SIAM J. MATH. ANAL.
Vol. 53, No. 3, pp. 2925–2956

ON THE 8π-CRITICAL-MASS THRESHOLD OF A PATLAK–KELLER–SEGEL–NAVIER–STOKES SYSTEM*

YISHU GONG† AND SIMING HE†

Abstract. In this paper, we proposed a coupled Patlak–Keller–Segel–Navier–Stokes system, which has dissipative free energy. On the plane $\mathbb{R}^2$, if the total mass of the cells is strictly less than 8π, classical solutions exist for any finite time, and their H^s-Sobolev norms are almost uniformly bounded in time. For the radially symmetric solutions, this 8π-mass threshold is critical. On the torus $\mathbb{T}^2$, the solutions are uniformly bounded in time under the same mass constraint.

Key words. Patlak–Keller–Segel–Navier–Stokes system, critical mass, long-time behavior

AMS subject classifications. 35B40, 35B45

DOI. 10.1137/20M1340629

1. Introduction.

We consider the following coupled Patlak–Keller–Segel–Navier–Stokes (PKS–NS) equation modeling chemotaxis in a moving fluid:

$$\begin{cases} \partial_t n + u\cdot\nabla n + \nabla\cdot(n\nabla c) = \Delta n, \\ -\Delta c = n, \\ \partial_t u + (u\cdot\nabla)u + \nabla p = \Delta u + n\nabla c, \quad \nabla\cdot u = 0, \\ n(t=0,x) = n_0(x), \quad u(t=0,x) = u_0(x), \quad x\in\mathbb{R}^2. \end{cases} \tag{1.1}$$

Here n, c denote the cell density and the chemical density, respectively. The divergence-free vector field u indicates the ambient fluid velocity. The first equation describes the time evolution of the cell density subject to chemotaxis-induced aggregation, diffusion caused by random Brownian motion, and transportation by ambient fluid flow u. Since the cells secrete the chemo-attractants, there exists a deterministic relation between the two distributions n and c. The second equation specifies this connection. The assumption behind this is that the chemo-attractant diffuses much faster than the fluid advection and cell aggregation and reaches equilibrium in a faster time-scale. The Newtonian potential is applied to determine c uniquely, i.e., $c = -\frac{1}{2\pi}\log|\cdot| * n$. The third equation on the divergence-free vector field u describes the fluid motion subject to forcing induced by the cells. The reasoning behind the coupling $n\nabla c$ is that in order to make the cells move without acceleration, the fluid exerts frictional force on the moving cells, so reaction forces act on the fluid. The force $n\nabla c$ in the Navier–Stokes equation matches the aggregation nonlinearity in the cell density evolution. The same forcing appears in the Nernst–Planck–Navier–Stokes system; see, e.g., [8].

If the ambient fluid velocity is identically equal to zero, i.e., $u \equiv 0$, the system (1.1) is the classical Patlak–Keller–Segel (PKS) equation, which was first derived by Patlak [25] and Keller and Segel [17]. The literature on the classical PKS model is

*Received by the editors May 26, 2020; accepted for publication (in revised form) February 17, 2021; published electronically May 20, 2021.
https://doi.org/10.1137/20M1340629
Funding: The work of the authors was partially supported by National Science Foundation (NSF) grant DMS-1848790. The work of the second author was also partially supported by NSF grant RNMS11-07444 (KINet).
†Department of Mathematics, Duke University, Durham, NC 27708 USA (yishu.gong@duke.edu, simhe@math.duke.edu).

extensive, and we refer the interested reader to the representative works [2], [4], [3], [14], and the references therein. The classical PKS model preserves the total mass $M := ||n(t)||_1 = ||n_0||_1$ and is L^1 critical. If the initial data n_0 has total mass M strictly less than 8π, then the smooth solution exists for all time, whereas if the initial data has total mass strictly larger than 8π and has a finite second moment, then the solution blows up in finite time; see, e.g., [4] and [16].

If the ambient fluid flow is not identically zero, i.e., $u \not\equiv 0$, the analysis of the long-time dynamics of the systems (1.1) is delicate. There is no heuristic argument to rule out global solutions with large masses. Moreover, the underlying fluid flow might suppress the potential chemotactic blow-up in the system. This assertion is based on a series of works on the suppression of chemotactic blow-up through *passive* fluid flows initiated by Kiselev and Xu [18]. To simplify the analysis, in these models, we note that the ambient fluid velocity fields u are assumed to be independent of the time evolution of the cell densities. In this series, there are two main fluid mechanisms for suppressing the blow-up. The first mechanism is the fluid-mixing-induced enhanced dissipation effect. The works in this direction are [1], [12], and [15]. The other mechanism for suppressing the blow-up is the fast splitting scenario introduced in the paper [13].

The model (1.1) takes into account the *active* chemotaxis-fluid interaction. The literature concerning coupled chemotaxis-fluid systems is vast. We refer the interested reader to the papers [23], [24], [22], [9], [11], [30], [31], [29], [7], [20], [32], [33], [34], [35], [36], and the references therein. A number of works are devoted to the study of parabolic-parabolic PKS equations subject to active fluid motions. The coupling between the chemotaxis and the fluid in these models is through the gravity-buoyancy relation. The closest models to ours are proposed by Lorz [24] and Kozono, Miura, and Sugiyama [20]. The chemical densities c in these models are also determined through elliptic-type equations. On the other hand, these models consider buoyancy forcing instead of the reaction force from the cells.

Another biologically relevant coupled PKS–NS model was introduced by Tuval et al. [30],

$$\begin{cases} \partial_t n + u\cdot\nabla n + \nabla\cdot(n\nabla c) = \Delta n, \\ \partial_t c + u\cdot\nabla c = \Delta c - nf(c), \\ \partial_t u + (u\cdot\nabla)u + \nabla p = \Delta u + n\nabla\phi, \quad \nabla\cdot u = 0. \end{cases}$$

Here the chemicals (oxygen) are transported by the fluid stream u and are consumed at a rate of $f(c) > 0$. Due to buoyancy, the cells exert force $n\nabla\phi$ on the fluid. Since the chemicals are consumed along the dynamics, one expects that the cell density will not concentrate to form finite-time singularities. However, the parabolic nature of the chemical evolution makes the analysis challenging. In the papers [32], [33], [34], [36], global regularity, long-time behavior, and the Leray structure of the system are explored in detail.

In this paper, we study the critical-mass threshold, below which the solutions of the system (1.1) are guaranteed to exist for all finite time. The main advantage of the proposed model (1.1) is that it possesses a naturally decreasing free energy,

$$E[n,u] := \int_{\mathbb{R}^2} n\log n - \frac{1}{2}nc + \frac{1}{2}|u|^2 dx. \tag{1.2}$$

Moreover, since the vector field u is divergence-free, the density equation for n possesses a divergence structure and hence preserves the L^1 norm.

For the whole plane, we prove the following theorem.

THEOREM 1 (plane $\mathbb{R}^2$ case). *Consider solutions* (n, u) *to* (1.1) *subject to initial conditions* $(n_0, u_0) \in H^s(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$, $s \geq 3$, *and* $n_0(1 + |x|^2) \in L^1(\mathbb{R}^2)$. *If the initial mass is strictly less than* 8π,

$$M := ||n_0||_{L^1(\mathbb{R}^2)} < 8\pi,$$

then there exists a constant C, *which depends on the initial data, such that the following estimate holds:*

$$||n(t)||_{H^s} + ||u(t)||_{H^s} \leq C(n_0, u_0, \delta)e^{\delta t} \quad \forall t \in [0, \infty), \tag{1.3}$$

where $0 < \delta$ *is an arbitrary small constant. Therefore, the strong solutions* (n, u) *exist on an arbitrary finite-time interval* $[0, T]$ $\forall$ $T < \infty$.

Remark 1. To our knowledge, this is the first critical-mass result in the coupled PKS–NS systems.

Remark 2. The exponential bounds stated in the theorem might not be optimal. We conjecture that the solutions subject to subcritical mass are uniformly bounded in time.

In the radially symmetric setting, the long-time behavior of the solutions is better understood. We will show that the chemotactic blow-up occurs if the initial density n_0 has total mass $||n_0||_{L^1} > 8\pi$ and has finite second moment (Corollary 1). On the other hand, if the total mass is strictly less than 8π, and the initial second moment is finite, then the L^2 norm of the solutions $(n, \text{curl}u)$ decay to zero as time approaches infinity with algebraic rate (Theorem 7).

Remark 3. Extending Theorem 1, which concerns the parabolic-elliptic Patlak–Keller–Segel–Navier–Stokes system, to the fully parabolic setting is both interesting and challenging.

One of the main obstacles to obtaining uniform-in-time bounds on the solutions is the lack of control over the second moment. To properly illustrate that this is the only obstacle, we choose to study the model (1.1) on torus $\mathbb{T}^2$ and show that under the same subcritical-mass constraint, the solutions are uniformly bounded in time. To this end, due to its compatibility with the boundary conditions involved, we have to adjust (1.1) accordingly. Here we specified the equation on the torus $\mathbb{T}^2$:

$$\begin{cases} \partial_t n + u \cdot \nabla n + \nabla \cdot (n \nabla c) = \Delta n, \\ -\Delta c = n - \overline{n}, \quad \overline{n} = \dfrac{1}{|\mathbb{T}^2|} \displaystyle\int_{\mathbb{T}^2} n dx, \\ \partial_t u + (u \cdot \nabla) u + \nabla p = \Delta u + n \nabla c, \quad \nabla \cdot u = 0, \\ n(t = 0, x) = n_0(x), \quad u(t = 0, x) = u_0(x), \quad x \in \mathbb{T}^2. \end{cases} \tag{1.4}$$

Without loss of generality, we assume that the size of the torus is $|\mathbb{T}| = 1$. The chemical c is determined by $c(x) = -\int_{\mathbb{T}^2} B_{\mathbb{T}^2}(x, y)n(y)dy$, where $B_{\mathbb{T}^2}(x, y)$ is the Green's function of the Laplacian Δ on the torus $\mathbb{T}^2$.

The second main theorem of the paper describes the global well-posedness of equations (1.4).

THEOREM 2 (torus $\mathbb{T}^2$ case). *Consider the solution to* (1.4) *subject to* H^s *initial data* $(n_0, u_0) \in H^s(\mathbb{T}^2) \times (H^s(\mathbb{T}^2))^2$, $s \geq 3$. *If the initial mass* $M := ||n_0||_{L^1(\mathbb{T}^2)}$ *is strictly less than* 8π, *i.e.,* $M < 8\pi$, *then the solution* (n, u) *has a uniform-in-time bounded* H^s *Sobolev norm, i.e.,*

$$||n||_{L_t^\infty([0,\infty);H^s)} + ||u||_{L_t^\infty([0,\infty);H^s)} \leq C_{H^s}(||n_0||_{H^s}, ||u_0||_{H^s}) < \infty.$$

Remark 4. We comment that similar uniform-in-time bounds are obtained in the parabolic-parabolic setting given that the total mass is small enough [35].

1.1. Ideas of the proof. We discuss the idea behind Theorem 1. Recall the free energy E for the system (1.1) and the second moment V,

$$V[n] := \int_{\mathbb{R}^2} n|x|^2 dx. \tag{1.5}$$

The existence of a decreasing free energy is crucial to obtaining sharp critical-mass results in PKS-type equations. We recall that for the classical PKS equation ($u \equiv 0$), there exists a dissipative free energy,

$$E_{\text{classic}} = \int_{\mathbb{R}^2} n \log n - \frac{1}{2} ncdx.$$

However, if the fluid transport structure is introduced in the cell density evolution equation, the classical free energy will no longer decay in general. This is one of the main difficulties in analyzing the coupled PKS–NS systems. However, our coupled system (1.1) possesses a new dissipative free energy (1.2). This is the main content of the next lemma.

LEMMA 1. *Consider regular solutions* (n, u) *to* (1.1)*. Further assume that* $(n, u) \in \text{Lip}_t([0, T]; H^s_x(\mathbb{R}^2) \times (H^s_x(\mathbb{R}^2))^2)$*,* $s \geq 3$*, and* $n(1 + |x|^2) \in L^\infty_t([0, T]; L^1_x(\mathbb{R}^2))$*. Then the free energy* (1.2) *is dissipated along the dynamics* (1.1)*, i.e.,*

$$\begin{aligned} &E[n(t), u(t)] \\ &= E[n_0, u_0] - \int_0^t \int_{\mathbb{R}^2} n|\nabla \log n - \nabla c|^2 dxds - \int_0^t \int_{\mathbb{R}^2} |\nabla u|^2 dxds \quad \forall t \in [0, T]. \end{aligned} \tag{1.6}$$

Proof. Direct calculation using integration by parts and a divergence-free condition of u yields that

$$\begin{aligned} \frac{d}{dt}E = &- \int_{\mathbb{R}^2} n|\nabla \log n - \nabla c|^2 dx - \int_{\mathbb{R}^2} nu \cdot \nabla cdx \\ &- \int_{\mathbb{R}^2} |\nabla u|^2 dx - \int_{\mathbb{R}^2} u \cdot ((u \cdot \nabla)u)dx - \int_{\mathbb{R}^2} u \cdot \nabla pdx + \int_{\mathbb{R}^2} nu \cdot \nabla cdx \\ = &- \int_{\mathbb{R}^2} n|\nabla \log n - \nabla c|^2 dx - \int_{\mathbb{R}^2} |\nabla u|^2 dx \leq 0. \end{aligned}$$

Here, in the last line we apply the relation that

$$\int_{\mathbb{R}^2} u \cdot ((u \cdot \nabla)u)dx = \int_{\mathbb{R}^2} u \cdot \nabla \left(\frac{|u|^2}{2}\right) dx = 0.$$

Now integration in time yields (1.6). ∎

Before utilizing the dissipative free energy to derive global well-posedness of the solutions, we present the following local well-posedness result, whose proof will be postponed to the appendix.

THEOREM 3 (local well-posedness). *Consider the solutions to* (1.1) *subject to* H^s *initial data, i.e.,* $(n_0, u_0) \in H^s(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$*,* $s \geq 3$*. There exists a small constant* $\epsilon = \epsilon(||n_0||_{L^1 \cap H^1}, ||u_0||_{H^1})$ *such that the Sobolev* H^s *norms of the solutions are bounded on the time interval* $[0, \epsilon]$*,*

$$||n(t)||_{H^s} + ||u(t)||_{H^s} < \infty \quad \forall t \in [0, \epsilon].$$

Next we recall from the classical PKS literature that the entropy bound of the solution is essential to propagating higher regularities of solutions; see, e.g., [4], [3]. We present here a similar criterion which guarantees propagation of regularity.

THEOREM 4. *Consider solution* (n, u) *to* (1.1) *subject to initial conditions* $(n_0, u_0) \in H^s(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$, $s \geq 3$, $n_0(1 + |x|^2) \in L^1(\mathbb{R}^2)$. *If the positive part of the entropy is bounded, i.e.,*

$$S^+[n(t)] := \int_{\mathbb{R}^2} n(t,x) \log^+ n(t,x) dx \leq C_{L\log L} < \infty \quad \forall t \in [0, T], \tag{1.7}$$

and the energy of the fluid u *is bounded, i.e.,*

$$||u(t)||_2^2 \leq C_{u;L^2}^2 < \infty \quad \forall t \in [0, T], \tag{1.8}$$

then the solution has bounded H^s, $s \geq 3$, *norms on the same time interval,*

$$||n(t)||_{H^s} + ||u(t)||_{H^s} \leq C_{H^s}(C_{L\log L}, C_{u;L^2}, ||n_0||_{H^s}, ||u_0||_{H^s}) < \infty \quad \forall t \in [0, T].$$

We recall the standard procedure for checking the criterion (1.7) for the classical PKS equations. In the subcritical regime, i.e., $||n_0||_1 < 8\pi$, combining the decaying free energy (1.6) and the logarithmic Hardy–Littlewood–Sobolev inequality (2.24) yields the uniform-in-time bound on the entropy,

$$\begin{aligned}
\sup_t S[n(t)] :=& \sup_t \int_{\mathbb{R}^2} n(t,x) \log n(t,x) dx \\
=& \sup_t \left(\int_{\mathbb{R}^2} n(t,x) \log^+ n(t,x) dx - \int_{\mathbb{R}^2} n(t,x) \log^- n(t,x) dx \right) \\
=:& \sup_t (S^+[n(t)] - S^-[n(t)]) < \infty.
\end{aligned}$$

Here $\log^+$, $\log^-$ denote the positive and the negative parts of the logarithmic function, respectively. As a result, we observe that as long as the negative component of the entropy $S^-[n]$ is bounded, the criterion (1.7) is checked. It is classical to apply the second moment V bound (1.5) to estimate the negative part of the entropy $S^-[n]$ (see, e.g., inequality (2.25)). We summarize the above heuristics in the next theorem, with consideration of our system.

THEOREM 5. *Consider solutions* (n, u) *to* (1.1) *on the time interval* $[0, T]$, *subject to initial conditions* $(n_0, u_0) \in (H^s(\mathbb{R}^2), (H^s(\mathbb{R}^2))^2)$, $s \geq 3$, $n_0(1 + |x|^2) \in L^1(\mathbb{R}^2)$. *If the initial mass is strictly less than* 8π,

$$M := ||n_0||_{L^1(\mathbb{R}^2)} < 8\pi,$$

and the second moment is bounded on the time interval $[0, T]$,

$$V[n(t)] \leq C_V < \infty \quad \forall t \in [0, T], \tag{1.9}$$

then the entropy bound (1.7) *and the energy bound* (1.8) *hold, i.e.,*

$$\int_{\mathbb{R}^2} n(t,x) \log^+ n(t,x) dx + ||u(t)||_2^2 \leq C(C_V, M, E[n_0, u_0]) < \infty \quad \forall t \in [0, T].$$

The condition (1.9) can be easily checked for the following two cases: (a) solutions on the bounded domain $\mathbb{T}^2$ (Theorem 2), and (b) radially symmetric solutions on $\mathbb{R}^2$.

COROLLARY 1 (plane $\mathbb{R}^2$, radially symmetric solutions). *Consider* (1.1) *subject to H^s radially symmetric initial data $(n_0, u_0) \in H^s(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$, $s \geq 3$. Further assume that the second moment is finite $\int_{\mathbb{R}^2} n_0|x|^2 dx < \infty$. If the initial mass $M := ||n_0||_{L^1(\mathbb{R}^2)}$ is strictly less than 8π, i.e., $M < 8\pi$, then the solution (n, u) has a bounded H^s Sobolev norm for any finite time $t < \infty$. On the other hand, if the total mass of the initial density n_0 is greater than 8π, i.e., $||n_0||_{L^1(\mathbb{R}^2)} > 8\pi$, then the solution (n, u) blows up in finite time.*

However, it is difficult to apply Theorem 5 to general solutions to (1.1) on the plane $\mathbb{R}^2$ because controlling second moment (1.9) requires $||u||_\infty$ information, which is typically missing in the a priori estimates. Here we develop a new method for checking criterion (1.7).

We modify the free energy E (1.2) so that the new negative component of the entropy $S^-[n]$ is bounded in terms of the L^1 norm of the density n. As a result, there is no need for the second moment control. To this end, we replace the logarithmic function by its degree two Taylor approximation when the argument n is smaller than the designated threshold. The drawback is that the modified free energy can potentially grow slowly. However, this is enough to derive the $S^+[n]$ bound for any finite time. As a result, we end up with the exponential bounds with arbitrarily small growth rate in the H^s Sobolev norms. Uniform-in-time bounds on the solutions are still open. Details of this modified free energy can be found in section 2.

THEOREM 6. *Consider regular solutions to* (1.1)*, subject to initial conditions $(n_0, u_0) \in H^s(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$, $s \geq 3$, $n_0(1 + |x|^2) \in L^1(\mathbb{R}^2)$. If the initial mass is strictly less than 8π,*

$$M := ||n_0||_{L^1(\mathbb{R}^2)} < 8\pi,$$

then the entropy bound (1.7) *and the energy bound* (1.8) *hold on any finite-time interval $[0, T] \subset [0, \infty)$. Moreover, for any small constant $\delta > 0$, there exists a constant $C(E[n_0, u_0], M, \delta)$ such that*

$$S^+[n(t)] + ||u(t)||_2^2 \leq C(E[n_0, u_0], M, \delta) + \delta t \quad \forall t \in [0, \infty).$$

From the linearly growing bound on the positive component of the entropy $S^+[n]$ and the energy $||u(t)||_2^2$, one can derive the exponential-in-time bound on the H^s-Sobolev norms (1.3) through standard energy estimates. This concludes the proof of Theorem 1.

In general, the long-time asymptotic behavior of the solution to (1.1) is not clear. However, for radially symmetric solutions, we have the following description.

THEOREM 7. *Consider radially symmetric solutions to* (1.1) *subject to the subcritical-mass constraint $||n_0||_1 < 8\pi$ and the conditions in Corollary* 1*. The L^2-norms of the solutions undergo polynomial decay in the sense that*

$$||n(t)||_{L^2}^2 + ||\mathrm{curl} u(t)||_{L^2}^2 \leq \frac{C}{1 + 2t} \quad \forall t \in [0, \infty),$$

where C is a constant depending on the initial data.

Remark 5. By applying the same argument as in the proof of Theorem 4, we obtain that the H^s norms of the solutions are uniformly bounded in time.

The paper is organized as follows. In section 2, we treat the planar case and prove Theorems 1, 4, 5, and 7 and Corollary 1. In section 3, we treat the torus case and prove Theorem 2.

Notation. Throughout the paper, the constants B, C are changing from line to line. However, the constants $C_{(\cdot)}$, e.g., C_{L^2}, $C_{L\log L}$, will be defined and fixed unless otherwise stated. An exception to this rule are the constants C_{GNS} and C_N; they are the constants appearing in the Gagliardo–Nirenberg–Sobolev inequalities and the Nash inequalities and are changing from line to line.

We denote $\mathbb{P}$ as the Leray projection, i.e.,

$$\mathbb{P}u = u - \nabla\Delta^{-1}(\nabla\cdot u). \tag{1.10}$$

Here the operator should be understood as the pseudodifferential operators. Explicitly speaking, for vector field $u = (u^1, u^2)$, we have

$$\mathbb{P}u^i(x) = \left(\sum_{j=1}^{2}\left(\delta^i_j - \frac{k_i k_j}{|k|^2}\right)\widehat{u^j}(k)\right)^\vee, \quad i \in \{1,2\},$$

where $\widehat{(\cdot)}$ and the $(\cdot)^\vee$ denote the Fourier transform and inverse transform on the plane $\mathbb{R}^2$ or the torus $\mathbb{T}^2$, respectively, and the δ^i_j is the Kronecker delta function. Further properties of the Leray projection are that it is a self-adjoint Fourier multiplier and a continuous map from L^2 to L^2. Now we define the Stokes operator as $\mathbb{P}(-\Delta)$. Furthermore, we define the bilinear form

$$B(u,v) = \mathbb{P}((u\cdot\nabla)v).$$

Properties of these operators can be found in classical literature; e.g., see Chapter 2 of [21].

The following multi-index notation is adopted:

$$\partial^\alpha_x = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}, \quad |\alpha| = |\alpha_1| + |\alpha_2|.$$

Moreover, we denote $\beta < \alpha$ if $\beta_1 \le \alpha_1$, $\beta_2 \le \alpha_2$, and at least one of the inequalities is strict.

Recall the classical L^p norms and Sobolev H^s norms,

$$||f||_{L^p_x} = ||f||_p = \left(\int |f|^p dx\right)^{1/p}; \quad ||f||_{L^q_t([0,T];L^p_x)} = \left(\int_0^T ||f(t,x)||^q_{L^p_x} dt\right)^{1/q};$$

$$||f||_{H^s_x} = \left(\sum_{|\alpha|\le s} ||\partial^\alpha_x f||^2_{L^2_x}\right)^{1/2}; \quad ||f||_{\dot{H}^s_x} = \left(\sum_{|\alpha|=s} ||\partial^\alpha_x f||^2_{L^2_x}\right)^{1/2};$$

$$||\nabla^i f||_{L^2} = \left(\sum_{|\alpha|=i} ||\partial^\alpha_x f||^2_{L^2}\right)^{1/2}.$$

2. Planar case: $\mathbb{R}^2$.

This section is organized as follows. We first prove Theorem 4. The proof will serve as a prototype for our later analysis on the torus $\mathbb{T}^2$. Next we prove Theorem 5, which assumes that the cell density n has a bounded second moment on the time interval $[0,T]$. Then we prove Corollary 1 by showing that the second moment bound (1.9) is checked in the radially symmetric setting. Finally, we introduce the modified free energy to prove Theorem 1.

Proof of Theorem 4. In this proof, we focus on deriving the a priori estimates for the H^s, $s \geq 3$, Sobolev norms of the solutions (n, u). Then by a standard limiting procedure and contraction mapping argument, one can deduce the existence and uniqueness of the solutions to (1.1). The proof is decomposed into steps.

Step 1. *L^p estimate of the density* n. First, we recall that due to the divergence structure of the cell density equation in (1.1), the total mass of the cells is conserved along the dynamics. Therefore, we set $M := ||n(t)||_1 = ||n_0||_1$. In order to estimate the L^p, $p > 1$, norm of the density n, we decompose it as follows:

$$n = (n - K)_+ + \min\{n, K\}, \quad K > 1.$$

Since $\min\{n, K\}$ has a bounded L^p norm, it is enough to estimate the size of $(n-K)_+$. To this end, define the following quantity:

$$\eta_K := \int_{\mathbb{R}^2} (n - K)_+ dx.$$

Since the positive part of the entropy is bounded on the interval $[0, T]$ (1.7), direct estimation yields that

$$\eta_K \leq \int_{\mathbb{R}^2} (n - K)_+ \frac{\log^+ n}{\log K} dx \leq \frac{C_{L \log L}}{\log K}. \tag{2.1}$$

As a result, if we choose the vertical cut-off level K large enough, the η_K can be made arbitrarily small. Next, we combine the smallness of η_K (2.1), the divergence-free condition of the fluid vector field u, the Gagliardo–Nirenberg–Sobolev inequality, and the Nash inequality to estimate the time evolution of the L^2 norm of the truncated density $(n - K)_+$ as follows:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||(n-K)_+||_2^2 \leq & -\int |\nabla (n-K)_+|^2 dx \\ & + \frac{1}{2}\int (n-K)_+^3 dx + \frac{3}{2}K\int (n-K)_+^2 dx + K^2 M \\ & \leq -(1 - C_{GNS}\eta_K)||\nabla(n-K)_+||_2^2 + 2K||(n-K)_+||_2^2 + K^2 M \\ & \leq -\frac{1}{2}||\nabla(n-K)_+||_2^2 + 2K||(n-K)_+||_2^2 + K^2 M \\ & \leq -\frac{1}{2C_N M^2}||(n-K)_+||_2^4 + 2K||(n-K)_+||_2^2 + K^2 M. \end{aligned} \tag{2.2}$$

As a result, we see that

$$\begin{aligned} ||n(t)||_2 & \leq ||(n(t) - K)_+||_2 + ||\min\{n(t), K\}||_2 \\ & \leq C(||n_0||_2, C_N, M, K) + K^{1/2} M^{1/2} \quad \forall t \in [0, T]. \end{aligned}$$

Since in the estimation above we choose K such that

$$\frac{C_{L \log L}}{\log K} \leq \frac{1}{2C_{GNS}},$$

we have that K can be any constant greater than $\exp\{2C_{GNS}C_{L \log L}\}$. To conclude, we have that

$$||n(t)||_2 \leq C_{L^2}(||n_0||_2, M, C_{L \log L}) < \infty \quad \forall t \in [0, T]. \tag{2.3}$$

Direct estimation of the time evolution of the L^4 norm of the cell density n with the L^2 bound on the cell density n (2.3), the Gagliardo–Nirenberg–Sobolev equality, and the Nash inequality yields

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}||n||_4^4 &\leq -\frac{3}{4}||\nabla(n^2)||_2^2 + \frac{3}{4}||n^2||_{5/2}^{5/2}\\ &\leq -\frac{3}{4}||\nabla(n^2)||_2^2 + C_{GNS}||\nabla(n^2)||_2^{1/2}||n^2||_2^2\\ &\leq -\frac{||n^2||_2^4}{C_N||n^2||_1^2} + C_{GNS}||n^2||_2^{8/3}\\ &\leq -\frac{||n||_4^8}{C_N C_{L^2}^4} + C_{GNS}||n||_4^{16/3}.\end{aligned}$$

Therefore we obtain that

$$||n(t)||_4 \leq C_{L^4}(||n_0||_4, C_{L^2}(||n_0||_2, M, C_{L\log L})) < \infty \quad \forall t \in [0,T].$$

Combining Morrey's inequality, the Calderon–Zygmund inequality, and the L^p bounds of the density n (2.5) yields that

$$||\nabla c(t)||_{L^\infty(\mathbb{R}^2)} \leq C\,\|n(t)\|_{L^3(\mathbb{R}^2)} \leq C_{\nabla c;\infty}(C_{L^4}, M) < \infty \quad \forall t \in [0,T]. \tag{2.4}$$

Since the vector field u is divergence-free, the fluid transport term $u \cdot \nabla n$ has no impact on the direct L^p energy estimate on the cell density n. Now by the standard Moser–Alikakos iteration, we have that there exists a finite constant $C_{1,\infty}$ such that the L^p norms are bounded as follows:

$$||n(t)||_{L^1\cap L^\infty} \leq C_{1,\infty}(||n_0||_{L^1\cap L^\infty}, C_{L\log L}) < \infty \quad \forall t \in [0,T]. \tag{2.5}$$

For the iteration argument in the classical PKS equation setting, we refer the readers to the Lemma 3.2 in [5] or the paper [19]. For the PKS equation subject to ambient divergence-free vector fields, we refer the reader to the appendix of [18].

Step 2. *H^s estimate of the density n and the velocity u.* Before estimating the $\dot{H}^1$ norms of the solutions (n,u), we present two estimates on the chemical gradient ∇c. Combining the L^p boundedness of the Riesz transform for $p \in (1,\infty)$ on $\mathbb{R}^2$ and the L^p bounds of the density n (2.5) yields that

$$\begin{aligned}||\nabla^2 c||_2 &= ||\nabla^2(-\Delta)n||_2 \leq C||n||_2 \leq CC_{1,\infty},\\ ||\nabla^2 c||_4 &= ||\nabla^2(-\Delta)n||_4 \leq C||n||_4 \leq CC_{1,\infty}.\end{aligned} \tag{2.6}$$

After these preparations, we first estimate the $\dot{H}^1$ norm of the velocity fields u. We apply the Leray projection $\mathbb{P}$ (1.10) on the fluid equation (1.1) to eliminate the pressure term and end up with

$$\partial_t u + B(u,u) = \Delta u + \mathbb{P}(n\nabla c), \quad B(u,u) := \mathbb{P}((u\cdot\nabla)u). \tag{2.7}$$

Here we use the fact that $\mathbb{P}u = u$ since u is divergence-free. Moreover, since the symbol of $\mathbb{P}$ is bounded, the projection $\mathbb{P}$ maps L^2 space to L^2 space. We also recall the following classical identity: for divergence-free $u \in L^2 \cap H^2$,

$$\int B(u,u)\cdot\Delta u dx = 0. \tag{2.8}$$

The proof of the identity, which involves the stream function of u, can be found in [21, Lemma 2.1.16]. The H^2-regularity required by this equality is guaranteed by the local well-posedness theorem 3. Now we estimate the time evolution of the $\dot{H}^1$ seminorm of the velocity u with the equality (2.8), the divergence-free condition of u, the self-adjoint property of $\mathbb{P}$, the Gagliardo–Nirenberg–Sobolev inequality, the chemical gradient estimates (2.4) and (2.6), and the L^p controls of the cell density n (2.5) as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{2}||\partial_{x_j}u||_2^2 =& -\sum_{j=1}^{2}\sum_{k=1}^{2}\int|\partial_{x_k}\partial_{x_j}u|^2dx\\ &-\sum_{j=1}^{2}\int\partial_{x_j}B(u,u)\cdot\partial_{x_j}udx+\sum_{j=1}^{2}\int\partial_{x_j}\mathbb{P}(n\nabla c)\cdot\partial_{x_j}udx\\ \leq& -\frac{1}{2}||\nabla^2u||_2^2+C||\nabla^2u||_2||n||_2||\nabla c||_\infty\\ \leq& -\frac{||\nabla u||_2^4}{2C_{GNS}||u||_2^2}+C||n||_2^2C_{1,\infty}^2.\end{aligned}$$

As a result, we recall the assumption (1.8) and obtain that

$$||\nabla u(t)||_{L_x^2}\leq C_{u;H^1}(C_{u;L^2},||\nabla u_0||_2,||n_0||_{L^1\cap L^\infty})\quad\forall t\in[0,T]. \tag{2.9}$$

Similarly, we estimate the time evolution of the $\dot{H}^1$ seminorm of n using the divergence-free property of u, the Gagliardo–Nirenberg–Sobolev inequality, the chemical gradient estimate (2.4), (2.6), the ∇u bound (2.9), and the L^2 bound of the density n (2.3) as follows:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\nabla n||_2^2\\ &\leq -\frac{1}{2}||\nabla^2n||_2^2+||\nabla n||_4^2||\nabla u||_2+||\nabla^2n||_2||\nabla n||_2||\nabla c||_\infty+||\nabla^2n||_2||n||_4||\nabla^2c||_4\\ &\leq -\frac{1}{2}||\nabla^2n||_2^2+C||\nabla^2n||_2||\nabla n||_2||\nabla u||_2\\ &\quad+||\nabla^2n||_2||\nabla n||_2||\nabla c||_\infty+C||\nabla^2n||_2||n||_2||\nabla n||_2\\ &\leq -\frac{1}{2}||\nabla^2n||_2^2+\frac{1}{4}||\nabla^2n||_2^2+C\left(||\nabla u||_2^2+||\nabla c||_\infty^2+||n||_2^2\right)||\nabla n||_2^2\\ &\leq -\frac{||\nabla n||_2^4}{4C_{GNS}||n||_2^2}+C\left(C_{u;H^1}^2+C_{\nabla c;\infty}^2+C_{L^2}^2\right)||\nabla n||_2^2.\end{aligned}$$

Now by standard ODE theory, we obtain that

$$||\nabla n(t)||_2^2\leq C\left(C_{u;H^1}^2+C_{\nabla c;\infty}^2+C_{L^2}^2\right)C_{L^2}^2+||\nabla n_0||_2^2\quad\forall t\in[0,T].$$

Combining this with (2.3), (2.4), and (2.9) yields

$$\begin{aligned}&||\nabla n(t)||_2+||\nabla u(t)||_2\\ &\leq C_{H^1}(C_{L\log L},C_{u;L^2},||n_0||_{L^1\cap L^\infty},||n_0||_{H^1},||u_0||_{H^1})<\infty\quad\forall t\in[0,T].\end{aligned} \tag{2.10}$$

An iteration argument yields the H^s ($s \geq 2, s \in \mathbb{N}$) estimates. To set up the iteration, we make the assumption

$$\begin{aligned}(2.11)\quad &||n(t)||_{H^{s-1}} + ||u(t)||_{H^{s-1}} \\ &\leq C_{H^{s-1}}(C_{L\log L}, C_{u;L^2}, ||n_0||_{L^1\cap L^\infty}, ||n_0||_{H^{s-1}}, ||u_0||_{H^{s-1}}) < \infty \quad \forall t \in [0,T],\end{aligned}$$

and prove that

$$\begin{aligned}&||n(t)||_{H^s} + ||u(t)||_{H^s} \\ &\leq C_{H^s}(C_{L\log L}, C_{u;L^2}, ||n_0||_{L^1\cap L^\infty}, ||n_0||_{H^s}, ||u_0||_{H^s}) < \infty \quad \forall t \in [0,T].\end{aligned}$$

Since we have already obtained the H^1 bound of the solution (n, u), by iterating this argument one can propagate any H^s-Sobolev norm as long as the conditions (1.7) and (1.8) are satisfied.

We focus on the estimate of the density n first. Applying the density equation (1.1), the time evolution of the $\dot{H}^s$ seminorm of n can be expressed using integration by parts as follows:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=s}||\partial_x^\alpha n||_2^2 + \sum_{|\alpha|=s}||\nabla\partial_x^\alpha n||_2^2 \\ (2.12)\quad &= -\sum_{|\alpha|=s}\int \partial_x^\alpha n \partial_x^\alpha(u\cdot\nabla n)dx - \sum_{|\alpha|=s}\int \partial_x^\alpha n\partial_x^\alpha \nabla\cdot(\nabla c n)dx =: \mathcal{I}_n + \mathcal{II}_n.\end{aligned}$$

Now we estimate the first term $\mathcal{I}_n$ in (2.12). We further decompose it into two parts:

$$\begin{aligned}&\mathcal{I}_n = \sum_{|\alpha|=s}\int(\partial_x^\alpha n)u\cdot\nabla(\partial_x^\alpha n)dx \\ (2.13)\quad &+\sum_{|\alpha|=s}\sum_{(0,0)<\beta\leq\alpha}\begin{pmatrix}\beta_1\\ \alpha_1\end{pmatrix}\begin{pmatrix}\beta_2\\ \alpha_2\end{pmatrix}\int\partial_x^\alpha n(\partial_x^\beta u)\cdot\nabla(\partial_x^{\alpha-\beta}n)dx =: \mathcal{I}_{n;1} + \mathcal{I}_{n;2}.\end{aligned}$$

The divergence-free property of the vector field u and integration by parts yield the vanishing of the first term $\mathcal{I}_{n;1}$ in (2.13), i.e.,

$$(2.14)\quad \mathcal{I}_{n;1} = \sum_{|\alpha|=s}\int u\cdot\nabla\left(\frac{|\partial_x^\alpha n|^2}{2}\right)dx = -\sum_{|\alpha|=s}\int(\nabla\cdot u)\left(\frac{|\partial_x^\alpha n|^2}{2}\right)dx = 0.$$

To estimate the second term $\mathcal{I}_{n;2}$ in (2.13), we first apply the Hölder inequality to obtain that

$$\begin{aligned}\mathcal{I}_{n;2} &\leq \sum_{\substack{(0,0)<\beta\leq\alpha,\\ |\alpha|=s}}\begin{pmatrix}\beta_1\\ \alpha_1\end{pmatrix}\begin{pmatrix}\beta_2\\ \alpha_2\end{pmatrix}\int\partial_x^\alpha n\nabla(\partial_x^{\alpha-\beta}n)\partial_x^\beta u dx \\ &\leq \sum_{\substack{(0,0)<\beta\leq\alpha,\\ |\alpha|=s}}\begin{pmatrix}\beta_1\\ \alpha_1\end{pmatrix}\begin{pmatrix}\beta_2\\ \alpha_2\end{pmatrix}||n||_{\dot{H}^s}||\nabla\partial_x^{\alpha-\beta}n||_{L^p}||\partial_x^\beta u||_{L^q}, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{2}.\end{aligned}$$

Applying the Gagliardo–Nirenberg–Sobolev inequalities yields the bounds

$$||\nabla\partial_x^{\alpha-\beta}n||_{L^p} \le C_{GNS}||n||_{\dot H^{s+1}}^{\theta_1}||n||_{\dot H^1}^{1-\theta_1}, \quad \theta_1 = \frac{|\alpha|-|\beta|+1-\frac{2}{p}}{s};$$
$$||\partial_x^\beta u||_{L^q} \le C_{GNS}||u||_{\dot H^{s+1}}^{\theta_2}||u||_{\dot H^1}^{1-\theta_2}, \quad \theta_2 = \frac{|\beta|-\frac{2}{q}}{s} = 1-\theta_1.$$

Combining these two estimates, we see that the H^1 estimate (2.10) with the previous estimation and the application of Young's inequality yield that

$$\mathcal{I}_{n;2} \le C_{GNS}||n||_{\dot H^s}\left(||n||_{\dot H^{s+1}}+||u||_{\dot H^{s+1}}\right)C_{H^1}.$$

Combining this inequality and the $\mathcal{I}_{n;1}$ estimate (2.14) and the decomposition (2.13) yields the estimate

$$\begin{aligned}\mathcal{I}_n &\le C||n||_{\dot H^s}\left(||n||_{\dot H^{s+1}}+||u||_{\dot H^{s+1}}\right)C_{H^1}\\ &\le \frac{1}{8}||n||_{\dot H^{s+1}}^2+\frac{1}{8}||u||_{\dot H^{s+1}}^2+C(C_{H^{s-1}})||n||_{\dot H^s}^2.\end{aligned} \tag{2.15}$$

This completes the estimation of the integral $\mathcal{I}_n$ in (2.12). Next we estimate the integral $\mathcal{II}_n$ in (2.12) as follows:

$$\mathcal{II}_n = \sum_{|\alpha|=s}\int \nabla(\partial_x^\alpha n)\cdot\partial_x^\alpha(n\nabla c)dx \le C||n||_{\dot H^{s+1}}||n\nabla c||_{\dot H^s}.$$

Now by the product estimate for Sobolev functions, the chemical gradient estimate (2.4), the L^p bound on the cell density (2.5), the assumption (2.11), and the L^2 boundedness of the Riesz transform, we have that

$$\begin{aligned}\mathcal{II}_n &\le C||n||_{\dot H^{s+1}}(||n||_{H^s}||\nabla c||_{L^\infty}+||\nabla c||_{H^s}||n||_{L^\infty})\\ &\le \frac{1}{8}||n||_{\dot H^{s+1}}^2+C(C_{1,\infty})||n||_{\dot H^s}^2+C(C_{H^{s-1}},C_{1,\infty}).\end{aligned} \tag{2.16}$$

Combining the $\mathcal{I}_n$ estimate (2.15), the $\mathcal{II}_n$ estimate (2.16), and equation (2.12), we obtain that there exists a constant C depending on the H^{s-1} norm of the solution (n,u) (2.11) and the L^p estimate of n (2.5) such that the following inequality holds:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=s}||\partial_x^\alpha n||_2^2+\frac{1}{2}\sum_{|\alpha|=s}||\nabla\partial_x^\alpha n||_2^2\\ &\le \frac{1}{8}||u||_{\dot H^{s+1}}^2+C(C_{H^{s-1}},C_{1,\infty})||n||_{\dot H^s}^2+C(C_{H^{s-1}},C_{1,\infty}).\end{aligned} \tag{2.17}$$

Next we focus on the H^s estimate of u. Direct calculation with the velocity equation (2.7) yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=s}||\partial_x^\alpha u||_2^2+\sum_{|\alpha|=s}||\nabla\partial_x^\alpha u||_2^2\\ &=-\sum_{|\alpha|=s}\int\partial_x^\alpha u\cdot\partial_x^\alpha B(u,u)dx+\int\partial_x^\alpha u\cdot\mathbb{P}\partial_x^\alpha(n\nabla c)dx =: \mathcal{I}_u+\mathcal{II}_u.\end{aligned} \tag{2.18}$$

Now we estimate each term in the decomposition (2.18). For the $\mathcal{I}_u$ term, we decompose it into three terms as follows:

$$\begin{aligned}
\mathcal{I}_u =& \sum_{|\alpha|=s}\int(\partial_x^\alpha u)\cdot((u\cdot\nabla)\partial_x^\alpha u)dx \\
&+\sum_{|\alpha|=s}\sum_{\substack{\beta<\alpha\\|\beta|\geq1}}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\int(\partial_x^\alpha u)\cdot((\partial_x^\beta u\cdot\nabla)\partial_x^{\alpha-\beta}u)dx \\
&+\sum_{|\alpha|=s}\int\partial_x^\alpha u\cdot((\partial_x^\alpha u\cdot\nabla)u)dx =: \mathcal{I}_{u;1}+\mathcal{I}_{u;2}+\mathcal{I}_{u;3}.
\end{aligned} \tag{2.19}$$

Now we estimate each term in the decomposition (2.19). For the first term, we apply the divergence-free property of the vector field u to obtain

$$\mathcal{I}_{u;1}=\sum_{|\alpha|=s}\int u\cdot\nabla\left(\frac{|\partial_x^\alpha u|^2}{2}\right)dx=0.$$

For the second term, direct application of the Hölder inequality yields that

$$\begin{aligned}
\mathcal{I}_{u;2}\leq&\sum_{|\alpha|=s}\sum_{\substack{\beta<\alpha,\\|\beta|\geq1}}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\int|\partial_x^\alpha u||\partial_x^\beta u||\nabla\partial_x^{\alpha-\beta}u|dx\\
\leq& C\sum_{|\alpha|=s}\sum_{\substack{\beta<\alpha,\\|\beta|\geq1}}||u||_{\dot{H}^s}||\partial_x^\beta u||_p||\nabla\partial_x^{\alpha-\beta}u||_q,\quad \frac{1}{p}+\frac{1}{q}=\frac{1}{2}.
\end{aligned}$$

Now we recall the following Gagliardo–Nirenberg–Sobolev inequalities:

$$\begin{aligned}
||\partial_x^\beta u||_{L^p}\leq C_{GNS}||u||_{\dot{H}^{s+1}}^{\theta_3}||u||_{\dot{H}^1}^{1-\theta_3},\quad &\theta_3=\frac{|\beta|-\frac{2}{p}}{s};\\
||\nabla\partial_x^{\alpha-\beta}u||_{L^q}\leq C_{GNS}||u||_{\dot{H}^{s+1}}^{\theta_4}||u||_{\dot{H}^1}^{1-\theta_4},\quad &\theta_4=\frac{(|\alpha|-|\beta|+1)-\frac{2}{q}}{s}=1-\theta_3.
\end{aligned}$$

Combining these inequalities and the estimation above yields that

$$\mathcal{I}_{u;2}\leq C_{GNS}||u||_{\dot{H}^{s+1}}||u||_{\dot{H}^s}||u||_{\dot{H}^1}.$$

Now we estimate the last term $\mathcal{I}_{u;3}$ in the decomposition (2.19) using the Hölder inequality and the Gagliardo–Nirenberg–Sobolev inequality as follows:

$$\mathcal{I}_{u;3}\leq C\sum_{|\alpha|=s}||u||_{\dot{H}^s}||\partial_x^\alpha u||_{L^4}||\nabla u||_{L^4}\leq C_{GNS}||u||_{\dot{H}^1}||u||_{\dot{H}^{s+1}}||u||_{\dot{H}^s}.$$

Combining the estimations of the $\mathcal{I}_{u;1}$, $\mathcal{I}_{u;2}$, and $\mathcal{I}_{u;3}$ terms above and the decomposition (2.19) and applying Young's inequality yield the following:

$$\mathcal{I}_u\leq\frac{1}{8}||u||_{\dot{H}^{s+1}}^2+C(C_{H^{s-1}},C_{1,\infty})||u||_{\dot{H}^s}^2+C(C_{H^{s-1}},C_{1,\infty}). \tag{2.20}$$

Now we estimate the term $\mathcal{II}_u$ in (2.18) with the product estimate for Sobolev functions, the chemical gradient estimate (2.4), the L^p bound on the cell density n (2.5), the iteration assumption (2.11), the divergence-free property of the vector field u, the fact that projection $\mathbb{P}$ is self-adjoint, and the L^2 boundedness of the Riesz transform as follows:

(2.21)
$$\begin{aligned}\mathcal{II}_u &\leq ||u||_{\dot{H}^s}||n\nabla c||_{\dot{H}^s} \leq C||u||_{\dot{H}^s}(||n||_{H^s}||\nabla c||_\infty + ||n||_\infty||\nabla c||_{H^s})\\ &\leq ||u||^2_{\dot{H}^s} + C||n||^2_{H^s}||\nabla c||^2_\infty + C||n||^2_\infty||\nabla c||^2_{H^s}\\ &\leq ||u||^2_{\dot{H}^s} + C(C_{1,\infty})||n||^2_{\dot{H}^s} + C(C_{H^{s-1}}, C_{1,\infty}).\end{aligned}$$

Combining the estimates for $\mathcal{I}_u$ (2.20) and $\mathcal{II}_u$ (2.21) with the decomposition (2.18), we end up with the estimate on the time evolution of the $\dot{H}^s$ seminorm of vector field u,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=s}||\partial_x^\alpha u||_2^2 + \frac{1}{2}\sum_{|\alpha|=s}||\nabla\partial_x^\alpha u||_2^2\\ &\leq C(C_{H^{s-1}}, C_{1,\infty})(||u||^2_{\dot{H}^s} + ||n||^2_{\dot{H}^s}) + C(C_{H^{s-1}}, C_{1,\infty}).\end{aligned} \tag{2.22}$$

Finally, combining the estimates (2.17) and (2.22), we have that

(2.23)
$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(||u||^2_{\dot{H}^s} + ||n||^2_{\dot{H}^s}\right)\\ &\leq -\frac{1}{4}||n||^2_{\dot{H}^{s+1}} - \frac{1}{4}||u||^2_{\dot{H}^{s+1}} + C(C_{H^{s-1}}, C_{1,\infty})(||n||^2_{\dot{H}^s} + ||u||^2_{\dot{H}^s}) + C(C_{H^{s-1}}, C_{1,\infty}).\end{aligned}$$

Applying the Gagliardo–Nirenberg–Sobolev inequality, we end up with the following:

$$-||f||^2_{\dot{H}^{s+1}} \leq -\frac{||f||^{2+\frac{2}{s}}_{\dot{H}^s}}{C_{GNS}||f||^{\frac{2}{s}}_{L^2}}.$$

Applying this upper bound on the dissipative terms appeared in (2.23) and recalling the L^p estimate (2.5) and the L^2 energy condition of the vector fields u (1.8), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(||u||^2_{\dot{H}^s} + ||n||^2_{\dot{H}^s}) &\leq -\frac{||n||^{2+\frac{2}{s}}_{\dot{H}^s}}{4C_{GNS}C^{\frac{2}{s}}_{1,\infty}} - \frac{||u||^{2+\frac{2}{s}}_{\dot{H}^s}}{4C_{GNS}C^{\frac{2}{s}}_{u;L^2}}\\ &+ C(C_{H^{s-1}}, C_{1,\infty})(||n||^2_{\dot{H}^s} + ||u||^2_{\dot{H}^s}) + C(C_{H^{s-1}}, C_{1,\infty}).\end{aligned}$$

Therefore we have that

$$||n(t)||_{H^s} + ||u(t)||_{H^s} \leq C_{H^s}(||n_0||_{H^s}, ||u_0||_{H^s}, C_{H^{s-1}}, C_{1,\infty}, C_{u;L^2}) < \infty \quad \forall t \in [0,T].$$

This concludes the proof. ∎

Next we prove Theorem 5.

Proof of Theorem 5. The proof involves two steps. First, we estimate the entropy

$$S[n] = \int_{\mathbb{R}^2} n\log n dx.$$

Then we estimate its negative part $S^-[n]$ through the second moment bound. Since $S^+[n] = S[n] + S^-[n]$, these estimates yield the bound on the positive part of the entropy $S^+[n]$.

To estimate the entropy, we combine the decay estimate of the free energy (1.2) and the following logarithmic Hardy–Littlewood–Sobolev inequality (see, e.g., [6]):

THEOREM 8 (logarithmic Hardy–Littlewood–Sobolev inequality). *For all nonnegative functions $f \in L^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1 + |x|^2)$ belong to $L^1(\mathbb{R}^2)$, there exists a constant $C(M)$ such that the following inequality holds:*

(2.24)
$$\int_{\mathbb{R}^2} f \log f dx + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)f(y) \log |x - y| dxdy \geq -C(M), \quad M = \int_{\mathbb{R}^2} f dx > 0.$$

Combining (2.24) and Lemma 1 yields that

$$\begin{aligned} E[n_0, u_0] \geq E[n, u] &= \left(1 - \frac{M}{8\pi}\right) \int_{\mathbb{R}^2} n \log n dx \\ &\quad + \frac{M}{8\pi} \left(\int_{\mathbb{R}^2} n \log n dx + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x) \log |x - y| n(y) dxdy \right) + \frac{||u||_2^2}{2} \\ &\geq \left(1 - \frac{M}{8\pi}\right) S[n] - \frac{M}{8\pi} C(M) + \frac{||u||_2^2}{2}. \end{aligned}$$

As a result, we obtain an a priori bound on the entropy $S[n]$ and on the L^2 norm of the velocity $||u||_2$ for any finite time

$$\frac{||u(t)||_2^2}{2(1 - \frac{M}{8\pi})} + S[n(t)] \leq \frac{E[n_0, u_0] + \frac{M}{8\pi} C(M)}{1 - \frac{M}{8\pi}} \leq C(M, E[n_0, u_0]) < \infty \quad \forall t \in [0, T].$$

Therefore, we obtain the bound on the entropy $S[n]$ and on the energy $||u||_2^2$.

Next we estimate the negative part of the entropy $S^-[n]$. To this end, we recall the inequality

$$\int_{\mathbb{R}^2} g \log^- g dx \leq \frac{1}{2} \int_{\mathbb{R}^2} g|x|^2 dx + \log(2\pi) \int_{\mathbb{R}^2} g dx + \frac{1}{e}, \quad g \geq 0, \tag{2.25}$$

whose proof can be found in Lemma 2.2 of [3]. Since the second moment is assumed to be bounded (1.9), direct application of the inequality yields the estimate

$$||u(t)||_2 + \int_{\mathbb{R}^2} n(t, x) \log^+ n(t, x) dx \leq C(C_V, E[n_0, u_0], M) < \infty,$$

on the interval $[0, T]$. Now all the conditions in Theorem 4 are checked, and this concludes the proof of Theorem 5. ∎

Proof of Corollary 1. It is enough to show that if the initial data $(n_0(x), u_0(x))$ is radially symmetric, then the second moment is bounded for any finite time, i.e.,

$$\int n(t, x)|x|^2 dx \leq \int n_0(x)|x|^2 dx + 4Mt. \tag{2.26}$$

Explicit calculation of the time evolution of the second moment yields that

$$\frac{d}{dt} \int n(t, x)|x|^2 dx = 4M - \frac{1}{2\pi} M^2 - \int x^2 \nabla \cdot (un) dx. \tag{2.27}$$

To estimate the last term in the above equality, we will rewrite it in a different form. To this end, we introduce the stream function of the velocity field u,

$$\phi := \Delta^{-1}\text{curl}u, \quad (-\partial_{x_2}, \partial_{x_1})\phi = u.$$

Since (1.1) preserves radial symmetry, the solutions (n, u) are radially symmetric. As a result, the stream functions ϕ are also radially symmetric, which implies $(x_1\partial_{x_2} - x_2\partial_{x_1})\phi \equiv 0$. Applying these facts, we rewrite the last term in the time evolution of the second moment in the following manner:

$$\int |x|^2\nabla\cdot(un)dx = -2\int x\cdot undx = -2\int x\cdot\nabla^{\perp}\phi ndx = 2\int(x_1\partial_{x_2} - x_2\partial_{x_1})\phi ndx = 0.$$

Combining this and (2.27) yields (2.26). Since the second moment condition (1.9) is checked, Theorem 5 can be applied. This completes the proof of the first part of Corollary 1.

If the total mass is greater than 8π, then by the same argument as above, we observe that

$$\frac{d}{dt}\int n(t,x)|x|^2dx = 4M - \frac{1}{2\pi}M^2 < 0.$$

Hence if the solution (n, u) is regular on the time interval

$$[0, T_\star],\ T_\star := \frac{8\pi}{4M(M-8\pi)}\int n_0|x|^2dx,$$

then the second moment becomes zero at time $T_\star$, which is impossible. Hence the solution must blow up on or before time $T_\star$. This concludes the proof of the second part of Corollary 1. ◻

Now we introduce the modified free energy E_Γ and its properties. We introduce the modified free energy

$$E_\Gamma[n, u] = \int n\Gamma(n) - \frac{nc}{2} + \frac{|u|^2}{2}dx, \tag{2.28}$$

where Γ is defined as

(2.29)
$$\Gamma(n) = \begin{cases} \log n, & n \geq \eta; \\ \log\eta + \eta^{-1}(n-\eta) - \frac{\eta^{-2}}{2}(n-\eta)^2, & n < \eta; \end{cases} \qquad \eta := \eta(\delta, M) = \min\left\{1, \frac{\delta}{M}\right\}.$$

The Γ function is chosen such that it matches log when n is large but is bounded from below when n is small. Here, we have replaced the function $\log(\eta + (n-\eta))$ by its degree two Taylor expansion centered at η when $n < \eta$ and use the original log function when $n \geq \eta$.

The next lemma states that the modified free energy (2.28) grows at most linearly under the dynamics (1.1).

Lemma 2. *The time derivative of the modified free energy $E_\Gamma[n, u]$, defined in* (2.28)*, satisfies the following estimate:*

$$\frac{d}{dt}E_\Gamma[n(t), u(t)] \leq \delta \quad \forall t \in [0, \infty). \tag{2.30}$$

Furthermore, the following quantity is bounded:

$$-\int_{n<1} n\Gamma(n)dx \leq \left(-\log\eta(\delta,M)+\frac{3}{2}\right)M. \tag{2.31}$$

Proof. Taking the time derivative of $E_\Gamma[n(t),u(t)]$ and applying the divergence-free condition of the vector field u and integration by parts yield

(2.32)

$$\begin{aligned}
&\frac{d}{dt}\left(\int n\Gamma(n)-\frac{nc}{2}+\frac{|u|^2}{2}dx\right)\\
&=\int (n)_t(\Gamma(n)-c)dx+\int n(\Gamma(n))_t dx+\int u\cdot u_t dx\\
&=-\int (n\nabla\log n-\nabla cn)\cdot(\Gamma'(n)\nabla n-\nabla c)dx-\int u\cdot\nabla n\Gamma(n)dx+\int \nabla\cdot(un)cdx\\
&-\int \nabla(n\Gamma'(n))\cdot(n\nabla\log n-\nabla cn)dx-\int u\cdot\nabla nn\Gamma'(n)dx-\int |\nabla u|^2dx+\int nu\cdot\nabla cdx\\
&=:\sum_{i=1}^{7}T_i.
\end{aligned}$$

Applying integration by parts, we have that the third term T_3 and the seventh term T_7 in (2.32) cancel each other. Now we consider the second term T_2 and the fifth term T_5. Since the Γ function is finite near the origin, we define the following functions:

$$\mathcal{E}(r)=\int_0^r\Gamma(s)ds,\quad \mathcal{G}(r)=\int_0^r s\Gamma'(s)ds.$$

The second term T_2 and fifth term T_5 can be explicitly calculated using the divergence-free condition $\nabla\cdot u=0$ and integration by parts as follows:

$$\begin{aligned}
T_2&=-\int u\cdot\nabla(\mathcal{E})dx=\int(\nabla\cdot u)\mathcal{E}dx=0;\\
T_5&=-\int u\cdot\nabla(\mathcal{G})dx=\int(\nabla\cdot u)\mathcal{G}dx=0.
\end{aligned}$$

Next we estimate the terms T_1+T_4. Applying the definition of Γ (2.29), the cut-off threshold $\eta=\min\{\frac{\delta}{M^2},1\}$, and the fact that $\Gamma'(n)=2\eta^{-1}-\eta^{-2}n$ for $n\leq\eta$, direct calculation yields the following equality:

$$\begin{aligned}
T_1+T_4=&-\int_{n\geq\eta}(n\nabla\log n-\nabla cn)\cdot\left(\frac{1}{n}\nabla n-\nabla c\right)dx\\
&-\int_{n<\eta}(n\nabla\log n-\nabla cn)\cdot\left(\left(2\eta^{-1}-\eta^{-2}n\right)\nabla n-\nabla c\right)dx\\
&-\int_{n<\eta}\left(2\eta^{-1}-\eta^{-2}n\right)\nabla n\cdot(n\nabla\log n-\nabla cn)dx\\
&+\int_{n<\eta}n\eta^{-2}\nabla n\cdot(n\nabla\log n-\nabla cn)dx.
\end{aligned}$$

Notice the inequality

$$\sup_{n<\eta}\sqrt{\left(-3\eta^{-2}n+4\eta^{-1}\right)n}\leq\frac{2}{\sqrt{3}}<2,$$

which implies

$$
\begin{aligned}
&T_1+T_4\\
&=-\int_{n\geq\eta} n|\nabla\log n-\nabla c|^2dx-\int_{n<\eta}\left(4\eta^{-1}-3\eta^{-2}n\right)|\nabla n|^2dx\\
&\quad+\int_{n<\eta}\sqrt{\left(-3\eta^{-2}n+4\eta^{-1}\right)n}\sqrt{\left(-3\eta^{-2}n+4\eta^{-1}\right)n}\nabla c\cdot\nabla n dx\\
&\quad-\int_{n<\eta} n|\nabla c|^2dx+\int_{n<\eta}\nabla n\cdot\nabla c dx\\
&\quad\leq-\int_{n\geq\eta} n|\nabla\log n-\nabla c|^2dx-\int_{n<\eta}\left(4\eta^{-1}-3\eta^{-2}n\right)|\nabla n|^2dx\\
&\quad+\frac{2}{\sqrt{3}}\int_{n<\eta}\sqrt{\left(-3\eta^{-2}n+4\eta^{-1}\right)n}|\nabla c||\nabla n|dx-\int_{n<\eta} n|\nabla c|^2dx+\int_{n<\eta}\nabla n\cdot\nabla c dx.
\end{aligned}
$$

Completing a square using the 2nd, 3rd, and 4th terms in the last line yields

$$
\begin{aligned}
T_1+T_4\leq&-\int_{n\geq\eta} n|\nabla\log n-\nabla c|^2dx-\frac{2}{3}\int_{n<\eta}\left(4\eta^{-1}-3\eta^{-2}n\right)|\nabla n|^2dx\\
&-\int_{n<\eta}\left(\sqrt{4\eta^{-1}-3\eta^{-2}n}\frac{1}{\sqrt{3}}|\nabla n|-\sqrt{n}|\nabla c|\right)^2dx+\int_{n<\eta}\nabla n\cdot\nabla c dx. \qquad (2.33)
\end{aligned}
$$

Claim. The following estimate holds:

$$\int_{n<\eta}\nabla n\cdot\nabla c dx\leq\delta.$$

To prove the claim, we make the qualitative assumption that $n\in C^\infty(\mathbb{R}^2)\cap H^s(\mathbb{R}^2)$, $s\geq 3$. However, the final estimate will be independent of the higher regularity norms of the densities n and c. We apply the choice of η (2.29) and integration by parts to obtain

$$\int_{n<\eta}\nabla n\cdot\nabla c dx=\int\nabla(\min\{n,\eta\})\cdot\nabla c dx=-\int\min\{n,\eta\}\Delta c dx\leq\int\eta n dx\leq\eta M\leq\delta.$$

Here we applied the equality $\nabla n\mathbf{1}_{n<\eta}=\nabla(\min\{n,\eta\})$ almost everywhere if $n\in W^{1,p}(\mathbb{R}^2)$, for $1<p<\infty$. This is a natural consequence of Exercise 17 in Evans [10, Chapter 5]. To explicitly justify integration by parts, one can use positive the C_c^∞ function to approximate the $W^{1,4/3}$ function $\min\{n,\eta\}$ and the $W^{1,4}$ function ∇c.

Therefore, combining the claim and estimate (2.33), we deduce that

$$T_1+T_4\leq-\int_{n\geq\eta} n|\nabla\log n-\nabla c|^2dx-\frac{2}{3}\int_{n<\eta}\left(4\eta^{-1}-3\eta^{-2}n\right)|\nabla n|^2dx+\delta\leq\delta.$$

This finishes the treatment of all the T_i's in (2.32). Therefore, the estimate (2.30) follows.

Estimate (2.31) follows from the fact that the function Γ is bounded from below by $\log\eta(\delta,M)-\frac{3}{2}$, which is a finite number. This finishes the proof of Lemma 2. ∎

Proof of Theorem 6. We rewrite the approximate free energy so that inequality (2.24) can be applied:

$$\begin{aligned}
E_\Gamma[n_0,u_0]+\delta t &\geq \int n\Gamma(n)dx - \int \frac{nc}{2}dx + \int \frac{1}{2}|u|^2dx\\
&= \int n\log^+ ndx + \int_{n<1} n\Gamma(n)dx\\
&\quad + \frac{1}{4\pi}\iint \log|x-y|n(x)n(y)dxdy + \frac{1}{2}||u||_2^2\\
&= \left(1-\frac{M}{8\pi}\right)\int n\log^+ ndx + \int_{n<1} n\Gamma(n)dx\\
&\quad + \frac{M}{8\pi}\left(\int n\log^+ ndx + \frac{2}{M}\iint \log|x-y|n(x)n(y)dxdy\right) + \frac{1}{2}||u||_2^2.
\end{aligned}$$

Applying the log-Hardy–Littlewood–Sobolev (2.24) and (2.31) yields

$$\begin{aligned}
E_\Gamma[n_0,u_0]+\delta t &\geq \left(1-\frac{M}{8\pi}\right)\int n\log^+ ndx + \int_{n<1} n\Gamma(n)dx - C(M)\frac{M}{8\pi} + \frac{1}{2}||u||_2^2\\
&\geq \left(1-\frac{M}{8\pi}\right)\int n\log^+ ndx - M\log\eta(\delta,M)^{-1} - \frac{3}{2}M - C(M)\frac{M}{8\pi} + \frac{1}{2}||u||_2^2,
\end{aligned}$$

which leads to a bound on the positive part of the entropy $S^+[n(t)]$ and on the fluid energy $||u||_2^2$ for any finite time, i.e.,

$$\left(1-\frac{M}{8\pi}\right)\int n\log^+ ndx + \frac{1}{2}||u||_2^2 \leq E_\Gamma[n_0,u_0] + \delta t + M\log\eta(\delta,M)^{-1} + \frac{3}{2}M + C(M)\frac{M}{8\pi}.$$

This yields that

$$S^+[n(t)] + ||u(t)||_2^2 < C(E_\Gamma[n_0,u_0],M,\delta) + \delta t < \infty \quad \forall t\in[0,\infty).$$

This concludes the proof of Theorem 6. ∎

Proof of Theorem 1. Now we highlight the adjustment in the remaining part of the proof of Theorem 1 and compare it to the proof of Theorem 4. The main adjustment takes place in the proof of the L^2 norm of the cell density (2.3). Since the positive part of the entropy $S^+[n(t)]$ is growing linearly with rate δ, the quantity $\eta_K = ||(n-K)_+||_1$ will not be uniformly bounded on an arbitrarily long interval as in (2.1). To overcome this difficulty, we adjust the vertical cut-off level K as time progresses. Specifically, we fix an arbitrary time interval $[0,T]$ and do estimation on it. First note that on this time interval, we have that

$$S^+[n(t)] + ||u(t)||_2^2 \leq C(E_\Gamma[n_0,u_0],M,\delta) + \delta T < \infty \quad \forall t\in[0,T].$$

Now we choose the vertical cut-off level $K(T)$ such that the quantity $\eta_{K(T)} := ||(n-K(T))_+||_1$ is small in the sense that

$$\eta_{K(T)} \leq \frac{C(E_\Gamma[n_0,u_0],M,\delta)+\delta T}{\log K(T)} \leq \frac{1}{8}C_{GNS},$$

where C_{GNS} is the universal constant that appeared in the L^2 energy estimate (2.2). The resulting $K(T)$ is larger than

$$K(T) \geq \exp\left\{\frac{8C(E_\Gamma[n_0,u_0],M,\delta)}{C_{GNS}}\right\}\exp\left\{\frac{8\delta T}{C_{GNS}}\right\}.$$

Now combining the size of $K(T)$ and a direct L^2 energy estimation on the quantity $(n-K(T))_+$, which is the same as (2.2), yields that

$$\begin{aligned}||n(t)||_2 &\le 2||\min\{n(t),K(T)\}||_2 + 2||(n(t)-K(T))_+||_2\\ &\le 2K(T)^{1/2}M^{1/2} + C(||n_0||_2, M)K(T)^{1/2}\\ &\le C(||n_0||_2, E[n_0,u_0], M, \delta)e^{\frac{4\delta}{C_{GNS}}T} \quad \forall t\in[0,T].\end{aligned}$$

Since the time T is arbitrary, we have that the L^2 norm of n can grow at most exponentially with rate $\frac{4\delta}{C_{GNS}}$. Since δ is arbitrarily small, we abuse the notation and still denote the rate as δ. The remaining part of the proof is similar to the proof of Theorem 4, so we omit the details. This concludes the proof of Theorem 1. ∎

Remark 6. With Theorem 1 proven, we make a comment on the second moment $V[n(t)]$ (1.5). We estimate the time evolution of the second moment as follows:

$$\begin{aligned}\frac{d}{dt}\int n|x|^2dx &= 4M - \frac{1}{2\pi}M^2 - \int x^2\nabla\cdot(un)dx = 4M - \frac{1}{2\pi}M^2 + 2\int x\cdot undx\\ &\le 4M + ||u||_\infty M^{1/2}\left(\int n|x|^2dx\right)^{1/2}.\end{aligned}$$

Since the $||u||_\infty$ is bounded on an arbitrary finite-time interval, the second moment is bounded for any finite time.

In the last part of this section, we consider the long-time behavior of the *radially symmetric solutions* to (1.1) and prove Theorem 7.

To prove Theorem 7, we first rewrite the equation of the velocity in the vorticity form and present some necessary lemmas. Recall that the vorticity

$$\omega = \nabla^\perp\cdot u = \partial_{x_1}u^2 - \partial_{x_2}u^1, \quad \nabla^\perp = (-\partial_{x_2}, \partial_{x_1}),$$

and the velocity u are related through the Biot–Savart law,

$$u(t,x) = \nabla^\perp\Delta^{-1}\omega(t,x) = \frac{1}{2\pi}\nabla^\perp\int_{\mathbb{R}^2}\log|x-y|\omega(t,y)dy =: \nabla^\perp\psi(t,x),$$

where ψ is the stream function. In the vorticity formulation, (1.1) has the following form:

$$\begin{cases}\partial_t n + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c), \quad -\Delta c = n,\\ \partial_t\omega + u\cdot\nabla\omega = \Delta\omega + \nabla^\perp\cdot(n\nabla c), \quad u = \nabla^\perp\Delta^{-1}\omega,\\ n(t=0,x) = n_0(x), \quad \omega(t=0,x) = \omega_0(x), \quad x\in\mathbb{R}^2.\end{cases} \tag{2.34}$$

To show long-time decay of the solution, it is classical to consider the solutions in the self-similar variables,

$$n(t,x) = \frac{1}{R^2(t)}N\left(\log R(t), \frac{x}{R(t)}\right), \; c(t,x) = C\left(\log R(t), \frac{x}{R(t)}\right), \; R(t) = (1+2t)^{1/2};$$

$$\omega(t,x) = \frac{1}{R^2(t)}\Omega\left(\log R(t), \frac{x}{R(t)}\right), \quad \psi(t,x) = \Psi\left(\log R(t), \frac{x}{R(t)}\right).$$

We further consider the new coordinate $\tau := \log R(t)$, $X := \frac{x}{R(t)}$, and rewrite (2.34) in the following form:

$$
\begin{cases} \partial_\tau N - \nabla\cdot(XN) = \Delta N - \nabla\cdot(N\nabla C) - \nabla^\perp\Psi\cdot\nabla N, \quad -\Delta C = N, \\ \partial_\tau \Omega - \nabla\cdot(X\Omega) = \Delta\Omega + \nabla^\perp\cdot(N\nabla C) - \nabla^\perp\Psi\cdot\nabla\Omega, \quad \Omega = \Delta\Psi, \\ N(0,X) = n_0(x), \quad \Omega(0,X) = \omega_0(x). \end{cases} \tag{2.35}
$$

If the solution is radially symmetric, then the second moment of N is uniformly bounded in time. This is the content of the next lemma.

Lemma 3. *Consider radially symmetric solutions* $(N,\Omega) \in \mathrm{Lip}_\tau([0, \frac{1}{2}\log(1+2T)]; H^s(\mathbb{R}^2))$, $s \geq 3$, *to* (2.37) *subject to initial constraints in Corollary* 1. *The second moment of the solution is bounded in time*

$$
\sup_{\tau\in[0,\infty)} \int_{\mathbb{R}^2} N(\tau,X)|X|^2 dX \leq C_{s;V} < \infty. \tag{2.36}
$$

Proof. The calculation is similar to the calculation in Corollary 1. Direct calculation yields that

$$
\begin{aligned} \frac{d}{d\tau}\int N(\tau,X)|X|^2 dX = {}& 4M - \frac{1}{2\pi}M^2 - 2\int N(\tau,X)|X|^2 dX \\ & - \int \nabla\cdot(\nabla^\perp\Psi(\tau,X)N(\tau,X))|X|^2 dX. \end{aligned}
$$

Since the solutions are radially symmetric, the last term is zero. Now we see that the second moment is uniformly bounded in time. ∎

For (2.35), if the solution (N,Ω) is radially symmetric, there is a dissipative free energy,

$$
E_S[N,\Omega] = \int_{\mathbb{R}^2} N\log N - \frac{1}{2}NC + \frac{1}{2}N|X|^2 - \frac{1}{2}\Psi\Omega dX. \tag{2.37}
$$

This is the content of the following lemma.

Lemma 4. *Consider radially symmetric solutions* $(N,\Omega) \in \mathrm{Lip}_\tau([0, \frac{1}{2}\log(1+2T)]; H^s(\mathbb{R}^2))$, $s \geq 3$, *to* (2.37) *subject to initial finite second moment constraint in Corollary* 1. *The free energy* (2.37) *is dissipative in the sense that* $\frac{d}{d\tau}E_S[N(\tau),\Omega(\tau)] \leq 0$.

Proof. Direct calculation yields that

$$
\begin{aligned}
(2.38)\quad &\frac{d}{d\tau}\int N\log N-\frac{1}{2}NC+\frac{1}{2}N|X|^2-\frac{1}{2}\Omega\Psi dX\\
&=\int N_\tau\left(\log N-C+\frac{1}{2}|X|^2\right)dX-\int\Omega_\tau\Psi dX\\
&=\int(\nabla\cdot(N\nabla\log N)-\nabla\cdot(N\nabla C)\\
&\quad+\nabla\cdot(NX)-\nabla\cdot(\nabla^\perp\Psi N))\left(\log N-C+\frac{1}{2}|X|^2\right)dX\\
&-\int(\Delta\Omega+\nabla\cdot(X\Omega)-\nabla\cdot(\nabla^\perp\Psi\Omega)+\nabla^\perp\cdot(N\nabla C))\Psi dX\\
&=-\int N|\nabla\log N-\nabla C+X|^2dX-\int\Omega^2dX\\
&+\int N\nabla^\perp\Psi\cdot(\nabla\log N-\nabla C+X)dX-\int\nabla\cdot(X\Omega)\Psi dX+\int N\nabla C\cdot\nabla^\perp\Psi dX\\
&=:-\int N|\nabla\log N-\nabla C+X|^2dX-\int\Omega^2dX+\sum_{\ell=1}^{3}T_{s;\ell}.
\end{aligned}
$$

By the fact that for the radially symmetric solutions, $\nabla^\perp\Psi$ is perpendicular to the vectors $X,\nabla C,\nabla N$, we have that $T_{s;1}=T_{s;3}=0$. For the $T_{s;2}$ term in (2.38), we have that

$$
\begin{aligned}
T_{s;2}=&\int X_1\partial_{X_1}\Psi(\partial_{X_1X_1}\Psi+\partial_{X_2X_2}\Psi)dX+\int X_2\partial_{X_2}\Psi(\partial_{X_1X_1}\Psi+\partial_{X_2X_2}\Psi)dX\\
=&\frac{1}{2}\int\partial_{X_1}(\partial_{X_1}\Psi)^2X_1dX-\frac{1}{2}\int\partial_{X_1}(\partial_{X_2}\Psi)^2X_1dX\\
&-\frac{1}{2}\int\partial_{X_2}(\partial_{X_1}\Psi)^2X_2dX+\frac{1}{2}\int\partial_{X_2}(\partial_{X_2}\Psi)^2X_2dX\\
=&\ 0.
\end{aligned}
$$

Combining these calculations and the relation (2.38), we have obtained that the free energy is decaying. ∎

Proof of Theorem 7. Since the free energy is bounded and the second moment is bounded (2.36), through a standard argument involving the logarithmic Hardy–Littlewood–Sobolev inequality, which is similar to arguments proving Theorems 5 and 4, we have that the solution N,Ω is uniformly bounded in time, i.e., $||N||_{L^\infty_\tau([0,\infty);L^2_X)}+||\Omega||_{L^\infty_\tau([0,\infty);L^2_X)}\le C_s<\infty$. Now by the relation between the L^2_x norm and the L^2_X norm, we have that

$$
\begin{aligned}
||n(t)||^2_{L^2_x}+||\omega(t)||^2_{L^2_x}\le\frac{1}{R^2(t)}&\left(||N||^2_{L^\infty_\tau([0,\infty);L^2_X)}+||\Omega||^2_{L^\infty_\tau([0,\infty);L^2_X)}\right)\\
&=\frac{1}{1+2t}C_s^2\quad\forall t\in[0,\infty).
\end{aligned}
$$

This concludes the proof of the theorem. ∎

3. Torus case: $\mathbb{T}^2$. Before we start the proof of Theorem 2, we first collect some useful facts. Without loss of generality, we assume that the average of the velocity u is zero, i.e.,

$$\frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2} u_0^i(x)dx = 0, \quad i = 1, 2.$$

The average-zero properties are propagated along the dynamics (1.4). To check this, we calculate the time evolution of the mean using the divergence-free condition of u and the elliptic equation of the chemical c as follows:

$$\begin{aligned}\frac{d}{dt}\int u^i dx &= -\sum_{j=1}^{2}\int u^j\partial_j u^i dx + \int n\partial_i c dx\\ &= \int (\nabla\cdot u)u^i dx + \int \left(-(\partial_1\partial_1 + \partial_2\partial_2)c + \overline{n}\right)\partial_i c dx\\ &= \int \partial_1 c\partial_i\partial_1 c dx + \int \partial_2 c\partial_i\partial_2 c dx = \sum_{j=1}^{2}\int \frac{1}{2}\partial_i(\partial_j c)^2 dx = 0.\end{aligned}$$

As a result, $\overline{u^i} = 0$, $i = 1, 2$, as long as the solution is smooth.

Now we study the 2D free energy of n on $\mathbb{T}^2$:

$$E_{\mathbb{T}^2}[n,u] = \int_{\mathbb{T}^2} n\log n - \frac{1}{2}(n-\overline{n})c + \frac{1}{2}|u|^2 dx. \tag{3.1}$$

LEMMA 5. *Consider the smooth solution to* (1.4)*, the free energy* $E_{\mathbb{T}^2}$ (3.1) *is dissipated along the dynamics, i.e.,*

$$E_{\mathbb{T}^2}[n(t),u(t)] \le E_{\mathbb{T}^2}[n_0,u_0] \quad \forall t\ge 0. \tag{3.2}$$

Proof. Direct calculation of the time derivative of $E[n]$ can be estimated as follows:

$$\begin{aligned}\frac{d}{dt}E_{\mathbb{T}^2}[n,u] &= \int n_t(\log n - c)dx - \int |\nabla u|^2 dx + \int u\cdot\nabla c n dx\\ &= -\int n(\nabla\log n - \nabla c)\cdot(\nabla \log n - \nabla c)dx\\ &\quad + \int \nabla\cdot(un)c dx - \int |\nabla u|^2 dx + \int u\cdot\nabla c n dx\\ &= -\int n|\nabla\log n - \nabla c|^2 dx - \int |\nabla u|^2 dx.\end{aligned}$$

□

The decaying free energy (3.2), together with a suitable logarithmic Hardy–Littlewood–Sobolev inequality, yields a uniform-in-time bound on the positive component of the entropy $S^+[n]$. To explicitly derive the bound, we recall the following logarithmic Hardy–Littlewood–Sobolev inequality on a compact manifold.

THEOREM 9 (see [27]). *Let $\mathcal{M}$ be a two-dimensional, Riemannian, compact manifold. For all $M>0$, there exists a constant $C(M)$ such that for all nonnegative functions $f\in L^1(\mathcal{M})$ such that $f\log f\in L^1$, if $\int_{\mathcal{M}} f dx = M$, then*

$$\int_{\mathcal{M}} f\log f dx + \frac{2}{M}\iint_{\mathcal{M}\times\mathcal{M}} f(x)f(y)\log d(x,y)dxdy \ge -C(M), \tag{3.3}$$

where $d(x,y)$ is the distance on the Riemannian manifold.

Since the logarithmic Hardy–Littlewood–Sobolev inequality (3.3) is stated with respect to the distance on the torus, we cannot directly combine it with the decaying free energy (3.2) here. To overcome this difficulty, we estimate the potential part of the free energy, i.e., $\frac{1}{2}\int(n-\overline{n})cdx$, from below. This is the main content of the next lemma.

LEMMA 6. *There exists a constant $B>0$, such that the following estimate holds:*

$$-\frac{1}{2}\int_{\mathbb{T}^2}(n-\overline{n})cdx = -\frac{1}{2}\int_{\mathbb{T}^2}(n-\overline{n})(-\Delta)^{-1}(n-\overline{n})dx \geq \frac{1}{4\pi}\iint_{\mathbb{T}^2\times\mathbb{T}^2}\log d(y,x)n(y)n(x)dydx - BM^2.$$

Proof. The proof of the lemma is the same as the parallel treatment in the paper [1]. For the sake of completeness, we provide the proof in the appendix. ∎

Combining Lemma 6 with (3.2) yields

$$\begin{aligned}&E_{\mathbb{T}^2}[n_0,u_0]\\ &\geq\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{T}^2}n\log n dx\\ &\quad+\frac{M}{8\pi}\left(\int_{\mathbb{T}^2}n\log ndx+\frac{2}{M}\iint_{\mathbb{T}^2\times\mathbb{T}^2}n(y)\log d(y,x)n(x)dydx\right)+\frac{||u||_2^2}{2}-BM^2.\end{aligned}$$

Applying (3.3) in the above estimate, we obtain

$$E_{\mathbb{T}^2}[n_0,u_0]\geq\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{T}^2}n\log ndx+\frac{||u||_2^2}{2}-C(M)-BM^2,$$

which results in

$$\int_{\mathbb{T}^2}n\log ndx+\frac{1}{2(1-\frac{M}{8\pi})}||u||_2^2\leq\frac{E_{\mathbb{T}^2}[n_0,u_0]+C(M)+BM^2}{1-\frac{M}{8\pi}}.$$

Since the function $s\log s$ is bounded from below, the negative part of the entropy $S^-[n]=\int_{\mathbb{T}^2}n\log^- ndx$ is bounded on the torus. Therefore, there exists a constant $C_{L\log L}$ depending only on the initial data such that the following estimate holds:

$$\int_{\mathbb{T}^2}n\log^+ ndx+||u||_2^2\leq C_{L\log L;L^2}(E_{\mathbb{T}^2}[n_0,u_0],M)<\infty.$$

The estimation above yields the following lemma.

LEMMA 7. *If the total mass is bounded in the sense that $||n_0||_{L^1}<8\pi$, there exists a constant $C_{L\log L}(n_0,u_0)$ such that*

$$\int_{\mathbb{T}^2}n(t,x)\log^+ n(t,x)dx+||u(t)||_{L_x^2}^2\leq C_{L\log L;L^2}(E_{\mathbb{T}^2}[n_0,u_0],M)<\infty\quad\forall t\in[0,\infty). \tag{3.4}$$

As in the plane case, the uniform-in-time bound on the positive part of the entropy $S^+[n]$ yields the bound on the L^p norms. This is the content of the next lemma.

LEMMA 8. *Assume that the entropy is bounded in the sense that (3.4) holds, then there exists a constant $C_{1,\infty}=C_{1,\infty}(n_0,u_0)$ such that the following estimate holds:*

$$||n(t)||_{L^1\cap L^\infty}\leq C_{1,\infty}(||n_0||_{L^1\cap L^\infty},E_{\mathbb{T}^2}[n_0,u_0])<\infty\quad\forall t\in[0,\infty). \tag{3.5}$$

The proof is a small variation of classical PKS techniques (see, e.g., [16], [4]). Before presenting the proof, we recall the following Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{T}^d$: Suppose $v \in H^1(\mathbb{T}^d), d \geq 2$, and $\int v dx = 0$. Assume that $q, r > 0, \infty > q > r$, and $\frac{1}{d} - \frac{1}{2} + \frac{1}{r} > 0$. Then

$$||v||_{L^q} \leq C(d,q)||\nabla v||_{L^2}^a ||v||_{L^r}^{1-a}, \quad \int_{\mathbb{T}^d} v dx = 0, \quad a = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{d} - \frac{1}{2} + \frac{1}{r}}. \tag{3.6}$$

For a fixed d, the constant $C(d,q)$ is bounded uniformly when q varies in any compact set in $(0,\infty)$.

Proof of Lemma 8. We focus on the L^2 estimate. Let $K > \max\{1, \overline{n}\}$ be a constant to be chosen later. Observe that (3.4) implies the following:

$$\int (n-K)_+ dx \leq \int_{n>K} n dx \leq \frac{1}{\log(K)} \int_{n>K} n \log^+(n) dx \leq \frac{C_{L\log L}}{\log(K)}. \tag{3.7}$$

Next, via (1.4) and the divergence-free property of the vector field u, there holds

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (n-K)_+^2 dx \\
&= \int (n-K)_+[\Delta n - \nabla\cdot(n\nabla c)]dx \\
&= -\int |\nabla((n-K)_+)|^2 dx + \frac{1}{2}\int (n-K)_+^3 dx \\
&\quad + \frac{3K-\overline{n}}{2}\int (n-K)_+^2 dx + K^2 - K\overline{n}\int (n-K)_+ dx \\
&\leq -\frac{7}{8}\int |\nabla((n-K)_+)|^2 dx + \frac{1}{2}\int (n-K)_+^3 dx \\
&\quad + \frac{3K-\overline{n}}{2}\int (n-K)_+^2 dx + (K^2 - K\overline{n})M.
\end{aligned} \tag{3.8}$$

We start with the second term in (3.8). Consider the average-zero function $(n-K)_+ - \overline{(n-K)_+}$ on $\mathbb{T}^2$, and apply the Gagliardo–Nirenberg–Sobolev inequality (3.6) to deduce

$$\begin{aligned}
\int |(n-K)_+|^3 dx &\leq C\left(\int |(n-K)_+ - \overline{(n-K)_+}|^3 dx + \overline{(n-K)_+}^3\right) \\
&\leq C_{GNS}\int |\nabla(n-K)_+|^2 dx \int (n-K)_+ dx + CM^3.
\end{aligned}$$

From (3.7), we choose K depending only on $C_{L\log L}$ such that

$$-\frac{7}{8}\int |\nabla((n-K)_+)|^2 dx + \frac{1}{2}\int (n-K)_+^3 dx \leq -\frac{1}{2}\int |\nabla((n-K)_+)|^2 dx + CM^3. \tag{3.9}$$

Plugging (3.9) into (3.8) yields the following for some universal constant $B > 0$:

$$\frac{1}{2}\frac{d}{dt}\int (n-K)_+^2 dx \leq -\frac{1}{2}\int |\nabla((n-K)_+)|^2 dx + KB\int (n-K)_+^2 dx + BK^2 M. \tag{3.10}$$

Recalling the Gagliardo–Nirenberg–Sobolev inequality (3.6), for a general function v on $\mathbb{T}^2$, the following Nash inequality holds:

$$||v||_{L^2(\mathbb{T}^2)}^2 \leq C_N||\nabla v||_{L^2(\mathbb{T}^2)}||v||_{L^1(\mathbb{T}^2)} + C_N||v||_{L^1(\mathbb{T}^2)}^2.$$

Applying the Nash inequality in the estimate (3.10) yields

$$\frac{1}{2}\frac{d}{dt}||(n-K)_+||_2^2 \leq -\frac{1}{2B}\frac{||(n-K)_+||_2^4}{C_N M^2} + \frac{3KB}{2}||(n-K)_+||_2^2 + BK^2M.$$

Further note that

$$\|n\|_{L^2} \leq \|(n-K)_+\|_{L^2} + \|\min\{n,K\}\|_{L^2} \leq \|(n-K)_+\|_{L^2} + K^{1/2}M^{1/2}.$$

The inequality (3.5) hence follows. As in the proof of Theorem 4, we apply energy estimates to derive the L^4 bound on the density n, which in turn implies the chemical gradient $||\nabla c||_\infty$ estimate through Morrey's inequality and the Calderon–Zygmund inequality. Now application of the Moser–Alikakos iteration yields that

$$||n(t)||_{L^\infty} \leq C_{1,\infty}(||n_0||_{L^1\cap L^\infty}, E_{\mathbb{T}^2}[n_0,u_0]) < \infty \quad \forall t \in [0,\infty).$$

This concludes the proof of the theorem. ∎

Next, we prove the higher regularity estimates using (3.5).

LEMMA 9. *Consider the solution to* (1.4). *The following* H^s, $2 \leq s \in \mathbb{N}$, *estimates hold on* $[0,\infty)$*:*

$$\begin{aligned}&||n(t)||_{H^s(\mathbb{T}^2)} + ||u(t)||_{H^s(\mathbb{T}^2)}\\ &\leq C_{H^s}(||n_0||_{H^s}, ||u_0||_{H^s}, C_{1,\infty}(||n_0||_{L^1\cap L^\infty}, E_{\mathbb{T}^2}[n_0,u_0])) < \infty \quad \forall t\in[0,\infty).\end{aligned}$$

Proof. Before proving the lemma, we collect the inequalities we are going to apply. The L^4 boundedness of the Riesz transform on $\mathbb{T}^d$ (see, e.g., [28, Chapter VII, section 3]) yields that

$$\begin{aligned}||\nabla^2 c||_2 &= ||\nabla^2(-\Delta)(n-\overline{n})||_2 \leq C||n-\overline{n}||_2;\\ ||\nabla^2 c||_4 &= ||\nabla^2(-\Delta)(n-\overline{n})||_4 \leq C||n-\overline{n}||_4.\end{aligned} \tag{3.11}$$

Combining Morrey's inequality and the Calderon–Zygmund inequality yields that

$$||\nabla c||_{L^\infty(\mathbb{T}^2)} \leq C\,\|n-\overline{n}\|_{L^3(\mathbb{T}^2)}. \tag{3.12}$$

Now we estimate the time evolution of the $\dot{H}^1$ norm of the velocity u with the identity (2.8), the Gagliardo–Nirenberg–Sobolev inequality, the chemical gradient estimates (3.12) and (3.11), and the L^p, $p \geq 1$, controls of the cell density n (3.5) as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla u||_2^2 &\leq -||\nabla^2 u||_2^2 + ||\nabla^2 u||_2||n||_2||\nabla c||_\infty\\ &\leq -\frac{||\nabla u||_2^4}{4C_{GNS}||u-\overline{u}||_2^2} + C||n||_{L^1\cap L^\infty}^4.\end{aligned}$$

Combining the estimates (3.4), (3.5) yields that

$$||\nabla u(t)||_{L^2} \leq C_{u;H^1}(C_{L\log L;L^2}, C_{1,\infty}, ||\nabla u_0||_2) \quad \forall t\in[0,\infty). \tag{3.13}$$

Similarly, we can estimate the time evolution of the $\dot{H}^1$ norm of n with estimates (3.5), (3.12), (3.13) as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla n||_2^2\\
&\leq -\frac{1}{2}||\nabla^2 n||_2^2 + ||\nabla n||_4^2||\nabla u||_2 + ||\nabla^2 n||_2||\nabla n||_2||\nabla c||_\infty + ||\nabla^2 n||_2||n||_4||\nabla^2 c||_4\\
&\leq -\frac{1}{2}||\nabla^2 n||_2^2 + C||\nabla^2 n||_2||\nabla n||_2||\nabla u||_2\\
&\quad + ||\nabla^2 n||_2||\nabla n||_2||\nabla c||_\infty + C||\nabla^2 n||_2||n||_4||n-\overline{n}||_4\\
&\leq -\frac{1}{2}||\nabla^2 n||_2^2 + \frac{1}{4}||\nabla^2 n||_2^2 + C\left(||\nabla u||_2^2 + ||\nabla c||_\infty^2 + ||n-\overline{n}||_2^2\right)||\nabla n||_2^2 + C\overline{n}^4\\
&\leq -\frac{||\nabla n||_2^4}{4C_{GNS}||n-\overline{n}||_2^2} + C\left(C_{u;H^1}^2 + C_{1,\infty}^2\right)||\nabla n||_2^2 + CM^4.
\end{aligned}$$

Combining the solution to the above differential inequality with (3.5), (3.12), and (3.13), we see that

$$\begin{aligned}
&||\nabla n(t)||_2^2 + ||\nabla u(t)||_2^2 \leq C_{H^1}(||n_0||_{H^1}, ||u_0||_{H^1}, ||n_0||_{L^1\cap L^\infty}, E_{\mathbb{T}^2}[n_0,u_0]) < \infty\\
&\qquad \forall t\in[0,\infty).
\end{aligned}$$

Further iterating this argument yields the H^s, $s \geq 3$, bound. This finishes the proof of Lemma 9. ∎

Appendix A.

Proof of Theorem 3. We prove the local *a priori estimates* of the H^s, $s \geq 3$, norms of the velocity field u and the density n assuming that the solution is smooth. These bounds can be justified through standard approximation procedure. Since the approximation step is classical, we refer the reader to Chapters 6 and 7 of [26] for further details.

We first derive the L^2 estimate of the density n. Recall the equation of the density

$$\partial_t n + u\cdot\nabla n + \nabla\cdot(\nabla c n) = \Delta n. \tag{A.1}$$

We multiply (A.1) by n and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\int n^2 dx + \int u\cdot\nabla\left(\frac{n^2}{2}\right)dx + \int n\nabla\cdot(\nabla c n)dx = -\int|\nabla n|^2 dx. \tag{A.2}$$

Since u is divergence-free, the second term on the left-hand side of (A.2) is zero. For the third term on the left-hand side of (A.2), direct integration by parts yields that

$$\int n\nabla\cdot(\nabla c n)dx = -\int n^3 dx + \int\nabla c\cdot\nabla\left(\frac{n^2}{2}\right)dx = -\frac{1}{2}\int n^3 dx.$$

Combining this equation with (A.2) and applying the Gagliardo–Nirenberg–Sobolev inequality $||f||_3 \leq C_{GNS}||f||_2^{\frac{2}{3}}||\nabla f||_2^{\frac{1}{3}}$ with $f = n$ and Young's inequality yield that

$$\begin{aligned}
\frac{d}{dt}||n||_2^2 &\leq -2\int|\nabla n|^2 dx + \int n^3 dx \leq -2||\nabla n||_2^2 + C_{GNS}||\nabla n||_2||n||_2^2\\
&\leq -\frac{3}{2}||\nabla n||_2^2 + C_{GNS}||n||_2^4 \leq C_{GNS}||n||_2^4.
\end{aligned}$$

By ODE theory, we obtain that there exists a small constant $\epsilon_2 = \epsilon_2(C_{GNS}, ||n_0||_2)$ such that for time t smaller than ϵ_2, i.e., $0 \le t < \epsilon_2$, the L^2 norm of the solution n is bounded:

$$||n(t)||_2^2 \le 2||n_0||_2^2 \quad \forall t \in [0, \epsilon_2]. \tag{A.3}$$

Once the L^2 bound of the density is achieved, we can estimate the L^4 norm of the chemical gradient ∇c on a short time interval $t \in [0, \epsilon_2]$. Applying the Hardy–Littlewood–Sobolev inequality, the Hölder inequality, and Young's inequality, we estimate the chemical gradient ∇c as follows:

$$||\nabla c(t)||_{L^4} \le C_{HLS}||n(t)||_{L^{4/3}} \le C_{HLS}(M + ||n(t)||_{L^2}) \le C(M, ||n_0||_2) \quad \forall t \in [0, \epsilon_2]. \tag{A.4}$$

Next we estimate the L^2 norm of the fluid velocity fields u. Recall the fluid equation after we apply the Leray projection operator $\mathbb{P}$,

$$\partial_t u + \mathbb{P}((u \cdot \nabla)u) = -\mathbb{P}(-\Delta)u + \mathbb{P}(n\nabla c). \tag{A.5}$$

Since the Leray projection is self-adjoint and the vector field u is divergence-free, multiplying (A.5) by u and integrating yield the equality

$$\frac{d}{dt}\frac{1}{2}\int |u|^2 dx + \int u \cdot ((u \cdot \nabla)u)dx = \int u\Delta u dx + \int u \cdot (n\nabla c)dx. \tag{A.6}$$

Due to the divergence-free property of the vector field u, we have that the second term on the left-hand side of (A.6) vanishes. Combining the estimates (A.3) and (A.4), we see that the Hölder inequality and the Gagliardo–Nirenberg–Sobolev inequality yield that the second term on the right-hand side of (A.6) is bounded on the time interval $[0, \epsilon_2]$,

$$\begin{aligned} &\int u(t,x) \cdot (n(t,x)\nabla c(t,x))dx \\ &\le ||\nabla c(t)||_4 ||u(t)||_4 ||n(t)||_2 \\ &\le C(M, ||n_0||_{L^2})||u(t)||_2^{1/2}||\nabla u(t)||_2^{1/2} \quad \forall t \in [0, \epsilon_2]. \end{aligned}$$

Combining these estimates with (A.6), we apply Young's inequality to obtain that

$$\frac{d}{dt}||u(t)||_2^2 \le -\frac{1}{2}||\nabla u||_2^2 + C(M, ||n_0||_{L^2})||u(t)||_2^{2/3} \quad \forall t \in [0, \epsilon_2].$$

Therefore, we have the following local control over $||u(t)||_2$:

$$||u(t)||_2 \le C(M, ||n_0||_{L^2}, ||u_0||_2) < \infty \quad \forall t \in [0, \epsilon_2].$$

Next we estimate the H^1 norm. Before estimating the $\dot{H}^1$ norms, we recall estimates (2.6),

$$||\nabla^2 c||_2 = ||\nabla^2(-\Delta)n||_2 \le C||n||_2, \quad ||\nabla^2 c||_4 = ||\nabla^2(-\Delta)n||_4 \le C||n||_4. \tag{A.7}$$

Now we estimate the time evolution of the $\dot{H}^1$ seminorm of the velocity u (2.7) with the divergence-free condition of u, the self-adjoint property of $\mathbb{P}$, the Gagliardo–Nirenberg–Sobolev inequality, and the chemical gradient estimates (A.4) and (A.7) as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{2}||\partial_j u||_2^2\\
&=-\sum_{j=1}^{2}\int|\nabla\partial_j u|^2dx-\sum_{j=1}^{2}\int\partial_j B(u,u)\cdot\partial_j udx+\sum_{j=1}^{2}\int\partial_j\mathbb{P}(\nabla cn)\cdot\partial_j udx\\
&\leq-\frac{1}{2}||\nabla^2 u||_2^2+C\left(||\nabla u||_2^4+||n||_4^2||\nabla c||_4^2\right)\\
&\leq C||\nabla u||_2^4+C||\nabla n||_2||n||_2(M^2+||n||_2^2).
\end{aligned}$$

Similarly, we estimate the time evolution of the $\dot{H}^1$ seminorm of n using the divergence-free property of u, the Gagliardo–Nirenberg–Sobolev inequality, and the chemical gradient estimates (A.4) and (A.7) as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla n||_2^2\\
&\leq-\frac{1}{2}||\nabla^2 n||_2^2+||\nabla n||_4^2||\nabla u||_2+||\nabla^2 n||_2||\nabla n||_4||\nabla c||_4+||\nabla^2 n||_2||n||_4||\nabla^2 c||_4\\
&\leq-\frac{1}{2}||\nabla^2 n||_2^2+C||\nabla^2 n||_2||\nabla n||_2||\nabla u||_2\\
&\quad+C||\nabla^2 n||_2^{3/2}||\nabla n||_2^{1/2}(M+||n||_2)+C||\nabla^2 n||_2||n||_2||\nabla n||_2\\
&\leq-\frac{1}{4}||\nabla^2 n||_2^2+C||\nabla n||_2^4+C||\nabla u||_2^4+C(1+M+||n||_2)^4||\nabla n||_2^2.
\end{aligned}$$

Combining these estimations on time evolution of $||n||_{\dot{H}^1}^2$ and $||u||_{\dot{H}^1}^2$ with the L^2 bound on the cell density n (2.5) and the assumption on the fluid velocity u (1.8) yields that there exists a universal constant C such that

$$\frac{1}{2}\frac{d}{dt}(||\nabla n||_2^2+||\nabla u||_2^2)\leq C||\nabla u||_2^4+C||\nabla n||_2^4+C(M,||n||_2)||\nabla n||_2^2+C(|n||_2,M).$$

Now by standard ODE theory and the L^2 bound (A.3), we obtain that

$$||\nabla n(t)||_2+||\nabla u(t)||_2\leq C_{H^1}(||n_0||_{L^1},||n_0||_{H^1},||u_0||_{H^1})<\infty\quad\forall t\in[0,\epsilon],$$

for some small enough $\epsilon=\epsilon(||n_0||_{L^1},||n_0||_{H^1},||u_0||_{H^1})$.

Now we can apply the similar procedure shown in the proof of Theorem 4 to gain control over H^s norms of u and n on the interval $[0,\epsilon]$. Following the arguments in Chapter 7 of [26], higher space-time regularity of the solutions can be obtained. This concludes the proof of the theorem. □

Proof of Lemma 6. Let $x\in\mathbb{T}^2$ be fixed. Define the cut-off function $\varphi_x(y)\in C^\infty$ such that

$$\begin{aligned}
\mathrm{supp}(\varphi_x)=&B(x,1/4),\\
\varphi_x(y)\equiv&1\quad\forall y\in B(x,1/8),\\
\mathrm{supp}(\nabla\varphi_x(y))\subset&\overline{B}(x,1/4)\backslash B(x,1/8).
\end{aligned}$$

By extending $n(y)$ and $c(y)$ periodically to $\mathbb{R}^2$, we can rewrite the equation $-\Delta c=n-\overline{n}$ on $\mathbb{T}^2$ such that it is posed on $\mathbb{R}^2$:

$$-\Delta_y(\varphi_x(y)c(y))=(n(y)-\overline{n})\varphi_x(y)-2\nabla_y\varphi_x(y)\cdot\nabla_y c(y)-\Delta_y\varphi_x(y)c(y).$$

Using the fundamental solution of the Laplacian on $\mathbb{R}^2$, we obtain

$$\begin{aligned}
c(x) &= c(x)\varphi_x(x)\\
&=-\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\Big((n(y)-\overline{n})\varphi_x(y)-2\nabla_y\varphi_x(y)\cdot\nabla_y c(y)-\Delta_y\varphi_x(y)c(y)\Big)dy\\
&=-\frac{1}{2\pi}\int_{|x-y|\le\frac14}\log|x-y|(n(y)-\overline{n})\varphi_x(y)dy\\
&\qquad-\frac{1}{\pi}\int_{|x-y|\le\frac14}\nabla_y\cdot(\log|x-y|\nabla_y\varphi_x(y))c(y)dy\\
&\quad+\frac{1}{2\pi}\int_{|x-y|\le\frac14}\log|x-y|\Delta_y\varphi_x(y)c(y)dy.
\end{aligned}$$

Due to the support of φ_x, we can identify the above with an analogous integral on $\mathbb{T}^2$ with $|x-y|$ replaced by $d(x,y)$. Therefore, we have the following estimate on the interaction energy:

$$\begin{aligned}
&-\frac12\int_{\mathbb{T}^2}(n(x)-\overline{n})c(x)dx\\
&\quad=\frac{1}{4\pi}\iint_{\substack{\mathbb{T}^2\times\mathbb{T}^2\\ d(x,y)\le\frac14}}\log d(x,y)(n(x)-\overline{n})(n(y)-\overline{n})\varphi_x(y)dydx\\
&\qquad+\frac{1}{2\pi}\iint_{\substack{\mathbb{T}^2\times\mathbb{T}^2\\ \frac18\le d(x,y)\le\frac14}}(n(x)-\overline{n})\nabla_y\cdot(\log d(x,y)\nabla_y\varphi_x(y))c(y)dydx\\
&\quad-\frac{1}{4\pi}\iint_{\substack{\mathbb{T}^2\times\mathbb{T}^2\\ \frac18\le d(x,y)\le\frac14}}(n(x)-\overline{n})\log d(x,y)\Delta_y\varphi_x(y)c(y)dydx\\
&=\frac{1}{4\pi}\iint_{d(x,y)\le\frac18}\log d(x,y)(n(x)-\overline{n})(n(y)-\overline{n})dydx\\
&\qquad+\frac{1}{4\pi}\iint_{\frac18\le d(x,y)\le\frac14}\log d(x,y)(n(x)-\overline{n})(n(y)-\overline{n})\varphi_x(y)dydx\\
&\quad+\frac{1}{2\pi}\iint_{\frac18\le d(x,y)\le\frac14}(n(x)-\overline{n})\nabla_y\cdot(\log d(x,y)\nabla_y\varphi_x(y))c(y)dydx\\
&\qquad-\frac{1}{4\pi}\iint_{\frac18\le d(x,y)\le\frac14}(n(x)-\overline{n})\log d(x,y)\Delta_y\varphi_x(y)c(y)dydx\\
&=\frac{1}{4\pi}\iint_{\mathbb{T}^2\times\mathbb{T}^2}\log d(x,y)n(x)n(y)dydx-\frac{1}{4\pi}\iint_{d(x,y)>\frac18}\log d(x,y)n(x)n(y)dydx\\
&\quad-\frac{1}{2\pi}\overline{n}\iint_{d(x,y)\le\frac18}\log d(x,y)n(x)dydx+\frac{1}{4\pi}\overline{n}^2\iint_{d(x,y)\le\frac18}\log d(x,y)dydx\\
&\qquad\quad+\frac{1}{4\pi}\iint_{\frac18\le d(x,y)\le\frac14}\log d(x,y)(n(x)-\overline{n})(n(y)-\overline{n})\varphi_x(y)dydx\\
&\qquad\quad+\frac{1}{2\pi}\iint_{\frac18\le d(x,y)\le\frac14}(n(x)-\overline{n})\nabla_y\cdot(\log d(x,y)\nabla_y\varphi_x(y))c(y)dydx\\
&\qquad\qquad-\frac{1}{4\pi}\iint_{\frac18\le d(x,y)\le\frac14}(n(x)-\overline{n})\log d(x,y)\Delta_y\varphi_x(y)c(y)dydx.
\end{aligned}$$

The 2nd, 3rd, 4th, and 5th terms in the last line are bounded below by $-BM^2$ for some constant $B > 0$. The 6th and 7th terms are bounded below by $-BM||c||_{L^1}$ for some constant $B > 0$, using the fact that $\nabla_y \cdot (\log|x-y|\nabla_y \varphi_x(y))$ and $\log|x-y|\Delta_y \varphi_x(y)$ are bounded in the region $\frac{1}{8} \le |x-y| \le \frac{1}{4}$. Denoting $K(y)$ to be the fundamental solution of the Laplacian on $\mathbb{T}^2$ we apply Young's inequality to obtain that

$$||c||_{L^1(\mathbb{T}^2)} = ||K * (n - \overline{n})||_{L^1(\mathbb{T}^2)} \le ||K||_{L^1(\mathbb{T}^2)} ||n - \overline{n}||_{L^1(\mathbb{T}^2)} \le BM. \qquad \square$$

Acknowledgments. The authors would like to thank Alexander Kiselev for many helpful discussions and careful reading of the first version of the draft. The second author would also like to thank Jian-Guo Liu for referring them to the paper [20] and for helpful discussions. The authors thank the anonymous referees for pointing out that the 8π-mass threshold is actually sharp on $\mathbb{R}^2$, which greatly improves the result.

REFERENCES

[1] J. Bedrossian and S. He, *Suppression of blow-up in Patlak–Keller–Segel via shear flows*, SIAM J. Math. Anal., 49 (2017), pp. 4722–4766, https://doi.org/10.1137/16M1093380; *Erratum*, SIAM J. Math. Anal., 50 (2018), pp. 6365–6372, https://doi.org/10.1137/18M1193293.

[2] P. Biler, G. Karch, P. Laurençot, and T. Nadzieja, *The* 8π*-problem for radially symmetric solutions of a chemotaxis model in the plane*, Math. Methods Appl. Sci., 29 (2006), pp. 1563–1583.

[3] A. Blanchet, J. Carrillo, and N. Masmoudi, *Infinite time aggregation for the critical Patlak-Keller-Segel model in* $\mathbb{R}^2$, Comm. Pure Appl. Math., 61 (2008), pp. 1449–1481.

[4] A. Blanchet, J. Dolbeault, and B. Perthame, *Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations, 2006, no. 44.

[5] V. Calvez and J. Carrillo, *Volume effects in the Keller-Segel model: Energy estimates preventing blow-up*, J. Math. Pures Appl., 86 (2006), pp. 155–175.

[6] E. Carlen and M. Loss, *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on* $\mathbb{S}^n$, Geom. Funct. Anal., 2 (1992), pp. 90–104.

[7] M. Chae, K. Kang, and J. Lee, *Existence of smooth solutions to coupled chemotaxis-fluid equations*, Discrete Contin. Dyn. Syst., 33 (2013), pp. 2271–2297.

[8] P. Constantin and M. Ignatova, *On the Nernst-Planck-Navier-Stokes system*, Arch. Ration. Mech. Anal., 232 (2019), pp. 1379–1428.

[9] R.-J. Duan, A. Lorz, and P. Markowich, *Global solutions to the coupled chemotaxis-fluid equations*, Comm. Partial Differential Equations, 35 (2010), pp. 1635–1673.

[10] L. Evans, *Partial Differential Equations*, Grad. Stud. Math. 19, American Mathematical Society, Providence, RI, 1998.

[11] M. D. Francesco, A. Lorz, and P. Markowich, *Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior*, Discrete Contin. Dyn. Syst. Ser. A, 28 (2010), pp. 1437–1453.

[12] S. He, *Suppression of blow-up in parabolic-parabolic Patlak-Keller-Segel via strictly monotone shear flows*, Nonlinearity, 31 (2018), pp. 3651–3688.

[13] S. He and E. Tadmor, *Suppressing chemotactic blow-up through a fast splitting scenario on the plane*, Arch. Ration. Mech. Anal., 232 (2019), pp. 951–986.

[14] D. Horstmann, *From* 1970 *until present: The Keller-Segel model in chemotaxis and its consequences* I, Jahresber. Deutsch. Math.-Verein, 105 (2003), pp. 103–165.

[15] G. Iyer, X. Xu, and A. Zlatoš, *Convection-Induced Singularity Suppression in the Keller-Segel and Other Non-linear PDEs*, preprint, https://arxiv.org/abs/1908.01941, 2019.

[16] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., 329 (1992), pp. 819–824.

[17] E. F. Keller and L. Segel, *Model for chemotaxis*, J. Theor. Biol., 30 (1971), pp. 225–234.

[18] A. Kiselev and X. Xu, *Suppression of chemotactic explosion by mixing*, Arch. Ration. Mech. Anal., 222 (2016), pp. 1077–1112.

[19] R. Kowalczyk, *Preventing blow-up in a chemotaxis model*, J. Math. Anal. Appl., 305 (2005), pp. 566–588.

[20] H. Kozono, M. Miura, and Y. Sugiyama, *Time global existence and finite time blow-up criterion for solutions to the Keller-Segel system coupled with the Navier-Stokes fluid*, J. Differential Equations, 267 (2019), pp. 5410–5492.

[21] S. Kuksin and A. Shirikyan, *Mathematics of Two-Dimensional Turbulence*, Cambridge Tracts Math. 194, Cambridge University Press, Cambridge, 2012.

[22] J.-G. Liu and A. Lorz, *A coupled chemotaxis-fluid model: Global existence*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28 (2011), pp. 643–652.

[23] A. Lorz, *Coupled chemotaxis fluid model*, Math. Models Methods Appl. Sci., 20 (2010) pp. 987–1004.

[24] A. Lorz, *A coupled Keller-Segel-Stokes model: Global existence for small initial data and blow-up delay*, Comm. Math. Sci., 10 (2012), pp. 555–574.

[25] C. S. Patlak, *Random walk with persistence and external bias*, Bull. Math. Biophys., 15 (1953), pp. 311–338.

[26] J. C. Robinson, J. L. Rodrigo, and W. Sadowski, *The Three-Dimensional Navier-Stokes Equations: Classical Theory*, Cambridge Stud. Adv. Math. 157, Cambridge University Press, Cambridge, 2016.

[27] I. Shafrir and G. Wolansky, *The logarithmic HLS inequality for systems on compact manifolds*, J. Funct. Anal., 227 (2006), pp. 200–226.

[28] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.

[29] Y. Tao and M. Winkler, *Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), pp. 157–178.

[30] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, and R. E. Goldstein, *Bacterial swimming and oxygen transport near contact lines*, Proc. Natl. Acad. Sci. USA, 102 (2005), pp. 2277–2282.

[31] M. Winkler, *Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops*, Comm. Partial Differential Equations, 37 (2012), pp. 319–351.

[32] M. Winkler, *Stabilization in a two-dimensional chemotaxis-Navier-Stokes system*, Arch. Ration. Mech. Anal., 211 (2014), pp. 455–487.

[33] M. Winkler, *Global weak solutions in a three-dimensional chemotaxis–Navier-Stokes system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), pp. 1329–1352.

[34] M. Winkler, *How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system?*, Trans. Amer. Math. Soc., 369 (2017), pp. 3067–3125.

[35] M. Winkler, *Small-mass solutions in the two-dimensional Keller–Segel system coupled to the Navier–Stokes equations*, SIAM J. Math. Anal., 52 (2020), pp. 2041–2080, https://doi.org/10.1137/19M1264199.

[36] M. Winkler, *Does Leray's structure theorem withstand buoyancy-driven chemotaxis-fluid interaction?*, J. Eur. Math. Soc. (JEMS), to appear.