Topology and its Applications 316 (2022) 108108

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Compactly supported $\mathbb{A}^1$-Euler characteristic and the Hochschild complex

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ARTICLE INFO

Article history:
Received 14 March 2020
Received in revised form 8 April 2021
Available online 7 April 2022

MSC:
primary 14F42
secondary 19E15, 13D03, 55M05

Keywords:
$\mathbb{A}^1$-Euler characteristic
Grothendieck–Witt group
Hochschild cohomology
Hermitian K-theory

ABSTRACT

We show the $\mathbb{A}^1$-Euler characteristic of a smooth, projective scheme over a characteristic 0 field is represented by its Hochschild complex together with a canonical bilinear form, and give an exposition of the compactly supported $\mathbb{A}^1$-Euler characteristic $\chi^c_{\mathbb{A}^1} : K_0(\mathbf{Var}_k) \longrightarrow \mathrm{GW}(k)$ from the Grothendieck group of varieties to the Grothendieck–Witt group of bilinear forms. We also provide example computations.

Published by Elsevier B.V.

1. Introduction

The Euler characteristic is one of the first combinatorial topological invariants. Leonhard Euler originally introduced it for polyhedra, claiming that for any Euclidean polyhedron it is the case that for any polyhedron

$$\#\text{ vertices} - \#\text{ edges} + \#\text{ faces} = 2.$$

Although this is true for any convex polyhedron, this is a surprisingly difficult fact to state formally and correctly, and depends intrinsically on what one means by "polyhedron". The Euler characteristic turns out to be a *topological* invariant, which is most classically defined for a finite CW complex X to be

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https://doi.org/10.1016/j.topol.2022.108108
0166-8641/Published by Elsevier B.V.

$$\chi(X) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q}).$$

To make the Euler characteristic well-defined for non-compact sets it is necessary to replace homology with cohomology with compact support; with this definition it follows that for a closed subspace Z of X with open complement U,

$$\chi(X) = \chi(Z) + \chi(U). \tag{1.1}$$

If instead of general topological spaces we consider only varieties (or schemes) over $\mathbb{C}$, the Euler characteristic of a variety X can be defined by the formula

$$\chi(X) \stackrel{\text{def}}{=} \chi(X(\mathbb{C})) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} H^i_c(X(\mathbb{C});\mathbb{Q}) \in \mathbb{Z}.$$

This invariant can be elevated by replacing $\mathbb{Q}$ with any field k, and retaining more cohomological information. Instead of taking dimensions, we can consider the vector spaces $H^i_c(X(\mathbb{C});k)$ as elements of $K_0(k)$, the Grothendieck group of the ground field. This gives the new definition

$$\chi_k(X) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} (-1)^i [H^i_c(X(\mathbb{C});k)] \in K_0(k).$$

As $K_0(k) \cong \mathbb{Z}$, with the isomorphism given by the dimension, this may not appear to be a useful observation.

However, in the case when X is a smooth, projective variety, $X(\mathbb{C})$ is a manifold and Poincaré duality states that there is a duality on the cohomology. By keeping track of the duality, we obtain an enriched Euler characteristic: such an Euler characteristic takes values in bilinear forms, rather than vector spaces. The form-valued Euler characteristic has beautiful applications in topology [28] [11] [9]. Stabilizing, we get an invariant taking values in the Grothendieck–Witt group, enriching the Euler characteristic valued in $K_0(k) \cong \mathbb{Z}$.

Definition 1.2. Let k be a field. The *Grothendieck–Witt group* $\mathrm{GW}(k)$ is the free abelian group generated by isomorphism classes of k-vector spaces equipped with a symmetric, nondegenerate bilinear form, under the relation that

$$[V,b] + [V',b'] = [V \oplus V', b \oplus b'].$$

To make this approach work correctly in the context of $\mathbb{A}^1$-homotopy theory, it is necessary to shift perspective. We will work with coherent cohomology, and the relevant additional structure is given by Grothendieck–Serre duality. Before describing the cohomological approach, we begin with the abstract definition of the categorical Euler characteristic. For the rest of this paper we restrict to fields of characteristic 0.

Definition 1.3. Let X be a smooth, projective variety over a field k. Then the motivic suspension spectrum $\Sigma^\infty_T(X_+)$ is dualizable in the stable motivic homotopy category $SH(k)$ [17, Theorem 5.22] [18, Appendix A] [20, Section 1 or ArXiv version 3 Section 1.1] [27] [34, Section 2]. Thus there exists an element

$$\chi^{\mathbb{A}^1}(X) \stackrel{\text{def}}{=} \left(1_k \longrightarrow \Sigma^\infty_T(X_+) \wedge_k D(\Sigma^\infty_T(X_+)) \longrightarrow D(\Sigma^\infty_T(X_+)) \wedge_k \Sigma^\infty_T(X_+) \longrightarrow 1_k\right)$$

in $\mathrm{End}_{SH(k)}(1_k)$, where 1_k denotes the motivic sphere spectrum, the first and last maps are coevaluation and evaluation, the middle is the symmetry swap isomorphism, and $D(-)$ denotes dualizing. See for example [16] or [20] for further discussion.

It is a beautiful theorem of F. Morel that $\mathrm{End}_{SH(k)}(1_k) \cong \mathrm{GW}(k)$, see for example [23] or [24, Theorem 1.23, Corollary 1.24], giving the categorical Euler characteristic $\chi^{\mathbb{A}^1}(X)$ in $\mathrm{GW}(k)$. So, one can hope to recover an enhancement of a cohomological construction by explicitly describing the map in Definition 1.3.

A description in terms of coherent duality was independently suggested by M.J. Hopkins, A. Raksit, and J.-P. Serre in oral communication. It was proven to equal $\chi^{\mathbb{A}^1}(X)$ by M. Levine and A. Raksit [21, Theorem 1.3]. This is the description that motivated the introduction above, and as we will use it to prove our main result, we give a more detailed account now.

For any vector space V over k, write $V[i]$ for the chain complex consisting of V concentrated in degree $-i$, with 0's elsewhere. Consider a smooth and proper variety X of pure dimension d.

Definition 1.4. Let

$$\mathrm{Hdg}(X/k) \stackrel{\mathrm{def}}{=} \bigoplus_{i,j}^{d} H^i(X, \Omega^j_{X/k})[j-i] \in \mathrm{Ch}_k\,. \tag{1.5}$$

This is equipped with a symmetric bilinear form via the cup product and the canonical trace map $\mathrm{Tr} : H^d(X; \Omega^d_{X/k}) \longrightarrow k$

$$H^i(X, \Omega^j_{X/k}) \otimes H^{d-i}(X, \Omega^{d-j}_{X/k}) \xrightarrow{\cup} H^d(X, \Omega^d_{X/k}) \xrightarrow{\mathrm{Tr}} k. \tag{1.6}$$

This structure produces a bilinear form Tr on $\mathrm{Hdg}(X/k)$.

See [21, Section 8D] for a review of the formalism of chain complexes with bilinear forms. The bilinear form Tr in fact recovers $\chi^{\mathbb{A}^1}$ as defined above.

Theorem 1.7 *([21, Theorem 1.3]). For X smooth and projective over a field k of characteristic not 2,*

$$\chi^{\mathbb{A}^1}(X/k) = (\mathrm{Hdg}(X/k), \mathrm{Tr}) \in \mathrm{GW}(k).$$

We remind the reader that in this paper k will be a field of characteristic 0. Another description is as follows.

Definition 1.8. Let $p_X : X \longrightarrow \mathrm{Spec}\, k$ denote the structure map of a smooth proper variety X. Let

$$\mathcal{T}_X := \oplus_{i=-d}^{0}(\wedge^{-i} T_X^*),$$

with zero differential. This is the Koszul complex of the tangent bundle T_X with respect to the zero section. It has a natural non-degenerate bilinear form

$$\beta_X : \mathcal{T}_X \otimes \mathcal{T}_X \longrightarrow \mathcal{T}_X \longrightarrow \wedge^d T_X^*[d]$$

given by composing the multiplication of forms with projection to the $-d$th term of $\mathcal{T}_X$.

Because X is smooth and proper, Serre duality allows us to pushforward a non-degenerate bilinear form valued in $\wedge^d T_X^*[d]$ and obtain a non-degenerate bilinear form valued in k [13, p. 7 Ideal Theorem c], [8, Theorem 4.2.9]:

$$\mathbf{R}p_{X*}\beta_X : \mathbf{R}p_{X*}\mathcal{T}_X \otimes \mathbf{R}p_{X*}\mathcal{T}_X \longrightarrow \mathbf{R}p_{X*}(\mathcal{T}_X \otimes \mathcal{T}_X) \longrightarrow \mathbf{R}p_{X*} \wedge^d T_X^*[d] \xrightarrow{\mathrm{Tr}} k \tag{1.9}$$

There is more discussion of this in [6, Section 2.2] and we will discuss a similar situation in Lemma 2.4.

Proposition 1.10 *([6, Proposition 2.4]). Let X be smooth and proper over k. With notation as above, $(\mathbf{R}p_{X*}\mathcal{T}_X, \mathbf{R}p_{X*}\beta_X) = (\mathrm{Hdg}(X/k), \mathrm{Tr})$ in $\mathrm{GW}(k)$.*

Each isomorphism class in $\mathrm{GW}(k)$ can be represented by a perfect chain complex over k with a nondegenerate symmetric bilinear form. This is a result of Hermitian K-theory; see [30] and [31] for background on Hermitian K-theory. A representing chain complex with duality provides the opportunity to do further homotopy theory. The example we have in mind will be described below. As a candidate representative, however, the Koszul complex has the following property: it depends on a section. There are situations in which one naturally has a section, e.g. [19], but the association of the Euler characteristic to a scheme is arguably not one of them: We chose the zero section above, but is this a good choice? Any section would in fact do [6, Proposition 2.4], and it may be desirable not to choose at all. The main result of this paper allows this by offering the Hochschild complex as an alternative.

There is a canonical pairing on the Hochschild homology HH, which appears in greater generality in work of Alonso-Jeremías-Lipman, [2]. This pairing can be viewed as coming from the connection between Grothendieck duality and Hochschild homology, due to Avramov, Lipman, and Iyengar, among others. We give a construction specific to our case of interest using work of Neeman in Section 2. We then show that this agrees with $(\mathrm{Hdg}(X/k), \mathrm{Tr})$ and $(\mathbf{R}p_{X*}\mathcal{T}_X, \mathbf{R}p_{X*}\beta_X)$ using the Hochschild–Kostant–Rosenberg theorem for smooth and proper schemes X over a field of characteristic 0.

Theorem 1. *Let X be a smooth, projective scheme over a field of characteristic 0. $\mathrm{HH}(X)$ together with a canonical pairing represents the categorical $\mathbb{A}^1$-Euler characteristic for smooth projective X.*

This is Theorem 2.11 in Section 2. The example alluded to above is the following enrichment of the $\mathbb{A}^1$-Euler characteristic. In joint as well as independent work, J. Campbell and the fifth-named author have constructed a spectrum whose π_0 is the Grothendieck group of varieties $K_0(\mathbf{Var}_k)$. See [7,35]. Independently, O. Röndigs [29] constructs a K-theory of varieties spectrum whose π_0 receives a surjective map from $K_0(\mathbf{Var}_k)$. (The Grothendieck group of varieties is by definition the universal cut-and-paste invariant, and is described in more detail in for example [4].) Bittner's presentation promotes the $\mathbb{A}^1$-Euler characteristic of a smooth, projective variety to a compactly supported $\mathbb{A}^1$-Euler characteristic for any variety over k. In more detail:

Theorem 1.11 *([4, Theorem 3.1]). Let k be a field of characteristic 0. The group $K_0(\mathbf{Var}_k)$ is isomorphic to the free abelian group generated by smooth projective varieties modulo the relation that for any smooth closed subvariety $Y \subseteq X$,*

$$[\mathrm{Bl}_Y X] - [E] = [X] - [Y],$$

where $\mathrm{Bl}_Y X$ is the blow-up of X at Y, and E is the exceptional divisor of the blowup.

Definition 1.12. Let X be a smooth scheme over a field k of characteristic 0. We define $\chi_c(X)$ as follows. For $\dim X = 0$ we define $\chi_c^{\mathbb{A}^1}(X) = \chi^{\mathbb{A}^1}(X)$. For $\dim X > 0$, write $[X] = \sum_{i=1}^n \epsilon_i [X_i]$ in $K_0(\mathbf{Var}_k)$, where each X_i is smooth projective and $\epsilon_i = \pm 1$. Then define

$$\chi_c^{\mathbb{A}^1}(X) \stackrel{\mathrm{def}}{=} \sum_{i=1}^n \epsilon_i \chi_c^{\mathbb{A}^1}(X_i).$$

Theorem 1.13. *$\chi_c^{\mathbb{A}^1}$ is well-defined, and induces a homomorphism $K_0(\mathbf{Var}_k) \longrightarrow \mathrm{GW}(k)$.*

This follows from [29, Theorem 5.2, Corollary 6.7]. We also provide an exposition in Theorem 2.13.

Question 1.14. *Does the compactly supported $\mathbb{A}^1$-Euler characteristic $\chi_c^{\mathbb{A}^1}$ lift to a map of spectra from a spectrum level version of $K_0(\mathbf{Var}_k)$ to Hermitian K-theory? What kind of information would such maps encode?*

O. Röndigs gives an affirmative answer to the first question in [29, Theorem 6.6]. He constructs his map using the categorical $\mathbb{A}^1$-Euler characteristic and Waldhausen's model of K-theory. We suggest an alternate approach these questions, studying the association of the Hochschild complex to a smooth, projective scheme, especially as related to Bittner's presentation. Section 3 provides sample computations.

Acknowledgments

We wish to thank Tom Bachmann and Marc Hoyois for useful discussions.

We likewise express our gratitude to the Women in Topology III program and the Hausdorff Research Institute for Mathematics, as well as support from the National Science Foundation NSF-DMS 1901795, NSF-HRD 1500481 - AWM ADVANCE grant and Foundation Compositio Mathematica. Niny Arcila-Maya was supported by St. John's college Itoko Muraoka fellowship. Candace Bethea was supported by the Southern Region Education Board under the Dissertation Award. Morgan Opie was partially supported by National Science Foundation Award DGE-1144152. Kirsten Wickelgren was partially supported by National Science Foundation Awards DMS-1406380 and CAREER-DMS-1552730. Inna Zakharevich was partially supported by National Science Foundation Award CAREER-1846767.

Notation

k will denote a field of characteristic 0.

By a *variety over k* we mean a reduced, irreducible, separated schemes of finite type over a field k. For a variety X over k, the map p_X denotes the structure map $p_X : X \longrightarrow \operatorname{Spec} k$. When X is clear from context we omit it from the notation, and write p for the structure map. Our chain complexes are cohomological and often concentrated in negative degrees. Write $\mathrm{hCoh}(X)$ for the derived category of coherent sheaves on X.

2. Hochschild complex represents the $\mathbb{A}^1$-Euler characteristic

We use the connection between Grothendieck duality and Hochschild homology to give a bilinear form on the absolute Hochschild complex in this section. This form appears in greater generality in work of Alonso-Jeremías-Lipman [2]. We then show this complex with duality represents the $\mathbb{A}^1$-Euler characteristic for a smooth, projective scheme over a field of characteristic 0, and use Bittner's presentation to give an exposition of the compactly supported $\mathbb{A}^1$-Euler characteristic. Recall that we are working over a field of characteristic 0.

In future work we hope to use this formalism to lift the compactly supported Euler characteristic to a spectrum-level construction from J. Campbell and the fifth-named author's spectrum of varieties [7] [35] landing in the Hermitian K-theory of k. As working in the derived category is presumably not sufficient to make such a construction well-defined, we keep track (and include in our notation) which functors are defined outside the derived category, as well. Thus, for example, given a map $f : X \longrightarrow Y$ we think of f^* as a functor $\mathrm{Coh}(Y) \longrightarrow \mathrm{Coh}(X)$ and denote the derived functor by $\mathbf{L}f^* : \mathrm{hCoh}(Y) \longrightarrow \mathrm{hCoh}(X)$.

Let k be a field. We will shortly restrict to the case where k is characteristic 0.

Definition 2.1. Let $p : X \longrightarrow \operatorname{Spec} k$ be a smooth separated scheme over k, and let $\mathrm{hCoh}(X)$ be the derived category of coherent sheaves on X. Write $\Delta : X \longrightarrow X \times X$ for the diagonal map, and define $\mathcal{O}_X^{\Delta} \overset{\text{def}}{=} \Delta_* \mathcal{O}_X$. This inherits a ring structure from $\mathcal{O}_X$; write $\mu_X : \mathcal{O}_X^{\Delta} \otimes \mathcal{O}_X^{\Delta} \longrightarrow \mathcal{O}_X^{\Delta}$ for the multiplication map.

Since Δ is affine, $\mathbf{R}\Delta_* = \Delta_*$, so this is in fact a derived construction. Define the *Hochschild complex* $\mathrm{HH}^X(X)$ in $\mathrm{hCoh}(X)$ to be

$$\mathrm{HH}^X(X) \stackrel{\mathrm{def}}{=} (\mathbf{L}\Delta^*)(\Delta_*\mathcal{O}_X),$$

which is equivalent to $\mathcal{O}_X^\Delta \otimes^{\mathbf{L}}_{X\times X} \mathcal{O}_X^\Delta$. Since $\mathbf{L}\Delta^*$ is strongly monoidal, the Hochschild complex inherits a multiplication

$$\mathbf{L}\Delta^*\mu_X : \mathrm{HH}^X(X) \otimes \mathrm{HH}^X(X) \longrightarrow \mathrm{HH}^X(X)$$

We then define the *absolute Hochschild complex* $\mathrm{HH}(X)$ in $\mathrm{hCoh}(k)$ to be

$$\mathrm{HH}(X) \stackrel{\mathrm{def}}{=} \mathbf{R}p_* \,\mathrm{HH}^X(X).$$

To see that the Hochschild homology is a good candidate to represent the $\mathbb{A}^1$-Euler characteristic, we first note that it is a representative in $K_0(k)$ when the characteristic of k is 0:

Theorem 2.2 *(Hochschild–Kostant–Rosenberg, [15] [3, p. 1]). Let k be a field of characteristic 0. For a smooth projective X, $[\mathrm{HH}(X)] = \mathrm{Hdg}(X/k)$ in $K_0(k)$.*

We would like the Euler characteristic to take values in $\mathrm{GW}(k)$, rather than $K_0(k)$. To accomplish this, we use the connection between Grothendieck duality and Hochschild homology found by Avramov and Iyengar [1], and particularly the work of Neeman [26] to construct a nondegenerate symmetric bilinear form on $\mathrm{HH}^X(X)$. This pairing appears in greater generality in Alonso-Jeremías-Lipman [2].

The multiplication on $\mathcal{O}_X^\Delta$ induces a multiplication

$$\widehat{\mu}_X : \mathrm{HH}(X) \otimes \mathrm{HH}(X) \longrightarrow \mathrm{HH}(X)$$

in the following manner. Define $\widehat{\mu}_X$ to be the composition

$$\mathrm{HH}(X) \otimes \mathrm{HH}(X) \longrightarrow \mathbf{R}p_*(\mathrm{HH}^X(X) \otimes \mathrm{HH}^X(X)) \xrightarrow{(\mathbf{R}p_*)(\mathbf{L}\Delta_X^*)\mu_X} \mathrm{HH}(X),$$

where the first map is given by the lax monoidal structure on $\mathbf{R}p_*$. Using this multiplication, $\mathrm{HH}(X)$ is equipped with the following canonical bilinear pairing.

Definition 2.3. Let X be proper, and let $p : X \longrightarrow \operatorname{Spec} k$ be the structure map.[1] Let $\pi_2 : X \times X \longrightarrow X$ be the projection onto the second coordinate, as in the following commutative diagram:

$$\begin{array}{ccccc} X & & & & \\ & \searrow^{\Delta} & & & \\ & & X \times X & \xrightarrow{\pi_1} & X \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p} \\ & & X & \xrightarrow{p} & \operatorname{Spec} k \end{array}$$

[1] Many of the constructions, and in particular the result of Neeman, are possible in much greater generality.

By [26, Proposition 3.3], there is a canonical isomorphism

$$\delta_X : (\mathbf{L}\Delta^*)\pi_2^!\mathcal{O}_X \longrightarrow p^!\mathcal{O}_k.$$

Inside the derived category, the adjoint to the weak equivalence $(\pi_2)_*\mathcal{O}_X^\Delta = (\pi_2)_*\Delta_*\mathcal{O}_X \xrightarrow{\sim} (\pi_2 \circ \Delta)_*\mathcal{O}_X = \mathcal{O}_X$, is

$$t : \mathcal{O}_X^\Delta \longrightarrow \pi_2^!\mathcal{O}_X.$$

Define α' to be the composition of $\mathbf{L}\Delta^* t$ with δ_X

$$\alpha' : \mathrm{HH}^X(X) = (\mathbf{L}\Delta^*)\mathcal{O}_X^\Delta \longrightarrow (\mathbf{L}\Delta^*)\pi_2^!\mathcal{O}_X \xrightarrow{\delta_X} p^!\mathcal{O}_k.$$

The map α' is a special case of the fundamental class of a flat map of Alonso-Jeremías-Lipman [2, p. 390]. Since X is proper, there is a natural equivalence $p_! \longrightarrow p_*$; the map α adjoint to α' can thus be written as

$$\alpha : \mathrm{HH}(X) \longrightarrow \mathcal{O}_k.$$

Define the bilinear form

$$B_X : \mathrm{HH}(X) \otimes \mathrm{HH}(X) \longrightarrow \mathcal{O}_k \qquad \text{by} \qquad B_X \overset{\text{def}}{=} \alpha \circ \widehat{\mu}_X.$$

Since μ_X is a commutative multiplication and $\mathbf{R}p_*$ is lax symmetric monoidal, $\widehat{\mu}_X$ is commutative; it follows that B_X is symmetric.

Lemma 2.4. *(Alonso-Jeremías-Lipman) B_X is nondegenerate.*

Lemma 2.4 is a special case of Alonso, Jeremías, and Lipman's [2, Corollary 4.8.4]. We provide a proof adapted to our level of generality to provide a self-contained exposition.

Proof. The map $\alpha' \circ (\mathbf{L}\Delta_X^*)\mu_X$ determines a map

$$\mathrm{HH}^X(X) \longrightarrow \mathbf{R}\mathrm{Hom}(\mathrm{HH}^X(X), p^!\mathcal{O}_k), \tag{2.5}$$

which we claim is an isomorphism. To see this, observe that (2.5) is equal to the composition

$$\mathbf{L}\Delta^*\mathcal{O}_X^\Delta \longrightarrow \mathbf{R}\mathrm{Hom}(\mathbf{L}\Delta^*\mathcal{O}_X^\Delta, \mathbf{L}\Delta^*\pi_2^!\mathcal{O}_X) \xrightarrow{\delta_X \circ} \mathbf{R}\mathrm{Hom}(\mathbf{L}\Delta^*\mathcal{O}_X^\Delta, p^!\mathcal{O}_k),$$

where the first map is induced from the multiplication. Since δ_X is an isomorphism, the second is as well, so it remains to show that

$$\mathbf{L}\Delta^*\mathcal{O}_X^\Delta \longrightarrow \mathbf{R}\mathrm{Hom}(\mathbf{L}\Delta^*\mathcal{O}_X^\Delta, \mathbf{L}\Delta^*\pi_2^!\mathcal{O}_X) \tag{2.6}$$

is an isomorphism. Since X is smooth, Δ is perfect [32, Section 37.54, Lemma 37.54.18.], and thus has finite Tor-dimension [32, Section 0685 Lemma 37.53.11]. Since Δ is proper and finite Tor-dimension,

$$\mathbf{L}\Delta^*\mathbf{R}\mathrm{Hom}(\mathcal{O}_X^\Delta, \pi_2^!\mathcal{O}_X) \simeq \mathbf{R}\mathrm{Hom}(\mathbf{L}\Delta^*\mathcal{O}_X^\Delta, \pi_2^!\mathcal{O}_X)$$

by [26, Lemma 4.3], and the morphism (2.6) is induced by $\mathbf{L}\Delta^*$ applied to

$$\mathcal{O}_X^{\Delta} \longrightarrow \mathbf{R}\mathrm{Hom}(\mathcal{O}_X^{\Delta}, \pi_2^! \mathcal{O}_X).$$

This is an isomorphism, since it equal to the composition

$$\Delta_* \mathcal{O}_X \cong \Delta_* \mathbf{R}\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \cong \Delta_* \mathrm{Hom}(\mathcal{O}_X, \Delta^! \pi_2^! \mathcal{O}_X) \cong \mathrm{Hom}(\Delta_* \mathcal{O}_X, \pi_2^! \mathcal{O}_X),$$

where the third isomorphism is by [26, (2.5.2)] and the fourth is by [26, p. 14].

By coherent duality [13, p. 7 Ideal Theorem c], the composition of the natural map

$$\mathbf{R}\mathrm{Hom}_X(\mathrm{HH}(X), p^! \mathcal{O}_k) \longrightarrow \mathbf{R}\mathrm{Hom}_Y(\mathbf{R}p_* \mathrm{HH}(X), \mathbf{R}p_* p^! \mathcal{O}_k)$$

with the map induced by the trace $\mathbf{R}p_* p^! \longrightarrow 1$ is isomorphism

$$\mathbf{R}p_* \mathbf{R}\mathrm{Hom}_X(\mathrm{HH}(X), p^! \mathcal{O}_k) \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_k(\mathrm{HH}(X), \mathcal{O}_k).$$

Since (2.5) is an isomorphism, it follows that the map $\mathrm{HH}(X) \longrightarrow \mathbf{R}\mathrm{Hom}_k(\mathrm{HH}(X), \mathcal{O}_k)$ induced by $\alpha \circ \widehat{\mu}_X \circ \zeta_X$ is an isomorphism. □

In order to show that $(\mathrm{HH}(X), B_X)$ gives the $\mathbb{A}^1$-Euler characteristic, we will use that the algebra structure on $\mathrm{HH}(X)$ is compatible with the algebra structure on differential forms: let $\mathcal{T}_X$ denote the Koszul complex as in Definition 1.8.

Theorem 2.7 *(Hochschild–Kostant–Rosenberg Theorem). Let k be a field of characteristic 0. Giving $\mathcal{H}^*(\mathrm{HH}^X(X))$ the algebra structure induced from $\mathbf{L}\Delta^* \mu_X$ and $\mathcal{T}_X$ the algebra structure from wedge product of forms, there is an algebra isomorphism $\mathcal{H}^*(\mathrm{HH}^X(X)) \longrightarrow \mathcal{T}_X$.*

See also [33, Corollaire 1.2], identifying the relevant Tor in commutative differential graded algebras.

Proof. Let I denote the ideal sheaf of the closed immersion Δ. There is an exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{X \times X} \longrightarrow \Delta_* \mathcal{O}_X \longrightarrow 0$$

of coherent sheaves on $X \times X$. Applying $\mathbf{L}\Delta^*$, we obtain the distinguished triangle

$$\mathbf{L}\Delta^* I \longrightarrow \mathbf{L}\Delta^* \mathcal{O}_{X \times X} \longrightarrow \mathrm{HH}^X(X) \longrightarrow \mathbf{L}\Delta^* I[1].$$

The associated long exact sequence on homology sheaves defines a map

$$\mathcal{HH}_1^X(X) \cong \mathcal{H}^1(\mathrm{HH}^X(X)) \longrightarrow \mathcal{H}^0 \mathbf{L}\Delta^* I \cong \Delta^* I \cong I/I^2 \cong \Omega_{X/k}, \tag{2.8}$$

where $\mathcal{HH}_1$ denotes the first Hochschild homology and $\mathcal{H}^1(\mathrm{HH}^X(X))$ denotes the first homology sheaf of the complex $\mathrm{HH}^X(X)$ (which is denoted with a superscript 1 because our complexes are graded cohomologically). $\mathcal{H}^1(\mathbf{L}\Delta^* \mathcal{O}_{X \times X}) = 0$ because $\mathcal{O}_{X \times X}$ is a flat $\mathcal{O}_{X \times X}$-module. Thus (2.8) is injective. Moreover the map $\mathcal{O}_X \cong \Delta^* \mathcal{O}_{X \times X} \longrightarrow \Delta^* \Delta_* \mathcal{O}_X \cong \mathcal{O}_X$ is the identity. We therefore have that (2.8) is an isomorphism.

The map $\mathbf{L}\Delta^* \mu_X$ induces a multiplication on $\mathcal{H}(\mathrm{HH}^X(X))$ compatible with the multiplication on open affines of [15]. By [15, Theorem 3.1], $\mathcal{H}(\mathrm{HH}^X(X))$ is the exterior algebra on $\mathcal{H}^1(\mathrm{HH}^X(X))$ after restricting to affine open subsets of X. Thus $\mathcal{H}(\mathrm{HH}^X(X))$ is the exterior algebra on $\mathcal{H}^1(\mathrm{HH}^X(X))$. Therefore, (2.8) extends to an algebra homomorphism $\mathcal{H}^*(\mathrm{HH}^X(X)) \longrightarrow \mathcal{T}_X$. Since $\mathcal{T}_X$ is the exterior algebra on $\Omega_{X/k}$, the homomorphism $\mathcal{H}^*(\mathrm{HH}^X(X)) \longrightarrow \mathcal{T}_X$ is an isomorphism. □

We now show that the $\mathbb{A}^1$-Euler characteristic is represented by the form on $\mathrm{HH}(X)$ just constructed. Given a bilinear form on a graded vector space $V^\bullet \times V^\bullet \longrightarrow k$, the 0-th graded piece V^0, and the direct sums $V^{(n)} := V^{-n} + V^n$ for $n > 0$ carry a nondegenerate bilinear form. We denote the corresponding elements of $\mathrm{GW}(k)$ by $[V^0]$ and $[V^{(n)}]$ respectively. The class in $\mathrm{GW}(k)$ determined by the form is the alternating sum $[V^0] + \sum_{n>0}(-1)^n[V^{(n)}]$.

Definition 2.9. Given a nondegenerate, symmetric $\beta : V^\bullet \times V^\bullet \longrightarrow \mathcal{O}_k$ in $\mathrm{hCoh}(k)$, let $[\beta]$ denote the class in $\mathrm{GW}(k)$ given by $[H^0(V^\bullet)] + \sum_{i>0}(-1)^i[H^*(V^\bullet)^{(i)}]$.

In the following theorem, we prove that $[B_X] = \chi^{\mathbb{A}^1}(X)$ by identifying the former with the pairing on $(\mathrm{Hdg}(X/k), \mathrm{Tr})$. It is proved analogously to the proof of [6, Proposition 2.4]. However, we also need a key duality result, which gives us a better understand of the constituents used to define B_X.

Theorem 2.10 *([2, Proposition 2.4.2][26, Theorem 3.5], [36, Proposition 4.6.3]). Let $d = \dim X$, and let $\alpha' : \mathrm{HH}^X(X) \longrightarrow p^!\mathcal{O}_k$ be as defined above. Then the map $\mathcal{H}^d(\alpha')$ induced on d-th cohomology sheaves over X is an isomorphism.*

With this result, we have the following:

Theorem 2.11. *Let X be smooth and projective over a field k of characteristic 0. Then the class of B_X in $\mathrm{GW}(k)$ is the categorical $\mathbb{A}^1$-Euler characteristic $\chi^{\mathbb{A}^1}(X)$ of X. When X is smooth and proper, the class of B_X is the Euler number of the tangent bundle.*

Remark 2.12. The Euler class of a vector bundle was defined by [5] and developed further by J. Fasel [10] and others. Under orientation conditions on the vector bundle which are always satisfied by the tangent bundle, the Euler class can be pushed forward to an Euler number. It is a theorem of M. Levine that for X smooth and projective, the categorical $\mathbb{A}^1$-Euler characteristic equals the Euler number of the tangent bundle [20, Theorem 3.1 or ArXiv version 3 Theorem 1.1], and in particular, X is dualizable in an appropriate sense, see [20, Theorem 3.1 or ArXiv version 3 Theorem 1.1]

Proof. Let $n = \dim X$. There is a hypercohomology spectral sequence

$$E_2^{i,j} = \mathbf{R}^i p_* H^j \,\mathrm{HH}^X(X) \Longrightarrow H^{i+j}\,\mathrm{HH}(X).$$

By the isomorphism $H^j\,\mathrm{HH}^X(X) \cong \Omega_X^{-j}$ of the Hochschild–Kostant–Rosenberg theorem, $E_2^{i,j} \cong H^i(X, \Omega_X^{-j})$. The hypercohomology spectral sequence is multiplicative, giving a bilinear form on the E_r-pages for $r \geq 2$ coming from the cup product

$$E_2^{i,j} \times E_2^{i',j'} \longrightarrow \mathbf{R}^{i+i'} p_* H^{j+j'}(\mathrm{HH}^X(X) \otimes \mathrm{HH}^X(X)).$$

Composing with $\alpha \circ (\mathbf{R}p_*(\widehat{\mu})) \circ \zeta_X$, we obtain a new form on the E_r-page which we denote β_r

$$\beta_r : E_r^{i,j} \times E_r^{n-i,n-j} \longrightarrow k.$$

Note that $E_2^{i,j}$ is the degree (i,j) summand of $(\mathrm{Hdg}(X/k), \mathrm{Tr})$, as defined in (1.5). By Theorems 2.10 and 2.7, β_2 is the cup product pairing, whence $[\beta_2] = [(\mathrm{Hdg}(X/k), \mathrm{Tr})]$. Recall that the notation $[\beta_2]$ was defined in 2.9. By [6, Lemma 2.7], it follows that β_r is symmetric, non-degenerate and $[\beta_r] = [\beta_2]$ for all $r \geq 2$, including $r = \infty$. By [6, Lemma 2.6], $[\beta_\infty] = B_X$. For X smooth and projective $[(\mathrm{Hdg}(X/k), \mathrm{Tr})] = \chi^{\mathbb{A}^1}(X)$ by [21, Theorem 1.3]. By [6, Theorem 1.1, second Corollary p. 3] $[(\mathrm{Hdg}(X/k), \mathrm{Tr})]$ is the Euler number of the tangent bundle. Combining, we see $[\beta_2] = [\beta_r] = B_X$, which completes the proof by the previous. □

Using Bittner's presentation, we now show that $[X] \longmapsto [B_X]$ defines a homomorphism out of the Grothendieck ring of varieties, and which agrees with the usual Euler characteristic. In future work we hope to prove this theorem using scdh-descent and Hochschild homology. It is interesting to note that, by contrast, Hochschild homology does not satisfy cdh-descent. The below proof is similar to one of O. Röndigs in that it uses cut and paste properties of the categorical Euler characteristic [29, Theorem 5.2] [25, 3 Remark 2.30]. The below is of a more computational flavor, relying on recent work of M. Levine.

Theorem 2.13. *Let k be a field of characteristic 0. The map $X \longmapsto [B_X]$ for X smooth and projective defines a homomorphism $K_0(\mathbf{Var}_k) \longrightarrow \mathrm{GW}(k)$ which agrees with the categorical $\mathbb{A}^1$-Euler characteristic.*

Proof. That $[B_X]$ is the categorical $\mathbb{A}^1$-Euler characteristic $\chi^{\mathbb{A}^1}(X)$ is Theorem 2.11. (Also see Remark 2.12 for context.)

Let

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Bl}_Y X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X, \end{array}$$

be the blow-up diagram of a smooth, proper k-variety X along a smooth closed subvariety Y of codimension c. It follows from [20, Proposition 1.4 (5) or ArXiv version 3 Proposition 1.10 (4)] that $\chi^{\mathbb{A}^1}(\mathrm{Bl}_Y X) = \chi^{\mathbb{A}^1}(X) + \sum_{i=1}^{c-1} \langle -1 \rangle^i \chi^{\mathbb{A}^1}(Y)$. The exceptional divisor E is the projectivization $\mathbb{P}N_Y X$ of the normal bundle of Y in X. By [20, Proposition 1.4 (4) or ArXiv version 3 Proposition 1.10 (3)], $\chi^{\mathbb{A}^1} E = \sum_{i=0}^{c-1} \langle -1 \rangle^i \chi^{\mathbb{A}^1}(Y)$. Thus

$$\chi^{\mathbb{A}^1}(\mathrm{Bl}_Y X) = \chi^{\mathbb{A}^1}(X) - \chi^{\mathbb{A}^1}(Y) + \chi^{\mathbb{A}^1}(E)$$

and the homomorphism is well-defined by Bittner's presentation (see Theorem 1.11). □

3. A direct perspective

In this section, we compute $\chi^{\mathbb{A}^1}(X) = \chi_c^{\mathbb{A}^1}(X)$ for the smooth, proper varieties $X = \mathrm{Spec}\, k$, $\mathbb{P}^1$, $\mathbb{P}^2$, and $\mathrm{Bl}_0 \mathbb{P}^2$ using the complex with duality $(\mathrm{Hdg}(X/k), \mathrm{Tr})$ described in Definition 1.4. In other words, we compute $\mathbb{A}^1$-Euler characteristics explicitly using coherent duality. This allows a direct verification that the association of the complex with duality $(\mathrm{Hdg}(X/k), \mathrm{Tr})$ to a smooth projective variety X satisfies the relation $[(\mathrm{Hdg}(\mathbb{P}^2/k), \mathrm{Tr})] + [(\mathrm{Hdg}(\mathbb{P}^1/k), \mathrm{Tr})] = [(\mathrm{Hdg}(\mathrm{Bl}_0 \mathbb{P}^2/k), \mathrm{Tr})] + [(\mathrm{Hdg}(k/k), \mathrm{Tr})]$ in $\mathrm{GW}(k)$ imposed on classes of smooth projective varieties in Bittner's presentation (see 1.11). It gives an alternate proof of the $n = 1, 2$ case of M. Hoyois's computation of $\chi^{\mathbb{A}^1}(\mathbb{P}^n)$ [16, Example 1.7], and an alternate proof of the case $X = \mathrm{Bl}_0 \mathbb{P}^2$ of M. Levine's computation of the $\mathbb{A}^1$-Euler characteristic of a blow up [20, Proposition 1.4 or ArXiv version 3 Proposition 1.10]. As before, we work over a field of characteristic 0.

The collapse of appropriate spectral sequences renders $(\mathrm{Hdg}(X/k), \mathrm{Tr})$ equivalent to the complex with duality $(\mathbf{R}p_{X*}\mathcal{T}_X, \mathbf{R}p_{X*}\beta_X)$ described in (1.9). Let $p_X : X \longrightarrow \mathrm{Spec}\, k$ denote the structure map of X; recall that

$$\mathcal{T}_X \overset{\mathrm{def}}{=} \oplus_{i=-d}^{0} (\wedge^{-i} T_X^*)$$

is equipped with the natural duality given by composing multiplication of forms with projection off of the top wedge power of T_X^*. After pushforward to $\mathrm{Spec}\, k$, this complex with duality represents $\chi^{\mathbb{A}^1}$ for smooth projective varieties, by [6, Proposition 2.4].

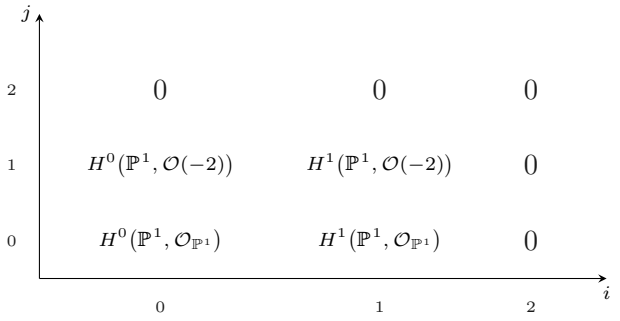

Fig. 1. (3.4) for $X = \mathbb{P}^1$.

Consider the blow-up square

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{i'} & \mathrm{Bl}_0\,\mathbb{P}^2 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k & \xrightarrow{i} & \mathbb{P}^2. \end{array} \tag{3.1}$$

The goal is to show the equality

$$[\mathbf{R}p_{\mathbb{P}^1 *}\mathcal{T}_{\mathbb{P}^1}] + [\mathbf{R}p_{\mathbb{P}^2 *}\mathcal{T}_{\mathbb{P}^2}] = [\mathbf{R}p_{\operatorname{Spec} k *}\mathcal{T}_{\operatorname{Spec} k}] + [\mathbf{R}p_{\mathrm{Bl}_0\,\mathbb{P}^2 *}\mathcal{T}_{\mathrm{Bl}_0\,\mathbb{P}^2}] \tag{3.2}$$

$\mathrm{GW}(k)$; we have omitted the dualities from the notation for legibility, but these too must be considered.

Remark 3.3. In this section we write p for the map $p_X : X \longrightarrow \operatorname{Spec} k$ whenever X is clear from context.

3.1. Verification in $K_0(k)$

In this subsection, k is any field. As a 1-category, $\mathrm{hCoh}(k)$ is equivalent to the category of graded k-vector spaces by taking homology. We now compute $\mathbf{R}p_*\mathcal{T}_{\operatorname{Spec} k}$, $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^1}$, $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^2}$, and $\mathbf{R}p_*\mathcal{T}_{\mathrm{Bl}_0\,\mathbb{P}^2}$ as k-vector spaces using the hypercohomology spectral sequence [22, Theorem 12.12]

$$E_2^{i,j} = H^i\big(X, \mathcal{T}_X^j\big) = H^i\big(X, \wedge^j T_X^*\big) \quad \Longrightarrow \quad H^{j-i}\big((p_X)_*\mathcal{T}_X\big). \tag{3.4}$$

Along the way, we verify (3.2) in $K_0(k)$.

Consider (3.2). The complex $\mathbf{R}p_*\mathcal{T}_{\operatorname{Spec} k}$ is k in degree zero and trivial otherwise.

The computations of $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^1}$, $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^2}$, and $\mathbf{R}p_*\mathcal{T}_{\mathrm{Bl}_0\,\mathbb{P}^2}$ as graded k-vector spaces are done in Propositions 3.5, 3.6, and 3.8, respectively; we compute the forms in Subsection 3.2. We will not explicitly discuss differentials in the spectral sequence, as each of the examples we compute will end up concentrated in a single diagonal, and therefore no differentials are possible.

Proposition 3.5. $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^1} \cong k^2$ *where* k^2 *is concentrated in degree* 0.

Proof. Consider the E_2-page of (3.4) when $X = \mathbb{P}^1$, illustrated in Fig. 1. Noting that $T_{\mathbb{P}^1}^* \cong \mathcal{O}(-2)$, Serre duality implies that

$$H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\big) = H^0\big(\mathbb{P}^1, \mathcal{O}(-2)\big) = 0$$

and

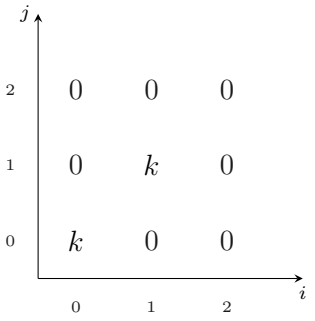

Fig. 2. Result of computation.

$$H^1\big(\mathbb{P}^1, \mathcal{O}(-2)\big) = H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\big) = k.$$

Thus the spectral sequence is as illustrated in Fig. 2. □

Proposition 3.6. $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^2} \cong k^3$ *where* k^3 *is concentrated in degree zero.*

Proof. The reader can refer to Fig. 3 for the E_2-page of (3.4) when $X = \mathbb{P}^2$.

The computation of $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^2}$ is similar to the computation of $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^1}$. In this case the Koszul complex takes the form

$$\mathcal{T}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2} \oplus T^*_{\mathbb{P}^2} \oplus \wedge^2 T^*_{\mathbb{P}^2}.$$

Thus we have possibly nonzero terms $E_2^{i,j}$ for $i, j \in \{0, 1, 2\}$.

Recall that $H^i\big(\mathbb{P}^k, \mathcal{O}(n)\big) = 0$ for all n and all $0 < i < k$. Note also that $\wedge^2 T^*_{\mathbb{P}^2} \cong \mathcal{O}(-3)$. It follows that $H^1\big(\mathbb{P}^2, \wedge^2 T^*_{\mathbb{P}^2}\big)$ and $H^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\big)$ are trivial. Moreover, Serre duality implies that

$$H^2\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\big) = H^0\big(\mathbb{P}^2, \mathcal{O}(-3)\big) = 0$$

and

$$H^2\big(\mathbb{P}^2, \mathcal{O}(-3)\big) = H^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\big) = k.$$

Since $T\mathbb{P}^n = \mathrm{Hom}\big(\mathcal{O}(-1), \mathcal{O}^{n+1}/\mathcal{O}(-1)\big)$, there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(-1)^{n+1} \to T\mathbb{P}^n \to 0.$$

The dual sequence is

$$0 \to T^*_{\mathbb{P}^n} \to \mathcal{O}(1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0. \tag{3.7}$$

From the long exact sequence associated to (3.7), it follows that

$$H^0\big(\mathbb{P}^2, T^*_{\mathbb{P}^2}\big) = H^2\big(\mathbb{P}^2, T^*_{\mathbb{P}^2}\big) = 0$$

and

$$H^1\big(\mathbb{P}^2, T^*_{\mathbb{P}^2}\big) = k.$$

These computations are summarized in Fig. 4. □

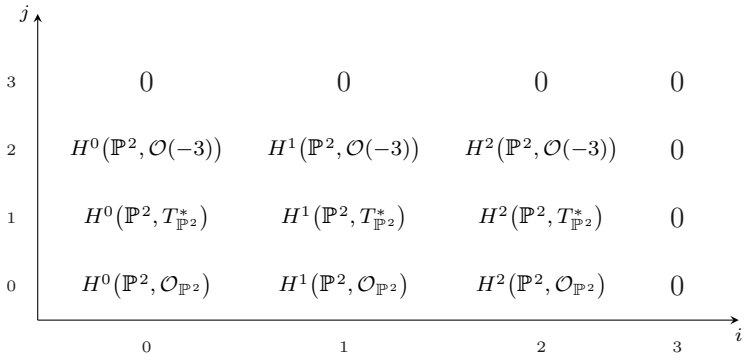

Fig. 3. (3.4) for $X = \mathbb{P}^2$.

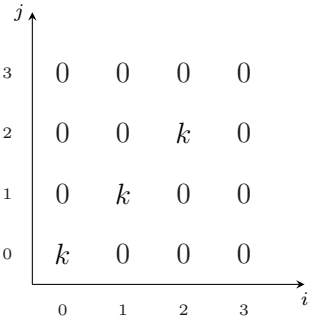

Fig. 4. Result of computation.

Proposition 3.8. $\mathbf{R}p_*\mathcal{T}_{\mathrm{Bl}_0\,\mathbb{P}^2} \cong k^4$ *where* k^4 *is concentrated in degree zero.*

In order to prove this we must first carry out two auxiliary computations in 3.9 and 3.10.

Lemma 3.9.

$$Hy^i\big(\mathrm{Bl}_0\,\mathbb{P}^2;\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}\big) \cong \begin{cases} k & i=1,\\ 0 & i\neq 1. \end{cases}$$

Proof. Let $i' : \mathbb{P}^1 \to \mathrm{Bl}_0\,\mathbb{P}^2$ be the inclusion of the exceptional divisor. Since Kähler differentials commute with pullback [32, Tag 01UM], the support of $\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}$ is contained in the exceptional divisor. In fact, the canonical map $\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2} \longrightarrow i'_*(i')^*\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}$ is an isomorphism. To see this: let $\mathcal{I}$ denote the sheaf of ideals associated to the closed immersion i'. By the proof of [32, Tag 01QY], it suffices to see that $\mathcal{I}\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2} = 0$. Since the support of $\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}$ is contained in the exceptional divisor, it thus suffices to see that $\mathcal{I}\Omega_{\mathrm{Bl}_0\,\mathbb{A}^2/\mathbb{A}^2} = 0$. This can be seen by direct computation. The blow-up

$$\mathrm{Bl}_0\,\mathbb{A}^2 \cong \mathrm{Proj}_{k[x,y]}\,\frac{k[x,y][Z,W]}{\langle xZ - yW\rangle}$$

is covered by two affine opens, one canonically isomorphic to $\mathrm{Spec}\,\frac{k[x,y,\frac{W}{Z}]}{\langle x-y\frac{W}{Z}\rangle}$ and the other canonically isomorphic to $\mathrm{Spec}\,\frac{k[x,y,\frac{Z}{W}]}{\langle x\frac{Z}{W}-y\rangle}$. The sheaf of relative differentials of, say, the first over $\mathrm{Spec}\,k[x,y]$ has a single generator $d\frac{W}{Z}$ with a single relation $-yd\frac{W}{Z}$. The sheaf of differentials of the second is computed similarly, showing that $\mathcal{I}\Omega_{\mathrm{Bl}_0\,\mathbb{A}^2/\mathbb{A}^2} = 0$. Thus $\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2} \cong i'_*(i')^*\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}$ as claimed.

It follows that $\Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2} \cong i'_* T^*_{\mathbb{P}^1}$ by another application of the commutativity of Kähler differentials and pullback [32, Tag 01UM]. Thus

$$H^i\big(\mathrm{Bl}_0\,\mathbb{P}^2; \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}\big) \cong H^i\big(\mathrm{Bl}_0\,\mathbb{P}^2; i'_* T^*_{\mathbb{P}^1}\big) \cong H^i\big(\mathbb{P}^1; T^*_{\mathbb{P}^1}\big).$$

The result then follows from Proposition 3.5. □

Lemma 3.10.

$$H^i\big(\mathrm{Bl}_0\,\mathbb{P}^2; f^* T^*_{\mathbb{P}^2}\big) \cong \begin{cases} k & i = 1, \\ 0 & i \neq 1. \end{cases}$$

Proof. Consider the blow-up square (3.1).

Claim 3.11. *This gives a distinguished triangle in the bounded derived category of* $\mathbb{P}^2$*:*

$$\mathcal{O}_{\mathbb{P}^2} \longrightarrow i_* \mathcal{O}_{\mathrm{Spec}\,k} \oplus \mathbf{R}f_* \mathcal{O}_{\mathrm{Bl}_0\,\mathbb{P}^2} \longrightarrow \mathbf{R}m_* \mathcal{O}_{\mathbb{P}^1},$$

where $m := f \circ i' = i \circ f'$.

Given the claim, note that $T^*_{\mathbb{P}^2}$ is a flat module (in fact, locally free) since $\mathbb{P}^2$ is smooth, so tensoring preserves exact sequences. Thus we get a distinguished triangle:

$$T^*_{\mathbb{P}^2} \longrightarrow (T^*_{\mathbb{P}^2} \otimes \mathbf{R}i_* \mathcal{O}_{\mathrm{Spec}\,k}) \oplus (T^*_{\mathbb{P}^2} \otimes \mathbf{R}f_* \mathcal{O}_{\mathrm{Bl}_0\,\mathbb{P}^2}) \longrightarrow T^*_{\mathbb{P}^2} \otimes \mathbf{R}m_* \mathcal{O}_{\mathbb{P}^1}.$$

Now, using a projection formula from [32, Theorem 20.49.2] we have that

$$T^*_{\mathbb{P}^2} \otimes \mathbf{R}f_* \mathcal{O}_{\mathrm{Bl}_0\,\mathbb{P}^2} \simeq \mathbf{R}f_*(f^* T^*_{\mathbb{P}^2}),$$

and similarly for other terms. Thus we have a distinguished triangle:

$$T^*_{\mathbb{P}^2} \longrightarrow \mathbf{R}i_*(i^* T^*_{\mathbb{P}^2}) \oplus \mathbf{R}f_*(f^* T^*_{\mathbb{P}^2}) \longrightarrow \mathbf{R}m_*(m^* T^*_{\mathbb{P}^2}).$$

Now, we apply $\mathbf{R}(p_{\mathbb{P}^2 *})$ to get a long exact sequence on cohomology.

$$\begin{aligned}
&H^0\big(\mathbb{P}^2; T^*_{\mathbb{P}^2}\big) \longrightarrow H^0\big(\mathrm{Spec}\,k; i^* T^*_{\mathbb{P}^2}\big) \oplus H^0\big(\mathrm{Bl}_0\,\mathbb{P}^2; f^* T^*_{\mathbb{P}^2}\big) \longrightarrow H^0\big(\mathbb{P}^1; (i \circ f')^* T^*_{\mathbb{P}^2}\big) \\
&H^1\big(\mathbb{P}^2; T^*_{\mathbb{P}^2}\big) \longrightarrow H^1\big(\mathrm{Spec}\,k; i^* T^*_{\mathbb{P}^2}\big) \oplus H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; f^* T^*_{\mathbb{P}^2}\big) \longrightarrow H^1\big(\mathbb{P}^1; (i \circ f')^* T^*_{\mathbb{P}^2}\big) \\
&H^2\big(\mathbb{P}^2; T^*_{\mathbb{P}^2}\big) \longrightarrow H^2\big(\mathrm{Spec}\,k; i^* T^*_{\mathbb{P}^2}\big) \oplus H^2\big(\mathrm{Bl}_0\,\mathbb{P}^2; f^* T^*_{\mathbb{P}^2}\big) \longrightarrow H^2\big(\mathbb{P}^1; (i \circ f')^* T^*_{\mathbb{P}^2}\big).
\end{aligned} \tag{3.12}$$

Considering dimensions, we see that

$$H^1\big(\mathrm{Spec}\,k; i^* T^*_{\mathbb{P}^2}\big) = H^2\big(\mathrm{Spec}\,k; i^* T^*_{\mathbb{P}^2}\big) = H^2\big(\mathbb{P}^1; (i \circ f')^* T^*_{\mathbb{P}^2}\big) = 0,$$

implying that $H^2(\mathrm{Bl}_0\,\mathbb{P}^2; f^* T^*_{\mathbb{P}^2}) = 0$.

From the proof Proposition 3.6 we know

$$H^j\big(\mathbb{P}^2; T^*_{\mathbb{P}^2}\big) = 0$$

unless $j = 1$, in which case it is k.

Recall that $i : \operatorname{Spec} k \hookrightarrow \mathbb{P}^2$ is the inclusion. By definition $i^*T^*_{\mathbb{P}^2} = i^{-1}T^*_{\mathbb{P}^2} \otimes_{i^{-1}\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_k$. Therefore

$$i^*T^*_{\mathbb{P}^2} = T^*_{\mathbb{P}^2}|_{\operatorname{Spec} k} \otimes_{\mathcal{O}_{\mathbb{P}^2}|_{\operatorname{Spec} k}} \mathcal{O}_k \cong T^*_{\mathbb{P}^2}|_{\operatorname{Spec} k} \cong \mathcal{O}_k^{\oplus 2}.$$

This implies

$$H^0\big(\operatorname{Spec} k; i^*T^*_{\mathbb{P}^2}\big) = H^0\big(\operatorname{Spec} k; \mathcal{O}_k^{\oplus 2}\big) = H^0\big(\operatorname{Spec} k; \mathcal{O}_k\big) \oplus H^0\big(\operatorname{Spec} k; \mathcal{O}_k\big) = k^2.$$

Observing that $(i \circ f')^*T^*_{\mathbb{P}^2} \cong (f')^*\mathcal{O}_k^{\oplus 2} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, we get that:

$$H^0\big(\mathbb{P}^1; (i \circ f')^*T^*_{\mathbb{P}^2}\big) = H^0\big(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}\big) = k^2,$$

and

$$H^1\big(\mathbb{P}^1; (i \circ f')^*T^*_{\mathbb{P}^2}\big) = H^1\big(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}\big) = H^1\big(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}\big) \oplus H^1\big(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}\big) = 0.$$

Plugging this in to (3.12) gives the exact sequence

$$0 \longrightarrow k^2 \oplus H^0\big(\operatorname{Bl}_0 \mathbb{P}^2; f^*T^*_{\mathbb{P}^2}\big) \longrightarrow k^2 \longrightarrow k \longrightarrow H^1\big(\operatorname{Bl}_0 \mathbb{P}^2; f^*T^*_{\mathbb{P}^2}\big) \longrightarrow 0.$$

The desired result follows. □

Proof of Claim 3.11. Consider the short exact sequence of $\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}$-modules

$$0 \longrightarrow \mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E) \longrightarrow \mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2} \longrightarrow i'_*\mathcal{O}_E \longrightarrow 0,$$

where $E \simeq \mathbb{P}^1$ is the exceptional divisor of the blow-up. Applying Rf_* gives an exact triangle in the derived category of $\mathbb{P}^2$

$$\mathbf{R}f_*\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E) \longrightarrow \mathbf{R}f_*\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2} \longrightarrow \mathbf{R}f_*(i'_*\mathcal{O}_E), \tag{3.13}$$

whose associated long exact sequence on cohomology gives

$$0 \longrightarrow f_*\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E) \longrightarrow f_*\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^2} \longrightarrow m_*\mathcal{O}_E \simeq i_*\mathcal{O}_{\operatorname{Spec} k} \longrightarrow \mathbf{R}^1 f_*\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E).$$

But in fact $\mathbf{R}^i f_*\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E) = 0$ for $i > 0$, as the following argument shows. Note first that $\mathbf{R}^i f_*\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E))$ is a coherent sheaf $\forall i \geq 0$ by [32, Theorem 30.19.1].

It is enough to check that the fibers are all zero. Combining [12, Theorem 3.2.1], [14, Exercise II.3.10], and [14, Corollary III.9.4], we see that the fiber over $p \in \mathbb{P}^2$ is

$$(\mathbf{R}^i f_*\mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E))(p) \cong H^i(\operatorname{Bl}_0 \mathbb{P}^2_p, \mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E)_{f^{-1}(p)}) \cong H^i(f^{-1}(p), \mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E)).$$

Consider the case where $p \in \mathbb{P}^2 - 0$. Since $f|_{\mathbb{P}^2 - 0}\colon \operatorname{Bl}_0 \mathbb{P}^2 - E \longrightarrow \mathbb{P}^2 - 0$ is an isomorphism, $f^{-1}(p)$ is a point and therefore affine. Thus $H^i(f^{-1}(p), \mathcal{O}_{\operatorname{Bl}_0 \mathbb{P}^2}(-E)) = 0$ for all $i > 0$, as desired.

Consider the case $p = 0$. Then we have $f^{-1}(p) \cong \mathbb{P}^1$, and

$$H^1(f^{-1}(p), \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}(-E)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0,$$

as desired. Therefore $\mathbf{R}^i f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}(-E) = 0$ for $i > 0$, and so we get a short exact sequence

$$0 \longrightarrow f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}(-E) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_* \mathcal{O}_{\mathrm{Spec}\, k} \longrightarrow 0,$$

which in fact represents a distinguished triangle

$$\mathbf{R} f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}(-E) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_* \mathcal{O}_{\mathrm{Spec}\, k},$$

in the derived category of $\mathbb{P}^2$ with the property that the shift map $i_* \mathcal{O}_{\mathrm{Spec}\, k} \longrightarrow \mathbf{R} f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}(-E)[1]$ is zero. By [32, Theorem 13.4.10], we get that

$$\mathcal{O}_{\mathbb{P}^2} = \mathbf{R} f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}(-E) \oplus i_* \mathcal{O}_{\mathrm{Spec}\, k} \tag{3.14}$$

in the derived category of $\mathbb{P}^2$.

Now, note that (3.13) becomes

$$\mathbf{R} f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}(-E) \longrightarrow \mathbf{R} f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2} \longrightarrow \mathbf{R} m_* \mathcal{O}_{\mathbb{P}^1}. \tag{3.15}$$

The modifications to the last term follow from recalling that $E \simeq \mathbb{P}^1$ and that the inclusion $i' \colon \mathbb{P}^1 \longrightarrow X'$ is affine. By [32, Theorem 36.5.3], $\mathbf{R} i'_* = i'_*$, and hence

$$\mathbf{R} f_* (i'_* \mathcal{O}_E) = \mathbf{R} f_* \mathbf{R} i'_* \mathcal{O}_{\mathbb{P}^1} = \mathbf{R} m_* \mathcal{O}_{\mathbb{P}^1}.$$

Moreover, we also have a trivial distinguished triangle:

$$i_* \mathcal{O}_{\mathrm{Spec}\, k} \longrightarrow i_* O_{\mathrm{Spec}\, k} \longrightarrow 0. \tag{3.16}$$

Using [32, Lemma 13.4.9] we get a distinguished triangle by summing (3.15) and (3.16):

$$\mathbf{R} f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}(-E) \oplus i_* \mathcal{O}_{\mathrm{Spec}\, k} \longrightarrow \mathbf{R} f_* \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2} \oplus i_* \mathcal{O}_{\mathrm{Spec}\, k} \longrightarrow \mathbf{R} m_* \mathcal{O}_{\mathbb{P}^1}.$$

But, appealing to (3.14), this is precisely the distinguished triangle claimed. □

We are now ready to prove Proposition 3.8.

Proof of Proposition 3.8. Consider the top row E^2-page of the spectral sequence (3.4) for $\mathrm{Bl}_0 \mathbb{P}^2$; this is given in Fig. 5. Following the proof of [14, Proposition V.3.4] we see that $H^2\big(\mathrm{Bl}_0 \mathbb{P}^2, \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}\big) = H^1\big(\mathrm{Bl}_0 \mathbb{P}^2, \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}\big) = 0$. Since $\mathrm{Bl}_0 \mathbb{P}^2$ is smooth and compact, Serre duality implies that

$$\begin{aligned}
H^2\big(\mathrm{Bl}_0 \mathbb{P}^2, \omega_{\mathrm{Bl}_0 \mathbb{P}^2}\big) &= H^0\big(\mathrm{Bl}_0 \mathbb{P}^2, \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}\big) = k, \\
H^1\big(\mathrm{Bl}_0 \mathbb{P}^2, \omega_{\mathrm{Bl}_0 \mathbb{P}^2}\big) &= H^1\big(\mathrm{Bl}_0 \mathbb{P}^2, \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}\big) = 0, \text{ and} \\
H^0\big(\mathrm{Bl}_0 \mathbb{P}^2, \omega_{\mathrm{Bl}_0 \mathbb{P}^2}\big) &= H^2\big(\mathrm{Bl}_0 \mathbb{P}^2, \mathcal{O}_{\mathrm{Bl}_0 \mathbb{P}^2}\big) = 0.
\end{aligned}$$

To compute the terms in the second row of the E_2-page, consider the blow-up map $f : \mathrm{Bl}_0 \mathbb{P}^2 \to \mathbb{P}^2$. There is an exact sequence of sheaves on $\mathrm{Bl}_0 \mathbb{P}^2$,

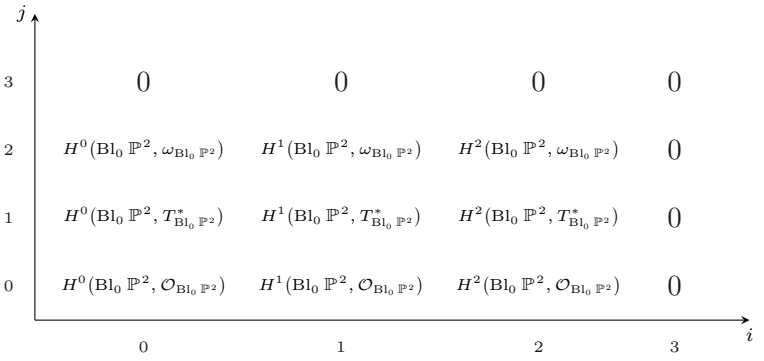

Fig. 5. (3.4) for $X = Bl_0\mathbb{P}^2$.

$$f^*T^*_{\mathbb{P}^2} \to T^*_{\mathrm{Bl}_0\,\mathbb{P}^2} \to \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2} \to 0.$$

In general this sequence is not left exact, but we will show that $f^*T^*_{\mathbb{P}^2} \to T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}$ is injective in this specific example. We can explicitly write

$$\mathrm{Bl}_0\,\mathbb{P}^2 = \mathrm{Proj}\left(\frac{k[x,y][X,Y]}{\langle Xy - xY\rangle}\right)$$

where x and y have degree 0 and X and Y have degree 1. We will show the map is injective on the affine open sets

$$U_X \overset{\text{def}}{=} \{X \neq 0\}$$

and

$$U_Y \overset{\text{def}}{=} \{Y \neq 0\}.$$

The situation is symmetric, so we will only give the argument on U_X. Observe

$$U_X = \mathrm{Spec}\left(\frac{k[x,y][X,Y][1/X]_0}{\langle Xy - xY\rangle}\right) = \mathrm{Spec}\left(\frac{k[x,y,Y/X]}{\langle Y - xY/X\rangle}\right).$$

Considering $f|_{U_X} : U_X \longrightarrow \mathrm{Spec}\,k[x,y] \subseteq \mathbb{P}^2$, we have an explicit description

$$f^*T^*_{\mathbb{P}^2}(U_X) = \mathcal{O}dx \oplus \mathcal{O}dy$$

and

$$T^*\,\mathrm{Bl}_0\,\mathbb{P}^2(U_X) = \mathcal{O}dx \oplus \mathcal{O}d(Y/X).$$

Under this identification, the map of sheaves $f^*T^*_{\mathbb{P}^2}(U_X) \longrightarrow T^*\,\mathrm{Bl}_0\,\mathbb{P}^2(U_X)$ is given in coordinates by

$$dx \mapsto dx$$

and

$$dy \mapsto d(x\frac{Y}{X}) = (dx)(\frac{Y}{X}) + xd(\frac{Y}{X}).$$

This is the map we want to show is injective.

Observe that given $(g,h) \in \mathcal{O} \oplus \mathcal{O}$,

$$(g,h) \mapsto (g + h\frac{Y}{X}, xh).$$

If $xh = 0 \in \frac{k[x,y,Y/X]}{\langle Y - xY/X \rangle}$, then $h = 0$. Therefore if $(0,0) = (g + h\frac{Y}{X}, xh) = (g,0)$, we must have $g = 0$. Thus,

$$\ker(f^*T^*\mathbb{P}^2(U_X) \longrightarrow T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}(U_X)) = \{(0,0)\},$$

as desired. Applying the same argument for U_Y, we conclude that $f^*T^*_{\mathbb{P}^2} \to T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}$ is injective.

It follows that there is a short exact sequence of coefficients

$$0 \to f^*T^*_{\mathbb{P}^2} \to T^*_{\mathrm{Bl}_0\,\mathbb{P}^2} \to \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2} \to 0,$$

that induces a long exact sequence on cohomology

$$\begin{array}{l} H^0\big(\mathrm{Bl}_0\,\mathbb{P}^2; f^*T^*_{\mathbb{P}^2}\big) \longrightarrow H^0\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) \longrightarrow H^0\big(\mathrm{Bl}_0\,\mathbb{P}^2; \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}\big) \\ \searrow \\ H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; f^*T^*_{\mathbb{P}^2}\big) \longrightarrow H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) \longrightarrow H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}\big) \\ \searrow \\ H^2\big(\mathrm{Bl}_0\,\mathbb{P}^2; f^*T^*_{\mathbb{P}^2}\big) \longrightarrow H^2\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) \longrightarrow H^2\big(\mathrm{Bl}_0\,\mathbb{P}^2; \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}\big) \to \cdots \end{array} \tag{3.17}$$

Substituting the results of Lemmas 3.9 and 3.10 implies that (3.17) takes the form

$$0 \to H^0\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) \to 0 \to k \to H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) \to k \to 0 \to H^2\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) \to 0.$$

Consequently,

$$H^0\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) = H^2\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) = 0$$

and

$$H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) = k^2.$$

Therefore, the second page of the spectral sequence takes the form shown in Fig. 6, which means that the spectral sequence collapses on this page. □

This completes the proof that equality (3.2) holds after applying the forgetful homomorphism $\mathrm{GW}(k) \longrightarrow K_0(k)$. It remains to show the relation is satisfied by forms.

3.2. Verification in $\mathrm{GW}(k)$

In this subsection, k denotes a field of characteristic not 2. We compute the bilinear form on $\mathbf{R}p_*\mathcal{T}_X$ for $X = \operatorname{Spec} k, \mathbb{P}^1, \mathbb{P}^2$ or $\mathrm{Bl}_0\,\mathbb{P}^2$, and complete the verification of equality (3.2) in $\mathrm{GW}(k)$. We begin by defining elements in $\mathrm{GW}(k)$ which we will use to express these classes.

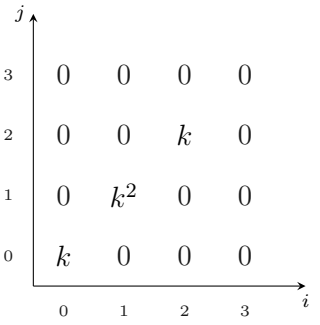

Fig. 6. Result of computation.

Definition 3.18. The hyperbolic bilinear form H is defined to be the rank 2 symmetric bilinear form with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 3.19. Let $a \in k$. Then we write $\langle a \rangle$ for the rank 1 bilinear form $(x, y) \mapsto axy$.

Remark 3.20. In $\mathrm{GW}(k)$ there is an equality between the class of H and $\langle 1 \rangle + \langle -1 \rangle$ in $\mathrm{GW}(k)$.

We start with computing the form for $\mathbf{R}p_*\mathcal{T}_{\mathrm{Spec}\, k}$: recall that the complex $\mathbf{R}p_*\mathcal{T}_{\mathrm{Spec}\, k}$ is k in degree zero. The following result is immediate:

Proposition 3.21. $\mathbf{R}p_*T_{\mathrm{Spec}\, k}$ *has the trivial duality* $\langle 1 \rangle$.

Next, consider $\mathbb{P}^1$.

Proposition 3.22. $\chi^{\mathbb{A}^1}(\mathbb{P}^1) = H$ *in* $\mathrm{GW}(k)$.

Remark 3.23. This provides an alternate proof of the $n = 1$ case of [16, Example 1.7].

Proof. Referring to the spectral sequences in Figs. 1 and 2, we recall that

$$\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^1} \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) \simeq k^2,$$

in degree 0. The cup product composed with the trace, which computes our bilinear form on $\mathbf{R}p_*\mathcal{T}_{\mathbb{P}^1}$, gives the Serre duality isomorphism $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \simeq H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1})$.

From this, we can express the bilinear form of interest in terms of evaluation on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \longrightarrow k$. Indeed, our form can be written as the composition:

$$\begin{aligned} &\big(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^*\big) \otimes \big(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^*\big) \longrightarrow \\ &\big(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^*\big) \oplus \big(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})\big) \longrightarrow k, \end{aligned}$$

where the first map is projecting onto the cross terms, and the second is given by

$$e \otimes f + f' \otimes e' \mapsto f(e) + f'(e').$$

From this formula, we readily verify that Gram matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \square$$

Proposition 3.24. $\chi^{\mathbb{A}^1}(\mathbb{P}^2) = H + \langle 1 \rangle$ *in* $\mathrm{GW}(k)$.

Remark 3.25. This provides an alternate proof of the $n = 2$ case of [16, Example 1.7].

Proof. We refer to the spectral sequences depicted in Figs. 3 and 4 for computations of underlying vector spaces. We first note that the rank 2 bilinear form on $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \oplus H^2(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2})$ is equal to H, by an argument virtually identical to that in Proposition 3.22.

To complete the proposition, we will show that the symmetric bilinear form

$$H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2}) \otimes H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2}) \xrightarrow{\cup} H^2(\mathbb{P}^2, \omega_{\mathbb{P}^2}) \xrightarrow{\mathrm{Tr}} k \tag{3.26}$$

is the map $k \times k \to k$ given by $(x, y) \mapsto xy$, which is $\langle 1 \rangle$ as defined in Definition 3.19.

We first find a basis for $H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2})$. To this end, consider the Euler sequence

$$0 \longrightarrow T^*_{\mathbb{P}^2} \xrightarrow{\psi} \mathcal{O}(-1)^3 \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0.$$

Say $\mathbb{P}^2 = \mathbb{P}(k[x, y, z])$. Let $f \in k[x, y, z]$, we denote the distinguished open set of f by $\mathbb{P}^2_f$. Following [14, Theorem II.8.13], the homomorphism ψ is defined on distinguished open sets as follows

$$\begin{aligned}
\psi\left(f_1 d\Big(\frac{y}{x}\Big) + f_2 d\Big(\frac{z}{x}\Big)\right) &= \left(-\frac{y}{x^2} f_1 - \frac{z}{x^2} f_2, \frac{f_1}{x}, \frac{f_2}{x}\right) \text{ on } \mathbb{P}^2_x, \\
\psi\left(g_1 d\Big(\frac{x}{y}\Big) + g_2 d\Big(\frac{z}{y}\Big)\right) &= \left(\frac{g_1}{y}, -\frac{x}{y^2} g_1 - \frac{z}{y^2} g_2, \frac{g_2}{y}\right) \text{ on } \mathbb{P}^2_y, \text{ and} \\
\psi\left(h_1 d\Big(\frac{x}{z}\Big) + h_2 d\Big(\frac{y}{z}\Big)\right) &= \left(\frac{h_1}{z}, \frac{h_2}{z}, -\frac{x}{z^2} h_1 - \frac{y}{z^2} h_2\right) \text{ on } \mathbb{P}^2_z,
\end{aligned}$$

with

$$\begin{aligned}
f_i &\overset{\text{def}}{=} f_i(y/x, z/x), \\
g_i &\overset{\text{def}}{=} g_i(x/y, z/y),
\end{aligned}$$

and

$$h_i \overset{\text{def}}{=} h_i(x/z, y/z)$$

for $i \in \{1, 2\}$. The homomorphism ϕ is defined by

$$\phi(s_0, s_1, s_2) = x s_0 + y s_1 + z s_2.$$

Let $\delta : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2})$ be the zeroth connecting homomorphism in the long exact sequence associated to the Euler sequence. Since δ is an isomorphism, $\delta(1)$ is a basis for $H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2}) \cong k$. We next use Čech cohomology to calculate $\delta(1)$.

Let $\mathcal{U}$ be the affine covering of $\mathbb{P}^2$ by distinguished open sets $\mathbb{P}^2_x$, $\mathbb{P}^2_y$ and $\mathbb{P}^2_z$. In what follows, $\big(C^*(\mathcal{U}, \mathcal{F}), d^*\big)$ denotes the Čech complex associated to $\mathcal{F}$ and $\mathcal{U}$, where $\mathcal{F}$ is a sheaf of abelian groups on $\mathbb{P}^2$.

We recall the definition of the connecting homomorphism. The Euler sequence induces a short exact sequence of cochain complexes, [14, Theorem III.4.5]:

$$0 \to C^*\big(\mathcal{U}, T^*_{\mathbb{P}^2}\big) \to C^*\big(\mathcal{U}, \mathcal{O}(-1)^3\big) \to C^*\big(\mathcal{U}, \mathcal{O}_{\mathbb{P}^2}\big) \to 0,$$

then δ is defined by diagram chasing in the diagram below,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0\big(\mathcal{U}, T^*_{\mathbb{P}^2}\big) & \longrightarrow & C^0\big(\mathcal{U}, \mathcal{O}(-1)^3\big) & \longrightarrow & C^0\big(\mathcal{U}, \mathcal{O}_{\mathbb{P}^2}\big) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^0} & & \\ 0 & \longrightarrow & C^1\big(\mathcal{U}, T^*_{\mathbb{P}^2}\big) & \longrightarrow & C^1\big(\mathcal{U}, \mathcal{O}(-1)^3\big) & \longrightarrow & C^1\big(\mathcal{U}, \mathcal{O}_{\mathbb{P}^2}\big) & \longrightarrow & 0. \end{array} \tag{3.27}$$

Take $(1,1,1) \in \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2_x) \times \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2_y) \times \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2_z)$ the generator of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \cong k$. By the definition of ϕ, d^0 and ψ we see that $\delta(1,1,1) = \Big(\frac{y}{x}d\big(\frac{x}{y}\big), \frac{z}{y}d\big(\frac{y}{z}\big), \frac{z}{x}d\big(\frac{x}{z}\big)\Big) \in H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2})$, as diagram (3.28) illustrates.

$$\begin{array}{ccc} & & \Big(\big(\frac{1}{x},0,0\big), \Big(0,\frac{1}{y},0\Big), \big(0,0,\frac{1}{z}\big)\Big) \longmapsto (1,1,1) \\ & & \big\downarrow \\ \Big(\frac{y}{x}d\big(\frac{x}{y}\big), \frac{z}{y}d\big(\frac{y}{z}\big), \frac{z}{x}d\big(\frac{x}{z}\big)\Big) & \longmapsto & \Big(\big(\frac{1}{x},-\frac{1}{y},0\big), \big(0,\frac{1}{y},-\frac{1}{z}\big), \big(\frac{1}{x},0,-\frac{1}{z}\big)\Big) \end{array} \tag{3.28}$$

Our next task is to calculate $Q(\delta(1)\otimes\delta(1))$. We first compute $\delta(1)\cup\delta(1)$, as $Q(\delta(1)\otimes\delta(1)) = \mathrm{Tr}(\delta(1)\cup\delta(1))$. Consider the diagram below

$$\begin{array}{ccccccc} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \otimes H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2}) & \xrightarrow{\delta\otimes id} & H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2}) \otimes H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2}) & & & & \\ \downarrow{\scriptstyle \cup} & & \downarrow{\scriptstyle \cup} & & & & \\ H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \otimes T^*_{\mathbb{P}^2}) & \xrightarrow{\partial} & H^2(\mathbb{P}^2, T^*_{\mathbb{P}^2} \otimes T^*_{\mathbb{P}^2}) & \xrightarrow{\wedge} & H^2(\mathbb{P}^2, \omega_{\mathbb{P}^2}) & \xrightarrow{\mathrm{Tr}} & k, \end{array} \tag{3.29}$$

where ∂ is the first connecting homomorphism in the long exact sequence associated to the short exact sequence

$$0 \longrightarrow T^*_{\mathbb{P}^2} \otimes T^*_{\mathbb{P}^2} \xrightarrow{\psi\otimes id} \mathcal{O}(-1)^3 \otimes T^*_{\mathbb{P}^2} \xrightarrow{\phi\otimes id} \mathcal{O}_{\mathbb{P}^2} \otimes T^*_{\mathbb{P}^2} \longrightarrow 0,$$

which is obtained by tensorizing the Euler sequence with $T^*_{\mathbb{P}^2}$. By commutativity of diagram (3.29), we have that $\delta(1) \cup \delta(1) = \partial(1 \cup \delta(1)) = \partial(\delta(1))$.

Claim 3.30. *Let α denote the Čech cocycle*

$$\frac{z^2}{xy} d\Big(\frac{x}{z}\Big) \wedge d\Big(\frac{y}{z}\Big) \in H^2(\mathbb{P}^2, \omega_{\mathbb{P}^2}).$$

We claim that the composite $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \otimes T^*_{\mathbb{P}^2}) \xrightarrow{\partial} H^2(\mathbb{P}^2, T^*_{\mathbb{P}^2} \otimes T^*_{\mathbb{P}^2}) \xrightarrow{\wedge} H^2(\mathbb{P}^2, \omega_{\mathbb{P}^2})$ *satisfies*

$$\delta(1) \mapsto -\alpha.$$

Given the claim, we complete the argument. The canonical trace map sends the form of [14, Remark 7.1.1] to 1. Since α differs from this by the permutation swapping x and z, which has sign -1, we deduce that α maps to -1 under the trace. By Claim 3.30 it follows that $\mathrm{Tr}(\delta(1) \otimes \delta(1)) = 1$. □

Proof of Claim 3.30. We calculate $\partial(\delta(1))$ via the commutative diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^1\big(\mathcal{U}, T^*_{\mathbb{P}^2}\otimes T^*_{\mathbb{P}^2}\big) & \longrightarrow & C^1\big(\mathcal{U}, \mathcal{O}(-1)^3\otimes T^*_{\mathbb{P}^2}\big) & \longrightarrow & C^1\big(\mathcal{U}, \mathcal{O}_{\mathbb{P}^2}\otimes T^*_{\mathbb{P}^2}\big) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^1} & & \\ 0 & \longrightarrow & C^2\big(\mathcal{U}, T^*_{\mathbb{P}^2}\otimes T^*_{\mathbb{P}^2}\big) & \longrightarrow & C^2\big(\mathcal{U}, \mathcal{O}(-1)^3\otimes T^*_{\mathbb{P}^2}\big) & \longrightarrow & C^2\big(\mathcal{U}, \mathcal{O}_{\mathbb{P}^2}\otimes T^*_{\mathbb{P}^2}\big) & \longrightarrow & 0. \end{array} \tag{3.31}$$

Note that

$$\delta(1) = \Big(\tfrac{y}{x}\otimes d\big(\tfrac{x}{y}\big), \tfrac{z}{y}\otimes d\big(\tfrac{y}{z}\big), \tfrac{z}{x}\otimes d\big(\tfrac{x}{z}\big)\Big) \in H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\otimes T^*_{\mathbb{P}^2}).$$

For ease of notation, we define

$$A \overset{\text{def}}{=} \left(\Big(\frac{y}{x^2}\otimes d\Big(\frac{x}{y}\Big), 0, 0\Big), \Big(0, \frac{z}{y^2}\otimes d\Big(\frac{y}{z}\Big), 0\Big), \Big(\frac{z}{x^2}\otimes d\Big(\frac{x}{z}\Big), 0, 0, \Big)\right)$$

and

$$B \overset{\text{def}}{=} \frac{z^2}{xy} d\Big(\frac{x}{z}\Big)\otimes d\Big(\frac{y}{z}\Big).$$

By definition of ϕ we have that $(\phi\otimes id)(A) = \delta(1)$. Moreover, $d^1(A)$ is equal to

$$\left(1\otimes\Big(\frac{y}{x^2}d\Big(\frac{x}{y}\Big) - \frac{z}{x^2}d\Big(\frac{x}{z}\Big)\Big), 1\otimes\frac{z}{y^2}d\Big(\frac{y}{z}\Big), 0\right) \in \big(\mathcal{O}(-1)^3\otimes T^*_{\mathbb{P}^2}\big)\,(\mathbb{P}^2_{xyz}). \tag{3.32}$$

To simplify (3.32) we use the equality

$$\frac{z}{x}d\Big(\frac{x}{z}\Big) = \frac{z}{x}\left(\frac{y}{z}d\Big(\frac{x}{y}\Big) + \frac{x}{y}d\Big(\frac{y}{z}\Big)\right) = \frac{y}{x}d\Big(\frac{x}{y}\Big) + \frac{z}{y}d\Big(\frac{y}{z}\Big). \tag{3.33}$$

Substituting (3.33) into the first coordinate of (3.32) yields

$$d^1(A) = \left(-\frac{z}{xy}, \frac{z}{y^2}, 0\right)\otimes d\Big(\frac{y}{z}\Big). \tag{3.34}$$

We now compute $(\psi\otimes id)(B)$ and $d^1(A) + (\psi\otimes id)(B)$. By definition of ψ we have

$$(\psi\otimes id)(B) = \left(\frac{z^2}{xy}\frac{1}{z}, 0, -\frac{z^2}{xy}\frac{x}{z^2}\right)\otimes d\Big(\frac{y}{z}\Big) = \left(\frac{z}{xy}, 0, -\frac{1}{y}\right)\otimes d\Big(\frac{y}{z}\Big), \tag{3.35}$$

and combining (3.34) and (3.35) we obtain

$$d^1(A) + (\psi\otimes id)(B) = \left(0, \frac{z}{y^2}, -\frac{1}{y}\right)\otimes d\Big(\frac{y}{z}\Big) = \psi\otimes id\left(-d\Big(\frac{z}{y}\Big)\otimes d\Big(\frac{y}{z}\Big)\right).$$

Note that we have actually proved that $\partial(\delta(1)) = d\Big(\frac{z}{y}\Big)\otimes d\Big(\frac{y}{z}\Big) - B$, as we can see in the diagram below.

$$\begin{array}{ccc} A & \longmapsto & \delta(1) \\ \big\downarrow & & \\ d\big(\frac{z}{y}\big)\otimes d\big(\frac{y}{z}\big) - B & \longmapsto & d^1(A) \end{array} \tag{3.36}$$

Since

$$d\Big(\frac{z}{y}\Big)\otimes d\Big(\frac{y}{z}\Big) = d\Big(\frac{z}{y}\Big)\otimes d\Big(\frac{y}{z}\Big) = -\frac{y^2}{z^2}d\Big(\frac{y}{z}\Big)\otimes d\Big(\frac{y}{z}\Big),$$

it follows that $d\big(\frac{z}{y}\big)\wedge d\big(\frac{y}{z}\big) = 0$. From this we obtain the claim:

$$\begin{array}{ccccc} H^1(\mathbb{P}^2, T^*_{\mathbb{P}^2}\otimes\mathcal{O}_{\mathbb{P}^2}) & \xrightarrow{\partial} & H^2(\mathbb{P}^2, T^*_{\mathbb{P}^2}\otimes T^*_{\mathbb{P}^2}) & \xrightarrow{\wedge} & H^2(\mathbb{P}^2,\omega_{\mathbb{P}^2}) \\ \delta(1) & \longmapsto & d\Big(\frac{z}{y}\Big)\otimes d\Big(\frac{y}{z}\Big) - B & \longmapsto & -\alpha. \end{array}$$

□

Proposition 3.37. $\chi^{\mathbb{A}^1}(\mathrm{Bl}_0\,\mathbb{P}^2) = 2H$ *in* $\mathrm{GW}(k)$.

Proof. As before, we first consider the pairing on $H^0(\mathrm{Bl}_0\,\mathbb{P}^2, \mathcal{O}_{\mathrm{Bl}_0\,\mathbb{P}^2}) \oplus H^2(\mathrm{Bl}_0\,\mathbb{P}^2, \Omega^2_{\mathrm{Bl}_0\,\mathbb{P}^2})$ and show it is H in $\mathrm{GW}(k)$. The argument here is again as in Proposition 3.22.

Now let Q' denote the class of the symmetric bilinear form

$$H^1(\mathrm{Bl}_0\,\mathbb{P}^2, T^*_{\mathrm{Bl}_0\,\mathbb{P}^2})\otimes H^1(\mathrm{Bl}_0\,\mathbb{P}^2, T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}) \xrightarrow{\cup} H^2(\mathrm{Bl}_0\,\mathbb{P}^2, \omega_{\mathrm{Bl}_0\,\mathbb{P}^2}) \xrightarrow{\mathrm{Tr}} k.$$

We will show that Q' isomorphic to H.

To do this it is enough to find a non-zero element v in $H^1(\mathrm{Bl}_0\,\mathbb{P}^2, T^*_{\mathrm{Bl}_0\,\mathbb{P}^2})$ such that the image of $v\cup v$ in $H^1(\mathrm{Bl}_0\,\mathbb{P}^2, \omega_{\mathrm{Bl}_0\,\mathbb{P}^2})$ is 0. To see this, note that we may extend v to a basis $\{v, w\}$ of $H^1(\mathrm{Bl}_0\,\mathbb{P}^2, T^*_{\mathrm{Bl}_0\,\mathbb{P}^2})$. Replacing w by $w - \frac{Q'(w,w)}{2Q'(v,w)}v$ (note that we use that the characteristic is not 2 here), we obtain a new basis $\{w', v\}$ so that the Gram matrix of Q' is

$$\begin{pmatrix} 0 & Q'(w',v) \\ Q'(w',v) & 0 \end{pmatrix}.$$

Rescaling v, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is H in $\mathrm{GW}(k)$.

The blow-up $T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}$ can be described as the projectivization of total space of the bundle $\mathcal{O}(-1)\oplus\mathcal{O}$ on $\mathbb{P}^1$:

$$T^*_{\mathrm{Bl}_0\,\mathbb{P}^2} \cong \mathrm{Proj}_{\mathrm{Proj}\,k[x,y,z]}\, k[x,y,z][S,T]/\langle Sx - Ty\rangle \cong \mathbb{P}_{\mathrm{Proj}\,k[S,T]}(\mathcal{O}(-1)\oplus\mathcal{O}).$$

Let $\pi : \mathrm{Bl}_0\,\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ denote the projection. The map π induces a map $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) \longrightarrow H^1(\mathrm{Bl}_0\,\mathbb{P}^2, \pi^*\Omega_{\mathbb{P}^1})$. Composing with the map $\pi^*\Omega_{\mathbb{P}^1} \longrightarrow \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2}$, we obtain

$$\pi^* : H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) \longrightarrow H^1(\mathrm{Bl}_0\,\mathbb{P}^2, \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2}).$$

Since $H^2(\mathbb{P}^2, \Omega_{\mathbb{P}^1}\otimes\Omega_{\mathbb{P}^1}) = 0$, any v in the image of π^* satisfies $Q'(v,v) = 0$ by naturally of the cup product.

We have thus reduced the problem to showing that π^* is nonzero. Since $\pi i'$ is the identity on $\mathbb{P}^1$, the composition of π^* with the map

$$H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; T^*_{\mathrm{Bl}_0\,\mathbb{P}^2}\big) \longrightarrow H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; \Omega_{\mathrm{Bl}_0\,\mathbb{P}^2/\mathbb{P}^2}\big) \cong H^1\big(\mathrm{Bl}_0\,\mathbb{P}^2; i'_*\Omega_{\mathbb{P}^1}\big) \cong H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1})$$

of Equation (3.17) is the identity. Since $H^1(\mathbb{P}^1; \Omega_{\mathbb{P}^1}) \cong k$, it follows that π^* is nonzero as claimed. □

Now we can verify the equality (3.2); that is, that $\chi^{\mathbb{A}^1}(\mathbb{P}^1) + \chi^{\mathbb{A}^1}(\mathbb{P}^2) = \chi^{\mathbb{A}^1}(\mathrm{Spec}\,k) + \chi^{\mathbb{A}^1}(\mathrm{Bl}_0\,\mathbb{P}^2)$. Substituting in the results of Propositions 3.21, 3.22, 3.24 and 3.37, we obtain the result.

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