

EXTERIOR POWERS AND TOR-PERSISTENCE

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ABSTRACT. A commutative Noetherian ring R is said to be Tor-persistent if, for any finitely generated R -module M , the vanishing of $\mathrm{Tor}_i^R(M, M)$ for $i \gg 0$ implies M has finite projective dimension. An open question of Avramov, et. al. asks whether any such R is Tor-persistent. In this work, we exploit properties of exterior powers of modules and complexes to provide several partial answers to this question; in particular, we show that every local ring $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ is Tor-persistent. As a consequence of our methods, we provide a new proof of the Tachikawa Conjecture for positively graded rings over a field of characteristic different from 2.

1. INTRODUCTION

Several conjectures and open questions on the rigidity of Ext and Tor have recently gained much attention. Among the most well-known of these is the Auslander-Reiten conjecture which poses that given a commutative Noetherian ring R and a finitely generated R -module M , the vanishing of $\mathrm{Ext}_R^i(M, M \oplus R)$ for all $i > 0$ forces M to be projective [1]. The Auslander-Reiten conjecture traces its roots to the representation theory of Artin algebras where it is intimately connected to Nakayama's conjecture and its generalized version. A significant special case of the Auslander-Reiten conjecture is the Tachikawa conjecture which posits that the Auslander-Reiten conjecture holds when R is Cohen-Macaulay and $M = \omega_R$ is a canonical module of R [4]. Inspired by work of Şega [16], Avramov et. al. introduce the following which can be thought of as a version of the Auslander-Reiten conjecture for Tor.

Question 1.1 ([5]). Let R be a commutative Noetherian ring. If, for a finitely generated R -module M , we have $\mathrm{Tor}_i^R(M, M) = 0$ for $i \gg 0$, must M have finite projective dimension?

Rings for which Question 1.1 has an affirmative answer are called *Tor-persistent*. Thus Question 1.1 can be rephrased to ask whether every commutative Noetherian ring is Tor-persistent. Several classes of rings are known to be Tor-persistent, for example, complete intersection rings, Golod rings, and rings of small embedding codimension or multiplicity [2, 5, 13, 14]. The complete intersection case depends on support theory, which is only available in this setting, while the other known results depend on conditions for the vanishing of $\mathrm{Tor}_i^R(M, N)$ for all $i \gg 0$ and every M and N , an approach that does not extend to the general case (see [5]).

The main purpose of this work is to provide evidence, and new insights, that this question may have an affirmative answer. Our first main result provides a new class of rings which are Tor-persistent (see Theorem 2.1).

Theorem A. *If $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m}^3 = 0$, then R is Tor-persistent.*

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Our second main result provides some restrictions in the graded case; here $e_R(M)$ denotes the Hilbert-Samuel multiplicity of the R -module M (see Theorem 3.5).

Theorem B. *Let $R = \bigoplus_{i \geq 0} R_i$ be a Noetherian standard graded algebra over a field $R_0 = k$ with $\text{char } k \neq 2$. Let M be a finitely generated graded R -module satisfying the following.*

- (1) $e_R(M) = e_R(R)$,
- (2) $\text{Tor}_i^R(M, M) = 0$ for $i > 0$, and
- (3) $M \otimes_R M$ has no embedded primes.

Then $M \cong R$.

In fact, we prove Theorem B under more general assumptions. As a consequence of this result, we provide a commutative algebra proof of the Tachikawa Conjecture for positively graded rings over a field of characteristic different from 2 (see Corollary 3.8). This result also follows from work of Zeng using techniques in representation theory of Artin algebras [17].

Our approach to both of our main results provides an explanation as to why the vanishing of $\text{Tor}_i^R(M, N)$ is special when $M = N$; namely, the vanishing of $\text{Tor}_i^R(M, M)$ has consequences for the exterior and symmetric powers of M . For the cases we consider, these consequences come in the form of numerical constraints on the exterior and symmetric squares, and are enough to conclude that the module in question is free.

We conclude this section with some notation that we use in the subsequent ones. Through the remainder of the paper, let $(R, \mathfrak{m}, k)$ be a commutative Noetherian ring which is either local or positively graded over the field k with maximal homogeneous ideal $\mathfrak{m}$. If R has a canonical module, it is denoted by ω_R . We let M be a finitely generated R -module; in the graded case we assume M is homogeneous. We use $v_R(M) = \beta_0^R(M)$ for the minimal number of generators of M and $l_R(M)$ for the length of M . We write $\Omega_i^R(M)$ for the i th syzygy of M and $\beta_i^R(M)$ for the i th Betti number. We let $\text{codim } R := v_R(\mathfrak{m}) - \dim R$ be the embedding codimension of R . We let $\iota_M : \bigwedge_R^2(M) \rightarrow M \otimes_R M$ be the antisymmetrization map defined on elementary wedges by $x_1 \wedge x_2 \mapsto x_1 \otimes x_2 - x_2 \otimes x_1$.

2. TOR-PERSISTENCE FOR RINGS WITH RADICAL CUBE ZERO

The following main result contains Theorem A from the introduction.

Theorem 2.1. *Assume $(R, \mathfrak{m}, k)$ is a local ring with $\mathfrak{m}^3 = 0$. If M is an R -module such that $\text{Tor}_i^R(M, M) = 0$ for $2 \leq i \leq 5$, then M is free.*

Proof. We may assume M is nonzero. As a notational convenience, we set $\gamma_R(M) = \frac{l_R(M)}{v_R(M)} - 1$. We note that $\gamma_R(M) \geq 0$ with equality if and only if $M \cong k^n$ for some n .

Suppose M is not free. Set $N = \Omega_1^R(M)$, $L = \Omega_2^R(M)$, and $b = v_R(N)$. Let φ be the map fitting in the natural exact sequence $0 \rightarrow L \xrightarrow{\varphi} R^b \rightarrow N \rightarrow 0$. Since $N \hookrightarrow \mathfrak{m}R^{v_R(M)}$ and since $\mathfrak{m}^3 = 0$, we have $\mathfrak{m}^2 N = 0$. Similarly, we have $\mathfrak{m}^2 L = 0$. By dimension shifting, $\text{Tor}_1^R(N, L) = 0$, and so we have $\mathfrak{m}(L \otimes_R L) = 0$ by [11, Lemma 1.4]. Further, we have $\text{Tor}_i^R(N, N) = 0$ for $i = 1, 2, 3$ so [11, Theorem 2.5] gives

- (1) $v_R(L) = \gamma_R(N)b$,
- (2) $v_R(\mathfrak{m}) = 2\gamma_R(N)$, and
- (3) $r(R) = \gamma_R(N)^2$, where $r(R) := \dim_k \text{Soc}(R)$ is the type of R .

The map $\iota_{L \otimes_R k} : \bigwedge_k^2(L \otimes_R k) \rightarrow (L \otimes_R k) \otimes_k (L \otimes_R k)$ is injective because $L \otimes_R k$ is a k -vector space, and this map is naturally identified with $\iota_L \otimes \text{id}_k : \bigwedge_R^2(L) \otimes_R k \rightarrow (L \otimes_R L) \otimes_R k$. As

$L \otimes_R L$ is a k -vector space, so is its quotient $\bigwedge_R^2(L)$, hence $\iota_L \otimes \text{id}_k$ is naturally identified with ι_L . In particular, ι_L is injective.

Now, we have $\varphi \otimes \varphi = (\varphi \otimes \text{id}_{R^b}) \circ (\text{id}_L \otimes \varphi)$. The map $\varphi \otimes \text{id}_{R^b}$ is injective since $\text{Tor}_1^R(N, R^b) = 0$, and $\text{id}_L \otimes \varphi$ is injective since $\text{Tor}_1^R(L, N) = \text{Tor}_4^R(M, M) = 0$. Thus $\varphi \otimes \varphi$ is also injective.

Next, we have the following commutative diagram

$$\begin{array}{ccc} \bigwedge_R^2(L) & \xrightarrow{\iota_L} & L \otimes_R L \\ \bigwedge_R^2(\varphi) \downarrow & & \downarrow \varphi \otimes \varphi \\ \bigwedge_R^2(R^b) & \xrightarrow{\iota_{R^b}} & R^b \otimes_R R^b \end{array}$$

Since $\varphi \otimes \varphi$ and ι_L are both injective, the commutativity of the diagram forces $\bigwedge_R^2(\varphi)$ to be injective. Since $\bigwedge_R^2(L)$ is a k -vector space, it must thus embed in the socle of $\bigwedge_R^2(R^b)$.

The vector space dimension of $\bigwedge_R^2(L)$ is

$$\binom{v_R(L)}{2} = \binom{\gamma_R(N)b}{2} = \frac{\gamma_R(N)b(\gamma_R(N)b-1)}{2}$$

while that of $\text{soc}(\bigwedge_R^2(R^b))$ is

$$r(R) \binom{b}{2} = \gamma_R(N)^2 \binom{b(b-1)}{2}.$$

It follows that we must have

$$\frac{\gamma_R(N)b(\gamma_R(N)b-1)}{2} \leq \gamma_R(N)^2 \binom{b(b-1)}{2}.$$

If $\gamma_R(N) = 0$, then N is a k -vector space which cannot be, since N has infinite projective dimension and $\text{Tor}_1^R(N, N) = 0$. Therefore, as $b \neq 0$, we have

$$\gamma_R(N)b-1 \leq \gamma_R(N)b - \gamma_R(N)$$

which forces $\gamma_R(N) = 1$. Thus R is Gorenstein with $v_R(\mathfrak{m}) = 2$, by items (1)–(2) above, and so R is also a complete intersection.

Let $\text{cx}_R(M)$ be the complexity of M , since R is a complete intersection, we have $\text{cx}_R(M) \leq \text{codim } R = 2$; see e.g. [12, Theorem 1.1 and subsequent paragraph]. Thus, [12, Proposition 2.3] forces $\text{Tor}_i^R(M, M) = 0$ for $i > 0$. By [2, Theorem 4.2], this contradicts the fact that M is not free. The result follows. $\square$

3. SOME RESTRICTIONS FOR TOR-PERSISTENCE

Unless otherwise stated, throughout this section we let $R = \bigoplus_{i \geq 0} R_i$ be a positively graded algebra over a field $R_0 = k$ and $\mathfrak{m} = \bigoplus_{i > 0} R_i$ its homogeneous maximal ideal. Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated graded R -module. The *Hilbert series* of M is

$$H_M(t) = \sum_{i \in \mathbb{Z}} (\dim_k M_i) t^i.$$

We recall that if $M \neq 0$, and if $\underline{x} = (x_1, \dots, x_{\dim M})$ is a homogeneous system of parameters on M with $\deg x_i = a_i$, then there exists a Laurent polynomial $\mathcal{E}_M^{\underline{x}}(t) \in \mathbb{Z}[t, t^{-1}]$ with $\mathcal{E}_M^{\underline{x}}(1) > 0$ such that $H_M(t)$ can be written as

$$H_M(t) = \frac{\mathcal{E}_M^{\underline{x}}(t)}{\prod_{i=1}^{\dim M} (1 - t^{a_i})}$$

[6, Proposition 4.4.1 and Remark 4.4.2].

We denote by $e_R(M)$ the *Hilbert-Samuel multiplicity* of M

$$e_R(M) = \lim_{n \rightarrow \infty} \frac{(\dim R)! l_R(M/\mathfrak{m}^n M)}{n^{\dim R}};$$

we also write $e(R)$ for $e_R(R)$.

For a graded complex of finite rank graded free R -modules

$$X = \cdots X_{i+1} \rightarrow X_i \rightarrow X_{i-1} \rightarrow \cdots$$

with $X_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{b_{i,j}}$, we denote by

$$P_X(t, z) = \sum_{i,j \in \mathbb{Z}} b_{i,j} t^i z^j$$

the (graded) *Poincaré series* of X . If F is a graded free resolution of M , then we set $P_M(t, z) := P_F(t, z)$. The additivity of length gives the following comparison of Hilbert and Poincaré series.

Fact 3.1. For R and M as above, we have $H_M(t) = H_R(t)P_M(t, -1)$.

We now describe a construction of Buchsbaum-Eisenbud [7], following the presentation of Frankild-Sather-Wagstaff-Taylor [9]. Assume for the remainder of this paragraph that $\text{char } k \neq 2$. Let X be as above, and let $\alpha^X : X \otimes_R X \rightarrow X \otimes X$ be the map defined on homogeneous generators by

$$\alpha^X(x \otimes x') = x \otimes x' - (-1)^{|x||x'|} x' \otimes x.$$

Let $S_R^2(X)$ be the complex $\text{Coker}(\alpha^X)$ and call it the *second symmetric power* of X .

In the following statement we summarize some important properties of $S_R^2(X)$. We remark that although the statements in [9] are in the local case, the arguments therein readily extend to account for the grading in R .

Fact 3.2 ([9, 3.8, 4.1, 3.12]). Assume $\text{char } k \neq 2$. Let X be a graded complex of finite rank graded free R -modules.

(1) The following exact sequences are split exact.

$$0 \rightarrow \text{Ker}(\alpha^X) \rightarrow X \otimes_R X \rightarrow \text{Im}(\alpha^X) \rightarrow 0$$

$$0 \rightarrow \text{Im}(\alpha^X) \rightarrow X \otimes_R X \rightarrow S_R^2(X) \rightarrow 0$$

(2) $H_0(S_R^2(X)) \cong S_R^2(H_0(X))$.

(3) $P_{S_R^2(X)}(t, z) = \frac{1}{2} [P_X(t, z)^2 + P_X(t^2, -z^2)]$

Now, assume $\text{char } k \neq 2$ and consider the antisymmetrization map $\iota_M : \bigwedge_R^2(M) \rightarrow M \otimes_R M$ defined in the introduction. This map is a split injection where the splitting map is given by $x \otimes y \mapsto \frac{1}{2} x \wedge y$. Since $\text{Coker}(\iota_M) = S_R^2(M)$, we have the following fact.

Fact 3.3. Assume $\text{char } k \neq 2$, $M \otimes_R M \cong S_R^2(M) \oplus \bigwedge_R^2(M)$

The following lemma is essential in the proof of our main result.

Lemma 3.4. Assume $\text{char } k \neq 2$. If M is a graded R -module such that $\text{Tor}_i^R(M, M) = 0$ for every $i > 0$, then we have

$$H_{S_R^2(M)}(t) = \frac{H_M^2(t)}{2H_R(t)} + \frac{H_M(t^2)H_R(t)}{2H_R(t^2)} \quad \text{and} \quad H_{\bigwedge_R^2(M)}(t) = \frac{H_M^2(t)}{2H_R(t)} - \frac{H_R(t)H_M(t^2)}{2H_R(t^2)}.$$

Proof. Let F be a minimal graded free resolution of M . By the vanishing of Tor assumption, the complex $F \otimes_R F$ is acyclic and therefore a minimal free resolution of $M \otimes_R M$. Therefore, Fact 3.2(1)–(2) imply that $S_R^2(F)$ is acyclic and a minimal free resolution of $S_R^2(M)$. From Facts 3.1 and 3.2(3) we obtain

$$\begin{aligned} H_{S_R^2(M)}(t) &= H_R(t)P_{S_R^2(F)}(t, -1) = \frac{H_R(t)}{2} [P_F(t, -1)^2 + P_F(t^2, -1)] \\ &= \frac{H_M^2(t)}{2H_R(t)} + \frac{H_R(t)H_R(t^2)P_F(t^2, -1)}{2H_R(t^2)} \\ &= \frac{H_M^2(t)}{2H_R(t)} + \frac{H_R(t)H_M(t^2)}{2H_R(t^2)}. \end{aligned}$$

We note that

$$H_{M \otimes_R M}(t) = H_R(t)P_{F \otimes_R F}(t, -1) = H_R(t)P_F^2(t, -1) = \frac{H_M^2(t)}{H_R(t)}.$$

Thus, it suffices to show $H_{M \otimes_R M}(t) = H_{S_R^2(M)}(t) + H_{\wedge_R^2(M)}(t)$, which follows from Fact 3.3. $\square$

We are now ready to prove Theorem B.

Theorem 3.5. *Let R be a Noetherian positively graded k -algebra with $\text{char } k \neq 2$. Let M be a finitely generated graded R -module such that $\dim(M) = \dim(R)$ and satisfying the following:*

- (1) *For some homogeneous system of parameters $\underline{x}$ of R , $\epsilon_M^{\underline{x}}(1) = \epsilon_R^{\underline{x}}(1)$.*
- (2) *$\text{Tor}_i^R(M, M) = 0$ for $i > 0$, and*
- (3) *$M \otimes_R M$ has no embedded primes.*

Then $M \cong R$.

Proof. We proceed by contradiction. Suppose that $\wedge_R^2(M) \neq 0$ so, by Fact 3.3, we have $M \otimes_R M \cong \wedge_R^2(M) \oplus S_R^2(M)$. Since $M \otimes_R M$ has no embedded primes, it follows that $\wedge_R^2(M)$ is has dimension $\dim(R)$. By Lemma 3.4, we have

$$H_R(t)H_R(t^2)H_{\wedge_R^2(M)}(t) = H_M(t)^2H_R(t^2) - H_M(t^2)H_R(t)^2.$$

As each module in question has maximal dimension, we may clear denominators to obtain a formula for multiplicity polynomials with respect to any system of parameters $\underline{x}$ of R

$$\epsilon_R^{\underline{x}}(t)\epsilon_R^{\underline{x}}(t^2)\epsilon_{\wedge_R^2(M)}^{\underline{x}}(t) = \epsilon_M^{\underline{x}}(t)^2\epsilon_R^{\underline{x}}(t^2) - \epsilon_M^{\underline{x}}(t^2)\epsilon_R^{\underline{x}}(t)^2.$$

Evaluating these at $t = 1$ shows that $\epsilon_{\wedge_R^2(M)}^{\underline{x}}(1) = 0$, a contradiction. Therefore, $\wedge_R^2(M) = 0$, and it follows that M is cyclic. Thus $M \cong R/I$ for some homogeneous ideal I . As $I/I^2 \cong \text{Tor}_1^R(R/I, R/I) \cong \text{Tor}_1^R(M, M) = 0$, it follows that $I = 0$, concluding the proof. $\square$

Remark 3.6. Hypothesis (1) in Theorem 3.5 follows from the condition that $e_R(M) = e(R)$ in a number of cases, e.g. if $\mathfrak{m}^n$ admits a homogeneous minimal reduction for some $n > 0$ (which occurs, for instance, if R is standard graded), if R is Artinian, if M has constant rank, or if R is Cohen-Macaulay and $M = \omega_R$ [6, Corollary 4.4.6]. When M has rank, hypothesis (1) in Theorem 3.5 is equivalent to M having rank 1. Frequently, such modules are ideals, e.g., if R is a domain and M is torsion-free.

In what follows, we set $(-)^{\vee} = \text{Hom}_R(-, \omega_R)$

Corollary 3.7. *Let R be a positively graded Cohen-Macaulay k -algebra with $\text{char } k \neq 2$. If M is a finitely generated graded maximal Cohen-Macaulay R -module such that:*

- (1) *For some homogeneous system of parameters $\underline{x}$ of R , $\epsilon_M^{\underline{x}}(1) = \epsilon_R^{\underline{x}}(1)$.*
- (2) *$\text{Ext}_R^i(M, M^\vee) = 0$ for $i > 0$.*

Then $M \cong R$.

Proof. From [14, Lemma 3.4 (1)], we have that $M \otimes_R M$ is maximal Cohen-Macaulay and that $\text{Tor}_i^R(M, M) = 0$ for all $i > 0$. The result then follows from Theorem 3.5. $\square$

As an immediate consequence of Corollary 3.7 (cf. [6, Corollary 4.4.6]), we prove the (commutative) graded case of the Tachikawa conjecture in characteristic different from 2, which also follows from work of Zeng. Notably, Zeng's approach requires passing to noncommutative algebras, whereas our proof uses only techniques in commutative algebra.

Corollary 3.8 ([17, Theorem 1.3]). *Let R be a positively graded Cohen-Macaulay k -algebra with $\text{char } k \neq 2$. If $\text{Ext}_R^i(\omega_R, R) = 0$ for every $i > 0$, then R is Gorenstein.*

For Artinian rings, condition (1) of Theorem 3.5 follows from the hypothesis that $l_R(M) = l(R)$ and condition (3) of Theorem 3.5 is automatically satisfied, so we obtain the following.

Corollary 3.9. *Let R be an Artinian positively graded k -algebra with $\text{char } k \neq 2$. If M is a finitely generated graded R -module such that $l_R(M) = l(R)$ and $\text{Tor}_i^R(M, M) = 0$ for every $i > 0$, then $M \cong R$.*

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REFERENCES

- [1] Maurice Auslander and Idun Reiten. On a generalized version of the nakayama conjecture. *Proceedings of the American Mathematical Society*, 52(1):69–74, 1975.
- [2] Luchezar L. Avramov and Ragnar-Olaf Buchweitz. Support varieties and cohomology over complete intersections. *Invent. Math.*, 142(2):285–318, 2000.
- [3] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, and Judith D Sally. Laurent coefficients and ext of finite graded modules. *Mathematische Annalen*, 307(3):401–415, 1997.
- [4] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, and Liana M. Şega. Extensions of a dualizing complex by its ring: commutative versions of a conjecture of Tachikawa. *J. Pure Appl. Algebra*, 201(1-3):218–239, 2005.
- [5] Luchezar L. Avramov, Srikanth B. Iyengar, Saeed Nasseh, and Sean K. Sather-Wagstaff. Persistence of homology over commutative noetherian rings. preprint (2020), [arxiv:2005.10808](https://arxiv.org/abs/2005.10808).
- [6] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.
- [7] David A. Buchsbaum and David Eisenbud. Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. *Amer. J. Math.*, 99(3):447–485, 1977.
- [8] Anders J. Frankild, Sean Sather-Wagstaff, and Amelia Taylor. Relations between semidualizing complexes. *J. Commut. Algebra*, 1(3):393–436, 2009.
- [9] Anders J. Frankild, Sean Sather-Wagstaff, and Amelia Taylor. Second symmetric powers of chain complexes. *B. Iran. Math. Soc.*, 37:39–75, 2011.

- [10] Douglas Hanes and Craig Huneke. Some criteria for the Gorenstein property. *J. Pure Appl. Algebra*, 201(1-3):4–16, 2005.
- [11] Craig Huneke, Liana M. Şega, and Adela N. Vraciu. Vanishing of Ext and Tor over some Cohen-Macaulay local rings. *Illinois J. Math.*, 48(1):295–317, 2004.
- [12] David A. Jorgensen. Complexity and Tor on a complete intersection. *J. Algebra*, 211(2):578–598, 1999.
- [13] David A. Jorgensen. A generalization of the Auslander-Buchsbaum formula. *J. Pure Appl. Algebra*, 144(2):145–155, 1999.
- [14] Justin Lyle and Jonathan Montaño. Extremal growth of betti numbers and trivial vanishing of (co)homology. *Transactions of the American Mathematical Society*, 2020. to appear.
- [15] S. Sather-Wagstaff. Semidualizing modules. in preparation <https://ssather.people.clemson.edu/DOCS/sdm.pdf>.
- [16] Liana M. Şega. Self-tests for freeness over commutative Artinian rings. *J. Pure Appl. Algebra*, 215(6):1263–1269, 2011.
- [17] Qiang Zeng. Vanishing of Hochschild’s cohomologies $H^i(A \otimes A)$ and gradability of a local commutative algebra A . *Tsukuba J. Math.*, 14(2):263–273, 1990.

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