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The bigerbes introduced here give a refinement of the notion of 2–gerbes, representing degree four integral cohomology classes of a space. Defined in terms of bisimplicial line bundles, bigerbes have a symmetry with respect to which they form "bundle 2–gerbes" in two ways; this structure replaces higher associativity conditions. We provide natural examples, including a Brylinski–McLaughlin bigerbe associated to a principal G–bundle for a simply connected simple Lie group. This represents the first Pontryagin class of the bundle, and is the obstruction to the lifting problem on the associated principal bundle over the loop space to the structure group consisting of a central extension of the loop group; in particular, trivializations of this bigerbe for a spin manifold are in bijection with string structures on the original manifold. Other natural examples represent "decomposable" 4–classes arising as cup products, a universal bigerbe on $K(\mathbb{Z}, 4)$ involving its based double loop space, and the representation of any 4–class on a space by a bigerbe involving its free double loop space. The generalization to "multigerbes" of arbitrary degree is also described.

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Introduction

Gerbes provide a (more or less) geometric representation of integral cohomology 3–classes on a space; see Giraud [8] and Brylinski [4]. Bundle gerbes, introduced by Murray in [16], are particularly geometric and have a well-known application in the

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form of the "lifting bundle gerbe", representing the obstruction to the extension of a principal G–bundle to a principal bundle with structure group a $\mathrm{U}(1)$ central extension of G. Here we present a direct extension of the notion of a bundle gerbe to obtain a similar representation of integral 4–classes. These *bigerbes* are special cases, in a sense more rigid, of the bundle 2–gerbes as defined by Stevenson [22], which in turn are a more geometric version of 2–gerbes as defined by Breen [3]. In particular, our bigerbes induce bundle 2–gerbes in two ways. One application of this notion is to Brylinski–McLaughlin (bi)gerbes [5], corresponding to the existence of an extension of the principal bundle over the loop space induced by a principal G–bundle over the original space, to a bundle with structure group a central extension of the loop group.

A gerbe may be defined as a simplicial object; see Murray and Stevenson [17; 22]. We work in the context of locally split maps, which is to say continuous maps $\pi\colon Y \to X$ with local right inverses over an open cover of the topological space X. Such a map determines an associated simplicial space, $Y^{[\bullet]}$, over X, formed from the fiber products $Y^{[k]} = Y \times_X \cdots \times_X Y$:

$$Y \leftleftarrows Y^{[2]} \; \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \; Y^{[3]} \cdots . \tag{1}$$

This constitutes a contravariant functor $\Delta \to \mathsf{Top}/X$, where Δ denotes the simplex category with objects the sets $\boldsymbol{n} = \{1, \dots, n\}$ for $n \in \mathbb{N}_0$ with morphisms the order-preserving maps between these, and Top/X denotes the category of spaces with commuting maps to X. Functions on $Y^{[\bullet]}$ admit a simplicial differential, denoted by d, by taking the alternating sum of the pullbacks, and this operation extends to line (or circle) bundles and sections thereof by taking the alternating tensor product of the pullbacks.

A bundle gerbe on X is specified in terms of the simplicial space (1) by the prescription of a complex line bundle L over $Y^{[2]}$ such that dL over $Y^{[3]}$ has a section s which pulls back to be the canonical section of d^2L over $Y^{[4]}$. The important special case of the lifting bundle gerbe is obtained when $\pi\colon E \to X$ is a principal G–bundle; then there is a natural map $E^{[2]} \to G$, and the line bundle is the pullback of the line bundle over G associated to a given central extension of G by $\mathbb{C}^*$ or $\mathrm{U}(1)$.

Our notion of a bigerbe is based on a *split square* of maps. This is a commutative square of locally split maps

$$\begin{array}{ccc} Y_2 & \longleftarrow & W \\ \downarrow & & \downarrow \\ X & \longleftarrow & Y_1 \end{array} \tag{2}$$

with the additional property that the induced map

$$W \to Y_1 \times_X Y_2$$

is also locally split (and in particular surjective).

Such a split square induces a bisimplicial space $W^{[\bullet,\bullet]}$ over X,

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
Y_2^{[3]} & \leftarrow & W^{[1,3]} & \Leftarrow & W^{[2,3]} & \Lleftarrow & W^{[3,3]} & \cdots\cdots \\
\downdownarrows\downarrow & & \downdownarrows\downarrow & & \downdownarrows\downarrow & & \downdownarrows\downarrow & \\
Y_2^{[2]} & \leftarrow & W^{[1,2]} & \Leftarrow & W^{[2,2]} & \Lleftarrow & W^{[3,2]} & \cdots\cdots \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows & \\
Y_2 & \leftarrow & W^{[1,1]} & \Leftarrow & W^{[2,1]} & \Lleftarrow & W^{[3,1]} & \cdots\cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
X & \longleftarrow & Y_1 & \Longleftarrow & Y_1^{[2]} & \Lleftarrow & Y_1^{[3]} & \cdots\cdots
\end{array}
\tag{3}
$$

where the left column and bottom row are the standard simplicial spaces as in (1) and the interior spaces are given inductively by

$$W^{[m,n]} = W^{[m,1]} \times_{Y_1^{[n]}} \times \cdots \times_{Y_1^{[n]}} W^{[m,1]} \cong W^{[1,n]} \times_{Y_2^{[m]}} \times \cdots \times_{Y_2^{[m]}} W^{[1,n]}. \tag{4}$$

The result is a commutative diagram in which all rows and columns are simplicial spaces. There are then two commuting simplicial differentials, d_1 and d_2, corresponding to the horizontal and vertical maps, respectively.

Definition A *bigerbe* on the bisimplicial space (3) corresponding to a locally split square (2) is specified by a (locally trivial) complex line bundle L over $W^{[2,2]}$ with $d_1 L$ and $d_2 L$, over $W^{[3,2]}$ and $W^{[2,3]}$, respectively, having trivializing sections s_i for $i = 1, 2$ such that ds_i is the canonical trivialization of $d_i^2 L$ and $d_2 s_1 = d_1 s_2$.

As for bundle gerbes, there are straightforward notions of products, inverses, pullbacks and morphisms of bigerbes, for which the characteristic 4–class defined below behaves naturally.

As noted above, among the natural examples is the Brylinski–McLaughlin bigerbe. Suppose that $E \to X$ is a principal G–bundle over a manifold with structure group a compact, connected, simply connected, simple Lie group. Then

$$
\begin{array}{ccc}
E & \longleftarrow & PE \\
\downarrow & & \downarrow \\
X & \longleftarrow & PX
\end{array}
\tag{5}
$$

is a split square, where PX and PE are the respective (based) path spaces, the vertical arrows are projections and the horizontal arrows are the endpoint maps. In the resulting bisimplicial space $W^{[2,1]} = \Omega E$ is the based loop space of E which is a principal bundle with structure group the based loop group ΩG of G. The central extensions

$$1 \to \mathrm{U}(1) \to \widehat{\Omega G} \to \Omega G \to 1 \tag{6}$$

are classified by $H^3(G;\mathbb{Z}) = H^3_G(G;\mathbb{Z}) = \mathbb{Z}$, and the associated line bundle for such a central extension pulls back over $W^{[2,2]} = \Omega E^{[2]}$ to a line bundle Q determining a bigerbe. Here the triviality of $d_1 Q$ is the multiplicativity of the central extension, as for a lifting gerbe, whereas the (consistent) triviality of $d_2 Q$ corresponds to the so-called "fusion" property of the central extension with respect to certain configurations of loops (see Stolz and Teichner [25], Waldorf [29] and Kottke and Melrose [11]) and which is equivalent to the gerbe property with respect to the path fibration $PE^{[2]} \to E^{[2]}$. Incorporation of an additional "figure-of-eight" condition as in Kottke and Melrose [11; 12] — a condition related to the simplicial space of products of X discussed in Section 3 — promotes this to a bigerbe involving free loops and paths, representing the obstruction of the lift of $LE \to LX$ to a "loop-fusion" $\widehat{LG}$–bundle, which is discussed further below.

There are various 2–gerbe versions of this in the literature. In [5], Brylinski and McLaughlin define a 2–gerbe in the sense of Breen by pulling back the canonical gerbe on G (corresponding to the given class in $H^3(G;\mathbb{Z})$) to $E^{[2]}$ by the difference map, particularly in the universal case where $X = BG$. Carey, Johnson, Murray, Stevenson and Wang [6], and later Waldorf [28], used a similar construction to produce a bundle 2–gerbe in the sense of Stevenson. Furthermore, in [5], the authors discuss a correspondence between the 2–gerbe and the problem of extending the structure group of the free loop space LX from LG to $\widehat{LG}$. The bigerbe above demonstrates this correspondence explicitly.

Returning to the simplicial space, (1), arising from any locally split map, the simplicial differentials extend to the Čech cochain spaces over the $Y^{[k]}$. We pass to the direct limits $\check{C}^\ell(X;A) = \lim_{\mathcal{U}} \check{C}^\ell_{\mathcal{U}}(X;A)$ of Čech cochains with values in a topological abelian group A, with respect to refinement of covers $\mathcal{U}$, so eliminating covers from the notation. Then the simplicial complex

$$0 \to \check{C}^\ell(X,A) \xrightarrow{d} \check{C}^\ell(Y,A) \xrightarrow{d} \check{C}^\ell(Y^{[2]},A) \xrightarrow{d} \cdots \tag{7}$$

is exact (see Proposition 2.3), with a homotopy inverse arising from local sections over an open cover. The simplicial differential commutes with the Čech differential resulting in a double complex.

For a bundle gerbe, the representative $c(L)$ of the Chern class of L can be chosen to be a pure cocycle: $\delta c(L) = dc(L) = 0$. From the exactness of the simplicial differential this class descends:

$$c(L) = -d\beta, \quad \delta\beta = d\alpha \quad \text{for some } \beta \in \check{C}^1(Y; \mathbb{C}^*),\ \alpha \in \check{C}^2(X; \mathbb{C}^*), \tag{8}$$

and then $\mathrm{DD}(L) \in \check{H}^3(X; \mathbb{Z})$, the image of $[\alpha] \in \check{H}^2(X; \mathbb{C}^*)$ under the Bockstein isomorphism, is the Dixmier–Douady class of the gerbe. This is not the original definition of the Dixmier–Douady class of a bundle gerbe as in [16; 17]; we show that it is equivalent below in Proposition 2.8 and use the simplicial characterization to prove that a locally split map $\pi: Y \to X$ supports a bundle gerbe with a given 3–class on X if and only if the class vanishes when pulled back to Y (Theorem 2.10), and also to classify trivializations of bundle gerbes (in Proposition 2.11).

For a general bigerbe there is a similar Čech analysis in terms of the triple complex, formed by the three commuting differentials δ, d_1 and d_2, on the Čech spaces over the bisimplicial space (3). Now the Chern class $c(L) \in \check{C}^1(W^{[2,2]}; \mathbb{C}^*)$ can again be chosen to be a pure cocycle: $\delta c(L) = d_1 c(L) = d_2 c(L) = 0$. As a consequence it descends to a cocycle on $Y^{[2]}$:

$$\begin{aligned} c(L) &= d_2\beta_1, & d_1\beta_1 &= 0, & \beta_1 &\in \check{C}^1(W^{[2,1]}; \mathbb{C}^*), \\ \delta\beta_1 &= -d_2\lambda_1, & d_1\lambda_1 &= 0, & \lambda_1 &\in \check{C}^2(Y_1^{[2]}; \mathbb{C}^*), \end{aligned} \tag{9}$$

essentially as for the gerbe. Thus the image of $[\lambda_1]$ in $\check{H}^3(Y^{[2]}; \mathbb{Z})$ (under the Bockstein isomorphism) is the Dixmier–Douady class of L as a gerbe over $Y_1^{[2]}$. Significantly, however, λ_1 is naturally a *simplicial* cocycle — a pure cocycle in the (δ, d_1) complex — and so $d\lambda_1 = 0$.

In view of this, the simplicial class further descends under d_1:

$$\lambda_1 = -d_1\mu_1, \quad \delta\mu_1 = d_1\gamma, \quad \delta\gamma = 0, \quad \mu_1 \in \check{C}^2(Y_1; \mathbb{C}^*),\ \gamma \in \check{C}^3(X; \mathbb{C}^*). \tag{10}$$

The Bockstein image $G(L) \in \check{H}^4(X; \mathbb{Z})$ of $[\gamma]$ is the characteristic 4–class associated to the bigerbe.

The symmetry of the bigerbe allows Y_1 and Y_2 to be interchanged, but this also reverses the sign of $G(L)$.

Theorems 4.13 and 4.15 (i) *The characteristic* 4*–class of a bigerbe is natural with respect to pullbacks, morphisms, products and inverses. It vanishes if and only if the bigerbe admits a trivialization, and two bigerbes have the same* 4*–class if and only if they are stably isomorphic.*

(ii) *The bisimplicial space generated by a split square, as in* (2)*, supports a bigerbe for a given class in* $\check{H}^4(X;\mathbb{Z})$ *if and only if this class lifts to the* Y_i *to be trivial, with primitives which when pulled back to* W *have exact difference.*

For the Brylinski–McLaughlin bigerbe Q associated to a principal bundle $E \to X$ with structure group a compact, connected, simply connected and simple Lie group G, the 4–class $G(Q) = [\gamma]$ is the transgression to X of the 3–class on G corresponding to a central extension $\widehat{LG}$ of LG:

$$X \leftarrow E \leftleftarrows E^{[2]} \xrightarrow{q} G, \quad \alpha \in H^3(G;\mathbb{Z}), \quad q^*\alpha = d\mu, \quad d\gamma = \delta\mu. \tag{11}$$

Theorems 5.9 and 5.11 *The Brylinski–McLaughlin bigerbe* $Q \to LE^{[2]}$ *has characteristic class* $G(Q) = p_1(E) \in \check{H}^4(X;\mathbb{Z})$*, the vanishing of which is equivalent to the existence of a "loop-fusion"* (meaning fusion and figure-of-eight; see Section 3) $\widehat{LG}$ *lift of the* LG*–bundle* $LE \to LX$*. Such lifts, which are equivalent to certain trivializations of* Q*, are classified by* $\check{H}^3(X;\mathbb{Z})$*.*

In particular, this applies to the spin frame bundle of a spin manifold. There it represents the obstruction to a lift of the principal loop spin bundle over the loop space to a loop-fusion principal bundle for the basic central extension of the loop spin group. The obstruction is then the Pontryagin class, usually denoted by $\frac{1}{2}p_1$ because of its relation to the Pontryagin class of the oriented orthogonal frame bundle, of the spin bundle [29; 6; 11]. Such loop-fusion lifts are, by the above theorem, in bijection with so-called "string structures" on the manifold (see Corollary 5.13).

In addition to the Brylinski–McLaughlin bigerbes, we provide other natural examples of bigerbes representing "decomposable" 4–classes which are the cup product of either 2–classes or a 1–class and a 3–class (see Section 5.1). Moreover, we show that, for a simply connected and locally contractible space X, every 4–class is represented by a bigerbe with respect to the locally split square in which the Y_i are the based path spaces PX, and $W = PPX$ is the based mapping space of the square into X (see Section 5.5). In particular, $K(\mathbb{Z},4)$ supports a universal bigerbe. Likewise, for X not necessarily simply connected, we show that every 4–class is represented by a bigerbe using the free path spaces; in particular, $W^{[2,2]} = LLX$ is the double free loop space in this case.

There is a direct extension of bigerbes to "multigerbes", higher versions of (bi)gerbes in which the locally split squares are replaced by n–cubes, where the line bundles satisfy simplicial conditions with respect to n commuting differentials; these represent cohomology classes of degree $n+2$. The symmetry of the (multi)simplicial conditions replaces the ever higher and more complex "associativity" conditions associated to higher gerbes. This extension is quite straightforward, and for this reason, and since we are unaware of examples apart from decomposable and path multigerbes, we only outline the theory briefly at the end of this paper.

In order to restrict attention to a simple category of topological spaces, and to avoid expanding the paper further, we do not develop the theory of connections on bigerbes here, though this will be done in future work. We also do not discuss here the bigerbe analogue of bundle gerbe modules or the related theory of generalized morphisms due to Waldorf [26].

Section 1 below contains a discussion of covers and locally split maps, which is the context for the rest of the paper, and our notation for Čech theory is introduced in Section 1.2.

Section 2 is a review of Murray's theory of bundle gerbes (without connections), as developed from the Čech-simplicial point of view, with the basic properties of bundle gerbes over split maps recalled in Section 2.1. The extension of the Čech cohomology complex to a bicomplex over the simplicial space of a split map in Section 2.2 leads to an alternative definition of the Dixmier–Douady class for a bundle gerbe in Section 2.3, the classification of gerbes over a given split map in Section 2.4, and the classification of trivializations in Section 2.5. Examples of bundle gerbes are recalled in Section 2.6.

A "product-simplicial" condition, which we refer to as "doubling", on bundle gerbes is defined in Section 3.1 with particular application to the free loop space, and the connection with results from [12] is described in Section 3.2.

In Section 4.1 the basic properties of locally split squares of maps are given, leading to the definition of bigerbes in Section 4.2. The characteristic 4–class of a bigerbe is obtained in Section 4.3 and, conversely, Section 4.4 contains a necessary and sufficient condition for representability of a 4–class over a given locally split square.

Section 5 is devoted to examples, with explicit bigerbes corresponding to decomposable classes constructed in Section 5.1, extending some of the results of Mathai, Melrose and Singer [14]. After a brief discussion of doubling for bigerbes in Section 5.2, our main application of bigerbes — the Brylinski–McLaughlin lifting gerbe — is discussed in Section 5.3 and its relation to string structures is discussed in Section 5.4. Further examples of path bigerbes can be found in Section 5.5 and, finally, we end with a brief account of multigerbes in Section 6.

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1 Čech theory

1.1 Covers and locally split maps

Since we will make substantial use of Čech theory, we start with some conventions on spaces, open covers and maps. We work throughout in the standard topologist's category of compactly generated Hausdorff spaces and continuous maps, without additional conditions unless explicitly noted.

An open cover of a topological space X is a collection of open sets, $\mathcal{U}$, for which $X = \bigcup_{U\in\mathcal{U}} U$. Such a cover defines an *étale space* by taking the disjoint union

$$\mathrm{Et}(\mathcal{U}) = \bigsqcup_{U\in\mathcal{U}} U \to X$$

with the map to X consisting of the inclusion map on each U.

Note that since the individual sets may not be connected, it is not generally possible to recover the collection $\mathcal{U}$ from $\mathrm{Et}(\mathcal{U})$ without specifying additional indexing information. We regard X as its own minimal cover.

If $\mathcal{U}$ and $\mathcal{V}$ are covers of X and Y, then a *map of covers* is a continuous map $g\colon \mathrm{Et}(\mathcal{U}) \to \mathrm{Et}(\mathcal{V})$, where each element $U \in \mathcal{U}$ is mapped to a specific element $V \in \mathcal{V}$. In particular, there is an underlying map of index sets.

Definition 1.1 A continuous map $\pi: Y \to X$ of topological spaces is *locally split* if it admits continuous local sections; thus π is surjective and there exists a cover $\mathcal{U}$ of X with respect to which the local sections constitute a continuous map of covers $s: \mathrm{Et}(\mathcal{U}) \to Y$ such that $\pi \circ s: \mathrm{Et}(\mathcal{U}) \to X$ is inclusion of the cover in X. In particular, the *inclusion map* of covers $\mathrm{Et}(\mathcal{U}) \to X$ is itself locally split.

If $\mathcal{U}$ and $\mathcal{V}$ are covers of the same space X, then a map of covers such that

(1-1)
$$\begin{array}{ccc} \mathrm{Et}(\mathcal{V}) & \longrightarrow & \mathrm{Et}(\mathcal{U}) \\ & \searrow & \downarrow \\ & & X \end{array}$$

commutes makes $\mathcal{V}$ a *refinement* of $\mathcal{U}$; often in the literature the underlying map of index sets is omitted but we always retain it, even if implicitly. In this way the covers of X define a category with refinements as morphisms. Observe also that if $\mathcal{V}$ is a cover of $\mathrm{Et}(\mathcal{U})$ considered as a space, then $\mathcal{V}$ is a cover of X as well, the composite map $\mathrm{Et}(\mathcal{V}) \to X$ is an inclusion map of covers, and $\mathcal{V}$ constitutes a refinement of $\mathcal{U}$.

If $f: Y \to X$ is a continuous map then the pullback

(1-2)
$$f^{-1}\mathcal{U} = \{f^{-1}(U) : U \in \mathcal{U}\}$$

is a cover of Y and f lifts to a well-defined map of covers

$$f: \mathrm{Et}(f^{-1}\mathcal{U}) \cong f^{-1}\mathrm{Et}(\mathcal{U}) \to \mathrm{Et}(\mathcal{U}).$$

If $\mathcal{U}$ and $\mathcal{V}$ are both covers of X, then

(1-3)
$$\mathcal{U} \cap \mathcal{V} = \{U \cap V : U \in \mathcal{U},\ V \in \mathcal{V}\}$$

is a cover of X mutually refining $\mathcal{U}$ and $\mathcal{V}$; indeed, this is the same thing as pulling back $\mathcal{V}$ to a cover of $\mathrm{Et}(\mathcal{U})$ by the inclusion map to X or vice versa, and $\mathrm{Et}(\mathcal{U} \cap \mathcal{V}) \cong \mathrm{Et}(\mathcal{U}) \times_X \mathrm{Et}(\mathcal{V})$. Note that

$$\mathcal{U}^{(\ell)} = \underbrace{\mathcal{U} \cap \cdots \cap \mathcal{U}}_{\ell \text{ times}}$$

gives the cover by ℓ–fold intersections of sets in $\mathcal{U}$ (we refer to $\mathrm{Et}(\mathcal{U}^{(\ell)})$ as a *Čech space*) and that there is a canonical identification

(1-4)
$$f^{-1}(\mathcal{U}^{(\ell)}) \cong (f^{-1}\mathcal{U})^{(\ell)}$$

of covers over Y whenever $\mathcal{U}$ is a cover of X and $f: Y \to X$ is continuous.

With these conventions, the collection of covers of X forms a directed category with (reverse) order given by refinement and upper bounds given by mutual refinement (1-3).

1.2 Čech cohomology

The standard definition of Čech cohomology proceeds by fixing an open cover $\mathcal{U}$ of X and taking the homology $\check{H}^\bullet_{\mathcal{U}}(X;A)$ of the cochain complex $(\check{C}^\bullet_{\mathcal{U}}(X;A),\delta) = (\Gamma(\mathcal{U}^{(\bullet+1)};A),\delta)$, where A is a sheaf of abelian groups on X, $\Gamma(\mathcal{U}^{(\bullet+1)};A)$ denotes the group of local sections of A on the intersection cover $\mathcal{U}^{(\bullet+1)}$ and $\delta\colon \Gamma(\mathcal{U}^{(\bullet)};A) \to \Gamma(\mathcal{U}^{(\bullet+1)};A)$ is given by the alternating sum of the pullbacks by the various inclusion maps $\mathcal{U}^{(\bullet+1)} \to \mathcal{U}^{(\bullet)}$. For our purposes, A will always be a fixed topological abelian group such as $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Z}$ or $\mathrm{U}(1)$ and we will work on the sheaf of continuous maps to A, so that

$$\check{C}^\bullet_{\mathcal{U}}(X;A) = \mathcal{C}(\mathrm{Et}(\mathcal{U}^{(\bullet+1)});A)$$

is a space of continuous maps to A from the étale space. The full Čech cohomology is defined as the direct limit

$$\check{H}^\bullet(X;A) = \varinjlim_{\mathcal{U}} \check{H}^\bullet_{\mathcal{U}}(X;A) \tag{1-5}$$

under refinement.

Direct limit being an exact functor, homology commutes with direct limits, so we will use the equivalent definition of $\check{H}^\bullet(X;A)$ as the homology of the direct limit of the cochain complex

$$\check{C}^\bullet(X;A) := \varinjlim_{\mathcal{U}} \check{C}^\bullet_{\mathcal{U}}(X;A).$$

This point of view will be convenient, as it allows us to suppress explicit notation for covers at various points. We use the standard terminology of cochains, cocycles and coboundaries for elements of $\check{C}^\bullet(X;A)$, and also borrow the adjectives *closed* and *exact* from de Rham theory for cocycles and coboundaries, respectively.

By taking the direct limit at the chain level, the pullback operation with respect to a continuous map becomes well defined on chains:

Proposition 1.2 *For a topological abelian group A, any continuous map $f\colon Y \to X$ of topological spaces induces a chain map*

$$f^*\colon \check{C}^\bullet(X;A) \to \check{C}^\bullet(Y;A),$$

which descends to the pullback functor $f^*\colon \check{H}^\bullet(X;A) \to \check{H}^\bullet(Y;A)$ on cohomology.

Proof As noted above, given a cover $\mathcal{U}$ of X, f lifts to a map $\mathrm{Et}(f^{-1}\mathcal{U}) \to \mathrm{Et}(\mathcal{U})$ of covers over Y and X, and using (1-4) there is a lift $\mathrm{Et}(f^{-1}\mathcal{U}^{(\ell)}) \to \mathrm{Et}(\mathcal{U}^{(\ell)})$ for each ℓ. This is well defined with respect to refinement; specifically, if $\mathrm{Et}(\mathcal{U}') \to \mathrm{Et}(\mathcal{U})$ is a refinement then so is $\mathrm{Et}(f^{-1}\mathcal{U}'^{(\ell)}) \to \mathrm{Et}(f^{-1}\mathcal{U}^{(\ell)})$ and

$$\begin{array}{ccc} \mathrm{Et}(f^{-1}\mathcal{U}'^{(\ell)}) & \xrightarrow{f} & \mathrm{Et}(\mathcal{U}'^{(\ell)}) \\ \downarrow & & \downarrow \\ \mathrm{Et}(f^{-1}\mathcal{U}^{(\ell)}) & \xrightarrow{f} & \mathrm{Et}(\mathcal{U}^{(\ell)}) \end{array}$$

commutes, so $f^*\colon \check{C}^\bullet_{\mathcal{U}}(X;A) \to \check{C}^\bullet_{f^{-1}\mathcal{U}}(Y;A)$ descends to the direct limit. Furthermore, the natural maps $\mathrm{Et}(f^{-1}\mathcal{U}^{(\ell)}) \to \mathrm{Et}(\mathcal{U}^{(\ell)})$ commute with the inclusions $\mathrm{Et}(\mathcal{U}^{(\ell)}) \hookrightarrow \mathrm{Et}(\mathcal{U}^{(\ell-1)})$ in each factor of the ℓ–fold intersections into the original covers, so f^* commutes with the Čech differential δ and defines a chain morphism. □

We proceed to define a more limited form of pullback on chains with respect to the sections of a locally split map. This is essential to the exactness of the simplicial complex (7) of Čech chains on the simplicial space induced from a locally split map mentioned in the introduction and discussed in detail in Section 2.2.

Lemma 1.3 *If $s\colon \mathrm{Et}(\mathcal{U}) \to Y$ is a collection of sections of a locally split map $\pi\colon Y \to X$, then any cover $\mathcal{V}$ of Y induces canonical refinements $\mathcal{U}' = s^{-1}\mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}' = \pi^{-1}\mathcal{U}' \cap \mathcal{V}$ of $\mathcal{V}$ such that s and π lift naturally to maps of covers $s\colon \mathrm{Et}(\mathcal{U}') \to \mathrm{Et}(\mathcal{V}')$ and $\pi\colon \mathrm{Et}(\mathcal{V}') \to \mathrm{Et}(\mathcal{U}')$ with $s\pi = \mathrm{Id}\colon \mathrm{Et}(\mathcal{U}') \to \mathrm{Et}(\mathcal{U}')$; this construction is natural under refinement of $\mathcal{V}$.*

Proof If $\mathcal{V}$ is any cover of Y and $s_U\colon U \to Y$ is the section of π over $U \in \mathcal{U}$, then the sets $s_U^{-1} V \subset U$ for $V \in \mathcal{V}$ define a cover of U, and hence together a refinement of $\mathcal{U}$ as a cover of X. This is the cover $\mathcal{U}' = s^{-1}\mathcal{V}$ defined above. Then s lifts to a naturally defined map of covers $\mathrm{Et}(\mathcal{U}') \to \mathrm{Et}(\mathcal{V})$ given by $s_U\colon s_U^{-1} V \to V$. Since it is natural, we denote this lift simply as s. The pullback $\pi^{-1}\mathcal{U}'$ of the new cover of X defines a refinement $\mathcal{V}' = \pi^{-1}\mathcal{U}' \cap \mathcal{V}$ of $\mathcal{V}$ to which π lifts as a map of covers $\pi\colon \mathrm{Et}(\mathcal{V}') \to \mathrm{Et}(\mathcal{U}')$. Moreover, as a consequence of the fact that $\pi s = \mathrm{Id}$ on each element $U' \in \mathcal{U}$, it follows that s lifts to a map of covers $s\colon \mathrm{Et}(\mathcal{U}') \to \mathrm{Et}(\mathcal{V}')$, and $s\pi = \mathrm{Id}$. □

Proposition 1.4 *Under the conditions of Lemma 1.3 the map of covers $s\colon \mathrm{Et}(\mathcal{U}') \to \mathrm{Et}(\mathcal{V}')$ induces a map of covers of $\mathrm{Et}(\mathcal{U}'^{(\ell)})$ to $\mathrm{Et}(\mathcal{V}'^{(\ell)})$ for each ℓ and a homomorphism*

$$s_\ell^*\colon \check{C}^\ell(Y;A) \to \check{C}^\ell(X;A) \quad \textit{satisfying } s_\ell^*\pi^* = \mathrm{Id}, \tag{1-6}$$

given by pullback under the map from the first factor

$$s_\ell : U_1' \cap \cdots \cap U_\ell' \to V_1' \cap \cdots \cap V_\ell', \quad x \mapsto s_{U_1'}(x). \tag{1-7}$$

Remark It is important that these s_ℓ^* do *not* generally commute with the Čech differential; in other words, we do not obtain a chain map, and, in particular, we do not claim that s_ℓ^* descends to cohomology. Indeed the lift, s_ℓ, of s in (1-7) corresponds to an (arbitrary) preference for the map corresponding to the first factor, U_1', in the ℓ–fold intersection.

Proof By definition, each $U_1' = U_1 \cap s_{U_1}^{-1}(V)$ for some elements $U_1 \in \mathcal{U}$ and $V \in \mathcal{V}$, respectively. Then $V_1' = V \cap \pi^{-1}(U_1')$ contains the image of U_1' under the lifted section and the other elements $V_j' = V \cap \pi^{-1}(U_j')$ contain the image of $U_1' \cap U_j'$ under $s_{U_1'}$. Then certainly $\pi s_\ell = \mathrm{Id}$ and $\iota_1 \circ s_\ell = s \circ \iota_1$, where ι_1 denotes the first factor inclusion map. That this holds for ι_1 and not for ι_j for $j \neq 1$ is ultimately the reason the s_ℓ^* constructed below do not commute with the Čech differential.

Omitting the abelian group A for notational convenience, the action, s_ℓ^*, on Čech cochains is defined by mapping $[\alpha] \in \check{C}^\ell(Y)$, with representative $\alpha \in \check{C}^\ell_{\mathcal{V}}(Y)$, to the image of α in $\check{C}^\ell_{\mathcal{V}'}(Y)$ (which we continue to denote by α), then to $s_{\ell+1}^*\alpha \in \check{C}^\ell_{\mathcal{U}'}(X)$, and finally to the image $[s_{\ell+1}^*\alpha] \in \check{C}^\ell(X)$.

That this is well defined is a consequence of naturality; it suffices to note that if $\mathrm{Et}(\mathcal{W}) \to \mathrm{Et}(\mathcal{V})$ is a refinement, then $s^{-1}\mathcal{W}$ and $\mathcal{W}' = \pi^{-1}(s^{-1}\mathcal{W}) \cap \mathcal{W}$ refine $\mathcal{U}'$ and $\mathcal{V}'$, respectively, and

$$\begin{array}{ccc} \mathrm{Et}(s^{-1}\mathcal{W}^{(\ell)}) & \xrightarrow{s_\ell} & \mathrm{Et}(\mathcal{W}^{(\ell)}) \\ \downarrow & & \downarrow \\ \mathrm{Et}(\mathcal{U}'^{(\ell)}) & \xrightarrow{s_\ell} & \mathrm{Et}(\mathcal{V}'^{(\ell)}) \end{array}$$

commutes. □

Remark It is worth highlighting the particular property of the cover $\mathcal{V}'$ of Y that was essential to the previous proof. If $\pi : Y \to X$ is locally split and $s : \mathrm{Et}(\mathcal{U}) \to Y$ is a fixed section over a cover $\mathcal{U}$ of X, then we may say a cover $\mathcal{V}'$ of Y is *admissible* with respect to s if there exists a refinement $\mathcal{U}'$ of $\mathcal{U}$ and

(i) a lift $\pi : \mathrm{Et}(\mathcal{V}') \to \mathrm{Et}(\mathcal{U}')$ of $\pi : Y \to X$;

(ii) for each $\ell \geq 1$, a lift $s_\ell\colon \mathrm{Et}(\mathcal{U}'^{(\ell)}) \to \mathrm{Et}(\mathcal{V}'^{(\ell)})$ of $s\colon \mathrm{Et}(\mathcal{U}) \to Y$ such that $\pi s_\ell = 1$ on $\mathrm{Et}(\mathcal{U}'^{(\ell)})$ and

$$\begin{array}{ccc} \mathrm{Et}(\mathcal{U}'^{(\ell)}) & \xrightarrow{s_\ell} & \mathrm{Et}(\mathcal{V}'^{(\ell)}) \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_1} \\ \mathrm{Et}(\mathcal{U}') & \xrightarrow{s_1} & \mathrm{Et}(\mathcal{V}') \end{array}$$

commutes, where ι_1 is the first factor inclusion on ℓ–fold intersections. (In fact, once such a lift exists for $\ell = 1, 2$ it automatically exists for every $\ell \geq 3$.)

An *admissible refinement* of an admissible cover $\mathcal{V}'$ of Y is a refinement $\mathcal{V}''$ of $\mathcal{V}'$ which is itself admissible and for which the obvious diagrams intertwining the s_ℓ commute.

Clearly the chain maps $s_\ell^*\colon \check{C}^\ell_{\mathcal{U}'}(X; A) \to \check{C}^\ell_{\mathcal{V}'}(Y; A)$ are defined whenever $\mathcal{V}'$ is an admissible cover of Y and they are natural with respect to admissible refinement. One consequence of the proof above is that admissible covers (and admissible refinements) are *final* in the directed set of all covers (ie any cover of Y has an admissible cover which refines it); in particular, the direct limit $\check{C}^\ell(Y; A) = \lim_{\mathcal{V}} \check{C}^\ell_{\mathcal{V}}(Y; A)$ is equivalent to the direct limit taken over admissible covers alone [13, Theorem 1, page 213]. The reader who is uncomfortable with the rather large chain complexes given by the direct limit over all covers may therefore wish to restrict attention to Čech cochains with respect to some fixed admissible covers.

A limited form of naturality holds for the chain maps of Proposition 1.4. Indeed, we say two locally split maps $\pi_i\colon Y_i \to X_i$ for $i = 1, 2$ are *compatible* if there exist continuous maps $f\colon X_1 \to X_2$ and $\tilde{f}\colon Y_1 \to Y_2$ which intertwine the π_i (so $\pi_2 \tilde{f} = f \pi_1$), and in addition intertwine some local sections; more precisely, there are open covers $\mathcal{U}_i$ of X_i for $i = 1, 2$ to which f lifts to a map of covers $f\colon \mathrm{Et}(\mathcal{U}_1) \to \mathrm{Et}(\mathcal{U}_2)$, as well as sections $s^i\colon \mathrm{Et}(\mathcal{U}_i) \to Y_i$ for $i = 1, 2$ such that $\tilde{f} s^1 = s^2 f\colon \mathrm{Et}(\mathcal{U}_1) \to Y_2$. This condition arises in the context of bigerbes in Section 4.2, though it will not be used in the interim.

Proposition 1.5 *If $\pi_i\colon Y_i \to X_i$ are locally split maps compatible under $\tilde{f}\colon Y_1 \to Y_2$ and $f\colon X_1 \to X_2$, then*

$$\begin{array}{ccc} \check{C}^\ell(Y_2; A) & \xrightarrow{\tilde{f}^*} & \check{C}^\ell(Y_1; A) \\ \downarrow{\scriptstyle s_\ell^{2*}} & & \downarrow{\scriptstyle s_\ell^{1*}} \\ \check{C}^\ell(X_2; A) & \xrightarrow{f^*} & \check{C}^\ell(X_1; A) \end{array}$$

commutes for every ℓ, where ${s^i_\ell}^*$ are the maps derived from the compatible local sections s^i of π_i.

Proof Given any cover $\mathcal{V}_2$ of Y_2 and proceeding as in the proof of Proposition 1.4, there is an admissible cover $\mathcal{V}_2'$ of Y_2 over a refinement $\mathcal{U}_2'$ of $\mathcal{U}_2$ on which π_2 and $s^2_\ell\colon \mathrm{Et}({\mathcal{U}_2'}^{(\ell)}) \to \mathrm{Et}({\mathcal{V}_2'}^{(\ell)})$ are defined.

It then follows by the commutativity hypotheses $\tilde f \circ s^1 = s^2 \circ f$ and $f \circ \pi_1 = \pi_2 \circ \tilde f$ that $\mathcal{V}_1' = \tilde f^{-1}\mathcal{V}_2'$ is an admissible cover of Y_1 over the refinement $\mathcal{U}_1' = f^{-1}\mathcal{U}_2'$ of $\mathcal{U}_1$ (where $f\colon \mathrm{Et}(\mathcal{U}_1) \to \mathrm{Et}(\mathcal{U}_2)$ is the given lift of f to the original covers), and

$$\begin{array}{ccc} \mathrm{Et}({\mathcal{V}_1'}^{(\ell)}) & \xrightarrow{\tilde f} & \mathrm{Et}({\mathcal{V}_2'}^{(\ell)}) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathrm{Et}({\mathcal{U}_1'}^{(\ell)}) & \xrightarrow{f} & \mathrm{Et}({\mathcal{U}_2'}^{(\ell)}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Et}({\mathcal{V}_1'}^{(\ell)}) & \xrightarrow{\tilde f} & \mathrm{Et}({\mathcal{V}_2'}^{(\ell)}) \\ {\scriptstyle s^1_\ell}\uparrow & & {\scriptstyle s^2_\ell}\uparrow \\ \mathrm{Et}({\mathcal{U}_1'}^{(\ell)}) & \xrightarrow{f} & \mathrm{Et}({\mathcal{U}_2'}^{(\ell)}) \end{array}$$

commute. Then

$$\begin{array}{ccc} \check{C}^\ell_{\mathcal{V}_2'}(Y_2; A) & \xrightarrow{\tilde f^*} & \check{C}^\ell_{\mathcal{V}_1'}(Y_1; A) \\ \downarrow{\scriptstyle {s^2_\ell}^*} & & \downarrow{\scriptstyle {s^1_\ell}^*} \\ \check{C}^\ell_{\mathcal{U}_2'}(X_2; A) & \xrightarrow{f^*} & \check{C}^\ell_{\mathcal{U}_1'}(X_1; A) \end{array}$$

commutes, and the result follows by passage to the direct limit over all covers. □

2 Bundle gerbes

2.1 Simplicial line bundles

We recall the notion of a bundle gerbe [16], which for our purposes is most efficiently defined in terms of simplicial line bundles.

Denote by $Y^{[k]}$ the k–fold fiber product $Y \times_\pi \cdots \times_\pi Y$, with projection maps $\pi_j\colon Y^{[k]} \to Y^{[k-1]}$ for $j = 0, 1, \dots, k-1$, where

$$\pi_j(y_0, \dots, y_{k-1}) = (y_0, \dots, \widehat{y}_j, \dots, y_{k-1})$$

omits the j^{th} factor enumerated from 0. Then

$$X \leftarrow Y \leftleftarrows Y^{[2]} \overset{\leftarrow}{\leftleftarrows} Y^{[3]} \cdots \tag{2-1}$$

is a *simplicial space* with face maps π_j and degeneracy maps the fiber diagonal maps $Y^{[k-1]} \to Y^{[k]}$ (of which we will not make use). More precisely, $Y^{[\bullet]}$ is a *simplicial space over* X, meaning that all maps commute with the projections $\pi\colon Y^{[k]} \to X$, and X itself may be regarded as an augmentation in (2-1). For notational convenience, we set $Y^{[1]} = Y$ and $Y^{[0]} = X$, with $\pi_0 = \pi\colon Y^{[1]} \to Y^{[0]}$.

Remark Our enumeration (which is geometrically natural here) differs unfortunately from the standard convention for simplicial spaces, under which one would typically write $Y_0 = Y$ (as the image of the 0 simplex), $Y_1 = Y^{[2]}$ (as the image of the 1–simplex), etc, augmented by $Y_{-1} = X$. For consistency we use this alternative convention throughout, and beg the pardon of readers who would prefer to use the standard one.

Given a complex line bundle $L \to Y^{[k]}$, its differential is defined to be the line bundle

$$dL := \bigotimes_{i=0}^{k} \pi_i^* L^{(-1)^i} \to Y^{[k+1]}. \tag{2-2}$$

Using the commutation relations between the π_j, it follows that $d^2L = d(dL)$ is canonically trivial over $Y^{[k+2]}$.

Remark While we will mostly work with complex line bundles, we could equivalently take L to be a principal $\mathbb{C}^*$– or $\mathrm{U}(1)$–bundle. At times we will use these objects interchangeably without further elaboration.

A *bundle gerbe* (L, Y, X) as defined by Murray is equivalent to a *simplicial line bundle* on the simplicial space $Y^{[\bullet]}$ in the sense of Brylinski and McLaughlin [5]; this consists of a complex line bundle $L \to Y^{[2]}$ along with a trivialization of the bundle

$$dL = \pi_0^* L \otimes \pi_1^* L^{-1} \otimes \pi_2^* L$$

over $Y^{[3]}$, which in turn induces the canonical trivialization of d^2L when pulled back over $Y^{[4]}$. The trivialization of $dL \to Y^{[3]}$ is equivalent to the "gerbe (or groupoid) product" isomorphism

$$\phi\colon \pi_2^* L \otimes \pi_0^* L \xrightarrow{\cong} \pi_1^* L,$$

which multiplies (composes) pairs of respective elements in the fibers L_{y_0,y_1} and L_{y_1,y_2} to get elements in L_{y_0,y_2}; the condition that the trivialization coincide with the canonical one on d^2L over $Y^{[4]}$ is equivalent to associativity of this product.

A bundle gerbe (L, Y, X) is *trivial* if there exists a bundle $L' \to Y$ and an isomorphism $L \cong dL'$ on $Y^{[2]}$; such an isomorphism is called a *trivialization* of L.

If (L, Y, X) is a bundle gerbe on X, then its *pullback* by a continuous map $f\colon X' \to X$ is the bundle gerbe $(\tilde{f}^*L, f^*Y, X')$; here we use the naturality $f^*(Y \times_X Y) \cong f^*(Y) \times_{X'} f^*(Y)$ and denote the resulting map $f^*Y^{[2]} \to Y^{[2]}$ by $\tilde{f}$. Likewise, the *product* of two bundle gerbes (L_i, Y_i, X) for $i = 1, 2$ on X is given by

$$(L_1 \otimes L_2, Y_1 \times_X Y_2, X),$$

where $L_1 \otimes L_2$ is shorthand for $\mathrm{pr}_1^* L_1 \otimes \mathrm{pr}_2^* L_2$ with $\mathrm{pr}_i\colon Y_1 \times_X Y_2 \to Y_i$ denoting the projections from the fiber product. It is straightforward to verify that this is a bundle gerbe, which we denote for simplicity as $L_1 \otimes L_2$. The definitions of product and pullback implicitly use the following standard result, which we record for later use:

Lemma 2.1 *Pullbacks and fiber products of locally split maps are locally split. More precisely, if $\pi\colon Y \to X$ is locally split and $f\colon X' \to X$ is continuous, then $f^*Y \to X'$ is locally split, and if $\pi'\colon Y' \to X$ is another locally split map, then $\pi \times \pi'\colon Y \times_X Y' \to X$ is locally split.*

More generally, a (*strong*) *morphism* $(L', Y', X') \to (L, Y, X)$ of bundle gerbes consists of a map $\tilde{f}\colon Y' \to Y$ covering a map $f\colon X' \to X$ (we do not require the local splittings to be compatible in the sense of Section 1.2) along with an isomorphism $L \cong L'$ over $\tilde{f}^{[2]}\colon Y'^{[2]} \to Y^{[2]}$ which intertwines the sections of dL and dL'; a (*strong*) *isomorphism* is a morphism for which $X = X'$ and $f = \mathrm{Id}$. In particular, a morphism $f\colon (L', Y', X') \to (L, Y, X)$ is equivalent to an isomorphism $(L', Y', X') \cong (f^*L, f^*Y, X')$.

Finally, two gerbes (L_i, Y_i, X) for $i = 1, 2$ are said to be *stably isomorphic* if $L_1 \otimes L_2^{-1}$ is trivial, or, equivalently, there exist trivial gerbes (T_i, Z_i, X) such that

$$L_1 \otimes T_1 \cong L_2 \otimes T_2,$$

in the sense of a strong isomorphism over a space $Z^{[2]}$ where $Z \to X$ admits maps to Y_1, Y_2, Z_1 and Z_2. This is strictly weaker than an isomorphism as defined above, and was introduced in [17] in order to obtain a classification of bundle gerbes up to stable isomorphism by their Dixmier–Douady class.

Remark There is a weaker notion of gerbe morphism due to Waldorf [26], which naturally incorporates the theory of gerbe modules, and has the property that the invertible morphisms are precisely the stable isomorphisms; moreover, a trivialization becomes the same thing as an isomorphism to a canonical trivial gerbe over X. Because we leave the generalization to bigerbes of Waldorf's morphisms to future work, we will not pursue this further here.

2.2 Simplicial Čech theory

A primary motivation for the definition of gerbes is that they represent a class in $\check{H}^3(X;\mathbb{Z}) \cong \check{H}^2(X;\mathbb{C}^*)$ — the Dixmier–Douady class; for a bundle gerbe (L,Y,X) this will be denoted by $\mathrm{DD}(L)$. This class is natural with respect to products, pullbacks and inverses, and it determines (L,Y,X) up to stable isomorphism [17]. We give an alternative (though not necessarily simpler) derivation of these facts based on a closer study of simplicial spaces; this approach is used in the generalization to bigerbes below. In doing so we identify the 3–classes on X which can be represented by a bundle gerbe with respect to a given locally split map $Y \to X$, and recover the classification of the trivializations of a trivial bundle gerbe.

Consider the simplicial space, (2-1), consisting of the fiber products of a locally split map $\pi\colon Y \to X$. Proposition 1.2 shows that the induced maps $\pi_j^*\colon \check{C}^\bullet(Y^{[k]};A) \to \check{C}^\bullet(Y^{[k+1]};A)$ are chain maps and the *simplicial differential* on Čech cochains is defined by

$$d = \sum_{j=0}^{k} (-1)^j \pi_j^*\colon \check{C}^\bullet(Y^{[k]};A) \to \check{C}^\bullet(Y^{[k+1]};A),$$

so $d^2 = 0$ and $d\delta = \delta d$.

Thus, $(\check{C}^\bullet(Y^{[\bullet]};A), d, \delta)$ forms a double complex

(2-3)
$$\begin{array}{ccccccc}
d\uparrow & & d\uparrow & & d\uparrow & & \\
\check{C}^0(Y^{[2]}) & \xrightarrow{\delta} & \check{C}^1(Y^{[2]}) & \xrightarrow{\delta} & \check{C}^2(Y^{[2]}) & \xrightarrow{\delta} & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\check{C}^0(Y) & \xrightarrow{\delta} & \check{C}^1(Y) & \xrightarrow{\delta} & \check{C}^2(Y) & \xrightarrow{\delta} & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\check{C}^0(X) & \xrightarrow{\delta} & \check{C}^1(X) & \xrightarrow{\delta} & \check{C}^2(X) & \xrightarrow{\delta} & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

where we have omitted the coefficient group from the notation. We take the row $\check{C}^\bullet(Y)$ to have vertical degree 0 in (2-3) (corresponding to its true simplicial degree), so the row $\check{C}^\bullet(X)$ has degree -1. This Čech-simplicial double complex appears more generally in algebraic geometry [7], and is also discussed in [5] in the context of simplicial gerbes.

Convention 2.2 Our convention for double (and higher) complexes is that the two differentials *commute*, as above. This necessitates the introduction of a sign (depending on an ordering of the differentials) in the total differential, which we take to be

$$D = \delta + (-1)^p d \quad \text{on } \check{C}^p(Y^{[q]}; A).$$

Another possible sign convention is given by changing the formal order of d and δ, namely $D' = d + (-1)^{q+1}\delta$ (recalling that $Y^{[q]}$ has vertical degree $q-1$). This is intertwined with D via the automorphism $(-1)^{p(q+1)}$ of the double complex.

In general, whenever we have a multicomplex $C^{p_1,\dots,p_k}$ with k commuting differentials $d_1, d_2, \dots, d_k$, the total differential will be defined inductively by

$$\begin{aligned} D_k &= D_{k-1} + (-1)^{p_1+\cdots+p_{k-1}} d_k \\ &= d_1 + (-1)^{p_1} d_2 + (-1)^{p_1+p_2} d_3 + \cdots + (-1)^{p_1+\cdots+p_{k-1}} d_k. \end{aligned}$$

Switching the order of the indices requires composing with an automorphism given in each degree by an appropriate power of -1 as above.

Proposition 2.3 *The simplicial chain complex*

$$0 \to \check{C}^\ell(X; A) \xrightarrow{d} \check{C}^\ell(Y; A) \xrightarrow{d} \check{C}^\ell(Y^{[2]}; A) \xrightarrow{d} \cdots \tag{2-4}$$

is exact. In particular, a collection of local sections $s\colon \mathrm{Et}(\mathcal{U}) \to Y$ determines a chain homotopy contraction via the maps in Proposition 1.4.

Compare the exactness of the de Rham complex in Section 8 of [16]. This may be understood as a manifestation of the fact that the geometric realization of the simplicial set $Y^{[\bullet]}$ is known to be homotopy equivalent to X.

Proof The assumption that $Y \to X$ is locally split in particular implies that $\pi_0\colon Y^{[k]} \to Y^{[k-1]}$ is locally split for each k; indeed, we may equip each $Y^{[k]}$ with the cover $\mathcal{U}_k := \pi^{-1}\mathcal{U} \to Y^{[k]}$ pulled back from the cover $\mathcal{U}$ of X and define $s\colon \mathcal{U}_{k-1} \to Y^{[k]}$ as $s \circ \pi \times 1 \times \cdots \times 1$:

$$s_{\pi^{-1}U}\colon \pi^{-1}U \ni (y_0, \dots, y_{k-2}) \mapsto \big(s_U(\pi(y_0)), y_0, \dots, y_{k-2}\big) \tag{2-5}$$

for each $\pi^{-1}U \in \mathcal{U}_{k-1}$, which manifestly satisfies $\pi_0 s = 1$. In fact this lift of s is a map of covers from $\mathcal{U}_{k-1}$ over $Y^{[k-1]}$ into $\mathcal{U}_k$ over $Y^{[k]}$, and a brief computation using (2-5), shows that

$$\pi_{j+1}s = s\pi_j : \mathcal{U}_k \to \mathcal{U}_k, \quad 1 \le j \le k-1, \tag{2-6}$$

as composed maps of covers on $\mathcal{U}_k$ for each k.

By Proposition 1.4, we obtain well-defined maps

$$s_\ell^* : \check{C}^\ell(Y^{[k+1]}; A) \to \check{C}^\ell(Y^{[k]}; A)$$

for all k and ℓ, which satisfy the identity

$$\pi_j^* s_\ell^* = \begin{cases} 1 & \text{for } j = 0, \\ s_\ell^* \pi_{j+1}^* & \text{for } j \ge 1, \end{cases} \tag{2-7}$$

in light of (2-6); indeed, this identity holds on $\check{C}^\ell_{\mathcal{U}_k}(Y^{[k]}; A)$ and, since all homomorphisms are well defined in the direct limit, the identity likewise descends.

The chain maps $s_\ell^* : \check{C}^\ell(Y^{[k]}; A) \to \check{C}^\ell(Y^{[k-1]}; A)$ determine the desired chain homotopy contraction of (2-4) since it follows from (2-7) that $ds_\ell^* + s_\ell^* d = 1$. □

2.3 Dixmier–Douady class of a gerbe

The setting for our analysis of the Čech cohomology of a bundle gerbe is the truncated complex

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ d\uparrow & & d\uparrow & & d\uparrow & & \\ \check{Z}^0(Y^{[2]}; A) & \xrightarrow{\delta} & \check{Z}^1(Y^{[2]}; A) & \xrightarrow{\delta} & \check{Z}^2(Y^{[2]}; A) & \xrightarrow{\delta} & \\ d\uparrow & & d\uparrow & & d\uparrow & & \\ \check{C}^0(Y; A) & \xrightarrow{\delta} & \check{C}^1(Y; A) & \xrightarrow{\delta} & \check{C}^2(Y; A) & \xrightarrow{\delta} & \end{array} \tag{2-8}$$

where we have omitted the bottom row of (2-3) and where

$$\begin{aligned} \check{Z}^\ell(Y^{[2]}; A) &:= \operatorname{Ker}\{d : \check{C}^\ell(Y^{[2]}; A) \to \check{C}^\ell(Y^{[3]}; A)\} \\ &= \operatorname{Im}\{d : \check{C}^\ell(Y; A) \to \check{C}^\ell(Y^{[2]}; A)\}. \end{aligned}$$

Denote by

$$\check{H}^\bullet_Z(Y^{[2]}; A) := H^\bullet(\check{Z}^\bullet(Y^{[2]}; A), \delta)$$

the Čech cohomology of the simplicially trivial classes on $Y^{[2]}$, or the horizontal cohomology of the top row in (2-8). For later use we note the following result:

Lemma 2.4 *There is a natural Bockstein isomorphism*

$$\check{H}^{\bullet}_{Z}(Y^{[2]};\mathbb{C}^*) \cong \check{H}^{\bullet+1}_{Z}(Y^{[2]};\mathbb{Z}).$$

Proof Regarding the chain complexes $\check{Z}^{\bullet}(Y^{[2]};A)$ for an abelian group A as the image under d of $\check{C}^{\bullet}(Y;A)$, it follows both that the coefficient sequence

$$0 \to \check{Z}^{\bullet}(Y^{[2]};\mathbb{Z}) \to \check{Z}^{\bullet}(Y^{[2]};\mathbb{C}) \xrightarrow{\exp(2\pi i\cdot)} \check{Z}^{\bullet}(Y^{[2]};\mathbb{C}^*) \to 0 \tag{2-9}$$

is short exact and that $\check{Z}^{\bullet}(Y^{[2]};\mathbb{C})$ is acyclic, from which the long exact sequence for (2-9) degenerates to the Bockstein isomorphism. □

Theorem 2.5 *The total cohomology of the double complex* (2-8) *is isomorphic to* $\check{H}^{\bullet}(X;A)$.

Proof Owing to exactness of the columns, the (d,δ) spectral sequence of (2-8) degenerates at the E_1 page to

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \\ 0 & & 0 & & 0 & & \\ \\ \check{C}^0(X) & \xrightarrow{\delta} & \check{C}^1(X) & \xrightarrow{\delta} & \check{C}^2(X) & \xrightarrow{\delta} & \end{array}$$

and therefore stabilizes at E_2 to the cohomology $\check{H}^{\bullet}(X;A)$. □

Next we will show that a bundle gerbe is represented by a pure cocycle in the double complex (2-8) concentrated at $\check{Z}^1(Y^{[2]};\mathbb{C}^*)$, so with $\delta c(L)=0$ and $dc(L)=0$, and this descends to the Dixmier–Douady class.

Proposition 2.6 *A bundle gerbe* (L,Y,X) *has Chern class represented by* $c(L) \in \check{Z}^1(Y^{[2]};\mathbb{C}^*)$; *in particular*,

$$c(L) \in \check{H}^1_{Z}(Y^{[2]};\mathbb{C}^*) \cong \check{H}^2_{Z}(Y^{[2]};\mathbb{Z}). \tag{2-10}$$

Conversely, *any such class determines a bundle gerbe*, *and* L *admits a trivialization if and only if* $[c(L)] \in d\check{H}^1(Y;\mathbb{C}^*)$.

Proof As a complex line bundle, L gives rise to a Chern cocycle $c(L) \in \check{C}^1(Y^{[2]},\mathbb{C}^*)$, and $dc(L) \in \check{C}^1(Y^{[3]},\mathbb{C}^*)$ represents the bundle dL on $Y^{[3]}$. The trivialization of dL is encoded by $\gamma \in \check{C}^0(Y^{[3]},\mathbb{C}^*)$ such that $\delta\gamma = dc(L)$, and the fact that this induces the canonical trivialization of d^2L on $Y^{[4]}$ means that $d\gamma = 0$. Thus, by exactness, we can alter $c(L)$ by δ applied to a d–preimage of γ to arrange that $dc(L)=0$.

Altering $c(L) \in \check{Z}^1(Y^{[2]}; \mathbb{C}^*)$ by $\delta\beta$ for $\beta \in \check{Z}^0(Y^{[2]}; \mathbb{C}^*)$ amounts to applying an automorphism to $L \to Y^{[2]}$ which does not change the trivialization of $dL \to Y^{[3]}$, so the Chern class in $\check{H}^1_Z(Y^{[2]}; \mathbb{C}^*)$ is well defined.

Conversely, given $\alpha \in \check{Z}^1_{\mathcal{V}}(Y^{[2]}; \mathbb{C}^*)$ representing a cocycle in $\check{Z}^1(Y^{[2]}; \mathbb{C}^*)$ on some fixed open cover $\mathcal{V} \to Y^{[2]}$, the usual construction uses α on $\mathcal{V}^{(2)}$ to assemble a line bundle $L \to Y^{[2]}$ out of trivial bundles on $\mathcal{V}$. Then, since $d\alpha = 0$, it follows that dL is assembled trivially out of trivial bundles on the open cover $\mathcal{V}' = \pi_0^{-1}\mathcal{V} \cap \pi_1^{-1}\mathcal{V} \cap \pi_2^{-1}\mathcal{V}$ of $Y^{[3]}$, and hence is globally trivial (with the trivialization agreeing with the canonical one on $d^2 L$).

Finally, L admits a trivialization $L \cong dQ$ for some $Q \to Y$, if and only if $c(L) = dc(Q) \in \check{H}^1_Z(Y^{[2]}; \mathbb{C}^*)$, where $c(Q) \in H^1(Y; \mathbb{C}^*)$ is the Chern class of Q. □

Definition 2.7 The *Dixmier–Douady class* of a bundle gerbe (L, Y, X) is the image $\mathrm{DD}(L) \in \check{H}^2(X; \mathbb{C}^*) \cong \check{H}^3(X; \mathbb{Z})$ of the hypercohomology class of $c(L) \in \check{Z}^1(Y^{[2]}; \mathbb{C}^*)$ in the double complex (2-8) and is obtained explicitly by a zigzag construction

(2-11)
$$\begin{array}{ccccc}
0 & & & & \\
\uparrow & & & & \\
c(L) & \longmapsto & 0 & & \\
{\scriptstyle -d}\uparrow & & \uparrow & & \\
\beta & \overset{\delta}{\longmapsto} & \delta\beta & \longmapsto & 0 \\
 & & {\scriptstyle d}\uparrow & & \uparrow \\
 & & \mathrm{DD}(L) & \longmapsto & 0
\end{array}$$

The sign in $-d\beta = c(L)$ arises from the fact that the total differential D involves the term $-d$ on that column according to Convention 2.2.

Note that if $\pi_i \colon Y_i \to X_i$ for $i = 1, 2$ are locally split maps which are intertwined by $f \colon X_1 \to X_2$ and $\tilde{f} \colon Y_1 \to Y_2$, even without requiring that the π_i are compatible by f and $\tilde{f}$ as in Section 1.1, then the Čech cochain maps determined by $\tilde{f}^{[k]} \colon Y_1^{[k]} \to Y_2^{[k]}$ as in Proposition 1.2 together form a morphism $\check{C}^\bullet(Y_2^{[\bullet]}; A) \to \check{C}^\bullet(Y_1^{[\bullet]}; A)$ of double complexes, in that the $(\tilde{f}^{[k]})^*$ commute with δ and d.

Proposition 2.8 *The Dixmier–Douady class as defined above coincides with the definition given by Murray and is natural with respect to inverse, product and pullback; it vanishes if and only if the gerbe is trivial.*

Proof The (well-known) naturality properties follow directly from the preceding remark. To see the coincidence of our definition with that of Murray given in [16], we first recall the latter.

Suppose $s:\mathcal{U}\to Y$ is a set of local sections of the locally split map, and consider the pullback $L'=(s^2)^*L$ to $\mathcal{U}^{(2)}$ of L via the map $s^2:\mathcal{U}^{(2)}\to Y^{[2]}$. Since L is locally trivial, this cover can be refined so that L' is trivial over each component, and so has a nonvanishing section $\sigma:\mathcal{U}^{(2)}\to L'$. The trivialization of $dL\to Y^{[3]}$ pulls back to give a trivialization of $\delta L'=(s^3)^*dL\to\mathcal{U}^{(3)}$ which allows $g:=\delta\sigma$ to be regarded as a cochain $g:\mathcal{U}^{(3)}\to\mathbb{C}^*$ and the associativity condition over $Y^{[4]}$ implies that g is closed, hence $[g]\in\check{H}^2_{\mathcal{U}}(X;\mathbb{C}^*)\cong H^3(X;\mathbb{Z})$ is defined to be the Dixmier–Douady class.

To see that this is equivalent to Definition 2.7, it suffices to show that $[g]$ represents the image of $c(L)$ in the total cohomology of the double complex $(\check{C}^\bullet(\mathrm{Et}(\mathcal{U}^{(\bullet)});\mathbb{C}^*),\delta,d)$, where we use $\mathrm{Et}(\mathcal{U})\to X$ itself as the locally split map. For convenience we suppose that $\mathcal{U}$ is a "good cover", meaning that each element of $\mathcal{U}^{(\ell)}$ is contractible for each ℓ. Note that, by this contractibility, the Čech cohomology $\check{H}^\bullet(\mathrm{Et}(\mathcal{U}^{(\ell)});\mathbb{C}^*)$ of the space $\mathrm{Et}(\mathcal{U}^{(\ell)})$ is trivial except in degree 0, where

$$\check{H}^0(\mathrm{Et}(\mathcal{U}^{(\ell)});\mathbb{C}^*)=\Gamma(\mathcal{U}^{(\ell)};\mathbb{C}^*)=\check{C}^{\ell-1}_{\mathcal{U}}(X;\mathbb{C}^*).$$

Thus the (δ,d) spectral sequence of the double complex

$$\begin{array}{ccccccc}
d\uparrow & & d\uparrow & & d\uparrow & & \\
\check{C}^0(\mathrm{Et}(\mathcal{U}^{(3)})) & \xrightarrow{\delta} & \check{C}^1(\mathrm{Et}(\mathcal{U}^{(3)})) & \xrightarrow{\delta} & \check{C}^2(\mathrm{Et}(\mathcal{U}^{(3)})) & \xrightarrow{\delta} & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\check{C}^0(\mathrm{Et}(\mathcal{U}^{(2)})) & \xrightarrow{\delta} & \check{C}^1(\mathrm{Et}(\mathcal{U}^{(2)})) & \xrightarrow{\delta} & \check{C}^2(\mathrm{Et}(\mathcal{U}^{(2)})) & \xrightarrow{\delta} & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\check{C}^0(\mathrm{Et}(\mathcal{U})) & \xrightarrow{\delta} & \check{C}^1(\mathrm{Et}(\mathcal{U})) & \xrightarrow{\delta} & \check{C}^2(\mathrm{Et}(\mathcal{U})) & \xrightarrow{\delta} &
\end{array}$$

degenerates at the E_1 page to

$$\begin{array}{ccc}
{\scriptstyle d=\delta}\uparrow & & \\
\check{H}^0(\mathrm{Et}(\mathcal{U}^{(3)}))=\check{C}^2_{\mathcal{U}}(X) & 0 & 0 \\
{\scriptstyle d=\delta}\uparrow & & \\
\check{H}^0(\mathrm{Et}(\mathcal{U}^{(2)}))=\check{C}^1_{\mathcal{U}}(X) & 0 & 0 \\
{\scriptstyle d=\delta}\uparrow & & \\
\check{H}^0(\mathrm{Et}(\mathcal{U}))=\check{C}^0_{\mathcal{U}}(X) & 0 & 0
\end{array}$$

with the simplicial differential now identified with the Čech differential on $\check{C}^\bullet_{\mathcal{U}}(X)$, and then stabilizes at E_2 to give $\check{H}^\bullet_{\mathcal{U}}(X;\mathbb{C}^*) \cong H^{\bullet+1}(X;\mathbb{Z})$. The image of $[L'] \in \check{C}^1(\mathrm{Et}(\mathcal{U}^{(2)});\mathbb{C}^*)$ in the total cohomology $H^3(X;\mathbb{Z})$ is therefore equivalently represented by its image in $\check{H}^0(\mathcal{U}^{(3)};\mathbb{C}^*) = \check{C}^2_{\mathcal{U}}(X;\mathbb{C}^*)$ on the E_1 page above, and Murray's construction gives an explicit zigzag

$$\begin{array}{ccccc} 0 & & & & \\ \uparrow & & & & \\ g & \longmapsto & 0 & & \\ \uparrow & & \uparrow & & \\ \sigma & \longmapsto & [L'] & \longmapsto & 0 \end{array}$$

realizing $[g]$ as $\mathrm{DD}(L)$. □

2.4 Representability of 3–classes

We proceed to give a characterization of the classes in $H^3(X;\mathbb{Z})$ which are represented by bundle gerbes (L,Y,X) for a given locally split map $Y\to X$. Note that the augmented double complex

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\ \check{Z}^0(Y^{[2]}) & \xrightarrow{\delta} & \check{Z}^1(Y^{[2]}) & \xrightarrow{\delta} & \check{Z}^2(Y^{[2]}) & \xrightarrow{\delta} & \\ {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\ \check{C}^0(Y) & \xrightarrow{\delta} & \check{C}^1(Y) & \xrightarrow{\delta} & \check{C}^2(Y) & \xrightarrow{\delta} & \\ {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\ \check{C}^0(X) & \xrightarrow{\delta} & \check{C}^1(X) & \xrightarrow{\delta} & \check{C}^2(X) & \xrightarrow{\delta} & \\ \uparrow & & \uparrow & & \uparrow & & \\ 0 & & 0 & & 0 & & \end{array} \tag{2-12}$$

has exact columns and therefore trivial total cohomology. Since the (δ,d) spectral sequence of this complex (beginning with the horizontal differential) must necessarily stabilize at the E_3 page (as there are only three rows), it follows that the E_2 differentials are necessarily isomorphisms, which we record in the following form:

Theorem 2.9 *There are isomorphisms*

$$\mathrm{Ker}\{\pi^*\colon \check{H}^{k+1}(X;A)\to\check{H}^{k+1}(Y;A)\} \cong \check{H}^k_Z(Y^{[2]};A)/\check{H}^k(Y;A) \tag{2-13}$$

for each $k \in \mathbb{N}$ *and coefficient group* A*; these isomorphisms are natural with respect to pullback by maps*

$$\begin{array}{ccc} Y_1 & \xrightarrow{\tilde{f}} & Y_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

of locally split spaces, and also with respect to the Bockstein isomorphisms

$$\check{H}^{k+1}_Z(Y^{[2]};\mathbb{Z}) \cong \check{H}^k_Z(Y^{[2]};\mathbb{C}^*) \quad \text{and} \quad \check{H}^{k+1}(X;\mathbb{Z}) \cong \check{H}^k(X;\mathbb{C}^*).$$

Remark It is reasonable to call the isomorphism (2-13) *transgression* from classes on X to (equivalence classes of) classes on $Y^{[2]}$; the map is realized at the chain level by the zigzag (2-11).

In particular, there is a natural isomorphism

$$\text{(2-14)} \qquad \begin{aligned} \operatorname{Ker}\{\pi^*\colon \check{H}^3(X;\mathbb{Z}) \to \check{H}^3(Y;\mathbb{Z})\} &\cong \check{H}^2_Z(Y^{[2]};\mathbb{Z})/\check{H}^2(Y;\mathbb{Z}) \\ &\cong \check{H}^1_Z(Y^{[2]};\mathbb{C}^*)/\check{H}^1(Y;\mathbb{C}^*) \end{aligned}$$

under which the Chern class $c(L) \in \check{H}^1_Z(Y^{[2]};\mathbb{C}^*)/d\check{H}^1(Y;\mathbb{C}^*)$ is the image of $\operatorname{DD}(L)$ in $\check{H}^3(X;\mathbb{Z})$. We see again that $\operatorname{DD}(L)=0$ if and only if $c(L)\in d\check{H}^1(Y;\mathbb{C}^*)$, which, by Proposition 2.6, holds precisely when L is trivial. In combination with Proposition 2.6, this proves the following result:

Theorem 2.10 *A class* $\alpha \in H^3(X;\mathbb{Z})$ *is represented by a bundle gerbe* (L, Y, X) *for a given locally split map* $\pi\colon Y \to X$ *if and only if* $\pi^*\alpha = 0 \in H^3(Y;\mathbb{Z})$.

Remark Another direct (and more geometric) way to show Theorem 2.10 is to use $B\mathrm{PU}(H)$ as a $K(\mathbb{Z},3)$, where H is an infinite-dimensional separable Hilbert space. Here $\mathrm{PU}(H) = \mathrm{U}(H)/\mathrm{U}(1)$ denotes the projective unitary group and, by Kuiper's theorem, $\mathrm{U}(H)$ is contractible, making $\mathrm{PU}(H)$ a $K(\mathbb{Z},2)$. Thus $\alpha \in H^3(X;\mathbb{Z})$ is classified by a map (up to homotopy) to $B\mathrm{PU}(H)$ and represented by a $\mathrm{PU}(H)$–bundle $E \to X$. If $\pi^*\alpha = 0 \in H^3(Y;\mathbb{Z})$, it follows that $\pi^*E \to Y$ admits a global section $s\colon Y \to \pi^*E$. Then, on $Y^{[2]}$, the shift map composed with $s^{[2]}$ determines a map $\chi\colon Y^{[2]} \to \mathrm{PU}(H)$, along which the universal line bundle can be pulled back to give a simplicial bundle $L = \chi^*\mathrm{U}(H) \to Y^{[2]}$ with $\operatorname{DD}(L) = \alpha$.

2.5 Classification of trivializations

Suppose (L, Y, X) is a trivial gerbe. There is an action on the set of trivializations of L by $H^2(X;\mathbb{Z})$ (in the form of equivalence classes of line bundles) as follows. Given a line bundle $P \to Y$ trivializing L, so $dP \cong L$, and $\alpha \in H^2(X;\mathbb{Z})$ representing a line bundle $Q \to X$, the bundle $P \otimes \pi^* Q = P \otimes dQ \to Y$ is another trivialization of L in light of the fact that $d^2 Q$ is canonically trivial.

Proposition 2.11 *Let (L, Y, X) be a trivial gerbe. Then the set of trivializations of L is a torsor for the group* $\mathrm{Im}\{\pi^*\colon H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})\}$.

Proof Clearly the action of $H^2(X;\mathbb{Z})$ factors through its image in $H^2(Y;\mathbb{Z})$; to see that this image acts transitively suppose $P \to Y$ and $P' \to Y$ are two trivializations of L, represented by classes $[P], [P'] \in \check{H}^2(Y;\mathbb{Z})$. Then $d([P]-[P']) = 0 \in H^2_Z(Y^{[2]};\mathbb{Z})$, and from the (δ, d) spectral sequence for (2-12), the E_2 term associated to $\check{C}^2(Y;\mathbb{Z})$ of which must vanish identically, it follows that

$$\mathrm{Ker}\{d\colon \check{H}^2(Y;\mathbb{Z}) \to \check{H}^2_Z(Y^{[2]};\mathbb{Z})\} = \mathrm{Im}\{\pi^*\colon \check{H}^2(X;\mathbb{Z}) \to \check{H}^2(Y;\mathbb{Z})\}, \tag{2-15}$$

and hence $[P]-[P'] = \pi^*[Q]$ for some $[Q] \in \check{H}^2(X;\mathbb{Z})$, represented by a line bundle $Q \to X$. □

2.6 Decomposable and universal gerbes

One consequence of Theorem 2.10 is the existence of the *decomposable gerbes* of [14]. Given a 3–class on X which is the cup product $\alpha \cup \beta$ of $\alpha \in H^2(X;\mathbb{Z})$ and $\beta \in H^1(X;\mathbb{Z})$, we may take $\pi : Y \to X$ to be the circle bundle with Chern class $c(Y) = \alpha$, and then, since Y is canonically trivial when pulled back to itself, it follows that $\pi^*(\alpha \cup \beta) = 0 \cup \pi^*\beta = 0 \in H^3(Y;\mathbb{Z})$, so by Theorem 2.10 the following is immediate:

Proposition 2.12 *For every $\alpha \in H^2(X;\mathbb{Z})$ and $\beta \in H^1(X;\mathbb{Z})$, the circle bundle $Y \to X$ with $c(Y) = \alpha$ supports a bundle gerbe (L, Y, X) with* $\mathrm{DD}(L) = \alpha \cup \beta$.

Remark In [14], the authors go further for X a smooth manifold by constructing a connection on the gerbe from a connection on Y and a function $u \in C^\infty(X; \mathrm{U}(1))$ representing β.

In fact, the image of $\alpha \in H^2(X;\mathbb{Z})$ in $H^1_Z(Y^{[2]};\mathbb{Z})/H^1(Y;\mathbb{Z})$ with respect to the isomorphism (2-14) has a geometric interpretation that will be of use in the construction of decomposable bigerbes in Section 5.1.

Lemma 2.13 *Let $\pi: Y \to X$ be a circle bundle with $c(Y) = \alpha \in H^2(X;\mathbb{Z})$. Then the image of α under the isomorphism* (2-14) *coincides with the pullback to $Y^{[2]}$ of the generator of $H^1(\mathrm{U}(1);\mathbb{Z})$ by the* ***shift map***

(2-16) $$\chi\colon Y^{[2]} \to \mathrm{U}(1), \qquad y_2 = \chi(y_1, y_2) y_1 \quad \text{for } (y_1, y_2) \in Y^{[2]}.$$

Proof This is easiest to see with U(1) coefficients. With respect to the isomorphism $H^1(\mathrm{U}(1);\mathbb{Z}) \cong H^0(\mathrm{U}(1);\mathrm{U}(1))$, the generator corresponds to the identity map, so it suffices to show that the image of $c(Y) \in H^1(X;\mathrm{U}(1))$ is represented by χ, itself regarded as a class in $H^0(Y^{[2]};\mathrm{U}(1))$.

Let $\alpha \in \check{C}^1(X;\mathrm{U}(1))$ represent $c(Y)$; explicitly, we may take α to be defined with respect to a cover $i: \mathcal{U} \to X$ with respect to which Y is (locally) trivialized by $h\colon i^*Y \to \mathcal{U} \times \mathrm{U}(1)$, and we may abuse notation to write $\alpha = \delta h$, meaning $\alpha: \mathcal{U}^{(2)} \to \mathrm{U}(1)$ is defined so that $\delta h = 1 \times \alpha: \mathcal{U}^{(2)} \times \mathrm{U}(1) \to \mathcal{U}^{(2)} \times \mathrm{U}(1)$. Now $\pi^* Y = Y^{[2]} \to Y$ is globally trivialized by $1 \times \chi$, to which $\pi^* h$ may be compared to write $\pi^* h = \gamma\chi$ for $\gamma \in \check{C}^0_{\pi^{-1}\mathcal{U}}(Y;\mathrm{U}(1))$ and then

$$d\alpha = \pi^*\alpha = \delta\pi^* h = \delta\gamma\delta\chi = \delta\gamma \in \check{C}^1_{\pi^{-1}\mathcal{U}}(Y;\mathrm{U}(1)).$$

Finally, a straightforward computation shows that $d\gamma = \chi^{-1}$ in $\check{C}^0_{\pi^{-1}\mathcal{U}}(Y^{[2]};\mathrm{U}(1))$, and then the result follows in observance of Convention 2.2. □

It also follows from Theorem 2.10 that for a connected, locally contractible space X, a gerbe can be constructed representing any integral 3–class using the (based) path fibration $PX \to X$. Indeed, the hypotheses on X imply that the endpoint map $PX \to X$ is locally split, and, since PX is contractible, any 3–class on X vanishes when lifted to PX. The fiber product $P^{[2]}X$ may be identified with the based loop space ΩX, and the isomorphism (2-14) takes the form

$$H^3(X;\mathbb{Z}) \cong \check{H}^1_Z(\Omega X;\mathbb{C}^*),$$

from which we recover the following well-known result:

Theorem 2.14 *For a connected, locally contractible space X, each $\alpha \in H^3(X;\mathbb{Z})$, corresponds to a unique bundle gerbe $L \to \Omega X$ (up to simplicial isomorphisms of the line bundle) with $\mathrm{DD}(L) = \alpha$.*

This "canonical gerbe" on the loop space goes back at least to Brylinski [4]. Murray defines a bundle gerbe version in [16] under the assumption that X is 2–connected, a hypothesis which is removed in [6].

In particular, since $K(\mathbb{Z},3)$ may be realized as a CW complex, its path space carries a universal gerbe.

The simplicial structure on ΩX coming from $P^{[k]}X$ is related to what has been called the *fusion product* in the literature [25; 27; 29; 12]. A point $\gamma=(\gamma_0,\gamma_1,\gamma_2)\in P^{[3]}X$ consists of three paths with common endpoints and so defines three loops, $\ell_i=\pi_i\gamma\in\Omega X=P^{[2]}X$ for $i=0,1,2$, by the simplicial maps, and we say $\ell_1=(\gamma_0,\gamma_2)$ is the *fusion product* of $\ell_2=(\gamma_0,\gamma_1)$ and $\ell_0=(\gamma_1,\gamma_2)$.

A *fusion structure* on a line bundle $L\to\Omega X$ is a collection of associative isomorphisms

$$L_{\ell_1}\cong L_{\ell_2}\otimes L_{\ell_0}$$

for all such triples, which is equivalent to a simplicial line bundle structure on L with respect to $P^{[\bullet]}X$. In this language, then, Theorem 2.14 shows that fusion line bundles on ΩX, which are equivalent to bundle gerbes (L,PX,X), are classified by $H^3(X;\mathbb{Z})$ (see also Waldorf's related results in [27]).

3 Doubling and the free loop space

3.1 Simplicial bundle gerbes and figure-of-eight

Replacing the simplicial line bundle in the definition of a bundle gerbe with a bundle gerbe over X_2 of a simplicial space $X_\bullet$ leads to the notion of a *simplicial bundle gerbe*, which has been defined by Stevenson [22] and is the setting for his definition of bundle 2–gerbes. Here we consider a more limited "product-simplicial" version, which we call simply *doubled*, of this theory, not yet to obtain a version of 2–gerbes as we shall do in Section 4 below, but rather to promote the examples of bundle gerbes involving the based loop space ΩX to those involving the free (unbased) loop space LX by satisfying an additional condition with respect to the simplicial space $\{X^k:k\in\mathbb{N}\}$ of products, with face maps the projections; this space is often denoted by EX in the literature.

While we specialize to this simplicial space of products below, we proceed for the moment in some generality for an arbitrary simplicial space $X_\bullet$, where we continue to use our unusual enumeration convention. Suppose then that (L,Y_2,X_2) is a bundle gerbe over X_2. Using products, inverses and pullbacks, we may define the gerbe

$$\partial L:=\pi_0^*L\otimes\pi_1^*L^{-1}\otimes\pi_2^*L$$

over X_3, where $\pi_j\colon X_3\to X_2$ for $j=0,1,2$ are the face maps of the simplicial space.

Definition 3.1 A *simplicial trivialization* of a bundle gerbe L over X_2 is a trivialization of the bundle gerbe ∂L over X_3. It follows by naturality that for such a gerbe the Dixmier–Douady class $\mathrm{DD}(L) \in H^3(X_2;\mathbb{Z})$ satisfies

$$\partial \mathrm{DD}(L) := \sum_{j=0}^{2} (-1)^j \pi_j^* \mathrm{DD}(L) = \mathrm{DD}(\partial L) = 0 \in H^3(X_3;\mathbb{Z}). \tag{3-1}$$

Note that the gerbe ∂L is defined a priori with respect to the locally split map

$$\pi_0^* Y_2 \times_{X_3} \pi_1^* Y_2 \times_{X_3} \pi_2^* Y_2 \to X_3. \tag{3-2}$$

However, using the notion of gerbe morphism, we may specialize to the setting in which there exists a locally split map $Y_3 \to X_3$ for some fixed space Y_3, along with lifts $\tilde{\pi}_j : Y_3 \to Y_2$ of the $\pi_j : X_3 \to X_2$ for $j = 0, 1, 2$. Indeed, it then follows that Y_3 maps through the product space (3-2), and we may require that

$$\tilde{\pi}_0^* L \otimes \tilde{\pi}_1^* L^{-1} \otimes \tilde{\pi}_2^* L \to Y_3^{[2]} \tag{3-3}$$

is trivial as a bundle gerbe over X_3, where we continue to denote the extensions of $\tilde{\pi}_j$ as maps from $Y_3^{[2]}$ to $Y_2^{[2]}$ by the same notation. When such data is available, we will abuse notation by referring to (3-3) itself as ∂L (as these are (strongly) isomorphic as bundle gerbes over X_3) and a trivialization of (3-3) as a simplicial trivialization of L. Explicitly, this then is the data of a line bundle $S \to Y_3$ such that $dS \cong \partial L$, as summarized in the diagram

$$\begin{array}{ccccccc} L & \longrightarrow & Y_2^{[2]} & \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} & Y_3^{[2]} & \longleftarrow & \partial L \cong dS \\ & & \downarrow\downarrow & & \downarrow\downarrow & & \\ & & Y_2 & \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} & Y_3 & \longleftarrow & S \\ & & \downarrow & & \downarrow & & \\ X_1 & \leftleftarrows & X_2 & \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} & X_3 & & \end{array} \tag{3-4}$$

By naturality of the Dixmier–Douady class, the conclusion (3-1) remains valid.

Remark We do not require that the split maps $Y_\bullet \to X_\bullet$ be compatible by the π_j in the sense of Section 1.2. We also do *not* require that $Y_\bullet$ extend to form (part of) a simplicial space over $X_\bullet$, as indeed our example of interest will not. By contrast, in the setting of the bigerbes defined in Section 4 below, we will employ a *bisimplicial space* of *compatible* locally split maps.

We now specialize to the case in which $X_\bullet = X^\bullet$ consists of products of a fixed space X. As a special case of the fiber product construction over the unique map $\pi : X \to *$ to a 1–point space, this map is globally split, with section $s : * \mapsto x_* \in X$ for any choice of $x_* \in X$. The Čech theory constructions of Sections 1.1 and 2.2 give a map $s^*\colon \check{C}^\bullet(X^k; A) \to \check{C}^\bullet(X^{k-1}; A)$, which in this case does commute with the Čech differential (since s is global), and hence descends to a chain homotopy contraction for each ℓ of the cohomology complex

$$0 \to \check{H}^\ell(X; A) \xrightarrow{\partial} \check{H}^\ell(X^2; A) \xrightarrow{\partial} \check{H}^\ell(X^3; A) \xrightarrow{\partial} \cdots, \tag{3-5}$$

which is therefore exact. (This is a reflection of the well-known fact that the geometric realization $|EX|$ of the simplicial set EX is contractible.) Indeed, denoting by $s = s \times 1 \times \cdots \times 1\colon X^k \to X^{k+1}$ the map $(x_0, \dots, x_{k-1}) \mapsto (x_*, x_0, \dots, x_{k-1})$, it follows that $s^*\partial + \partial s^* = 1$ on $\check{H}^\bullet(X^k; A)$. Note that throughout this section and below, we denote this product simplicial differential by $\partial = \sum_{j=0}^{k-1} (-1)^j \pi_j^*$ rather than d to avoid confusion whenever both appear together.

As a consequence of (3-5) and (3-1), we have the following result:

Proposition 3.2 *For a gerbe (L, Y_2, X^2) with simplicial trivialization over X^3, the Dixmier–Douady class of L descends from X^2 to X itself, so*

$$\mathrm{DD}(L) \in H^3(X; \mathbb{Z})$$

is well defined.

We refer to such a gerbe as a *doubled gerbe*.

Remark When $X_\bullet = X^{[\bullet]}$ is a more general simplicial space of fiber products of a locally split map $X = X_1 \to X_0$, Stevenson in [22] defines additional conditions for a *simplicial gerbe*, including higher associativity conditions over X_4 and X_5 under which the class of a bundle gerbe further descends to a degree four cohomology class on X_0 and such an object is defined to be a *bundle 2–gerbe* on X_0. Here we only use the simplicial condition to descend the 3–class from $X_2 = X^2$ to $X_1 = X$, and will not make use of these additional conditions.

The locally split map of present interest consists of the free (unbased) path space

$$IX = \mathcal{C}([0,1]; X) \to X^2, \quad \gamma \mapsto (\gamma(0), \gamma(1)),$$

mapping to X^2 by the evaluation map on both endpoints. Instead of the fiber products of the pullbacks of IX to X^3 we will take $Y_3 = IX$ also, with evaluation map

$$IX \to X^3, \quad \gamma \mapsto \big(\gamma(0), \gamma(\tfrac{1}{2}), \gamma(1)\big),$$

mapping to the midpoint as well as the endpoints. For disambiguation, we will often distinguish these two incarnations of the free path space by writing them as $I_2 X$

and I_3X, respectively. The three liftings $\widetilde{\pi}_i$ of the projection maps $\pi_j\colon X^3 \to X^2$ taking $\gamma \in I_3X$ to I_2X are obtained by reparametrizing to obtain the three paths

$$\widetilde{\pi}_1\gamma(t) = \gamma(t), \quad \widetilde{\pi}_2\gamma(t) = \gamma(\tfrac{1}{2}t) \quad \text{and} \quad \widetilde{\pi}_0\gamma(t) = \gamma\big(\tfrac{1}{2}(1+t)\big).$$

Remark While it is possible to continue to the right, with $Y_n = I_nX$ mapping to X^n by evaluating along n points, the need to reparametrize paths to define the lifts $\widetilde{\pi}_i$ means that these do not satisfy the simplicial relations, so we do not in fact obtain simplicial spaces $Y_\bullet^{[k]}$ over $X_\bullet$. In particular, the associated maps ∂ on line bundles do not form a complex, ie $\partial^2 L$ is not canonically trivial, except at the bottom level.

Observe that $Y_2^{[2]} = I_2^{[2]}X$ may be naturally identified with the free loop space, $LX = \mathcal{C}(\mathbb{R}/2\pi\mathbb{Z}; X)$, while the space $Y_3^{[2]} = I_3^{[2]}X$ consists of pairs of paths which coincide at their midpoint in addition to their endpoints. The latter may be identified with those loops ℓ in LX for which $\ell\big(\tfrac{1}{2}\pi\big) = \ell\big(\tfrac{3}{2}\pi\big)$, which we call *figure-of-eight* loops, and we accordingly denote the *figure-of-eight loop* (*sub*)*space* by

$$L_8X \cong I_3^{[2]}X.$$

In fact, in this case the product doubling condition for a gerbe over X^2 can be strengthened.

Lemma 3.3 *A gerbe (L, IX, X^2) or (L, IX, X^3) is trivial if and only if $L \to LX$ (resp. $L \to L_8X$) is trivial as a line bundle. In particular, a gerbe (L, IX, X^2) is doubled if and only if $\partial L \to L_8X$ is a trivial line bundle.*

Proof Retraction of paths onto their initial points determines a deformation retract of I_kX onto X, with respect to which the two simplicial maps

$$I_k^{[2]}X \rightrightarrows I_kX$$

both become identified with the evaluation map $I_k^{[2]}X \to X$ at a single parameter value. Thus every line bundle $P \to I_kX$ is isomorphic to a bundle Q pulled back from X, and then $dP \cong Q \otimes Q^{-1} \to I_k^{[2]}X$ is isomorphic to a trivial bundle. This result is independent of k.

Alternatively, we may use the equality (2-15) proved in Proposition 2.11, which here takes the form

$$\begin{aligned}\operatorname{Ker}\{d\colon H^2(I_kX;\mathbb{Z}) \cong H^2(X;\mathbb{Z}) \to H^2_Z(I_k^{[2]}X;\mathbb{Z})\} \\ = \operatorname{Im}\{\pi^* \cong \Delta^*\colon H^2(X^k;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \cong H^2(I_kX;\mathbb{Z})\} \\ = H^2(X;\mathbb{Z}),\end{aligned}$$

since, with respect to the retraction $IX \simeq X$, the map $\pi\colon I_k X \to X^k$ is identified with the diagonal map $\Delta\colon X \to X^k$. Since Δ^* is surjective on cohomology, it follows that $d \equiv 0\colon H^2(I_k X;\mathbb{Z}) \to H^2_Z(I_k^{[2]};\mathbb{Z})$, so every trivial gerbe is in fact trivial as a line bundle. □

Remark In other words, there are no "nontrivial trivial gerbes" with respect to the path spaces. This seems at first confusing in light of Proposition 2.11, since the classifying set $\operatorname{Im}\{\pi^*\colon H^2(X^2;\mathbb{Z}) \to H^2(IX;\mathbb{Z})\} \cong H^2(X;\mathbb{Z})$ of gerbe trivializtions may well be nontrivial, yet these facts are not inconsistent. Indeed, while the only trivial gerbe with respect to $IX \to X^2$ is the equivalence class of the trivial line bundle on LX, the set of *gerbe trivializations* of this trivial gerbe may itself be nontrivial.

On the other hand, we could restrict consideration to *doubled trivializations*, meaning line bundles $P \to I_2 X$ with $dP = L$ such that $\partial P \to I_3 X$ is a trivial bundle. The set of these doubled trivializations is indeed trivial, since, under the retractions $I_k X \simeq X$, the reparametrization maps $\tilde{\pi}_j\colon I_3 X \to I_2 X$ become the identity, and the operator $\partial \cong \mathrm{Id}^* - \mathrm{Id}^* + \mathrm{Id}^*$ likewise becomes the identity, so triviality of ∂P implies triviality of P itself.

The extension of this notion of doubling will be important in the setting of the Brylinski–McLaughlin bigerbe in Section 5.3.

From the point of view of fusion line bundles on loop space, the doubling property corresponds to the "figure-of-eight" condition, as defined in [12; 11]. The following definition is therefore just a repackaging of the above in a different language:

Definition 3.4 A *loop-fusion* structure on a line bundle $L \to LX$ is a fusion structure, meaning a trivialization of $dL \to I_2^{[3]} X$ inducing the canonical trivialization of $d^2 L \to I_2^{[4]} X$, along with the *figure-of-eight* condition that $\partial L \to L_8 X \cong I_3^{[2]} X$ is trivial as a line bundle. An isomorphism of loop-fusion line bundles is a line bundle isomorphism which intertwines the fusion structures.

Theorem 3.5 *The following are naturally in bijection*:

(i) *The set of doubled gerbes* (L, IX, X^2) *up to strong isomorphism.*

(ii) *The set of loop-fusion line bundles on* LX *up to isomorphism.*

(iii) $H^3(X;\mathbb{Z})$.

Proof Equivalence of the first two is a consequence of Lemma 3.3 and Definition 3.4. Doubled gerbes (L, IX, X^2) are classified by their Dixmier–Douady class, which descends to X, as was noted above, and that every element in $H^3(X;\mathbb{Z})$ is represented by a doubled gerbe (equivalently, loop-fusion line bundle) $L \to LX$ follows from Theorem 3.6 below. □

Remark The figure-of-eight structure is weaker than other conditions that have been considered in the literature, such as thin homotopy equivariance in [27], or reparametrization equivariance in [11], which likewise identify categories of fusion line bundles on LX with gerbes on X.

3.2 Loop-fusion cohomology

In fact, applying the above considerations to Čech theory in place of line bundles leads to a general result, which recovers the main theorem in our previous paper [12]. There we defined *loop-fusion cohomology* on LX, which in the present language is equivalent to the group

$$\check{H}^{\ell}_{\mathrm{lf}}(LX;A) = \operatorname{Ker}\{\partial\colon \check{H}^{\ell}_{Z}(LX;A) \to \check{H}^{\ell}_{Z}(L_8X;A)\}.$$

In particular, the set $\check{H}^2_{\mathrm{lf}}(LX;\mathbb{Z})$ classifies loop-fusion line bundles up to isomorphism.

Theorem 3.6 [12] *For each $\ell \in \mathbb{N}$ and topological abelian group A, there is an isomorphism*

$$\check{H}^{\ell}(X;A) \cong \check{H}^{\ell-1}_{\mathrm{lf}}(LX;A).$$

It is additionally shown in [12] that the isomorphism descends via the forgetful map $\check{H}^{\bullet}_{\mathrm{lf}}(LX;A) \to \check{H}^{\bullet}(LX;A)$ to the transgression homomorphism $\check{H}^{\ell}(X;A) \to \check{H}^{\ell-1}(LX;A)$; recall that the latter is defined by composing the pullback along the evaluation map $S^1 \times LX \to X$ with the pushforward along the projection $S^1 \times LX \to LX$ (given by cap product with the fundamental class of S^1).

Proof The result follows from naturality of the isomorphism (2-13), and exactness of (3-5). Applied to the three maps from $I_3^{[k]}X$ to $I_2^{[k]}X$ this yields an isomorphism (omitting the coefficient group for brevity)

$$\begin{aligned}(3\text{-}6)\quad &\operatorname{Ker}\partial \cap \operatorname{Ker}\pi^* \\ &\subset \check{H}^{\ell}(X^2) \\ &\cong \operatorname{Ker}\{\partial\colon \check{H}^{\ell-1}_{Z}(I_2^{[2]}X)/d\check{H}^{\ell-1}(I_2X) \to \check{H}^{\ell-1}_{Z}(I_3^{[2]}X)/d\check{H}^{\ell-1}(I_3X)\}.\end{aligned}$$

However, as noted in the proof of Lemma 3.3, the deformation retraction of the free path spaces $I_k X$ onto X implies that $d\colon \check{H}^{\ell-1}(I_k X; A) \to \check{H}^{\ell-1}(I_k^{[2]} X; A)$ is trivial, so the quotients in (3-6) disappear. Moreover, by exactness of (3-5), the kernel of ∂ in $\check{H}^{\ell}(X^2; A)$ is the image of $\check{H}^{\ell}(X; A)$ and this is automatically in the kernel of $\pi^*\colon \check{H}^{\ell}(X^2; A) \to \check{H}^{\ell}(I_2 X; A)$ under the retraction $I_2 X \simeq X$, so (3-6) simplifies to

$$\check{H}^{\ell}(X; A) \cong \operatorname{Ker}\{\partial\colon \check{H}_Z^{\ell-1}(I_2^{[2]} X; A) \to \check{H}_Z^{\ell-1}(I_3^{[2]} X; A)\} = \check{H}_{\mathrm{lf}}^{\ell-1}(LX; A),$$

as claimed. □

4 Bundle bigerbes

4.1 Locally split squares

Bigerbes as introduced below are based on the following notion.

Definition 4.1 A commutative diagram

$$\begin{array}{ccc} Y_2 & \longleftarrow & W \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ X & \xleftarrow{\ \pi_1\ } & Y_1 \end{array} \tag{4-1}$$

is a *locally split square* if $Y_i \to X$ for $i = 1, 2$ and the induced map $W \to Y_1 \times_X Y_2$ are locally split.

There is manifest symmetry in the definition.

Lemma 4.2 *If (W, Y_2, Y_1, X) is a locally split square, then the fiber projections $Y_1 \times_X Y_2 \to Y_i$ for $i = 1, 2$, and hence all four maps in* (4-1), *are locally split, and the horizontal maps are compatible with respect to the vertical ones in the sense of Section 1.2.*

Proof Let $s^1\colon \mathrm{Et}(\mathcal{U}_X) \to Y_1$ be a collection of sections of π_1 over a cover $\mathcal{U}_X$ of X. Then $\mathcal{U}_2 = \pi_2^{-1}\mathcal{U}_X$ is a cover of Y_2 and $s^1 \times \mathrm{Id}\colon \mathrm{Et}(\mathcal{U}_2) \to Y_1 \times_X Y_2$ is a collection of sections of the projection $Y_1 \times_X Y_2 \to Y_2$, which is therefore locally split.

From the definition, there is a cover $\mathcal{V}$ of $Y_1 \times_X Y_2$ and a section $t\colon \mathrm{Et}(\mathcal{V}) \to W$ of the induced map $p\colon W \to Y_1 \times_X Y_2$. Passing to the refinement $\mathcal{V} \cap \pi_2^{-1}\mathcal{U}_2$ we may arrange that π_2 is a map of covers from $\mathrm{Et}(\mathcal{V})$ to $\mathrm{Et}(\mathcal{U}_2)$. Passing to the refinement $\mathcal{U}_2' = \mathcal{U}_2 \cap (s^1 \times \mathrm{Id})^{-1}\mathcal{V}$ of $\mathcal{U}_2$, on which $\tilde{s}^1 = t \circ (s^1 \times \mathrm{Id})$ determines a local section of the composition $W \to Y_2$, it follows that the latter is locally split.

Moreover, it follows from the fact that $p \circ t$ coincides with the map of covers $\mathcal{V} \to Y_1 \times_X Y_2$ that the diagram

$$\begin{array}{ccc} \mathrm{Et}(\mathcal{U}_2') & \xrightarrow{\tilde{s}^1 = t\circ(s^1\times 1)} & W \\ \downarrow & & \downarrow \\ \mathrm{Et}(\mathcal{U}_X) & \xrightarrow{\quad s^1 \quad} & Y_1 \end{array}$$

commutes, so the horizontal locally split maps are compatible with respect to the vertical ones. □

As in Section 2.1, let $Y_i^{[k]}$ be the k–fold fiber product $Y_i \times_X \cdots \times_X Y_i$ for $i = 1, 2$. Then $Y_1^{[\bullet]}$ and $Y_2^{[\bullet]}$ each form simplicial spaces over X, giving the bounding column and row in (4-2) below.

Setting $W^{[1,1]} = W$ and

$$W^{[1,k]} = W \times_{Y_1} \cdots \times_{Y_1} W \quad \text{and} \quad W^{[k,1]} = W \times_{Y_2} \cdots \times_{Y_2} W$$

with projection maps $W^{[1,k]} \to W^{[1,k-1]}$ and $W^{[k,1]} \to W^{[k-1,1]}$ gives simplicial spaces over Y_1 and Y_2 extending above and to the right of $W^{[1,1]}$ in (4-2).

That the rest of the quadrant can then be filled out unambiguously by fiber products is a consequence of the following result:

Proposition 4.3 *For each n and m, there is a natural isomorphism*

$$W^{[m,n]} := \overbrace{W^{[m,1]} \times_{Y_1^{[m]}} \cdots \times_{Y_1^{[m]}} W^{[m,1]}}^{n \text{ times}} \cong \overbrace{W^{[1,n]} \times_{Y_2^{[n]}} \cdots \times_{Y_2^{[n]}} W^{[1,n]}}^{m \text{ times}}.$$

Proof Both sides may be identified with the set of tuples $(w_{i,j} : 1 \le i \le m,\ 1 \le j \le n) \in W^{mn}$ such that for each i, $(w_{i,1}, \dots, w_{i,n})$ all map to a fixed $y_{1,i} \in Y_1$ and, for each j, $(w_{1,j}, \dots, w_{m,j})$ all map to a fixed $y_{2,j} \in Y_2$, and where every $y_{1,i}$ and $y_{2,j}$ sit over a fixed $x \in X$. □

The spaces $W^{[\bullet,\bullet]}$ in the resulting diagram

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
Y_2^{[3]} & \longleftarrow & W^{[1,3]} & \leftleftarrows & W^{[2,3]} & \Lleftarrow & W^{[3,3]} & \cdots\cdots \\
\downdownarrows\downarrow & & \downdownarrows\downarrow & & \downdownarrows\downarrow & & \downdownarrows\downarrow & \\
Y_2^{[2]} & \longleftarrow & W^{[1,2]} & \leftleftarrows & W^{[2,2]} & \Lleftarrow & W^{[3,2]} & \cdots\cdots \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows & \\
Y_2 & \longleftarrow & W^{[1,1]} & \leftleftarrows & W^{[2,1]} & \Lleftarrow & W^{[3,1]} & \cdots\cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
X & \longleftarrow & Y_1 & \leftleftarrows & Y_1^{[2]} & \Lleftarrow & Y_1^{[3]} & \cdots\cdots
\end{array}
\tag{4-2}
$$

form a *bisimplicial space* over X, meaning a functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Top}/X$, where Δ is the simplex category. In particular, $W^{[m,\bullet]}$ and $W^{[\bullet,n]}$ are simplicial spaces over $Y_1^{[m]}$ and $Y_2^{[n]}$, respectively, and the squares commute for consistent choices of maps. For notational convenience, we also set $W^{[k,0]} = Y_1^{[k]}$, $W^{[0,k]} = Y_2^{[k]}$ and $W^{[0,0]} = X$.

4.2 Bigerbes

If $L \to W^{[m,n]}$ is a line bundle over one of the spaces in (4-2) then its two simplicial differentials are

$$
d_1 L = \bigotimes_{i=0}^{m} (\pi_i^1)^* L^{(-1)^i} \to W^{[m+1,n]}, \quad d_2 L = \bigotimes_{i=0}^{n} (\pi_i^2)^* L^{(-1)^i} \to W^{[m,n+1]},
$$

where $\pi_j^1 \colon W^{[m+1,n]} \to W^{[m,n]}$ for $0 \le j \le m$ and $\pi_j^2 \colon W^{[m,n+1]} \to W^{[m,n]}$ for $0 \le j \le n$ denote the fiber projection maps. The bundles $d_1 d_1 L$ and $d_2 d_2 L$ are canonically trivial, and there is a natural isomorphism $d_1 d_2 L \cong d_2 d_1 L$.

Definition 4.4 A *bigerbe* consists of a locally split square (W, Y_2, Y_1, X), a line bundle $L \to W^{[2,2]}$ and trivializations of $d_1 L$ and $d_2 L$, which induce the same trivialization of $d_1 d_2 L$ and which induce the canonical trivializations of $d_1^2 L$ and $d_2^2 L$. We denote the bigerbe by (L, W, Y_2, Y_1, X) or simply L.

For reasons that will become clear below, the order of the spaces Y_1 and Y_2, or equivalently the orientation of the square (4-1), is part of the data of the bigerbe.

Definition 4.5 If $Q_1 \to W^{[1,2]}$ and $Q_2 \to W^{[2,1]}$ are line bundles which are simplicial with respect to d_2 and d_1, respectively — so $d_2 Q_1$ over $W^{[1,3]}$ is equipped with a trivialization inducing the canonical trivialization of $d_2^2 Q_1$ and similarly for Q_2 — then $d_1 Q_1 \otimes d_2 Q_2^{-1}$ has a canonical bigerbe structure. A bigerbe L is said to be *trivial* if

$$L \cong d_1 Q_1 \otimes d_2 Q_2^{-1}$$

for Q_1 and Q_2 as above, with the isomorphism identifying the bigerbe structure on L with the canonical one on $d_1 Q_1 \otimes d_2 Q_2^{-1}$; such an isomorphism is referred to as a *trivialization* of L.

In particular, L is trivial if either

(i) $L \cong d_1 P$, where $P \to W^{[1,2]}$ is a line bundle with trivialization $d_2 P \cong \mathbb{C}$ inducing the canonical trivialization of $d_2^2 P$, or

(ii) $L \cong d_2 Q$, where $Q \to W^{[2,1]}$ is a line bundle with trivialization $d_1 Q \cong \mathbb{C}$ inducing the canonical trivialization of $d_1^2 Q$,

as in either case we can take the trivial bundle on the other factor.

As for ordinary bundle gerbes, we proceed to define pullbacks and products of bigerbes.

Lemma 4.6 *If (W, Y_2, Y_1, X) and (W', Y_2', Y_1', X) are locally split squares over X and $f\colon X' \to X$ is a continuous map, then*

(i) *$(f^*(W), f^*(Y_2), f^*(Y_1), X')$ is a locally split square over X', and*

(ii) *$(W \times_X W', Y_2 \times_X Y_2', Y_1 \times_X Y_1', X)$ is a locally split square over X.*

Proof By hypothesis there are covers $\mathcal{U}_i \to X$ admitting sections $s^i\colon \mathcal{U}_i \to Y_i$ of $\pi_i\colon Y_i \to X$ and a cover $\mathcal{V} \to Y_1 \times_X Y_2$ admitting a section $t\colon \mathcal{V} \to W$ of the universal map $p\colon W \to Y_1 \times_X Y_2$.

Pullback of these by f gives covers $f^{-1}\mathcal{U}_i \to X'$ and sections $f^* s^i\colon f^{-1}\mathcal{U}_i \to f^* Y_i$, with $f^* s^i = 1 \times s^i \circ f\colon f^{-1}\mathcal{U}_i \to X' \times_X Y_i = f^* Y_i$, where the section is composed with the lift $f\colon f^{-1}\mathcal{U}_i \to \mathcal{U}_i$ and 1 denotes the inclusion map $f^{-1}\mathcal{U}_i \to X'$ of covers. Similarly, if $\tilde{f}\colon f^*(Y_1 \times_X Y_2) \to Y_1 \times_X Y_2$ denotes the natural lift over f, then $\tilde{f}^{-1}\mathcal{V} \to f^*(Y_1 \times_X Y_2)$ supports the section $\tilde{f}^* t\colon \tilde{f}^{-1}\mathcal{V} \to f^* W$ of the natural map $f^* W \to f^*(Y_1 \times_X Y_2) \cong (f^* Y_1) \times_{X'} (f^* Y_2)$, proving (i).

For the fiber product, the cover $\mathcal{U}_i \cap \mathcal{U}_i' \to X$ admits sections $s^i \times (s')^i$ of $Y_i \times_X Y_i'$, and then $(Y_1 \times_X Y_1') \times_X (Y_2 \times_X Y_2') \cong (Y_1 \times_X Y_2) \times_X (Y_1' \times_X Y_2')$ may be equipped with the cover $\mathcal{V} \times_X \mathcal{V}'$, which admits the section $t \times t'$ to $W \times_X W'$, proving (ii). □

Definition 4.7 If $L \to W^{[2,2]}$ is a bigerbe with respect to the locally split square (W, Y_2, Y_1, X) and $f\colon X' \to X$ is a continuous map, then the *pullback* of L is the bigerbe $\tilde{f}^* L \to f^*(W^{[2,2]})$ with respect to the locally split square

$$(f^*(W), f^*(Y_2), f^*(Y_1), X').$$

If $L = (L, W, Y_2, Y_1, X)$ and $L' = (L', W', Y_2', Y_1', X)$ are bigerbes on X, then the *product* of L and L' is the bigerbe

$$(L \otimes L', W \times_X W', Y_2 \times_X Y_2', Y_1 \times_X Y_1', X).$$

Next we define (strong) morphisms and stable isomorphisms for bigerbes. A *morphism* of locally split squares $(W', Y_2', Y_1', X') \to (W, Y_2, Y_1, X)$ is a collection of maps from each space in the first square to the corresponding space in the second, with each of the relevant squares commuting. As for bundle gerbes we do not require compatibility in the sense of Section 1.2 of the locally split maps of the first square with those of the second. By naturality of fiber products, these maps extend to maps $W'^{[m,n]} \to W^{[m,n]}$ for each $(m,n) \in \mathbb{N}_0^2$ commuting with the various fiber projections in both directions. By abuse of notation we will denote all such maps by a single letter, say $f\colon W'^{[m,n]} \to W^{[m,n]}$.

Definition 4.8 If (L, W, Y_2, Y_1, X) and (L', W', Y_2', Y_1', X') are bigerbes over X and X', respectively, then a (*strong*) *morphism* from L' to L consists of a morphism $f\colon (W', Y_2', Y_1', X') \to (W, Y_2, Y_1, X)$ and an isomorphism $L' \cong f^* L$ over $W'^{[2,2]}$ which intertwines the sections of $d_i L$ and $d_i f^* L$ for $i = 1, 2$. A (*strong*) *isomorphism* is a morphism for which $X = X'$ and $f = \mathrm{Id}\colon X \to X'$.

Finally, a *stable isomorphism* of bigerbes L and L' over X is a (strong) isomorphism

$$L \otimes T \cong L' \otimes T,$$

where T and T' are trivial bigerbes.

4.3 The 4–class of a bigerbe

The abelian groups of Čech cochains $\check{C}^\ell(W^{[j,k]}; A) = \lim_{\mathcal{U}} \check{C}^\ell_{\mathcal{U}}(W^{[j,k]}; A)$ are defined for each ℓ, j and $k \in \mathbb{N}_0$, by passage to the direct limit over covers of $W^{[j,k]}$. Using Proposition 1.2 we may define

$$d_1 = \sum_{j=0}^{m} (-1)^j (\pi_j^1)^* \colon \check{C}^\ell(W^{[m,n]}; A) \to \check{C}^\ell(W^{[m+1,n]}; A),$$

$$d_2 = \sum_{j=0}^{n} (-1)^j (\pi_j^2)^* \colon \check{C}^\ell(W^{[m,n]}; A) \to \check{C}^\ell(W^{[m,n+1]}; A),$$

which are differentials commuting with one another and with the Čech differential δ. Thus $(\check{C}^\bullet(W^{[\bullet,\bullet]}; A), \delta, d_1, d_2)$ forms a triple complex, and combining Proposition 1.4 and Lemma 4.2 leads to the following result:

Proposition 4.9 *Compatible sections of $Y_i \to X$ and $W \to Y_i$ determine homotopy contractions for the d_i which commute with the other simplicial differential and for each fixed ℓ and k, the subcomplex*

$$(\operatorname{Ker}\{d_1 \colon \check{C}^\ell(W^{[k,\bullet]}) \to \check{C}^\ell(W^{[k+1,\bullet]})\}, d_2)$$

is exact, and similarly with indices reversed.

Proof It follows from the fact that $W^{[j,k]}$ is naturally isomorphic to the j–fold fiber product of $W^{[1,k]}$ over $W^{[0,k]}$ as well as to the k–fold fiber product of $W^{[j,1]}$ over $W^{[j,0]}$ that the local sections s^1 and s^2 of Lemma 4.2 induce, as in the proof of Proposition 2.3, compatible sections of the projections π_0^i (denoted by dashed arrows to avoid writing covers), such that

$$\begin{array}{ccc} W^{[j,k+1]} & \overset{s^1}{\dashrightarrow} & W^{[j+1,k+1]} \\ \big\downarrow \pi_0^2 & & \big\downarrow \pi_0^2 \\ W^{[j,k]} & \overset{s^1}{\dashrightarrow} & W^{[j+1,k]} \end{array} \qquad \begin{array}{ccc} W^{[j,k+1]} & \overset{\pi_0^1}{\longleftarrow} & W^{[j+1,k+1]} \\ s^2 \, \big\uparrow & & s^2 \, \big\uparrow \\ W^{[j,k]} & \overset{\pi_0^1}{\longleftarrow} & W^{[j+1,k]} \end{array}$$

commute.

The claim then follows if we define chain homotopy contractions by $s_\ell^{i\,*}$ for the d_i directions of the triple complex, respectively. □

Just as a bundle gerbe has a Dixmier–Douady class in $H^3(X; \mathbb{Z})$, a bigerbe determines a characteristic class in $H^4(X; \mathbb{Z})$. To see this, consider the truncation of the triple complex $(\check{C}^\bullet(W^{[\bullet,\bullet]}; A), \delta, d_1, d_2)$, which we denote by $(\check{Z}^\bullet(W^{[\bullet,\bullet]}; A), \delta, d_1, d_2)$, where

$$\check{Z}^\ell(W^{[j,k]}; A) = \begin{cases} \check{C}^\ell(W^{[j,k]}; A), & \text{if } [j,k] = [1,1], \\ \operatorname{Ker} d_1 \subset \check{C}^\ell(W^{[2,1]}; A) & \text{if } [j,k] = [2,1], \\ \operatorname{Ker} d_2 \subset \check{C}^\ell(W^{[1,2]}; A) & \text{if } [j,k] = [1,2], \\ \operatorname{Ker} d_1 \cap \operatorname{Ker} d_2 \subset \check{C}^\ell(W^{[2,2]}; A) & \text{if } [j,k] = [2,2], \\ 0 & \text{otherwise.} \end{cases}$$

Suppressing the Čech direction, we may depict the truncated complex as

$$
\begin{array}{ccccc}
0 & & 0 & & \\
d_2\uparrow & & d_2\uparrow & & \\
\check{Z}^\bullet(W^{[1,2]};A) & \xrightarrow{d_1} & \check{Z}^\bullet(W^{[2,2]};A) & \xrightarrow{d_1} & 0 \\
d_2\uparrow & & d_2\uparrow & & \\
\check{C}^\bullet(W^{[1,1]};A) & \xrightarrow{d_1} & \check{Z}^\bullet(W^{[2,1]};A) & \xrightarrow{d_1} & 0
\end{array} \tag{4-3}
$$

In particular, the leftmost column and bottom row of (4-3) are taken to have d_i–degree 0. Then, following Convention 2.2, the total differential on (4-3) is

$$D = \delta + (-1)^\ell d_1 + (-1)^{\ell+m+1} d_2 \quad \text{on } \check{Z}^\ell(W^{[m,n]};A) \tag{4-4}$$

since $\check{C}^\bullet(W^{[m,n]})$ occupies the $(m-1,n-1)$ coordinate in the (d_1,d_2) plane.

Employing a spectral sequence argument twice immediately yields the following result:

Proposition 4.10 *The triple complex $(\check{Z}^\bullet(W^{[\bullet,\bullet]};A),\delta,d_1,d_2)$ has total cohomology isomorphic to the ordinary cohomology $\check{H}^\bullet(X;A)$ of X.*

Proof The total differential (4-4) of the (δ,d_1,d_2) triple complex can be written as $D = D_1 + (-1)^{\ell+m+1}d_2$ on $\check{C}^\ell(W^{[m,n]};A)$, where $D_1 = \delta + (-1)^\ell d_1$ is the total differential of the (δ,d_1) double complex. By exactness of d_2, the total cohomology of the (D_1,d_2) double complex is isomorphic to the cohomology of the D_1 (double) complex $\check{C}^\bullet(Y_1^{[\bullet]};A)$, which in turn is isomorphic to $\check{H}^\bullet(X;A)$ as in Theorem 2.5. □

Lemma 4.11 *The line bundle $L \to W^{[2,2]}$ of a bigerbe determines a pure cocycle $c(L) \in \check{Z}^1(W^{[2,2]};\mathbb{C}^*)$ in the triple complex* (4-3), *and conversely any line bundle with $c(L) \in \check{Z}^1(W^{[2,2]};\mathbb{C}^*)$ determines a bigerbe. Moreover, the pure cocycle $c(L) \in \check{Z}^1(W^{[2,2]};\mathbb{C}^*)$ is a coboundary if and only if L admits a trivialization.*

Proof The line bundle L is represented on some cover by its "transition" Chern class and hence by an element $c(L) \in \check{C}^1(W^{[2,2]};\mathbb{C}^*)$ such that $\delta c(L) = 0$. The simplicial trivializations of d_iL for $i = 1,2$ are represented by elements $\alpha_1 \in \check{C}^0(W^{[3,2]})$ and $\alpha_2 \in \check{C}^0(W^{[2,3]})$ such that $d_i\alpha_i = 0$, $\delta\alpha_i = d_ic(L)$ and $\partial(d_2\alpha_1 - d_1\alpha_2) = 0$. In other words, the triple $(c(L),-\alpha_1,\alpha_2)$ forms a cocycle in the triple complex $(\check{C}^\bullet(W^{[\bullet,\bullet]};\mathbb{C}^*),\delta,d_1,d_2)$. Now, by exactness, we may obtain d_i–preimages β_i of the α_i, and then $c(L)$ can be altered by the image under δ of the β_i to obtain a pure cocycle, which we again denote by $c(L) \in \check{Z}^1(W^{[2,2]};\mathbb{C}^*)$.

A coboundary for $c(L)$ in the triple complex consists of a triple (α,β,γ), where $\alpha\in\check{Z}^0(W^{[2,2]})$, $\beta\in\check{Z}^1(W^{[1,2]})$ and $\gamma\in\check{Z}^1(W^{[2,1]})$ such that $\delta\beta=0$ and $\delta\gamma=0$, and

$$D(\alpha,\beta,\gamma)=\delta\alpha-d_1\beta+d_2\gamma=c(L). \tag{4-5}$$

But this amounts precisely to saying that β and γ determine d_1 and d_2 simplicial line bundles $Q\to W^{[2,1]}$, and $P\to W^{[1,2]}$, such that L is isomorphic (with isomorphism determined by α) to $d_1P\otimes d_2Q^{-1}$, ie L is trivial. Conversely, a trivialization of the bigerbe L determines such a coboundary (4-5). □

Definition 4.12 Let (L,W,Y_2,Y_1,X) be a bigerbe over X. The *characteristic* 4–*class* of L is the image $G(L)\in H^4(X;\mathbb{Z})\cong H^3(X;\mathbb{C}^*)$ of the hypercohomology class of $c(L)\in\check{Z}^1(W^{[2,2]};\mathbb{C}^*)$ in the triple complex (4-3).

For an explicit zigzag construction of $G(L)$ from $c(L)$, see (4-8) and (4-9) in the proof of Theorem 4.15 below.

Because of the need to introduce signs in the (δ,d_1,d_2) total complex following Convention 2.2, the sign of the class $G(L)\in H^4(X;\mathbb{Z})$ depends on the order of Y_1 and Y_2, which is to say the orientation of the locally split square. In particular, reversing the roles of Y_1 and Y_2 while keeping the bundle $L\to W^{[2,2]}$ fixed determines a bundle gerbe L' with class $G(L')=-G(L)$. Indeed, this follows from the fact that the total differential $D'=\delta+(-1)^\ell d_2+(-1)^{\ell+n+1}d_1$ on $\check{C}^\ell(W^{[m,n]})$, where we have interchanged the roles of d_1 and d_2, is intertwined with $D=\delta+(-1)^\ell d_2+(-1)^{\ell+m+1}d_2$ by the automorphism $(-1)^{(m+1)(n+1)}$ of the triple complex. In particular, this amounts to multiplication by -1 on $\check{Z}^\bullet(W^{[2,2]})$, exchanging $c(L)$ and $-c(L)$.

Alternatively, from the explicit zigzag (4-8) and (4-9) it follows that $c(L)$ is the double transgression of $G(L)\in H^4(X;\mathbb{Z})$ (in the sense of the isomorphism (2-13)) first to $H^3_Z(Y_1^{[2]};\mathbb{Z})$ and then to $H^2_Z(W^{[2,2]};\mathbb{Z})$. The transgression the other way, first to $Y_2^{[2]}$ and then to $W^{[2,2]}$, has the opposite image $-c(L)$.

Theorem 4.13 *The characteristic* 4*–class* $G(L)$ *vanishes if and only if* L *is trivial as a bigerbe, and is natural with respect to pullback, product and inverses in that*

$$G(f^*L)=f^*G(L),\quad G(L_1\otimes L_2)=G(L_1)+G(L_2),\quad G(L^{-1})=-G(L).$$

A morphism $f:(L',W',Y_2',Y_1',X')\to(L,W,Y_2,Y_1,X)$ *of bigerbes induces an equality* $f^*G(L)=G(L')$*, and two bigerbes* L *and* L' *over* X *satisfy* $G(L)=G(L')$ *if and only if they are stably isomorphic.*

Proof That $G(L) = 0$ if and only if L admits a trivialization was proved in Lemma 4.11. The pullback of a locally split square over X by a continuous map $f\colon X' \to X$ induces natural maps $f^*W^{[m,n]} \to W^{[m,n]}$ commuting with each π^i_j, and thus a map $f^*\check{C}^\bullet(W^{[\bullet,\bullet]}) \to \check{C}^\bullet(f^*W^{[\bullet,\bullet]})$ of triple complexes. The naturality of G with respect to pullbacks and morphisms is then a consequence of the naturality of the spectral sequences which identify the total cohomology of the triple complex with the cohomology of X and X', respectively. Naturality with respect to products and inverses is a direct consequence of the fact that we can take $[L^{-1}] = -c(L)$ and $[L_1 \otimes L_2] = [\mathrm{pr}_1^* L_1] + [\mathrm{pr}_2^* L_2]$ as representatives. Finally, if L and L' are stably isomorphic, then $G(L) = G(L')$ by triviality and products, and, conversely, if $G(L) = G(L')$, then $L^{-1} \otimes L' = T$ is trivial, from which a stable isomorphism $L \otimes T \cong L'$ may be constructed. □

4.4 Representability of 4–classes

To characterize those 4–classes which are represented by bigerbes over a given locally split square, we follow a similar argument to that in Section 2.4, though it is necessary in this case to go further in a spectral sequence for the triple complex. Consider the augmented triple complex

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
{\scriptstyle d_2}\uparrow & & {\scriptstyle d_2}\uparrow & & {\scriptstyle d_2}\uparrow & & \\
\check{Z}^\bullet(Y_2^{[2]}) & \xrightarrow{d_1} & \check{Z}^\bullet(W^{[1,2]}) & \xrightarrow{d_1} & \check{Z}^\bullet(W^{[2,2]}) & \xrightarrow{d_1} & 0 \\
{\scriptstyle d_2}\uparrow & & {\scriptstyle d_2}\uparrow & & {\scriptstyle d_2}\uparrow & & \\
\check{C}^\bullet(Y_2) & \xrightarrow{d_1} & \check{C}^\bullet(W^{[1,1]}) & \xrightarrow{d_1} & \check{Z}^\bullet(W^{[2,1]}) & \xrightarrow{d_1} & 0 \\
{\scriptstyle d_2}\uparrow & & {\scriptstyle d_2}\uparrow & & {\scriptstyle d_2}\uparrow & & \\
\check{C}^\bullet(X) & \xrightarrow{d_1} & \check{C}^\bullet(Y_1) & \xrightarrow{d_1} & \check{Z}^\bullet(Y_1^{[2]}) & \xrightarrow{d_1} & 0
\end{array}
\tag{4-6}
$$

with the leftmost column and bottom row considered as degree -1 for d_1 and d_2, respectively.

Lemma 4.14 *Fix $\ell \geq 1$ and an abelian group A. Suppose $[\alpha] \in H^\ell(X; A)$ satisfies $\pi_i^*[\alpha] = 0 \in \check{H}^\ell(Y_i; A)$ for $i = 1, 2$. Then there is a well-defined* ***transgression class*** *defined by*

$$
\mathrm{Tr}[\alpha] = [d_1\beta_2 - d_2\beta_1] \in \check{H}^{\ell-1}(W; A)/(H^{\ell-1}(Y_1; A) \oplus H^{\ell-1}(Y_2; A)),
\tag{4-7}
$$

where $\beta_i \in \check{C}^{\ell-1}(Y_i; A)$ are any elements satisfying $\delta\beta_i = \pi_i^*\alpha \in \check{C}^{\ell}(Y_i; A)$ for a representative $\alpha \in \check{C}^{\ell}(X; A)$.

Remark This transgression can be understood as the $W^{[1,1]}$ component of the E_2 page differential of the (δ, D_{12}) spectral sequence of $(\check{C}^\bullet(W^{[\bullet,\bullet]}), \delta, D_{12})$ applied to $[\alpha]$, where we have rolled up d_1 and d_2 into a total differential $D_{12} = d_1 \pm d_2$.

In fact, to observe the sign convention discussed in Convention 2.2, we should properly define $\mathrm{Tr}[\alpha]$ as the class $[(-1)^{\ell+1} d_2\beta_1 + (-1)^{\ell+1} d_1\beta_2]$, where $\delta\beta_1 = (-1)^{\ell} d_1\alpha$ and $\delta\beta_2 = (-1)^{\ell+1} d_2\alpha$, but then cancellation of the two factors of $(-1)^{\ell+1}$ and exchanging β_1 with $-\beta_1$ makes this equivalent to the definition given above.

Proof With α, β_1 and β_2 as above, it follows that $d_1\beta_2 - d_2\beta_1$ is a cocycle since

$$\delta(d_1\beta_2 - d_2\beta_1) = d_1\pi_2^*\alpha - d_2\pi_1^*\alpha = d_1 d_2\alpha - d_2 d_1\alpha = 0.$$

Another choice of representative $\alpha' = \alpha + \delta\gamma$ can be incorporated as a different choice $\beta_i' = \beta_i + d_i\gamma$ of the β_i; moreover, if $\beta_i' \in \check{C}^{\ell-1}(Y_i)$ are another choice of bounding chains for $\pi_i\alpha$, then $\delta(\beta_i - \beta_i') = 0$ and

$$(d_1\beta_2 - d_2\beta_1) - (d_1\beta_2' - d_2\beta_1') = d_1(\beta_2 - \beta_2') + d_2(\beta_1' - \beta_1)$$

is in the image under $\begin{bmatrix} d_2 & d_1 \end{bmatrix}$ of $\check{H}^{\ell-1}(Y_1) \oplus H^{\ell-1}(Y_2)$. □

Theorem 4.15 *A locally split square (W, Y_2, Y_1, X) supports a bigerbe with a given class $[\alpha] \in H^4(X; \mathbb{Z})$ if and only if*

(i) $\pi_i^*[\alpha] = 0 \in H^4(Y_i, \mathbb{Z})$ *for $i = 1, 2$, and*

(ii) $\mathrm{Tr}[\alpha] = 0 \in H^3(W; \mathbb{Z})/(H^3(Y_1; \mathbb{Z}) \oplus H^3(Y_2; \mathbb{Z}))$.

Proof By naturality of the Bockstein isomorphism, it suffices to work one degree lower with $\mathbb{C}^*$ coefficients. Thus suppose $\alpha \in \check{C}^3(X; \mathbb{C}^*)$ represents $[\alpha]$. Since by hypothesis $\mathrm{Tr}[\alpha]$ vanishes, there exist representatives $\beta_i \in \check{C}^2(Y_i; \mathbb{C}^*)$ such that $[d_1\beta_2 - d_2\beta_1] = 0 \in \check{H}^2(W; \mathbb{C}^*)$; thus $d_1\beta_2 - d_2\beta_1 = \delta\gamma$ for $\gamma \in \check{C}^1(W; \mathbb{C}^*)$. Then we claim $d_1 d_2\gamma = d_2 d_1\gamma \in \check{Z}^1(W^{[2,2]}; \mathbb{C}^*)$ is a pure cocycle and that a bigerbe $L \to W^{[2,2]}$ with

$$c(L) = -d_1 d_2\gamma$$

satisfies $G(L) = [\alpha]$. Indeed, it is obvious that $d_i(d_1 d_2\gamma) = 0$ for $i = 1, 2$; moreover, $\delta d_1 d_2\gamma = d_1 d_2\delta\gamma = d_1 d_2(d_1\beta_2 - d_2\beta_1) = 0$ as well, so, by Lemma 4.11, $d_1 d_2\gamma = -c(L) \in \check{Z}^1(W^{[2,2]}; \mathbb{C}^*)$ for a bigerbe $L \to W^{[2,2]}$.

To see that $G(L)=\alpha$, we follow the proof of Proposition 4.10, carefully observing the sign convention (4-4) and observe that

(4-8)
$$\begin{array}{ccc} c(L) & & \\ {\scriptstyle d_2}\uparrow & & \\ -d_1\gamma & \xmapsto{D_1=\delta-d_1} & (d_2d_1\beta_1,0) \\ & & {\scriptstyle -d_2}\uparrow \\ & & (-d_1\beta_1,0) \end{array}$$

is a zigzag which identifies $-d_1\beta_1\in\check{Z}^2(Y_1^{[2]};\mathbb{C}^*)$ as a pure cocycle representing the image of $c(L)$ in the E_1 page of the $(d_2, D_1=\delta\pm d_1)$ spectral sequence of the triple complex (4-3) which collapses to the D_1 cohomology of $\check{Z}^\bullet(Y_1^{[\bullet]};\mathbb{C}^*)$. Then, as in (2-11),

(4-9)
$$\begin{array}{ccc} -d_1\beta_1 & & \\ {\scriptstyle d_1}\uparrow & & \\ -\beta_1 & \xmapsto{\delta} & -d_1\alpha \\ & & {\scriptstyle -d_1}\uparrow \\ & & \alpha \end{array}$$

is a further zigzag which identifies $\alpha\in\check{C}^3(X;\mathbb{C}^*)$ as the image of $c(L)$ in the E_1 page of the (d_1,δ) spectral sequence of $\check{Z}^\bullet(Y_1^{[\bullet]};\mathbb{C}^*)$ representing the class $G(L)=[\alpha]$.

Conversely, to show necessity of this condition, suppose that $L\to W^{[2,2]}$ is a bigerbe. As shown in Lemma 4.11, this generates a Čech cocycle, $\lambda=c(L)\in\check{Z}^1(W^{[2,2]};\mathbb{C}^*)$, with values in $\mathbb{C}^*$, which is a pure cocycle in the triple complex:

(4-10)
$$\delta\lambda=d_1\lambda=d_2\lambda=0.$$

Using the exactness of the simplicial complexes we may pull this back under the two homotopy contractions $(s_1^1)^*$ and $(s_2^2)^*$, giving

(4-11)
$$\gamma\in\check{C}^1(W;\mathbb{C}^*),\quad d_1d_2\gamma=d_2d_1\gamma=\lambda.$$

Consider the Čech differential $\delta\gamma\in\check{C}^2(W;\mathbb{C}^*)$. The images of this,

$$d_1\delta\gamma=\delta d_1\gamma\in\check{C}^2(W^{[2,1]},\mathbb{C}^*)\quad\text{and}\quad d_2\delta\gamma\in\check{C}^2(W^{[1,2]};\mathbb{C}^*),$$

are pure cocycles in the triple complex, since λ is closed. Thus $d_2\delta\gamma$ descends to a uniquely defined Čech cocycle $\mu_2\in\check{C}^2(Y_2^{[2]},\mathbb{C}^*)$ with $d_1\mu_2=d_2\delta\gamma$. Note that $\delta\mu_2=0$ by injectivity of d_1 at the bottom level. Under $(s_2^2)^*$, this in turn pulls back to $\beta_2 \in \check{C}^2(Y_2;\mathbb{C}^*)$ with $d_2\beta_2 = \mu_2$. Now $d_2(\delta\gamma - d_1\beta_2) = 0$ by construction, so there is a unique $\beta_1 \in \check{C}^2(Y_1;\mathbb{C}^*)$ such that

$$-d_2\beta_1 = \delta\gamma - d_1\beta_2. \tag{4-12}$$

It follows that $\mu_1 = d_1\beta_1$ satisfies $d_1\mu_1 = 0$ and $\delta\mu_1 = 0$ (by injectivity of d_2 on $\check{C}^2(Y_1^{[2]};\mathbb{C}^*)$ and the fact that $\delta d_2\mu_1 = \delta d_2 d_1\beta_1 = -\delta^2 d_1\gamma = 0$).

Thus $\delta\beta_2$ and $\delta\beta_1$ descend, from Y_2 and Y_1, respectively, to define cocycles in $\check{C}^3(X;\mathbb{C}^*)$; moreover, these must be the same cocycle $\alpha \in \check{C}^3(X;\mathbb{C}^*)$ by injectivity of d_1 and d_2 and the fact that $d_1\delta\beta_2 = d_2\delta\beta_1$, so this represents the 4–class of the bigerbe. This shows that the difference $d_1\beta_2 - d_2\beta_1$ is exact and the criterion therefore holds. □

There is an analogue of Proposition 2.11 classifying trivializations of bundle gerbes.

Proposition 4.16 *The trivializations of a bundle bigerbe (L, W, Y_2, Y_1, X) form a torsor for the group*

$$\operatorname{Im}\{d_2 \colon \check{H}^2_Z(Y_1^{[2]};\mathbb{Z}) \to \check{H}^2_Z(W^{[2,1]};\mathbb{Z})\} \oplus \operatorname{Im}\{d_1 \colon \check{H}^2_Z(Y_2^{[2]};\mathbb{Z}) \to \check{H}^2_Z(W^{[1,2]};\mathbb{Z})\}, \tag{4-13}$$

where $\check{H}^\bullet_Z$ denotes the δ cohomology of the associated space in the diagram (4-6).

In specific cases, as for the Brylinski–McLaughlin bigerbe in Section 5.3, this may be simplified further.

Proof If (Q_1, Q_2) and (Q_1', Q_2') are two trivializations of a bigerbe L, then $P_1 = Q_1' \otimes Q_1^{-1} \to W^{[1,2]}$ and $P_2 = Q_2' \otimes Q_2^{-1} \to W^{[2,1]}$ are line bundles represented by Čech cocycles $\alpha_i = c(P_i)$ satisfying $d_1\alpha_i = d_2\alpha_i = \delta\alpha_i = 0$ for $i = 1, 2$. Exactness of the rows and columns of (4-3) gives the existence of $\beta_1 \in \check{Z}^1(Y_2^{[2]};\mathbb{C}^*)$ and $\beta_2 \in \check{Z}^1(Y_1^{[2]};\mathbb{C}^*)$ satisfying $d_2\beta_1 = \alpha_1$ and $d_1\beta_2 = \alpha_2$, $d_i\beta_i = 0$ and (by injectivity) $\delta\beta_i = 0$. It is straightforward to see that

$$[\alpha_1] \in \operatorname{Im}\{d_1 \colon \check{H}^1_Z(Y_2^{[2]};\mathbb{C}^*) \to \check{H}^1_Z(W^{[1,2]};\mathbb{C}^*)\}$$

is well defined independent of choices, and similarly for $[\alpha_2]$. Conversely, given elements in (4-13) corresponding to line bundles on $W^{[1,2]}$ and $W^{[2,1]}$ coming from simplicial line bundles $Y_2^{[2]}$ and $Y_1^{[2]}$, respectively, a trivialization (Q_1, Q_2) may be altered to give a different trivialization of the same bigerbe. □

5 Examples of bigerbes

5.1 Decomposable bigerbes

As for the decomposable bundle gerbes discussed in Section 5.1, we consider the special classes of bigerbes corresponding to decomposable classes in $H^4(X;\mathbb{Z})$. These are either of the form $\alpha_1 \cup \alpha_2$ with the $\alpha_i \in H^2(X;\mathbb{Z})$ or of the form $\rho \cup \alpha$ with $\rho \in H^1(X;\mathbb{Z})$ and $\alpha \in H^3(X;\mathbb{Z})$. Stuart Johnson, in his PhD thesis [9], makes related constructions in the setting of 2–gerbes.

From Theorem 4.15 it follows that if, for $i = 1, 2$, $\alpha_i \in H^2(X;\mathbb{Z})$ and $\pi_i \colon Y_i \to X$ are locally split maps such that $\pi_i^*\alpha_i = 0 \in H^2(Y_i;\mathbb{Z})$, then the cup product $\alpha_1 \cup \alpha_2$ is represented by a bigerbe over the locally split square $(Y_1 \times_X Y_2, Y_2, Y_1, X)$. Indeed, in Čech theory, if $\rho_i \in \check{C}^1(Y_i;\mathbb{Z})$ are primitives for the $\pi_i^*\alpha_i$ then $\rho_1 \cup \alpha_2$ and $\alpha_1 \cup \rho_2$ are primitives for $\alpha_1 \cup \alpha_2$ on Y_1 and Y_2, respectively, and, pulled back to $Y_1 \times_X Y_2$, their difference $\alpha_1 \cup \rho_2 - \rho_1 \cup \alpha_2$ has primitive $\rho_1 \cup \rho_2$.

If the spaces Y_i are the total spaces of circle bundles representing the 2–classes, the bigerbe is given quite explicitly in terms of the classifying line bundle for decomposed 2–forms over the torus.

Lemma 5.1 *The fundamental line bundle on* $\mathbb{T}^2$ *(with Chern class that generates* $H^2(\mathbb{T}^2;\mathbb{Z}) = \mathbb{Z}$*) has a "bimultiplicative" representative* $S \to \mathbb{T}^2$*, meaning there are natural isomorphisms between fibers* $S_{\theta_1+\theta_2,\zeta} \cong S_{\theta_1,\zeta} \otimes S_{\theta_2,\zeta}$ *and* $S_{\theta,\zeta_1+\zeta_2} \cong S_{\theta,\zeta_1} \otimes S_{\theta,\zeta_1}$ *such that*

$$\begin{array}{ccc} S_{\theta_1+\theta_2,\zeta_1+\zeta_2} & \longrightarrow & S_{\theta_1,\zeta_1+\zeta_2} \otimes S_{\theta_2,\zeta_1+\zeta_2} \\ \downarrow & & \downarrow \\ S_{\theta_1+\theta_2,\zeta_1} \otimes S_{\theta_1+\theta_2,\zeta_2} & \longrightarrow & S_{\theta_1,\zeta_1} \otimes S_{\theta_2,\zeta_1} \otimes S_{\theta_1,\zeta_2} \otimes S_{\theta_2,\zeta_2} \end{array} \tag{5-1}$$

commutes.

Proof Line bundles over $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ are naturally identified with $\mathbb{Z}^2$–invariant line bundles over the universal cover, $\mathbb{R}^2$. We equip the trivial bundle $\mathbb{R}^2 \times \mathbb{C}$ with the $\mathbb{R}^2$–action covering translation via

$$(s,t) \cdot (x,y,z) = (x+s, y+t, e^{2\pi i(s+t)}z), \quad (s,t) \in \mathbb{R}^2,\ (x,y,z) \in \mathbb{R}^2 \times \mathbb{C},$$

and it is clear that the restricted $\mathbb{Z}^2 \subset \mathbb{R}^2$–action is trivial, so this descends to a line bundle $S \to \mathbb{T}^2$.

The bimultiplicative property follows from the natural identifications

$$S_{\zeta,\theta} \ni [(\zeta,\theta,z)] = [(0,0,e^{-2\pi i(\zeta+\theta)}z)] \in S_{0,0} \quad \text{for every } (\zeta,\theta) \in \mathbb{T}^2 \tag{5-2}$$

coming from the $\mathbb{R}^2$–action upstairs. Moreover, the identification (5-2) is equivalent to parallel transport in S along the linear path from $(0,0)$ to (ζ,θ) with respect to the connection 1–form $2\pi i(\zeta d\theta + \theta d\zeta)$, the curvature of which is the fundamental class in $H^2(\mathbb{T}^2;\mathbb{Z})$. □

Proposition 5.2 *For a decomposed* 4*–class* $\alpha_1 \cup \alpha_2 \in H^4(X;\mathbb{Z})$, *with the* $\alpha_i \in H^2(X;\mathbb{Z})$ *represented by circle bundles* $Y_i \to X$, *the pullback under the product of the difference maps* $\chi_i\colon Y_i^{[2]} = Y_i \otimes Y_i \to \mathrm{U}(1)$,

$$L = (\chi_1 \times \chi_2)^* S \to W^{[2,2]} = Y_1^{[2]} \times_X Y_2^{[2]},$$

defines a bigerbe $(L, Y_1 \times_X Y_2, Y_2, Y_1, X)$ *with characteristic class* $G(L) = \alpha_1 \cup \alpha_2$.

Proof The bimultiplicative relations of Lemma 5.1 correspond under pullback by $\chi_1 \times \chi_2$ to the bisimplicial conditions for L, and that $G(L) = \alpha_1 \cup \alpha_2$ is a consequence of Lemma 2.13. □

Similarly, if $\rho \in H^1(X;\mathbb{Z})$ and $\alpha \in H^3(X;\mathbb{Z})$, the representability condition is satisfied by the fiber product square given by any locally split maps $\pi_i\colon Y_i \to X$ for $i = 1,2$ such that $\pi_1^*\rho = 0 \in H^1(Y_1;\mathbb{Z})$ and $\pi_2^*\alpha = 0 \in H^3(Y_2;\mathbb{Z})$.

Taking the "logarithmic" covering $\widetilde{X} \to X$ corresponding to ρ, meaning the pullback of the universal cover of $\mathrm{U}(1)$ by a homotopy class of maps $X \to \mathrm{U}(1)$ representing ρ and a bundle gerbe (L, Y, X) with $\mathrm{DD}(L) = \alpha \in H^3(X;\mathbb{Z})$, there is again a direct construction of a bigerbe for the fiber product square.

Proposition 5.3 *If* (L, Y, X) *is a bundle gerbe with Dixmier–Douady class* $\alpha \in H^3(X;\mathbb{Z})$ *and* $\widetilde{X} \to X$ *is the logarithmic cover corresponding to a class* $[\rho] \in H^1(X;\mathbb{Z})$ *represented by* $\rho\colon X \to \mathrm{U}(1)$, *then the line bundle*

$$L^\chi \to \widetilde{X}^{[2]} \times_X Y^{[2]},$$

where $\chi\colon \widetilde{X} \times_X \widetilde{X} \to \mathbb{Z}$ *is the fiber-shift map, defines a bigerbe*

$$(L^\chi, \widetilde{X} \times_X Y, Y, \widetilde{X}, X) \quad \text{with } G(L^\chi) = \rho \cup \alpha \in H^4(X;\mathbb{Z}).$$

Proof We view the covering space $\widetilde{X} \to X$ as a principal $\mathbb{Z}$–bundle, and then the shift map

$$\chi\colon \widetilde{X}^{[2]} \to \mathbb{Z}$$

defines the collective bundle L^χ on $\widetilde{X}^{[2]} \times_X Y^{[2]}$ given by the tensor product L^n over $\chi^{-1}(n)$.

The bisimplicial space is

$$W^{[m,n]} = \widetilde{X}^{[m]} \times_X Y^{[n]}.$$

and the line bundle L^χ is simplicial in the d_2 direction, with trivializing section of $d_2(L^\chi) = (d_2 L)^\chi$ over $W^{[2,3]}$ given by s^χ, and the d_1 differential of L^χ is given by

$$d_1(L^\chi) = L^{d_1 \chi} = L^0$$

so is canonically trivial. Thus this is indeed a bigerbe.

To see that $G(L^\chi) = \rho \cup \alpha$, observe that representative cocycles $c(L) \in \check{C}^2(Y^{[2]}; \mathbb{Z})$ and $\chi \in \check{C}^0(\widetilde{X}^{[2]}; \mathbb{Z})$ pull back to $\check{C}^\bullet(\widetilde{X}^{[2]} \times_X Y^{[2]}; \mathbb{Z})$ by the fiber product projections, and their cup product $\chi \cup c(L) \in \check{C}^2(\widetilde{X}^{[2]} \times_X Y^{[2]}; \mathbb{Z})$ represents the transgression image of $\rho \cup \alpha$ from X. Then $\chi \cup c(L) = nc(L) = [L^n]$ locally on components $\chi^{-1}(n)$, so the result follows. □

5.2 Doubling for bigerbes

As in Section 3, we may incorporate an additional simplicial structure with respect to the space of products $EX_\bullet = X^\bullet$ in order to promote examples of bigerbes involving based loop spaces to examples involving free loop spaces.

Definition 5.4 A bigerbe L on X^2 will be said to be *double* if the bigerbe

$$\partial L = \pi_0^* L \otimes \pi_1^* L^{-1} \otimes \pi_2^* L$$

is trivial on X^3 with respect to the three projection maps $\pi_i\colon X^3 \to X^2$. In the absence of additional data, ∂L is defined with respect to the bisimplicial space over X^3 obtained by the fiber products of the three pullbacks of the bisimplicial space $W_2^{[\bullet,\bullet]}$ over X^2. However, as for gerbes above, it will typically be the case that X^3 carries a natural split square and induced bisimplicial space $W_3^{[\bullet,\bullet]}$ along with maps $W_3^{[\bullet,\bullet]} \mathrel{\substack{\to\\[-0.3em]\to\\[-0.3em]\to}} W_2^{[\bullet,\bullet]}$ over the projections $X^3 \mathrel{\substack{\to\\[-0.3em]\to\\[-0.3em]\to}} X^2$. In this case, it suffices that $\partial L \to W_3^{[2,2]}$, defined by pulling back along the three maps $W_3^{[2,2]} \to W_2^{[2,2]}$ and taking the alternating product, admits a bigerbe trivialization.

As a special case relevant in our primary example, a bigerbe is a double if ∂L is itself trivial as a line bundle over $W_3^{[2,2]}$. Naturality of the bigerbe characteristic class and exactness of the sequence (3-5) together lead to the following analogue of Proposition 3.2:

Proposition 5.5 *The characteristic* 4*–class,* $G(L)$*, of a double bigerbe* L *on* X^2*, descends from* $H^4(X^2;\mathbb{Z})$ *to* $H^4(X;\mathbb{Z})$*.*

In Section 5.5 we will also consider a similar condition with respect to the bisimplicial space of products $X^{m,n} = X^{mn}$, with the two sets of projections $\pi_j^1\colon X^{m,n} \to X^{m-1,n}$ and $\pi_j^2\colon X^{m,n} \to X^{m,n-1}$. A bigerbe L over $X^4 = X^{2,2}$ is *quadruple* if $\partial_1 L$ and $\partial_2 L$ are respectively trivial over $X^{1,2} = X^2$ and $X^{2,1} = X^2$. The natural differentials ∂_1 and ∂_2, defined on cohomology $\check{H}^{\ell}(X^{\bullet,\bullet}, A)$, commute, and from exactness of these we obtain the following result:

Proposition 5.6 *For a quadruple bigerbe* L *on* X^4*, the characteristic* 4*–class* $G(L)$ *descends from* $H^4(X^4;\mathbb{Z})$ *to* $H^4(X;\mathbb{Z})$*.*

5.3 Brylinski–McLaughlin bigerbes

Next we turn to our main application. The loop space of a principal G–bundle over a manifold is a principal bundle over the loop space with structure group the loop group of G. The Brylinski–McLaughlin bigerbe captures the obstruction to lifting this bundle to a (loop-fusion) principal bundle for a central extension of the loop group. While the version involving based path and loop spaces is simpler, we focus from the beginning on the doubled version involving free path and loop spaces, as this gives the results of primary interest. Note that this theory most naturally involves $\mathrm{U}(1)$ principal bundles in place of line bundles, which we shall use for the remainder of the section without further comment.

Let G be a compact, simple, connected and simply connected group. As is well known (see for instance [19]), there is a classification of $\mathrm{U}(1)$ central extensions

$$1 \to \mathrm{U}(1) \to \widehat{LG} \to LG \to 1 \tag{5-3}$$

of the loop group LG by $H^3(G;\mathbb{Z}) \cong H^3_G(G;\mathbb{Z}) \cong \mathbb{Z}$. These extensions descend to the quotient $LG/G \cong \Omega G$ and so the classification of central extensions of the based loop group ΩG is equivalent.

Forgetting the group structure for the moment, such a central extension may be viewed as a circle bundle over $LG \cong I^{[2]}G$, the Chern class $c(\widehat{LG}) \in H^2(LG;\mathbb{Z})$ of which is the transgression of the defining class in $H^3(G;\mathbb{Z})$. As such, it follows from Theorem 3.6 that $c(\widehat{LG})$ has a loop-fusion refinement. In the equivalent language of the loop-fusion structures of Definition 3.4, we may restate this as follows:

Theorem 5.7 *As a* $\mathrm{U}(1)$*–bundle,* $\widehat{LG} \to LG$ *has a canonical loop-fusion structure, meaning a trivialization of* $d\,\widehat{LG} \to I^{[3]}G$ *inducing the canonical trivialization of* $d^2\,\widehat{LG} \to I^{[4]}G$ *and a trivialization of* $\partial\,\widehat{LG} \to L_8 G$*.*

Remark In fact, the additional structure that promotes a general $\mathrm{U}(1)$–principal bundle over LG to a central extension is also a simplicial one. Indeed, as noted by Brylinski and McLaughlin in [5] and attributed to Grothendieck, a $\mathrm{U}(1)$ central extension of any group H is equivalent to a simplicial circle bundle with respect to the simplicial space $BH_\bullet$ defined by $BH_k = H^{k-1}$ with the face maps $H^{k+1} \to H^k$ given by

$$\pi_i \colon (h_0, h_1, \dots, h_k) \mapsto \begin{cases} (h_1, \dots, h_k) & \text{if } i = 0, \\ (h_0, \dots, h_{i-1}h_i, h_{i+1}, \dots, h_k) & \text{if } 1 \le i \le k, \\ (h_0, \dots, h_{k-1}) & \text{if } i = k. \end{cases}$$

Thus, given a circle bundle $Q \to H = BH_2$, a trivialization of

$$\partial Q = \pi_0^* Q \otimes \pi_1^* Q^{-1} \otimes \pi_2^* Q$$

inducing the canonical trivialization of $d^2 Q \to H^3$ equips Q with the (associative) multiplicative structure of a $\mathrm{U}(1)$ central extension of H and vice versa.

For the groups under consideration, we believe it can be shown that the classes in $H^3_G(G;\mathbb{Z}) = H^4(|BG|;\mathbb{Z})$ are represented by cohomology classes

$$\alpha \in H^3(G = BG_2;\mathbb{Z})$$

satisfying $\partial\alpha = 0 \in H^3(G^2 = BG_3;\mathbb{Z})$, and that the corresponding gerbe (W, PG, G), with circle bundle $W \to \Omega G$, admits a simplicial structure with respect to $B\Omega G_\bullet$, and thus a central extension of ΩG. Further considering a doubled structure with respect to $G_\bullet = G^\bullet$ gives rise to the central extensions of LG. For reasons of space, and since the theory of central extensions of LG is already well known, we will not elaborate further on this point.

To define the Brylinski–McLaughlin bigerbe, let X be a connected manifold with principal G–bundle $E \to X$. One case of particular interest is the spin frame bundle over a spin manifold of dimension ≥ 5.

Lemma 5.8 *With vertical maps projections and evaluation at end- and midpoints in the horizontal directions, the diagrams*

$$\begin{array}{ccc} E^k & \xleftarrow{\pi} & I_k E \\ \downarrow & & \downarrow \\ X^k & \xleftarrow{\pi} & I_k X \end{array} \tag{5-4}$$

for $k \geq 2$ are locally split squares.

Proof The maps are locally trivial fiber bundles, so all maps are locally split, and $I_k E$ is likewise a fiber bundle over the fiber product $I_k X \times_{X^k} E^k$, which is the space of paths in X along with prescribed points in E over the endpoints (for $k = 2$) and midpoint (for $k = 3$) of the path. A connection on E gives a horizontal lift of each path segment in X given an initial point in E, and from the connectedness of G this can be concatenated with a path in the fiber from the endpoint of the lifted path segment to any other prescribed point in the same fiber; this can be done for each segment defined between the k marked points of the path on X. This construction can be carried out locally continuously, so giving a local section of $I_k E$ over $E^k \times_{X^k} I_k X$. □

In the resulting bisimplicial diagrams we may write $IE^{[2]}$, etc, without risk of confusion in light of the canonical isomorphisms

$$(IE)^{[2]} = IE \times_{IX} IE \cong I(E^{[2]}) = I(E \times_X E),$$
$$(LE)^{[2]} = LE \times_{LX} LE \cong L(E^{[2]}) = L(E \times_X E),$$

etc. Filling out the bisimplicial space for $k = 2$ by fiber products leads to the diagram

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \\ (E^{[3]})^2 & \leftarrow & IE^{[3]} & \leftleftarrows & LE^{[3]} & \Lleftarrow & I^{[3]}E^{[3]} & \cdots\cdots & \\ \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \\ (E^{[2]})^2 & \leftarrow & IE^{[2]} & \leftleftarrows & LE^{[2]} & \Lleftarrow & I^{[3]}E^{[2]} & \cdots\cdots & \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \\ E^2 & \longleftarrow & IE & \leftleftarrows & LE & \Lleftarrow & I^{[3]}E & \cdots\cdots & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X^2 & \longleftarrow & IX & \leftleftarrows & LX & \Lleftarrow & I^{[3]}X & \cdots\cdots & \end{array} \tag{5-5}$$

The third column of (5-5) is the simplicial space generated by the fibration $LE \to LX$, itself a principal bundle with structure group LG, and thus supports a lifting bundle gerbe

$$Q = \chi^* \widehat{LG} \to LE^{[2]},$$

where

$$\chi\colon LE^{[2]}LG, \quad (l_1(\theta), l_2(\theta)) \mapsto \ell(\theta), \quad l_2(\theta) = \ell(\theta) l_1(\theta),$$

is the shift map of the principal bundle, and we consider $\widehat{LG} \to LG$ as a $\mathrm{U}(1)$–bundle. The other columns are likewise the simplicial spaces of principal bundles, with structure groups G and $I^{[k]}G$ for $k \geq 1$, and we denote their associated shift maps by the same letter.

Theorem 5.9 *Given a central extension* (5-3) *of level* $\ell \in \mathbb{Z} = H^3_G(G;\mathbb{Z})$, *the lifting bundle gerbe* $Q \to LE^{[2]}$ *is the double bigerbe* (Q, IE, E^2, IX, X^2) *with characteristic class*

$$G(Q) = \ell p_1(E) \in H^4(X;\mathbb{Z}), \tag{5-6}$$

where $p_1(E)$ *is the first Pontryagin class of* E.

The bigerbe (Q, IE, E^2, IX, X^2) will be called the *Brylinski–McLaughlin* bigerbe.

Proof It follows immediately from the lifting gerbe construction that Q is vertically simplicial, and the simplicial condition in the horizontal direction follows from naturality of the shift map and Theorem 5.7.

Indeed, unwinding the definitions reveals that $d_1 \chi^* \widehat{LG} = \chi^* d\, \widehat{LG}$, which admits the trivialization noted in Theorem 5.7 inducing the canonical trivialization of $\chi^* d^2\, \widehat{LG} = d_1^2 \chi^* \widehat{LG}$.

For doubling, we define ∂Q with respect to the locally split square (5-4) for $k = 3$, the induced bisimplicial space of which sits in the diagram

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
(E^{[3]})^3 & \leftarrow & IE^{[3]} & \leftleftarrows & L_8 E^{[3]} & \Lleftarrow & I^{[3]}E^{[3]} & \cdots & \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \\
(E^{[2]})^3 & \leftarrow & IE^{[2]} & \leftleftarrows & L_8 E^{[2]} & \Lleftarrow & I^{[3]}E^{[2]} & \cdots & \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \\
E^3 & \longleftarrow & IE & \leftleftarrows & L_8 E & \Lleftarrow & I^{[3]}E & \cdots & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
X^3 & \longleftarrow & IX & \leftleftarrows & L_8 X & \Lleftarrow & I^{[3]}X & \cdots &
\end{array} \tag{5-7}$$

with the obvious maps to (5-5), and where we have omitted the subscript 3 on the path spaces and made the identification $L_8 X \cong I_3^{[2]} X$, etc. Once again, it follows that $\partial Q = \partial \chi^* \widehat{LG} = \chi^* \partial \widehat{LG}$ with respect to the shift map for the principal $L_8 G$–bundle $L_8 E \to L_8 X$, which admits a trivialization in light of Theorem 5.7. Thus Q is product simplicial in the sense of Definition 5.4 and its characteristic class descends to $H^4(X;\mathbb{Z})$.

Observe that this characteristic class can be obtained in two steps, first by regressing the Chern class $c(Q)$ from $H^2_Z(LE^{[2]};\mathbb{Z})$ to $H^3_Z(LX;\mathbb{Z})$ and then to $H^4(X;\mathbb{Z})$. The image of $c(Q)$ in $H^3_Z(LX;\mathbb{Z})$ is, essentially by definition, the Dixmier–Douady class of the lifting bundle gerbe (or more precisely, its loop-fusion refinement), and it is well known that this class is the transgression of the Pontryagin class of E on X (multiplied by ℓ in the case of higher levels), so from Theorem 3.6 we obtain (5-6). □

Without the vertical simplicial condition, the gerbe $(Q, LE^{[2]}, LX)$ represents the obstruction to lifting the LG–bundle $LE \to LX$ to an $\widehat{LG}$–bundle $\widehat{LE} \to LX$. The enhancement of this data to a bigerbe carries additional information, which is formalized in the following definition:

Definition 5.10 Let $E \to X$ be a principal G–bundle for G a simple, connected and simply connected Lie group, and fix a central extension (5-3) of LG of level $\ell \in \mathbb{Z} = H^3_G(G;\mathbb{Z})$. A *loop-fusion* $\widehat{LG}$ *lift* of $LE \to LX$ is a principal $\widehat{LG}$–bundle $\widehat{LE} \to LX$ lifting LE with the property that $\widehat{LE} \to LE$ is loop-fusion as a U(1)–bundle; in other words, $d\,\widehat{LE} \to I^{[3]}E$ has a trivialization inducing the canonical trivialization of $d^2\,\widehat{LE} \to I^{[4]}E$ and $\partial \widehat{LE} \to L_8 E$ admits a trivialization.

Without the additional figure-of-eight structure, such fusion lifts have been considered by Waldorf in [29], and with stronger conditions (high regularity and equivariance with respect to diffeomorphisms of S^1) by the authors in [11].

Theorem 5.11 *Loop fusion $\widehat{LG}$ lifts of $LE \to LX$ are in bijection with doubled trivializations of the Brylinski–McLaughlin bigerbe; they exist if and only if $p_1(E)$ vanishes, and then form a torsor for $H^3(X;\mathbb{Z})$.*

Proof Here by a doubled trivialization we mean a trivialization (P_1, P_2) in the sense of Definition 4.5 with the additional property that ∂P_i is trivial on the bisimplicial space (5-7). As argued in the proof of Lemma 3.3, the retractions for any space Y of $I_k Y$ onto Y itself for each k give a homotopy with respect to which $\partial = \mathrm{Id}$ as an operator from line bundles on $I_2 Y$ to those $I_3 Y$; in particular, the condition that ∂P_1

is trivial for $P_1 \to I_3 E^{[2]}$ means that P_1 itself is trivial. Thus doubled trivializations for the bigerbe in question are reduced to loop-fusion line bundles $P \to LE$ satisfying $d_2 P \cong Q$. On the one hand, these are clearly equivalent to loop-fusion $\widehat{LG}$ lifts of $LE \to LX$, and on the other, they are classified by those classes in $\check{H}^2_{\mathrm{lf}}(LE;\mathbb{Z})$ with image $c(Q)$ under d_2.

By Lemma 5.12 below, the difference of any two such classes descends to a class in $\check{H}^2_{\mathrm{lf}}(LX;\mathbb{Z})$, and so doubled trivializations form a torsor for the image of $\check{H}^2_{\mathrm{lf}}(LX;\mathbb{Z})$ in $\check{H}^2_{\mathrm{lf}}(LE;\mathbb{Z})$, which by Theorem 3.6 is equivalent to the image of $H^3(X;\mathbb{Z})$ in $H^3(E;\mathbb{Z})$. Finally, given the conditions on G, it follows by the Serre spectral sequence for $E \to X$ that $H^3(X;\mathbb{Z}) \to H^3(E;\mathbb{Z})$ is an isomorphism in this case, so the trivializations are classified simply by $H^3(X;\mathbb{Z})$. □

It remains to show that the exactness of d_2 in the Čech-simplicial double complex is consistent with ∂, which is a consequence of the following.

Lemma 5.12 *The homotopy chain contraction for d_2 in the triple complex*

$$(\check{Z}^\bullet(W^{[\bullet,\bullet]}), \delta, d_1, d_2)$$

for the locally split squares $(I_k E, E^k, I_k X, X^k)$ commutes with the product simplicial operator ∂.

Proof This follows ultimately from the existence of local sections of $E^k \to X^k$ (resp. $IE \to IX$) which are compatible with respect to the three projection maps $E^3 \to E^2$ and $X^3 \to X^2$ (resp. $I_3 E \to I_2 E$ and $I_3 X \to I_2 X$), which we proceed to demonstrate. In the first case, we may fix an admissible pair of covers $(\mathcal{V}, \mathcal{U})$ for (E, X) and then equip E^k and X^k with the covers $\mathcal{V}^k$ and $\mathcal{U}^k$ along with the induced sections $\mathrm{Et}(\mathcal{U}^k) \to \mathrm{Et}(\mathcal{V}^k)$, which are then automatically compatible by the projection maps.

For the path spaces, we begin with the fact that (IE, IX, E^3, X^3) is a locally split square, so $I_3 E$ and $I_3 X$ admit covers $\mathcal{W}$ and $\mathcal{Y}$ and sections $\mathrm{Et}(\mathcal{Y}) \to \mathrm{Et}(\mathcal{W})$ lying over the sections $\mathrm{Et}(\mathcal{U}^3) \to \mathrm{Et}(\mathcal{V}^3)$. In general, the reparametrization maps $I_3 Y \rightrightarrows I_2 Y$ are open, so we may equip $I_2 E$ and $I_2 X$ with the union of the three image covers $\mathcal{W}' = \tilde{\pi}_0(\mathcal{W}) \cup \tilde{\pi}_1(\mathcal{W}) \cup \tilde{\pi}_2(\mathcal{W})$ and $\mathcal{Y}' = \tilde{\pi}_0(\mathcal{Y}) \cup \tilde{\pi}_1(\mathcal{Y}) \cup \tilde{\pi}_2(\mathcal{Y})$, along with the induced section

$$\tilde{s}' \colon \mathrm{Et}(\mathcal{Y}') \to \mathrm{Et}(\mathcal{W}'),$$

giving a set of local sections of $I_2 E \to I_2 X$ which is compatible with the three reparametrization maps $\tilde{\pi}_i$, and which covers the local sections of $E^2 \to X^2$. □

There is a simpler version of this bigerbe using based path and loop spaces, starting with the locally split square (PE, E, PX, X), pulling back a central extension $\widehat{\Omega G} \to \Omega G$ to $\Omega E^{[2]}$, and omitting the doubling conditions. We leave the details as an exercise to the reader.

5.4 Loop spin structures

There is a well-known relationship between string structures on a spin manifold X of dimension $2n > 4$, and (loop) spin structures on its loop space LX.

Here, a *string structure* is a lift of the principal $\mathrm{Spin}(2n)$–bundle $E \to X$ to a principal bundle with structure group $\mathrm{String}(2n)$, a 3–connected topological group covering $\mathrm{Spin}(2n)$ in the sequence of ever more connected groups that form the Whitehead tower for $O(2n)$; see for instance [23]. The string group cannot be a finite-dimensional Lie group (having a subgroup with the topology of $K(\mathbb{Z}, 2)$), though there are various realizations as a 2–group [2; 21]. The obstruction to lifting the structure group is $\frac{1}{2}p_1(X) \in H^4(X;\mathbb{Z})$ (the Pontryagin class of the Spin–bundle being a refinement of the Pontryagin class of the oriented frame bundle), and, if unobstructed, string structures are classified by $H^3(X;\mathbb{Z})$ [24].

As originally defined by Killingback in [10] and further developed by McLaughlin [15], a spin structure on LX is a lift of the $L\mathrm{Spin}$–bundle $LE \to LX$ to the structure group $\widehat{L\mathrm{Spin}}$, the fundamental $\mathrm{U}(1)$ central extension of $L\mathrm{Spin}$. (By analogy, as originally suggested by Atiyah in [1], an *orientation* on LX is a refinement of the $L\mathrm{SO}(2n)$–bundle $LE_{\mathrm{SO}} \to LX$ to have structure group the connected component of the identity, $L_+\mathrm{SO}(2n) \cong L\mathrm{Spin}(2n)$, and therefore is typically related to a spin structure on X.) The obstruction to this lift is the 3–class on LX obtained by transgression of $\frac{1}{2}p_1(X) \in H^4(X;\mathbb{Z})$.

As defined, string structures on X and spin structures on LX are not necessarily in bijection [18]. In fact, it was Stolz and Teichner in [25] who first noted the importance of the fusion structure on LX and showed that string structures on X were in correspondence with what they called "stringor bundles" on (the piecewise smooth loop space) LX, essentially bundles associated to a lift $\widehat{LE}$ along with a fusion condition. It was further proved by the authors in [11] that string structures in the sense of Redden [20] correspond with spin structures on the smooth loop space LX which are both fusion and equivariant for the group $\mathrm{Diff}^+(S^1)$ of oriented diffeomorphisms of the loop parameter, and it was independently proved by Waldorf in [29] that string structures on X exist if and only if fusion spin structures on (piecewise smooth) LX exist, using a transgression theory relating the 2–gerbe obstructing string structures of [6] and fusion gerbes on loop space. Waldorf did not obtain a complete correspondence between string structures and fusion loop spin structures, noting that this would necessitate additional conditions such as equivariance with respect to thin homotopy; in the version considered here, it is the figure-of-eight (ie doubling) condition that provides the remedy.

In any case, the bigerbe formulation here leads to the following result:

Corollary 5.13 *There are natural bijections between the following sets*:

(i) *The set of string structures on a spin manifold X of dimension $2n > 4$.*

(ii) *The set of loop-fusion spin structures on LX, meaning lifts of $LE \to LX$ to the structure group $\widehat{L\mathrm{Spin}}$ such that the resultant $\mathrm{U}(1)$–bundle $\widehat{LE} \to LE$ is a loop-fusion bundle according to Definition 3.4.*

(iii) *The set of doubled trivializations of the Brylinski–McLaughlin bigerbe*

$$(Q, IE, E^2, IX, X^2).$$

The sets are empty unless $\frac{1}{2}p_1(E) = 0 \in H^4(X;\mathbb{Z})$ and otherwise are torsors for $H^3(X;\mathbb{Z})$.

5.5 Path bigerbes

If X is a path-connected and simply connected space with basepoint b, from the based double path space

$$QX = PPX = \{u\colon [0,1]^2 \to X : u|_{\{0\}\times[0,1]} = u|_{[0,1]\times\{0\}} = b\} \tag{5-8}$$

there are two surjective restriction maps

$$f_i\colon QX \to PX, \quad f_1 u = u|_{[0,1]\times\{1\}} \quad \text{and} \quad f_2 u = u|_{\{1\}\times[0,1]}. \tag{5-9}$$

Theorem 5.14 *On a connected, simply connected and locally contractible space the endpoint maps and restriction maps in* (5-9) *form a locally split square*

$$\begin{array}{ccc} PX & \xleftarrow{f_1} & QX \\ \downarrow & & \downarrow{\scriptstyle f_2} \\ X & \longleftarrow & PX \end{array} \tag{5-10}$$

and any class $\gamma \in H^4(X,\mathbb{Z})$ arises from a bigerbe corresponding to (5-10).

Proof The fiber product of the two copies of PX is the based loop space of X. The simple-connectedness of X implies the fiber product of the two f_i is surjective and, from local contractibility, it is locally split. Since PX and QX are both contractible, Theorem 4.15 applies to any 4–class on X. □

Since the Eilenberg–Mac Lane spaces can be represented by CW complexes, Theorem 5.14 applies in particular to $K(\mathbb{Z},4)$.

Theorem 5.15 *There exists a universal bigerbe over $K(\mathbb{Z},4)$ with respect to the locally split square* (5-10) *with $X = K(\mathbb{Z},4)$.*

Note the structure of the bisimplicial space in this case, in which $*$ represents a contractible space:

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ K(\mathbb{Z},3) & \leftarrow & * & \leftleftarrows & K(\mathbb{Z},2) \;\cdots \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ * & \longleftarrow & * & \leftleftarrows & * \;\cdots\cdots \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbb{Z},4) & \leftarrow & * & \leftleftarrows & K(\mathbb{Z},3) \;\cdots \end{array}$$

Finally, incorporation of a *product-bisimplicial* condition allows any 4–class to be represented as a bigerbe on any connected, locally contractible space X, whether simply connected or not. Indeed, consider the locally split square (IIX, IX^2, IX^2, X^4), where $IIX = \{u\colon [0,1]^2 \to X\}$ is the free double path space and the projection maps are given by evaluation at both endpoints of a given path factor. The induced bisimplicial space becomes

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ LX^2 & \leftarrow & ILX & \leftleftarrows & LLX \;\cdots \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ IX^2 & \longleftarrow & IIX & \leftleftarrows & LIX \;\cdots \\ \downarrow & & \downarrow & & \downarrow \\ X^4 & \longleftarrow & IX^2 & \leftleftarrows & LX^2 \;\cdots \end{array} \tag{5-11}$$

and, in particular, $W^{[2,2]} = LLX$ is the double free loop space of X. We may view X^4 at the bottom as the factor $X^{2,2}$ in the bisimplicial space $X^{m,n} = X^{mn}$ of products as discussed in Section 5.2. Over $X^{3,2}$ consider the locally split square and induced bisimplicial space

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ LX^3 & \longleftarrow & I_3LX & \Lleftarrow & L_8LX \; \cdots\cdots \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ I_2X^3 & \leftarrow & I_3I_2X & \Lleftarrow & L_8I_2X \; \cdots\cdots \\ \downarrow & & \downarrow & & \downarrow \\ X^{3,2} & \longleftarrow & I_3X^2 & \Lleftarrow & L_8X^2 \; \cdots\cdots \end{array} \tag{5-12}$$

and likewise for $X^{2,3}$ with factors reversed. The bisimplicial spaces (5-12) map to (5-11) over the product maps $X^{3,2} \rightrightarrows X^{2,2}$ and $X^{2,3} \rightrightarrows X^{2,2}$, and there are associated operators ∂_1 and ∂_2 on line bundles.

Theorem 5.16 *For a connected, locally contractible space X, every class in $H^4(X;\mathbb{Z})$ is represented by a product-bisimplicial bigerbe with respect to* (5-11), *that is, a bigerbe $(L, IIX, IX^2, IX^2, X^4)$ having in addition trivializations of the line bundles $\partial_1 L \to L_8LX$ and $\partial_2 L \to LL_8X$.*

Proof Such a bigerbe has characteristic class $G(L) \in H^4(X^{2,2};\mathbb{Z})$ satisfying

$$\partial_1 G(L) = 0 \in H^4(X^{3,2};\mathbb{Z}) \quad \text{and} \quad \partial_2 G(L) = 0 \in H^4(X^{2,3};\mathbb{Z}),$$

hence by Proposition 5.6 this descends to a well-defined class

$$G(L) \in H^4(X;\mathbb{Z}).$$

Conversely, given any $\alpha \in H^4(X;\mathbb{Z})$, let $\beta = \partial_1\partial_2\alpha \in H^4(X^{2,2};\mathbb{Z})$. This evidently satisfies $\partial_i\beta = 0$ for $i = 1, 2$, and moreover, denoting by $\Delta_i : X^2 \hookrightarrow X^{2,2}$ for $i = 1, 2$ the diagonal inclusions, satisfies $\Delta_i^*\beta = 0 \in H^4(X^2;\mathbb{Z})$. Under the deformation retractions $IX^2 \simeq X^2$, the evaluation maps $IX^2 \to X^{2,2}$ become identified with these diagonal inclusions, so it follows that β lifts to vanish in $H^4(IX^2;\mathbb{Z})$. Then, since $IIX \simeq *$ is contractible, Theorem 4.15 applies and it follows that β is represented by a bigerbe $(L, IIX, IX^2, IX^2, X^4)$ such that $\partial_i L$ is trivial (as a bigerbe) for $i = 1, 2$. As a consequence of Lemma 3.3, which applies to both horizontal and vertical directions in the diagram (5-12), bigerbe triviality of $\partial_i L$ is equivalent to triviality of $\partial_i L$ as a line bundle. □

6 Multigerbes

We end by sketching out the theory of *multigerbes*, the higher degree generalization of bigerbes. By contrast to bundle gerbes, this generalization to higher degree is straightforward, with symmetry of the simplicial conditions replacing the need for higher and ever more complicated associativity conditions.

Fix a degree $n \in \mathbb{N}$, where $n = 1$ and $n = 2$ correspond to bundle gerbes and bigerbes, respectively. To establish notation, let $e_j = (0, \dots, 1, \dots, 0)$ denote the j^{th} standard basis vector, and for each integer k let $\underline{k} = (k, \dots, k)$ denote the vector with constant entries. For a multiindex $\alpha \in \mathbb{N}^n$ we let $|\alpha| = \alpha_1 + \dots + \alpha_n \in \mathbb{N}$, and we distinguish the sets of natural numbers starting at 1 and 0, respectively, by $\mathbb{N}_0$ and $\mathbb{N}_1$.

Definition 6.1 By a *locally split n–cube* we mean a set of spaces X_α indexed by $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1\}^n$ along with continuous maps

$$X_\alpha \to X_{\alpha - e_j} \quad \text{whenever } \alpha_j = 1$$

such that each diagram

$$\begin{array}{ccc} X_{\alpha-e_j} & \longleftarrow & X_\alpha \\ \downarrow & & \downarrow \\ X_{\alpha-e_j-e_k} & \longleftarrow & X_{\alpha-e_k} \end{array}$$

is a locally split square in the sense of Definition 4.1.

In particular, a locally split 1–cube is just a locally split map $X_1 \to X_0$ and a locally split 2–cube is a locally split square.

Lemma 6.2 *A locally split n–cube extends naturally by taking fiber products to a set of spaces $\{X_\alpha : \alpha \in \mathbb{N}_0^n\}$ such that $X_{\bullet \geq \underline{1}} = \{X^\alpha : \alpha \in \mathbb{N}_1^n\}$ is an n–fold multisimplicial space over $X := X_{\underline{0}}$; in particular, for each fixed $\alpha = (\alpha_1, \dots, 0, \dots, \alpha_n)$ with vanishing j^{th} coordinate, the sequence*

$$X_\alpha \leftarrow X_{\alpha+e_j} \leftleftarrows X_{\alpha+2e_j} \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} X_{\alpha+3e_j} \cdots$$

is the simplicial space of fiber products of the map $X_{\alpha+e_j} \to X_\alpha$.

Proof The proof is by induction on n, the case $n = 2$ having been proved as Proposition 4.3. Assuming therefore that the result holds for $n-1$, the "hypersurfaces" $\{X_\alpha : \alpha_j \equiv 0\}$ are well defined for $1 \leq j \leq n$, and for general $\alpha \in \mathbb{N}_1^n$ define X_α as a subspace of $X_{\underline{1}}^{\alpha_1 \cdots \alpha_n}$ as follows. For each j, there are α_j projection maps $X_{\underline{1}}^{\alpha_1 \cdots \alpha_n} \to X_{\underline{1}}^{\alpha_1 \cdots \widehat{\alpha}_j \cdots \alpha_n}$, where the caret denotes omission, and these may be composed with the structure map

$$X_{\underline{1}}^{\alpha_1 \cdots \widehat{\alpha}_j \cdots \alpha_n} \to X_{\underline{1}-e_j}^{\alpha_1 \cdots \widehat{\alpha}_j \cdots \alpha_n}.$$

Introducing the notation $\alpha(j)$ to mean the multiindex obtained from α by setting $\alpha_j = 0$, we may view $X_{\alpha(j)}$ as a subspace of $X^{\alpha_1\cdots\widehat{\alpha}_k\cdots\alpha_n}_{\underline{1}-e_j}$, and then X_α is well defined as the subspace of $X^{\alpha_1\cdots\alpha_n}_{\underline{1}}$ in the mutual preimage of $X_{\alpha(j)}$ under the α_j maps $X^{\alpha_1\cdots\alpha_n}_{\underline{1}} \to X^{\alpha_1\cdots\widehat{\alpha}_j\cdots\alpha_n}_{\underline{1}-e_j}$ for each j. □

We denote the face maps of the multisimplicial space by $\pi^j_\bullet$ for $1 \le j \le n$. Then, for each $\alpha \in \mathbb{N}^n_0$, there are n simplicial differentials defined on a line bundle $L \to X_\alpha$ by

$$d_j L = \bigotimes_{k=0}^{\alpha_j} (\pi^j_k)^* L^{(-1)^k} \to X_{\alpha+e_j}, \quad 1 \le j \le n,$$

with the property that $d_j^2 L \to X_{\alpha+2e_j}$ is canonically trivial.

Definition 6.3 A *bundle n–multigerbe*, or simply *multigerbe*, defined with respect to a locally split n–cube X_α is a multisimplicial line bundle $L \to X_{\underline{2}}$, meaning $d_j L$ is given a trivializing section s_j inducing the canonical trivialization of $d_j^2 L$ for each j, and the induced trivializations of $d_j d_k L \cong d_k d_j L$ are consistent for each pair $j \ne k$.

A set $(P_1, \dots, P_n)$ of line bundles $P_j \to X_{\underline{2}-e_j}$ such that each P_j is multisimplicial with respect to d_i for $i \ne j$ determines a multisimplicial line bundle $\bigotimes_{j=1}^n d_j P_j^{(-1)^j} \to X_{\underline{2}}$ in the obvious way, and a *trivialization* of an n–multigerbe L consists of a set $(P_1, \dots, P_n)$ as above along with an isomorphism

$$L \cong \bigotimes_{j=1}^n d_j P_j^{(-1)^j}$$

intertwining the multisimplicial structures of both sides.

Pullbacks, products and morphisms of multigerbes are defined by generalizing in the obvious way those same operations for bigerbes, and making use of the following result, which follows immediately from Lemma 4.6:

Lemma 6.4 *The pullback by a continuous map $X' \to X_{\underline{0}}$ of a locally split n–cube X_α is a locally split n–cube over X'. Likewise, if X_α and X'_α are locally split n–cubes over the same base $X := X_{\underline{0}} = X'_{\underline{0}}$, then the fiber products $X_\alpha \times_X X'_\alpha$ form a locally split n–cube.*

The characteristic class is defined as before in terms of the total cohomology of a Čech-simplicial multicomplex.

Lemma 6.5 *The simplicial differentials*

$$d_j = \sum_{k=0}^{\alpha_j} (-1)^k (\pi^j)_k^* \colon \check{C}^\ell(X_\alpha; A) \to \check{C}^\ell(X_{\alpha+e_j}; A)$$

on Čech cochains form, along with the Čech differential δ, *an* $(n+1)$*–multicomplex* $(\check{C}^\bullet(X_\bullet; A), \delta, d_1, \dots, d_n)$ *with the following properties:*

(i) *For each* j, *fixing all other indices, the complex* $(\check{C}^\ell(X_{\bullet_j}; A), d_j)$ *is exact, and, for each* $k \neq j$, *admits a homotopy chain contraction commuting with* d_k.

(ii) *The total cohomology of*

$$(\check{C}^\bullet(X_{\bullet \geq \underline{1}}; A), \delta, d_1, \dots, d_n)$$

is isomorphic to $\check{H}^\bullet(X; A)$, *where* $X = X_{\underline{0}}$.

(iii) *The Chern class of the line bundle* $L \to X_{\underline{2}}$ *of a multigerbe is represented by a cocycle* $c(L) \in \check{C}^1(X_{\underline{2}}; \mathbb{C}^*)$ *with* $d_j c(L) = 0$ *for each* j, *and such a multigerbe is trivial if and only if* $c(L)$ *is a coboundary in the total* $(\delta, d_1, \dots, d_n)$ *complex.*

Proof Here (i) is a consequence of Proposition 4.9, since, for each pair $j \neq k$, the bisimplicial space obtained from X_α by freezing all but the j^{th} and k^{th} indices is equivalent to the one obtained from a locally split square.

Part (ii) follows by induction, rolling up the $(n+1)$–multicomplex into the double complex (d_n, D_{n-1}), where D_{n-1} denotes the total differential associated to $(\delta, d_1, \dots, d_{n-1})$. By exactness of d_n, this total cohomology is isomorphic to the total D_{n-1} cohomology of the complex $\check{C}^\bullet(X_{\bullet(n) \geq \underline{1}(n)}; A)$, where again $\alpha(n)$ denotes the index obtained from α by setting $\alpha_n = 0$.

Finally, (iii) is proved by a straightforward generalization of the proof of Lemma 4.11. □

Definition 6.6 The *characteristic class* of a multigerbe (L, X_α) is the class

$$G(L) \in H^{n+3}(X; \mathbb{Z}), \quad X = X_{\underline{0}},$$

given by the Bockstein image of $[c(L)] \in H^{n+2}(\check{C}^\bullet(X_{\bullet \geq \underline{1}}; \mathbb{C}^*), \delta, d_1, \dots, d_n)$ in $\check{H}^{n+2}(X; \mathbb{C}^*)$ with respect to the isomorphism of Lemma 6.5(ii).

Proposition 6.7 *The characteristic class is natural with respect to pullback, product and inverse operations on multigerbes, and a morphism of multigerbes induces an equality of the (pulled back) characteristic classes on the base spaces. It vanishes if and* *only if the multigerbe admits a trivialization. Moreover, $G(L)$ transforms according to the sign representation of the symmetric group Σ_n acting by permutation of the indices of the locally split n–cube X_α.*

Proof sketch As for (bi)gerbes, the naturality of the characteristic class is a consequence of the naturality of the multicomplex $(\check{C}^\bullet(X_\bullet; A), \delta, d_1, \dots, d_n)$ and the naturality of the Chern class of the line bundle $L \to X_{\underline{2}}$, and the equivalence between vanishing of $G(L)$ and multigerbe triviality of L follows from Lemma 6.5(iii). Finally, that $G(L)$ is odd with respect to permutations of the n–cube is a consequence of the sign convention, Convention 2.2, since changing the order of the differentials in the multicomplex by a permutation σ involves multiplying the complex by powers of -1, and, in particular, the sign $(-1)^{\mathrm{sgn}(\sigma)}$ on the term $\check{C}^1(X_{\underline{2}}; \mathbb{C}^*)$. □

The question of representability of a given $(n+2)$–class by a multigerbe supported by a given locally split n–cube can be addressed along similar lines as for bigerbes in Section 4.4. Consider the multicomplex $(\check{C}^\bullet(X_\bullet; \mathbb{Z}), \delta, d_1, \dots, d_n)$ truncated to involve only the spaces in the n–cube, so the X_α with $\alpha \in \{0,1\}^n$. The $(\delta, D_{1,\dots,n})$ spectral sequence of this complex (with the d_i rolled up into a single differential) has E_1 page consisting of the cohomology complexes

$$H^k(X;\mathbb{Z}) \xrightarrow{D_{1,\dots,n}} \bigoplus_{|\alpha|=1} H^k(X_\alpha;\mathbb{Z}) \xrightarrow{D_{1,\dots,n}} \bigoplus_{|\alpha|=2} H^k(X_\alpha;\mathbb{Z}) \xrightarrow{D_{1,\dots,n}} \cdots$$

for each $k \in \mathbb{N}$. At the bottom level, the $D_{1,\dots,n}$ differential of a class in $H^{n+2}(X;\mathbb{Z})$ is just the sum of the pullbacks along the n–cube face maps to $\bigoplus_{|\alpha|=1} H^{n+2}(X_\alpha;\mathbb{Z})$, and if this vanishes, then we say the class *survives to the E_2 page*. In this case the E_2 differential maps the class into the quotient $\bigoplus_{|\alpha|=2} H^{n+1}(X_\alpha;\mathbb{Z}) / \bigoplus_{|\alpha|=1} H^{n+1}(X_\alpha;\mathbb{Z})$ (the cohomology of the E_1 page), and we say the class *survives to the E_3 page* if this E_2 differential vanishes and so on. Provided the class survives to the E_n page, the associated differential maps it into the quotient of $\bigoplus_{|\alpha|=n} H^2(X_\alpha;\mathbb{Z}) = H^2(X_{\underline{1}};\mathbb{Z})$ by some complicated subgroup, and this is the last nontrivial differential of the spectral sequence, which therefore stabilizes at $E_{n+1} = E_\infty$. We say the class *survives to E_∞*, or simply *stabilizes*, if it survives to E_n and has vanishing E_n differential.

Proposition 6.8 *A given locally split n–cube X_α supports an n–multigerbe representing a given class $\alpha \in H^{n+2}(X;\mathbb{Z})$ if and only if α stabilizes in the above sense.*

We leave the details of the proof, which we claim is a relatively straightforward generalization of the proof of Theorem 4.15, as an exercise.

6.1 Examples

We end with some simple examples of multigerbes which are straightforward generalizations of the $(3+1)$–decomposable bigerbes of Section 5.1 and the path bigerbes of Section 5.5.

First, suppose (L, X_α) is an n–multigerbe over $X = X_{\underline{0}}$ with characteristic class $\alpha = G(L) \in H^{n+2}(X;\mathbb{Z})$, and let $[\rho] \in H^1(X;\mathbb{Z}) \cong \check{H}^0(X;\mathrm{U}(1))$ be a given 1–class represented by a homotopy class of maps $\rho\colon X \to \mathrm{U}(1)$. We proceed to construct a "decomposable" $(n+1)$–multigerbe representing the class $[\rho] \cup G(L)$. With $\widetilde{X} \to X$ the "logarithmic" $\mathbb{Z}$–covering of X associated to ρ as in Section 5.1, define the $(n+1)$–cube $\widetilde{X}_\beta$ by

$$\widetilde{X}_{(\alpha,0)} = X_\alpha, \quad \widetilde{X}_{(\alpha,1)} = \widetilde{X} \times_X X_\alpha.$$

Then, in the induced multisimplicial simplicial space, $\widetilde{X}_{\underline{2}} = \widetilde{X}^{[2]} \times_X X_{\underline{2}}$ and we define the line bundle by

$$L^\chi = (\mathrm{pr}_2{}^* L)^{\otimes \mathrm{pr}_1^* \chi} \to \widetilde{X}^{[2]} \times_X X_{\underline{2}},$$

where $\chi\colon \widetilde{X}^{[2]} \to \mathbb{Z}$ is the fiber shift map with $\widetilde{X} \to X$ thought of as a principal $\mathbb{Z}$–bundle.

Proposition 6.9 *With notation as above, $(L^\chi, \widetilde{X}_\beta)$ is an $(n+1)$–multigerbe with characteristic class*

$$G(L^\chi) = [\rho] \cup G(L) \in H^{n+3}(X;\mathbb{Z}).$$

For the generalization of the path bigerbes of Section 5.5, let X be connected, simply connected and locally contractible with a chosen basepoint. Then, with notation $P^1 Y = PY$ and $P^0 Y = Y$, the iterated (based) path spaces

$$X_\alpha = P^{\alpha_1} \cdots P^{\alpha_n} X = P^{|\alpha|} X$$

with evaluation maps $X_\alpha \to X_{\alpha - e_j}$ form a locally split n–cube over X. Indeed, the only obstruction to the locally split condition occurs at the bottom level, with $PPX \to PX \times_X PX \cong \Omega X$ surjective by the simple-connectedness of X; all other split squares have the form (PPY, PY, PY, Y) with $Y = P^{k>0} X$ contractible. Since in this case all the X_α in the n–cube for $\alpha \neq \underline{0}$ are contractible spaces, every class in $H^{n+2}(X;\mathbb{Z})$ survives to the E_∞ page in the $(\delta, D_{1,\dots,n})$ spectral sequence of the n–cube, so, in light of Proposition 6.8, we conclude the following:

Proposition 6.10 *For X connected, simply connected and locally contractible, every class in $H^{n+2}(X;\mathbb{Z})$ is represented by an n–multigerbe supported on the iterated path n–cube $X_\alpha = P^{|\alpha|}X$; in particular, the multisimplicial line bundle of the multigerbe lives on the iterated loop space $X_{\underline{2}} = \Omega^n X$.*

Finally, as in Section 5.5 there is a free path/loop version of this multigerbe obtained at the cost of imposing product-multisimplicial conditions. Indeed, the set

$$\{X^{m_1,\dots,m_n} = X^{m_1\cdots m_n} : (m_1,\dots,m_n) \in \mathbb{N}^n\}$$

of n–fold iterated products of X along with projections forms a multisimplicial space, with induced "differentials" $\partial_1,\dots,\partial_n$ defined on functions, line bundles, gerbes, multigerbes, etc. A *product-multisimplicial* multigerbe is a multigerbe L over $X^{2,\dots,2} = X^{2^n}$ such that $\partial_i L$ is a trivial multigerbe for $1 \le i \le n$, and then its characteristic class descends from $H^{n+2}(X^{2^n};\mathbb{Z})$ to $H^{n+2}(X;\mathbb{Z})$. Again leaving the details of the generalization of Theorem 5.16 as an exercise, we claim the following result:

Proposition 6.11 *If X is connected and locally contractible, then every class in $H^{n+2}(X;\mathbb{Z})$ is represented by a product-multisimplicial n–multigerbe supported by the iterated free path n–cube $X_\alpha = I^{|\alpha|}X^{(2-|\alpha|)^n}$ with $X_{\underline{0}} = X^{2,\dots,2} = X^{2^n}$; in particular, the line bundle of the multigerbe lives on the free loop space $L^n X$, where it satisfies an n–fold fusion condition as well as the multi-figure-of-eight condition that $\partial_i L \to L \cdots L_8 \cdots LX$ are trivial for each i.*

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