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Dynamics and asymptotic profiles of a nonlocal dispersal SIS epidemic model with bilinear incidence and Neumann boundary conditions

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Abstract

This paper is concerned with a nonlocal (convolution) dispersal susceptible-infected-susceptible (SIS) epidemic model with bilinear incidence and Neumann boundary conditions. First we establish the existence and uniqueness of stationary solutions by reducing the system to a single equation. Then we study the asymptotic profiles of the endemic steady states for large and small diffusion rates to illustrate the persistence or extinction of the infectious disease. The lack of regularity of the endemic steady state makes it more difficult to obtain the limit function of the sequence of endemic steady states. We also observe the concentration phenomenon which occurs when the diffusion rate of the infected individuals tends to zero. Our analytical results demonstrate that limiting the movement of susceptible individuals is not effective in eliminating the infectious disease unless the total population size is relatively small.

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1. Introduction

Since the pioneer work of Kermack and McKendrick [24], various diffusive epidemic models have been proposed to describe the spatial spread of infectious diseases, earlier studies include Bailey [3], Bartlett [4], Busenberg and Travis [7], de Monttoni et al. [17], Kendall [22], Mollison [30], Noble [32], and so on. We refer to the monograph of Murray [31] and surveys by Fitzgibbon and Langlais [19], Ruan [38], Ruan and Wu [39] and the references cited therein. In order to understand the effects of movement of infected individuals and heterogeneity of the environment on the spatial spread of infectious diseases, Capasso [8] and Webb [45] proposed susceptible-infectious (SI) epidemic models with diffusion and spatial heterogeneity and investigated the dynamical properties of these SI epidemic models with diffusion and spatial heterogeneity. To include the loss of immunity, Kuperman and Wio [26], Beardmore and Beardmore [6] and Huang et al. [20] constructed susceptible-infectious-susceptible (SIS) epidemic models with diffusion and studied the dynamics of these models such as the existence of traveling waves.

In most of the above mentioned models, the coefficients are constants. To incorporate spatial heterogeneity, Allen et al. [1] proposed a frequency-dependent SIS reaction-diffusion model with space dependent coefficients and investigated the impact of spatial heterogeneity of environment and movements of individuals on the persistence and extinction of infectious diseases. Since then, the model proposed in [1] has attracted much attention. For example, Peng and Liu [35] studied global stability of the steady states, Peng [34] and Peng and Yi [36] gave the asymptotic profiles of endemic steady states. Moreover, Cui et al. [12] investigated the impacts of diffusion and advection on asymptotic profiles of endemic steady states and concluded that advection can help to speed up the elimination of infectious diseases. Cui and Lou [14] considered the effects of diffusion and advection rates on the stability of the disease-free steady state. Cui et al. [13] and Kuto et al. [27] considered the concentration behavior of endemic steady states, see also Cui [11], Zhang and Cui [52], Tong and Lei [44], Sun and Cui [43], Zhang and Cui [53] and so on.

For bilinear incidence, Deng and Wu [15,16] considered an SIS diffusive epidemic model and studied the existence and global attractivity of the steady states in term of the basic reproduction number. For the same model, Wu and Zou [47] further investigated the asymptotic profiles of the endemic steady states for small and large diffusion rates; Wen et al. [46] and Castellano and Salako [9] improved some results of [47] when the diffusion rates go to zero. See Li et al. [28] and Lei et al. [29] for related studies.

Now it is well-understood that nonlocal convolution operators can better capture long-range dispersal of species including humans (Andreu-Vaillo et al. [2] and Fife [18]). Nonlocal epidemic models have been extensively studied since the classical work of Kendall [22,23], in which he generalized the Kermack-McKendrick model to a space-dependent integro-differential equation and used the integral term $\beta S(x,t)\int_{-\infty}^{\infty} K(x-y)I(y,t)dy$ to describe how infectious individuals $I(y,t)$ at location y disperse to infect susceptible individuals $S(x,t)$ at location x. See also the studies of Busenberg and Travis [7], de Monttoni et al. [17], and Noble [32]. For further results on nonlocal epidemic models, we refer to the monograph of Rass and Radcliffe [37] and a survey by Ruan [38].

Recently, Yang, Li and Ruan [51] considered a nonlocal dispersal SIS epidemic model with Neumann boundary conditions in Ω in which the Laplacian operator is replaced by a nonlocal convolution operator. They showed the existence, uniqueness and stability of steady states and obtained the asymptotic profiles of endemic steady states for large diffusion rates. For the same model under the Dirichlet boundary condition with $\Omega = \mathbb{R}^n$, Yang and Li [50] established the existence, uniqueness and global attractivity of the disease-free and endemic steady states. For other nonlocal dispersal epidemic models, we refer to Kuniya and Wang [25], Xu et al. [49] and references cited therein.

Motivated by Deng and Wu [15], Wu and Zou [47], and Yang et al. [51], in this paper we aim to investigate the dynamics and asymptotic profiles of the following nonlocal (convolution) dispersal susceptible-infected-susceptible (SIS) epidemic model with bilinear incidence and Neumann boundary conditions

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \int_\Omega J(x-y)[S(y,t)-S(x,t)]\,dy - \beta(x)SI + \gamma(x)I, & x\in\Omega,\ t>0, \\ \frac{\partial I}{\partial t} = d_I \int_\Omega J(x-y)[I(y,t)-I(x,t)]\,dy + \beta(x)SI - \gamma(x)I, & x\in\Omega,\ t>0, \\ S(x,0)=S_0(x),\ I(x,0)=I_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain; $S(x,t)$ and $I(x,t)$ represent the density of susceptible and infectious individuals at location $x\in\Omega$ and time $t>0$, respectively; positive constants d_S and d_I are diffusion coefficients for susceptible and infectious individuals, respectively; $\beta(x)$ and $\gamma(x)$ are positive continuous functions on Ω which denote the transmission rate of susceptible individuals and the recovery rate of infectious individuals at $x\in\Omega$, respectively. The convolution integrals describe the nonlocal dispersal of individuals. More specifically, $\int_\Omega J(x-y)S(y,t)dy$ and $\int_\Omega J(x-y)I(y,t)dy$ represent the rates at which susceptible and infectious individuals are arriving at position x from other places, while $\int_\Omega J(x-y)S(x,t)dy$ and $\int_\Omega J(x-y)I(x,t)dy$ are the rates at which susceptible and infectious individuals are leaving location x for other locations, respectively. Since integrals are taken over the domain Ω, we assume that diffusion takes places only in Ω. Individuals may not enter or leave the domain Ω. This is analogous to the homogeneous Neumann boundary condition in the literature, we also call it *Neumann boundary condition*, meaning that all the involved integrals are taken over the domain Ω (see the definition in Andreu-Vaillo et al. [2]). Throughout the whole paper, we assume that the dispersal kernel function J satisfies

(J) $J(\cdot)\in C(\mathbb{R}^n),\ J(0)>0,\ J(x)=J(-x)\geq 0,\ \int_{\mathbb{R}^n} J(x)\,dx=1, \int_\Omega J(x-y)\,dy\leq 1$ for any $x\in\Omega$ and $\int_\Omega J(x-y)\,dy\not\equiv 1$,

and the initial data satisfy

(H1) $S_0(x)$ and $I_0(x)$ are nonnegative continuous functions in $\bar{\Omega}$, and the total number of initial infectious individuals is positive; that is, $\int_\Omega I_0(x)\,dx>0$;

(H2) $\int_\Omega (S_0(x)+I_0(x))\,dx\equiv N>0$.

Compared with the frequency-dependent SIS epidemic model considered in Yang, Li and Ruan [51], the bilinear incidence in (1.1) induces new challenges and phenomena. To find the steady states of system (1.1), the method of upper and lower solutions cannot be applied directly to get the endemic steady state. We transform its stationary system to a single equation and then combine the method of upper and lower solutions with some auxiliary problems to obtain the existence of endemic steady states. It should be pointed out that we also need to overcome the difficulty caused by the fact that the nonlocal eigenvalue problems do not admit principal eigenvalues in general (see [5,40–42]). All these difficulties bring challenges for us. Finally, we consider the asymptotic profiles of endemic steady states corresponding to system (1.1), which imply that the infectious disease is persistent or goes to extinction when the diffusion rates are large or small. The lack of regularity of the endemic steady state makes the limit function of the sequence of endemic steady states hard to get. In particular we would like to point out that it is difficult and complex to obtain the asymptotic profile of the endemic steady state of (1.1) as $d_I \to 0$. We also observe the concentration phenomenon in which the infected individuals concentrate on the sites

$$\mathcal{S} = \left\{ x_* \in \bar{\Omega} : \frac{\gamma(x_*)}{\beta(x_*)} = \min_{x \in \bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \right\},$$

and this special phenomenon takes place because of the bilinear incidence. Moreover, as d_S goes to zero and d_I is fixed, we find that the infectious disease may vanish through taking the limit of the endemic steady state of (1.1).

This paper is organized as follows. In Section 2, we list the main results of this paper including not only the existence, uniqueness and global stability of the disease-free steady state and endemic steady state of system (1.1) but also the asymptotic profile of the endemic steady state of system (1.1) for small and large diffusion rates. In Section 3, we give some preliminary results involving the properties of principal eigenvalues corresponding to the nonlocal eigenvalue problems. In Section 4, we present proofs of the main results stated in Section 2. In Section 5, we give some biological implications of our analytical results and provide some strategies for disease control.

2. Main results

In this section, we state the main results of this paper. Define

$$\mathcal{M}[u](x) := d_I \int_{\Omega} J(x-y)(u(y)-u(x))\,dy - \gamma(x)u(x).$$

It is well-known that $\mathcal{M}$ can generate a uniformly continuous semigroup, denoted by $\{T(t)\}_{t\geq 0}$. Denote

$$\mathcal{L}[\phi](x) := \frac{N}{|\Omega|}\beta(x)\int_0^{\infty} T(t)\phi\,dt.$$

We define the basic reproduction number of system (1.1) as follows

$$R_0 = r(\mathcal{L}),$$

where $r(\mathcal{L})$ represents the spectral radius of $\mathcal{L}$.

First we present the existence and stability of the disease-free steady state and endemic steady state of (1.1).

Theorem 2.1. *Suppose $R_0 > 1$. Then system (1.1) admits a unique endemic steady state $(S(x), I(x)) \in C(\bar{\Omega}) \times C(\bar{\Omega})$.*

Theorem 2.2. *Suppose $d_S = d_I$. Then the following statements hold:*

(i) *If $R_0 \le 1$, then all positive solutions of (1.1) converge to the disease-free steady state $\left(\frac{N}{|\Omega|}, 0\right)$ as $t \to +\infty$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$;*
(ii) *If $R_0 > 1$, then all positive solutions of (1.1) converge to the endemic steady state (S, I) as $t \to +\infty$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$.*

Next we state results on the asymptotic profile of the endemic steady state for large and small diffusion rates.

Theorem 2.3. *Suppose that $R_0 > 1$. For any fixed $d_I > 0$, there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to +\infty$ such that the corresponding endemic steady state of (1.1) satisfies $(S_n, I_n) \to (S^*, I^*)$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$, where S^* is a positive function and I^* is a nonnegative constant. Furthermore, either*

(i) $(S^*, I^*) = \left(\frac{\gamma(x)}{\beta(x)}, \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma(x)}{\beta(x)}\,dx\right)$,

or

(ii) *$I^* = 0$ and S^* is the solution of the following problem*

$$\begin{cases} \int_\Omega J(x-y)(S(y)-S(x))\,dy + \psi(-\beta S + \gamma) = 0, & x \in \Omega, \\ \int_\Omega S\,dx = N, \end{cases}$$

where ψ is some positive continuous function on $\bar{\Omega}$ satisfying

$$d_I \int_\Omega J(x-y)(\psi(y)-\psi(x))\,dy + \psi(\beta S^* - \gamma) = 0, \quad x \in \Omega. \tag{2.1}$$

Theorem 2.4. *Suppose $R_0 > 1$ and d_I is fixed. If $d_S \to 0$, then the corresponding endemic steady state of (1.1) satisfies*

$$(S, I) \to \left(\frac{\gamma(x)}{\beta(x)}, \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma(x)}{\beta(x)}\,dx\right) \text{ in } C(\bar{\Omega}) \times C(\bar{\Omega}),$$

provided one of the following assumptions holds:

(i) *β is a positive constant with $N - \frac{1}{\beta}\int_\Omega \gamma(x)\,dx > 0$;*
(ii) $N - \int_\Omega \frac{\gamma(x)}{\beta(x)}\,dx > \frac{1}{2} d_I \int_\Omega\int_\Omega J(x-y)\left(\sqrt{\frac{1}{\beta(y)}} - \sqrt{\frac{1}{\beta(x)}}\right)^2 dy\,dx$;
(iii) $\frac{N}{|\Omega|} > \frac{\gamma(x)}{\beta(x)}$ *on* $\bar{\Omega}$.

Theorem 2.5. *Assume $N > \int_\Omega \frac{\gamma(x)}{\beta(x)}\,dx$. There exists an endemic steady state (S, I) of (1.1) for $0 < d_S \ll 1$ satisfying*

$$(S, I) \to \left(\frac{\gamma(x)}{\beta(x)}, \frac{N}{|\Omega|} - \frac{1}{|\Omega|} \int_\Omega \frac{\gamma(x)}{\beta(x)} dx \right) \ as \ d_S \to 0$$

uniformly on $\bar{\Omega}$.

Taking the same definition as Wu and Zou [47], we denote the high-risk region and low-risk region respectively by

$$\Omega^+ = \left\{ x \in \Omega : \frac{N}{|\Omega|}\beta(x) - \gamma(x) > 0 \right\}$$

and

$$\Omega^- = \left\{ x \in \Omega : \frac{N}{|\Omega|}\beta(x) - \gamma(x) < 0 \right\}.$$

The domain Ω is called a high-risk domain if $\int_\Omega (\frac{N}{|\Omega|}\beta(x) - \gamma(x))\, dx > 0$ and a low-risk domain if $\int_\Omega (\frac{N}{|\Omega|}\beta(x) - \gamma(x))\, dx < 0$. Inspired by Wu and Zou [47,48], we focus on the asymptotic profile of the endemic steady state as $d_I \to 0$. As stated in the following theorem, we observe the concentration phenomenon which shows that restricting the movement of infected individuals only cannot eradicate the infectious disease. Set

$$\theta(x) = \frac{\gamma(x)}{\beta(x)}, \ \ \theta_{min} = \min_{x \in \bar{\Omega}} \frac{\gamma(x)}{\beta(x)},$$

and

$$\mathcal{S} = \left\{ x_* \in \bar{\Omega} : \theta(x_*) = \theta_{min} \right\}.$$

Clearly, $\mathcal{S}$ is nonempty. We say $I \to \mu$ weakly in the sense of

$$\int_\Omega I(x)\zeta(x)\, dx \to \int_\Omega \zeta(x)\mu(dx) \ \text{ for all } \ \zeta(\cdot) \in C(\bar{\Omega}), \tag{2.2}$$

where μ is a Radon measure with nonempty support contained in $\mathcal{S}$.

Theorem 2.6. *Suppose that Ω^+ is nonempty and d_S is fixed. If $d_I \to 0$, then the corresponding endemic steady state (S, I) of (1.1) satisfies $S \to \theta_{min}$ uniformly on $\bar{\Omega}$ and $\int_\Omega I\, dx \to k > 0$ with $k = N - |\Omega|\theta_{min}$. Moreover, the following conclusions hold:*

(i) *If $\mathcal{S} = \{x_0\}$, then $I \to 0$ locally uniformly on $\bar{\Omega} \setminus \{x_0\}$ and $I \to (N - |\Omega|\theta_{min})\delta(x_0)$ weakly in the sense of (2.2), where $\delta(x_0)$ is the Dirac measure centered at x_0;*
(ii) *If $\mathcal{S} = K$ for some closed subset $K \subset \bar{\Omega}$ with positive measure, then we have $I \to 0$ uniformly on $\overline{\Omega \setminus K}$ and $I \to \check{I}$ uniformly on K, where $\check{I}$ is the unique positive solution of*

$$\begin{cases} \int_{\mathring{K}} J(x-y)\check{I}(y)\,dy - \int_{\Omega} J(x-y)\,dy\,\check{I}(x) + \frac{\beta(x)}{d_S}(\alpha - \check{I})\check{I} = 0, & x \in \mathring{K}, \\ \check{I} = 0, & x \in \Omega \setminus \mathring{K}, \\ \int_{\mathring{K}} \check{I}\,dx = N - |\Omega|\theta_{min}, \end{cases} \tag{2.3}$$

in which the positive constant α is uniquely determined by the third equation of (2.3) and $\mathring{K}$ is the interior of K.

Remark 2.7.

(i) If $\mathcal{S} = \{x_1, \cdots, x_j\}$ for some $j \geq 2$, then following a similar argument as in the proof of Theorem 2.6 we have $S \to \theta_{min}$ uniformly on $\bar{\Omega}$ and $I \to 0$ locally uniformly on $\bar{\Omega} \setminus \left(\bigcup_{i=1}^{j} \{x_i\}\right)$, and $I(x) \to \sum_{i=1}^{j} c_i \delta(x_i)$ weakly in the sense of (2.2) as $d_I \to 0$, where constants $c_i \geq 0$ satisfy $\sum_{i=1}^{j} c_i = N - |\Omega|\theta_{min}$.

(ii) If $\mathcal{S} = \left(\bigcup_{i=1}^{j_1} K_i\right) \bigcup \left(\bigcup_{i=0}^{j_2} \{x_i\}\right)$ with K_i being some closed subset of $\bar{\Omega}$ and having a positive measure for some $j_1 \geq 1$ and $j_2 \geq 0$, then $S \to \theta_{min}$ uniformly on $\bar{\Omega}$ as $d_I \to 0$ and we guess that I converges to some nonnegative function uniformly on $\bar{\Omega}$. On the assumption that $I \to \check{I}$ uniformly on $\bar{\Omega}$ as $d_I \to 0$, we readily obtain that $\check{I} = 0$ on $\bar{\Omega} \setminus \bigcup_{i=1}^{j_1} K_i$ and either $\check{I} = 0$ or $\check{I} > 0$ on K_i. Assuming that $\check{I} > 0$ on $\bigcup_{i=1}^{j_1^*} K_i$ for some $1 \leq j_1^* \leq j_1$, then by Theorem 2.6, in such K_i, $\check{I}$ satisfies

$$\begin{cases} \sum_{i=1}^{j_1^*} \int_{\mathring{K}_i} J(x-y)\check{I}(y)\,dy - \int_{\Omega} J(x-y)\,dy\,\check{I}(x) + \frac{\beta(x)}{d_S}(\alpha - \check{I})\check{I} = 0, & x \in \bigcup_{i=1}^{j_1^*} \mathring{K}_i, \\ \check{I} = 0, & x \in \Omega \setminus \bigcup_{i=1}^{j_1^*} \mathring{K}_i, \end{cases}$$

and $\sum_{i=1}^{j_1^*} \int_{\mathring{K}_i} \check{I}\,dx = N - |\Omega|\theta_{min}$.

As a consequence, in this case, it is vital to show that $I \to \check{I}$ uniformly on $\bar{\Omega}$ as $d_I \to 0$.

Theorem 2.8. *Suppose that Ω^+ is nonempty.*

(i) *If $d_I \to 0$ and $\frac{d_I}{d_S} \to d \in (0, +\infty)$, then the corresponding endemic steady state (S, I) of (1.1) converges to (S^*, I^*) in $C(\bar{\Omega}) \times C(\bar{\Omega})$, where I^* is the unique nonnegative solution of*

$$\left(\frac{N}{|\Omega|}\beta - \gamma - \frac{(1-d)\beta}{|\Omega|}\int_\Omega I^* \, dx\right)^+ - d\beta I^* = 0, \quad x \in \bar{\Omega}$$

and

$$S^* = \frac{N}{|\Omega|} - \frac{(1-d)}{|\Omega|}\int_\Omega I^* \, dx - dI^*.$$

(ii) *If* $d_S \to 0$ *and* $\frac{d_I}{d_S} \to 0$, *then the corresponding endemic steady state* (S, I) *of* (1.1) *satisfies* $S \to \theta_{min}$ *uniformly on* $\bar{\Omega}$ *and* $\int_\Omega I \, dx \to k > 0$ *with* $k = N - |\Omega|\theta_{min}$. *Moreover, the following conclusions hold:*
 (a) *If* $\mathcal{S} = \{x_0\}$, *then* $I \to 0$ *locally uniformly on* $\bar{\Omega} \setminus \{x_0\}$ *and* $I \to (N - |\Omega|\theta_{min})\delta(x_0)$ *weakly in the sense of* (2.2), *where* $\delta(x_0)$ *is the Dirac measure centered at* x_0.
 (b) *If* $\mathcal{S} = K$ *for some closed subset* $K \subset \bar{\Omega}$ *with positive measure, then we have* $I \to 0$ *uniformly on* $\overline{\Omega \setminus K}$ *and* $I \to \nu$ *uniformly on* K *with* $\nu = \frac{N - |\Omega|\theta_{min}}{|K|}$.

(iii) *If* $N < \int_\Omega \frac{\gamma(x)}{\beta(x)} dx$, *then the corresponding endemic steady state* (S, I) *of* (1.1) *converges to* $(S^*, 0)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$ *as* $d_I \to 0$ *and* $\frac{d_S}{d_I} \to 0$, *where* $S^* = \min\left\{\vartheta_*, \frac{\gamma}{\beta}\right\}$ *and* ϑ_* *is the unique positive number satisfying*

$$\max\left\{\frac{N}{|\Omega|}, \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}\right\} < \vartheta_* < \max_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$$

and

$$N = \vartheta_* \int_\Omega \left[1 - \left(1 - \frac{\gamma}{\vartheta_*\beta}\right)^+\right] dx = \int_\Omega \min\left\{\vartheta_*, \frac{\gamma}{\beta}\right\} dx.$$

Moreover, there exist constants $0 < d_0 \ll 1$, C_1 *and* C_2 *such that*

$$C_1 \frac{d_S}{d_I} \le \|I\|_{L^\infty(\Omega)} \le C_2 \frac{d_S}{d_I} \ \textit{for all} \ 0 < d_I, \frac{d_S}{d_I} < d_0. \tag{2.4}$$

(iv) *If* $N > \int_\Omega \frac{\gamma(x)}{\beta(x)} dx$, *then there is* $0 < d_0 \ll 1$ *such that for every* $0 < d_I < d_0$ *and* $d_{S,I} > 0$, (1.1) *has an endemic steady state* (S, I) *with* $d_S = d_{S,I}$ *and*

$$(S, I) \to \left(\frac{\gamma}{\beta}, \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma}{\beta} dx\right)$$

uniformly on $\bar{\Omega}$ *as* $d_I \to 0$ *and* $\lim_{d_I \to 0} \frac{d_{S,I}}{d_I} = 0$.

For Theorem 2.8 (ii), we give the following remark.

Remark 2.9.

(i) If $\mathcal{S} = \{x_1, \cdots, x_j\}$ for some $j \geq 2$, then a similar discussion as in the proof of Theorem 2.8 (ii) yields that $S \to \theta_{min}$ uniformly on $\bar{\Omega}$ and $I \to 0$ locally uniformly on $\bar{\Omega} \setminus \left(\bigcup_{i=1}^{j} \{x_i\}\right)$, and $I(x) \to \sum_{i=1}^{j} c_i \delta(x_i)$ weakly in the sense of (2.2) as $d_S \to 0$ and $\frac{d_I}{d_S} \to 0$, where constants $c_i \geq 0$ satisfy $\sum_{i=1}^{j} c_i = N - |\Omega|\theta_{min}$.

(ii) If $\mathcal{S} = \left(\bigcup_{i=1}^{j_1} K_i\right) \bigcup \left(\bigcup_{i=0}^{j_2} \{x_i\}\right)$ with K_i being some closed subset of $\bar{\Omega}$ and having a positive measure for some $j_1 \geq 1$ and $j_2 \geq 0$, then $S \to \theta_{min}$ uniformly on $\bar{\Omega}$ as $d_S \to 0$ and $\frac{d_I}{d_S} \to 0$. However, due to the effect of nonlocal dispersal and bilinear incidence, it is difficult to get the limit of I in this case and we leave it for the further study.

Theorem 2.10. *The following statements hold:*

(i) *Suppose that Ω is a high-risk domain. If $d_S \to +\infty$ and $d_I \to +\infty$, then the endemic steady state of (1.1) satisfies*

$$(S, I) \to \left(\frac{\int_\Omega \gamma(x)\,dx}{\int_\Omega \beta(x)\,dx}, \frac{N}{|\Omega|} - \frac{\int_\Omega \gamma(x)\,dx}{\int_\Omega \beta(x)\,dx}\right) \ \ in \ \ C(\bar{\Omega}) \times C(\bar{\Omega}).$$

(ii) *Suppose that Ω is a high-risk domain. If d_S is fixed, then there exists a sequence $\{d_{I_n}\}$ with $d_{I_n} \to +\infty$ as $n \to +\infty$ such that the corresponding endemic steady state $(S_n, I_n) \to (S^*, I^*)$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$ as $n \to +\infty$, where I^* is a positive constant and S^* is the positive solution of*

$$\begin{cases} d_S \int_\Omega J(x-y)(\tilde{S}(y) - \tilde{S}(x))\,dy + (-\beta\tilde{S} + \gamma)I^* = 0, & x \in \Omega, \\ \int_\Omega \tilde{S}\,dx = N - I^*|\Omega|. \end{cases} \tag{2.5}$$

Moreover, there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to +\infty$ such that the corresponding solution (S_n^*, I_n^*) of (2.5) satisfies $(S_n^*, I_n^*) \to (\tilde{S}^*, \tilde{I}^*)$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$ as $n \to +\infty$, where $\tilde{S}^*$ is a positive function and $\tilde{I}^*$ is a nonnegative constant satisfying either

(a) $(\tilde{S}^*, \tilde{I}^*) = \left(\frac{\gamma(x)}{\beta(x)}, \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma(x)}{\beta(x)}\,dx\right)$,

or

(b) $\tilde{I}^* = 0$ *and $\tilde{S}^*$ is the solution of the following problem*

$$\begin{cases} \int_\Omega J(x-y)(S(y) - S(x))\,dy + \psi(-\beta S + \gamma) = 0, & x \in \Omega, \\ \int_\Omega S\,dx = N, \end{cases}$$

where ψ is some positive continuous function on $\bar{\Omega}$ satisfying

$$d_I \int_\Omega J(x-y)(\psi(y)-\psi(x))\,dy + \psi(\beta\tilde{S}^* - \gamma) = 0, \quad x \in \Omega.$$

(iii) *Assume that $R_0 > 1$. If d_I is fixed and $d_S \to +\infty$, then the endemic steady state of (1.1) $(S, I) \to (S^*, I^*)$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$, where I^* is the unique positive solution of*

$$d_I \int_\Omega J(x-y)(I(y)-I(x))\,dy + I\left(\frac{N\beta}{|\Omega|} - \gamma - \frac{\beta}{|\Omega|}\int_\Omega I\,dx\right) = 0 \tag{2.6}$$

and

$$S^* = \frac{N - \int_\Omega I^*\,dx}{|\Omega|}.$$

3. Preliminaries

By the standard semigroup theory of linear bounded operators (Pazy [33]), it follows that there exists a unique nonnegative solution of system (1.1) (see Kao, Lou and Shen [21]). Set $X = C(\bar{\Omega})$.

Proposition 3.1. *Suppose that $(S_0(\cdot), I_0(\cdot)) \in X \times X$. Then system (1.1) admits a unique solution $(S(x,t), I(x,t))$ for all $x \in \Omega$ and $t \in (0, T_{max})$ with $T_{max} > 0$ satisfying either $T_{max} = +\infty$ or $\lim_{t \to T_{max}^-} \|(S(\cdot,t), I(\cdot,t))\|_{X \times X} = +\infty$.*

We claim that $T_{max} = +\infty$. In fact, by the maximum principle, we have $S(x,t) > 0$, $I(x,t) > 0$ in $\bar{\Omega} \times [0, T_{max})$. Choose $M_1 = \max\left\{\max_{x\in\bar{\Omega}} S_0(x), \max_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}\right\}$. We can see that 0 and M_1 are lower and upper solutions of the first equation of (1.1), respectively. It follows from the comparison principle that $S(x,t) \le M_1$ in $\bar{\Omega} \times [0, T_{max})$. Note that

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_I \int_\Omega J(x-y)[u(y,t) - u(x,t)]\,dy + (\beta(x)M_1 - \gamma(x))u(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = I_0(x), & x \in \Omega \end{cases}$$

has a unique solution $u(x,t)$ for all $x \in \bar{\Omega}$ and $t > 0$. The comparison principle yields that $I(x,t) \le u(x,t)$ for all $x \in \bar{\Omega}$ and $t > 0$.

By the assumption (H2), adding up the two equations of (1.1) and integrating on Ω yields that the total population size is constant; that is,

$$\int_\Omega (S(x,t) + I(x,t))\,dx = N \quad \text{for all } t \ge 0.$$

We give the existence and uniqueness of the disease-free steady state of system (1.1). That is, we consider the following stationary problem

$$\begin{cases} d_S \int_\Omega J(x-y)(S(y)-S(x))\,dy - \beta(x)SI + \gamma(x)I = 0, & x \in \Omega, \\ d_I \int_\Omega J(x-y)(I(y)-I(x))\,dy + \beta(x)SI - \gamma(x)I = 0, & x \in \Omega. \end{cases} \tag{3.1}$$

The stationary solutions of (3.1) also satisfy

$$\int_\Omega (S(x)+I(x))\,dx = N. \tag{3.2}$$

Lemma 3.2. *System* (1.1) *admits a unique disease-free steady state* $(\frac{N}{|\Omega|}, 0)$.

Proof. Letting I be identically equal to zero in (3.1) yields

$$\int_\Omega J(x-y)(S(y)-S(x))\,dy = 0 \;\text{ in } \Omega.$$

It follows from Andreu-Vaillo et al. [2, Proposition 3.3] that $S(x)$ is a constant. Combining $\int_\Omega S(x)\,dx = N$, we know that $S(x) = \frac{N}{|\Omega|}$ on $\bar{\Omega}$. The proof is completed. □

Consider the eigenvalue problem

$$\mathcal{A}[u](x) := d_I \int_\Omega J(x-y)(u(y)-u(x))\,dy + \frac{N}{|\Omega|}\beta(x)u(x) - \gamma(x)u(x) = -\lambda u(x). \tag{3.3}$$

Define

$$\lambda_p(d_I) := \inf_{\substack{\varphi \in L^2(\Omega)\\ \varphi \neq 0}} \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,dy\,dx + \int_\Omega (\gamma(x) - \frac{N}{|\Omega|}\beta(x))\varphi^2(x)\,dx}{\int_\Omega \varphi^2(x)\,dx}.$$

We list some results including the relation between R_0 and $\lambda_p(d_I)$. The proof of the following several lemmas can be found in Yang, Li and Ruan [51].

Lemma 3.3. *Set*

$$m(x) = -d_I \int_\Omega J(x-y)\,dy + \frac{N}{|\Omega|}\beta(x) - \gamma(x).$$

Suppose there is some $x_0 \in \mathrm{Int}(\Omega)$ *satisfying that* $m(x_0) = \max_{\bar{\Omega}} m(x)$, *and the partial derivatives of* $m(x)$ *up to order* $n-1$ *at* x_0 *are zero. Then* $\lambda_p(d_I)$ *is the unique principal eigenvalue of* (3.3) *and its corresponding eigenfunction* φ *is positive and continuous on* $\bar{\Omega}$.

Remark 3.4. $\lambda_p(d_I)$ is continuous on J, $\beta(x)$ and $\gamma(x)$, see the proof in Coville [10].

In the following, we always denote $\alpha(x) := \frac{N}{|\Omega|}\beta(x) - \gamma(x)$.

Lemma 3.5. *Suppose that $\lambda_p(d_I)$ is the principal eigenvalue of (3.3). Then the following statements hold:*

(i) *If $\alpha(x)$ is constant, then $\lambda_p(d_I) = -\alpha(x)$ for every $d_I > 0$;*
(ii) *If $\alpha(x)$ is nonconstant, then $\lambda_p(d_I)$ is strictly monotone increasing in d_I. In addition, $\lambda_p(d_I) \to \min\limits_{x\in\bar{\Omega}}(-\alpha(x))$ as $d_I \to 0$ and $\lambda_p(d_I) \to -\frac{1}{|\Omega|}\int_\Omega \alpha(x)\,dx$ as $d_I \to +\infty$.*

Lemma 3.6. *$\lambda_p(d_I)$ has the same sign as $1 - R_0$.*

Lemma 3.7. *The following statements hold:*

(i) *Suppose $\alpha(x)$ is constant. Then $\alpha(x)$ has the same sign as $R_0 - 1$;*
(ii) *Suppose $\alpha(x)$ is nonconstant.*
  (a) *If $\alpha(x_0) > 0$ for some $x_0 \in \Omega$ and $\int_\Omega \alpha(x)dx < 0$, then there exists some $d_* > 0$ such that $R_0 > 1$ for all $0 < d_I < d_*$ and $R_0 < 1$ for $d_I > d_*$;*
  (b) *If $\int_\Omega \alpha(x)dx \geq 0$, then $R_0 > 1$ for any $d_I > 0$. Further, if $\alpha(x) < 0$ for all $x \in \Omega$, then $R_0 < 1$ for all $d_I > 0$.*

In order to establish the existence and uniqueness of the endemic steady state of (1.1), we show the following preliminary results.

Lemma 3.8. *The pair (S, I) is a nonnegative solution of (3.1) if and only if it is a nonnegative solution of the following problem:*

$$\begin{cases} d_I \int_\Omega J(x-y)(I(y)-I(x))\,dy + I\left[\frac{N}{|\Omega|}\beta - \gamma - \left(1-\frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega I\,dx - \frac{d_I\beta}{d_S}I\right] = 0, & x \in \Omega, \\ S = \frac{N}{|\Omega|} - \left(1-\frac{d_I}{d_S}\right)\frac{1}{|\Omega|}\int_\Omega I\,dx - \frac{d_I}{d_S}I, & x \in \Omega. \end{cases} \tag{3.4}$$

Proof. It can be easily verified that (S, I) is a nonnegative solution of (3.1) if and only if it is a nonnegative solution of the following problem:

$$\begin{cases} d_S S + d_I I = K, & x \in \bar{\Omega}, \\ d_I \int_\Omega J(x-y)(I(y)-I(x))\,dy + I(\beta S - \gamma) = 0, & x \in \Omega, \\ \int_\Omega (S(x)+I(x))\,dx = N, \end{cases} \tag{3.5}$$

where K is some positive constant independent of $x \in \Omega$. So it suffices to show that problems (3.4) and (3.5) are equivalent. We first assume that (S, I) is a nonnegative solution of (3.5). By virtue of the first equation of (3.5), we obtain $S = \frac{K - d_I I}{d_S}$. Substituting it into the third equation of (3.5) yields that

$$K = \frac{1}{|\Omega|}\left[d_S N - (d_S - d_I)\int_\Omega I\,dx\right].$$

Thus, we have

$$S = \frac{K - d_I I}{d_S} = \frac{N}{|\Omega|} - \left(1 - \frac{d_I}{d_S}\right) \frac{1}{|\Omega|} \int_\Omega I \, dx - \frac{d_I}{d_S} I.$$

Substituting this S into the second equation of (3.5) yields the first equation of (3.4).

Now assume that (S, I) is a nonnegative solution of (3.4). It follows from the second equation of (3.4) that

$$\frac{d_I}{d_S} I = \frac{N}{|\Omega|} - \left(1 - \frac{d_I}{d_S}\right) \frac{1}{|\Omega|} \int_\Omega I \, dx - S.$$

Substituting this into the first equation of (3.4) yields the second equation of (3.5). Integrating both sides of the second equation of (3.4) gives the third equation of (3.5). We derive from the second equation of (3.4) that

$$d_S \int_\Omega J(x-y)(S(y) - S(x)) \, dy = -d_I \int_\Omega J(x-y)(I(y) - I(x)) \, dy,$$

which implies that $d_S S + d_I I$ is a constant. Combining the third equation of (3.5), we know that this constant must be positive. The proof is completed. □

Lemma 3.9. *If $I \in C(\bar{\Omega})$ is a nonnegative solution of the first equation of* (3.4)*, then we have*

$$\left(1 - \frac{d_I}{d_S}\right) \frac{1}{|\Omega|} \int_\Omega I \, dx + \frac{d_I}{d_S} I(x) < \frac{N}{|\Omega|} \text{ for all } x \in \bar{\Omega}.$$

Proof. If $I \equiv 0$, then the conclusion is quickly obtained. If I is not identically zero on $\bar{\Omega}$, suppose on the contrary that the conclusion is false. It follows from the continuity of I that there exists $x_0 \in \bar{\Omega}$ such that $I(x_0) = \max_{x \in \bar{\Omega}} I(x) > 0$. In view of the above assumption, there must be

$$\left(1 - \frac{d_I}{d_S}\right) \frac{1}{|\Omega|} \int_\Omega I \, dx + \frac{d_I}{d_S} I(x_0) \geq \frac{N}{|\Omega|}.$$

We derive from the first equation of (3.4) that

$$\begin{aligned} &d_I \int_\Omega J(x_0 - y)(I(y) - I(x_0)) \, dy \\ &\quad + I(x_0) \left[\frac{N}{|\Omega|} \beta(x_0) - \gamma(x_0) - \left(1 - \frac{d_I}{d_S}\right) \frac{\beta(x_0)}{|\Omega|} \int_\Omega I(x) \, dx - \frac{d_I \beta(x_0)}{d_S} I(x_0) \right] = 0, \end{aligned}$$

which implies that

$$\int_\Omega J(x_0 - y)(I(y) - I(x_0)) \, dy > 0.$$

This contradicts the fact that

$$\int_\Omega J(x_0 - y)(I(y) - I(x_0))\,dy \le 0$$

due to $I(x_0) = \max_{x\in\bar\Omega} I(x)$. The proof is completed. $\square$

By virtue of Lemma 3.9, if $I \in C(\bar\Omega)$ is a nonnegative solution of the first equation of (3.4), then S defined by the second equation of (3.4) is positive. In order to obtain the existence of positive solutions of (3.4), we introduce a lemma from Yang, Li and Ruan [51] firstly. Consider

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\int_\Omega J(x-y)(u(y,t) - u(x,t))\,dy + (r(x) - c(x)u)u, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.6}$$

where d is a positive constant and $u_0(x)$ is a bounded continuous function.

Lemma 3.10. *Assume that $r(\cdot)$, $c(\cdot) \in C(\bar\Omega)$ and $c(x) > 0$ on $\bar\Omega$. Then the positive stationary solution u_* of (3.6) is unique if and only if $\lambda_p(d) < 0$, in which*

$$\lambda_p(d) = \inf_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \frac{\frac{d}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,dy\,dx - \int_\Omega r(x)\varphi^2(x)\,dx}{\int_\Omega \varphi^2(x)\,dx}.$$

Moreover, u_* is globally asymptotically stable.

Set

$$f(\tau, I) = \frac{N}{|\Omega|}\beta - \gamma - \left(1 - \frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\tau - \frac{d_I\beta}{d_S}I$$

and define

$$F(\tau, I) := d_I \int_\Omega J(x-y)(I(y) - I(x))\,dy + If(\tau, I).$$

Lemma 3.11. *Suppose that $R_0 > 1$ and $d_S > d_I$. Then there exists $(\tau, I_\tau) \in \mathbb{R}^+ \times X$ such that $F(\tau, I_\tau) = 0$. In addition, some $\tau^* > 0$ exists such that $I_\tau(x) > 0$ for all $x \in \bar\Omega$ and $\tau \in [0, \tau^*)$, and $I_{\tau^*} = 0$. Furthermore, I_τ is decreasing and continuous on τ in $(0, \tau^*)$.*

Proof. Since $R_0 > 1$, we know from Lemma 3.6 that $\lambda_p(d_I) < 0$. Then there exists $\tau^* > 0$ such that $\tilde\lambda_\tau < 0$ for all $\tau \in [0, \tau^*)$, and $\tilde\lambda_{\tau^*} = 0$ in which

$$\tilde\lambda_\tau = \inf_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,dy\,dx + \int_\Omega\left[\gamma - \frac{N}{|\Omega|}\beta + \left(1 - \frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\tau\right]\varphi^2(x)\,dx}{\int_\Omega \varphi^2(x)\,dx}.$$

It follows from Lemma 3.10 that $F(\tau, I) = 0$ admits a unique positive solution $I_\tau \in X$ for each given $\tau \in [0, \tau^*)$ and $F(\tau^*, I_{\tau^*}) = 0$ with $I_{\tau^*} = 0$. In fact, following the same discussion as $\tilde{I}$ below, we obtain either $I_{\tau^*} = 0$ or $I_{\tau^*} > 0$ for the nonnegative solution of $F(\tau^*, I_{\tau^*}) = 0$. If $I_{\tau^*} > 0$, it follows from Lemma 3.10 that $\tilde{\lambda}_{\tau^*} < 0$, which is a contradiction.

Assume $\tau_1, \tau_2 \in (0, \tau^*)$ and $\tau_1 < \tau_2$. We derive from $d_S > d_I$ that $F(\tau_1, I_{\tau_2}) > 0$, which implies that I_{τ_2} is a lower solution of $F(\tau_1, I) = 0$. Obviously, a sufficiently large number is an upper solution. By the method of upper and lower solutions and the uniqueness of a positive solution of $F(\tau_1, I) = 0$, we have $I_{\tau_1} > I_{\tau_2}$. Since I_τ is decreasing with respect to τ, we have $I_\tau(x) \le \tilde{I}_0(x) \le \max_{x\in\bar{\Omega}} \tilde{I}_0(x)$ implying that I_τ is uniformly bounded, where $\tilde{I}_0(x)$ satisfies $F(0, \tilde{I}_0) = 0$. Denote

$$a(x) = -\frac{d_I}{d_S}\beta(x), \quad H_\tau(x) = d_I \int_\Omega J(x-y) I_\tau(y)\, dy,$$

$$G_\tau(x) = -d_I \int_\Omega J(x-y)\, dy + \frac{N}{|\Omega|}\beta(x) - \gamma(x) - \left(1 - \frac{d_I}{d_S}\right)\frac{\beta(x)}{|\Omega|}\tau.$$

Then, we have

$$I_\tau(x) = \frac{-G_\tau(x) - \sqrt{G_\tau^2(x) - 4a(x)H_\tau(x)}}{2a(x)}.$$

A simple compactness argument gives that

$$\lim_{\tau\to\tilde{\tau}} I_\tau(x) = \tilde{I}(x) \ \text{ uniformly on } \ \bar{\Omega}, \quad \text{for any } \ \tilde{\tau} \in (0, \tau^*),$$

in which $\tilde{I}$ is some nonnegative function. In addition, $\tilde{I}$ satisfies $F(\tilde{\tau}, \tilde{I}) = 0$. As a consequence, we know either $\tilde{I} \equiv 0$ or $\tilde{I} > 0$. In fact, if $\tilde{I}$ is neither strictly positive nor identically vanishing; that is, there exists some $x_0 \in \bar{\Omega}$ such that $\tilde{I}(x_0) = \min_{x\in\bar{\Omega}} \tilde{I}(x) = 0$, then we derive from $F(\tilde{\tau}, \tilde{I}) = 0$ that $\int_\Omega J(x_0 - y)\tilde{I}(y)\, dy = 0$. Thus, $\tilde{I}(x) = 0$ for all $x \in \bar{\Omega}$, which is a contradiction.

On the other hand, the positivity of I_τ implies that $\hat{\lambda}_\tau = 0$, in which $\hat{\lambda}_\tau$ is defined by

$$\hat{\lambda}_\tau := \inf_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \left\{ \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\, dy\, dx + B_\tau(\varphi)}{\int_\Omega \varphi^2(x)\, dx} \right\}$$

and

$$B_\tau(\varphi) := \int_\Omega \left[\gamma(x) - \frac{N}{|\Omega|}\beta(x) + \left(1 - \frac{d_I}{d_S}\right)\frac{\beta(x)}{|\Omega|}\tau + \frac{d_I}{d_S}\beta(x) I_\tau(x)\right]\varphi^2(x)\, dx.$$

If $\tilde{I} \equiv 0$, letting $\tau \to \tilde{\tau}$ yields that $\tilde{\lambda}_{\tilde{\tau}} = 0$, which contradicts the fact that $\tilde{\lambda}_{\tilde{\tau}} < 0$ due to $\tilde{\tau} \in (0, \tau^*)$. Thus, $\tilde{I} > 0$. Note that $\tilde{I}$ satisfies $F(\tilde{\tau}, \tilde{I}) = 0$. The uniqueness of a positive solution of $F(\tilde{\tau}, u) = 0$ gives that $\tilde{I} = I_{\tilde{\tau}}$. As a result, I_τ is continuous on τ in $(0, \tau^*)$. The proof is completed. □

Now we give two results inspired by Castellano and Salako [9].

Lemma 3.12. *Suppose that $d_I > 0$ and $d_S > 0$ are fixed. The following conclusions hold:*

(i) *Let (S, I) be a nonnegative solution of (3.1). Then the function*

$$\kappa = d_S S + d_I I \quad x \in \Omega \tag{3.7}$$

is a constant function. Furthermore, by letting

$$\tilde{S} = \frac{S}{\kappa} \quad \text{and} \quad \tilde{I} = \frac{I}{\kappa}, \tag{3.8}$$

$(\kappa, \tilde{S}, \tilde{I})$ satisfies

$$\tilde{S} = \frac{1}{d_S}\left(1 - d_I \tilde{I}\right), \tag{3.9}$$

$$\frac{\kappa}{d_S}\int_\Omega \left[\left(1 - d_I \tilde{I}\right) + d_S \tilde{I}\right] dx = N \tag{3.10}$$

and

$$\begin{cases} d_I \int_\Omega J(x-y)[\tilde{I}(y) - \tilde{I}(x)]\,dy + \left[\frac{\kappa\beta}{d_S}\left(1 - d_I \tilde{I}\right) - \gamma\right]\tilde{I} = 0, & x \in \Omega, \\ 0 \le \tilde{I} < \frac{1}{d_I}, & x \in \Omega. \end{cases} \tag{3.11}$$

(ii) *If $(\kappa, \tilde{S}, \tilde{I})$ solves (3.9), (3.10) and (3.11), then $(S, I) = (\kappa\tilde{S}, \kappa\tilde{I})$ is a nonnegative solution of (3.1).*

Proof. (i) Note that κ satisfies

$$\int_\Omega J(x-y)[\kappa(y) - \kappa(x)]\,dy = 0.$$

It follows from Andreu-Vaillo et al. [2, Proposition 3.3] that $\kappa(x)$ is a constant. Dividing both sides of (3.7) by κ, we obtain (3.9). (3.10) is derived by (3.2). And substituting $S = \kappa\tilde{S} = \frac{\kappa}{d_S}(1 - d_I \tilde{I})$ in the second equation of (3.1) yields (3.11).

(ii) It readily follows by inspection that $(S, I) = (\kappa\tilde{S}, \kappa\tilde{I})$ is a nonnegative solution of (3.1) whenever $(\kappa, \tilde{S}, \tilde{I})$ solves (3.9), (3.10) and (3.11). The proof is completed. □

Let $l > 0$ be a real number. Consider

$$d_I \int_\Omega J(x-y)[u(y) - u(x)]\,dy + [l\beta(1 - d_I u) - \gamma]u = 0, \quad x \in \Omega. \tag{3.12}$$

Lemma 3.13. *Suppose $R_0 > 1$ and $l > \frac{N}{|\Omega|}$. Then (3.12) admits a unique positive solution u_l satisfying $0 < u_l < \frac{1}{d_I}$. In addition,*

$$\lim_{l\to+\infty} u_l = \frac{1}{d_I} \quad \text{and} \quad \lim_{l\to+\infty} l(1-d_I u_l) = \frac{\gamma}{\beta} \tag{3.13}$$

uniformly on $\bar{\Omega}$.

Proof. By virtue of Lemma 3.10, (3.12) admits a unique positive solution if and only if $\lambda_l < 0$, in which

$$\lambda_l = \inf_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,dy\,dx - \int_\Omega (l\beta(x)-\gamma(x))\varphi^2(x)\,dx}{\int_\Omega \varphi^2(x)\,dx}.$$

Since $R_0 > 1$, we have $\lambda_p(d_I) < 0$. Then, $\lambda_l < 0$ due to $l > \frac{N}{|\Omega|}$. Therefore, (3.12) admits a unique positive solution u_l. Set

$$m_l(x) = -d_I \int_\Omega J(x-y)\,dy + l\beta(x) - \gamma(x).$$

Finding a sequence $\{v_n\}$ with

$$\|v_n - m_l\|_{L^\infty(\Omega)} \to 0 \text{ as } n \to +\infty,$$

the eigenvalue problem

$$d_I \int_\Omega J(x-y)\varphi_n(y)\,dy + v_n(x)\varphi_n(x) = -\lambda\varphi_n(x) \text{ in } \Omega$$

admits a principal eigenpair $(\lambda_l^n, \varphi_n(x))$. There exists $n_1 > 0$ large enough such that

$$\lambda_l^n \le \frac{1}{2}\lambda_l - \|v_n - m_l\|_{L^\infty(\Omega)} \text{ for all } n \ge n_1.$$

Constructing $\underline{u}(x) = \delta\varphi_n(x)$ for some $\delta > 0$ and taking a direct computation, we have

$$\begin{aligned}
& d_I \int_\Omega J(x-y)(\underline{u}(y)-\underline{u}(x))\,dy + \underline{u}\Big(l\beta - \gamma - d_I l\beta\underline{u}\Big) \\
&= -\delta\lambda_l^n\varphi_n(x) + \delta\varphi_n(x)(m_l(x)-v_n(x)) - d_I l\beta(x)\delta^2\varphi_n^2(x) \\
&\ge -\frac{1}{2}\lambda_l\delta\varphi_n(x) - d_I l\beta(x)\delta^2\varphi_n^2(x) \\
&\ge 0,
\end{aligned}$$

provided δ small enough. Obviously, $\frac{1}{d_I}$ is an upper solution. Take $\delta > 0$ sufficiently small such that $\underline{u} \le \frac{1}{d_I}$ on $\bar{\Omega}$. The method of upper and lower solutions and the uniqueness of the positive solution of (3.12) give that $0 < \delta\varphi_n \le u_l \le \frac{1}{d_I}$ on $\bar{\Omega}$ for each $n \ge n_1$.

We claim that $u_l(x) \ne \frac{1}{d_I}$ for all $x \in \bar{\Omega}$. On the contrary, assume that there is $x_0 \in \mathrm{Int}(\Omega)$ such that $u_l(x_0) = \frac{1}{d_I}$. Thus, (3.12) yields that

$$\frac{1}{d_I}\gamma(x_0) = d_I \int_\Omega J(x_0 - y)(u_l(y) - u_l(x_0))\, dy \le 0,$$

which is a contradiction. On the other hand, if $x_0 \in \partial\Omega$, we can find a point sequence $\{x_n\} \subset \Omega$ such that $x_n \to x_0$ and $u_l(x_n) \to u_l(x_0)$ as $n \to +\infty$. The same arguments can lead to a contradiction.

Set

$$g_l(x) = \beta(x) - \frac{\gamma(x)}{l} - \frac{d_I \int_\Omega J(x-y)\, dy}{l}.$$

We derive from (3.12) that

$$u_l(x) = \frac{g_l(x) + \sqrt{g_l^2(x) + 4d_I\beta(x)\frac{d_I}{l}\int_\Omega J(x-y)u_l(y)\, dy}}{2d_I\beta(x)},$$

which implies that $u_l \to \frac{1}{d_I}$ uniformly on $\bar{\Omega}$ as $l \to +\infty$.

Note that the function $w_l = l(1 - d_I u_l)$ satisfies

$$w_l(x) = \frac{\gamma(x)u_l(x) - d_I \int_\Omega J(x-y)[u_l(y) - u_l(x)]\, dy}{\beta(x)u_l(x)},$$

which implies that $w_l(x) \to \frac{\gamma(x)}{\beta(x)}$ uniformly on $\bar{\Omega}$ as $l \to +\infty$. The proof is completed. □

4. Proof of main results

Proof of Theorem 2.1. In view of Lemmas 3.8 and 3.9, it suffices to prove that there exists a unique continuous positive solution of the first equation of (3.4). Since $R_0 > 1$, we know from Lemma 3.6 that $\lambda_p(d_I) < 0$. We proceed with the proof by considering the following three cases in turn.

Case I. $d_S = d_I$. We derive from Lemma 3.10 that the claim is true.

Case II. $d_S > d_I$. By Lemma 3.11, there exists $(\tau, I_\tau) \in [0, \tau^*) \times X$ such that $F(\tau, I_\tau) = 0$. By virtue of the definition of F, I_τ is a solution of the first equation of (3.4) if $\tau = \int_\Omega I_\tau\, dx$. Since

$$0 < \int_\Omega \tilde{I}_0\, dx \ \text{ and } \ \tau^* > \int_\Omega I_{\tau^*}\, dx = 0,$$

where $\tilde{I}_0(x)$ satisfies $F(0, \tilde{I}_0) = 0$, it follows from the continuity and monotonicity of I_τ on τ that there exists a unique $\tau_0 \in (0, \tau^*)$ such that $\tau_0 = \int_\Omega I_{\tau_0}\,dx$. Thus, the first equation of (3.4) admits a unique continuous positive solution.

Case III. $d_S < d_I$. We apply the method of upper and lower solutions. Recall that

$$m(x) = -d_I \int_\Omega J(x-y)\,dy + \frac{N}{|\Omega|}\beta(x) - \gamma(x).$$

The continuity of $m(x)$ on $\bar\Omega$ implies that there exists some $x_0 \in \bar\Omega$ such that $m(x_0) = \max_{x\in\bar\Omega} m(x)$. Define

$$m_n(x) = \begin{cases} m(x_0), & x \in B_{x_0}\left(\frac{1}{n}\right), \\ m_{n,1}(x), & x \in \left(B_{x_0}\left(\frac{2}{n}\right) \setminus B_{x_0}\left(\frac{1}{n}\right)\right), \\ m(x), & x \in \Omega \setminus B_{x_0}\left(\frac{2}{n}\right), \end{cases}$$

where $B_{x_0}\left(\frac{1}{n}\right) = \left\{x \in \Omega : |x - x_0| < \frac{1}{n}\right\}$, $m_{n,1}(x)$ satisfies $m_{n,1}(x) \le m(x_0)$, and $m_{n,1}(x)$ is continuous in Ω. Indeed, $m_{n,1}(x)$ exists if only we take n large enough, denoted by $n \ge n_0 > 0$. It follows from Lemma 3.3 that the eigenvalue problem

$$d_I \int_\Omega J(x-y)\phi(y)\,dy + m_n(x)\phi(x) = -\lambda\phi(x)$$

admits a principal eigenpair, denoted by $(\lambda_p^n(d_I), \phi_n)$. As a consequence of Remark 3.4, some $n_1 \ge n_0$ exists such that

$$\lambda_p^n(d_I) \le \frac{1}{2}\lambda_p(d_I) - \|m_n - m\|_{L^\infty(\Omega)} \text{ for any } n \ge n_1.$$

Normalizing ϕ_n by $\|\phi_n\|_{L^\infty(\Omega)} = 1$ and letting $\underline{I} = \delta\phi_n$ for some $\delta > 0$, a direct computation yields that

$$\begin{aligned} & d_I \int_\Omega J(x-y)(\underline{I}(y) - \underline{I}(x))\,dy + \underline{I}\left[\frac{N}{|\Omega|}\beta - \gamma - \left(1 - \frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega \underline{I}\,dx - \frac{d_I\beta}{d_S}\underline{I}\right] \\ &= \delta\phi_n(x)[-\lambda_p^n(d_I) + (m(x) - m_n(x))] - \delta^2\phi_n(x)\left[\left(1 - \frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega \phi_n(x)\,dx + \frac{d_I\beta}{d_S}\phi_n(x)\right] \\ &\ge -\frac{1}{2}\lambda_p(d_I)\delta\phi_n(x) - \delta^2\phi_n(x)\left[\left(1 - \frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega \phi_n(x)\,dx + \frac{d_I\beta}{d_S}\phi_n(x)\right] \\ &\ge 0, \end{aligned}$$

provided δ small enough. Let $\bar{I} = \frac{N}{|\Omega|}$. Then, we have

$$
\begin{aligned}
&d_I \int_\Omega J(x-y)(\bar I(y)-\bar I(x))\,dy + \bar I\left[\frac{N}{|\Omega|}\beta-\gamma-\left(1-\frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega \bar I\,dx - \frac{d_I\beta}{d_S}\bar I\right]\\
&=-\gamma(x)\frac{N}{|\Omega|}<0,
\end{aligned}
$$

which implies that $\bar I$ is an upper solution. Take $\delta>0$ sufficiently small such that $\underline{I}\le \bar I$ on Ω. The method of upper and lower solutions implies that the first equation of (3.4) admits a positive solution $I\in L^2(\Omega)$.

Denote

$$
\begin{aligned}
&a(x)=-\frac{d_I}{d_S}\beta(x),\quad H(x)=d_I\int_\Omega J(x-y)I(y)\,dy,\\
&G(x)=-d_I\int_\Omega J(x-y)\,dy+\frac{N}{|\Omega|}\beta-\gamma-\left(1-\frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega I\,dx.
\end{aligned}
$$

By the first equation of (3.4), we have

$$
I(x)=\frac{-G(x)-\sqrt{G^2(x)-4a(x)H(x)}}{2a(x)},
$$

which implies that $I(x)$ is continuous on $\bar\Omega$. By virtue of the second equation of (3.4), we obtain $S(x)$ is continuous on $\bar\Omega$.

Finally, we prove the uniqueness of the positive solution of (3.4). Define

$$
\tilde F(x,u)=d_I\int_\Omega J(x-y)I(y)\,dy-d_I\int_\Omega J(x-y)\,dy\,u+\tilde f(x,u),\quad u\in\mathbb{R},
$$

in which

$$
\tilde f(x,u)=\left[\frac{N}{|\Omega|}\beta-\gamma-\left(1-\frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega I\,dx-\frac{d_I\beta}{d_S}u\right]u.
$$

A direct computation gives that

$$
\tilde F_u(x,u)=-d_I\int_\Omega J(x-y)\,dy+\frac{N}{|\Omega|}\beta-\gamma-\left(1-\frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega I\,dx-\frac{2d_I\beta}{d_S}u.
$$

Obviously, $\tilde F(x,I)=0$ and

$$
\tilde F_u(x,I)=-d_I\int_\Omega J(x-y)\frac{I(y)}{I(x)}\,dy+\frac{d_I\beta}{d_S}I-\frac{2d_I\beta}{d_S}I
$$

$$= -d_I \int_\Omega J(x-y)\frac{I(y)}{I(x)}\,dy - \frac{d_I\beta}{d_S} I$$

$$< 0.$$

Then, the Implicit Function Theorem implies that there exists a unique function $u(x)$ defined on a neighborhood of x such that $\tilde{F}(x, u(x)) = 0$. Note that $\tilde{F}(x, I(x)) = 0$. By the uniqueness of $u(x)$, we have $I(x) = u(x)$ on the above neighborhood of x. The arbitrariness of x implies the uniqueness of positive solution of (3.4). The proof is completed. □

Remark 4.1. For the case $d_S > d_I$, it is worth mentioning that the method of upper and lower solutions is not applicable due to the term $\int_\Omega I(x)\,dx$. In fact, the iterative sequence of functions constructed by the upper and lower solutions is not monotone in this case.

Proof of Theorem 2.2. Suppose $R_0 < 1$. By the same discussion as in the proof of [51, Theorem 3.10], we know that $S(x,t) + I(x,t) \to \frac{N}{|\Omega|}$ uniformly on $\bar{\Omega}$ as $t \to +\infty$. For any small ε, there exists $T > 0$ such that $S(x,t) \le \frac{N}{|\Omega|} + \varepsilon - I(x,t)$ for all $x \in \bar{\Omega}$ and $t > T$. By the second equation of (1.1), we have

$$\frac{\partial I}{\partial t} \le d_I \int_\Omega J(x-y)(I(y,t) - I(x,t))\,dy + I\left[\left(\frac{N}{|\Omega|} + \varepsilon\right)\beta - \gamma - \beta I\right], \quad x \in \Omega,\ t \in (T, +\infty).$$

It follows from the comparison principle that $I(x,t) \le \tilde{I}(x,t)$ for all $x \in \bar{\Omega}$ and $t > T$, where $\tilde{I}$ is the solution of

$$\begin{cases} \frac{\partial \tilde{I}}{\partial t} = d_I \int_\Omega J(x-y)(\tilde{I}(y,t) - \tilde{I}(x,t))\,dy & \\ \qquad\qquad + \tilde{I}\left[\left(\frac{N}{|\Omega|} + \varepsilon\right)\beta - \gamma - \beta\tilde{I}\right], & x \in \Omega,\ t \in (T, +\infty), \\ \tilde{I}(x,T) = I(x,T), & x \in \Omega. \end{cases} \tag{4.1}$$

Since $R_0 < 1$, we have $\lambda_p(d_I) > 0$, which implies that $\lambda_p(d_I, \varepsilon) > 0$ if ε is small enough, where $\lambda_p(d_I, \varepsilon)$ is obtained by replacing $\frac{N}{|\Omega|}$ in $\lambda_p(d_I)$ by $\frac{N}{|\Omega|} + \varepsilon$. Thus, $\tilde{I}(x,t) \to 0$ uniformly for $x \in \bar{\Omega}$ as $t \to +\infty$, which implies that $I(x,t) \to 0$ uniformly for $x \in \bar{\Omega}$ as $t \to +\infty$. As a consequence, $S(x,t) \to \frac{N}{|\Omega|}$ uniformly for $x \in \bar{\Omega}$ as $t \to +\infty$.

If $R_0 = 1$, then $\lambda_p(d_I) = 0$ and $\lambda_p(d_I, \varepsilon) < 0$. By virtue of Lemma 3.10, $\tilde{I}(x,t) \to \tilde{I}_*(x)$ uniformly on $\bar{\Omega}$ as $t \to +\infty$, where $\tilde{I}_*(x)$ is the positive steady state of (4.1). There exists some $x_0 \in \bar{\Omega}$ such that $\tilde{I}_*(x_0) = \max_{x \in \bar{\Omega}} \tilde{I}_*(x)$. By (4.1), we have $\tilde{I}_*(x_0) \le \frac{N}{|\Omega|} + \varepsilon$. Since ε is small enough, without loss of generality, we assume $\varepsilon < 1$. Then, $\tilde{I}_*(x)$ is uniformly bounded with respect to small ε. Suppose $\varepsilon_1 < \varepsilon_2$ and let $\tilde{I}_*^{\varepsilon_1}(x)$ and $\tilde{I}_*^{\varepsilon_2}(x)$ be the corresponding positive steady states of (4.1). A simple calculation yields that

$$
\begin{aligned}
d_I \int_\Omega & J(x-y)(\tilde{I}_*^{\varepsilon_2}(y) - \tilde{I}_*^{\varepsilon_2}(x))\,dy + \tilde{I}_*^{\varepsilon_2}\left[\left(\frac{N}{|\Omega|}+\varepsilon_1\right)\beta - \gamma - \beta \tilde{I}_*^{\varepsilon_2}\right] \\
&= (\varepsilon_1 - \varepsilon_2)\beta \tilde{I}_*^{\varepsilon_2} \\
&< 0.
\end{aligned}
$$

It follows from the uniqueness of the positive steady state of (4.1) that $\tilde{I}_*^{\varepsilon_1}(x) < \tilde{I}_*^{\varepsilon_2}(x)$. As a result, $\tilde{I}_*(x)$ is monotone with respect to ε. One can derive from Lemma 3.10 that $\tilde{I}_*(x) \to 0$ as $\varepsilon \to 0$ due to $\lambda_p(d_I) = 0$. Thus, $I(x,t) \to 0$ and $S(x,t) \to \frac{N}{|\Omega|}$ uniformly for $x \in \bar{\Omega}$ as $t \to +\infty$.

Now we are in a position to consider the case $R_0 > 1$ implying $\lambda_p(d_I) < 0$. Note that there exists $T > 0$ such that

$$
\frac{N}{|\Omega|} - \varepsilon - I(x,t) \le S(x,t) \le \frac{N}{|\Omega|} + \varepsilon - I(x,t) \ \text{ for all } \ x \in \bar{\Omega} \ \text{ and } \ t > T.
$$

By the second equation of (1.1), I satisfies

$$
\begin{aligned}
I\left[\left(\frac{N}{|\Omega|} - \varepsilon\right)\beta - \gamma - \beta I\right] &\le \frac{\partial I}{\partial t} - d_I \int_\Omega J(x-y)(I(y,t) - I(x,t))\,dy \\
&\le I\left[\left(\frac{N}{|\Omega|} + \varepsilon\right)\beta - \gamma - \beta I\right]
\end{aligned}
$$

for all $x \in \bar{\Omega}$ and $t > T$. The comparison principle yields that $\hat{I}(x,t) \le I(x,t) \le \tilde{I}(x,t)$ for all $x \in \bar{\Omega}$ and $t > T$, where $\tilde{I}$ is the solution of (4.1) and $\hat{I}$ is the solution of

$$
\begin{cases}
\frac{\partial \hat{I}}{\partial t} = d_I \int_\Omega J(x-y)(\hat{I}(y,t) - \hat{I}(x,t))\,dy \\
\qquad\qquad + \hat{I}\left[\left(\frac{N}{|\Omega|} - \varepsilon\right)\beta - \gamma - \beta \hat{I}\right], & x \in \Omega,\ t \in (T, +\infty), \\
\hat{I}(x,T) = I(x,T), & x \in \Omega.
\end{cases} \tag{4.2}
$$

Note that $\lambda_p(d_I, \pm\varepsilon) < 0$ with $\varepsilon > 0$ small enough. As a consequence of Lemma 3.10, we have

$$
\hat{I}(x,t) \to \hat{I}_\varepsilon(x) \ \text{ and } \ \tilde{I}(x,t) \to \tilde{I}_\varepsilon(x) \ \text{ uniformly on } \ \bar{\Omega} \ \text{ as } \ t \to +\infty,
$$

where $\hat{I}_\varepsilon(x)$ and $\tilde{I}_\varepsilon(x)$ are respectively the unique steady states of (4.2) and (4.1). The same argument as in the proof of [51, Theorem 3.10] gives that $I(x,t) \to I_1(x)$ uniformly on $\bar{\Omega}$ as $t \to +\infty$, where $I_1(x)$ satisfies

$$
d_I \int_\Omega J(x-y)(I_1(y) - I_1(x))\,dy + I_1\left(\frac{N}{|\Omega|}\beta - \gamma - \beta I_1\right) = 0 \ \text{ in } \ \Omega.
$$

Thus, $S(x,t) \to \frac{N}{|\Omega|} - I_1(x)$ uniformly on $\bar{\Omega}$ as $t \to +\infty$. The uniqueness of the positive solution of the first equation of (3.4) yields $I_1(x) = I(x)$. Hence,

$$
S(x,t) \to \frac{N}{|\Omega|} - I(x) = S(x) \ \text{ uniformly on } \ \bar{\Omega} \ \text{ as } \ t \to +\infty.
$$

The proof is completed. □

Proof of Theorem 2.3. In view of $R_0 > 1$, we derive from Theorem 2.1 that there exists a unique endemic steady state (S, I) of (1.1) for any $d_S > 0$ with R_0 independent of d_S.

Since S is continuous on $\bar{\Omega}$, there exist $x_0, y_0 \in \bar{\Omega}$ such that $S(x_0) = \min_{x\in\bar{\Omega}} S(x)$ and $S(y_0) = \max_{x\in\bar{\Omega}} S(x)$. By the first equation of (3.1), we have $-\beta(x_0)S(x_0) + \gamma(x_0) \leq 0$ and $-\beta(y_0)S(y_0) + \gamma(y_0) \geq 0$. Then, there hold

$$S(x_0) \geq \frac{\gamma(x_0)}{\beta(x_0)} \geq \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$$

and

$$S(y_0) \leq \frac{\gamma(y_0)}{\beta(y_0)} \leq \max_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}.$$

Thus, we have

$$\min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \leq S(x) \leq \max_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}. \tag{4.3}$$

Note that $\int_\Omega I\,dx \leq N$. Thus, a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to +\infty$ exists such that the corresponding endemic steady state (S_n, I_n) satisfies $\int_\Omega I_n\,dx \to k$ for some $k \geq 0$. We claim that

$$I_n \to \frac{k}{|\Omega|} \text{ uniformly on } \bar{\Omega} \text{ as } n \to +\infty.$$

Easily, one can get

$$F_n \equiv \left(\frac{N}{|\Omega|}\beta - \gamma\right)d_{S_n} + (d_I - d_{S_n})\frac{\beta}{|\Omega|}\int_\Omega I_n\,dx \to d_I\beta\frac{k}{|\Omega|} \text{ as } n \to +\infty.$$

Then for any $\varepsilon > 0$, there exists $n_1 > 0$ such that

$$\frac{d_I\beta}{|\Omega|}(k-\varepsilon) \leq F_n \leq \frac{d_I\beta}{|\Omega|}(k+\varepsilon) \text{ for all } n > n_1.$$

Note that

$$d_{S_n}d_I\int_\Omega J(x-y)(I_n(y) - I_n(x))\,dy + I_n(F_n - d_I\beta I_n) = 0, \quad x \in \Omega.$$

Thus, I_n is a lower solution of

$$d_{S_n}d_I\int_\Omega J(x-y)(\hat{I}(y) - \hat{I}(x))\,dy + \hat{I}\left[\frac{d_I\beta}{|\Omega|}(k+\varepsilon) - d_I\beta\hat{I}\right] = 0, \quad x \in \Omega, \tag{4.4}$$

and an upper solution of

$$d_{S_n} d_I \int_\Omega J(x-y)(\hat{I}(y)-\hat{I}(x))\,dy + \hat{I}\left[\frac{d_I\beta}{|\Omega|}(k-\varepsilon) - d_I\beta\hat{I}\right] = 0, \quad x \in \Omega. \tag{4.5}$$

Obviously, $\frac{k+\varepsilon}{|\Omega|}$ and $\frac{k-\varepsilon}{|\Omega|}$ are the solutions of (4.4) and (4.5), respectively. Then, we have

$$\frac{k-\varepsilon}{|\Omega|} \le I_n \le \frac{k+\varepsilon}{|\Omega|} \text{ for all } x \in \bar{\Omega} \text{ and } n > n_1.$$

The arbitrariness of $\varepsilon > 0$ implies that $I_n \to \frac{k}{|\Omega|}$ uniformly on $\bar{\Omega}$ as $n \to +\infty$.

If $k > 0$, the first equation of (3.1) gives

$$S_n = \frac{d_{S_n}\int_\Omega J(x-y)S_n(y)\,dy + \gamma I_n}{d_{S_n}\int_\Omega J(x-y)\,dy + \beta I_n}, \tag{4.6}$$

which implies that $S_n \to \frac{\gamma}{\beta}$ in $C(\bar{\Omega})$ as $n \to +\infty$. By (3.2), we have

$$I_n \to \frac{k}{|\Omega|} = \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma(x)}{\beta(x)}\,dx \text{ uniformly on } \bar{\Omega} \text{ as } n \to +\infty.$$

Now, we are in a position to consider the case $k = 0$. Up to a subsequence if needed, one of the following three statements must hold:

(a) $\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}} \to 0$ as $n \to +\infty$;
(b) $\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}} \to +\infty$ as $n \to +\infty$;
(c) $\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}} \to C_0$ with C_0 being a positive constant as $n \to +\infty$.

If (a) holds, then $\frac{\int_\Omega I_n\,dx}{d_{S_n}} \to 0$ as $n \to +\infty$. Let $\hat{I}_n = \frac{I_n}{d_{S_n}}$. Then, $\hat{I}_n$ satisfies

$$d_I \int_\Omega J(x-y)(\hat{I}_n(y)-\hat{I}_n(x))\,dy + \hat{I}_n\left[\frac{N}{|\Omega|}\beta - \gamma + (d_I - d_{S_n})\frac{\beta}{|\Omega|}\frac{\int_\Omega I_n\,dx}{d_{S_n}} - d_I\beta\hat{I}_n\right] = 0,$$
$$x \in \Omega.$$

We show that $\hat{I}_n \to \hat{I}$ in $C(\bar{\Omega})$ as $n \to +\infty$, where $\hat{I}$ is the unique positive solution of

$$d_I \int_\Omega J(x-y)(\hat{I}(y)-\hat{I}(x))\,dy + \hat{I}\left(\frac{N}{|\Omega|}\beta - \gamma - d_I\beta\hat{I}\right) = 0, \quad x \in \Omega. \tag{4.7}$$

Since $R_0 > 1$, we have $\lambda_p(d_I) < 0$. The existence and uniqueness of the positive solution $\hat{I}$ of (4.7) are obtained by Lemma 3.10. There exists $n_0 > 0$ large enough such that

$$(d_I - d_{S_n})\frac{\beta}{|\Omega|}\frac{1}{d_{S_n}}\int_\Omega I_n\,dx > 0 \ \text{ for all } \ n \ge n_0.$$

Thus, $\hat{I}_n$ is an upper solution of (4.7) for each given $n \ge n_0$. Recall that

$$m(x) = -d_I\int_\Omega J(x-y)\,dy + \frac{N}{|\Omega|}\beta(x) - \gamma(x).$$

Finding a sequence $\{m_n\}$ with

$$\|m_n - m\|_{L^\infty(\Omega)} \to 0 \ \text{ as } \ n \to +\infty,$$

the eigenvalue problem

$$d_I\int_\Omega J(x-y)\varphi_n(y)\,dy + m_n(x)\varphi_n(x) = -\lambda\varphi_n(x) \ \text{ in } \ \Omega$$

admits a principal eigenpair $(\lambda_p^n(d_I), \varphi_n(x))$. There exists $n_1 > 0$ large enough such that

$$\lambda_p^n(d_I) \le \frac{1}{2}\lambda_p(d_I) - \|m_n - m\|_{L^\infty(\Omega)} \ \text{ for all } \ n \ge n_1.$$

Constructing $\underline{\hat{I}}(x) = \delta\varphi_n(x)$ for some $\delta > 0$ and taking a direct computation, we have

$$\begin{aligned}
&d_I\int_\Omega J(x-y)(\underline{\hat{I}}(y) - \underline{\hat{I}}(x))\,dy + \underline{\hat{I}}\Big(\frac{N}{|\Omega|}\beta - \gamma - d_I\beta\underline{\hat{I}}\Big)\\
&= -\delta\lambda_p^n(d_I)\varphi_n(x) + \delta\varphi_n(x)(m(x) - m_n(x)) - d_I\beta(x)\delta^2\varphi_n^2(x)\\
&\ge -\frac{1}{2}\lambda_p(d_I)\delta\varphi_n(x) - d_I\beta(x)\delta^2\varphi_n^2(x)\\
&\ge 0
\end{aligned}$$

provided δ small enough. Take $\delta > 0$ sufficiently small such that $\underline{\hat{I}} \le \hat{I}_n$ on $\bar{\Omega}$. Set $\tilde{n} = \max\{n_0, n_1\}$. The method of upper and lower solutions and the uniqueness of the positive solution of (4.7) give that $\delta\varphi_n \le \hat{I} \le \hat{I}_n$ on $\bar{\Omega}$ for each $n \ge \tilde{n}$. Since

$$(d_I - d_{S_n})\frac{\beta}{|\Omega|}\frac{1}{d_{S_n}}\int_\Omega I_n\,dx \to 0 \ \text{ as } \ n \to +\infty,$$

for any $\varepsilon > 0$, there exists $n^* > \tilde{n}$ such that

$$0 < (d_I - d_{S_n})\frac{\beta}{|\Omega|}\frac{1}{d_{S_n}}\int_\Omega I_n\,dx < \varepsilon \ \text{ for all } \ n \ge n^*.$$

We can see that $\hat{I}_n$ is a lower solution of

$$d_I \int_\Omega J(x-y)(\tilde{I}(y)-\tilde{I}(x))\,dy + \tilde{I}\left(\frac{N}{|\Omega|}\beta - \gamma + \varepsilon - d_I\beta\tilde{I}\right) = 0, \quad x \in \Omega. \tag{4.8}$$

Following the same discussion as in the proof of Theorem 2.2, we get that (4.8) admits a unique positive solution $\tilde{I}_\varepsilon \geq \hat{I}_n$ and $\tilde{I}_\varepsilon \to \hat{I}$ as $\varepsilon \to 0$. Combining $\delta\varphi_n \leq \hat{I} \leq \hat{I}_n$, we obtain $\hat{I}_n \to \hat{I}$ in $C(\bar{\Omega})$ as $n \to +\infty$. But

$$(d_I - d_{S_n})\frac{\beta}{|\Omega|}\int_\Omega \frac{I_n}{d_{S_n}}\,dx \to d_I\frac{\beta}{|\Omega|}\int_\Omega \hat{I}\,dx > 0,$$

a contradiction. Hence, the case $\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}} \to 0$ as $n \to +\infty$ is impossible to occur.

Next, we prove that if the case (b) or (c) holds, then $\frac{I_n}{\|I_n\|_{L^\infty(\Omega)}} \to \check{I}$ in $C(\bar{\Omega})$ as $n \to +\infty$ with $\check{I}$ being some positive function. Note that the first equation of (3.1) gives that

$$S_n(x) = \frac{\int_\Omega J(x-y)S_n(y)dy + \gamma(x)\frac{I_n(x)}{d_{S_n}}}{\int_\Omega J(x-y)dy + \beta(x)\frac{I_n(x)}{d_{S_n}}}.$$

Substituting this $S_n(x)$ into the second equation of (3.1) and then dividing both sides by $\|I_n\|_{L^\infty(\Omega)}$ yield that

$$\frac{I_n}{\|I_n\|_{L^\infty(\Omega)}} = \frac{-G_n(x) + \sqrt{G_n^2(x) + 4a(x)H_n(x)}}{2a(x)}, \tag{4.9}$$

where

$$a(x) = d_I\beta(x)\int_\Omega J(x-y)\,dy,$$

$$H_n(x) = d_I\frac{d_{S_n}}{\|I_n\|_{L^\infty(\Omega)}}\int_\Omega J(x-y)\,dy\int_\Omega J(x-y)\frac{I_n(y)}{\|I_n\|_{L^\infty(\Omega)}}\,dy,$$

$$\begin{aligned} G_n(x) = \frac{d_{S_n}}{\|I_n\|_{L^\infty(\Omega)}}&\left[d_I\left(\int_\Omega J(x-y)\,dy\right)^2 - \beta(x)\int_\Omega J(x-y)S_n(y)\,dy\right. \\ &\left.+\gamma(x)\int_\Omega J(x-y)\,dy\right] - d_I\beta(x)\int_\Omega J(x-y)\frac{I_n(y)}{\|I_n\|_{L^\infty(\Omega)}}\,dy. \end{aligned}$$

Since

$$\|S_n\|_{L^\infty(\Omega)} \leq \max_{x\in\bar{\Omega}}\frac{\gamma(x)}{\beta(x)} \quad \text{and} \quad \left\|\frac{I_n}{\|I_n\|_{L^\infty(\Omega)}}\right\|_{L^\infty(\Omega)} = 1,$$

there exist nonnegative functions S^* and $\check{I}$ such that up to a subsequence,

$$S_n \to S^* \text{ and } \frac{I_n}{\|I_n\|_{L^\infty(\Omega)}} \to \check{I} \text{ weakly in } L^2(\Omega) \text{ as } n \to +\infty.$$

If the case (b) or (c) holds, by (4.9), we obtain that

$$\frac{I_n}{\|I_n\|_{L^\infty(\Omega)}} \to \check{I} \text{ in } C(\bar{\Omega}) \text{ as } n \to +\infty.$$

Since $\|\check{I}\|_{L^\infty(\Omega)} = 1$, we have $\check{I} \geq 0$ but $\check{I} \not\equiv 0$. We claim that $\check{I} > 0$ in $\bar{\Omega}$. In fact, if case (b) holds, it follows from (4.9) that $\check{I}$ satisfies

$$\int_\Omega J(x-y)\,dy\,\check{I}^2(x) - \check{I}(x)\int_\Omega J(x-y)\check{I}(y)\,dy = 0. \tag{4.10}$$

There exists some $x_1 \in \bar{\Omega}$ such that $\check{I}(x_1) = \max_{x\in\bar{\Omega}} \check{I}(x) > 0$. By virtue of (4.10), we obtain

$$\int_\Omega J(x_1-y)[\check{I}(y) - \check{I}(x_1)]\,dy = 0.$$

Since $J(0) > 0$, we have $\check{I}(y) = \check{I}(x_1)$ in $B_\delta(x_1)$ where $B_\delta(x_1)$ is a ball with radius δ centered at x_1. Now for any $x_2 \in B_{\frac{\delta}{2}}(x_1)$,

$$\int_\Omega J(x_2-y)[\check{I}(y) - \check{I}(x_2)]\,dy = 0.$$

Thus, we obtain

$$\check{I}(y) = \check{I}(x_1), \quad \forall y \in \bigcup_{x\in B_{\frac{\delta}{2}}(x_2)} B_{\frac{3\delta}{4}}(x).$$

Repeating the above procedures yields $\check{I}(y) = \check{I}(x_1)$ for all $y \in \bar{\Omega}$. Since $\|\check{I}\|_{L^\infty(\Omega)} = 1$, we have $\check{I} \equiv 1$. If case (c) holds, then we derive from (4.9) that $\check{I}$ satisfies

$$\begin{aligned}
&d_I\beta(x)\int_\Omega J(x-y)\,dy\,\check{I}^2(x) - \frac{d_I}{C_0}\int_\Omega J(x-y)\,dy\int_\Omega J(x-y)\check{I}(y)\,dy \\
&+\check{I}(x)\left[\frac{d_I}{C_0}\left(\int_\Omega J(x-y)\,dy\right)^2 - d_I\beta(x)\int_\Omega J(x-y)\check{I}(y)\,dy\right. \\
&\left.-\frac{\beta(x)}{C_0}\int_\Omega J(x-y)S^*(y)\,dy + \frac{\gamma(x)}{C_0}\int_\Omega J(x-y)\,dy\right] = 0,
\end{aligned}$$

from which we have $\int_\Omega J(x_0-y)\check{I}(y)\,dy=0$ on the assumption that there exists some $x_0\in\bar{\Omega}$ such that $\check{I}(x_0)=0$. Hence, the same arguments as above give $\check{I}\equiv 0$, which is a contradiction. Thus, $\check{I}>0$ in $\bar{\Omega}$.

At present, suppose that case (b) holds. By the above arguments, we know in this case,

$$\frac{I_n}{\|I_n\|_{L^\infty(\Omega)}}\to 1 \ \text{ in } \ C(\bar{\Omega}) \ \text{ as } \ n\to+\infty.$$

By the first equation of (3.1), we have

$$S_n(x)=\frac{\frac{d_{S_n}}{\|I_n\|_{L^\infty(\Omega)}}\int_\Omega J(x-y)S_n(y)\,dy+\gamma(x)\frac{I_n(x)}{\|I_n\|_{L^\infty(\Omega)}}}{\frac{d_{S_n}}{\|I_n\|_{L^\infty(\Omega)}}\int_\Omega J(x-y)\,dy+\beta(x)\frac{I_n(x)}{\|I_n\|_{L^\infty(\Omega)}}}.$$

Letting $n\to+\infty$ in the above equality gives $S_n(x)\to\frac{\gamma(x)}{\beta(x)}$ in $C(\bar{\Omega})$ as $n\to+\infty$. By (3.2), we have

$$I_n\to\frac{N}{|\Omega|}-\frac{1}{|\Omega|}\int_\Omega\frac{\gamma(x)}{\beta(x)}\,dx \ \text{ uniformly on } \ \bar{\Omega} \ \text{ as } \ n\to+\infty.$$

If $N-\int_\Omega\frac{\gamma(x)}{\beta(x)}\,dx\neq 0$, a contradiction occurs due to $I_n\to 0$ uniformly on $\bar{\Omega}$ as $n\to+\infty$. If $N-\int_\Omega\frac{\gamma(x)}{\beta(x)}\,dx=0$, we have

$$(S_n(x),I_n(x))\to\left(\frac{\gamma(x)}{\beta(x)},\frac{N}{|\Omega|}-\frac{1}{|\Omega|}\int_\Omega\frac{\gamma(x)}{\beta(x)}\,dx\right),$$

which is the conclusion (i) of Theorem 2.3.

Finally, we assume the case (c) holds. As above, we know that $S_n(x)\to S^*(x)$ weakly in $L^2(\Omega)$ as $n\to+\infty$. Dividing both sides of the first equation of (3.1) by d_{S_n} gives

$$\begin{aligned}&\int_\Omega J(x-y)(S_n(y)-S_n(x))\,dy-\beta(x)S_n(x)\frac{I_n(x)}{\|I_n\|_{L^\infty(\Omega)}}\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}}\\&\quad+\gamma(x)\frac{I_n(x)}{\|I_n\|_{L^\infty(\Omega)}}\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}}=0,\end{aligned}$$

which implies that

$$S_n(x)=\frac{\int_\Omega J(x-y)S_n(y)\,dy+\gamma(x)\frac{I_n(x)}{\|I_n\|_{L^\infty(\Omega)}}\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}}}{\int_\Omega J(x-y)\,dy+\beta(x)\frac{I_n(x)}{\|I_n\|_{L^\infty(\Omega)}}\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}}}.$$

Thus, $S_n(x)\to S^*(x)$ in $C(\bar{\Omega})$ as $n\to+\infty$. Note that

$$\frac{I_n(x)}{\|I_n\|_{L^\infty(\Omega)}}\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}} \to C_0\check{I} \text{ in } C(\bar{\Omega}) \text{ as } n\to+\infty.$$

Then the conclusion (ii) of the theorem holds with $\psi = C_0\check{I}$. The proof is completed. $\square$

Proof of Theorem 2.4. At first, we assume that (i) holds. Dividing both sides of the first equation of (3.4) by I and integrating it on Ω yield that

$$d_I\int_\Omega\int_\Omega J(x-y)\left(\frac{I(y)}{I(x)}-1\right)dy\,dx + N\beta - \int_\Omega \gamma(x)\,dx - \beta\int_\Omega I(x)\,dx = 0.$$

Since J is symmetric, we have

$$\int_\Omega\int_\Omega J(x-y)\left(\frac{I(y)}{I(x)}-1\right)dy\,dx = \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\left(\sqrt{\frac{I(y)}{I(x)}}-\sqrt{\frac{I(x)}{I(y)}}\right)^2 dy\,dx \geq 0.$$

Then

$$\int_\Omega I(x)\,dx \geq N - \int_\Omega \frac{\gamma(x)}{\beta}\,dx. \tag{4.11}$$

By Theorem 2.3, there exists a sequence $\{d_{S_n}\}$ with $d_{S_n}\to 0$ as $n\to+\infty$ such that the corresponding endemic steady state satisfies $I_n \to I^*$ in $C(\bar{\Omega})$ as $n\to+\infty$, where I^* is a nonnegative constant. It follows from (4.11) that

$$I^* \geq \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma(x)}{\beta}\,dx > 0.$$

As a result, the first conclusion of Theorem 2.3 holds.

Next, as for assumptions (ii) and (iii), suppose on the contrary that the second conclusion of Theorem 2.3 holds. Set $w_n = \frac{I_n}{d_{S_n}}$. By the proof of Theorem 2.3, we derive

$$\frac{I_n}{\|I_n\|_{L^\infty(\Omega)}} \to \check{I} \text{ and } \frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}} \to C_0 \text{ as } n\to+\infty.$$

Thus,

$$w_n = \frac{I_n}{\|I_n\|_{L^\infty(\Omega)}}\frac{\|I_n\|_{L^\infty(\Omega)}}{d_{S_n}} \to C_0\check{I} := \hat{w} \text{ as } n\to+\infty.$$

In view of the positivity of C_0 and $\check{I}$, which is stated in the proof of Theorem 2.3, we have $\hat{w}>0$. In addition, $\hat{w}$ satisfies

$$d_I\int_\Omega J(x-y)(\hat{w}(y)-\hat{w}(x))\,dy + \hat{w}\left(\frac{N}{|\Omega|}\beta - \gamma + \frac{d_I\beta}{|\Omega|}\int_\Omega \hat{w}\,dx - d_I\beta\hat{w}\right) = 0. \tag{4.12}$$

Dividing both sides of the equation (4.12) by $\beta\hat{w}$ and integrating it on Ω give that

$$-d_I \int_\Omega \int_\Omega \frac{J(x-y)}{\beta(x)\hat{w}(x)} \left(\hat{w}(y) - \hat{w}(x)\right) dy\, dx = \int_\Omega \left(\frac{N}{|\Omega|} - \frac{\gamma(x)}{\beta(x)}\right) dx.$$

A direct computation yields that

$$\begin{aligned}
& - d_I \int_\Omega \int_\Omega \frac{J(x-y)}{\beta(x)\hat{w}(x)} \left(\hat{w}(y) - \hat{w}(x)\right) dy\, dx \\
&= -\frac{1}{2} d_I \int_\Omega \int_\Omega J(x-y) \left(\sqrt{\frac{\hat{w}(y)}{\beta(x)\hat{w}(x)}} - \sqrt{\frac{\hat{w}(x)}{\beta(y)\hat{w}(y)}}\right)^2 dy\, dx \\
&\quad - d_I \int_\Omega \int_\Omega J(x-y) \frac{1}{\sqrt{\beta(x)\beta(y)}} dy\, dx + d_I \int_\Omega \int_\Omega J(x-y) \frac{1}{\beta(x)} dy\, dx \\
&= -\frac{1}{2} d_I \int_\Omega \int_\Omega J(x-y) \left(\sqrt{\frac{\hat{w}(y)}{\beta(x)\hat{w}(x)}} - \sqrt{\frac{\hat{w}(x)}{\beta(y)\hat{w}(y)}}\right)^2 dy\, dx \\
&\quad + \frac{1}{2} d_I \int_\Omega \int_\Omega J(x-y) \left(\sqrt{\frac{1}{\beta(y)}} - \sqrt{\frac{1}{\beta(x)}}\right)^2 dy\, dx \\
&\le \frac{1}{2} d_I \int_\Omega \int_\Omega J(x-y) \left(\sqrt{\frac{1}{\beta(y)}} - \sqrt{\frac{1}{\beta(x)}}\right)^2 dy\, dx,
\end{aligned}$$

which contradicts the assumption (ii).

On the other hand, when $b = \min_{\bar{\Omega}} \left\{\frac{N}{|\Omega|} - \frac{\gamma(x)}{\beta(x)}\right\} > 0$, $\hat{w}$ is an upper solution of

$$d_I \int_\Omega J(x-y)(u(y) - u(x))\, dy + \beta u \left(b + \frac{d_I}{|\Omega|} \int_\Omega u\, dx - d_I u\right) = 0. \tag{4.13}$$

Choose a sufficiently small constant ϵ such that $0 < \epsilon \le \min\left\{\min_{\bar{\Omega}} \hat{w}, \frac{b}{d_I}\right\}$. Then ϵ is a lower solution of (4.13). Using the same arguments as in the proof of [46, Theorem A], it can be verified that the method of upper and lower solutions implies that (4.13) admits a positive solution u satisfying $\epsilon \le u \le \hat{w}$ on $\bar{\Omega}$. Set

$$G(x) = -d_I \int_\Omega J(x-y)\, dy + \beta(x) \left(b + \frac{d_I}{|\Omega|} \int_\Omega u(x)\, dx\right),$$

$$H(x) = d_I \int_\Omega J(x-y)u(y)\,dy.$$

Note that

$$u(x) = \frac{-G(x) - \sqrt{G^2(x) + 4d_I\beta(x)H(x)}}{-2d_I\beta(x)}.$$

We know that u is continuous on $\bar{\Omega}$. Then there exist $x_0, x_1 \in \bar{\Omega}$ such that $u(x_0) = \min_{\bar{\Omega}} u(x)$ and $u(x_1) = \max_{\bar{\Omega}} u(x)$. By (4.13), we get

$$\begin{aligned}
&\frac{b}{d_I} + \frac{1}{|\Omega|}\int_\Omega u(x)\,dx \le u(x_0) \le u(x) \ \text{ for all } \ x \in \bar{\Omega},\\
&\frac{b}{d_I} + \frac{1}{|\Omega|}\int_\Omega u(x)\,dx \ge u(x_1) \ge u(x) \ \text{ for all } \ x \in \bar{\Omega},
\end{aligned}$$

which imply that $u \equiv \frac{b}{d_I} + \frac{1}{|\Omega|}\int_\Omega u(x)\,dx$. Substituting this u into (4.13) yields $b = 0$, which implies a contradiction due to $b > 0$. Thus, we exclude the second conclusion in Theorem 2.3. The proof is completed. □

Proof of Theorem 2.5. Since $N > \int_\Omega \frac{\gamma}{\beta}\,dx$, Lemma 3.7 gives $R_0 > 1$ for all small d_I. Fix such a d_I. By Lemma 3.13, equation (3.12) admits a unique positive solution $0 < u_l < \frac{1}{d_I}$ for every $l > \frac{N}{|\Omega|}$. Define

$$d_{S_l} := \frac{N - l\int_\Omega (1 - d_I u_l)\,dx}{l\int_\Omega u_l\,dx}.$$

In view of (3.13) and the assumption $N > \int_\Omega \frac{\gamma}{\beta}\,dx$, there exists $l_0 \gg 1$ such that $d_{S_l} > 0$ for $l > l_0$. Set

$$S_l = l(1 - d_I u_l) \ \text{ and } \ I_l = \left(N - \int_\Omega S_l\,dx\right)\frac{u_l}{\int_\Omega u_l\,dx} \ \text{ for every } \ l > l_0.$$

Direct verification yields that (S_l, I_l) is an endemic steady state of (1.1) for every $l > l_0$. Moreover,

$$(S_l, I_l) \to \left(\frac{\gamma}{\beta}, \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma}{\beta}\,dx\right) \ \text{ uniformly on } \ \bar{\Omega} \ \text{ as } \ l \to +\infty.$$

And $d_{S_l} \to 0$ as $l \to +\infty$. The proof is completed. □

Proof of Theorem 2.6. Since Ω^+ is nonempty, Lemma 3.7 implies that $R_0 > 1$ for all small d_I. Then by Theorem 2.1, the endemic steady state (S, I) exists for small d_I. Since $\int_\Omega I\,dx \le N$, there exists some sequence $\{d_{I_n}\}$ with $d_{I_n} \to 0$ as $n \to +\infty$ such that the corresponding solution (S_n, I_n) of (3.1) satisfies $\int_\Omega I_n\,dx \to k$ as $n \to +\infty$ for some nonnegative constant k.

Since $d_{I_n} \to 0$ as $n \to +\infty$, we can assume that $d_{I_n} \le d_S$ for all n. Set

$$\kappa_n = d_S S_n + d_{I_n} I_n, \quad \tilde{S}_n = \frac{S_n}{\kappa_n} \text{ and } \tilde{I}_n = \frac{I_n}{\kappa_n}.$$

We derive from (3.10) that

$$\kappa_n = \frac{d_S N}{\int_\Omega (1 - d_{I_n}\tilde{I}_n + d_S \tilde{I}_n)\,dx} = \frac{d_S N}{|\Omega| + (d_S - d_{I_n})\int_\Omega \tilde{I}_n\,dx} \le \frac{d_S N}{|\Omega|} \text{ for all } n.$$

Combining with (4.3) yields

$$d_S \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \le \kappa_n \le \frac{d_S N}{|\Omega|} \text{ for all } n. \tag{4.14}$$

We claim that $\lim\limits_{n\to+\infty} \kappa_n = d_S \min\limits_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$. Assume on the contrary that this is false. Then there exists a subsequence $\{d_{I_{n_l}}\}$ of $\{d_{I_n}\}$ such that

$$d_S \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} < \vartheta := \lim_{l\to+\infty} \kappa_{n_l} \le \frac{d_S N}{|\Omega|}. \tag{4.15}$$

Set $w_l = d_{I_{n_l}} \tilde{I}_{n_l}$. Then w_l satisfies

$$d_{I_{n_l}} \int_\Omega J(x-y)[w_l(y) - w_l(x)]\,dy + \left[\frac{\beta\kappa_{n_l}}{d_S}(1 - w_l) - \gamma\right] w_l, \quad x \in \Omega. \tag{4.16}$$

Note that $0 \le w_l < 1$ by the second inequality of (3.11). Let

$$g_l(x) = \frac{\beta(x)\kappa_{n_l}}{d_S} - \gamma(x) - d_{I_{n_l}} \int_\Omega J(x-y)\,dy.$$

We derive from (4.16) that

$$w_l(x) = \frac{g_l(x) + \sqrt{g_l^2(x) + \frac{4\beta(x)\kappa_{n_l}}{d_S} d_{I_{n_l}} \int_\Omega J(x-y) w_l(y)\,dy}}{2\frac{\beta(x)\kappa_{n_l}}{d_S}}, \tag{4.17}$$

which implies that $w_l \to \left(1 - \frac{d_S\gamma}{\vartheta\beta}\right)^+$ in $C(\bar{\Omega})$ as $l \to +\infty$. On the other hand,

$$\int_\Omega w_l\,dx = \frac{d_{I_{n_l}}}{\kappa_{n_l}}\int_\Omega I_{n_l}\,dx \le d_{I_{n_l}}\frac{N}{\kappa_{n_l}} \le d_{I_{n_l}}\frac{N}{d_S\min\limits_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)}} \to 0 \text{ as } l\to+\infty.$$

As a result, we have $\left(1-\frac{d_S\gamma(x)}{\vartheta\beta(x)}\right)^+=0$ for all $x\in\Omega$ implying that $\vartheta \le d_S\min\limits_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)}$. This contradicts (4.15). Hence, $\lim\limits_{n\to+\infty}\kappa_n = d_S\min\limits_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)}$. Recalling (4.3) yields

$$\min_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)} \le S_n(x) \le \frac{\kappa_n}{d_S} \to \min_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)} \text{ as } n\to+\infty,$$

which implies that $S_n\to\min\limits_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)}$ in $C(\bar\Omega)$ as $n\to+\infty$. As a result, $k=N-|\Omega|\min\limits_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)}$. If $k=0$, then $\frac{\gamma(x)}{\beta(x)}\ge\min\limits_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)}=\frac{N}{|\Omega|}$ for all $x\in\bar\Omega$, contradicting the assumption that Ω^+ is nonempty. Hence, $k>0$.

Now we aim to prove (i). Since $\int_\Omega I_n\,dx\le N$, passing to a subsequence if necessary, we have $I_n\to\mu$ weakly as $n\to+\infty$ in the sense of

$$\int_\Omega I_n(x)\zeta(x)\,dx \to \int_\Omega \zeta(x)\mu(dx) \text{ for all } \zeta(x)\in C(\bar\Omega), \tag{4.18}$$

where μ is a Radon measure. Using the same argument in the proof of [48, Theorem 4.5], we can prove that the support of μ is contained in $\mathcal{S}$. Now, assuming $\mu(\bar\Omega)=0$ and letting $\zeta\equiv1$ in (4.18) yield that

$$\lim_{n\to+\infty}\int_\Omega I_n\,dx = \int_\Omega\mu(dx)=\mu(\bar\Omega)=0,$$

which is a contradiction due to $\lim\limits_{n\to+\infty}\int_\Omega I_n\,dx=k>0$. Hence, $\mu(\bar\Omega)>0$. Since $\beta(x)S_n(x)-\gamma(x)\to\beta(x)\theta_{min}-\gamma(x)$ uniformly on $\bar\Omega$ as $n\to+\infty$ and $\beta(x)\theta_{min}-\gamma(x)<0$ for $x\in\bar\Omega\setminus\{x_0\}$, there exists $n_*>0$ large enough such that $\beta(x)S_n(x)-\gamma(x)<0$ for $x\in\bar\Omega\setminus\{x_0\}$ and all $n\ge n_*$. We derive from the first equation of (3.1) that

$$I_n(x)=\frac{d_S\int_\Omega J(x-y)[S_n(y)-S_n(x)]\,dy}{\beta(x)S_n(x)-\gamma(x)},$$

which implies that $I_n(x)\to0$ locally uniformly on $\bar\Omega\setminus\{x_0\}$ as $n\to+\infty$. Note that

$$\begin{aligned}\lim_{\varepsilon\to0}\lim_{n\to+\infty}\int_{B_\varepsilon(x_0)} I_n(x)\,dx &= N-\lim_{n\to+\infty}\int_\Omega S_n(x)\,dx-\lim_{\varepsilon\to0}\lim_{n\to+\infty}\int_{\Omega\setminus B_\varepsilon(x_0)} I_n(x)\,dx\\ &= N-|\Omega|\theta_{min}.\end{aligned} \tag{4.19}$$

Given $\zeta\in C(\bar\Omega)$, for any small $\varepsilon>0$, it is obvious to see that

$$
\begin{aligned}
\int_{\Omega} I_n(x)\zeta(x)\,dx &= \int_{\Omega\setminus B_\varepsilon(x_0)} I_n(x)\zeta(x)\,dx + \int_{B_\varepsilon(x_0)} I_n(x)\zeta(x)\,dx \\
&= \int_{\Omega\setminus B_\varepsilon(x_0)} I_n(x)\zeta(x)\,dx + \zeta(x_\varepsilon)\int_{B_\varepsilon(x_0)} I_n(x)\,dx,
\end{aligned}
\tag{4.20}
$$

where $B_\varepsilon(x_0)$ is a ball centered at x_0 with radius ε and $x_\varepsilon \in \overline{B_\varepsilon(x_0)}$. Taking $n \to +\infty$ first and $\varepsilon \to 0$ second in equality (4.20) and combining (4.19), we derive

$$
\int_{\Omega} I_n(x)\zeta(x)\,dx \to (N - |\Omega|\theta_{min})\zeta(x_0) \ \text{ as } \ n \to +\infty.
$$

Thus, $I_n(x) \to (N - |\Omega|\theta_{min})\delta(x_0)$ weakly as $n \to +\infty$ in the sense of (4.18).

Finally, we focus on the proof of (ii). By the same discussion as in the proof of (i), we have $I_n(x) \to 0$ uniformly on $\overline{\Omega \setminus K}$ as $n \to +\infty$. In view of the first equation of (3.4), we have

$$
\begin{aligned}
&\int_{\Omega} J(x-y)(I_n(y) - I_n(x))\,dy \\
&+ \frac{\beta(x)I_n}{d_S}\left[\frac{d_S}{d_{I_n}}\left(\frac{N}{|\Omega|} - \frac{\gamma(x)}{\beta(x)}\right) - \left(\frac{d_S}{d_{I_n}} - 1\right)\frac{\int_\Omega I_n\,dx}{|\Omega|} - I_n\right] = 0.
\end{aligned}
\tag{4.21}
$$

Obviously, $\frac{\gamma(x)}{\beta(x)} = \theta_{min}$ for all $x \in K$. Set

$$
\alpha_n = \frac{d_S}{d_{I_n}}\left(\frac{N}{|\Omega|} - \theta_{min}\right) - \left(\frac{d_S}{d_{I_n}} - 1\right)\frac{\int_\Omega I_n\,dx}{|\Omega|}.
$$

In view of (3.2) and (4.3), we obtain $\alpha_n > 0$ for all n. Since $\int_\Omega J(x-y)I_n(y)\,dy > 0$ for all $x \in \Omega$, by (4.21), we have

$$
-\frac{d_S}{\beta(x)}\int_{\Omega} J(x-y)\,dy + \alpha_n - I_n(x) \le 0 \ \text{ for all } \ x \in K,
$$

which implies that

$$
\begin{aligned}
\alpha_n &\le \frac{d_S}{|K|}\int_K \frac{1}{\beta(x)}\int_\Omega J(x-y)\,dy\,dx + \frac{1}{|K|}\int_K I_n(x)\,dx \\
&\le \frac{d_S}{\min\limits_{x\in\bar{\Omega}}\beta(x)} + \frac{N}{|K|}.
\end{aligned}
\tag{4.22}
$$

As a result, $\{\alpha_n\}$ is a bounded sequence and then $\{\alpha_n\}$ owns a convergent subsequence, still denoted by itself.

Since $I_n(x) \to 0$ uniformly on $\overline{\Omega \setminus K}$ as $n \to +\infty$, we have $\int_{\Omega\setminus \mathring{K}} J(x-y)I_n(y)\,dy \to 0$ uniformly on $\bar{\Omega}$ as $n \to +\infty$. Then there exists some constant $M_* > 0$ such that

$$\left\| \int_{\Omega\setminus\mathring{K}} J(x-y)I_n(y)\,dy \right\|_{L^\infty(\Omega)} \le M_* \text{ for all } n. \tag{4.23}$$

For all $x \in K$, rewrite (4.21) as

$$\begin{aligned} &\int_{\mathring{K}} J(x-y)(I_n(y)-I_n(x))\,dy + \int_{\Omega\setminus\mathring{K}} J(x-y)I_n(y)\,dy \\ &\quad + \frac{\beta(x)I_n}{d_S}\left[-\frac{d_S}{\beta(x)}\int_{\Omega\setminus\mathring{K}} J(x-y)\,dy + \alpha_n - I_n\right] = 0. \end{aligned} \tag{4.24}$$

There exists $x_n^* \in K$ such that $I_n(x_n^*) = \max_{x\in K} I_n(x)$. By (4.24), we have

$$\frac{\beta(x_n^*)I_n^2(x_n^*)}{d_S} - \left[-\frac{d_S}{\beta(x_n^*)}\int_{\Omega\setminus\mathring{K}} J(x_n^*-y)\,dy + \alpha_n\right]\frac{\beta(x_n^*)I_n(x_n^*)}{d_S} - \int_{\Omega\setminus\mathring{K}} J(x_n^*-y)I_n(y)\,dy \le 0.$$

Hence, combining (4.22) with (4.23), there exists some constant $M^* > 0$ such that $I_n(x_n^*) \le M^*$. That is, $\|I_n\|_{L^\infty(K)} \le M^*$. Then there exists a subsequence of $\{I_n\}$, still denoted by itself, and a nonnegative function $\check{I}$ such that $I_n \to \check{I}$ weakly in $L^2(K)$ as $n \to +\infty$. It is well-known that

$$\int_{\mathring{K}} J(x-y)I_n(y)\,dy \to \int_{\mathring{K}} J(x-y)\check{I}(y)\,dy \text{ uniformly on } \bar{\Omega} \text{ as } n \to +\infty.$$

Thus,

$$\int_{\Omega} J(x-y)I_n(y)\,dy \to \int_{\mathring{K}} J(x-y)\check{I}(y)\,dy \text{ uniformly on } \bar{\Omega} \text{ as } n \to +\infty.$$

Set

$$G_n(x) = -\int_{\Omega} J(x-y)\,dy + \frac{\beta(x)}{d_S}\alpha_n.$$

By virtue of (4.21), we have

$$I_n(x) = \frac{-G_n(x) - \sqrt{G_n^2(x) + 4\frac{\beta(x)}{d_S}\int_\Omega J(x-y)I_n(y)\,dy}}{-2\frac{\beta(x)}{d_S}}, \quad x \in K.$$

We can see that $I_n(x) \to \check{I}(x)$ uniformly on K as $n \to +\infty$. In addition, $\check{I}(x)$ satisfies

$$\int_{\mathring{K}} J(x-y)(\check{I}(y)-\check{I}(x))\,dy + \frac{\beta(x)}{d_S}\check{I}(x)\left[\alpha - \frac{d_S}{\beta(x)}\int_{\Omega\setminus\mathring{K}} J(x-y)\,dy - \check{I}(x)\right] = 0, \quad (4.25)$$

where α is the limit of α_n as $n \to +\infty$. If there exists some $x_2 \in K$ such that $0 = \check{I}(x_2) = \min_{x\in K} \check{I}(x)$, then we derive from (4.25) that $\int_{\mathring{K}} J(x_2-y)\check{I}(y)\,dy = 0$. Thus, $\check{I}(x) = 0$ for all $x \in K$, contradicting $\int_{\mathring{K}} \check{I}\,dx = k > 0$. Hence, $\check{I}(x) > 0$ for all $x \in K$. The uniqueness of the positive solution of (4.25) can be obtained by the Implicit Function Theorem as in the proof of Theorem 2.1. Now denote the nonnegative solution of (4.25) by $\check{I}_\alpha$. By Lemma 3.10, there exists a maximal value $\alpha_* \geq 0$ such that $\check{I}_{\alpha_*}(x) = 0$ for all $x \in K$. The positivity of $\check{I}_\alpha$ gives $\alpha > \alpha_*$. And (4.22) gives $\alpha \leq \alpha^* := \frac{d_S}{\min_{x\in\bar{\Omega}}\beta(x)} + \frac{N}{|K|}$. By using the same arguments as in the proof of Lemma 3.11, we derive that $\check{I}_\alpha$ is strictly increasing and continuous on $\alpha \in (\alpha_*, \alpha^*)$. In view of (4.25), it is easy to get

$$\check{I}_{\alpha^*} \geq \min_{x\in K} \check{I}_{\alpha^*}(x) \geq \frac{N}{|K|} \quad \text{in} \quad K.$$

Thus, $\int_{\mathring{K}} \check{I}_{\alpha_*}\,dx < k$ and $\int_{\mathring{K}} \check{I}_{\alpha^*}\,dx > k$. Then there exists a unique α such that $\int_{\mathring{K}} \check{I}_\alpha\,dx = k$. As a result, the limit of I is independent of any chosen subsequence. The proof is completed. □

Next, we are devoted to the case $d_S \to 0$ and $d_I \to 0$. To this end, we first state the following lemma from Wu and Zou [47].

Lemma 4.2. *Assume that Ω^+ is nonempty and d is a positive constant. Then the following equation*

$$\left(\frac{N}{|\Omega|}\beta - \gamma - \frac{(1-d)\beta}{|\Omega|}\int_\Omega I^*\,dx\right)^+ - d\beta I^* = 0, \quad x \in \bar{\Omega} \quad (4.26)$$

has a unique nonnegative solution.

Proof of Theorem 2.8. Since Ω^+ is nonempty, Lemma 3.7 implies that $R_0 > 1$ for all small d_I. Then by Theorem 2.1, the endemic steady state (S, I) exists for small d_I.

(i) By (3.2), there exist two sequences $\{d_{I_n}\}$ and $\left\{\frac{d_{I_n}}{d_{S_n}}\right\}$ with $d_{I_n} \to 0$ and $\frac{d_{I_n}}{d_{S_n}} \to d$ as $n \to +\infty$ such that $\int_\Omega I_n\,dx \to k \in [0, N]$. In the following, we prove that

$$\int_\Omega I_n\,dx \to \int_\Omega I^*\,dx \quad \text{as} \quad n \to +\infty,$$

where I^* is the unique nonnegative solution of (4.26). Since I_n is continuous on $\bar{\Omega}$, there exists some $x_n \in \bar{\Omega}$ such that $I_n(x_n) = \max_{x\in\bar{\Omega}} I_n(x)$. We derive from the first equation of (3.4) that

$$\frac{N}{|\Omega|}\beta(x_n) - \gamma(x_n) - \left(1 - \frac{d_{I_n}}{d_{S_n}}\right)\frac{\beta(x_n)}{|\Omega|}\int_\Omega I_n\,dx - \frac{d_{I_n}\beta(x_n)}{d_{S_n}}I_n(x_n) \ge 0. \tag{4.27}$$

Since $\frac{d_{I_n}}{d_{S_n}} \to d$ as $n \to +\infty$, some $n_0 \in \mathbb{N}$ exists such that $\frac{d}{2} < \frac{d_{I_n}}{d_{S_n}} \le d+1$ for all $n > n_0$. Then, by (4.27), we have

$$\begin{aligned} I_n(x_n) &\le \frac{d_{S_n}}{d_{I_n}\beta(x_n)}\left[\frac{N}{|\Omega|}\beta(x_n) - \gamma(x_n) - \left(1 - \frac{d_{I_n}}{d_{S_n}}\right)\frac{\beta(x_n)}{|\Omega|}\int_\Omega I_n\,dx\right] \\ &\le \frac{N}{|\Omega|}\left(\frac{2}{d}+1\right), \end{aligned}$$

which implies that $\|I_n\|_{L^\infty(\Omega)} \le \frac{N}{|\Omega|}\left(\frac{2}{d}+1\right)$ for all $n > n_0$. Thus, there is some subsequence of $\{I_n\}$, still denoted by itself, converges weakly to some nonnegative function in $L^2(\Omega)$ as $n \to +\infty$. In view of the first equation of (3.4), we have

$$I_n(x) = \frac{-G_n(x) - \sqrt{G_n^2(x) - 4a_n(x)H_n(x)}}{2a_n(x)}, \tag{4.28}$$

in which

$$\begin{aligned} &a_n(x) = -\frac{d_{I_n}\beta(x)}{d_{S_n}}, \quad H_n(x) = d_{I_n}\int_\Omega J(x-y)I_n(y)\,dy, \\ &G_n(x) = -d_{I_n}\int_\Omega J(x-y)\,dy + \frac{N}{|\Omega|}\beta(x) - \gamma(x) - \left(1 - \frac{d_{I_n}}{d_{S_n}}\right)\frac{\beta(x)}{|\Omega|}\int_\Omega I_n(x)\,dx. \end{aligned}$$

Letting $n \to +\infty$ in (4.28) yields that

$$\lim_{n\to+\infty} I_n(x) = \left(\frac{\frac{N}{|\Omega|}\beta(x) - \gamma(x) - (1-d)\frac{\beta(x)}{|\Omega|}k}{d\beta(x)}\right)^+ \quad \text{uniformly on } \bar{\Omega}.$$

Furthermore,

$$k = \lim_{n\to+\infty}\int_\Omega I_n\,dx = \int_\Omega \frac{1}{d}\left(\frac{N}{|\Omega|} - \frac{\gamma}{\beta} - (1-d)\frac{k}{|\Omega|}\right)^+ dx.$$

In view of Lemma 4.2, we have $k = \int_\Omega I^*\,dx$. Setting $k = \int_\Omega I^*\,dx$ in the previous arguments yields

$$\lim_{n\to+\infty} I_n = \left(\frac{\frac{N}{|\Omega|}\beta - \gamma - (1-d)\frac{\beta}{|\Omega|}\int_\Omega I^*\,dx}{d\beta}\right)^+ = I^*.$$

By the second equation of (3.4), we have

$$S_n \to \frac{N}{|\Omega|} - \frac{(1-d)}{|\Omega|}\int_\Omega I^*\,dx - dI^* \ \text{ uniformly on } \bar{\Omega} \ \text{ as } \ n \to +\infty.$$

(ii) Note that

$$\min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \le \frac{\kappa}{d_S} = \frac{N}{\int_\Omega (1-d_I\tilde{I}) + d_S\tilde{I}\,dx} \le \frac{N}{|\Omega|} \ \text{ for all } \ 0 < d_I < d_S. \tag{4.29}$$

By the same arguments as in the proof of Theorem 2.6, we can obtain $\frac{\kappa}{d_S} \to \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$ and $S \to \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$ uniformly on $\bar{\Omega}$ as $d_S \to 0$ and $\frac{d_I}{d_S} \to 0$. Then, $\int_\Omega I\,dx \to k$ with $k = N - |\Omega| \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$. If $k = 0$, then $\frac{\gamma(x)}{\beta(x)} \ge \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} = \frac{N}{|\Omega|}$ for all $x \in \bar{\Omega}$, contradicting the assumption that Ω^+ is nonempty. Hence, $k > 0$. By the same proof of Theorem 2.6 (i), we obtain (a).

Next we prove (b). Following the same proof of Theorem 2.6 (i), we have $I(x) \to 0$ uniformly on $\overline{\Omega \setminus K}$ as $d_S \to 0$ and $\frac{d_I}{d_S} \to 0$. Set

$$\kappa = d_S S + d_I I, \ \ \tilde{S} = \frac{S}{\kappa}, \ \ \tilde{I} = \frac{I}{\kappa} \ \text{ and } \ \hat{I} = d_S\tilde{I}.$$

Then,

$$d_S \int_{\Omega\setminus\mathring{K}} J(x-y)\hat{I}(y)\,dy = d_S\frac{d_S}{\kappa}\int_{\Omega\setminus\mathring{K}} J(x-y)I(y)\,dy \to 0 \ \text{ uniformly on } \ \bar{\Omega}$$

as $d_S \to 0$ and $\frac{d_I}{d_S} \to 0$. Then there exist two positive constants d_* and M_* such that

$$\left\| d_S \int_{\Omega\setminus\mathring{K}} J(x-y)\hat{I}(y)\,dy \right\|_{L^\infty(\Omega)} \le M_* \ \text{ for all } \ 0 < d_S, \frac{d_I}{d_S} < d_*. \tag{4.30}$$

By (3.11), we have $\hat{I}$ satisfies

$$d_S \int_\Omega J(x-y)[\hat{I}(y) - \hat{I}(x)]\,dy + \left[\frac{\frac{\kappa\beta}{d_S} - \gamma}{\frac{d_I}{d_S}} - \frac{\kappa\beta}{d_S}\hat{I} \right]\hat{I} = 0 \ \text{ in } \ \Omega. \tag{4.31}$$

Clearly, $\frac{\gamma(x)}{\beta(x)} = \theta_{min}$ for all $x \in K$. Since $\int_\Omega J(x-y)\hat{I}(y)\,dy > 0$ for all $x \in \Omega$, we derive from (4.31) that

$$-d_S \int_\Omega J(x-y)\,dy + \beta(x)\frac{\frac{\kappa}{d_S} - \theta_{min}}{\frac{d_I}{d_S}} - \frac{\kappa\beta(x)}{d_S}\hat{I}(x) \le 0 \ \text{ for all } \ x \in K,$$

which implies that

$$0 \le \frac{\frac{\kappa}{d_S} - \theta_{min}}{\frac{d_I}{d_S}} \le \frac{d_S}{|K|} \int_K \frac{1}{\beta(x)} \int_\Omega J(x-y)\,dy\,dx + \frac{1}{|K|}\frac{\kappa}{d_S}\int_K \hat{I}(x)\,dx$$
$$\le d_* \frac{1}{\min\limits_{x\in\bar{\Omega}} \beta(x)} + \frac{N^2}{|K||\Omega|} \text{ for all } 0 < d_S, \frac{d_I}{d_S} < d_*. \tag{4.32}$$

Thus, $\frac{\frac{\kappa}{d_S} - \theta_{min}}{\frac{d_I}{d_S}}$ has a convergent subsequence still denoted by itself, and denote the limit of it by ν. For all $x \in K$, rewrite (4.31) as

$$d_S \int_{\mathring{K}} J(x-y)[\hat{I}(y) - \hat{I}(x)]\,dy + d_S \int_{\Omega\setminus\mathring{K}} J(x-y)\hat{I}(y)\,dy$$
$$+ \left[-d_S \int_{\Omega\setminus\mathring{K}} J(x-y)\,dy + \beta \frac{\frac{\kappa}{d_S} - \theta_{min}}{\frac{d_I}{d_S}} - \frac{\kappa\beta}{d_S}\hat{I} \right] \hat{I} = 0. \tag{4.33}$$

There exists $x_* \in K$ such that $\hat{I}(x_*) = \max\limits_{x\in K} \hat{I}(x)$. We derive from (4.33) that

$$\frac{\kappa\beta(x_*)\hat{I}^2(x_*)}{d_S} - \left[-d_S \int_{\Omega\setminus\mathring{K}} J(x_* - y)\,dy + \beta(x_*)\frac{\frac{\kappa}{d_S} - \theta_{min}}{\frac{d_I}{d_S}} \right] \hat{I}(x_*)$$
$$- d_S \int_{\Omega\setminus\mathring{K}} J(x_* - y)\hat{I}(y)\,dy \le 0.$$

So, combining (4.29) with (4.30) and (4.32) gives that some constant $M^* > 0$ exists such that $\hat{I}(x_*) \le M^*$. That is, $\|\hat{I}\|_{L^\infty(K)} \le M^*$ for all $0 < d_S, \frac{d_I}{d_S} < d_*$. Thus,

$$d_S \int_\Omega J(x-y)\hat{I}(y)\,dy = d_S \int_{\Omega\setminus\mathring{K}} J(x-y)\hat{I}(y)\,dy + d_S \int_{\mathring{K}} J(x-y)\hat{I}(y)\,dy \to 0 \text{ uniformly on } \bar{\Omega}$$

as $d_S \to 0$ and $\frac{d_I}{d_S} \to 0$. Set

$$g(x) = -d_S \int_\Omega J(x-y)\,dy + \beta(x)\frac{\frac{\kappa}{d_S} - \theta_{min}}{\frac{d_I}{d_S}}.$$

We derive from (4.33) that

$$\hat{I}(x) = \frac{g(x) + \sqrt{g^2(x) + 4\frac{\kappa\beta(x)}{d_S} d_S \int_\Omega J(x-y)\hat{I}(y)\,dy}}{2\frac{\kappa\beta(x)}{d_S}},$$

which implies that $\hat{I}(x) \to \frac{\nu}{\theta_{min}}$ uniformly on K as $d_S \to 0$ and $\frac{d_I}{d_S} \to 0$. Then, $I = \frac{\kappa}{d_S}\hat{I} \to \nu$ uniformly on K as $d_S \to 0$ and $\frac{d_I}{d_S} \to 0$. Therefore, $\int_K \nu\,dx = N - |\Omega|\theta_{min}$. That is, $\nu = \frac{N-|\Omega|\theta_{min}}{|K|}$. Now we have ν is uniquely determined by this. As a result, the limit of I is independent of any chosen subsequence.

The proofs of (iii) and (iv) are inspired by Castellano and Salako [9].

(iii) Set

$$\kappa = d_S S + d_I I, \quad \tilde{S} = \frac{S}{\kappa} \text{ and } \tilde{I} = \frac{I}{\kappa}.$$

We first claim that

$$\vartheta_0 := \liminf_{\max\left\{d_I, \frac{d_S}{d_I}\right\} \to 0} \int_\Omega \left(1 - d_I\tilde{I}\right)\,dx > 0. \tag{4.34}$$

Otherwise, there exist two sequences $\{d_{I_n}\}$ and $\left\{\frac{d_{S_n}}{d_{I_n}}\right\}$ with $d_{I_n} \to 0$ and $\frac{d_{S_n}}{d_{I_n}} \to 0$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} \int_\Omega \left(1 - d_{I_n}\tilde{I}_n\right)\,dx = 0.$$

Combining with $d_{I_n}\tilde{I}_n < 1$ yields that

$$\lim_{n\to\infty} \frac{\kappa_n}{d_{S_n}} = \lim_{n\to+\infty} \frac{N}{\int_\Omega \left(1 - d_{I_n}\tilde{I}_n\right)\,dx + \frac{d_{S_n}}{d_{I_n}} \int_\Omega d_{I_n}\tilde{I}_n\,dx} = +\infty. \tag{4.35}$$

Recalling $S_n = \frac{\kappa_n}{d_{S_n}}\left(1 - d_{I_n}\tilde{I}_n\right)$ and (4.3), we have

$$\frac{d_{S_n}}{\kappa_n} \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \le \left(1 - d_{I_n}\tilde{I}_n\right) \le \frac{d_{S_n}}{\kappa_n} \max_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \text{ for all } n,$$

which implies $d_{I_n}\tilde{I}_n \to 1$ uniformly on $\bar{\Omega}$ as $n \to +\infty$ due to (4.35). Note that S_n satisfies

$$d_{I_n}\frac{d_{S_n}}{\kappa_n} \int_\Omega J(x-y)[S_n(y) - S_n(x)]\,dy + (-\beta S_n + \gamma)\left(d_{I_n}\tilde{I}_n\right) = 0 \text{ in } \Omega,$$

from which we get

$$S_n(x) = \frac{d_{I_n}\frac{d_{S_n}}{\kappa_n} \int_\Omega J(x-y)S_n(y)\,dy + \gamma(x)d_{I_n}\tilde{I}_n(x)}{d_{I_n}\frac{d_{S_n}}{\kappa_n} \int_\Omega J(x-y)\,dy + \beta(x)d_{I_n}\tilde{I}_n(x)}.$$

Therefore, $S_n \to \frac{\gamma}{\beta}$ uniformly on $\bar{\Omega}$ as $n \to +\infty$ and then

$$N \geq \lim_{n\to+\infty} \int_\Omega S_n \, dx = \int_\Omega \frac{\gamma}{\beta} \, dx \quad \text{as} \quad n \to +\infty,$$

contradicting the assumption that $N < \int_\Omega \frac{\gamma}{\beta} \, dx$. Thus, (4.34) holds.

Note that

$$\limsup_{\max\left\{d_I, \frac{d_S}{d_I}\right\}\to 0} \frac{\kappa}{d_S} = \limsup_{\max\left\{d_I, \frac{d_S}{d_I}\right\}\to 0} \frac{N}{\int_\Omega \left(1 - d_I \tilde{I}\right) dx + d_S \int_\Omega \tilde{I} \, dx} \leq \frac{N}{\vartheta_0}.$$

Then there exist constants $0 < d_0 \ll 1$ and $C_* > 0$ such that $\frac{\kappa}{d_S} \leq C_*$ for all $0 < d_I, \frac{d_S}{d_I} < d_0$. Hence,

$$0 \leq I = \kappa \tilde{I} = \frac{\kappa}{d_I} d_I \tilde{I} \leq \frac{\kappa}{d_I} \leq C_* \frac{d_S}{d_I} \quad \text{for all} \quad x \in \bar{\Omega} \quad \text{and} \quad 0 < d_I, \frac{d_S}{d_I} < d_0, \tag{4.36}$$

which implies that $I \to 0$ uniformly on $\bar{\Omega}$ as $d_I \to 0$ and $\frac{d_S}{d_I} \to 0$.

Next we prove (2.4). Claim that $\vartheta_1 := \liminf_{\max\left\{d_I, \frac{d_S}{d_I}\right\}\to 0} \|d_I \tilde{I}\|_{L^\infty(\Omega)} > 0$. Assume on the contrary that it is false. Then there exist two sequences $\{d_{I_m}\}$ and $\left\{\frac{d_{S_m}}{d_{I_m}}\right\}$ with $d_{I_m} \to 0$ and $\frac{d_{S_m}}{d_{I_m}} \to 0$ as $m \to +\infty$ such that $d_{I_m} \tilde{I}_m \to 0$ as $m \to +\infty$. Observe that

$$\min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \leq S_m \leq S_m + \frac{d_{I_m}}{d_{S_m}} I_m = \frac{\kappa_m}{d_{S_m}} \leq C_*. \tag{4.37}$$

Up to a subsequence, we have $\frac{\kappa_m}{d_{S_m}} \to \vartheta_2$ as $m \to +\infty$ for some $\vartheta_2 \in \left[\min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}, C_*\right]$. Note that $d_{I_m} \tilde{I}_m$ satisfies (4.16). By the same arguments as in the proof of Theorem 2.6, we can prove that $d_{I_m} \tilde{I}_m \to \left(1 - \frac{\gamma}{\vartheta_2 \beta}\right)^+$ as $m \to +\infty$. Combining with $d_{I_m} \tilde{I}_m \to 0$ as $m \to +\infty$, we conclude that $\left(1 - \frac{\gamma}{\vartheta_2 \beta}\right)^+ \equiv 0$; that is, $\vartheta_2 \leq \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$. This together with (4.37) yields

$$S_m \to \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \quad \text{uniformly on} \quad \bar{\Omega} \quad \text{and} \quad \int_\Omega I_m \, dx \to N - |\Omega| \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \quad \text{as} \quad m \to +\infty.$$

In view of (4.36), we have $I_m \to 0$ uniformly on $\bar{\Omega}$ as $m \to +\infty$. Therefore, $\frac{N}{|\Omega|} = \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$, which contradicts the assumption that Ω^+ is nonempty. As a result, the claim $\vartheta_1 > 0$ holds, which implies that there exists a constant $C^* > 0$ such that $\|d_I \tilde{I}\|_{L^\infty(\Omega)} \geq C^*$ for all $0 < d_I$, $\frac{d_S}{d_I} < d_0$. Thus,

$$\|I\|_{L^\infty(\Omega)} = \|\kappa \tilde{I}\|_{L^\infty(\Omega)} = \frac{\kappa}{d_I} \|d_I \tilde{I}\|_{L^\infty(\Omega)} \geq C^* \frac{\kappa}{d_S} \frac{d_S}{d_I} \geq C^* \frac{d_S}{d_I} \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$$

for all $0 < d_I, \frac{d_S}{d_I} < d_0$. This together with (4.36) gives (2.4).

In fact, we obtain from the above arguments that up to a subsequence $\frac{\kappa}{d_S} \to \vartheta \in \left[\min_{x\in\bar\Omega} \frac{\gamma}{\beta}, C_*\right]$, and up to a further subsequence, $d_I \tilde I \to \left(1 - \frac{\gamma}{\vartheta\beta}\right)^+$ uniformly on $\bar\Omega$ as $d_I \to 0$ and $\frac{d_S}{d_I} \to 0$. In addition, $\left\|\left(1 - \frac{\gamma}{\vartheta\beta}\right)^+\right\|_{L^\infty(\Omega)} > 0$ due to $\vartheta_1 > 0$. This gives $\vartheta > \min_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)}$. Note that up to a subsequence,

$$S = \frac{\kappa}{d_S}(1 - d_I \tilde I) \to \vartheta \left[1 - \left(1 - \frac{\gamma}{\vartheta\beta}\right)^+\right] \text{ uniformly on } \bar\Omega \text{ as } d_I \to 0 \text{ and } \frac{d_S}{d_I} \to 0. \quad (4.38)$$

If $\vartheta \geq \max_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)}$, that is, $\left(1 - \frac{\gamma}{\vartheta\beta}\right)^+ = 1 - \frac{\gamma}{\vartheta\beta}$ on $\bar\Omega$, then we get that up to a subsequence $S \to \frac{\gamma}{\beta}$ as $d_I \to 0$ and $\frac{d_S}{d_I} \to 0$. Since $I \to 0$ uniformly on $\bar\Omega$ as $d_I \to 0$ and $\frac{d_S}{d_I} \to 0$, we have $N - \int_\Omega \frac{\gamma}{\beta}\, dx = 0$, contradicting the assumption $N < \int_\Omega \frac{\gamma}{\beta}\, dx$. Hence, $\vartheta < \max_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)}$ holds. In view of $\int_\Omega S + I\, dx = N$, we obtain

$$N = \int_\Omega \vartheta \left[1 - \left(1 - \frac{\gamma}{\vartheta\beta}\right)^+\right] dx = \int_\Omega \min\left\{\vartheta, \frac{\gamma}{\beta}\right\} dx. \quad (4.39)$$

Observe that $\int_\Omega \min\left\{\vartheta, \frac{\gamma}{\beta}\right\} dx$ is strictly decreasing with respect to $\vartheta \in \left(\min_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)}, \max_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)}\right)$. Then ϑ is uniquely determined by the equation $N = \int_\Omega \min\left\{\vartheta, \frac{\gamma}{\beta}\right\} dx$, which implies that the limit of S in (4.38) is independent of any chosen subsequence. In the end, by virtue of (4.39) and $\min_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)} < \vartheta < \max_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)}$, we conclude that $N < \vartheta|\Omega|$.

(iv) Since $N > \int_\Omega \frac{\gamma}{\beta}\, dx$, by Lemma 3.7, there exists $0 < d_0 \ll 1$ such that $R_0 > 1$ and $\frac{1}{d_I} > \frac{N}{|\Omega|}$ for every $0 < d_I \leq d_0$. For every $0 < d_I \leq d_0$, with $l_I = \frac{1}{d_I}$, we derive from Lemma 3.13 that (3.12) admits a unique positive solution $0 < u_I < \frac{1}{d_I}$. Note that $w_I = l_I(1 - d_I u_I)$ satisfies

$$\frac{1}{l_I} \int_\Omega J(x-y)[w_I(y) - w_I(x)]\, dy + (\gamma - \beta w_I) u_I = 0 \text{ in } \Omega. \quad (4.40)$$

There exist $x_1, x_2 \in \bar\Omega$ such that $w_I(x_1) = \min_{x\in\bar\Omega} w_I(x)$ and $w_I(x_2) = \max_{x\in\bar\Omega} w_I(x)$. By (4.40), $\gamma(x_1) - \beta(x_1) w_I(x_1) \leq 0$ and $\gamma(x_2) - \beta(x_2) w_I(x_2) \geq 0$. Then we have

$$\min_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)} \leq w_I \leq \max_{x\in\bar\Omega} \frac{\gamma(x)}{\beta(x)} \text{ for all } 0 < d_I < d_0 \quad (4.41)$$

and

$$d_I \min_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)} = \frac{1}{l_I}\min_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)} \le 1 - d_I u_I \le \frac{1}{l_I}\max_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)} = d_I \max_{x\in\bar\Omega}\frac{\gamma(x)}{\beta(x)} \quad \text{for all } \ 0<d_I<d_0,$$

from which we get $d_I u_I \to 1$ uniformly on $\bar\Omega$ as $d_I \to 0$. Multiplying (4.40) by d_I yields

$$\frac{d_I}{l_I}\int_\Omega J(x-y)[w_I(y)-w_I(x)]\,dy + (\gamma-\beta w_I)d_I u_I = 0 \ \text{ in } \ \Omega, \tag{4.42}$$

and (4.42) gives that

$$w_I(x) = \frac{\frac{d_I}{l_I}\int_\Omega J(x-y)w_I(y)\,dy + \gamma(x)d_I u_I(x)}{\frac{d_I}{l_I}\int_\Omega J(x-y)\,dy + \beta(x)d_I u_I(x)}.$$

Recalling (4.41) and $d_I u_I \to 1$ as $d_I \to 0$, we conclude that $w_I \to \frac{\gamma}{\beta}$ uniformly on $\bar\Omega$ as $d_I \to 0$ and

$$N - \int_\Omega w_I\,dx \to N - \int_\Omega \frac{\gamma}{\beta}\,dx > 0 \ \text{ as } \ d_I \to 0.$$

Hence, there exists $0<d_1 \ll d_0$ such that $N - \int_\Omega w_I\,dx > 0$ for every $0<d_I<d_1$. Define

$$d_{S,I} := \frac{N - \int_\Omega w_I\,dx}{l_I\int_\Omega u_I\,dx} \ \text{ for every } \ 0<d_I<d_1.$$

Then, $d_{S,I} > 0$ for every $0<d_I<d_1$ and

$$\frac{d_{S,I}}{d_I} = \frac{N - \int_\Omega w_I\,dx}{l_I\int_\Omega d_I u_I\,dx} = d_I\frac{\left(N - \int_\Omega w_I\,dx\right)}{\int_\Omega d_I u_I\,dx} \to 0 \ \text{ as } \ d_I \to 0.$$

Now define

$$S_I := w_I \ \text{ and } \ I_I := \left(N - \int_\Omega w_I\,dx\right)\frac{d_I u_I}{\int_\Omega d_I u_I\,dx} \ \text{ for every } \ 0<d_I<d_1.$$

It is easily verified that (S_I, I_I) is an endemic steady state of (1.1) with $d_S = d_{S,I}$. In addition, recalling $d_I u_I \to 1$ and $w_I \to \frac{\gamma}{\beta}$ uniformly on $\bar\Omega$ as $d_I \to 0$ gives

$$S_I \to \frac{\gamma}{\beta} \ \text{ and } \ I_I \to \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma}{\beta}\,dx \ \text{ uniformly on } \ \bar\Omega \ \text{ as } \ d_I \to 0.$$

The proof is completed. □

Now, we are in a position to investigate the asymptotic profile of the endemic steady state for large diffusion rates.

Proof of Theorem 2.10. In view of Lemma 3.7 and Theorem 2.1, there exists a unique endemic steady state (S, I). We first give the proof of (i). Choose sequences $\{d_{S_n}\}$ and $\{d_{I_n}\}$ such that $d_{S_n} \to +\infty$ and $d_{I_n} \to +\infty$ as $n \to +\infty$. Denote (S_n, I_n) the corresponding positive solution of (3.1). Up to a subsequence if needed, one of the following three statements must hold:

(A1) $\frac{d_{I_n}}{d_{S_n}} \to 0$;
(A2) $\frac{d_{I_n}}{d_{S_n}} \to +\infty$;
(A3) $\frac{d_{I_n}}{d_{S_n}} \to C$ with C being a positive constant.

Since I_n is continuous, there exists some $x_n \in \bar{\Omega}$ such that $I_n(x_n) = \max_{x\in\bar{\Omega}} I_n(x)$. By the first equation of (3.4), we have

$$I_n(x_n) \leq \frac{d_{S_n}}{d_{I_n}} \frac{N}{|\Omega|} + \frac{1}{|\Omega|} \int_\Omega I_n \, dx.$$

If case (A2) or (A3) holds, then $\|I_n\|_{L^\infty(\Omega)} \leq M$ with M being some positive constant. Recall $\|S_n\|_{L^\infty(\Omega)} \leq \max_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$. Then by the same arguments as in the proof of [51, Theorem 4.1], the proof of (i) can be obtained.

If case (A1) holds, without loss of generality, we assume $d_{S_n} > d_{I_n}$ for all n. Set

$$\kappa_n = d_{S_n} S_n + d_{I_n} I_n, \quad \tilde{S}_n = \frac{S_n}{\kappa_n} \quad \text{and} \quad \tilde{I}_n = \frac{I_n}{\kappa_n}.$$

The second inequality of (3.11) gives $\tilde{I}_n \to 0$ uniformly on $\bar{\Omega}$ due to $d_{I_n} \to +\infty$. Recalling $\|S_n\|_{L^\infty(\Omega)} \geq \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)}$, (3.10) and (3.7) yields

$$\min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \leq \frac{\kappa_n}{d_{S_n}} = \frac{N}{\int_\Omega \left[\left(1 - d_{I_n}\tilde{I}_n\right) + d_{S_n}\tilde{I}_n\right] dx} \leq \frac{N}{|\Omega|}.$$

Since $\|S_n\|_{L^\infty(\Omega)}$ is bounded, there exists a subsequence of $\{S_n\}$ still denoted by $\{S_n\}$, such that $S_n \to S^*$ weakly in $L^2(\Omega)$ as $n \to +\infty$ for some nonnegative function S^*. By the first equation of (3.1), we have

$$\int_\Omega J(x-y)[S_n(y) - S_n(x)]\, dy + \frac{\kappa_n}{d_{S_n}}[-\beta(x)S_n(x) + \gamma(x)]\tilde{I}_n(x) = 0. \tag{4.43}$$

This gives that

$$S_n(x) = \frac{\int_\Omega J(x-y)S_n(y)\, dy + \frac{\kappa_n}{d_{S_n}}[-\beta(x)S_n(x) + \gamma(x)]\tilde{I}_n(x)}{\int_\Omega J(x-y)\, dy},$$

implying that $S_n \to S^*$ in $C(\bar{\Omega})$ as $n \to +\infty$. And letting $n \to +\infty$ in (4.43) leads to

$$\int_\Omega J(x-y)[S^*(y)-S^*(x)]\,dy=0.$$

Then S^* is a constant.

On the other hand, up to a subsequence if needed, one of the following three statements must hold:

(B1) $\kappa_n\|\tilde{I}_n\|_{L^\infty(\Omega)}\to 0$;
(B2) $\kappa_n\|\tilde{I}_n\|_{L^\infty(\Omega)}\to +\infty$;
(B3) $\kappa_n\|\tilde{I}_n\|_{L^\infty(\Omega)}\to \hat{C}$ with $\hat{C}$ being a positive constant.

Note that $\frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}$ satisfies

$$d_{I_n}\int_\Omega J(x-y)\left[\frac{\tilde{I}_n(y)}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}-\frac{\tilde{I}_n(x)}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}\right]dy+\left[\frac{\kappa_n\beta}{d_{S_n}}(1-d_{I_n}\tilde{I}_n)-\gamma\right]\frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}=0$$

and $\left\|\left[\frac{\kappa_n\beta}{d_{S_n}}(1-d_{I_n}\tilde{I}_n)-\gamma\right]\frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}\right\|_{L^\infty(\Omega)}\le\tilde{C}$ for some positive constant $\tilde{C}$. Then the same arguments as in the proof of [51, Theorem 4.1] give that $\frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}\to C_*$ in $C(\bar{\Omega})$ as $n\to+\infty$ for some nonnegative constant C_*. Observing that $\left\|\frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}\right\|_{L^\infty(\Omega)}=1$, we have $C_*=1$. If case (B2) holds, then

$$\int_\Omega I_n\,dx=\kappa_n\|\tilde{I}_n\|_{L^\infty(\Omega)}\int_\Omega\frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}\,dx\to+\infty\ \text{ as }\ n\to+\infty,$$

which is a contradiction. If case (B1) or (B3) holds, then

$$I_n=\kappa_n\|\tilde{I}_n\|_{L^\infty(\Omega)}\frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}\to C_1\ \text{ as }\ n\to+\infty,$$

where C_1 is a nonnegative constant. The remaining proof is the same as in the proof of [51, Theorem 4.1]. So we omit it.

Now we are devoted to the proof of (ii). Recalling (4.3) gives that there exist a sequence $\{d_{I_n}\}$ with $d_{I_n}\to+\infty$ as $n\to+\infty$ and some nonnegative function S^* such that $S_n(x)\to S^*(x)$ weakly in $L^2(\Omega)$ as $n\to+\infty$. Since $d_{I_n}\to+\infty$ as $n\to+\infty$, we can assume that $d_{I_n}>1$ for all n. In view of the second equation of (3.4), we have

$$\begin{aligned}I_n(x)&=\frac{d_S}{d_{I_n}}\left[\frac{N}{|\Omega|}-\left(1-\frac{d_{I_n}}{d_S}\right)\frac{1}{|\Omega|}\int_\Omega I_n\,dx-S_n(x)\right]\\&\le\frac{d_S}{d_{I_n}}\frac{N}{|\Omega|}+\frac{1}{|\Omega|}\int_\Omega I_n\,dx\le\frac{N}{|\Omega|}(1+d_S),\end{aligned}$$

implying that $\|I_n\|_{L^\infty(\Omega)} \le \frac{N}{|\Omega|}(1+d_S)$ for all n. Recalling (4.3) and using a similar argument as in the proof of [51, Theorem 4.1], up to a subsequence, we know that $I_n \to I^*$ as $n \to +\infty$, where I^* is a nonnegative constant. Note that

$$\int_\Omega J(x-y)S_n(y)\,dy \to \int_\Omega J(x-y)S^*(y)\,dy \ \text{ as } \ n \to +\infty.$$

By the first equation of (3.1), we obtain

$$S_n(x) = \frac{d_S \int_\Omega J(x-y)S_n(y)\,dy + \gamma(x)I_n(x)}{d_S \int_\Omega J(x-y)\,dy + \beta(x)I_n(x)},$$

which implies that $S_n(x) \to S^*(x)$ in $C(\bar{\Omega})$ as $n \to +\infty$. The same argument as in the proof of [51, Theorem 4.1] gives that $S^*(x) > 0$ and $I^* > 0$. Obviously, $(S^*(x), I^*)$ satisfies (2.5).

There exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to +\infty$ such that the corresponding solution (S_n^*, I_n^*) of (2.5) satisfies one of the following three cases:

(D1) $\frac{d_{S_n}}{I_n^*} \to 0$ as $n \to +\infty$;
(D2) $\frac{d_{S_n}}{I_n^*} \to +\infty$ as $n \to +\infty$;
(D3) $\frac{d_{S_n}}{I_n^*} \to C^*$ with C^* being some positive constant as $n \to +\infty$.

If (D1) holds, dividing both sides of the first equation of (2.5) by I_n^* and letting $n \to +\infty$ yield that $(S_n^*, I_n^*) \to (\tilde{S}^*, \tilde{I}^*)$ in $C(\bar{\Omega})$ as $n \to +\infty$, where $(\tilde{S}^*, \tilde{I}^*)$ satisfies the conclusion (i) of Theorem 2.3. If (D2) holds, the same discussion in the proof of [51, Theorem 4.1] yields $(S_n^*, I_n^*) \to \left(\frac{N}{|\Omega|}, 0\right)$ in $C(\bar{\Omega})$ as $n \to +\infty$. Integrating the first equation of (2.5) gives

$$\int_\Omega \left(-\beta(x)S_n^*(x) + \gamma(x)\right)\,dx = 0.$$

Letting $n \to +\infty$ yields

$$\int_\Omega \left(-\frac{N}{|\Omega|}\beta(x) + \gamma(x)\right)\,dx = 0,$$

which is a contradiction. For the case (D3), noting that

$$S_n^* = \frac{\frac{d_{S_n}}{I_n^*}\int_\Omega J(x-y)S_n^*(y)\,dy + \gamma(x)}{\beta(x) + \frac{d_{S_n}}{I_n^*}\int_\Omega J(x-y)\,dy},$$

we have $(S_n^*, I_n^*) \to (\tilde{S}^*, \tilde{I}^*)$ in $C(\bar{\Omega})$ as $n \to +\infty$, where $(\tilde{S}^*, \tilde{I}^*)$ satisfies the conclusion (ii) of Theorem 2.3. The proof of (ii) is completed.

Finally we prove (iii). Since $\int_\Omega I\,dx \le N$, there exists some sequence $\{d_{S_n}\}$ with $d_{S_n} \to +\infty$ as $n \to +\infty$ such that the corresponding endemic steady state (S_n, I_n) satisfies $\int_\Omega I_n\,dx \to k$ for some constant $k \ge 0$ as $n \to +\infty$.

Set

$$\kappa_n = d_{S_n} S_n + d_I I_n, \quad \tilde{S}_n = \frac{S_n}{\kappa_n} \text{ and } \tilde{I}_n = \frac{I_n}{\kappa_n}.$$

By the second inequality of (3.11), up to a subsequence, we derive that there exist some constant $\tilde{k} \ge 0$ and a nonnegative function $\tilde{I}_*$ such that $\int_\Omega \tilde{I}_n\,dx \to \tilde{k}$ and $\tilde{I}_n \to \tilde{I}_*$ weakly in $L^2(\Omega)$ as $n \to +\infty$. If $\tilde{k} > 0$, (3.10) gives that

$$\frac{\kappa_n}{d_{S_n}} = \frac{N}{\int_\Omega \left[\left(1 - d_I \tilde{I}_n\right) + d_{S_n} \tilde{I}_n\right] dx} \to 0 \text{ as } n \to +\infty.$$

Note that

$$\int_\Omega J(x-y)\tilde{I}_n(y)\,dy \to \int_\Omega J(x-y)\tilde{I}_*(y)\,dy \text{ in } C(\bar{\Omega}) \text{ as } n \to +\infty. \tag{4.44}$$

We derive from the first equation of (3.11) that

$$\tilde{I}_n(x) = \frac{d_I \int_\Omega J(x-y)\tilde{I}_n(y)\,dy}{d_I \int_\Omega J(x-y)\,dy - \left[\frac{\kappa_n \beta(x)}{d_{S_n}}(1 - d_I \tilde{I}_n) - \gamma(x)\right]},$$

which implies that $\tilde{I}_n \to \tilde{I}_*$ in $C(\bar{\Omega})$ as $n \to +\infty$. In addition, $0 \le \tilde{I}_* \le \frac{1}{d_I}$ satisfies

$$d_I \int_\Omega J(x-y)[\tilde{I}_*(y) - \tilde{I}_*(x)]\,dy - \gamma(x)\tilde{I}_*(x) = 0. \tag{4.45}$$

There exists $x_1 \in \bar{\Omega}$ such that $\tilde{I}_*(x_1) = \max_{x\in\bar{\Omega}} \tilde{I}_*(x)$. Then, $\tilde{I}_*(x_1) \le 0$ due to (4.45). Hence, $\tilde{I}_*(x) = 0$ for all $x \in \bar{\Omega}$, contradicting $\tilde{k} > 0$. As a result, $\tilde{k} = 0$. (3.8) and (3.10) give that

$$\frac{\kappa_n}{d_{S_n}} = \frac{N - \int_\Omega I_n\,dx}{\int_\Omega (1 - d_I \tilde{I}_n)\,dx} \to \frac{N-k}{|\Omega|} \text{ as } n \to +\infty. \tag{4.46}$$

Observe that $N - k = \lim_{n\to+\infty} \int_\Omega S_n\,dx \ge |\Omega| \min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)} > 0$. Set

$$g_n(x) = \frac{\kappa_n \beta(x)}{d_{S_n}} - \gamma(x) - d_I \int_\Omega J(x-y)\,dy.$$

We obtain from the first equation of (3.11) that

$$\tilde{I}_n(x) = \frac{g_n(x) + \sqrt{g_n^2(x) + \frac{4d_I^2 \kappa_n \beta(x)}{d_{S_n}} \int_\Omega J(x-y)\tilde{I}_n(y)\,dy}}{\frac{2d_I \kappa_n \beta(x)}{d_{S_n}}},$$

which combined with (4.44) implies that $\tilde{I}_n \to \tilde{I}_*$ in $C(\bar{\Omega})$ as $n \to +\infty$. Then, $\tilde{I}_* \equiv 0$ due to $\tilde{k} = 0$. Now we have $\frac{I_n}{d_{S_n}} = \tilde{I}_n \frac{\kappa_n}{d_{S_n}} \to 0$ in $C(\bar{\Omega})$ as $n \to +\infty$. In view of the second equation of (3.4), $S_n \to \frac{N-k}{|\Omega|}$ in $C(\bar{\Omega})$ as $n \to +\infty$.

We claim that $k > 0$. Set

$$A_n(x) := \frac{N}{|\Omega|}\beta(x) - \gamma(x) - \left(1 - \frac{d_I}{d_{S_n}}\right)\frac{\beta(x)}{|\Omega|}\int_\Omega I_n(x)\,dx - \frac{d_I}{d_{S_n}}\beta(x)I_n(x).$$

Since I_n satisfies the first equation of (3.4), the positivity of I_n implies that 0 is the principal eigenvalue of the following eigenvalue problem

$$d_I \int_\Omega J(x-y)[\varphi_n(y) - \varphi_n(x)]\,dy + A_n(x)\varphi_n(x) = -\lambda\varphi_n(x) \ \text{ in } \ \Omega. \tag{4.47}$$

It is well-known that if λ_n is the principal eigenvalue of (4.47), then $\lambda_n = \lambda_n^*$, where λ_n^* is defined by

$$\lambda_n^* := \inf_{\substack{\varphi \in L^2(\Omega) \\ \varphi \neq 0}} \left\{ \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,dy\,dx - \int_\Omega A_n(x)\varphi^2(x)\,dx}{\int_\Omega \varphi^2(x)\,dx} \right\}.$$

Thus, $\lambda_n^* = 0$. If $k = 0$, combining with $\frac{I_n}{d_{S_n}} \to 0$ in $C(\bar{\Omega})$ as $n \to +\infty$ and letting $n \to +\infty$ in the equality $\lambda_n^* = 0$ yield $\lambda_p(d_I) = 0$. This contradicts $\lambda_p(d_I) < 0$ due to $R_0 > 1$.

Set $w_n = \frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}$. Then w_n satisfies

$$d_I \int_\Omega J(x-y)\,[w_n(y) - w_n(x)]\,dy + \left[\frac{\kappa_n \beta}{d_{S_n}}(1 - d_I\tilde{I}_n) - \gamma\right] w_n = 0 \ \text{ in } \ \Omega. \tag{4.48}$$

We claim that

$$d_I \int_\Omega J(x-y)\,dy - \frac{N-k}{|\Omega|}\beta(x) + \gamma(x) > 0 \ \text{ for all } \ x \in \Omega. \tag{4.49}$$

If there exists $x_* \in \Omega$ such that

$$d_I \int_\Omega J(x_*-y)\,dy - \frac{N-k}{|\Omega|}\beta(x_*) + \gamma(x_*) = 0,$$

then (4.48) gives

$$d_I \int_\Omega J(x_* - y) w_n(y)\, dy + \beta(x_*) \left[-\frac{N-k}{|\Omega|} + \frac{\kappa_n}{d_{S_n}}(1 - d_I \tilde{I}_n(x_*)) \right] w_n(x_*) = 0. \tag{4.50}$$

Note that $\|w_n\|_{L^\infty(\Omega)} = 1$. There exist a subsequence of $\{w_n\}$, still denoted by itself, and a non-negative function w_* such that $w_n \to w_*$ weakly in $L^2(\Omega)$ as $n \to +\infty$. Combining (4.46) and $\tilde{I}_n \to 0$ as $n \to +\infty$, we conclude from (4.50) that $\int_\Omega J(x_* - y) w_*(y)\, dy = 0$. By the same arguments as in the proof of Theorem 2.3, we get $w_*(y) = 0$ almost everywhere in Ω, contradicting the fact that $\|w_*\|_{L^\infty(\Omega)} = 1$. Hence, (4.49) holds. We derive from (4.48) that

$$w_n(x) = \frac{d_I \int_\Omega J(x-y) w_n(y)\, dy}{d_I \int_\Omega J(x-y)\, dy - \frac{\kappa_n \beta(x)}{d_{S_n}}(1 - d_I \tilde{I}_n(x)) + \gamma(x)},$$

which gives that $w_n \to w_*$ in $C(\bar{\Omega})$ as $n \to +\infty$. In addition, w_* satisfies

$$d_I \int_\Omega J(x-y)[w_*(y) - w_*(x)]\, dy + \left[\frac{N-k}{|\Omega|}\beta(x) - \gamma(x) \right] w_*(x) = 0. \tag{4.51}$$

On the other hand, up to a subsequence if needed, one of the following three statements must hold:

(E1) $\kappa_n \|\tilde{I}_n\|_{L^\infty(\Omega)} \to 0$;
(E2) $\kappa_n \|\tilde{I}_n\|_{L^\infty(\Omega)} \to +\infty$;
(E3) $\kappa_n \|\tilde{I}_n\|_{L^\infty(\Omega)} \to C_2$ with C_2 being a positive constant.

If case (E1) holds, then

$$I_n = \kappa_n \|\tilde{I}_n\|_{L^\infty(\Omega)} \frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}} \to 0 \ \text{ in } \ C(\bar{\Omega}) \ \text{ as } \ n \to +\infty,$$

contradicting $k > 0$. If case (E2) holds, then

$$\int_\Omega I_n\, dx = \kappa_n \|\tilde{I}_n\|_{L^\infty(\Omega)} \int_\Omega \frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}}\, dx \to +\infty \ \text{ as } \ n \to +\infty,$$

which is a contradiction. If case (E3) holds, then

$$I_n = \kappa_n \|\tilde{I}_n\|_{L^\infty(\Omega)} \frac{\tilde{I}_n}{\|\tilde{I}_n\|_{L^\infty(\Omega)}} \to C_2 w_* \ \text{ in } \ C(\bar{\Omega}) \ \text{ as } \ n \to +\infty.$$

By (4.51), $I^* := C_2 w_*$ satisfies

$$d_I \int_\Omega J(x-y)[I^*(y) - I^*(x)]\, dy + \left(\frac{N\beta}{|\Omega|} - \gamma - \frac{\beta}{|\Omega|} \int_\Omega I^*\, dx \right) I^* = 0. \tag{4.52}$$

Finally we prove the uniqueness of the positive solution of (2.6). Suppose that I_1 and I_2 are two positive solutions of (2.6). Since J is symmetric, we have

$$\int_\Omega \int_\Omega J(x-y)(I_1(y)-I_1(x))I_2(x)\,dy\,dx = \int_\Omega \int_\Omega J(x-y)(I_2(y)-I_2(x))I_1(x)\,dy\,dx.$$

Then a simple computation yields that

$$\left(\int_\Omega I_1(x)\,dx - \int_\Omega I_2(x)\,dx\right)\int_\Omega \frac{\beta(x)}{|\Omega|} I_1(x)I_2(x)\,dx = 0,$$

which implies that

$$\int_\Omega I_1(x)\,dx = \int_\Omega I_2(x)\,dx. \tag{4.53}$$

The positivity of I_1 yields that 0 is the principal eigenvalue of the following eigenvalue problem

$$d_I \int_\Omega J(x-y)(\psi(y)-\psi(x))\,dy + \psi\left(\frac{N\beta}{|\Omega|} - \gamma - \frac{\beta}{|\Omega|}\int_\Omega I_1\,dx\right) = -\lambda\psi.$$

Let ψ be an eigenfunction corresponding to the principal eigenvalue of the above eigenvalue problem. Then, we have

$$\psi(x) = c_1 I_1(x) = c_2 I_2(x), \quad x \in \Omega,$$

in which c_i $(i = 1, 2)$ are some constants. Following (4.53), one can get that

$$c_1 = c_2 = \frac{\int_\Omega \psi(x)dx}{\int_\Omega I_1(x)dx} = \frac{\int_\Omega \psi(x)dx}{\int_\Omega I_2(x)dx},$$

which implies that $I_1(x) = I_2(x)$ for all $x \in \Omega$. As a consequence, the positive solution of (2.6) is unique. The proof is completed. □

5. Discussion

Taking nonlocal dispersal and heterogeneity into account, in this paper we proposed a nonlocal (convolution) dispersal SIS epidemic model. On the basis of the existence and uniqueness of the endemic steady state, we focused on the impact of small and large diffusion rates of susceptible or infectious individuals on the disease transmission. In the following, we give some biological implications of our analytical results and provide some strategies for disease control.

Theorem 2.3 tells us that limiting the movements of susceptible individuals cannot help to eliminate the infectious disease modeled by (1.1) unless the total population satisfies $N < \int_\Omega \frac{\gamma(x)}{\beta(x)}dx$. Theorem 2.6 presents a particular phenomenon that the infectious individuals concentrate on the site

$$\mathcal{S} = \left\{ x_* \in \bar{\Omega} : \frac{\gamma(x_*)}{\beta(x_*)} = \min_{x \in \bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \right\}$$

by limiting the movement of infectious individuals. Theorem 2.8 indicates that limiting the movement of susceptible individuals sufficiently smaller than that of infectious individuals can help to eliminate the infectious disease modeled by (1.1) if $N < \int_\Omega \frac{\gamma(x)}{\beta(x)}dx$. As a result, when the total population N is large, limiting the movement of susceptible or infectious individuals does not work effectively, which reveals that the infectious disease outbreak in an area with relatively small total population size is easier to be controlled, and looking for other strategies to control the disease is crucial when N is large. But if no other measures are taken effect, limiting the movement of infectious individuals can prevent the infectious disease from spreading throughout the whole region. Theorem 2.10 demonstrates that enlarging infinitely the movement of susceptible individuals makes the density of susceptible individuals positive and spatially homogeneous, and the density of infectious individuals positive and spatially heterogeneous. Enlarging infinitely the movement of infectious individuals yields analogous conclusions. So, large diffusion rate of susceptible or infectious individuals is inadvisable. In practice, when an infectious disease such as COVID-19 breaks out in a region, people are required to reduce their activities in order to prevent spreading the disease to other areas.

It is also interesting to consider the asymptotic profile of the endemic steady state of the SIS epidemic model taking into account the constant recruitment of the susceptible individuals with nonlocal dispersal, as the diffusion rate of infectious individuals tends to zero. However, we find that the analytical process will be rather difficult and complicated. Particularly, different dispersal kernel functions describing the dispersal strategies for susceptible and infectious individuals are more realistic and worth consideration. In this case, (1.1) may not be reduced to a single equation to get the existence of positive stationary solutions by the method of lower and upper solutions. We look forward to finding a new method to prove the existence of stationary solutions of systems with nonlocal dispersal and leave this for further study.

Data availability

No data was used for the research described in the article.

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