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Model category structures on multicomplexes

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We present a family of model structures on the category of multicomplexes. There is a cofibrantly generated model structure in which the weak equivalences are the morphisms inducing an isomorphism at a fixed stage of an associated spectral sequence. Corresponding model structures are given for truncated versions of multicomplexes, interpolating between bicomplexes and multicomplexes. For a fixed stage of the spectral sequence, the model structures on all these categories are shown to be Quillen equivalent.

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1. Introduction

We provide a family of model structures on the category of multicomplexes of R-modules, for R a commutative unital ring. A multicomplex is an algebraic structure generalizing the notion of a (graded) chain complex and that of a bicomplex. The structure involves a family of higher "differentials" indexed by the non-negative integers. The terms twisted chain complex and D_∞-module are also used. Multicomplexes have arisen in many different places and play an important role in homotopical and homological algebra. A multicomplex has an associated total complex, with filtration, and thus an associated spectral sequence.

For each $r \geq 0$, we show that there is a cofibrantly generated model structure on the category of multicomplexes in which the weak equivalences are the morphisms inducing an isomorphism at the $(r+1)$-th page of the spectral sequence. The fibrations are explicitly specified via surjectivity conditions.

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We also provide such models for certain truncated versions of these structures, the n-multicomplexes. We write n-mC_R for the category of n-multicomplexes. The case $n = 2$ gives the category of bicomplexes and the results here recover those of [2]. Multicomplexes can be thought of as the case $n = \infty$ and we make frequent use of this notational device.

A key ingredient of the model structures is the explicit description of the spectral sequence associated to a multicomplex in [8]. The main techniques imitate the work of [2], using representable versions of r-cycles and r-boundaries to provide generating (trivial) cofibrations for the model structures. One difference, however, is that we describe the representing objects for cycles via an iterated pushout process, as a direct description would be cumbersome. The model structures appear in Theorems 3.28 and 3.30, the latter being a minor variant of the former.

We introduce a graded associative algebra $\mathcal{C}_n$ in the category of vertical bicomplexes such that n-multicomplexes can be viewed as $\mathcal{C}_n$-modules in vertical bicomplexes. This allows us to set up, for fixed r, Quillen adjunctions relating the model structures of Theorem 3.30 on n-multicomplexes as n varies. Indeed, the functors can be viewed as restriction and extension of scalars. We show that these adjunctions are Quillen equivalences for $n \geq 2$. Multicomplexes can be viewed as the *homotopy-coherent version* of bicomplexes [9,7], so that one would expect ∞-mC_R and 2-mC_R to have equivalent homotopy theories. Our work confirms that this is the case for the r-model structure for each r and that the same is true for all the intermediate categories of n-multicomplexes.

Our results can be summarized as follows. We have a chain of adjunctions:

$$1\text{-}\mathrm{mC}_R \rightleftarrows 2\text{-}\mathrm{mC}_R \rightleftarrows 3\text{-}\mathrm{mC}_R \rightleftarrows \cdots \rightleftarrows n\text{-}\mathrm{mC}_R \rightleftarrows \cdots \rightleftarrows \infty\text{-}\mathrm{mC}_R.$$

Apart from at the far left, we may fix any $r \geq 0$ and endow the categories with the r-model structure of Theorem 3.30. Equipped with these model structures, each adjoint pair, apart from the leftmost one, gives a Quillen equivalence. The category 1-mC_R is the category of vertical bicomplexes, where the objects have only one non-trivial structure map, a vertical differential. In this case, we only have the $r = 0$ model structure, corresponding to the usual projective model structure on cochain complexes. Indeed, in this case the associated spectral sequence degenerates at the E_1 page and the notions of equivalence in our hierarchy all coincide. The leftmost adjoint pair gives a Quillen adjunction for $r = 0$, but it is not a Quillen equivalence.

In the category of n-multicomplexes with the r-model structure of Theorem 3.30, let the weak equivalences be denoted $\mathcal{E}_r^n$, the fibrations Fib_r^n and the cofibrations Cof_r^n. Then for all $n \geq 2$ and $r \geq 0$ we have

$$\mathcal{E}_r^n \subseteq \mathcal{E}_{r+1}^n, \qquad Fib_{r+1}^n \subseteq Fib_r^n, \qquad Fib_r^n \cap \mathcal{E}_r^n \subseteq Fib_{r+1}^n \cap \mathcal{E}_{r+1}^n, \qquad Cof_{r+1}^n \subseteq Cof_r^n.$$

The paper is arranged as follows. In Section 2 we give the necessary background on multicomplexes and related categories. Section 3 presents the r-model category structure on these categories for each $r \geq 0$. In Section 4 we describe the relationships between these model structures, setting up the Quillen adjunctions and equivalences. Section 5 considers analogues of the previously obtained model category structures for bounded multicomplexes. Section 6 gives various examples of cofibrations and cofibrant replacements for our model category structures, in both the bounded and unbounded cases.

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2. Notations and preliminaries

In this section, we summarize the definitions and results on multicomplexes that are required for the rest of the paper, and fix sign and grading conventions. Throughout this paper, R will denote a commutative unital ground ring and R-Mod will denote the category of R-modules.

Definition 2.1. A *multicomplex* or ∞-*multicomplex* A is a $(\mathbb{Z},\mathbb{Z})$-bigraded R-module $A = \{A^{p,q}\}_{p,q\in\mathbb{Z}}$ endowed with a family of maps $\{d_i \colon A \to A\}_{i\geq 0}$ of bidegree $(-i, 1-i)$ satisfying for all $l \geq 0$,

$$\sum_{i+j=l} (-1)^i d_i d_j = 0. \tag{1}$$

Let $n \geq 1$ be an integer. An n-*multicomplex* is a multicomplex with $d_i = 0$ for all $i \geq n$.

For $1 \leq n \leq \infty$, a (*strict*) *morphism* of n-multicomplexes is a map f of bigraded R-modules of bidegree $(0,0)$ satisfying $d_i f = f d_i$ for all $i \geq 0$. We denote by $n\text{-mC}_R$ the category of n-multicomplexes and strict morphisms.

For example, a 1-multicomplex is a vertical bicomplex, as defined in [7, Section 2.1], that is, a $(\mathbb{Z},\mathbb{Z})$-bigraded R-module endowed with a differential d_0 of bidegree $(0,1)$. The category 1-mC_R will also be denoted by vbC_R when we need to emphasize vertical bicomplexes.

A 2-multicomplex is a bicomplex, with the convention $d_0 d_1 = d_1 d_0$; thus the chosen sign convention agrees with [2, Definition 2.10].

As observed in [8, Remark 2.2], the above choice of sign convention for multicomplexes gives an isomorphic category to the version without signs in the relations.

By [3, Lemma 3.3], the category of multicomplexes is symmetric monoidal, where the monoidal structure is given by the bifunctor

$$\otimes \colon \infty\text{-mC}_R \times \infty\text{-mC}_R \to \infty\text{-mC}_R$$

which on objects is given by $((A, d_i^A), (B, d_i^B)) \mapsto (A \otimes B, d_i^A \otimes 1 + 1 \otimes d_i^B)$ and on strict morphisms is given by $(f,g) \mapsto f \otimes g$. The symmetry isomorphism is given by the morphism of multicomplexes

$$\tau_{A\otimes B} \colon A \otimes B \to B \otimes A$$

defined by

$$a \otimes b \mapsto (-1)^{\langle a,b\rangle} b \otimes a.$$

Here for a, b of bidegree (a_1, a_2), (b_1, b_2) respectively, we let $\langle a, b\rangle = a_1 b_1 + a_2 b_2$.

This functor also describes a symmetric monoidal structure on $n\text{-mC}_R$ for each $n \geq 1$ by restriction.

For the rest of this section, let $r \geq 0$ be an integer. We consider the spectral sequence $E_r^{*,*}(A)$ associated to the multicomplex A as described in [8, Proposition 2.8]. The following is a reformulation of the description in [8, Definition 2.6] to make the notation consistent with [2] in the case of bicomplexes.

Proposition 2.2 *([2, Lemma 2.13]). Let $(A, d_0, d_1, \dots, d_n, \dots)$ be a multicomplex. Then*

$$E_r^{p,q}(A) \cong Z_r^{p,q}(A)/B_r^{p,q}(A)$$

where the cycles are $Z_0^{p,q}(A) := A^{p,q}$ and for all $r \geq 1$,

$$Z_r^{p,q}(A) := \big\{a_0 \in A^{p,q} \mid \textit{for all } 0 \le l \le r-1, \\ \textstyle\sum_{i+j=l}(-1)^i d_i a_j = 0 \textit{ for some } a_j \in A^{p-j,q-j},\ 1 \le j \le r-1\big\},$$

and the boundaries are $B_0^{p,q}(A) := 0$, $B_1^{p,q}(A) = A^{p,q} \cap \operatorname{im} d_0$, *and for all* $r \ge 2$,

$$\begin{aligned} B_r^{p,q}(A) := \big\{x \in A^{p,q} \mid \textit{there exist } b_i \in A^{p+r-1-i,q+r-2-i} \textit{ for } 0 \le i \le r-1 \textit{ such that} \\ x = \sum_{i=0}^{r-1}(-1)^i d_i b_{r-i-1}, \\ \textit{and} \sum_{i=0}^{l}(-1)^i d_i b_{l-i} = 0 \textit{ for } 0 \le l \le r-2\big\}. \end{aligned} \tag{2}$$

The differential $\Delta_r\colon Z_r^{p,q}(A)/B_r^{p,q}(A) \to Z_r^{p-r,q+1-r}(A)/B_r^{p-r,q+1-r}(A)$ *is given by*

$$\Delta_r([a_0]) = \left[\sum_{i=1}^{r}(-1)^i d_i a_{r-i}\right]. \quad \square$$

Definition 2.3. Let $2 \le n \le \infty$. A morphism of n-multicomplexes $f\colon A \to B$ is said to be an E_r*-quasi-isomorphism* if the morphism $E_r(f)\colon E_r(A) \to E_r(B)$ at the r-stage of the associated spectral sequence is a quasi-isomorphism of r-bigraded complexes (that is, $E_{r+1}(f)$ is an isomorphism).

Denote by $\mathcal{E}_r^n$ the class of E_r-quasi-isomorphisms of n-mC_R. This class contains all isomorphisms of n-mC_R, satisfies the two-out-of-three property and is closed under retracts.

Finally, we recall from [3] the definition of r-homotopies of multicomplexes in the context of strict morphisms.

Definition 2.4. [3, Proposition 3.18] Let $f, g\colon A \to B$ be two strict morphisms of multicomplexes. An r-*homotopy* h from f to g is a collection of maps $h_m\colon A \to B$ of bidegree $(-m+r, -m+r-1)$ satisfying for all $m \ge 0$,

$$\sum_{i+j=m}(-1)^{i+r} d_i h_j + (-1)^i h_i d_j = \begin{cases} g-f & \text{if } m = r, \\ 0 & \text{if } m \ne r. \end{cases}$$

We write $f \simeq_r g$ if there is an r-homotopy from f to g. A morphism $f\colon A \to B$ is an r*-homotopy equivalence* if there exists a morphism $g\colon B \to A$ such that $f \circ g \simeq_r 1_B$ and $g \circ f \simeq_r 1_A$. A multicomplex A is r*-contractible* if $1_A \simeq_r 0$.

Any r-homotopy equivalence is an E_r-quasi-isomorphism by [3, Proposition 3.24].

3. Model structures on multicomplexes and n-multicomplexes, for $n \ge 2$

We now describe our model category structures on n-multicomplexes, for $2 \le n \le \infty$. In the case $n = 2$, the model category structures here are precisely those of bicomplexes obtained in [2], and indeed, the proofs for general n-multicomplexes are essentially the same. Just like for bicomplexes, a key idea in the proof is to show that the spectral sequence admits a description in terms of certain *witness* functors that have the advantage of being representable. Our presentation here differs from [2, Sections 4.1–4.2] in that we show the representing objects for the witness functors can be defined recursively; this is helpful for avoiding notational difficulties in the general multicomplex case.

3.1. Cofibrantly generated model categories

We collect some definitions and results on cofibrantly generated model categories from [6].

Definition 3.1. Let $\mathcal{C}$ be a category with all small colimits and limits and I be a class of maps in $\mathcal{C}$.

(1) A morphism is called *I-injective* (resp. *I-projective*) if it has the right (resp. left) lifting property with respect to morphisms in I. We write

$$I\text{-inj} := \mathrm{RLP}(I) \text{ and } I\text{-proj} := \mathrm{LLP}(I).$$

(2) A morphism is called an *I-fibration* (resp. *I-cofibration*) if it has the right (resp. left) lifting property with respect to I-projective (resp. I-injective) morphisms. We write

$$I\text{-fib} := \mathrm{RLP}(I\text{-proj}) \text{ and } I\text{-cof} := \mathrm{LLP}(I\text{-inj}).$$

(3) A map is a *relative I-cell complex* if it is a transfinite composition of pushouts of elements of I. We denote by I-cell the class of relative I-cell complexes.

Definition 3.2. A model category $\mathcal{C}$ is said to be *cofibrantly generated* if there are sets I and J of maps such that the following conditions hold.

(1) The domains of the maps of I are small relative to I-cell.
(2) The domains of the maps of J are small relative to J-cell.
(3) The class of fibrations is J-inj.
(4) The class of trivial fibrations is I-inj.

The set I is called the *set of generating cofibrations*, and J the *set of generating trivial cofibrations*.

The following is a consequence of Kan's Theorem (cf. [5, Theorem 11.3.1] or [6, Theorem 2.1.19]).

Theorem 3.3 *(D. M. Kan). Suppose $\mathcal{C}$ is a category with all small colimits and limits. Let $\mathcal{W}$ be a subcategory of $\mathcal{C}$ and I and J be sets of maps in $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

(1) *The subcategory $\mathcal{W}$ satisfies the two-out-of-three property and is closed under retracts.*
(2) *The domains of I are small relative to I-cell.*
(3) *The domains of J are small relative to J-cell.*
(4) *$J\text{-cof} \subseteq \mathcal{W}$.*
(5) *$I\text{-inj} = \mathcal{W} \cap J\text{-inj}$.*

Note that the categories of n-multicomplexes we will consider satisfy the assumptions of this theorem as well as conditions (1), (2) and (3).

3.2. Witness cycles and witness boundaries in multicomplexes

We begin by defining the witness cycles and witness boundaries functors and showing that they can be used to describe the spectral sequence of a multicomplex.

Definition 3.4. Let A be a multicomplex and $r \geq 0$.

Define the *witness r-cycles* $ZW_r^{p,q}(A)$ to be the bigraded R-modules $ZW_0^{p,q}(A) = A^{p,q}$ and for $r \geq 1$,

$$ZW_r^{p,q}(A) = \{(a_0, a_1, \ldots, a_{r-1}) \mid a_i \in A^{p-i,q-i} \text{ for } 0 \leq i \leq r-1 \text{ such that } \sum_{i+j=l} (-1)^i d_i a_j = 0 \text{ for } 0 \leq l \leq r-1\}.$$

There is a natural map of bigraded R-modules

$$z_r \colon ZW_r^{p,q}(A) \to Z_r^{p,q}(A)$$

given by $z_0 = 1_A$, and for $r \geq 1$,

$$z_r(a_0, \ldots, a_{r-1}) = a_0.$$

Define the *witness r-boundaries* to be the bigraded R-modules $BW_0^{p,q}(A) = 0$, $BW_1^{p,q}(A) = A^{p,q}$ and for $r \geq 2$,

$$BW_r^{p,q-1}(A) = ZW_{r-1}^{p+r-1,q+r-2}(A) \oplus A^{p,q-1} \oplus ZW_{r-1}^{p-1,q-1}(A).$$

Writing elements of $BW_r^{*,*}(A)$ as $(b_0, \ldots, b_{r-2}; a; c_0, \ldots, c_{r-2})$ with $a \in A^{*,*}$ and $(b_0, \ldots, b_{r-2}), (c_0, \ldots, c_{r-2}) \in ZW_{r-1}^{*,*}(A)$, there is a natural bidegree $(0, 1)$ map of bigraded R-modules

$$\beta_r \colon BW_r^{p,q-1}(A) \to B_r^{p,q}(A)$$

given by $\beta_0 = 0$, $\beta_1 = d_0$ and for $r \geq 2$,

$$(b_0, \ldots, b_{r-2}; a; c_0, \ldots, c_{r-2}) \longmapsto d_0 a + \sum_{i=1}^{r-1} (-1)^i d_i b_{r-i-1}.$$

We note that the maps z_r and β_r are surjective.

The final ingredient we need here is a map from witness boundaries to witness cycles. The following lemma is a check necessary for the definition of this map.

Lemma 3.5. *For $r \geq 2$, the map of bigraded R-modules specified by*

$$(b_0, \ldots, b_{r-2}) \longmapsto \Big(\sum_{i=1}^{r-1} (-1)^i d_i b_{r-1-i}, -\sum_{i=2}^{r} (-1)^i d_i b_{r-i}, \ldots, (-1)^{r-1} \sum_{i=r}^{2r-2} (-1)^i d_i b_{2r-2-i}\Big),$$

gives a map from $ZW_{r-1}^{p+r-1,q+r-2}(A)$ to $ZW_r^{p,q}(A)$.

Proof. Let $\underline{b} = (b_0, \ldots, b_{r-2}) \in ZW_{r-1}^{p+r-1,q+r-2}(A)$ and for $0 \leq j \leq r-1$ let

$$\alpha_j = (-1)^j \sum_{k=j+1}^{r+j-1} (-1)^k d_k b_{r+j-1-k}.$$

Proving that $(\alpha_0, \ldots, \alpha_{r-1}) \in ZW_r^{p,q}(A)$ amounts to computing, for $0 \leq l \leq r-1$:

$$\begin{aligned}\sum_{i+j=l}(-1)^i d_i\alpha_j &= \sum_{t=0}^{r-2}(-1)^{r-1-t}\left(\sum_{i=0}^{l}(-1)^i d_i d_{r+l-1-t-i}\right)b_t\\ &= \sum_{t=0}^{r-2}(-1)^{r-t}\left(\sum_{i=l+1}^{r+l-1-t}(-1)^i d_i d_{r+l-1-t-i}\right)b_t\\ &\qquad\text{by the multicomplex relations}\\ &= \sum_{i=l+1}^{l+r-1}(-1)^{l-1}d_i\left(\sum_{t=0}^{r+l-1-i}(-1)^{l+r-1-t-i}d_{r+l-1-t-i}b_t\right)\\ &= 0 \qquad \text{since } \underline{b}\in ZW_{r-1}^{p+r-1,q+r-2}(A).\end{aligned}$$

Thus the image of $(b_0,\dots,b_{r-2})$ lies in $ZW_r^{p,q}(A)$, as required. □

Definition 3.6. The bidegree $(0,1)$ map of bigraded R-modules

$$w_r\colon BW_r^{p,q-1}(A)\to ZW_r^{p,q}(A)$$

is given by $w_0=0$, $w_1=d_0$ and for $r\geq 2$,

$$\begin{aligned}&(b_0,\dots,b_{r-2};a;c_0,\dots,c_{r-2})\\ &\quad\xmapsto{w_r}\Big(d_0a+\sum_{i=1}^{r-1}(-1)^i d_i b_{r-1-i},\, d_1a-\sum_{i=2}^{r}(-1)^i d_i b_{r-i}+c_0,\, d_2a+\sum_{i=3}^{r+1}(-1)^i d_i b_{r+1-i}+c_1,\dots,\\ &\qquad\qquad d_{r-1}a+(-1)^{r-1}\sum_{i=r}^{2r-2}(-1)^i d_i b_{2r-2-i}+c_{r-2}\Big).\end{aligned}$$

This is well-defined as it is the sum of the map defined in Lemma 3.5 and the maps

$$A^{p,q-1}\longrightarrow ZW_r^{p,q}(A),\ a\longmapsto(d_0a,d_1a,d_2a,\dots,d_{r-1}a)$$

and

$$ZW_{r-1}^{p-1,q-1}(A)\longrightarrow ZW_r^{p,q}(A),\ (c_0,\dots,c_{r-2})\longmapsto(0,c_0,\dots,c_{r-2})$$

which are well-defined due to the definition of multicomplexes and of ZW_r.

All these definitions extend naturally to functors from multicomplexes to R-modules and natural transformations. By abuse of notation we will also denote by $ZW_r^{p,q}$, $BW_r^{p,q}$ the restriction of these functors to the category of n-multicomplexes.

Proposition 3.7 *([2, Proposition 4.3]). For every $r\geq 0$, for every $p,q\in\mathbb{Z}$, and for $2\leq n\leq\infty$ there is a commutative diagram of natural transformations of functors n-$\mathrm{mC}_R\to R$-Mod*

$$\begin{tikzcd} BW_r^{p,q-1} \arrow[r, "w_r"] \arrow[d, "\beta_r"'] & ZW_r^{p,q} \arrow[d, "z_r"] & \\ B_r^{p,q} \arrow[r, hook, "\iota_r"] & Z_r^{p,q} \arrow[r, two heads, "\pi"] & E_r^{p,q} \end{tikzcd}$$

and the natural transformation $\pi_r = \pi \circ z_r \colon ZW_r^{p,q} \to E_r^{p,q}$ *induced by the above diagram satisfies*

$$\ker \pi_r(A) = \operatorname{im} w_r(A)$$

for every n*-multicomplex* A*. In particular,* $E_r^{p,q}(A) \cong ZW_r^{p,q}(A)/w_r(BW_r^{p,q-1}(A))$*.*

Under this isomorphism, the differential on the r*-page of the spectral sequence*

$$\delta_r \colon ZW_r^{p,q}(A)/w_r(BW_r^{p,q-1}(A)) \to ZW_r^{p-r,q+1-r}(A)/w_r(BW_r^{p-r,q-r}(A))$$

is given by

$$[(a_0, a_1, \ldots, a_{r-1})] \mapsto \left[(\sum_{i=1}^{r} (-1)^i d_i a_{r-i}, \sum_{i=1}^{r} (-1)^i d_{i+1} a_{r-i}, \ldots, \sum_{i=1}^{r} (-1)^i d_{i+r-1} a_{r-i}) \right].$$

Proof. The result is trivial for $r = 0$ and $r = 1$, so we consider $r \geq 2$. It is straightforward to check that the diagram commutes. We next show that $\ker \pi_r \subseteq \operatorname{im} w_r$. If $\underline{a} = (a_0, a_1, \ldots, a_{r-1}) \in ZW_r^{p,q}(A)$ satisfies $\pi_r(\underline{a}) = 0$, this means that $z_r(\underline{a}) \in B_r^{p,q}(A)$, i.e., $a_0 \in B_r^{p,q}(A)$. By (2), there exists $(b_0, b_1, \ldots, b_{r-2}; b_{r-1}) \in ZW_{r-1}^{p+r-1,q+r-2}(A) \oplus A^{p,q-1}$ such that $a_0 = \sum_{i=0}^{r-1} (-1)^i d_i b_{r-i-1}$.

Let us compute

$$(a_0, \ldots, a_{r-1}) - w_r(b_0, \ldots, b_{r-2}; b_{r-1}; 0, \ldots, 0) = \\ \underbrace{(0, a_1 + \sum_{i=1}^{r} (-1)^i d_i b_{r-i}, a_2 - \sum_{i=2}^{r+1} (-1)^i d_i b_{r+1-i}, \ldots, a_{r-1} - (-1)^{r-1} \sum_{i=r-1}^{2r-2} (-1)^i d_i b_{2r-2-i})}_{=:(0, c_0, c_1, \ldots, c_{r-2})}.$$

A computation shows that $(c_0, c_1, \ldots, c_{r-2}) \in ZW_{r-1}^{p-1,q-1}(A)$ so that

$$\underline{a} = w_r(b_0, \ldots, b_{r-2}; b_{r-1}; c_0, \ldots, c_{r-2}) \in \operatorname{im} w_r.$$

Conversely we have

$$\pi_r \circ w_r = \pi \circ z_r \circ w_r = \pi \circ \iota_r \circ \beta_r = 0,$$

so that $\operatorname{im} w_r \subseteq \ker \pi_r$.

Another calculation shows that the claimed differential δ_r gives a well-defined map on $ZW_r(A)/w_r(BW_r(A))$. Indeed, if $\underline{a} = w_r(b_0, \ldots, b_{r-2}; b_{r-1}; c_0, \ldots, c_{r-2})$, then

$$\delta_r(\underline{a}) = w_r(c_0, \ldots, c_{r-2}; \beta_{r-1}; \gamma_0, \ldots, \gamma_{r-2}),$$

where $\beta_{r-1} = \sum_{l=0}^{r-1} (-1)^{r+l} d_{2r-1-l} b_l$ and $\gamma_j = \sum_{i=0}^{j} \sum_{k=1}^{r} (-1)^{k+1} d_i d_{r+j+k-i} b_{r-k}$, for $0 \leq j \leq r-2$.

It is straightforward to check that δ_r corresponds to the differential Δ_r under the isomorphism. □

Lemma 3.8. *Let* $A \in n\text{-mC}_R$*. For* $r \geq 1$*, the kernel of the map* $w_r \colon BW_r^{p,q-1}(A) \to ZW_r^{p,q}(A)$ *is isomorphic to* $ZW_r^{p+r-1,q+r-2}(A)$*, via the map* $(\underline{b}; a; \underline{c}) \mapsto (\underline{b}, a)$*.*

Proof. This is clear from the definition of w_r: the element $(\underline{b}; a; \underline{c})$ being in $\ker w_r$ means that $\underline{c}$ is completely determined in terms of $(\underline{b}, a)$ and that $d_0 a = -\sum_{i=1}^{r-1} (-1)^i d_i b_{r-1-i}$. Together with $\underline{b} \in ZW_{r-1}^{p+r-1,q+r-2}(A)$, this gives exactly that $(\underline{b}, a) \in ZW_r^{p+r-1,q+r-2}(A)$. □

The following result is straightforward.

Lemma 3.9. *The following commutative diagrams are pullback squares in the category of R-modules for every $r \geq 2$ and every n-multicomplex A.*

$$\begin{tikzcd} ZW_1^{p,q}(A) \arrow[r, hook] \arrow[d] & ZW_0^{p,q}(A) \arrow[d, "d_0"] \\ 0 \arrow[r] & ZW_0^{p,q+1}(A) \end{tikzcd} \qquad \begin{tikzcd} ZW_r^{p,q}(A) \arrow[r, "\pi_r"] \arrow[d, "\rho_r"'] & ZW_0^{p-r+1,q-r+1}(A) \arrow[d, "d_0"] \\ ZW_{r-1}^{p,q}(A) \arrow[r, "D_{r-1}"] & ZW_0^{p-r+1,q-r+2}(A) \end{tikzcd}$$

Here π_r is the projection onto the last coordinate, ρ_r is the projection onto the first $r-1$ components, and

$$D_{r-1} = \sum_{i=1}^{r-1} (-1)^{i+1} d_i \colon (a_0, \ldots, a_{r-2}) \longmapsto \sum_{i=1}^{r-1} (-1)^{i+1} d_i a_{r-1-i}. \quad \square$$

The maps π_r, ρ_r, d_0 and D_{r-1} define natural transformations between the functors $ZW_r^{p,q}$, and as a consequence we obtain the following proposition.

Proposition 3.10. *The following commutative diagrams are pullback squares in the functor category* $\mathrm{Fun}(n\text{-}\mathrm{mC}_R, R\text{-}\mathrm{Mod})$ *for every $r \geq 2$.*

$$\begin{tikzcd} ZW_1^{p,q} \arrow[r, hook] \arrow[d] & ZW_0^{p,q} \arrow[d, "d_0"] \\ 0 \arrow[r] & ZW_0^{p,q+1} \end{tikzcd} \qquad \begin{tikzcd} ZW_r^{p,q} \arrow[r, "\pi_r"] \arrow[d, "\rho_r"'] & ZW_0^{p-r+1,q-r+1} \arrow[d, "d_0"] \\ ZW_{r-1}^{p,q} \arrow[r, "D_{r-1}"] & ZW_0^{p-r+1,q-r+2} \end{tikzcd}$$

Proof. Since R-Mod is complete, limits in the functor category exist and they are computed objectwise, so the result follows directly from Lemma 3.9. $\square$

Remark 3.11. Similarly to [2, Remark 4.5], for $2 \leq n \leq \infty$, if $f \colon A \to B$ is a morphism of n-multicomplexes and $r \geq 1$, then the following are equivalent.

(1) The maps $ZW_r(f)$, $ZW_{r-1}(f)$ and f are surjective.
(2) The maps $E_r(f)$ and $ZW_{r-1}(f)$ and f are surjective.

3.3. Representing elements

We now describe suitable representing objects for the witness cycles and boundaries previously defined.

Definition 3.12. Let $2 \leq n \leq \infty$. The *n-disk at place (p, q)*, denoted $\mathbb{D}^n(p, q)$, is the n-multicomplex freely generated by a single element x in bidegree (p, q), in the sense of satisfying the following universal construction. For any n-multicomplex A, every map of bigraded sets $\{x\} \to A$ extends uniquely to an n-multicomplex morphism $\mathbb{D}^n(p, q) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} \{x\} \arrow[r, hook] \arrow[dr] & \mathbb{D}^n(p,q) \arrow[d] \\ & A \end{tikzcd}$$

By definition the n-multicomplex $\mathbb{D}^n(p,q)$ freely generated by x in bidegree (p,q) is the quotient of the free bigraded R-module generated by all finite words $d_{i_1}d_{i_2}\dots d_{i_k}(x)$, $k \geq 0$, $0 \leq i_1,\dots,i_k \leq n-1$, by the relations

$$\sum_{i+j=l}(-1)^i d_i d_j(x) = 0 \text{ for } l \geq 0,$$

with differential

$$d_i(d_{i_1}d_{i_2}\dots d_{i_k}(x)) = d_i d_{i_1}d_{i_2}\dots d_{i_k}(x).$$

Remark 3.13. Using the relations, one can rewrite any word $d_{i_1}d_{i_2}\dots d_{i_k}(x)$ by swapping any occurrence of d_0 with all the higher structure maps to its left. The rewriting process and the relation $d_0^2 = 0$ ensure that every word is a linear combination of words of the form

$$d_0^i d_{i_1}d_{i_2}\dots d_{i_k}(x), \text{ for } i \in \{0,1\},\ k \geq 0,\ 0 < i_1,\dots,i_k \leq n-1. \tag{3}$$

It is clear from this description that the d_0-homology of an n-disk is 0.

The words listed in (3) above form a basis for the ∞-disk; see [10, Definition 5.4] for an explicit description of the ∞-disk for multicomplexes concentrated in the right half-plane. For n finite, the words listed in (3) are not necessarily distinct or nonzero, so do not form a basis for the n-disk.

Example 3.14. The 3-multicomplex $\mathbb{D}^3(p,q)$ can be depicted as follows.

$$\begin{tikzcd}
\cdots & \bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[dll] & \bullet \arrow[l] \arrow[dll] & \bullet \arrow[l] \arrow[dll] \\
\cdots & \bullet\bullet \arrow[u] & \bullet\bullet \arrow[l] \arrow[u] & \bullet\bullet \arrow[l] \arrow[u] \arrow[dll] & \bullet \arrow[l] \arrow[u] \arrow[dll] & * \arrow[l] \arrow[u] \arrow[dll] \\
\cdots & \bullet \arrow[u] & \bullet \arrow[l] \arrow[u] & \bullet \arrow[l] \arrow[u] & &
\end{tikzcd}$$

Here each vertex marked $\bullet$ represents the ring R, each vertex marked $\bullet\bullet$ represents $R \oplus R$, the vertex marked $*$ represents R in bidegree (p,q) and the arrows are

$$\begin{aligned}
d_0 &: \quad \bullet \xrightarrow{1} \bullet, \quad \bullet \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} \bullet\bullet, \quad \bullet\bullet \xrightarrow{\left(\begin{smallmatrix}1&0\end{smallmatrix}\right)} \bullet, \\
d_1 &: \quad \bullet \xrightarrow{1} \bullet, \quad \bullet \xrightarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} \bullet\bullet, \quad \bullet\bullet \xrightarrow{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \bullet\bullet, \\
d_2 &: \quad \bullet \xrightarrow{1} \bullet, \quad \bullet \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \bullet\bullet, \quad \bullet\bullet \xrightarrow{\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)} \bullet.
\end{aligned}$$

Definition 3.15. Let $2 \leq n \leq \infty$. Define the n-multicomplex $\mathcal{ZW}_0^n(p,q) = \mathbb{D}^n(p,q)$, define $\mathcal{ZW}_1^n(p,q)$ to be the pushout

$$\begin{tikzcd}
\mathcal{ZW}_0^n(p,q+1) \arrow[r, "d_0^*"] \arrow[d] & \mathcal{ZW}_0^n(p,q) \arrow[d, "j_0"] \\
0 \arrow[r] & \mathcal{ZW}_1^n(p,q)
\end{tikzcd}$$

in the category of n-multicomplexes, and for $r \geq 2$, define $\mathcal{ZW}_r^n(p,q)$ recursively to be the pushout

$$\begin{tikzcd}
\mathcal{ZW}_0^n(p-r+1,q-r+2) \arrow[r, "d_0^*"] \arrow[d, "D_{r-1}^*"'] & \mathcal{ZW}_0^n(p-r+1,q-r+1) \arrow[d, "j_{r-1}"] \\
\mathcal{ZW}_{r-1}^n(p,q) \arrow[r, "i_{r-1}"] & \mathcal{ZW}_r^n(p,q)
\end{tikzcd}$$

in the category of n-multicomplexes. Here, for all $r \geq 1$, writing x and a_{r-1} for the generators of $\mathcal{ZW}_0^n(p-r+1,q-r+2)$ and $\mathcal{ZW}_0^n(p-r+1,q-r+1)$ respectively, the morphism d_0^* is

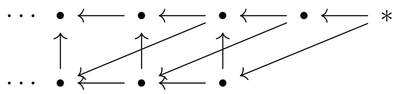
$$d_0^*(x) = d_0 a_{r-1}.$$

By abuse of notation, we also denote the element $j_0(a_0)$ in $\mathcal{ZW}_1^n(p,q)$ by a_0. For $r \geq 2$, we recursively define the morphism D_{r-1}^* to be

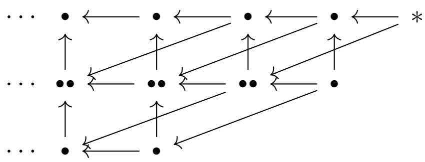
$$D_{r-1}^*(x) = \sum_{i=1}^{r-1}(-1)^{i+1} d_i a_{r-1-i},$$

and again by abuse of notation, we denote the elements $i_{r-1}(a_s)$ $(0 \leq s \leq r-2)$ and $j_{r-1}(a_{r-1})$ in $\mathcal{ZW}_r^n(p,q)$ by a_s $(0 \leq s \leq r-2)$ and a_{r-1} respectively.

Example 3.16. The 3-multicomplex $\mathcal{ZW}_1^3(p,q)$ can be depicted as:



The 3-multicomplex $\mathcal{ZW}_2^3(p,q)$ can be depicted as:



Definition 3.17. Define the n-multicomplexes

$$\mathcal{BW}_0^n(p,q-1) = 0, \qquad \mathcal{BW}_1^n(p,q-1) = \mathbb{D}^n(p,q-1)$$

and for $r \geq 2$, define the n-multicomplex $\mathcal{BW}_r^n(p,q-1)$ to be

$$\mathcal{BW}_r^n(p,q-1) = \mathcal{ZW}_{r-1}^n(p+r-1,q+r-2) \oplus \mathbb{D}^n(p,q-1) \oplus \mathcal{ZW}_{r-1}^n(p-1,q-1).$$

Lemma 3.18. *Let $r \geq 0$ and let $p,q \in \mathbb{Z}$ and $2 \leq n \leq \infty$.*

(1) *Giving a morphism of n-multicomplexes $\mathcal{ZW}_r^n(p,q) \to A$ is equivalent to giving an element in $ZW_r^{p,q}(A)$.*
(2) *Giving a morphism of n-multicomplexes $\mathcal{BW}_r^n(p,q) \to A$ is equivalent to giving an element in $BW_r^{p,q}(A)$.*

Furthermore, these statements are functorial, so that $\mathcal{ZW}_r^n(p,q)$, $\mathcal{BW}_r^n(p,q)$ are representing n-multicomplexes for the functors $ZW_r^{p,q}$, $BW_r^{p,q} \colon n\text{-mC}_R \to R\text{-Mod}$ respectively.

Proof. The case $r = 0$ in part (1) is immediate from the definition of $\mathcal{ZW}_0^n(p,q) = \mathbb{D}^n(p,q)$. For $r \geq 1$ we proceed inductively: assume $ZW_{r-1}^{p,q} = n\text{-mC}_R(\mathcal{ZW}_{r-1}^n(p,q), -)$ as functors $n\text{-mC}_R \to R\text{-Mod}$. It is an easy check that the n-multicomplex morphisms d_0^* and D_{r-1}^* correspond to the natural transformations d_0 and D_{r-1} in Proposition 3.10 under the Yoneda embedding $\mathcal{Y}\colon n\text{-mC}_R \to \text{Fun}(n\text{-mC}_R, R\text{-Mod})^{\text{op}}$. Furthermore, $\mathcal{Y}$ takes pushout squares in n-mC$_R$ to pullback squares in Fun(n-mC$_R$, R-Mod), hence $ZW_r^{p,q} = n\text{-mC}_R(\mathcal{ZW}_r^n(p,q), -)$ by Proposition 3.10. Part (2) is now immediate from the definition of $BW_r^{p,q}(A)$. $\square$

Lastly, for $r \geq 0$, define $\iota_r\colon \mathcal{ZW}_r^n(p,q) \to \mathcal{BW}_r^n(p,q-1)$ to be the n-multicomplex morphism corresponding to the natural transformation $w_r\colon BW_r^{p,q-1} \to ZW_r^{p,q}$ under the Yoneda embedding $\mathcal{Y}$. Under these correspondences, a commutative diagram of n-multicomplexes of the form

$$\begin{tikzcd}
\mathcal{ZW}_r^n(p,q) \arrow[r] \arrow[d, "\iota_r"'] & A \arrow[d, "f"] \\
\mathcal{BW}_r^n(p,q-1) \arrow[r] & B
\end{tikzcd}$$

corresponds to a pair (a,b), $a \in ZW_r^{p,q}(A)$, $b \in BW_r^{p,q-1}(B)$ such that $ZW_r(f)(a) = w_r(b)$.

The following two results will be useful for constructing our model category structures.

Lemma 3.19. *Let $2 \leq n \leq \infty$. For $r \geq 1$ the n-multicomplex $\mathcal{ZW}_0^n(p,q-1)$ is a retract of $\mathcal{BW}_r^n(p,q-1)$ and for $r \geq 2$ the n-multicomplex $\mathcal{ZW}_{r-1}^n(p-1,q-1)$ is a retract of $\mathcal{BW}_r^n(p,q-1)$.*

Proof. Immediate from the definition of $\mathcal{BW}_r^{p,q}$. $\square$

Lemma 3.20. *Let $2 \leq n \leq \infty$. For $r \geq 1$, the diagram*

$$\begin{tikzcd}
\mathcal{ZW}_r^n(p,q) \arrow[r] \arrow[d, "\iota_r"'] & 0 \arrow[d] \\
\mathcal{BW}_r^n(p,q-1) \arrow[r] & \mathcal{ZW}_r^n(p+r-1,q+r-2)
\end{tikzcd}$$

is a pushout diagram in n-multicomplexes.

Proof. By Lemma 3.8, the following diagram is a pullback square in the functor category Fun(n-mC$_R$, R-Mod) for $r \geq 1$.

$$\begin{tikzcd}
ZW_r^{p+r-1,q+r-2} \arrow[r] \arrow[d] & BW_r^{p,q-1} \arrow[d, "w_r"] \\
0 \arrow[r] & ZW_r^{p,q}
\end{tikzcd}$$

The result now follows by Yoneda's lemma. $\square$

3.4. Model category structures

In this section, we present the model structures on n-multicomplexes. We are able to exploit the r-cone defined for the case of bicomplexes.

We denote by C_r the bicomplex $\mathcal{ZW}_r^2(0,0)$. We recall from [2] that for $r = 0$ it is depicted as a square, and for $r \geq 1$, it is depicted as a staircase graph with r horizontal steps as follows, where each vertex marked $\bullet$ represents R, each arrow represents the identity map and the vertex marked $*$ represents R in bidegree $(0,0)$.

$$\begin{tikzcd}
 & & & \bullet & \ast \arrow[l] \\
 & & \bullet & \bullet \arrow[l] \arrow[u] & \\
 & & \bullet \arrow[u] & & \\
 \bullet & \bullet \arrow[l] \arrow[ur, dotted, no head] & & &
\end{tikzcd}$$

We may write $C_r = \bigoplus_{k=0}^{r-1} R\beta_{-k,-k} \oplus \bigoplus_{k=0}^{r-1} R\beta_{-k-1,-k}$ with the differentials d_0, d_1 indicated by the graph, and $\beta_{i,j}$ a generator of bidegree (i,j). Since C_r is a bicomplex, we may also view it as an n-multicomplex, for $2 \leq n \leq \infty$. Then for any n-multicomplex A, with $2 \leq n \leq \infty$, we have that $C_r \otimes A$ is an n-multicomplex, using the symmetric monoidal structure on n-mC_R.

Proposition 3.21. *Let $2 \leq n \leq \infty$. Let A be an n-multicomplex and $r \geq 0$. Then $E_{r+1}(C_r \otimes A) = 0$.*

Proof. Proposition 4.29 of [2] proves that C_r is r-contractible in the sense that the identity map of C_r is r-homotopic (in the category of bicomplexes but also in that of multicomplexes) to 0. As a corollary, for any multicomplex (and thus for any n-multicomplex A), $C_r \otimes A$ is r-contractible, hence by [3, Proposition 3.24], $E_{r+1}(C_r \otimes A) = 0$. □

Proposition 3.22. *Let $p, q \in \mathbb{Z}$ and $2 \leq n \leq \infty$. Let A be an n-multicomplex and $r \geq 0$. The projection morphism $\phi_r : C_r \otimes A \to A$ has the property that $ZW_k^{p,q}(\phi_r)$ is surjective for $0 \leq k \leq r$.*

Proof. The case $r = 0$ is trivial. Let us assume $r \geq 1$. Let $(a_0, a_1, \dots, a_{r-1})$ be an element of $ZW_r^{p,q}(A)$, with $a_i \in A^{p-i,q-i}$. We have

$$\sum_{i+j=l} (-1)^i d_i a_j = 0 \text{ for } 0 \leq l \leq r-1.$$

For $0 \leq k \leq r-1$, we define the element

$$X_k = \sum_{i=0}^{k} \beta_{-i,-i} \otimes a_{k-i} \in (C_r \otimes A)^{p-k,q-k}.$$

Let us prove that $(X_0, \dots, X_{r-1})$ is an element of $ZW_r(C_r \otimes A)$. Fix $0 \leq l \leq r-1$ and compute

$$\begin{aligned}
\sum_{i=0}^{l} (-1)^i d_i X_{l-i} &= \sum_{i=0}^{l} (-1)^i d_i \Big(\sum_{j=0}^{l-i} \beta_{-j,-j} \otimes a_{l-i-j} \Big) \\
&= \sum_{j=0}^{l} (d_0 \beta_{-j,-j}) \otimes a_{l-j} - \sum_{j=0}^{l-1} (d_1 \beta_{-j,-j}) \otimes a_{l-1-j} \\
&\quad + \sum_{j=0}^{l} (-1)^j \beta_{-j,-j} \otimes \Big(\sum_{i=0}^{l-j} (-1)^i d_i a_{l-i-j} \Big) \\
&= \sum_{j=1}^{l} \beta_{-j,-j+1} \otimes a_{l-j} - \sum_{j=0}^{l-1} \beta_{-j-1,-j} \otimes a_{l-1-j} = 0.
\end{aligned}$$

Hence, the induced map $ZW_r(\phi_r)$ on $ZW_r(C_r \otimes A)$ satisfies

$$ZW_r(\phi_r)(X_0, \dots, X_{r-1}) = (a_0, \dots, a_{r-1}).$$

Note that since $(X_0, \dots, X_k) \in ZW_k(C_r \otimes A)$ is defined from the data $(a_0, \dots, a_k)$, the same proof applies to $ZW_k(\phi_r)$, for $0 \le k \le r$. $\square$

Remark 3.23. Let C_r^∞ be the multicomplex $Re_{0,0} \oplus Re_{-r,1-r}$ with only non trivial differential $d_r(e_{0,0}) = e_{-r,1-r}$. We have that C_r^∞ is an r-contractible multicomplex, with $h_0(e_{-r,1-r}) = e_{0,0}$ satisfying $d_r h_0 + h_0 d_r = 1_{C_r^\infty}$. Hence, for any multicomplex Y, $C_r^\infty \otimes Y$ is r-contractible. In addition the projection $\pi\colon C_r^\infty \otimes Y \to Y$ induced by the projection of C_r^∞ onto $Re_{0,0}$ satisfies $ZW_s(\pi)$ is surjective for all $0 \le s \le r$: it is easy to see that if $(a_0, \dots, a_{s-1}) \in ZW_s(Y)$ then $(e_{0,0} \otimes a_0, \dots, e_{0,0} \otimes a_{s-1}) \in ZW_s(C_r^\infty \otimes Y)$.

Definition 3.24. Let $2 \le n \le \infty$. For $r \ge 0$, consider the sets of morphisms of n-multicomplexes

$$I_r^n = \left\{ \mathcal{ZW}_{r+1}^n(p,q) \xrightarrow{\iota_{r+1}} \mathcal{BW}_{r+1}^n(p,q-1) \right\}_{p,q\in\mathbb{Z}} \text{ and } J_r^n = \left\{ 0 \longrightarrow \mathcal{ZW}_r^n(p,q) \right\}_{p,q\in\mathbb{Z}}.$$

Proposition 3.25. *For each $r \ge 0$, a map f is J_r^n-injective if and only if $ZW_r(f)$ is surjective.*

Proof. This follows from (1) of Lemma 3.18. $\square$

Proposition 3.26. *For all $r \ge 0$ and $2 \le n \le \infty$, we have I_r^n-inj $= \mathcal{E}_r^n \cap J_0^n$-inj $\cap J_r^n$-inj.*

Proof. The proof proceeds exactly like that of [2, Proposition 4.35], the corresponding result in the bicomplex case $n = 2$, using Lemmas 3.8, 3.18, 3.19, 3.20 and Remark 3.11. $\square$

Proposition 3.27. *For all $r \ge 0$, $2 \le n \le \infty$ and all $0 \le k \le r$ we have J_k^n-cof $\subseteq \mathcal{E}_r^n$.*

Proof. Let $r \ge 0$ and $0 \le k \le r$ and $f\colon X \to Y \in J_k^n$-cof. Consider the following diagram.

$$\begin{tikzcd} X \arrow[r, "\begin{pmatrix} 1_X \\ 0 \end{pmatrix}"] \arrow[d, "f"'] & X \oplus (C_r \otimes Y) \arrow[d, "(f\ \phi_r)"] \\ Y \arrow[r, "="] & Y \end{tikzcd}$$

From Propositions 3.22 and 3.25 the right-hand vertical map is J_k^n-injective so there is a lift in the diagram. From Proposition 3.21 one has $E_{r+1}(C_r \otimes Y) = 0$. Applying the functor E_{r+1} to the diagram, we see that $E_{r+1}(f)$ is an isomorphism. Note that in the case $n = \infty$ the proof also holds using C_r^∞ (instead of C_r) and Remark 3.23. $\square$

Theorem 3.28. *For every $r \ge 0$ and $2 \le n \le \infty$, the category n-mC_R admits a right proper cofibrantly generated model structure, where:*

(1) *weak equivalences are E_r-quasi-isomorphisms,*
(2) *fibrations are morphisms of n-multicomplexes $f\colon A \to B$ such that f and $ZW_r(f)$ are bidegree-wise surjective, and*
(3) *I_r^n and $J_0^n \cup J_r^n$ are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Proof. The proof is standard (see, for example, the proof of [2, Theorem 3.14]) and uses Proposition 3.25, Proposition 3.27 and Proposition 3.26. $\square$

As in the bicomplex case, in certain situations it may be easier to characterize fibrations if they are described in terms of surjectivity of E_i instead of ZW_r.

Definition 3.29. Let $(I_r^n)'$ and $(J_r^n)'$ be the sets of morphisms of n-mC$_R$ given by

$$(I_r^n)' := \cup_{k=1}^{r-1} J_k^n \cup I_r^n \text{ and } (J_r^n)' := \cup_{k=0}^{r} J_k^n.$$

The proof of the following result is analogous to that for bicomplexes [2, Theorem 4.39].

Theorem 3.30. *For every $r \geq 0$ and $2 \leq n \leq \infty$, the category n-mC$_R$ admits a right proper cofibrantly generated model structure, denoted $(n\text{-mC}_R)_r$, where:*

(1) *weak equivalences are E_r-quasi-isomorphisms,*
(2) *fibrations are morphisms of n-multicomplexes $f : A \to B$ such that $E_i(f)$ is bidegree-wise surjective for every $0 \leq i \leq r$, and*
(3) *$(I_r^n)'$ and $(J_r^n)'$ are the sets of generating cofibrations and generating trivial cofibrations respectively.* □

Definition 3.31. We refer to the model structure $(n\text{-mC}_R)_r$ of Theorem 3.30 as the *r-model structure*. The terms *r-fibrant*, *r-cofibrant* and *r-trivial* all refer to the corresponding notions in this model structure.

Remark 3.32. Note that the generating (trivial) cofibrations of the model structure of Theorem 3.28 form a subclass of the generating (trivial) cofibrations of the model structure of Theorem 3.30. Moreover, these two model structures have the same weak equivalences. Thus, for each n with $2 \leq n \leq \infty$ and each $r \geq 0$, the identity functors give a Quillen equivalence between $(n\text{-mC}_R)_r$ and n-mC$_R$ with the model structure of Theorem 3.28.

4. Relationships between model category structures

In order to compare our model structures on n-multicomplexes as n varies, in this section we reinterpret n-multicomplexes as modules over a graded associative algebra in the category of vertical bicomplexes.

4.1. Monoids in vertical bicomplexes

Recall from Section 2 that the category 1-mC$_R$ = vbC$_R$ has as objects vertical bicomplexes, and that it is a symmetric monoidal category. A monoid (M, δ_0) in this category is a vertical bicomplex endowed with a unital and associative multiplication $M \otimes M \to M$ compatible with the differential δ_0. In other words, it is a unital bigraded (associative, not necessarily commutative) R-algebra, endowed with a derivation of algebras δ_0 of bidegree (0,1) such that $\delta_0^2 = 0$. For simplicity, we call such an object a *dg algebra*. This is only a slight abuse of terminology – this differs from the usual notion just by having an extra grading.

Consider $R\langle d_1, d_2, \ldots, d_i, \ldots\rangle$, the free bigraded associative R-algebra generated by the bigraded set $\{d_i, i \geq 1\}$, with d_i of bidegree $(-i, 1-i)$.

For $k \geq 1$, we consider the following element of $R\langle d_1, d_2, \ldots, d_i, \ldots\rangle$:

$$S_k = \sum_{\substack{i+j=k \\ i,j \geq 1}} (-1)^{i+1} d_i d_j,$$

in bidegree $(-k, 2-k)$, with the convention that $S_1 = 0$.

Since $R\langle d_1, d_2, \ldots, d_i, \ldots\rangle$ is a free associative algebra, a derivation δ_0 on $R\langle d_1, d_2, \ldots, d_i, \ldots\rangle$ is determined by its values on the generators d_i. Set, for $i \geq 1$,

$$\delta_0(d_i) = S_i.$$

Proving that $\delta_0^2 = 0$ amounts to proving that $\delta_0(S_i) = 0$, which is standard.

For $n \geq 1$, let I_n be the two sided ideal of $R\langle d_1, d_2, \ldots, d_i, \ldots\rangle$ generated by the elements S_k and d_k for $k \geq n$. The definition of δ_0 shows that this ideal is compatible with the differential. By convention $I_\infty = \{0\}$.

Definition 4.1. Let $\mathcal{C}_\infty$ be the dg algebra $R\langle d_1, d_2, \ldots, d_i, \ldots\rangle$ endowed with differential δ_0 as above. And for $n \geq 1$, let $\mathcal{C}_n$ be the dg algebra

$$\mathcal{C}_n = (\mathcal{C}_\infty / I_n, \delta_0).$$

For $1 \leq n \leq l \leq \infty$, we have $I_l \subset I_n$ and thus a surjective morphism of dg algebras

$$\Phi_{l,n} \colon \mathcal{C}_l \to \mathcal{C}_n.$$

Proposition 4.2. *If $2 \leq n \leq l \leq \infty$, then $\Phi_{l,n} \colon \mathcal{C}_l \to \mathcal{C}_n$ is a quasi-isomorphism of vertical bicomplexes.*

Proof. For $2 \leq m \leq n \leq l \leq \infty$ we have $\Phi_{l,m} = \Phi_{n,m} \circ \Phi_{l,n}$. By the two-out-of-three property of quasi-isomorphisms, it is enough to prove that the maps $\Phi_{\infty,n} \colon \mathcal{C}_\infty \to \mathcal{C}_n$ are quasi-isomorphisms for all $2 \leq n < \infty$.

For $n = 2$, we have $\mathcal{C}_2 = R\langle d_1\rangle/(d_1^2)$ with $\delta_0(d_1) = 0$. Hence it is enough to prove that the induced map on the homology with respect to δ_0, $H^{*,*}(\Phi_{\infty,2}) \colon H^{*,*}(\mathcal{C}_\infty) \to H^{*,*}(\mathcal{C}_2)$ is an isomorphism. In order to do so we build a homotopy $h \colon \mathcal{C}_\infty \to \mathcal{C}_\infty$. Any element in $\mathcal{C}_\infty$ is a linear combination with coefficients in R of words of the form $d_{i_1} \ldots d_{i_k}$ with $i_j \geq 1$. The empty word corresponds to 1_R and we define $h(1_R) = 0$. Let h be the R-linear map determined by

$$h(d_{i_1} \ldots d_{i_k}) = \begin{cases} 0, & \text{if } k = 1 \text{ or } i_1 > 1, \\ d_{i_2+1} \ldots d_{i_k}, & \text{if } k > 1 \text{ and } i_1 = 1. \end{cases}$$

Note that for any word w we have $h(S_i w) = d_i w$ for $i \geq 2$. Let us compute:

$$\begin{aligned} &(\delta_0 h + h\delta_0)(1_R) = 0, \quad (\delta_0 h + h\delta_0)(d_1) = 0, \\ &(\delta_0 h + h\delta_0)(d_i) = h(S_i) = d_i, \qquad \text{for } i \geq 2. \end{aligned}$$

For $k \geq 2$,

$$\begin{aligned} &(\delta_0 h + h\delta_0)(d_1 d_{i_2} \ldots d_{i_k}) \\ &\quad = \delta_0(d_{i_2+1} d_{i_3} \ldots d_{i_k}) + \sum_{j=2}^{k} (-1)^{i_2+1+\ldots+i_{j-1}+1} h(d_1 d_{i_2} \ldots \delta_0(d_{i_j}) \ldots d_{i_k}) \\ &\quad = \sum_{j=3}^{k} (-1)^{i_2+2+\ldots+i_{j-1}+1} d_{i_2+1} \ldots S_{i_j} \ldots d_{i_k} + \sum_{j=3}^{k} (-1)^{i_2+1+\ldots+i_{j-1}+1} d_{i_2+1} \ldots S_{i_j} \ldots d_{i_k} \\ &\qquad + S_{i_2+1} d_{i_3} \ldots d_{i_k} + \sum_{\substack{u+v=i_2 \\ u,v\geq 1}} (-1)^{u+1} d_{u+1} d_v d_{i_3} \ldots d_{i_k} \\ &\quad = d_1 d_{i_2} d_{i_3} \ldots d_{i_k}, \end{aligned}$$

and for $i_1 > 1$

$$(\delta_0 h + h\delta_0)(d_{i_1} d_{i_2} \dots d_{i_k}) = h(S_{i_1} d_{i_2} \dots d_{i_k}) = d_{i_1} d_{i_2} \dots d_{i_k}.$$

So $H^{p,q}(\mathcal{C}_\infty) = 0$ for every $(p, q) \notin \{(0, 0), (-1, 0)\}$. In addition

$$(\mathcal{C}_\infty)^{0,0} = R,\ \mathcal{C}_\infty^{-1,0} = Rd_1,\ \mathcal{C}_\infty^{0,-1} = \mathcal{C}_\infty^{-1,-1} = 0,$$

hence $\Phi_{\infty,2}$ is a quasi-isomorphism.

Let us prove that for any $n \geq 3$ we have $h(I_n) \subseteq I_n$. Any element in I_n is a sum of elements of the form abc, with $a, c \in \mathcal{C}_\infty$ and $b = S_k$ or $b = d_k$ for some $k \geq n$. If $a \neq 1_R$ and $a \neq d_1$ then $h(abc) = h(a)bc \in I_n$. Assume $a = 1_R$ and let $k \geq n \geq 3$.

- If $b = d_k$ then $h(d_k c) = 0$.
- If $b = S_k$ then $h(S_k c) = d_k c \in I_n$.

Assume $a = d_1$.

- If $b = d_k$ then $h(d_1 d_k c) = d_{k+1} c \in I_n$, since $k + 1 \geq n + 1 \geq n$.
- If $b = S_k$ then

$$h(d_1 S_k c) = \sum_{\substack{u+v=k \\ u,v\geq 1}} (-1)^{u+1} h(d_1 d_u d_v c) = \sum_{\substack{u+v=k \\ u,v\geq 1}} (-1)^{u+1} d_{u+1} d_v c = -S_{k+1} c + d_1 d_{k+1} c \in I_n.$$

The quotient map $\Phi_{\infty,n} \colon \mathcal{C}_\infty \to \mathcal{C}_n$ has kernel I_n and $h \colon I_n \to I_n$ is a homotopy from the identity of I_n to 0. Hence I_n is contractible and the morphism is a quasi-isomorphism. □

Remark 4.3. The morphism $\Phi_{\infty,2} \colon \mathcal{C}_\infty \to \mathcal{C}_2$ corresponds to the Koszul resolution of the operad of dual numbers $\mathcal{C}_2$ (see for example [9, 10.3.16]), hence it is a quasi-isomorphism. The proof given here via the homotopy h is not a consequence of this result, however, and this method has been chosen because it allows us to treat the case of general n.

4.2. Quillen equivalences

Proposition 4.4. *For $1 \leq n \leq \infty$, the category of $\mathcal{C}_n$-modules in vertical bicomplexes is isomorphic to the category of n-multicomplexes.*

Proof. In the category of vertical bicomplexes a (left) $\mathcal{C}_\infty$-module is a bigraded R-module M endowed with a differential d_0^M of bidegree $(0, 1)$ together with an action $\lambda \colon \mathcal{C}_\infty \otimes M \to M$ compatible with the differentials δ_0 and d_0^M. Since $\mathcal{C}_\infty$ is free as a bigraded R-algebra the action is determined by its values on $d_i, i \geq 1$. We denote by $d_i^M \colon M \to M$ the map that associates $\lambda(d_i \otimes m)$ to m. The compatibility with the differentials gives that

$$d_0^M d_n^M = \sum_{i+j=n, i,j\geq 1} (-1)^{i+1} d_i^M d_j^M + (-1)^{1-n} d_n^M d_0^M,$$

that is, M is a multicomplex. In addition morphisms of $\mathcal{C}_\infty$-modules are morphisms of multicomplexes. This completes the proof for $n = \infty$. A (left) $\mathcal{C}_n$-module is a (left) $\mathcal{C}_\infty$-module M such that $d_i^M = 0$ for all $i \geq n$, hence an n-multicomplex. □

As a corollary, the dg algebra morphisms $\Phi_{l,n} \colon \mathcal{C}_l \to \mathcal{C}_n$, for $1 \leq n \leq l \leq \infty$ induce pairs of adjoint functors

$$n\text{-mC}_R = \mathcal{C}_n\text{-Mod} \underset{i_{l,n}}{\overset{p_{l,n}}{\leftrightarrows}} \mathcal{C}_l\text{-Mod} = l\text{-mC}_R$$

where the right adjoint $i_{l,n}$ is the restriction of scalars functor and the left adjoint $p_{l,n}(M) = \mathcal{C}_n \otimes_{\mathcal{C}_l} M$ is the extension of scalars functor. Note that if M is an n-multicomplex, then $i_{l,n}(M)$ is the l-multicomplex M with $d_n = \ldots = d_{l-1} = 0$.

Recall that we write $(n\text{-mC}_R)_r$ for the category of n-multicomplexes with the r-model structure of Theorem 3.30.

Theorem 4.5. *For $2 \leq n \leq l \leq \infty$ and $r \geq 0$ the adjunction*

$$(n\text{-mC}_R)_r \underset{i_{l,n}}{\overset{p_{l,n}}{\leftrightarrows}} (l\text{-mC}_R)_r$$

is a Quillen equivalence.

Proof. It is a Quillen adjunction from Theorem 3.30, for the right adjoint preserves fibrations and trivial fibrations. Note that the right adjoint reflects weak equivalences and that all objects are fibrant. Hence to establish a Quillen equivalence it is enough to prove that for any r-cofibrant object M in l-mC$_R$, the unit of the adjunction $M \to i_{l,n}p_{l,n}M$ is an E_r-quasi-isomorphism (see [6, Corollary 1.3.16]).

Recall that any r-cofibrant object is 0-cofibrant. Thus, if the unit of the adjunction is an E_0-quasi-isomorphism for any 0-cofibrant object, then it is an E_r-quasi-isomorphism for any r-cofibrant object, and it is enough to treat the case $r = 0$.

Let us prove that the adjunction is a Quillen equivalence for $r = 0$. The model category structure $(n\text{-mC}_R)_0$ corresponds to the transferred model category structure along the adjunction

$$n\text{-mC}_R = \mathcal{C}_n\text{-Mod} \underset{U_n}{\overset{\mathcal{C}_n\otimes -}{\leftrightarrows}} \text{vbC}_R,$$

where the right adjoint U_n is the forgetful functor and the model category structure on vbC$_R$ coincides with the projective model structure on $\mathbb{Z}$-graded cochain complexes, that is, weak equivalences are quasi-isomorphisms with respect to the bidegree $(0, 1)$ differential d_0, fibrations are bidegreewise surjective morphisms. A standard result (see [4, Proposition 11.2.10]) states that a morphism of dg algebras $\alpha\colon R \to S$ induces a Quillen adjunction between the categories of R-modules and S-modules (with the transferred model structure from vbC$_R$ as seen above) through the restriction and extension of scalars functors, and this is a Quillen equivalence if (and only if) α is a quasi-isomorphism. Hence, Proposition 4.2 implies that the Quillen adjunction

$$(n\text{-mC}_R)_0 \underset{i_{l,n}}{\overset{p_{l,n}}{\leftrightarrows}} (l\text{-mC}_R)_0$$

is a Quillen equivalence, that is, $M \to i_{l,n}p_{l,n}M$ is an E_0-quasi-isomorphism for every 0-cofibrant object M in l-mC$_R$. □

Remark 4.6. In the previous proof the model category structure considered in vbC$_R$ is precisely $(1\text{-mC}_R)_0$. The adjunction

$$(1\text{-mC}_R)_0 \underset{i_{2,1}}{\overset{p_{2,1}}{\leftrightarrows}} (2\text{-mC}_R)_0$$

is a Quillen adjunction, however it is not a Quillen equivalence. Indeed, $\mathcal{ZW}_1^2(0, 0)$ is 0-cofibrant in 2-mC$_R$ and the unit of the adjunction for this object is the projection onto the $(0, 0)$-coordinate

$$\mathcal{ZW}_1^2(0,0) \to R^{0,0}$$

which is not an E_0-quasi-isomorphism.

5. Model structures on bounded multicomplexes

In this section, we will apply the transfer theorem to give model structures on certain categories of bounded n-multicomplexes. We obtain such transferred model structures on $(-\mathbb{N}, \mathbb{Z})$-graded n-multicomplexes for all $r \geq 0$ and on $(\mathbb{Z}, \mathbb{N})$-graded multicomplexes for $r = 0$. Our exposition of the transfer principle follows [1, Sections 2.5–2.6].

Theorem. *Let $\mathcal{M}$ be a model category cofibrantly generated by the sets I and J of generating cofibrations and generating trivial cofibrations respectively. Let $\mathcal{C}$ be a category with finite limits and small colimits. Let*

$$\mathcal{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{C}$$

be a pair of adjoint functors. Define a map f in $\mathcal{C}$ to be a weak equivalence (respectively fibration) if $R(f)$ is a weak equivalence (respectively fibration). These two classes determine a model category structure on $\mathcal{C}$ cofibrantly generated by $L(I)$ and $L(J)$ provided that:

(1) *The sets $L(I)$ and $L(J)$ permit the small object argument.*
(2) *$\mathcal{C}$ has a functorial fibrant replacement and a functorial path object for fibrant objects.*

Furthermore, with this model structure on $\mathcal{C}$, the adjunction $L \dashv R$ becomes a Quillen adjunction.

Recall that a path object for X is a factorisation of the diagonal map $X \longrightarrow X \times X$ into a weak equivalence followed by a fibration $X \xrightarrow{\sim} P(X) \twoheadrightarrow X \times X$. To apply the transfer theorem, we first need to show the existence of r-path objects for n-multicomplexes. For this, we adapt [3, Section 5] to our context.

5.1. Path objects for n-multicomplexes

As with the r-cone, we start with constructions for bicomplexes and then extend to n-multicomplexes using the tensor product.

Definition 5.1 *([2]).* For $r = 0$, we define the 0-*path* Λ_0 as the bicomplex

$$\begin{tikzcd} R^{0,1} \\ (R \oplus R)^{0,0}. \arrow[u, "{\left(\begin{smallmatrix} -1 & 1 \end{smallmatrix}\right)}"'] \end{tikzcd}$$

For $r \geq 1$, define the r-*path* Λ_r as the bicomplex whose underlying bigraded module is $R^{0,0} \oplus \mathcal{ZW}_r^2(0,0)$ and whose differentials coincide with those of $\mathcal{ZW}_r^2(0,0)$ except for $d_1^{0,0}$ which is:

$$\mathcal{ZW}_r^2(0,0)^{-1,0} = R^{-1,0} \xleftarrow{\left(\begin{smallmatrix} -1 & 1 \end{smallmatrix}\right)} (R \oplus \mathcal{ZW}_r^2(0,0))^{0,0} = (R \oplus R)^{0,0} \ .$$

Example 5.2. The 1-path Λ_1 is the bicomplex given by

$$R^{-1,0} \xleftarrow{\left(\begin{smallmatrix} -1 & 1 \end{smallmatrix}\right)} (R \oplus R)^{0,0}.$$

The 2-path Λ_2 is given by

$$\begin{tikzcd} & R^{-1,0} & (R\oplus R)^{0,0} \arrow[l, "(-1\ 1)"'] \\ R^{-2,-1} & R^{-1,-1} \arrow[l, "1"'] \arrow[u, "1"'] & \end{tikzcd}$$

More generally, we write

$$\Lambda_r = R\beta_- \oplus \bigoplus_{i=0}^{r-1} R\beta_{-i,-i} \bigoplus_{i=0}^{r-1} R\beta_{-i-1,-i}$$

where $\beta_{u,v}$ has bidegree (u, v) and β_- has bidegree $(0, 0)$, with nonzero differentials given by

$$d_1(\beta_{0,0}) = -d_1(\beta_-) = \beta_{-1,0},\ d_0(\beta_{-i,-i}) = \beta_{-i,1-i},\ d_1(\beta_{-i,-i}) = \beta_{-i-1,-i},$$

for $1 \leq i \leq r-1$.

Lemma 5.3. *For $r \geq 1$, there is an isomorphism of bicomplexes $\varphi_r \colon \Lambda_r \to R^{0,0} \oplus C_r$ where $C_r = \mathcal{ZW}_r^2(0,0)$ has been defined in Section 3.4.*

Proof. Let us keep the notation $\beta_{u,v}$ for both the generators of Λ_r and C_r and let e be a generator of $R^{0,0}$. The map of bigraded modules $\varphi_r \colon \Lambda_r \to R^{0,0} \oplus C_r$ which associates $e - \beta_{0,0}$ to β_- and $\beta_{u,v}$ to $\beta_{u,v}$ for $(u, v) \in \{(-i, -i), (-i-1, -i), 0 \leq i \leq r-1\}$ is an isomorphism of bicomplexes since $\varphi_r d_0(\beta_-) = 0 = d_0(e - \beta_{0,0})$ and $\varphi_r d_1(\beta_-) = -\beta_{-1,0} = d_1(e - \beta_{0,0})$. □

Let us consider the following morphisms of bicomplexes

$$R^{0,0} \xrightarrow{\ \iota\ } \Lambda_r \xrightarrow{\pi=\partial_-+\partial_+} (R\oplus R)^{0,0}$$

where ι sends e to $\beta_- + \beta_{0,0}$ and ∂_- is the projection onto $R\beta_-$ and ∂_+ is the projection onto $R\beta_{0,0}$.

Proposition 5.4. *For $r \geq 0$,*

$$\iota \colon R^{0,0} \to \Lambda_r$$

is an r-homotopy equivalence.

Proof. If $r \geq 1$, since an isomorphism is an r-homotopy equivalence, it is enough to prove that the composite $\varphi_r\iota = 1_R \oplus 0 \colon R^{0,0} \oplus 0 \to R^{0,0} \oplus C_r$ is a r-homotopy equivalence, which is a direct consequence of the r-contractibility of C_r proven in Proposition 4.29 of [2]. Similarly, if $r = 0$, then the bicomplex

$$\begin{tikzcd} R^{0,1} \\ R^{0,0} \arrow[u, "1"'] \end{tikzcd}$$

is 0-contractible and the proof follows. □

Definition 5.5. For A an n-multicomplex, the *r-path object* $P_r(A)$ is the n-multicomplex $\Lambda_r \otimes A$. We denote by ι_A and π_A the maps $\iota \otimes 1_A$ and $\pi \otimes 1_A$ so that the diagonal of A factors as

$$A \xrightarrow{\iota_A} P_r(A) \xrightarrow{\pi_A} A \oplus A \ .$$

This construction is functorial, with $P_r(f) = 1_{\Lambda_r} \otimes f\colon P_r(A) \to P_r(B)$, for $f\colon A \to B$ a morphism of n-multicomplexes.

Remark 5.6. As a bigraded module we have

$$\begin{aligned} P_0(A)^{p,q} =& A^{p,q} \oplus A^{p,q-1} \oplus A^{p,q} \\ P_r(A)^{p,q} =& A^{p,q} \oplus \bigoplus_{i=0}^{r-1} A^{p+i,q+i} \oplus \bigoplus_{i=0}^{r-1} A^{p+i+1,q+i}, \quad \text{for } r \geq 1. \end{aligned}$$

Proposition 5.7. *Let A be an n-multicomplex and $r \geq 0$. The path object $P_r(A)$ is an r-path object for A. Indeed, the map $\iota_A\colon A \longrightarrow P_r(A)$ is an r-homotopy equivalence, hence an E_r-quasi-isomorphism and the map $\pi_A\colon P_r(A) \to A \oplus A$ is an r-fibration in the model structure of Theorem 3.30.*

Proof. That ι_A is an r-homotopy equivalence is a direct consequence of Proposition 5.4. For the second assertion, the case $r = 0$ is trivial and for $r \geq 1$, we consider the following commutative diagram of n-multicomplexes

$$\begin{tikzcd} P_r(A) \arrow[r, "\pi_A"] \arrow[d, "\varphi_r \otimes 1_A"'] & A \oplus A \arrow[d, "\begin{pmatrix} 1_A & 0 \\ -1_A & 1_A \end{pmatrix}"] \\ A \oplus (C_r \otimes A) \arrow[r, "1_A \oplus \phi_r"] & A \oplus A \end{tikzcd}$$

The vertical maps are isomorphisms, hence π_A is an r-fibration if and only if $1_A \oplus \phi_r$ is an r-fibration, which is so by Proposition 3.22 together with Remark 3.11. □

Remark 5.8. A path object for n-multicomplexes when $n = \infty$ is given in [3, Section 3.4].

5.2. Model structures on bounded n-multicomplexes

For $2 \leq n \leq \infty$, recall that n-mC_R denotes the category of $(\mathbb{Z}, \mathbb{Z})$-graded n-multicomplexes of R-modules. The categories of $(-\mathbb{N}, \mathbb{Z})$-graded (left half-plane) and $(\mathbb{Z}, \mathbb{N})$-graded (upper half-plane) n-multicomplexes of R-modules will be denoted by n-$\mathrm{mC}_{\text{-}\mathbb{N},\mathbb{Z}}$ and n-$\mathrm{mC}_{\mathbb{Z},\mathbb{N}}$, respectively.

By Proposition 4.4, the category of n-multicomplexes is isomorphic to the category of $\mathcal{C}_n$-modules in vertical bicomplexes, previously denoted $\mathcal{C}_n$-Mod. In this section, we will write $(\mathcal{C}_n\text{-Mod})_{\mathbb{Z},\mathbb{Z}}$ when we want to emphasize the $(\mathbb{Z}, \mathbb{Z})$-grading.

Similarly, the category n-$\mathrm{mC}_{\text{-}\mathbb{N},\mathbb{Z}}$ is isomorphic to the category of $\mathcal{C}_n$-modules M in vertical bicomplexes concentrated in bidegrees lying in the left half-plane (i.e., with $M^{p,q} = 0$ if $p > 0$), where the latter is denoted by $(\mathcal{C}_n\text{-Mod})_{\text{-}\mathbb{N},\mathbb{Z}}$.

We show that the inclusion functor from n-$\mathrm{mC}_{\text{-}\mathbb{N},\mathbb{Z}}$ to n-mC_R has a left adjoint by showing that the corresponding inclusion functor from $(\mathcal{C}_n\text{-Mod})_{\text{-}\mathbb{N},\mathbb{Z}}$ to $(\mathcal{C}_n\text{-Mod})_{\mathbb{Z},\mathbb{Z}}$ has a left adjoint.

Let $2 \leq n \leq \infty$ and let (M, d_0^M, λ) be a $\mathcal{C}_n$-module, where $\lambda\colon \mathcal{C}_n \otimes M \longrightarrow M$ denotes the module action. Let $M_{\leq 0}$ and $M_{>0}$ denote the vertical bicomplexes given by

$$M^{p,q}_{\leq 0} = \begin{cases} 0 & \text{if } p > 0 \\ M^{p,q} & \text{if } p \leq 0, \end{cases} \qquad \text{and} \qquad M^{p,q}_{>0} = \begin{cases} M^{p,q} & \text{if } p > 0 \\ 0 & \text{if } p \leq 0. \end{cases}$$

It is clear that $M_{\leq 0}$ is a $\mathcal{C}_n$-submodule of M, that $M_{>0}$ is not, but $\lambda(\mathcal{C}_n \otimes M_{>0})$ is. Hence the intersection $\lambda(\mathcal{C}_n \otimes M_{>0}) \cap M_{\leq 0}$ is a $\mathcal{C}_n$-submodule of $M_{\leq 0}$.

Lemma 5.9. *The projection $\pi\colon M \to M_{\leq 0}/(\lambda(\mathcal{C}_n \otimes M_{>0}) \cap M_{\leq 0})$ which maps m to 0 if $m \in M_{>0}$ and to its class if $m \in M_{\leq 0}$ is a morphism of $\mathcal{C}_n$-modules.*

Proof. For $m \in M$ and $x \in \mathcal{C}_n$, let us write $x \cdot m$ for $\lambda(x \otimes m)$.

Assume $m \in M_{>0}$. If $x \cdot m \in M_{>0}$, then $\pi(x \cdot m) = 0 = x \cdot \pi(m)$. If $x \cdot m \in M_{\leq 0}$, then $x \cdot m \in \lambda(\mathcal{C}_n \otimes M_{>0}) \cap M_{\leq 0}$, hence $\pi(x \cdot m) = 0 = x \cdot \pi(m)$.

Assume $m \in M_{\leq 0}$. Since $M_{\leq 0}$ is a $\mathcal{C}_n$-submodule of M, $\pi(x \cdot m) = x \cdot \pi(m)$. ◻

Proposition 5.10. *The natural inclusion functor $i\colon (\mathcal{C}_n\text{-Mod})_{\text{-}\mathbb{N},\mathbb{Z}} \longrightarrow (\mathcal{C}_n\text{-Mod})_{\mathbb{Z},\mathbb{Z}}$ has a left adjoint t given on objects by*

$$t(M) = M_{\leq 0}/(\lambda(\mathcal{C}_n \otimes M_{>0}) \cap M_{\leq 0}) \ \ \textit{for a } \mathcal{C}_n\textit{-module } M,$$

and on morphisms by sending a map of $\mathcal{C}_n$-modules to the induced map on the subquotient.

Proof. Let $M \in (\mathcal{C}_n\text{-Mod})_{\mathbb{Z},\mathbb{Z}}$ and $N \in (\mathcal{C}_n\text{-Mod})_{\text{-}\mathbb{N},\mathbb{Z}}$. Given a morphism $\tilde{f}\colon t(M) \longrightarrow N$ in $(\mathcal{C}_n\text{-Mod})_{\text{-}\mathbb{N},\mathbb{Z}}$, consider the composite

$$f\colon M \xrightarrow{\pi} t(M) \xrightarrow{\tilde{f}} i(N) = N,$$

where π is the morphism of $\mathcal{C}_n$-modules defined in Lemma 5.9, so that f is a morphism of $\mathcal{C}_n$-modules. On the other hand, if $f\colon M \longrightarrow i(N) = N$ is a morphism of $\mathcal{C}_n$-modules, then $M_{>0} \subseteq \ker f$ and $\lambda(\mathcal{C}_n \otimes M_{>0}) \cap M_{\leq 0} \subseteq \ker f$. Hence, f induces a morphism $\tilde{f}\colon t(M) \longrightarrow N$ such that $f = \tilde{f}\pi$. ◻

Theorem 3.30 shows that for each $r \geq 0$, there is a cofibrantly generated model structure on $n\text{-mC}_{\mathbb{Z},\mathbb{Z}}$ where a map f is a weak equivalence if it is an E_r-quasi-isomorphism, and a fibration if $E_i(f)$ is surjective for $0 \leq i \leq r$. The generating cofibrations and generating trivial cofibrations are denoted $(I^n_r)'$ and $(J^n_r)'$ respectively. An application of the transfer theorem gives the following.

Proposition 5.11. *For each $r \geq 0$, there is a cofibrantly generated model structure on $n\text{-mC}_{\text{-}\mathbb{N},\mathbb{Z}}$, where*

(1) *weak equivalences are E_r-quasi-isomorphisms,*
(2) *fibrations are morphisms of n-multicomplexes $f\colon A \to B$ such that $E_i(f)$ is bidegree-wise surjective for every $0 \leq i \leq r$, and*
(3) *the generating cofibrations and generating trivial cofibrations are $t(I^n_r)'$ and $t(J^n_r)'$ respectively.*

Proof. We apply the transfer theorem to the adjunction $t \dashv i$ of Proposition 5.10. The descriptions of the weak equivalences and fibrations are immediate as long as the transfer theorem holds. We check the conditions (1) and (2) in the transfer theorem. Every n-multicomplex is r-fibrant, so the first part of (2) trivially holds. Condition (1) holds as the functor t preserves small objects. It remains to find functorial path objects for $(-\mathbb{N},\mathbb{Z})$-graded n-multicomplexes. These exist because if $A \in n\text{-mC}_{\text{-}\mathbb{N},\mathbb{Z}}$, then $P_r(A) \in n\text{-mC}_{\text{-}\mathbb{N},\mathbb{Z}}$ by Remark 5.6. ◻

It is also possible to transfer the model category structure to the upper half-plane in the case $r = 0$. Similarly to above, we prove that the inclusion functor from $n\text{-mC}_{\mathbb{Z},\mathbb{N}}$ to $n\text{-mC}_R$ has a left adjoint.

Proposition 5.12. *For $A \in n\text{-mC}_R$, there is a $(\mathbb{Z}, \mathbb{N})$-graded n-multicomplex given by*

$$(t'A)^{p,q} = \begin{cases} A^{p,q} & q > 0 \\ A^{p,0}/d_0(A^{p,-1}) & q = 0 \\ 0 & q < 0, \end{cases}$$

with structure maps d_i induced from those of A. Furthermore, this construction is functorial and there exists an adjunction

$$n\text{-mC}_R \underset{i}{\overset{t'}{\rightleftarrows}} n\text{-mC}_{\mathbb{Z},\mathbb{N}}$$

where i is the natural inclusion functor and the functor t' is its left adjoint.

Proof. We check that for any $A \in n\text{-mC}_R$, $t'(A)$ is an n-multicomplex. Consider A as a $\mathcal{C}_n$-module (A, d_0, λ) in a natural way (see Proposition 4.4). Let $A_{*,-1}$ and $A_{q<0}$ denote the following bigraded R-modules

$$A^{p,q}_{*,-1} = \begin{cases} A^{p,-1} & \text{if } q = -1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad A^{p,q}_{q<0} = \begin{cases} A^{p,q} & \text{if } q < 0 \\ 0 & \text{otherwise.} \end{cases}$$

These are not vertical bicomplexes in general, but $A_{q<0} \oplus d_0(A_{*,-1})$ is. Furthermore, this is a $\mathcal{C}_n$-submodule of A and the quotient $A/(A_{q<0} \oplus d_0(A_{*,-1}))$ is a $\mathcal{C}_n$-module which corresponds to $t'(A)$. Hence $t'(A)$ is an n-multicomplex.

The functor t' is a left adjoint. Let $\pi \colon A \longrightarrow t'(A)$ be the projection in $n\text{-mC}_R$. For $B \in n\text{-mC}_{\mathbb{Z},\mathbb{N}}$, a morphism $f \colon A \to i(B)$ in $n\text{-mC}_R$ satisfies $d_0 f(A_{*,-1}) = f d_0(A_{*,-1}) = 0$ and $f(A_{q<0}) = 0$. Hence $A_{q<0} \oplus d_0(A_{*,-1})$ is contained in $\ker f$ which implies that f corresponds to a well defined morphism $\tilde{f} \colon t'(A) \longrightarrow B$ such that $f = \tilde{f}\pi$. $\square$

Proposition 5.13. *For $r = 0$, there is a cofibrantly generated model structure on $n\text{-mC}_{\mathbb{Z},\mathbb{N}}$, where*

(1) *weak equivalences are E_0-quasi-isomorphisms,*
(2) *fibrations are morphisms of n-multicomplexes $f \colon A \to B$ such that f is bidegreewise surjective, and*
(3) *the generating cofibrations and generating trivial cofibrations are $t'I^n_0$ and $t'J^n_0$ respectively.*

Proof. The proof proceeds in the same way as that of Proposition 5.11, using the existence of a functorial path object $P_0(A)$ for the category $n\text{-mC}_{\mathbb{Z},\mathbb{N}}$ when $r = 0$ (see Remark 5.6). $\square$

6. Examples of cofibrancy and cofibrant replacement

In this section we give some examples of cofibrant and non-cofibrant objects. We will see that all the objects appearing in our generating (trivial) cofibrations for the model structures of Theorem 3.30 have trivial total homology. This leads naturally to the question of how one can build cofibrant objects with non-trivial total homology and we explore this here. In particular, we note that the ground ring R concentrated in a single bidegree is not a cofibrant object and we describe a cofibrant replacement in n-multicomplexes. For example, in the case of bicomplexes, this is an “infinite staircase”. We also consider briefly what happens under transfer of model structures to bounded versions.

Example 6.1. For any $p, q \in \mathbb{Z}$, the n-multicomplex $R^{p,q}$ is not cofibrant in $(n\text{-mC}_R)_r$ for $2 \le n \le \infty$ and $r \ge 0$. Consider the "corner" bicomplex, $C(p, q)$, pictured below.

$$\begin{tikzcd} R^{p-1,q} & R^{p,q} \arrow[l, "1"'] \\ R^{p-1,q-1} \arrow[u, "1"] & \end{tikzcd}$$

We can view this as an n-multicomplex for $2 \le n \le \infty$. Define the map of n-multicomplexes $\pi \colon C(p, q) \to R^{p,q}$ to be the identity on R in bidegree (p, q) and zero in all other bidegrees. Then π is clearly bidegreewise surjective, so a 0-fibration. Also, $E_1(C(p, q)) = R^{p,q}$ and $E_1(\pi)$ is the identity map of $R^{p,q}$. Thus π is a trivial 0-fibration.

Now we can test against this trivial 0-fibration to see that $R^{p,q}$ is not 0-cofibrant. Indeed we find that there is no lift $R^{p,q} \to C(p, q)$ in the diagram of n-multicomplexes

$$\begin{tikzcd} & C(p,q) \arrow[d, "\pi"] \\ R^{p,q} \arrow[ur, dotted, "\nexists"] \arrow[r, "1"'] & R^{p,q} \end{tikzcd}$$

Any such lift f would have to take the generator $\mathbb{1}_R$ to the generator $\mathbb{1}_R$ in bidegree (p, q) in $C(p, q)$, but then for f to be a map of bicomplexes it would have to satisfy $0 = f(d_1\mathbb{1}_R) = d_1 f(\mathbb{1}_R) = \mathbb{1}_R$, giving a contradiction.

Since $R^{p,q}$ is not 0-cofibrant, it is not r-cofibrant for any r.

Proposition 6.2. *For $p, q \in \mathbb{Z}$, $r, s \ge 0$ and $2 \le n \le \infty$, $\mathcal{ZW}_s^n(p, q)$ is cofibrant in $(n\text{-mC}_R)_r$.*

Proof. Fix $p, q \in \mathbb{Z}$, $r, s \ge 0$ and n with $2 \le n \le \infty$. Note that a lift exists in the diagram of n-multicomplexes

$$\begin{tikzcd} & A \arrow[d, "f"] \\ \mathcal{ZW}_s^n(p,q) \arrow[ur, dotted] \arrow[r] & B \end{tikzcd}$$

if and only if $ZW_s(f)$ is surjective in bidegree (p, q). Now suppose that f is an r-trivial r-fibration. Then $E_i(f)$ is surjective for all $i \ge 0$. Using Remark 3.11, it follows that $ZW_s(f)$ is surjective for all s. So the required lift exists. □

Remark 6.3. If we use the r-model structure of Theorem 3.28 instead, the same line of argument shows that $\mathcal{ZW}_s^n(p, q)$ is r-cofibrant for $s \ge r$.

Corollary 6.4. *For every $p, q \in \mathbb{Z}$, $s \ge 0$ and $2 \le n \le \infty$, we have*

$$E_i(\mathcal{ZW}_s^n(p,q)) = \begin{cases} R^{p,q} \oplus R^{p-s,q-s+1} & \text{if } 1 \le i \le s \\ 0 & \text{if } i \ge s+1. \end{cases}$$

Proof. The n-multicomplex $\mathcal{ZW}_s^n(p, q)$ is r-cofibrant for any $r \ge 0$ by Proposition 6.2. We claim that $p_{n,2}(\mathcal{ZW}_s^n(p, q)) = \mathcal{ZW}_s^2(p, q)$ (see Section 4.2 for the definition of $p_{n,2}$). This follows from the definition of $\mathcal{ZW}_s^n(p, q)$ via successive pushout (Definition 3.15) and the fact that $p_{n,2}$ is a left adjoint and so preserves pushouts, together with the initial cases $p_{n,2}(\mathcal{ZW}^n_0(p,q)) = \mathcal{ZW}^2_0(p,q)$ and $p_{n,2}(d^*_0) = d^*_0$. By Theorem 4.5, since $\mathcal{ZW}^n_s(p,q)$ is r-cofibrant, the unit of the adjunction $\mathcal{ZW}^n_s(p,q) \to \mathcal{ZW}^2_s(p,q)$ is an E_r-quasi-isomorphism, for each $r \geq 0$, in particular an E_0-quasi-isomorphism. For the staircase bicomplex $\mathcal{ZW}^2_s(p,q)$ it is easy to read off the pages of the spectral sequence directly:

$$E_i(\mathcal{ZW}^2_s(p,q)) = \begin{cases} R^{p,q} \oplus R^{p-s,q-s+1} & \text{if } 1 \leq i \leq s \\ 0 & \text{if } i \geq s+1, \end{cases}$$

as required. $\square$

Definition 6.5. Let $p, q \in \mathbb{Z}$ and $2 \leq n \leq \infty$. We define $\mathcal{ZW}^n_\infty(p,q) = \varinjlim_s \mathcal{ZW}^n_s(p,q)$, where the colimit is taken over the maps $\mathcal{ZW}^n_s(p,q) \to \mathcal{ZW}^n_{s+1}(p,q)$ representing the projection maps $ZW^n_{s+1} \to ZW^n_s$.

Example 6.6. When $n = 2$, the map $\mathcal{ZW}^2_s(p,q) \to \mathcal{ZW}^2_{s+1}(p,q)$ is the inclusion of a staircase with s-horizontal steps into a staircase with $s+1$-horizontal steps and $\mathcal{ZW}^2_\infty(p,q)$ is the infinite (downwards to the left) staircase bicomplex, with top right entry in bidegree (p,q):

$$\begin{tikzcd}
 & & & \bullet & \ast \arrow[l] \\
 & & \bullet & \bullet \arrow[l] \arrow[u] & \\
 & \bullet & \bullet \arrow[l] \arrow[u] & & \\
 & \bullet \arrow[u] & & & \\
 {} \arrow[ur, dotted, dash] & & & &
\end{tikzcd}$$

Proposition 6.7. *Let $p, q \in \mathbb{Z}$ and $2 \leq n \leq \infty$. Then $\mathcal{ZW}^n_\infty(p,q) \to R^{p,q}$ given by projection to $R^{p,q}$ is an r-cofibrant replacement of $R^{p,q}$ for all $r \geq 0$.*

Proof. First we check that $\mathcal{ZW}^n_\infty(p,q)$ is r-cofibrant for all $r \geq 0$. The relevant lift exists for $\mathcal{ZW}^n_\infty(p,q)$ if and only if compatible lifts exist for each $\mathcal{ZW}^n_s(p,q)$. Such lifts do exist for each $\mathcal{ZW}^n_s(p,q)$ by Proposition 6.2 and it is straightforward to check that they are compatible.

The map $E_r(\mathcal{ZW}^n_s(p,q) \to \mathcal{ZW}^n_{s+1}(p,q))$ is the projection to $R^{p,q}$ if $1 \leq r \leq s$ and 0 otherwise, so we see that $E_r(\mathcal{ZW}^n_\infty(p,q)) = R^{p,q}$ for all $r \geq 1$. And the projection $\mathcal{ZW}^n_\infty(p,q) \to R^{p,q}$ induces an isomorphism on E_r for all $r \geq 1$, that is, it is an E_r-quasi-isomorphism for all $r \geq 0$. $\square$

6.1. Upper half-plane versions

We consider the $r = 0$ model structure on upper half-plane n-multicomplexes from Proposition 5.13. The generating cofibrations and generating trivial cofibrations are given by $t'I_0$ and $t'J_0$. The interesting new thing that appears is the cotruncation of $\iota_1 \colon \mathcal{ZW}_1(p,0) \to \mathcal{BW}_1(p,-1)$, which is $t'\iota_1 = 0 \colon \mathcal{ZW}_1(p,0) \to 0$. This allows one to see that $R^{p,0}$ is 0-cofibrant, since we have a pushout diagram

$$\begin{tikzcd}
\mathcal{ZW}_1(p-1,0) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{ZW}_1(p,0) \arrow[r] & R^{p,0}
\end{tikzcd}$$

where the top horizontal map is a cofibration and $\mathcal{ZW}_1(p, 0)$ is cofibrant. On the other hand, $R^{p,q}$ for $q > 0$ is not 0-cofibrant, just as in Example 6.1. This shows (unsurprisingly) that in the 0-model structure on upper half-plane n-multicomplexes, cofibrancy is not preserved under vertical shift.

References

[1] C. Berger, I. Moerdijk, Axiomatic homotopy theory for operads, Comment. Math. Helv. 78 (4) (2003) 805–831.
[2] J. Cirici, D. Egas Santander, M. Livernet, S. Whitehouse, Model category structures and spectral sequences, Proc. R. Soc. Edinb., Sect. A, Math. 150 (2020) 2815–2848.
[3] J. Cirici, D. Egas Santander, M. Livernet, S. Whitehouse, Derived A-infinity algebras and their homotopies, Topol. Appl. 235 (2018) 214–268.
[4] B. Fresse, Modules over Operads and Functors, Lecture Notes in Mathematics, vol. 1967, Springer-Verlag, Berlin, 2009.
[5] P.S. Hirschhorn, Model Categories and Their Localizations, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.
[6] M. Hovey, Model Categories, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.
[7] M. Livernet, C. Roitzheim, S. Whitehouse, Derived A-infinity algebras in an operadic context, Algebraic Geom. Topol. 13 (2013) 409–440.
[8] M. Livernet, S. Whitehouse, S. Ziegenhagen, On the spectral sequence associated to a multicomplex, J. Pure Appl. Algebra 224 (2) (2020) 528–535.
[9] J. Loday, B. Vallette, Algebraic Operads, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 346, Springer, Heidelberg, 2012.
[10] F. Muro, C. Roitzheim, Homotopy theory of bicomplexes, J. Pure Appl. Algebra 223 (5) (2019) 1913–1939.