal{F}(k)$ implies that the composition $\gamma_{\mathcal{C}\mathrm{act}^1}$ on $\mathcal{C}\mathrm{act}^1$ can be interpreted as a map in $\mathcal{F}$, written as

$$\begin{aligned}
&\mathcal{F}(k) \times (\mathcal{F}(m_1) \times \cdots \times \mathcal{F}(m_k)) \\
&\quad \xrightarrow{S} \mathcal{F}_{(m_1,\dots,m_k)}(k) \times \big(\mathcal{F}_{(1,\dots,1)}(m_1) \times \cdots \times \mathcal{F}_{(1,\dots,1)}(m_k)\big) \\
&\quad \xrightarrow{\gamma} \mathcal{F}_{(1,\dots,1)}(\sum_i m_i) \\
&\quad \xrightarrow{N} \mathcal{F}_{\frac{1}{\sum m_i},\dots,\frac{1}{\sum m_i}}(\sum_i m_i) = \mathcal{F}(\sum_i m_i)
\end{aligned}$$

where S and N are scaling and normalizing maps and the map labeled γ is the insertion map.

Our task is to write the composition $\gamma_{\mathcal{C}\mathrm{act}^1}$ in terms of the operad MS^+. To do this, we will use scaling maps inside $Mon^+(I, \partial I)$. More precisely, define a map

$$\mathcal{F}(k) \times (\mathcal{F}(m_1) \times \cdots \times \mathcal{F}(m_k)) \xrightarrow{G} MS^+(k) \times (MS^+(m_1) \times \cdots \times MS^+(m_k))$$

that takes $(x; y_1, \dots, y_k)$ to $((x, g^{-1}); (y_1, \mathrm{id}), \dots, (y_k, \mathrm{id}))$ where $g = g(x; m_1, \dots, m_k) \in Mon^+(I, \partial I)$ is the map in equation (5.4). The statement we want to prove is that $(\gamma_{\mathcal{C}\mathrm{act}^1}, \mathrm{id})$ can be written as the composition

$$\mathcal{F}(k) \times (\mathcal{F}(m_1) \times \cdots \times \mathcal{F}(m_k)) \xrightarrow{G} MS^+(k) \times (MS^+(m_1) \times \cdots \times MS^+(m_k))$$
$$\xrightarrow{\gamma_{MS^+}} MS^+(\sum_i m_i).$$

In particular, we claim that the resulting element of $MS^+(\sum_i m_i)$ is in the image of $\mathcal{C}\mathrm{act}^1$, that is, of the form (z, id).

To prove this, we start by expressing G as a composition $G' \circ S$, where S is the scaling map in the description $\gamma_{\mathcal{C}\mathrm{act}^1} = N \circ \gamma \circ S$ given above and

$$G' \colon \mathcal{F}_{(m_1,\dots,m_k)}(k) \times (\mathcal{F}_{(1,\dots,1)}(m_1) \times \cdots \times \mathcal{F}_{(1,\dots,1)}(m_k)) \longrightarrow MS^+(k) \times (MS^+(m_1) \times \cdots \times MS^+(m_k))$$

is the map that takes a tuple $(x; y_1, \dots, y_k)$ to the tuple $((N(x), g^{-1}); (N(y_1), \mathrm{id}), \dots, (N(y_k), \mathrm{id}))$, with N the normalization also as above. In order to compare $\gamma_{\mathcal{C}\mathrm{act}^1} = N \circ \gamma \circ S$ with $\gamma_{MS^+} \circ G' \circ S$, we have to show that the diagram

$$\begin{array}{ccc} \mathcal{F}_{(m_1,\dots,m_k)}(k) \times (\mathcal{F}_{(1,\dots,1)}(m_1) \times \cdots \times \mathcal{F}_{(1,\dots,1)}(m_k)) & \xrightarrow{\gamma} & \mathcal{F}_{(1,\dots,1)}(\sum_i m_i) \\ \downarrow{\scriptstyle G'} & & \downarrow{\scriptstyle (N,id)} \\ MS^+(k) \times (MS^+(m_1) \times \cdots \times MS^+(m_k)) & \xrightarrow{\gamma_{MS^+}} & MS^+(\sum_i m_i) \end{array}$$

commutes, where the right vertical map takes z to $(N(z), \mathrm{id})$. To see this, let $(x; y_1, \dots, y_k)$ be an element in the top left corner of the square. Its image $\gamma_{MS^+} \circ G'(x; y_1, \dots, y_k)$ along the bottom composition is the element of $MS^+(m_1 + \cdots + m_k)$ given by the following composition:

$$S^1 \xrightarrow{g^{-1}} S^1 \xrightarrow{c_x} (S^1)^k \xrightarrow{c_{y_1} \times \cdots \times c_{y_k}} (S^1)^{m_1 + \cdots + m_k}$$

since we consider $MS^+(m_1 + \cdots + m_k)$ as a subspace of $\mathrm{CoEnd}(m_1 + \cdots + m_k)$ and use the composition in (5.3). The jth factor S^1 in the above $(S^1)^k$ is subdivided into submanifolds $I_s(y_j)$ according to c_{y_j}.

Their inverse image $g \circ (c_x)_j^{-1}(I_s(y_j))$ in the source S^1 of the composition is thus taken to the $(m_1 + \cdots + m_{j-1} + s)$th factor S^1 in $(S^1)^{m_1 + \cdots + m_k}$, being first scaled by a factor $\frac{\sum m_i}{k m_j}$ (using g^{-1}), then by a factor k (via the jth component of c_x) and finally by a factor m_j (via the sth component of c_{y_j}). So in total the composition takes $g \circ (c_x)_j^{-1}(I_s(y_j))$ to $S^1 = I/\partial I$ linearly by a factor $\sum m_i$, and is constant on the connected components of the complement of $g \circ (c_x)_j^{-1}(I_s(y_j))$. In particular, $g \circ (c_x)_j^{-1}(I_s(y_j))$ has length $\frac{1}{\sum m_i}$, which is independent of j and s. Thus we see that the resulting element does indeed live in the image of $\mathcal{C}\mathrm{act}^1$. As the scaling is always independent of s and j, the proportion of each $g \circ (c_x)_j^{-1}(I_s(y_j))$ inside the source S^1 is always as dictated by c_{y_j}, with each $g^{-1}(I_j(x))$ having total length $\frac{m_j}{\sum_i m_i}$. Hence the composition is the same as following the other side of the square, which inserts $I_s(y_j)$ inside $I_j(x)$, scaling each $I_j(x)$ to length m_j, then scales it by $\frac{1}{\sum m_i}$ to be inside MS^+. □

Therefore up to scaling in accordance with the homeomorphism $\mathcal{C}\mathrm{act}^1 \cong \mathcal{F}$ in Lemma 5.4, we have shown that both $\mathcal{C}\mathrm{act}^1$ and its composition are contained within the operad MS^+, but not as a suboperad.

5.2. $\mathcal{C}\mathrm{act}^1$ is a $B\mathcal{O}$–algebra

Here we will construct an action of $B\mathcal{O}$ on normalized cacti using the fact that $\mathcal{C}\mathrm{act}^1 \subset MS^+$, and that its composition can also be described in terms of the composition in MS^+. In Theorem 5.12, we show that the quasi-operad structure on normalized cacti $\mathcal{C}\mathrm{act}^1 = \{\mathcal{C}\mathrm{act}^1(k)\}_{k\geq 0}$ is part of a $B\mathcal{O}$-algebra structure. In Corollary 5.13, we conclude that $\mathcal{C}\mathrm{act}^1$ determines a dendroidal Segal space $X \in \mathcal{S}^{\Omega^{op}}$ with $X(C_v) = \mathcal{C}\mathrm{act}^1(|v|)$.

Recall from Section 3.3 that a $B\mathcal{O}$–algebra is a symmetric sequence with $\circ_i$–operations that are homotopy associative up to all higher homotopies. Elements from $B\mathcal{O}$ are (T, σ, τ, B, t) where T is a planar tree equipped with bijections $\sigma : |V(T)| \to V(T)$ and $\tau : |L(T)| \to L(T)$, and (B, t) is a weighted bracketing of T.

Let

$$\mathcal{R}\colon MS^+ \longrightarrow \mathcal{F} \tag{5.5}$$

denote the projection map that forgets the $Mon^+(I, \partial I)$ component, $\mathcal{R}(x, f) = x$. This is a map of symmetric sequences. If we think of elements of MS^+ as cacti, the map $\mathcal{R}$ has the effect of *renormalizing*, that is, rescaling the lobes so that they all have the same length. Since MS^+ is an operad, it is an $\mathcal{O}$-algebra. The $\mathcal{O}$-action

$$\lambda_{MS^+} : \mathcal{O}(k; m_1, \ldots, m_k) \times MS^+(m_1) \times \ldots \times MS^+(m_k) \longrightarrow MS^+(\sum_i m_i)$$

takes a sequence of elements

$$((T, \sigma, \tau), (x_1, f_1), \ldots, (x_k, f_k)) \in \mathcal{O}(k; m_1, \ldots, m_k) \times MS^+(m_1) \times \ldots \times MS^+(m_k)$$

to the composition of the elements $(x_1, f_1), \ldots, (x_k, f_k)$ according to γ_{MS^+}, in the order prescribed by the labeled tree (T, σ), acting by the permutation τ on the resulting element of $MS^+(\sum_i m_i)$. This composition can be depicted by labeling the ith vertex of (T, σ, τ) by $(x_i, f_i) \in MS^+(m_i)$. This action is compatible with the composition in $\mathcal{O}$ because MS^+ is an operad. We will use this existing $\mathcal{O}$-algebra structure to define the $B\mathcal{O}$-algebra structure of $\mathcal{C}\mathrm{act}^1$ by representing the $\mathcal{C}\mathrm{act}^1$-composition by $\mathcal{R} \circ \lambda_{MS^+}$.

Theorem 5.12. *The $\mathcal{C}\mathrm{act}^1$-composition (5.1) is part of a $B\mathcal{O}$–algebra structure.*

Proof. In order to construct a $B\mathcal{O}$–algebra structure on the sequence $\{\mathcal{C}\mathrm{act}^1(n)\}_{n\geq 0}$, we want to define a map

$$B\mathcal{O}(k; m_1, \ldots, m_k) \times \mathcal{C}\mathrm{act}^1(m_1) \times \ldots \times \mathcal{C}\mathrm{act}^1(m_k) \longrightarrow \mathcal{C}\mathrm{act}^1(\sum_i m_i)$$

that restricts to the Σ_n–action on $\mathcal{C}\mathrm{act}^1(n)$, which permutes the labels on the lobes, and its already defined $\circ_i$–compositions. Using the homeomorphism $\mathcal{C}\mathrm{act}^1 \cong \mathcal{F}$ from Lemma 5.4, we will equivalently construct a map

$$\lambda : B\mathcal{O}(k; m_1, \ldots, m_k) \times \mathcal{F}(m_1) \times \ldots \times \mathcal{F}(m_k) \longrightarrow \mathcal{F}(\sum_i m_i).$$

Firstly, the Σ_n-action on the n–space of a $B\mathcal{O}$–algebra is encoded by the labeled corollas

$$(C_n, 1, \tau, \emptyset, \emptyset) \in B\mathcal{O}(n; n) \cong \mathcal{O}(n; n) \cong \Sigma_n,$$

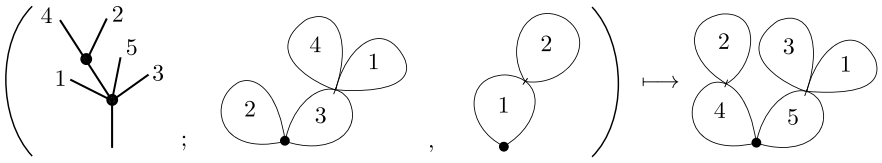


Fig. 21. Example of the $B\mathcal{O}$-action on $\mathcal{C}\mathrm{act}^1$.

where τ labels the leaves of the corollas C_n, which are thought of as elements of the symmetric group Σ_n, and the identity corresponds to the planar ordering. This fixes the action of such elements of $B\mathcal{O}$ as we have already fixed the Σ_n–action on $\mathcal{C}\mathrm{act}^1(n)$.

The $\mathcal{C}\mathrm{act}^1$ $\circ_i$-composition is encoded in $B\mathcal{O}$ by the trees with exactly two vertices, one attached to the ith incoming edge of the other. These trees admit no non-trivial bracketings so such elements of $B\mathcal{O}$ have the form

$$(T, \sigma, \tau, \emptyset, \emptyset) \in B\mathcal{O}(m+n-1; m, n)$$

where σ labels the two vertices of T and τ labels its $n+m-1$ leaves. The compatibility with the prechosen operadic composition of $\mathcal{C}\mathrm{act}^1$ dictates the action of such elements of $B\mathcal{O}$: $(T, \sigma, \tau, \emptyset, \emptyset)$ acts on $x_1 \in \mathcal{C}\mathrm{act}^1(m)$ and $x_2 \in \mathcal{C}\mathrm{act}^1(n)$ by taking their $\circ_i$-composition, as dictated by the tree, and then acting by τ on the lobes of the resulting element of $\mathcal{C}\mathrm{act}^1(m+n-1)$. Fig. 21 illustrates an example of this action.

By Lemma 5.11, this $\mathcal{C}\mathrm{act}^1$-composition can be defined in terms of the MS^+ composition:

$$\mathcal{R} \circ \lambda_{MS^+}\big((T, \sigma, \tau); (x_1, g_1), (x_2, g_2)\big)$$

where $\mathcal{R}$ is the projection map (5.5), and $g_1 = g(x_1, 1, \ldots, k_i, \ldots, 1)$ and $g_2 = g(x_2, 1, \ldots, l_j, \ldots, 1)$ are the rescaling maps of (5.4), with $k_i = n$ and $l_j = 1$ if first vertex is the bottom vertex and the second is attached to its ith input, or $k_i = 1$ and $l_j = m$ if the second vertex is the bottom vertex with the first attached to its jth input. Let $(y_T, f_T) \in MS^+$ denote the element $\lambda_{MS^+}\big((T, \sigma, \tau); (x_1, g_1), (x_2, g_2)\big)$.

We will now extend this definition of the $B\mathcal{O}$–action of trees with at most two vertices to an action of the whole operad. We start by defining an explicit expression for the action of bracketings of trees $(T, \sigma, \tau, B, \underline{1})$ with brackets of weight 1, and afterwards extend this definition to the remaining elements of $B\mathcal{O}$, whose brackets have weight strictly between 0 and 1.

Let $\mathbf{T} = (T, \sigma, \tau, B, \underline{1})$ be an element of $B\mathcal{O}(n; m_1, \ldots, m_k)$ with all brackets of weight 1, and let $x_i \in \mathcal{F}(m_i) \cong \mathcal{C}\mathrm{act}^1(m_i)$ for each $1 \leq i \leq k$. We first construct scaling maps $g \in Mon^+(I, \partial I)$ as in (5.4). Recall from Definition 3.1 that a bracketing $B = \{S_j\}_{j \in J}$ consists of large, nested proper subtrees of T. Here we allow B to be empty. Recall that σ orders the vertices of T. For a fixed $i \in \{1, \ldots, k\}$, let $S \in B$ be the smallest bracket that contains the vertex $\sigma(i)$, allowing $S = T$ if there are no such bracket. Recall that $in(\sigma(i))$ is the set of incoming edges of $\sigma(i)$, and $L(S)$ is the set of leaves of the bracket S. We define a map

$$\xi : in(\sigma(i)) \longrightarrow \mathbb{N} \tag{5.6}$$

by setting

(i) $\xi(e) = 1$ if $e \in L(S)$;
(ii) $\xi(e) = |L(S')|$ if e is the root of a bracket $S' \subset S$ in B, with $S' \subset S$ the largest such bracket;
(iii) $\xi(e) = |w|$ if $e \in iE(S)$ is not the root of any $S' \in B$, where $|w|$ denotes the arity of the vertex $w \in V(S)$ for which e is the outgoing edge.

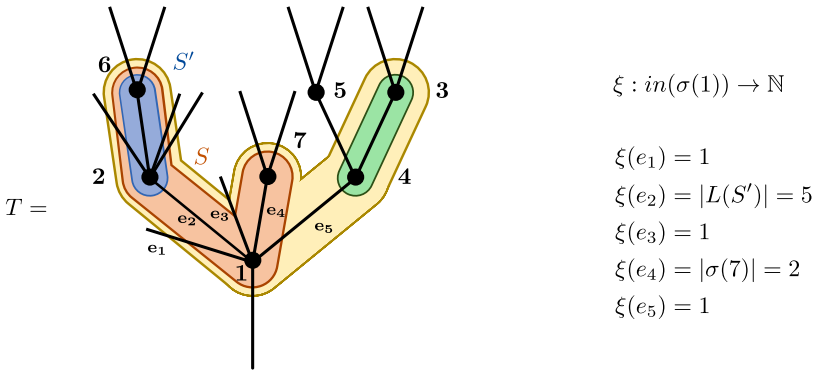


Fig. 22. An example of ξ.

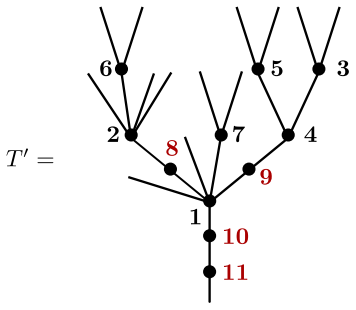


Fig. 23. An example of T' for the bracketing of T in Fig. 22.

Fig. 22 shows an example of the map ξ. We then set

$$g_i := g(x_i; \xi(e_1), \dots, \xi(e_{m_i})) \tag{5.7}$$

for $e_1, \dots, e_m$ the incoming edges of $\sigma(i)$ ordered by the planar ordering of T.

We define the action of $B\mathcal{O}$ inductively on the size of the bracketing B.

If B is empty, then we define

$$\lambda(\mathbf{T}; x_1, \dots, x_k) := \mathcal{R} \circ \lambda_{MS^+}((T, \sigma, \tau); (x_1, g_1), \dots, (x_k, g_k))$$

and use $(y_T, f_T) \in MS^+$ to denote the image of λ_{MS^+}. Note that when $k = 1$ or 2, this is the same as the $B\mathcal{O}$-structure already defined above.

If B is not empty, then we define additional scaling maps for each bracket, using the inductive hypothesis that the action has already been defined action on subtrees with fewer brackets.

Let T' be the tree obtained from T by adding a binary vertex at the root of each bracket $S_j \in B$. Extend the order σ of the vertices of T to an order σ' of vertices of T' by setting the $|J| = |B|$ new vertices last. An example of T' is shown in Fig. 23. We will use each additional vertex of T' to assign a scaling map to the associated bracket.

Let $w_j \in V(T') \setminus V(T)$ be the jth vertex of T' not in T, according to the chosen order σ'. Let $S_j \in B$ be the bracket associated to w_j. Since the number of brackets of B that lie inside S_j is less than $|B|$, we have an element

$$(y_{S_j}, f_{S_j}) \in MS^+$$

defined by the inductive assumption by restricting $\mathbf{T} = (T, \sigma, \tau, B, \underline{1})$ to the subtree S_j. Consider the tree T/S_j in which all vertices in S_j are identified and internal edges between them are collapsed. The tree T/S_j has a vertex $[S_j]$ associated to the collapsed tree S_j. We have an induced bracketing $\tilde{B}$ of T/S_j from the bracketing B of T, and thus can define a map $\xi_j : in([S_j]) \to \mathbb{N}$ as in (5.6) by replacing (T, B) by $(T/S_j, \tilde{B})$. Then define

$$h_j := g(y_{S_j}; \xi_j(e_1), \dots, \xi_j(e_l)) \circ f_{S_j}^{-1} \tag{5.8}$$

for $e_1, \dots, e_l$ the incoming edges of $[S_j]$ in T/S_j. We define the action of $B\mathcal{O}$ by setting

$$\lambda(\mathbf{T}; x_1, \dots, x_k) := \mathcal{R} \circ \lambda_{MS^+}\big((T', \sigma', \tau); (x_1, g_1), \dots, (x_k, g_k), (1, h_1), \dots, (1, h_{|B|})\big) \tag{5.9}$$

for the rescaling maps g_i and h_j defined above.

We claim that the formula for the action (5.9) is indeed compatible with composition of bracketings of trees of weight 1. It is enough to check this for a $\circ_i$-composition in $B\mathcal{O}$, so consider $\mathbf{T_1} = (T_1, \sigma_1, \tau_1, B_1, \underline{1})$ and $\mathbf{T_2} = (T_2, \sigma_2, \tau_2, B_2, \underline{1})$ in $B\mathcal{O}$. We need to check that

$$\lambda(\mathbf{T_1}; x_1, \dots, x_{i-1}, \lambda(\mathbf{T_2}; x_i, \dots, x_{i+l-1}), x_{i+1}, \dots, x_{k+l-1}) = \lambda(\mathbf{T_1} \circ_i \mathbf{T_2}; x_1, \dots, x_{k+l-1}). \tag{5.10}$$

From the above definition, we have $\lambda(\mathbf{T_1}; x_1, \dots, x_{i-1}, \lambda(\mathbf{T_2}; x_i, \dots, x_{i+l-1}), x_{i+1}, \dots, x_{k+l-1}) =$

$$\mathcal{R} \circ \lambda_{MS^+}\big((T_1', \sigma_1', \tau_1); (x_1, g_1), \dots, (x_{i-1}, g_{i-1}), (y_{T_2}, g_i), (x_{i+l}, g_{i+l}), \dots, (x_k, g_k), (1, h_1), \dots, (1, h_{|B_1|})\big)$$

for

$$y_{T_2} = \mathcal{R} \circ \lambda_{MS^+}\big((T_2', \sigma_2', \tau_2); (x_i, g_1'), \dots, (x_{i+l-1}, g_l'), (1, h_1'), \dots, (1, h_{|B_2|}')\big)$$

where the maps g_i and h_i are those associated to (T_1, B_1) and the maps g_i' and h_i' associated to (T_2, B_2). In the above notation, we also have

$$(y_{T_2}, f_{T_2}) = \lambda_{MS^+}\big((T_2', \sigma_2', \tau_2); (x_i, g_i'), \dots, (x_{i+l-1}, g_{i+l-1}'), (1, h_1'), \dots, (1, h_{|B_2|}')\big).$$

Note that one can change the $Mon^+(I, \partial I)$ component of an element of MS^+ by doing a $\circ_1$–composition in the operad. In particular,

$$(y_{T_2}, g_i) = (1, g_i \circ f_{T_2}^{-1}) \circ_1 (y_{T_2}, f_{T_2})$$

in MS^+ ane we can rewrite the left hand side of (5.10) as the first component of the MS^+–composition

$$\begin{aligned}\lambda_{MS^+}\big((T_1', \sigma_1', \tau_1) \circ_i (\bar{T}_2', \bar{\sigma}_2', \tau_2); (x_1, g_1), \dots, (x_{i-1}, g_{i-1}),\\ (x_i, g_i'), \dots, (x_{i+l-1}, g_{i+l-1}'), (1, h_1'), \dots, (1, h_{|B_2|}'), (1, g_i \circ f_{T_2}^{-1}),\\ (x_{i+l}, g_{i+l}), \dots, (x_k, g_k), (1, h_1), \dots, (1, h_{|B_1|})\big)\end{aligned}$$

where $\bar{T}_2'$ has an extra vertex at the bottom of the tree to encode the change of $Mon^+(I, \partial I)$–component for the T_2 composition. If T_2 is large, this extra vertex corresponds exactly to the extra bracket T_2 arising in the $B\mathcal{O}$–composition, and one checks that the corresponding scaling map h defined by the formula (5.8) is precisely the map $g_i \circ f_{T_2}^{-1}$. As the other labels of the vertices of the composed tree agree with those of the right hand side, we see that we recover the right hand side of (5.10). If T_2 is not large, then there is no

such additional bracket in the $B\mathcal{O}$–composition, but in this case $f_{T_2} = \mathrm{id}$ and the left and right hand side agree directly.

Recall from Remark 3.10 that we may consider $B\mathcal{O}$ as the geometric realization of the simplicial operad of bracket trees. Then the above definition of λ on bracketings of weight 1 defines the action of the vertices of $B\mathcal{O}$. We finally extend this action to all bracketings of a tree T by linear interpolation on the rescaling maps g_i. For a fixed tree T and a point $((B_0 \subset \cdots \subset B_r), \underline{t})$ in the realization of the poset $\mathcal{B}(T)$, let $g_i(T, B_j)$ denote the definition of the rescaling map g_i with respect to the bracketing B_j on T in (5.7), and likewise for the maps f_j in (5.8). We set

$$g_i = t_0 g_i(T, B_0) + \ldots + t_r g_i(T, B_r).$$

This is well-defined as $Mon^+(I, \partial I)$ is convex. Also note that this is continuous in $B\mathcal{O}$ as going to the lth face of the simplex $(B_0 \subset \cdots \subset B_r)$ corresponds to t_l going to 0, that is, dropping the bracket B_l. Then we define (y_T, f_T) and $\lambda(T, \sigma, \tau, B, t) := \mathcal{R}(y_T, f_T)$ as in (5.9) but with this definition of g_i instead.

This defines the action of $B\mathcal{O}$ on $\mathcal{C}\mathrm{act}^1$. It is compatible under composition because the composition in $B\mathcal{O}$ is the realization of the composition in the poset operad, and we have already checked the compatibility under composition there. □

Given that normalized cacti, together with the cactus composition (5.1), forms a $B\mathcal{O}$-algebra we can now use the rectification results from Proposition 4.10 to define an ∞-operad.

Corollary 5.13. *Normalized cacti define dendroidal spaces of the following two flavors:*

(i) *There exists a reduced homotopy dendroidal space $X \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$, satisfying the strict Segal condition, such that $X(C_n) = \mathcal{C}\mathrm{act}^1(n)$ and with value on the inner face maps ∂_e given by the $\mathcal{C}\mathrm{act}^1$–composition.*

(ii) *There exists a weakly reduced dendroidal space $Y \in \mathcal{S}^{\Omega^{op}}$, satisfying the weak Segal condition (in the sense of Proposition 4.10), such that $Y(C_n) \simeq \mathcal{C}\mathrm{act}^1(n)$ and with value on the inner face maps ∂_e homotopic to the $\mathcal{C}\mathrm{act}^1$–composition.*

Proof. Theorem 5.12 shows that $\mathcal{C}\mathrm{act}^1$ is a $B\mathcal{O}$-algebra. Applying the construction from Theorem 4.8, we define a homotopy dendroidal space $X := \Phi(\mathcal{C}\mathrm{act}^1) \in \mathcal{S}^{\widetilde{\Omega}_0^{op}}$. By construction, $\Phi(\mathcal{C}\mathrm{act}^1)(C_n) = \mathcal{C}\mathrm{act}^1(n)$, and by the theorem it is a reduced homotopy dendroidal space satisfying the strict Segal condition. The evaluation of $\Phi(\mathcal{C}\mathrm{act}^1)$ on an inner edge is the $\circ_i$ composition, as encoded by the $B\mathcal{O}$-structure, which in the present case is the $\mathcal{C}\mathrm{act}^1$–composition by Theorem 5.12. This proves (i) in the statement.

For (ii), we set $Y := p_! X = p_! \Phi(\mathcal{C}\mathrm{act}^1) \in \mathcal{S}^{\Omega^{op}}$ to be the rectification of X, as constructed in Proposition 4.10. By Lemma 4.9, $Y(C_w) = p_! \Phi(\mathcal{C}\mathrm{act}^1)(C_w) \simeq \Phi(\mathcal{C}\mathrm{act}^1)(C_w) = \mathcal{C}\mathrm{act}^1(|w|)$ and the value of Y on inner face maps identifies under these homotopy equivalences with the value of X on inner face maps, and hence identifies with the $\mathcal{C}\mathrm{act}^1$–composition. By the proposition, Y is weakly reduced and satisfies the weak Segal condition. □

6. Relation between the operads $B\mathcal{O}$ and $W\mathcal{O}$

The Boardman-Vogt W-construction is a construction on operads with the property that, for any topological operad $\mathcal{P}$, algebras over $W\mathcal{P}$ are "up-to-homotopy" or "weak" $\mathcal{P}$-algebras. A lax operad [8] is an algebra over the operad $W\mathcal{O}$, the Boardman-Vogt W–construction applied to the operad of operads $\mathcal{O}$ (Definition 2.8), and is a notion of a "weak" or "infinity" operad. It is known that there exists a zig-zag of Quillen equivalences between the category of $W\mathcal{O}$-algebras and the category of reduced dendroidal spaces by, for example, combining Theorem 4.1 of [1] with either Theorem 1.1 of [3] or a restriction of Theorem 8.15 of [12].

Here we show how the operad $B\mathcal{O}$ can be identified with a variant W_0 of the W-construction of the operad $\mathcal{O}$ of operads (see Theorem 6.4). From this, it will follow that $B\mathcal{O}$–algebras are lax operads that are strictly symmetric and with a strict identity (see Example 6.2). We start by recalling the W–construction.

6.1. The W-construction

The Boardman-Vogt W-construction is an enlargement of the free operad construction. Given an operad $\mathcal{P}$, there are canonical morphisms of topological operads

$$F\mathcal{P} \hookrightarrow W\mathcal{P} \xrightarrow{\sim} \mathcal{P},$$

where the map $p : W\mathcal{P} \to \mathcal{P}$ is a surjective homotopy equivalence. Algebras for $W\mathcal{P}$ are up-to-homotopy $\mathcal{P}$-algebras. We briefly recall the construction here and refer the reader to [8, Section 17] or [1, Section 3] for full details.

Definition 6.1. Let $\mathcal{P}$ be a $\mathfrak{C}$–colored (discrete or topological) operad. The operad $W\mathcal{P}$ is a topological operad with the same set of colors $\mathfrak{C}$, built from the free operad $F(\mathcal{P})$ (Definition 2.3) by adding length in $[0,1]$ to the internal edges of the trees that define the elements of $F(\mathcal{P})$. More precisely, for each list of colors $c; c_1, \dots, c_k$ in $\mathfrak{C}$, we have

$$W\mathcal{P}(c; c_1, \dots, c_k) = \Big(\coprod_{(\mathbb{T}, f, \lambda)} \big([0,1]^{|iE(\mathbb{T})|} \times \prod_{v \in V(\mathbb{T})} \mathcal{P}(out(v); in(v))\big)\Big) / \sim$$

where the disjoint union, as for the free operad, runs over the isomorphim classes of leaf-labeled $\mathfrak{C}$–colored planar trees

$$(\mathbb{T}, f\colon E(\mathbb{T}) \to \mathfrak{C}, \lambda\colon \{1, \dots, k\} \to L(\mathbb{T}))$$

with k leaves such that $f(\lambda(i)) = c_i$, $f(R(\mathbb{T})) = c$. The equivalence relation is generated by the relation $(*)$ in Definition 2.3 in addition to the following additional relations that capture "weak" operadic composition and units:

(1) any tree with an internal edge of length of zero is identified with the tree where that edge has been collapsed and the operations labeling its end vertices composed;
(2) any tree that has a vertex with only one input and one output, both colored by $c \in \mathfrak{C}$, labeled by the identity in $\iota_c \in \mathcal{P}(c; c)$, is identified with the tree where that vertex is deleted. The resulting new edge, if internal, has length the maximum length of the two original internal edges connected to the deleted vertex.

See [8, p 75] for a pictorial version of these relations. The symmetric group acts on $W\mathcal{P}$ by relabeling the leaves, as for the free operad. Composition is by grafting, giving length 1 to the newly created internal edge.

We will denote elements of $W\mathcal{P}$ by $(\mathbb{T}, f, \lambda, s, p)$, where $\mathbb{T}$ is a planar tree, $f : E(\mathbb{T}) \to \mathfrak{C}$ is the map coloring its edges, $\lambda : \{1, \dots, k\} \to L(\mathbb{T})$ is the bijection labeling its leaves, $s \in [0,1]^{|iE(\mathbb{T})|}$ is a collection of weights, and $p = (p_v)_{v \in V(\mathbb{T})}$ is a labeling of the vertices by operations in $\mathcal{P}$. An example is shown in Fig. 24. There is a canonical projection map $\pi : W\mathcal{P} \to \mathcal{P}$ defined by sending all the edge lengths to 0 and composing the operations of $\mathcal{P}$ as dictated by the trees.

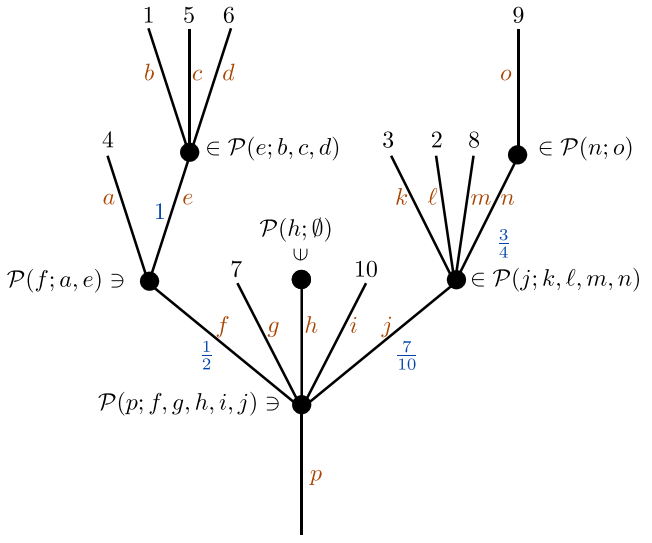


Fig. 24. Example of an element of $W\mathcal{P}(p; a, b, c, d, g, i, k, \ell, m, o)$.

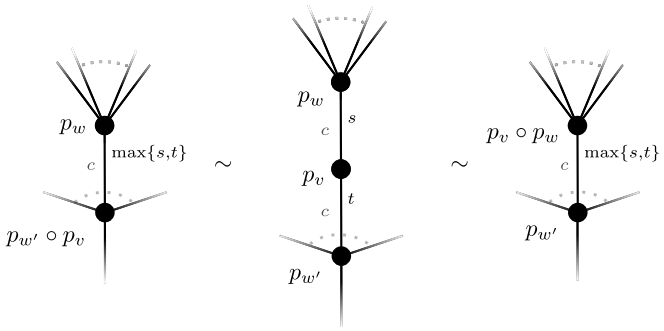


Fig. 25. Local representation of the relation (2′) on a tree.

6.2. A variant on the W-construction

Given a (discrete or topological) $\mathfrak{C}$-colored operad $\mathcal{P}$, the topological operad $W_0\mathcal{P}$ is defined as the quotient of $W\mathcal{P}$ by replacing relation (2) in Definition 6.1 by the following stronger relations for arity one vertices, as well as a version for arity zero vertices:

(2′) any tree that has a vertex v with only one input and one output both colored by c, adjacent to at least one other vertex w, with v labeled by *any* element $\mathcal{P}(c; c)$, is identified with the tree where the vertex v is deleted, and the label of v and w are composed in $\mathcal{P}$ (Fig. 25).
If the resulting new edge is internal, then its length is the maximum length of the two original (then necessarily internal) edges adjacent to v.

(3’) any tree that has a vertex v with no input, adjacent to another vertex w, with v labeled by *any* element of $\mathcal{P}(c; \emptyset)$, is identified with the tree where the vertex v and the edge between v and w are deleted, and the labels of v and w are composed in $\mathcal{P}$ (Fig. 26).

So a $W_0\mathcal{P}$–algebra is a weak $\mathcal{P}$–algebra ($W\mathcal{P}$–algebra) for which the nullary and unary operation are strict. And in particular, one has that $W_0\mathcal{P}(c; c) = \mathcal{P}(c; c)$ and $W_0\mathcal{P}(c; \emptyset) = \mathcal{P}(c; \emptyset)$ for any color c. Also, one can always choose representatives of elements of $W_0\mathcal{P}$ using trees with no valence 0 or 1 vertices (unless it only has 0 or 1 vertex). In a tree that defines an element of $W_0\mathcal{P}$, an arity one vertex lying in between two other

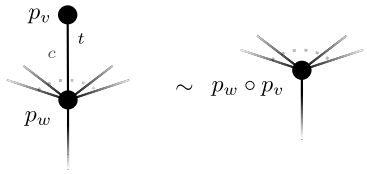


Fig. 26. Local representation of the relation (3') on a tree.

vertices can be slid up or down to either of its neighboring vertices, composing its label with that of the chosen vertex, while an arity zero vertex can be "pushed down" to the vertex it is attached to.

Example 6.2. The example relevant to us here is when we set $\mathcal{P} = \mathcal{O}$ is the operad of operads. In this case, $\mathfrak{C} = \mathbb{N}$ is the natural numbers and an $\mathcal{O}$-algebra is a (monochrome) operad. The nullary operations in $\mathcal{O}(1;\emptyset)$ encode the identity operation in the $\mathcal{O}$–algebra, while the unary operations in $\mathcal{O}(n;n)$ encode the action of the symmetric groups. It follows that a $W_0\mathcal{O}$–algebra is a strictly symmetric weak operad with a strict identity.

By construction, the canonical projection $p : W\mathcal{P} \to \mathcal{P}$ factors through the quotient map $q : W\mathcal{P} \to W_0\mathcal{P}$. Moreover, both $W\mathcal{P}$ and $W_0\mathcal{P}$ are homotopy equivalent to $\mathcal{P}$:

Proposition 6.3. *There are operad maps $W\mathcal{P} \to W_0\mathcal{P} \to \mathcal{P}$, inducing homotopy equivalences*

$$W\mathcal{P}(c;c_1,\dots,c_n) \xrightarrow[\sim]{q} W_0\mathcal{P}(c;c_1,\dots,c_n) \xrightarrow[\sim]{p_0} \mathcal{P}(c;c_1,\dots,c_n)$$

for each $n \geq 0$ and each $c;c_1,\dots,c_n$ in $\mathfrak{C}$.

Proof. For each $n \geq 0$ and $c;c_1,\dots,c_n$ the map

$$q : W\mathcal{P}(c;c_1,\dots,c_n) \to W_0\mathcal{P}(c;c_1,\dots,c_n)$$

is the projection on to the quotient. It is an operad map because if elements of $W\mathcal{P}$ are equivalent in $W_0\mathcal{P}$ before being composed, they are necessarily also equivalent in $W_0\mathcal{P}$ after composition. The map $p_0 : W_0\mathcal{P} \to \mathcal{P}$ contracts the remaining edges in the trees of $W_0\mathcal{P}$ by sending the lengths to 0 (and composing the operations in $\mathcal{P}$). This map is well-defined, as it is compatible with the relations (2') and (3'), and respects the operad structure. These maps induce homotopy equivalences, with homotopy inverses given by including $\mathcal{P}(c_1,\dots,c_n;c)$ as labeled corollas in $W_0\mathcal{P}(c;c_1,\dots,c_n)$ or $W\mathcal{P}(c;c_1,\dots,c_n)$. □

6.3. B𝒪-algebras are strictly symmetric lax operads

In this section we prove that there is an isomorphism of topological operads $B\mathcal{O} \cong W_0\mathcal{O}$.

Theorem 6.4. *The operads $W_0\mathcal{O}$ and $B\mathcal{O}$ are isomorphic.*

The combination of Theorem 6.4 with Example 6.2, describes $B\mathcal{O}$–algebras as strictly symmetric lax operads with strict identity. Moreover, the isomorphism $B\mathcal{O} \cong W_0\mathcal{O}$ allows us to observe that any $B\mathcal{O}$–algebra will receive a canonical $W\mathcal{O}$–structure via the quotient map $q : W\mathcal{O} \longrightarrow W_0\mathcal{O}$. In fact, Theorem 6.4 together with Proposition 6.3 gives homotopy equivalences

$$W\mathcal{O} \xrightarrow{\simeq} B\mathcal{O} \xrightarrow{\simeq} \mathcal{O},$$

where $\mathcal{O} \cong \pi_0 W\mathcal{O} \cong \pi_0 B\mathcal{O}$ and the latter equivalence is the one already considered in Remark 3.10. Each of the topological operads $W\mathcal{O}$, $B\mathcal{O}$ and $\mathcal{O}$ are admissible in the sense of [1, Theorem 2.1] and thus the natural projection maps induce Quillen adjunctions

$$W\mathcal{O}-\mathrm{Alg} \rightleftarrows B\mathcal{O}-\mathrm{Alg} \rightleftarrows \mathcal{O}-\mathrm{Alg}.$$

We expect that these adjunctions induce Quillen equivalences, though the operad $B\mathcal{O}$ is not Σ-cofibrant so we can not immediately apply [1, Theorem 4.1]. We nonetheless have the following:

Corollary 6.5. *Every $B\mathcal{O}$-algebra is homotopy equivalent to a strict operad.*

Proof. Every $B\mathcal{O}$-algebra is canonically a $W\mathcal{O}$-algebra via the quotient map $q : W\mathcal{O} \to B\mathcal{O}$, and every $W\mathcal{O}$-algebra is homotopic to a strict operad, i.e. an $\mathcal{O}$-algebra, using e.g. Theorem 4.1 of [1] that gives a Quillen equivalence between the categories $W\mathcal{O}-\mathrm{Alg}$ and $\mathcal{O}-\mathrm{Alg}$ (using that the operads $W\mathcal{O}$ and $\mathcal{O}$ are well-pointed, Σ-cofibrant admissible operads).

Given a $B\mathcal{O}$-algebra $\mathcal{P}$, one can alternatively directly construct such a strict operad $\mathcal{P}'$ homotopic to $\mathcal{P}$ using the following double bar construction:

$$\mathcal{P} \xleftarrow{\simeq} B(B\mathcal{O}, B\mathcal{O}, \mathcal{P}) \xrightarrow{\simeq} B(\mathcal{O}, B\mathcal{O}, \mathcal{P}) =: \mathcal{P}',$$

where $\mathcal{P}'$ is an $\mathcal{O}$-algebra and hence a strict operad. In the above zig-zag, we use the notation of e.g. [28, Const 9.6 and Thm 9.10], identifying an operad with its associated monad. The leftmost map is an equivalence by properties of the bar construction and the rightmost because of the equivalence $B\mathcal{O} \xrightarrow{\simeq} \mathcal{O}$ of Remark 3.10. □

Combining Theorem 4.8 with Theorem 6.4, we can also relate $B\mathcal{O}$-algebras with reduced, strictly Segal $\widetilde{\Omega}_0^{op}$-diagrams.

Corollary 6.6. *There exist isomorphisms of categories*

$$W_0 O-\mathrm{Alg}_{\mathcal{S}} \cong (\mathcal{S}^{\widetilde{\Omega}_0^{op}})_{strict}.$$

The proof of Theorem 6.4 will be given in Section 6.4. Though not saying this explicitly, the proof uses the natural association of a bracketing to a *clustering tree*, which is described for instance in [38, Definition 2.7].

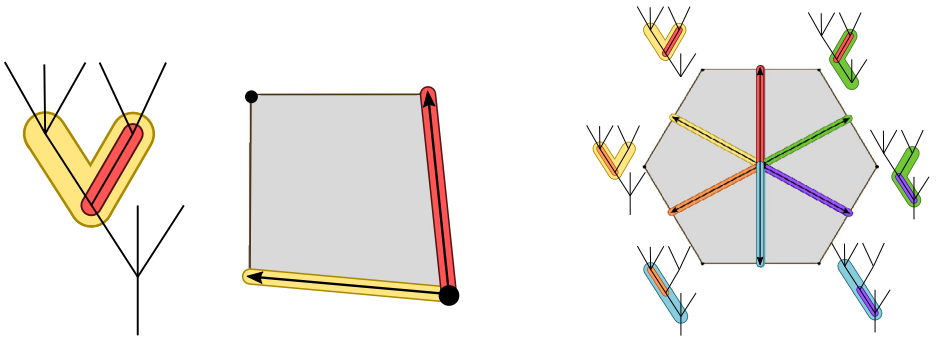
Since the W-construction is built out of cubes, to prepare for the proof, we start by giving an alternative description of $B\mathcal{O}$ in terms of cubes as well.

Definition 6.7. We can define a *weighted bracketing* of a tree T to be a pair (B, t) with bracketing $B = \{S_j\}_{j\in J}$ of T and $t \in [0,1]^J$. The jth coordinate $t_j \in t$ is the *weight* of S_j. The addition of weights associates to each bracketing a cube $[0,1]^{|B|}$. These cubes fit together to form a space:

$$\mathbb{B}(T) = \coprod_{B\in\mathcal{B}(T)} [0,1]^{|B|}/_{\sim}$$

where the equivalence relation is by identifying any bracketings with weights that only differ by a bracket of weight 0 (see Fig. 27(b)).

Recall from Definition 3.2 the poset $\mathcal{B}(T)$ of bracketings of a tree T under the inclusion relation.



(a) A bracketing B of T with 2 brackets and its associated cube $[0,1]^{|B|}$ in $\mathbb{B}(T)$.

(b) All bracketings of T and their associated cubes assembled to make $\mathbb{B}(T)$ a hexagon.

Fig. 27. A tree T and its corresponding space of bracketings $\mathbb{B}(T)$.

Lemma 6.8. *Let T be a tree. There is a homeomorphism $|\mathcal{B}(T)| \cong \mathbb{B}(T)$, between the realization of the nerve of the poset $\mathcal{B}(T)$ and the cubical space $\mathbb{B}(T)$.*

Proof. We consider the topological k–simplex as the space

$$\Delta^k = \{(s_1, \dots, s_k) \in \mathbb{R}^k : 1 = s_0 \geq s_1 \geq \cdots \geq s_k \geq 0\}.$$

Fix a tree T and let σ denote a k–simplex $B_0 \subseteq \cdots \subseteq B_k$ of the nerve of the poset $\mathcal{B}(T)$. To each σ we associate a map

$$\chi_\sigma : \Delta^k \longrightarrow \mathbb{B}(T)$$

where $\chi_\sigma(s_1, \dots, s_k)$ is the weighted bracketing of T in which all trees of B_0 have weight $1 = s_0$ and all trees of $B_i \backslash B_{i-1}$ have weight s_i for $i \geq 1$.

The maps χ_σ assemble into a continuous map

$$\chi : |\mathcal{B}(T)| = \Big(\coprod_{k \geq 0} \mathcal{B}(T)_k \times \Delta^k \Big) / \sim \ \longrightarrow \ \mathbb{B}(T) = \coprod_{B \in \mathcal{B}(T)} [0,1]^{|B|} / \sim .$$

This map is a homeomorphism with inverse defined by mapping a cube $[0,1]^{|B|}$ in $\mathbb{B}(T)$ to the realization of the sub-poset $\mathcal{B}_{\leq B}$, which is a cube whose dimension is the cardinality $|B|$ of the bracketing. Explicitly, given an element $(B, t) \in \mathbb{B}(T)$ with $B = (B_0, \dots, B_k)$, we order the coordinates of $t = (t_1, \dots, t_k)$ so that they are in decreasing order

$$1 = t_{\sigma(1)} = \cdots = t_{\sigma(r_1)} > t_{\sigma(r_1+1)} = \cdots = t_{\sigma(r_1+r_2)} > \cdots > t_{\sigma(r_1+\cdots+r_{l+1}+1)} = \cdots = t_{\sigma(r_1+\cdots+r_{l+2})} = 0.$$

This defines an l–simplex $\bar{B}_0 \subset \bar{B}_1 \subset \cdots \subset \bar{B}_l$ by setting

$$\bar{B}_i = B_{\sigma(1)} \cup \cdots \cup B_{\sigma(r_1+\cdots+r_{i+1})}. \quad \square$$

6.4. Proof of Theorem 6.4

In order to prove $B\mathcal{O} \cong W_0\mathcal{O}$, we first recall some definitions. Recall that elements $(\mathbb{T}, f, \lambda, s, p) \in W_0\mathcal{O}(n; m_1, \dots, m_k)$ are represented by a planar tree $\mathbb{T}$ with k leaves ordered by the bijection λ :

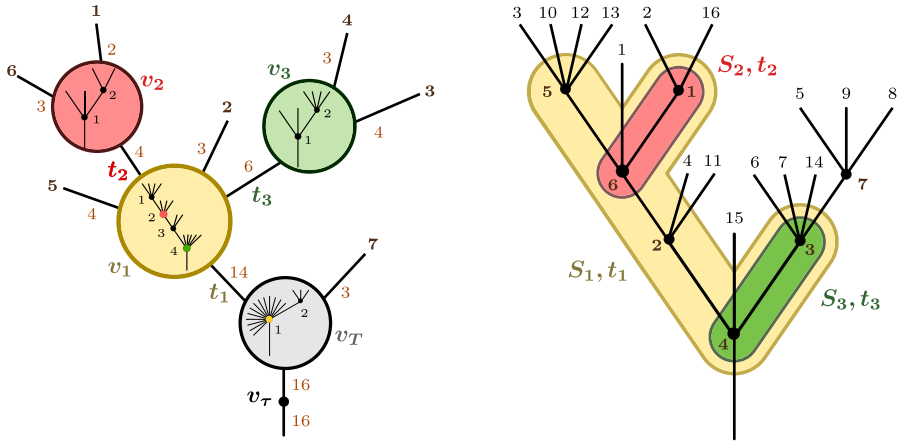


Fig. 28. Element of $W_0\mathcal{O}$ and corresponding element of $B\mathcal{O}(16; 2, 3, 4, 3, 4, 3, 3)$.

$\{1, \dots, k\} \to L(\mathbb{T})$ and with an edge coloring $f : E(\mathbb{T}) \to \mathbb{N}$ that, in particular, colors the leaves by $m_1, \dots, m_k$. In addition, $\mathbb{T}$ is equipped with a collection of lengths $s \in [0,1]^{|iE(\mathbb{T})|}$, and a decoration of the vertices $p = (p_v)_{v \in V(\mathbb{T})}$ by operations by p_v in $\mathcal{O}(out(v); in(v))$.

We call a representative $(\mathbb{T}, f, \lambda, s, p)$ *reduced* if the tree $\mathbb{T}$ has no vertices of arity zero or one, unless such a vertex cannot be removed using the equivalence relation in $W_0\mathcal{O}$, i.e. if $\mathbb{T}$ is the corolla C_0 or C_1. In particular, every element of $W_0\mathcal{O}$ has a reduced representative, which in general is not unique. It greatly simplifies the proof of Lemma 6.9 to work with reduced representative.

For a given tree $\mathbb{T}$, and vertices $v, w \in V(\mathbb{T})$, we say that w is *above* v if the unique shortest path between w and the root of the tree goes through v. In this case v is *below* w. Every other vertex of $\mathbb{T}$ is above the *root vertex* v_0 whose outgoing edge is the root of $\mathbb{T}$.

Lemma 6.9. *There is a map of topological operads $\Psi : W_0\mathcal{O} \to B\mathcal{O}$.*

The map Ψ is illustrated in Fig. 28.

Proof. Given a reduced element $(\mathbb{T}, f, \lambda, s, p) \in W_0\mathcal{O}(n; m_1, \dots, m_k)$, we construct

$$\Psi(\mathbb{T}, f, \lambda, s, p) = (T, \sigma, \tau, B, t) \in B\mathcal{O}(n; m_1, \dots, m_k),$$

where (B, t) is a weighted bracketing on the labeled tree

$$(T, \sigma, \tau) = p_0(\mathbb{T}, f, \lambda, s, p) \in \mathcal{O}(n; m_1, \dots, m_k)$$

that is the image of $(\mathbb{T}, f, \lambda, s, p)$ under the canonical projection $p_0 \colon W_0\mathcal{O} \to \mathcal{O}$.

The bracketing B is constructed from the set of vertices of $\mathbb{T}$. If $\mathbb{T}$ has at most one vertex, then set $B = \emptyset$ to be the trivial bracketing, in which case there are no weights to choose so t is the empty map.

Otherwise, since $(\mathbb{T}, f, \lambda, s, p)$ is reduced, and $\mathbb{T}$ is not a corolla, all its vertices have arity ≥ 2. Let v_0 be the root vertex of $\mathbb{T}$. For each $v \in V(\mathbb{T}) \backslash \{v_0\}$, let

$$(S_v, \sigma_v, \tau_v) = p_0(\mathbb{T}_v, f|_{\mathbb{T}_v}, \lambda|_{\mathbb{T}_v}, p|_{\mathbb{T}_v}),$$

where $\mathbb{T}_v$ is the subtree of $\mathbb{T}$ with v as its root vertex, and containing all the vertices above v. Observe, in particular, that, since $v \neq v_0$, the outgoing edge e_v of v – that is the root of S_v – is internal in T. Since the

vertices of $\mathbb{T}$ have arity at least 2, each S_v is a large proper subtree of T, and because composition in $\mathcal{O}$ is by substitution,

$$B = \{S_v \ : \ v \in V(\mathbb{T})\backslash\{v_0\}\}$$

is a collection of nested subtrees, and hence a bracketing.

To define the weight function t of B, we associate, to each S_v the weight $t_v = s(e_v)$, the length of $e_v \in iE(\mathbb{T})$. This completes the definition of $\Psi(\mathbb{T}, f, \lambda, s, p)$.

We need to check that the defined bracketing is independent of our choice of (reduced) representative $(\mathbb{T}, f, \lambda, s, p) \in W_0\mathcal{O}(n; m_1, \dots, m_k)$, and continuous. In particular, we must check that it is compatible with the relations (1), (2') and (3') in Definition 6.1 and Section 6.2.

To prove that Ψ is well defined with respect to relation (1), and hence also continuous, let s_j be the length of an internal edge of $\mathbb{T}$ with end vertices v, w, where v is above w. Then if s_j goes to 0 in $W_0\mathcal{O}$, the vertices v, w are identified and their labels are composed in $\mathcal{O}$. Applying Ψ, this will precisely have the effect of taking the weight of the bracketing S_v to 0, which is equivalent to simply forgetting the bracketing S_v in $B\mathcal{O}$.

Relation (2') allows that a vertex v with only 1 input in $\mathbb{T}$, labeled by a permutation $\alpha \in \mathcal{O}(n; n) \cong \Sigma_n$, to be composed to either of the vertices it shares an edge with. So suppose $\mathbb{T}$ is the reduction of a tree $\widetilde{\mathbb{T}}$ with an arity one vertex v attached to two vertices w and w', with w' below w. We may assume that w and w' both have arity at least two. We let $\mathbb{T}$ be the tree obtained from $\widetilde{\mathbb{T}}$ by collapsing the edge between w and v and let $\mathbb{T}'$ be the tree obtained from $\widetilde{\mathbb{T}}$ by collapsing the edge between v and w'. We need to check that the brackets S_w and $S_{w'}$ are the same if computed using the representative $(\mathbb{T}, f, \lambda, s, p)$ associated to $\mathbb{T}$ or $(\mathbb{T}', f', \lambda, s', p')$ associated to $\mathbb{T}'$. This is immediate for the bracket $S_{w'}$ because w' is below v and thus $p_0(\mathbb{T}_{w'}, f|_{\mathbb{T}_{w'}}, \lambda|_{\mathbb{T}_{w'}}, p|_{\mathbb{T}_{w'}}) = p_0(\mathbb{T}'_{w'}, f'|_{\mathbb{T}'_{w'}}, \lambda|_{\mathbb{T}'_{w'}}, p'|_{\mathbb{T}'_{w'}})$. For the vertex w, the two representatives in general do not have the same image under p_0, but if $p_0(\mathbb{T}'_w, f'|_{\mathbb{T}'_w}, \lambda|_{\mathbb{T}'_w}, p'|_{\mathbb{T}'_w}) = (S'_w, \sigma'_w, \tau'_w)$, we still have that $S'_w = S_w$. In fact, only τ'_w might differ from τ_w as the vertex v is a permutation $\alpha \in \mathcal{O}(n; n) = \Sigma_n$ that acts on a labeling $p \in \mathcal{O}(n; k_1, \dots, k_l)$ by permuting the leaves of the labeled tree representing p.

For relation (3') in the definition of W_0, the relation gives a unique way to reduce a tree if an arity zero vertex is attached to another vertex, so the representative with no arity 0 vertices is unique and nothing needs to be checked.

Finally, we check that Ψ is a map of operads. Consider a composition $(\mathbb{T}_1, f_1, \lambda_1, s_1, p_1) \circ_i (\mathbb{T}_2, f_2, \lambda_2, s_2, p_2)$ of reduced representatives in $W_0\mathcal{O}$, and let $\Psi(\mathbb{T}_j, f_j, \lambda_j, s_j, p_j) = (T_j, \sigma_j, \tau_j, B_j, t_j)$ for $j = 1, 2$. Composition in $W_0\mathcal{O}$ is induced by grafting a tree $\mathbb{T}_2$ onto the ith leaf of $\mathbb{T}_1$, creating a new internal edge of length 1. If $\mathbb{T}_2$ has at least one vertex of arity 2, this corresponds exactly under Ψ to adding a new bracket T_2 of weight 1 in the composed tree $T_1 \bullet_{\sigma_1(i),\tau_2} T_2$ where the composition here is by insertion. If not, then, since $(\mathbb{T}_2, f_2, \lambda_2, s_2, p_2)$ is reduced, $\mathbb{T}_2$ has either no vertices or a single arity 1 vertex, so T_2 is either the exceptional tree η or a corolla C_n. In each case, the newly added edge in the composed tree $\mathbb{T}_1 \circ_i \mathbb{T}_2$ will be collapsed when going to a reduced tree, corresponding under Ψ to a composition in $B\mathcal{O}$ where no extra bracket is added. This finishes the proof. □

Lemma 6.10. *For every $(n; m_1, \dots, m_k)$ the map $\Psi : W_0\mathcal{O}(n; m_1, \dots, m_k) \longrightarrow B\mathcal{O}(n; m_1, \dots, m_k)$ is a bijection.*

Proof. We start by checking that Ψ is surjective. So let (T, σ, τ, B, t) of $B\mathcal{O}(n; m_1, \dots, m_k)$ with $B = \{S_j\}_{j\in J}$ and $t \in [0, 1]^J$. We may always choose a representative where all brackets have non-zero weight, so we assume that $t_j \neq 0$ for any $j \in J$. We will construct an element $(\mathbb{T}, f, \lambda, s, p) \in W_0\mathcal{O}(n; m_1, \dots, m_k)$ in the preimage of (T, σ, τ, B, t).

If $B = \emptyset$ is the empty bracketing then define $\mathbb{T}$ to be the corolla with k leaves, with f coloring its leaves $m_1, \ldots, m_k$ in the ordering given by λ, and the root by n and p labeling the unique vertex by (T, σ, τ). The weights s are trivial in this case. By definition, Ψ takes this element to $(T, \sigma, \tau, \emptyset, 0)$ as required.

We now assume that $B = \{S_j\}_{j \in J}$ is non-empty. To encode the leaf labeling τ on T, it is convenient to choose a non-reduced representative of its preimage, using a tree $\mathbb{T}$ with one valence 1 vertex at its root. We define $\mathbb{T}$ as follows: we set $V(\mathbb{T}) = \{v_\tau, v_T\} \cup \{v_j\}_{j \in J}$, where the vertex v_j corresponds to the bracket $S_j \in B$, v_T corresponds to an additional "trivial bracket" $S_T := T$, and v_τ will be associated to the permutation τ. To construct the tree, we set v_i above v_j if $S_i \subset S_j$, connecting the two vertices by an edge e_i if there is no $k \in J$ such that $S_i \subsetneq S_k \subsetneq S_j$, where we allow $S_j = S_T$. This edge e_i is colored by the number $|L(S_i)|$ of leaves of the smaller tree S_i and we define its length by setting $s_i = t_i$ is the weight of the corresponding bracket. The nesting condition on the brackets implies that no cycles are formed this way. We also connect v_T and v_τ by an edge of length 1, colored by $n = |L(T)|$, which is also the color of the root of the tree.

Finally for each vertex v of T, we attach a leaf l_v to the vertex $v_i \in V(\mathbb{T})$ if S_i is the smallest tree of the bracketing containing v, attaching it to v_T if v is contained in no bracket. This leaf is colored by the arity of v in T. This defines the tree $\mathbb{T}$, with edge lengths s and edge coloring f.

We pick some planar structure for $\mathbb{T}$. (Recall that elements of $W_0\mathcal{O}$ are only defined up to non-planar isomorphism, which is why there is some freedom here.) Note that the leaves of $\mathbb{T}$ correspond exactly to the vertices of T. The ordering $\lambda : \{1, \ldots, k\} \to L(\mathbb{T})$ is determined by σ and this identification. This defines the tuple $(\mathbb{T}, f, \lambda, s)$.

All that remains is to define the decoration p of the vertices of $\mathbb{T}$ by elements of $\mathcal{O}$. We need to have that (T, σ, τ) is given by the composition of the elements of the vertices of $\mathbb{T}$ so to determine the decorations in $\mathbb{T}$, what we need is to "undo" the compositions in T marked by the bracketings.

Let $v_j \in V(\mathbb{T})$. We define $p(v_j)$ to be the element $(S_j/\sim, \sigma_j, \tau_j) \in \mathcal{O}(out(v_j); in(v_j))$ where $S_j/\sim$ is the planar tree S_j with each subtree $S_i \subsetneq S_j$ collapsed to a corolla with the same set of leaves, σ_j orders the vertices according to the above chosen planar ordering of $\mathbb{T}$, where we note that the incoming edges of v_j correspond precisely to the vertices of $S_j/\sim$, and τ_j labels the leaves of $S_j/\sim$, which are also the leaves of S_j, in the order given by the planar embedding of T. (Here it is important that the chosen planar structure of $\mathbb{T}$ is compatible with the chosen order σ_j of $V(S_j/\sim)$. On the other hand, the chosen order τ_j of $L(S_j/\sim)$ is not important, as we will fix it below using the vertex v_τ.) This determines p uniquely on all vertices $\{v_j\}_{j \in J} \cup \{v_T\}$. Finally, the vertex v_τ is labeled by the permutation $\tau \in \Sigma_n$, considered as an element of $\mathcal{O}(n; n)$.

This finishes the construction of $(\mathbb{T}, f, \lambda, s, p)$. To compute its image under Ψ, we have to pass to a reduced representative, which means collapsing the edge between v_τ and v_T and composing their labeling. (The length of that edge is forgotten.) We have that $\Psi(\mathbb{T}, f, \lambda, s, p) = (T, \sigma, \tau, B, t)$, by our choice of p for the tree T and its leaf-labeling τ, our choice of λ for the ordering σ of the vertices, our choice of vertices of $\mathbb{T}$ for B, and with a direct correspondence between the length s_i of the edge e_i and the weight t_i of B_i.

To finish the proof, we check that Ψ is injective. We will check that, up to the equivalence relations defining $W_0\mathcal{O}$, there is a unique reduced $(\mathbb{T}', f', \lambda', s', p')$ in the preimage of (T, σ, τ, B, t). Note that the number of vertices of such a reduced representative is determined by the tree T and the cardinality of B. We consider first the cases where $\mathbb{T}'$ has 0 or 1 vertex.

If $\mathbb{T}'$ has no vertices, then $\mathbb{T}' = \eta$ representing the identity element in $\mathcal{O}(1; 1)$, $B = \emptyset$, and, up to the equivalence relations of $W_0\mathcal{O}$, there is only one possibility for $(\mathbb{T}', f', \lambda', s', p')$.

Suppose now that $\mathbb{T}' = C_k$ has exactly one vertex of arity k. The leaves of $\mathbb{T}'$ are in one-to-on correspondence with the vertices of T, with λ' ordering its leaves, and f' coloring them $m_1, \ldots, m_k, n$, with m_i the color of $\lambda'(i)$. We can choose a representative of $(\mathbb{T}', f', \lambda', s', p')$ so that the planar structure of $\mathbb{T}' = C_k$ is given by the ordering σ of the vertices of T. Then the labeling p of the vertex is necessarily precisely (T, σ, τ). So there is only one possibility for $(\mathbb{T}', f', \lambda', s', p')$.

Finally, if $\mathbb{T}'$ has at least two vertices, then it must have precisely $|B|+1$ vertices arranged in a tree according to the nested structure of the bracket, and k leaves, with each leaf attached to the vertex corresponding to the appropriate bracket. The root vertex of $\mathbb{T}$ corresponds to the whole tree T. The coloring of the edges is determined by the arity of the vertices and brackets in T, and the labeling of the leaves λ is determined by the ordering σ. The vertices are decorated by tuples (T_j, σ_j, τ_j), with T_j determined by the bracketing B, σ_j determined by the nesting of the bracketing once a planar structure for $\mathbb{T}'$ is chosen. Choosing a different planar structure will give an equivalent element of $W_0\mathcal{O}$ (in fact also of $W\mathcal{O}$). The ordering τ_j is likewise not uniquely determined by the situation, but a different choice that does yield the same tuple (T, σ, τ, B, s) under Ψ will be equivalent in $W_0\mathcal{O}$, using relation $(2')$. This finishes the proof of injectivity. □

We are now ready to prove our main result in this section, namely that $W_0\mathcal{O}$ and $B\mathcal{O}$ are isomorphic as topological operads.

Proof of Theorem 6.4. In Lemma 6.9 we constructed a map of topological operads

$$\Psi : W_0\mathcal{O} \longrightarrow B\mathcal{O}.$$

Combining this with Lemma 6.10 we know that, for each tuple $(n; m_1, \dots, m_k)$, the map

$$\Psi : W_0\mathcal{O}(n; m_1, \dots, m_k) \longrightarrow B\mathcal{O}(n; m_1, \dots, m_k)$$

is a continuous bijection. As the source of this map is a compact space ($\pi_0 W_0\mathcal{O}(n; m_1, \dots, m_k) = \mathcal{O}(n; m_1, \dots, m_k)$ is finite and there are finitely many reduced representatives $(\mathbb{T}, f, \lambda, s, p)$ defining a cube in each component), and the target is a Hausdorff space, Ψ is therefore a local homeomorphism and hence an isomorphism of topological operads. □

Remark 6.11. A corollary of the result we just proved is that $W_0\mathcal{O}$ is the realization of an operad in posets, namely the operad $B\mathcal{O}$. The operad $W\mathcal{O}$ can likewise be seen as the realization of an operad in posets, namely the poset of elements of the free operad $F\mathcal{O}$, with poset structure generated by edge collapses. The map of operads $q : W\mathcal{O} \to W_0\mathcal{O}$ is the realization of a map of posets. Indeed, the map $q\colon W\mathcal{O} \to W_0\mathcal{O} \cong B\mathcal{O}$ respects the poset structure because collapsing an edge in $\mathbb{T}$, which defines the poset structure underlying $W\mathcal{O}$, corresponds under the map q to forgetting a bracket, which defines the poset structure underlying $W_0\mathcal{O} = B\mathcal{O}$.

Appendix A. The explosion category of Ω

In Section 4.1 we introduced an enriched version of the dendroidal category $\widetilde{\Omega}_0$ which is closely related to the category of $B\mathcal{O}$-algebras. As mentioned in the introduction of Section 4, the idea of the category $\widetilde{\Omega}_0$ is to encode homotopy coherent Ω–diagram, and hence $\widetilde{\Omega}_0$ should be connected to the *explosion category* of Ω, as defined by Leitch [25] and Segal [37, Appendix B].

In this appendix we describe the explosion category of Ω, denoted $\widetilde{\Omega}$, and show that our topological category $\widetilde{\Omega}_0$ sits between $\widetilde{\Omega}$ and Ω in the sense that there exist equivalences of topological categories

$$\widetilde{\Omega} \xrightarrow[q]{} \widetilde{\Omega}_0 \underset{p}{\overset{\tilde{p}}{\rightleftarrows}} \Omega \ .$$

The explosion construction and the W–construction are very closely related in spirit. One might thus expect a relationship between Segal $\widetilde{\Omega}$–diagrams and $W\mathcal{O}$–algebras, similar to the relationship between

Segal dendroidal spaces (Ω–diagrams) and $\mathcal{O}$–algebras, and between Segal homotopy dendroidal spaces ($\widetilde{\Omega}_0$–diagrams) and $W_0\mathcal{O}$– or $B\mathcal{O}$–algebras. Theorem A.6 below will show that such a relationship exists, but without being as close as in the other cases: $W\mathcal{O}$–algebras identify with a full subcategory of the category of reduced strict Segal $\widetilde{\Omega}$–diagrams.

A.1. The explosion of Ω

For each morphism $g : S \to T$ in Ω, we define a *poset of paths* $\mathrm{Path}_\Omega(S,T)_g$ whose objects are the factorizations of $g : S \to T$ in Ω

$$S \xrightarrow[g_1]{} T_1 \xrightarrow[g_2]{} \dots \longrightarrow T_{n-1} \xrightarrow[g_n]{} T, \qquad (g)$$

where we identify two factorizations if they differ only by identity morphisms. In particular, each such factorization $(g_1, \dots, g_n)$ has a unique reduced representative containing no identity morphisms unless $n = 1$ and g is the identity on S. (Such a factorization can be thought of as a path in the nerve of Ω.) The poset structure is by refinement of factorization: $(g_1, \dots, g_n) \leq (g'_1, \dots, g'_m)$ if $n \leq m$ and there is a monotone map $\alpha : \{0, \dots, n\} \to \{0, \dots, m\}$ such that $\alpha(0) = 0$, $\alpha(n) = (m)$, and $g_i = g'_{\alpha(i)} \circ \dots \circ g'_{\alpha(i-1)+1}$ for each $1 \leq i \leq n$.

We denote the geometric realization of this poset by

$$K_g := |\mathrm{Path}_\Omega(S,T)_g|.$$

Definition A.1. The topological category $\widetilde{\Omega}$ has the same objects as Ω. Morphism spaces in $\widetilde{\Omega}$ are defined as

$$\mathrm{Hom}_{\widetilde{\Omega}}(S,T) = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} K_g = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} |\mathrm{Path}_\Omega(S,T)_g|.$$

Composition of morphisms of $\widetilde{\Omega}$ is given by concatenation of factorizations.

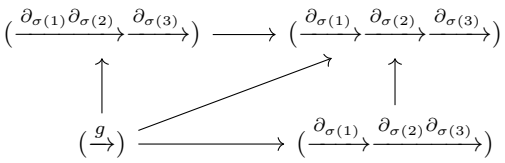
Example A.2. Fix a tree T with $|L(T)| = n$ leaves and three inner edges: e_1, e_2, e_3. Recall that C_n denotes the corolla with n leaves. Let ∂_{e_1}, ∂_{e_2}, ∂_{e_3} denote the inner face maps in Ω associated to each inner edge, and let $g = \partial_{e_1}\partial_{e_2}\partial_{e_3} : C_n \to T$ be their composition. Then g admits a factorization

$$C_n \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T, \qquad (g)$$

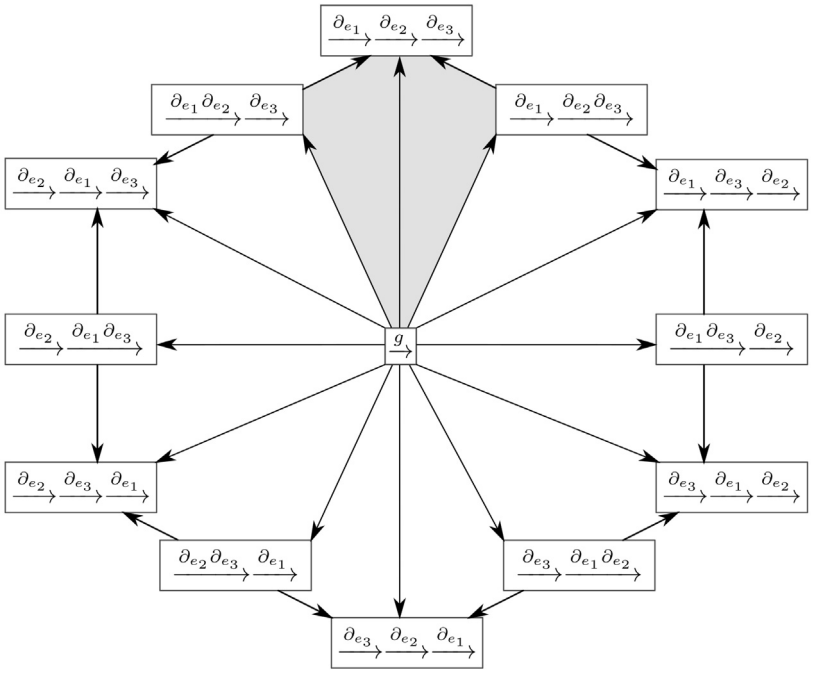
as a composition of three inner face maps for each permutation of $\{1, 2, 3\}$. The elements of $\mathrm{Path}_\Omega(C_n, T)_g$ that involve only these three inner face maps form a subposet with $([1], \xrightarrow{g})$ as minimum, and for each permutation $\sigma \in \Sigma_3$ the elements

$$([3], \xrightarrow{\partial_{\sigma(1)}} \xrightarrow{\partial_{\sigma(2)}} \xrightarrow{\partial_{\sigma(3)}}) \qquad ([2], \xrightarrow{\partial_{\sigma(1)}} \xrightarrow{\partial_{\sigma(2)}\partial_{\sigma(3)}}) \qquad ([2], \xrightarrow{\partial_{\sigma(1)}\partial_{\sigma(2)}} \xrightarrow{\partial_{\sigma(3)}}).$$

Each permutation σ this way contributes to a square



in this subposet, and the dendroidal identities tell us that these squares together form the following hexagon inside $|\mathrm{Path}_\Omega(C_n, T)_g|$:



Additional elements of $\mathrm{Path}_\Omega(C_n, T)_g$ can be obtained by inserting tree isomorphisms. This example should be compared to Examples 3.4 and 3.5 which can be interpreted as computing morphism spaces in the category $\widetilde{\Omega}_0$ likewise associated to trees with three internal edges, where in one case a pentagon occurs, and in the other it is a hexagon.

Lemma A.3. *For each $g \in \mathrm{Hom}_\Omega(S, T)$ the space $K_g = |Path_\Omega(S, T)_g|$ is contractible.*

Proof. The poset $\mathrm{Path}_\Omega(S, T)_g$ has the trivial factorization $S \xrightarrow{g} T$ as a minimal element. □

Let $\tilde{p} : \widetilde{\Omega} \to \Omega$ be the functor that is the identity on objects and projects each morphism space K_g to g. Considering Ω as a discrete topological category, the lemma immediately gives the following proposition.

Proposition A.4. *The functor $\tilde{p} : \widetilde{\Omega} \to \Omega$ induces a homotopy equivalence on morphism spaces.*

Note that the proposition identifies Ω with the "path component category" $\pi_0\widetilde{\Omega}$, which has the same objects as $\widetilde{\Omega}$ and $\mathrm{Hom}_{\pi_0\widetilde{\Omega}}(S, T) := \pi_0(\mathrm{Hom}_{\widetilde{\Omega}}(S, T))$.

A.2. The relationship between $\widetilde{\Omega}$ and $\widetilde{\Omega}_0$

The category $\widetilde{\Omega}_0$ sits between $\widetilde{\Omega}$ and Ω in the sense of the following proposition.

Proposition A.5. *There is a functor $q : \widetilde{\Omega} \to \widetilde{\Omega}_0$, which is the identity on objects and induces a homotopy equivalence on each morphism space. Moreover, the composition $p \circ q = \tilde{p} : \widetilde{\Omega} \to \Omega$ is the projection functor of Proposition A.4.*

Proof. Fix two objects $S, T \in \widetilde{\Omega}$. Recall from Definition A.1 that

$$\mathrm{Hom}_{\widetilde{\Omega}}(S,T) = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} K_g$$

for $K_g = |\mathrm{Path}_\Omega(S,T)_g|$ is the realization of the poset of factorizations of g, and K_g is contractible. Likewise by Definition 4.4

$$\mathrm{Hom}_{\widetilde{\Omega}_0}(S,T) = \coprod_{g \in \mathrm{Hom}_\Omega(S,T)} L_g$$

and $L_g = \prod_{v \in V(S)} |\mathcal{B}(g(C_v))|$ is the realization of the poset $\mathcal{L}_g$ of bracketings of the trees $g(C_v)$, with L_g likewise contractible. So to prove the proposition, it is enough to produce a functor q which is the identity on objects and takes K_g to L_g for each g. We will define the functor by defining a poset map

$$q_g : \mathrm{Path}_\Omega(S,T)_g \to \mathcal{L}_g$$

and show that it is compatible with composition.

Fix a map $g : S \to T$ in Ω. An object of $\mathrm{Path}_\Omega(S,T)_g$ is a factorization $(g_1, \dots, g_n)$ of g and to such a factorization of g, for each $v \in V(S)$, we associate a bracketing of $g(C_v)$ as follows: set

$$B_v = \{S_w = g_n \circ \cdots \circ g_{i+1}(C_w)\}_{1 \le i \le n-1,\ w \in V(g_i \circ \cdots \circ g_1(C_v)),\ S_w \subsetneq g(C_v) \text{ large}}$$

This is a (possibly empty) bracketing as these sets are by definition nested. We then define $q_g(g_1, \dots, g_n) = (B_v)_{v \in V(S)}$. Note that this association is a map of posets as refining a factorization will correspond under q_g to an inclusion of bracketings.

We are left to check that the maps q_g assemble to define a functor, i.e. that they are compatible with composition in $\widetilde{\Omega}$ and $\widetilde{\Omega}_0$. Let $f : R \to S$ be another morphism in Ω. We need to check that

$$\begin{array}{ccc} \mathrm{Path}_\Omega(S,T)_g \times \mathrm{Path}_\Omega(R,S)_f & \longrightarrow & \mathrm{Path}_\Omega(R,T)_{g\circ f} \\ \downarrow{\scriptstyle q_g \times q_f} & & \downarrow{\scriptstyle q_{g\circ f}} \\ \mathcal{L}_g^{op} \times \mathcal{L}_f^{op} & \longrightarrow & \mathcal{L}_{g\circ f}^{op} \end{array}$$

commutes. Because the target is a poset, it is enough to check that it commutes on objects. Let $(g_1, \dots, g_n)$ and $(f_1, \dots, f_m)$ be objects of $\mathrm{Path}_\Omega(S,T)_g$ and $\mathrm{Path}_\Omega(R,S)_f$. By definition, their composition is $(f_1, \dots, f_m, g_1, \dots, g_n) \in \mathrm{Path}_\Omega(R,T)_{g\circ f}$. We have $q_f(f_1, \dots, f_m) = (B_x^f)_{x \in V(R)}$ and $q_g(g_1, \dots, g_n) = (B_v^g)_{v \in V(S)}$ with

$$B_x^f = \{S_y = f_m \circ \cdots \circ f_{i+1}(C_y)\}_{1 \le i \le m-1,\ y \in f_i \circ \cdots \circ f_1(C_x),\ S_y \subsetneq f(C_x) \text{ large}}$$

and bracketing of $f(C_x) \subset S$, and

$$B_v^g = \{S_w = g_n \circ \cdots \circ g_{i+1}(C_w)\}_{1 \leq i \leq n-1,\ w \in g_i \circ \cdots \circ g_1(C_v),\ S_w \subsetneq g(C_v) \text{ large}}$$

a bracketing of $g(C_v) \subset T$. By definition, $q_g(g_1, \ldots, g_n) \circ q_f(f_1, \ldots, f_m)$ is the collection $(\bar{B}_x)_{x \in V(R)}$ of bracketings of each tree $g \circ f(C_x) \subset T$ defined by

$$\bar{B}_x = \left(\bigcup_{v \in f(C_x)} B_v^g \right) \cup \left(\bigcup_{\substack{v \in f(C_x) \\ g(C_v) \subsetneq g \circ f(C_x) \text{ large}}} \{g(C_v)\} \right) \cup \left(\bigcup_{\substack{S_y \in B_x^f \\ g(S_y) \text{ large}}} \{g(S_y)\} \right),$$

where B_v^g is considered as a bracketing of $g \circ f(C_x)$ via the inclusion $g(C_v) \subset g \circ f(C_x)$. Now we see that this is exactly the bracketing of $g \circ f(C_x)$ defined by the factorization $(f_1, \ldots, f_m, g_1, \ldots, g_m)$, which indeed is the union of the sets

$$\{g(S_y) = g \circ f_m \circ \cdots \circ f_{i+1}(C_y)\}_{\substack{1 \leq i \leq m-1 \\ y \in f_i \circ \cdots \circ f_1(C_x) \\ g(S_y) \subsetneq g \circ f(C_x) \text{ large}}} \cup \{S_v = g(C_v)\}_{\substack{v \in f(C_x) \\ S_v \subsetneq g \circ f(C_x) \text{ large}}}$$
$$\cup \{S_w = g_n \circ \cdots \circ g_{i+1}(C_w)\}_{\substack{1 \leq i \leq n-1 \\ w \in g_i \circ \cdots \circ g_1 \circ f(C_x) \\ S_w \subsetneq g \circ f(C_x) \text{ large}}}.$$

Hence the poset maps q_g assemble to define a functor $q : \widetilde{\Omega} \to \widetilde{\Omega}_0$ as claimed. Moreover, one readily checks that the composition with the projection $p : \widetilde{\Omega}_0 \to \Omega$ is the canonical projection $\tilde{p} : \widetilde{\Omega} \to \Omega$. □

A.3. $W\mathcal{O}$–algebras as $\widetilde{\Omega}$–diagrams

In Section 4.2 we showed that $B\mathcal{O}$-algebras describe dendroidal Segal spaces. For completeness, we now show how homotopy dendroidal spaces $\mathcal{S}^{\widetilde{\Omega}^{op}}$ are related to $W\mathcal{O}$-algebras.

We will only need to consider $\widetilde{\Omega}$–diagrams $X : \widetilde{\Omega}^{op} \to \mathcal{S}$ that are reduced, i.e. such that $X(\eta) = *$. Recall that in this case, for $X : \Omega \to \mathcal{S}$, the Segal map becomes the map

$$X(T) \xrightarrow{\chi} \prod_{v \in V(T)} X(C_v)$$

induced by the restriction maps $T \to C_v$ in Ω^{op}. Considering these morphisms as morphisms of $\widetilde{\Omega}$, we likewise have a Segal map for $X : \widetilde{\Omega} \to \mathcal{S}$ in the reduced case.

In analogy to the case of dendroidal and homotopy dendroidal spaces, let $(\mathcal{S}^{\widetilde{\Omega}^{op}})_{strict}$ denote the full subcategory of $\mathcal{S}^{\widetilde{\Omega}^{op}}$ of $\widetilde{\Omega}$–diagrams $X : \widetilde{\Omega}^{op} \to \mathcal{S}$ such that $X(\eta) = *$ and such that the Segal map χ as above is an isomorphism for every $T \neq \eta$. We have the following:

Theorem A.6. *There exists a functor*

$$\Psi : W\mathcal{O}\text{–Alg} \to (\mathcal{S}^{\widetilde{\Omega}^{op}})_{strict}$$

that embeds the category of $W\mathcal{O}$-algebras as a full subcategory of the category of reduced $\widetilde{\Omega}$–diagrams satisfying the strict Segal condition.

As we will see in the proof, $\widetilde{\Omega}$–diagrams are governed by a version of $W\mathcal{O}$ where the trees $\mathbb{T}$ have an additional level structure, and $W\mathcal{O}$–algebras identify them as the subcategory of diagrams where this level structure does not matter. If one wished to describe a category of homotopy dendroidal spaces which is isomorphic to $W\mathcal{O}$-algebras, one could use this observation to take an appropriate quotient of $\widetilde{\Omega}$. As this is particularly messy, and not the main focus of this article, we have elected not to include such a construction.

Proof. The proof is similar to that of Theorem 4.8 treating the case of $B\mathcal{O}$–algebras. We start with the definition of the functor Ψ. Let $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ be a $W\mathcal{O}$–algebra with structure maps

$$\alpha_{\mathcal{P}} : W\mathcal{O}(n; m_1, \dots, m_k) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_k) \longrightarrow \mathcal{P}(n).$$

We associate to this data an $\widetilde{\Omega}$–diagram

$$\Psi(\mathcal{P}) = \Psi(\mathcal{P}, \alpha_{\mathcal{P}}) : \widetilde{\Omega}^{op} \to \mathcal{S}$$

as follows. Set $\Psi(\mathcal{P})(\eta) = *$ and, for $T \neq \eta$ in $\widetilde{\Omega}$, set

$$\Psi(\mathcal{P})(T) = \prod_{w\in V(T)} \mathcal{P}(|w|).$$

For every morphism $g : S \to T$ in Ω, we need to define maps

$$\Psi(\mathcal{P})(g) : K_g \times \prod_{w\in V(T)} \mathcal{P}(|w|) \longrightarrow \prod_{v\in V(S)} \mathcal{P}(|v|).$$

As in Theorem 4.8, we do this one vertex of S at a time.

Recall that K_g is the realization of the poset $\mathrm{Path}_{\Omega}(S,T)_g$ of factorizations

$$S \xrightarrow{g_1} T_1 \xrightarrow{g_2} \dots \xrightarrow{g_{n-1}} T_{n-1} \xrightarrow{g_n} T$$

of g in Ω. For each $v \in V(S)$, we consider the restriction of these maps to $C_v \in S$:

$$C_v \xrightarrow{g_1} g_1(C_v) \xrightarrow{g_2} \dots \xrightarrow{g_{n-1}} g_{n-1} \circ \cdots \circ g_1(C_v) \xrightarrow{g_n} g(C_v) \subset T. \qquad (*)$$

Recall from Remark 6.11 that $W\mathcal{O}$ is the realization of an operad in posets, whose elements are those of the free operad $F\mathcal{O}$ (identifying elements of $F\mathcal{O}$ with elements of $W\mathcal{O}$ in which all weights of internal edges are 1). We will now use the restriction $(*)$ of $(g_1, \dots, g_n)$ to C_v to construct a labeled planar tree

$$(\mathbb{T}, f, \lambda, p) \in F\mathcal{O}(|v|; (|w|)_{w\in V(g(C_v))})$$

by induction on the height of the tree:

Starting at the root, we attach a vertex $\bar{v}$ of valence $|V(g_1(C_v))|$. The incoming edges of $\bar{v}$ are labeled in accordance with $(g_1(C_v), \sigma_v, \tau_v)$, where σ_v is a chosen ordering of the vertices of the tree $g_1(C_v)$, and τ_v is induced by the planar structure of $g_1(C_v) \subset T_1$. Specifically, the incoming edges of $\bar{v}$ are labeled by the vertices of $g_1(C_v)$ and ordered via the map σ_v.

For each vertex $w \in g_1(C_v)$, which is now an incoming edge of $\bar{v}$, we can attach a vertex $\bar{w}$ of valence $|V(g_2(C_w))|$. These incoming edges are labeled with the tuple $(g_2(C_w), \sigma_w, \tau_w)$, as in the previous case.

More generally, for vertices with height $2 \leq i \leq n$, the tree $\mathbb{T}$ has a vertex $\bar{y}$ for every vertex y in $(g_{i-1} \circ \cdots \circ g_1)(C_v)$, attached to the previously constructed vertex $\bar{x}$ associated to the vertex $x \in (g_{i-2} \circ \cdots \circ g_1)(C_v)$

satisfying that $y \in g_{i-1}(C_x)$. We label $\bar{y}$ by the tuple $(g_i(\bar{y}), \sigma_y, \tau_y)$ with τ_y induced by the planar structure of T_i, giving $C_{\bar{y}}$ the planar structure dictated by the chosen σ_y.

We now set $f : E(\mathbb{T}) \to \mathbb{N}$ to be the unique meaningful coloring which makes $\mathbb{T}$ an element of $F\mathcal{O}(|v|; |w_1|, \ldots, |w_k|)$. We set the ordering σ of the vertices $w_1, \ldots, w_k$ of $g(C_v)$ in accordance to the resulting planar structure on $\mathbb{T}$. As the set of vertices of $g(C_v)$ is also the set of leaves of the tree $\mathbb{T}$, this also defines λ. We note that the tree constructed this way is in no way reduced and will, a priori, have many arity one vertices labeled by identities. We can use relations defining $F\mathcal{O}$, however, to remove such vertices and give an equivalent element in $F\mathcal{O}$.

This assignment of the restriction of a factorization $(*)$ to a labeled tree $\mathbb{T}$ respects the poset structure of $\mathrm{Path}_\Omega(S,T)_g$ and $W\mathcal{O}$ as refining the factorization corresponds to undoing the collapse of edges, namely if $(g_1, \ldots, g_n) \leq (g'_1, \ldots, g'_m)$, then the image $(\mathbb{T}, f, \lambda, p)$ of the first factorization can be obtained from the image $(\mathbb{T}', f', \lambda', p')$ of the second by collapsing the edges corresponding to the added levels, as collapsing level in the tree correspond in this construction to composing consecutive maps g_i.

In this way we can apply the structure map $\alpha_{\mathcal{P}}$ one vertex at a time and define a map

$$\alpha_v : |\mathrm{Path}_\Omega(S,T)_g| \times \prod_{w_i \in V(g(C_v))} \mathcal{P}(|w_i|) \longrightarrow \mathcal{P}(|v|) \tag{A.1}$$

and we can define

$$\Psi(\mathcal{P})(g) = (\alpha_v)_{v \in V(S)}.$$

By construction, the action of $\Psi(P)$ on morphisms commutes with composition in $\widetilde{\Omega}$, and thus $\Psi(\mathcal{P}) : \widetilde{\Omega}^{op} \to \mathcal{S}$ defines a functor. That the Segal map for $\Psi(\mathcal{P})$ is an isomorphism for every $T \neq \eta$ follows immediately from our definition of $\Psi(\mathcal{P})$.

The assignment $\mathcal{P} \mapsto \Psi(\mathcal{P})$ requires only the data of underlying symmetric sequence of $\mathcal{P}$ and the algebra structure maps $\alpha_{\mathcal{P}}$. This data is natural under maps of $W\mathcal{O}$-algebras and thus

$$\Psi : W\mathcal{O}\text{–Alg}_{\mathcal{S}} \longrightarrow \mathcal{S}^{\widetilde{\Omega}^{op}}$$

is a functor.

It remains to check that Ψ is an embedding of a full subcategory. Injectivity on objects follows from the fact that if $\mathcal{P}$ and $\mathcal{Q}$ satisfy that $\Psi(\mathcal{P}) = \Psi(\mathcal{Q})$, then we necessarily have that $\mathcal{P}(n) = \mathcal{Q}(n)$ for each n, as given by the value at the corolla, with agreeing symmetric group actions as given by the isomorphisms of corollas, and the structure maps $\alpha_{\mathcal{P}}$ and $\alpha_{\mathcal{Q}}$ likewise must agree as the value of the structure map on every element of $W\mathcal{O}$ is the value of the functor $\Psi(\mathcal{P}) = \Psi(\mathcal{Q})$ on an associated morphism of $\widetilde{\Omega}$ obtained by choosing a level structure on the tree and interpreting the collapse of each level of the tree as a morphism in Ω. As morphisms of $W\mathcal{O}$–algebras are determined by what they do on spaces $\mathcal{P}(n)$, we see that the functor is faithful. It is also full as natural transformations between diagrams originating from $W\mathcal{O}$–algebras, will necessarily respect the $W\mathcal{O}$-algebra structure of their values at the corollas. □

Remark A.7. The reader might be tempted to compare the functor $\Psi(-)$ from Theorem A.6 with the *homotopy coherent nerve* of a topological operad $\mathcal{P}$. This is a functor

$$w^* : \mathcal{O}\text{–Alg} \to \mathsf{Set}^{\Omega^{op}}$$

defined by

$$(w^*\mathcal{P})(T) = \mathrm{Hom}_{\mathsf{Op}}(W\Omega(T), \mathcal{P}),$$

where $W\Omega(T)$ denotes the Boardman-Vogt W-construction applied to the free operad generated by a tree T (Example 2.4). The functors Ψ and w^* are not equivalent on operads, though if one has a $W\mathcal{O}$-algebra $\mathcal{P}$ which happens to be an operad then one can define a dendroidal space $X_{\mathcal{P}} \in \mathcal{S}^{\Omega^{op}}/w^*\mathcal{P}$, where the later denotes the slice category. For more on this point of view, see [32, Remark 6.2] or [5, Corollary 1.7].

References

[1] Clemens Berger, Ieke Moerdijk, Resolution of coloured operads and rectification of homotopy algebras, in: Categories in Algebra, Geometry and Mathematical Physics, in: Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 31–58.
[2] Clemens Berger, Ieke Moerdijk, On an extension of the notion of reedy category, Math. Z. 269 (3) (Dec 2011) 977–1004.
[3] Julia E. Bergner, Philip Hackney, Group actions on Segal operads, Isr. J. Math. 202 (1) (2014) 423–460.
[4] Pedro Boavida de Brito, Geoffroy Horel, Marcy Robertson, Operads of genus zero curves and the Grothendieck-Teichmüller group, Geom. Topol. 23 (1) (2019) 299–346.
[5] Pedro Boavida de Brito, Ieke Moerdijk, Dendroidal spaces, Γ-spaces and the special Barratt-Priddy-Quillen theorem, J. Reine Angew. Math. 760 (2020) 229–265.
[6] Pedro Boavida de Brito, Michael Weiss, Spaces of smooth embeddings and configuration categories, J. Topol. 11 (1) (2018) 65–143.
[7] B.H. Bowditch, D.B.A. Epstein, Natural triangulations associated to a surface, Topology 27 (1) (1988) 91–117.
[8] Micheal Brinkmeier, On Operads, PhD thesis, Universität Osnabrück, 2000.
[9] Michael Carr, Satyan Devadoss, Coxeter complexes and graph-associahedra, Topol. Appl. 153 (2006) 2155.
[10] Hongyi Chu, Rune Haugseng, Gijs Heuts, Two models for the homotopy theory of ∞-operads, J. Topol. 11 (4) (2018) 857–873.
[11] Denis-Charles Cisinski, Ieke Moerdijk, Dendroidal Segal spaces and ∞-operads, J. Topol. 6 (3) (2013) 675–704.
[12] Denis-Charles Cisinski, Ieke Moerdijk, Dendroidal sets and simplicial operads, J. Topol. 6 (3) (2013) 705–756.
[13] Satyan L. Devadoss, A realization of graph associahedra, Discrete Math. 309 (1) (2009) 271–276.
[14] Satyan L. Devadoss, Stefan Forcey, Stephen Reisdorf, Patrick Showers, Convex polytopes from nested posets, Eur. J. Comb. 43 (2015) 229–248.
[15] Kosta Došen, Zoran Petrić, Hypergraph polytopes, Topol. Appl. 158 (12) (2011) 1405–1444.
[16] W.G. Dwyer, D.M. Kan, Equivalences between homotopy theories of diagrams, in: Algebraic Topology and Algebraic K-Theory, Princeton, N.J., 1983, in: Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 180–205.
[17] Daniela Egas Santander, Comparing fat graph models of moduli space, arXiv preprint arXiv:1508.03433, 2015.
[18] Eva Maria Feichtner, Bernd Sturmfels, Matroid polytopes, nested sets and Bergman fans, Port. Math. New Ser. 62 (4) (2005) 437–468.
[19] Véronique Godin, Higher string topology operations, Preprint arXiv:0711.4859, 2007.
[20] Philip Hackney, Configuration spaces form a Segal semi-dendroidal space, High. Struct. 2 (1) (2018) 234–240.
[21] John L. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface, Invent. Math. 84 (1) (1986) 157–176.
[22] Gijs Heuts, Vladimir Hinich, Ieke Moerdijk, On the equivalence between Lurie's model and the dendroidal model for infinity-operads, Adv. Math. 302 (2016) 869–1043.
[23] Philip S. Hirschhorn, Model Categories and Their Localizations, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.
[24] Ralph M. Kaufmann, On several varieties of cacti and their relations, Algebraic Geom. Topol. 5 (2005) 237–300.
[25] R.D. Leitch, The homotopy commutative cube, J. Lond. Math. Soc. (2) 9 (1974/75) 23–29.
[26] Jacob Lurie, Higher Topos Theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.
[27] Martin Markl, Operads and PROPs, in: Handbook of Algebra. Vol. 5, in: Handb. Algebr., vol. 5, Elsevier/North-Holland, Amsterdam, 2008, pp. 87–140.
[28] J.P. May, The Geometry of Iterated Loop Spaces, Lectures Notes in Mathematics, vol. 271, Springer-Verlag, Berlin-New York, 1972.
[29] James E. McClure, Jeffrey H. Smith, A solution of Deligne's Hochschild cohomology conjecture, in: Recent Progress in Homotopy Theory, Baltimore, MD, 2000, in: Contemp. Math., vol. 293, Amer. Math. Soc., Providence, RI, 2002, pp. 153–193.
[30] James E. McClure, Jeffrey H. Smith, Cosimplicial objects and little n-cubes. I, Am. J. Math. 126 (5) (2004) 1109–1153.
[31] Ieke Moerdijk, Bertrand Toën, Simplicial Methods for Operads and Algebraic Geometry, in: Carles Casacuberta, Joachim Kock (Eds.), CRM Barcelona, in: Advanced Courses in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2010.
[32] Ieke Moerdijk, Ittay Weiss, Dendroidal sets, Algebraic Geom. Topol. 7 (2007) 1441–1470.
[33] Jovana Obradović, Combinatorial homotopy theory for operads, arXiv e-prints, page arXiv:1906.06260, Jun 2019.
[34] R.C. Penner, The decorated Teichmüller space of punctured surfaces, Commun. Math. Phys. 113 (2) (1987) 299–339.
[35] Alexander Postnikov, Permutohedra, associahedra, and beyond, Int. Math. Res. Not. 6 (2009) 1026–1106.
[36] Paolo Salvatore, The topological cyclic Deligne conjecture, Algebraic Geom. Topol. 9 (1) (2009) 237–264.
[37] Graeme Segal, Categories and cohomology theories, Topology 13 (1974) 293–312.
[38] Dev P. Sinha, Manifold-theoretic compactifications of configuration spaces, Sel. Math. New Ser. 10 (3) (2004) 391–428.
[39] Alexander A. Voronov, Notes on universal algebra, in: Graphs and Patterns in Mathematics and Theoretical Physics, in: Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, pp. 81–103.

[40] Nathalie Wahl, Infinite loop space structure(s) on the stable mapping class group, Topology 43 (2) (2004) 343–368.
[41] Nathalie Wahl, Craig Westerland, Hochschild homology of structured algebras, Adv. Math. 288 (2016) 240–307.
[42] Donald Yau, Mark W. Johnson, A Foundation for PROPs, Algebras, and Modules, Mathematical Surveys and Monographs, vol. 203, American Mathematical Society, Providence, RI, 2015.