

# THE SPACE $L_1(L_p)$ IS PRIMARY FOR $1 < p < \infty$

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ABSTRACT. The classical Banach space  $L_1(L_p)$  consists of measurable scalar functions  $f$  on the unit square for which

$$\|f\| = \int_0^1 \left( \int_0^1 |f(x, y)|^p dy \right)^{1/p} dx < \infty.$$

We show that  $L_1(L_p)$  ( $1 < p < \infty$ ) is primary, meaning that, whenever  $L_1(L_p) = E \oplus F$  then either  $E$  or  $F$  is isomorphic to  $L_1(L_p)$ . More generally we show that  $L_1(X)$  is primary, for a large class of rearrangement invariant Banach function spaces.

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## 1. INTRODUCTION

The decomposition of normed linear spaces into direct sums and the analysis of the associated projection operators is central to important chapters in the theory of modern and classical Banach spaces. In a seminal paper J. Lindenstrauss [19] set forth an influential research program, aiming at detailed investigations of complemented subspaces and operators on Banach spaces.

The main question addressed by J. Lindenstrauss was this: Which are the spaces  $X$  that cannot be further decomposed into two “essentially different, infinite dimensional subspaces? That is to say, which are the Banach spaces  $X$  that are not isomorphic to the direct sum of two infinite dimensional spaces  $Y$  and  $Z$ , where neither,  $Y$  nor  $Z$ , are isomorphic to  $X$ ? This condition would be satisfied if  $X$  were indecomposable, i.e., for any decomposition of  $X$  into two spaces, one of them has to be finite dimensional. Separately, such a space could be primary, meaning that for any decomposition of  $X$  into two spaces, one of them has to be isomorphic to  $X$ . The first example of an indecomposable Banach spaces was constructed by T. Gowers and B. Maurey [14] who also showed that their space  $X_{\text{GM}}$  is not primary—indeed, the infinite dimensional component of  $X_{\text{GM}} \sim X \oplus Y$  is not isomorphic to the whole space.

While indecomposable spaces play a tremendous role ([3, 14, 23]) in the present day study of non classical Banach spaces, a wide variety of Banach function spaces may usually be decomposed, for instance by restriction to subsets, or by taking conditional expectations etc. This provides the background for the program set forth by J. Lindenstrauss to determine the “classical” spaces that are primary.

**1.1. Background and History.** The term classical Banach space—while not formally defined—applies certainly to the space  $C[0, 1]$  and to scalar and vector valued Lebesgue spaces. The space of continuous functions was shown to be primary by J. Lindenstrauss and A. Pełczyński [20], who posed the corresponding problem for scalar valued  $L_p$  spaces. Its elegant solution, by P. Enflo via B. Maurey [22], introduced a groundbreaking method of proof which applies equally well to each of the  $L_p$  spaces, ( $1 \leq p < \infty$ ). Later alternative proofs were given by D. Alspach P. Enflo E. Odell [1] for  $L_p$  in the reflexive range  $1 < p < \infty$  and by P. Enflo and T. Starbird [13] for  $L_1$ .

Exceptionally deep results on the decomposition of Bochner-Lebesgue spaces  $L_p(X)$  are due to M. Capon [9, 8] who obtained that those spaces are is primary in the following cases.

- $X$  is a Banach space with a symmetric basis, and  $1 \leq p < \infty$ .
- $X = L_q$  where  $1 < q < \infty$  and  $1 < p < \infty$ .

This leaves the spaces  $L_1(L_p)$  and  $L_p(L_1)$  among the most prominent examples of classical Banach spaces for which primariness is open.

The purpose of the present paper is to prove that  $L_1(L_p)$  is primary. Our proof works equally well for real and complex valued functions. Before we turn to describing our work, we review in some detail the development of methods pertaining to the spaces  $L_p$ , and more broadly to rearrangement invariant spaces.

Projections on those spaces are studied effectively alongside the Haar system and the reproducing properties of its block bases. The methods developed for proving that a particular Lebesgue space  $L_p$ , is primary may be divided into two basic classes, depending on whether the Haar system is an unconditional Schauder basis, or not.

In case of unconditionality, the most flexible method goes back to the work of D. Alspach, P. Enflo, and E. Odell [1]. For a linear operator  $T$  on  $L_p$  it yields a block basis of the Haar system  $\tilde{h}_I$  and a bounded sequence of scalars  $a_I$  forming an approximate eigensystem of  $T$  such that

$$(1) \quad T\tilde{h}_I = a_I\tilde{h}_I + \text{a small error}$$

and  $\tilde{h}_I$  spans a complemented copy of the space  $L_p$ . Thus, when restricted to  $\text{span}\{\tilde{h}_I\}$ , the operator  $T$  acts as a bounded Haar multiplier. Since the Haar basis is unconditional, the Haar multiplier is invertible if  $|a_I| > \delta$  for some  $\delta > 0$ .

D. Alspach, P. Enflo, and E. Odell [1] arrive at (1) by ensuring that, for  $\varepsilon_{I,J} > 0$  sufficiently small, the following linearly ordered set of constraints holds true,

$$(2) \quad |\langle T\tilde{h}_I, \tilde{h}_J \rangle| + |\langle \tilde{h}_I, T^*\tilde{h}_J \rangle| \leq \varepsilon_{I,J} \quad \text{for } I \prec J,$$

where the relation  $\prec$  refers to the lexicographic order on the collection of dyadic intervals. Utilizing that the independent  $\{-1, +1\}$ -valued Rademacher system  $\{r_n\}$  is a weak null sequence in  $L_p$ , ( $1 \leq p < \infty$ ), D. Alspach, P. Enflo, and E. Odell [1] obtain, *by induction* along  $\prec$ , the block basis  $\tilde{h}_I$  satisfying (2).

The Alspach-Enflo-Odell method provides the basic model for the study of operators on function spaces in which the Haar system is unconditional; this applies in particular to rearrangement invariant spaces in the work of W. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri [16], and D. Dosev, W. Johnson, and G. Schechtman [12].

In  $L_1$  the Haar system is a Schauder basis but fails to be unconditional. The basic methods for proving that  $L_1$  is primary are due to P. Enflo via B. Maurey [22] on the one hand side and P. Enflo and T. Starbird [13] on the other hand side. For operators  $T$  on  $L_1$  the Enflo-Maurey method yields a block basis of the Haar basis  $\tilde{h}_I$  and a bounded measurable function  $g$ , such that

$$(3) \quad (Tf)(t) = g(t)f(t) + \text{a small error},$$

for  $f \in \text{span}\{\tilde{h}_I\}$ , and  $\tilde{h}_I$  spans a copy of  $L_1$ . Thus the restricted operator  $T$  acts as a bounded multiplication operator and is invertible if  $|g| > \delta$  for

some  $\delta > 0$ . The full strength of the proof by Enflo-Maurey is applied to show that the representation (3) holds true.

Enflo-Maurey [22], exhibit in their proof of (3) a sequence of bounded scalars  $a_I$  such that

$$(4) \quad T\tilde{h}_I = a_I\tilde{h}_I + \text{a very small error}$$

Since the Rademacher system  $\{r_n\}$  is a weakly null sequence in  $L_1$ , (4) may be obtained directly by choosing a block basis for which the constraints (2) and

$$(5) \quad |\langle T\tilde{h}_I, \tilde{h}_J \rangle| \leq 2\varepsilon_{I,J} \quad \text{for } I \neq J,$$

hold true. Remarkably, *until very recently* [18], eigensystem representations such as (4) were not exploited in the context of  $L^1$ , where the Haar system is not unconditional.

The powerful precision of  $L_1$ -constructions with dyadic martingales and block basis of the Haar system is in full display in [15] and [30]. W. Johnson, B. Maurey and G. Schechtman determined in [15] a normalized weakly null sequence in  $L_1$  such that each of its infinite subsequences contains in its span a block basis of the Haar system  $\tilde{h}_I$ , spanning a copy of  $L_1$ . Thus  $L_1$  fails to satisfy the unconditional subsequence property, a problem posed by B. Maurey and H. Rosenthal [24]. By contrast M. Talagrand [30] constructed a dyadic martingale difference sequence  $g_{n,k}$  such that neither  $X = \overline{\text{span}}^{L_1}\{g_{n,k}\}$  nor  $L_1/X$  contain a copy of  $L_1$ .

The investigation of complemented subspaces in Bochner Lebesgue spaces was initiated by M. Capon [9, 8] who pushed hard to further the development of the scalar methods, and proved that  $L_p(X)$  ( $1 \leq p < \infty$ ) is primary when  $X$  is a Banach space with a symmetric basis, say  $(x_k)$ . Specifically, M. Capon [9] showed, that for an operator  $T$  on  $L_p(X)$ , there exists a block basis of the Haar basis  $\tilde{h}_I$ , a subsequence of the symmetric basis  $(x_{k_n})$  and a bounded measurable  $g$  such that

$$(T(f \otimes x_{k_n}))(t) = g(t)f(t) \otimes x_{k_n} + \text{a small error},$$

for  $f \in \text{span}\{\tilde{h}_I\}$ . Thus on  $\text{span}\{\tilde{h}_I\} \otimes \text{span}\{x_{k_n}\}$  the operator  $T$  acts like  $M_g \otimes Id$  where  $M_g$  is the multiplication operator induced by  $g$ . Simultaneously, M. Capon shows that the tensor products form an approximate eigensystem,

$$T(\tilde{h}_I \otimes x_{k_n}) = a_I\tilde{h}_I \otimes x_{k_n} + \text{a small error}$$

where  $a_I$  is a bounded sequence of scalars and  $\tilde{h}_I$  spans a copy of  $L_p$ .

In the mixed norm space  $L_p(L_q)$  where  $1 < q < \infty$  and  $1 < p < \infty$  the bi-parameter Haar system forms an unconditional basis. Displaying extraordinary combinatorial strength, M. Capon [8] exhibited a so called local product block basis  $k_{I \times J}$ , spanning a complemented copy of  $L_p(L_q)$ , such that

$$Tk_{I \times J} = a_{I \times J}k_{I \times J} + \text{a small error}.$$

**1.2. The present paper.** Now we turn to describing the main ideas in the approach of the present paper.

Introducing a transitive relation between operators  $S, T$  on a Banach space  $X$ , we say that  $T$  is a projectional factor of  $S$  if there exist transfer operators  $A, B : X \rightarrow X$  such that

$$(6) \quad S = ATB \quad \text{and} \quad BA = Id_X.$$

If merely  $S = ATB$ , without the additional constraint  $BA = Id_X$ , we say that  $T$  is a factor of  $S$ , or equivalently that  $S$  factors through  $T$ .

Clearly, if  $T$  is a projectional factor of  $S$  and  $S$  one of  $R$  then  $T$  is a projectional factor of  $R$ , i.e., being a projectional factor is a transitive relation. Given any operator  $T : L_1(L_p) \rightarrow L_1(L_p)$  the goal is to show that either  $T$  or  $Id - T$  is a factor of the identity  $Id : L_1(L_p) \rightarrow L_1(L_p)$ . In section 2.1 we expand on the quantitative aspects of the transitive relation (6) and the role it plays in providing a step-by-step reduction of the problem, allowing for the replacement of a given operator with a simpler one, that is easier to work with.

Let  $T : L_1(L_p) \rightarrow L_1(L_p)$  be a bounded linear operator. It is represented by a matrix  $T = (T^{I,J})$  of operators  $T^{I,J} : L_1 \rightarrow L_1$ , indexed by pairs of dyadic intervals  $(I, J)$ , that is, on  $f \in L_1(L_p)$  with Haar expansion

$$(7) \quad f = \sum x_J h_J / |J|^{1/p}, \quad x_J \in L_1,$$

the operator  $T$  acts by

$$(8) \quad Tf = \sum_I \left( \sum_J T^{I,J} x_J \right) h_I / |I|^{1/p}.$$

Theorem 6.1, the main result of this paper, asserts that there exists a bounded operator  $T^0 : L_1 \rightarrow L_1$  such that

$$T \text{ is a projectional factor of } T^0 \otimes Id_{L_p},$$

meaning that there exist bounded transfer operators  $A, B : L_1(L_p) \rightarrow L_1(L_p)$  such that  $BA = Id_{L_1(L_p)}$  and

$$(9) \quad \begin{array}{ccc} L_1(L_p) & \xleftarrow{A} & L_1(L_p) \\ T \downarrow & & \downarrow T^0 \otimes Id \\ L_1(L_p) & \xrightarrow{B} & L_1(L_p) \end{array}$$

The ideas involved in the proof of Theorem 6.1, are based on the interplay of topological, geometric, and probabilistic principles. Specifically we build on compact families of  $L_1$ -operators, extracted from  $span\{T^{I,J}\}$ , and large deviation estimates for empirical processes:

- (a) (Compactness.) We utilize the Semenov-Uksusov characterization [29] of Haar multipliers on  $L_1$  and uncover compactness properties of the operators  $T^{I,J} : L_1 \rightarrow L_1$ . See Theorem 3.2 and Theorem 3.4.

- (b) (Stabilization.) Large deviation estimates for the empirical distribution method gave rise to a novel connection between factorization problems on  $L_1(L_p)$  and the concentration of measure phenomenon. See Lemma 5.3 and Lemma 5.4. .

**Step 1.** We say that  $T$  is a diagonal operator if  $T^{I,J} = 0$  for  $I \neq J$ , in which case we put  $T^L = T^{L,L}$ . The first step provides the reduction to diagonal operators. Specifically, Theorem 4.1 asserts, that for any operator  $T = (T^{I,J})$  there exists a diagonal operator  $T_{\text{diag}} = (T^L)$  such that

$$(10) \quad T \text{ is a projectional factor of } T_{\text{diag}} = (T^L).$$

The reduction (10) results from compactness properties for the family of  $L_1$  operators  $T^{I,J}$  established in Theorem 3.2 and Theorem 3.4. Specifically, if  $f \in L_1$  then the set

$$(11) \quad \{T^{I,J}f : I, J \in \mathcal{D}\} \subset L_1 \text{ is weakly relatively compact;}$$

if, moreover,  $T^{I,J}$  satisfies uniform off-diagonal estimates

$$(12) \quad \sup_{I,J} |\langle T^{I,J}h_L, h_M \rangle| < \varepsilon_{L,M}, \quad \text{for } L \neq M.$$

then, for  $\eta > 0$ , there exists a stopping time collection of dyadic intervals  $\mathcal{A}$  satisfying  $|\limsup \mathcal{A}| > 1 - \eta$  such that the set of operators

$$(13) \quad \{T^{I,J}P_{\mathcal{A}} : I, J \in \mathcal{D}\} \subset L(L_1) \text{ is relatively norm-compact.}$$

Recall that  $\mathcal{A} \subseteq \mathcal{D}$  is a stopping time collection if for  $K, L \in \mathcal{A}$  and  $J \in \mathcal{D}$  the assumption  $K \subset J \subset L$  implies that  $J \in \mathcal{A}$ . By Theorem 2.6, the orthogonal projection

$$P_{\mathcal{A}}(f) = \sum_{I \in \mathcal{A}} \langle f, h_I \rangle h_I / |I|,$$

is bounded on  $L_1$  when  $\mathcal{A}$  is a stopping time collection of dyadic intervals.

**Step 2.** Next we show that it suffices to prove the factorization (9) for diagonal operators satisfying uniform off-diagonal estimates. We say that  $T = (R^L)$  is a reduced diagonal operator if the  $R^L : L_1 \rightarrow L_1$  satisfy

$$(14) \quad \sup_L |\langle R^L h_I, h_J \rangle| < \varepsilon_{I,J}, \quad \text{for } I \neq J.$$

Proposition 5.6 asserts that, there exists a reduced diagonal operator  $T_{\text{diag}}^{\text{red}} = (R^L)$  satisfying (14), such that

$$(15) \quad T_{\text{diag}} = (T^L) \text{ is a projectional factor of } T_{\text{diag}}^{\text{red}} = (R^L).$$

To prove (15) we utilize the compactness properties of  $T_{\text{diag}} = (T^L)$  together with measure concentration estimates [6, 28] associated to the empirical distribution method. See Lemma 5.3 and Lemma 5.4.

**Step 3.** Next we show that we may replace reduced diagonal operators by stable diagonal operators. We say that  $T_{\text{diag}}^{\text{stbl}} = (S^L)$  is a stable diagonal

operator if

$$(16) \quad \|S^L - S^M\| < \varepsilon_M,$$

for dyadic intervals  $M, L$  satisfying  $L \subseteq M$ . We obtain in Proposition 5.2, that for any reduced diagonal operator  $T_{\text{diag}}^{\text{red}}$  there exists a stable diagonal operator  $T_{\text{diag}}^{\text{stbl}}$  such that

$$(17) \quad T_{\text{diag}}^{\text{red}} = (R^L) \text{ is a projectional factor of } T_{\text{diag}}^{\text{stbl}} = (S^L).$$

We verify (17) exploiting again the compactness properties of  $T_{\text{diag}}^{\text{red}} = (R^L)$  in tandem with the probabilistic estimates of Lemma 5.3 and Lemma 5.4.

**Step 4.** Proposition 6.2 provides the final step of the argument. It asserts that for any stable diagonal operator  $T_{\text{diag}}^{\text{stbl}} = (S^L)$  there exists a bounded operator  $T^0 : L_1 \rightarrow L_1$  such that,

$$(18) \quad T_{\text{diag}}^{\text{stbl}} \text{ is a projectional factor of } T^0 \otimes Id_X.$$

To prove (18) we set up a telescoping chain of operators connecting any of the  $S^L$  to  $S^{[0,1]}$  and invoke the stability estimates (16) available for the operators  $S^I$  when  $L \subset I \subset [0, 1]$ . Thus we may finally take  $T^0 = S^{[0,1]}$ .

**Step 5.** Retracing our steps, taking into account that the notion of projectional factors forms a transitive relation, yields (9).

## 2. PRELIMINARIES

**2.1. Factors and Projectional Factors up to Approximation.** A common strategy in proving primariness of spaces such as  $L_p$  is to study the behavior of a bounded linear operator on a  $\sigma$ -subalgebra on a subset of  $[0, 1]$  of positive measure. This process may have to be repeated several times. We introduce some language that will make this process notationally easier.

**Definition 2.1.** Let  $X$  be a Banach space,  $T, S : X \rightarrow X$  be bounded linear operators and let  $C \geq 1$ ,  $\varepsilon \geq 0$ .

- (a) We say that  $T$  is a  $C$ -factor of  $S$  with error  $\varepsilon$  if there exist  $A, B : X \rightarrow X$  with  $\|BTA - S\| \leq \varepsilon$  and  $\|A\|\|B\| \leq C$ . We may also say that  $S$   $C$ -factors through  $T$  with error  $\varepsilon$ .
- (b) We say that  $T$  is a  $C$ -projectional factor of  $S$  with error  $\varepsilon$  if there exists a complemented subspace  $Y$  of  $X$  that is isomorphic to  $X$  with associated projection and isomorphism  $P, A : X \rightarrow Y$  (i.e.,  $A^{-1}PA$  is the identity on  $X$ ), so that  $\|A^{-1}PTA - S\| \leq \varepsilon$  and  $\|A\|\|A^{-1}P\| \leq C$ . We may also say that  $S$   $C$ -projectionally factors through  $T$  with error  $\varepsilon$ .

When the error is  $\varepsilon = 0$  we will simply say that  $T$  is a  $C$ -factor or  $C$ -projectional factor of  $S$ .

**Remark 2.2.** If  $T$  is a  $C$ -projectional factor of  $S$  with error  $\varepsilon$  then  $I - T$  is a  $C$ -projectional factor of  $I - S$  with error  $\varepsilon$ . Indeed, if  $P$  and  $A$  witness Definition (b), then  $PA = A$  and therefore  $A^{-1}P(I - T)A = I - A^{-1}PTA$ , i.e.,  $\|A^{-1}P(I - T)A - (I - S)\| \leq \varepsilon$ .

In a certain sense, being an approximate factor or projectional factor is a transitive property.

**Proposition 2.3.** Let  $X$  be a Banach space and  $R, S, T : X \rightarrow X$  be bounded linear operators.

- (a) If  $T$  is a  $C$ -factor of  $S$  with error  $\varepsilon$  and  $S$  is a  $D$ -factor of  $R$  with error  $\delta$  then  $T$  is a  $CD$ -factor of  $R$  with error  $D\varepsilon + \delta$ .
- (b) If  $T$  is a  $C$ -projectional factor of  $S$  with error  $\varepsilon$  and  $S$  is a  $D$ -projectional factor of  $R$  with error  $\delta$  then  $T$  is a  $CD$ -projectional factor of  $R$  with error  $D\varepsilon + \delta$ .

*Proof.* The first statement is straightforward and thus we only provide a proof of the second one. Let  $Y$  and  $Z$  be complemented subspaces of  $X$  which are isomorphic to  $X$ . Let  $P : X \rightarrow Y$  and  $Q : X \rightarrow Z$ , be the associated projections, and  $A : X \rightarrow Y$  and  $B : X \rightarrow Z$  the associated isomorphisms satisfying  $\|A\|\|A^{-1}P\| \leq C$ ,  $\|B\|\|B^{-1}Q\| \leq D$ .  $\|A^{-1}PT - S\| \leq \varepsilon$  and  $\|B^{-1}QSB - R\| \leq \delta$ .

We define  $\tilde{P} = AQA^{-1}P$  and  $\tilde{A} = AB$ . Then,  $\tilde{P}$  is a projection onto  $\tilde{A}[X]$  and  $\|\tilde{P}\|\|\tilde{A}^{-1}P\| \leq CD$ . We obtain

$$\|B^{-1}Q(A^{-1}PTA)B - B^{-1}QSB\| \leq \|B^{-1}Q\|\|B\|\|A^{-1}PTA - S\| \leq D\varepsilon$$

and thus  $\|B^{-1}QA^{-1}PTAB - R\| \leq D\varepsilon + \delta$ . Finally, observe that

$$\tilde{A}^{-1}\tilde{P} = B^{-1}A^{-1}AQA^{-1}P = B^{-1}QA^{-1}P$$

and thus  $\|\tilde{A}^{-1}\tilde{P}T\tilde{A} - R\| \leq D\varepsilon + \delta$ .  $\square$

The following explains the relation between primariness and approximate projectional factors.

**Proposition 2.4.** Let  $X$  be a Banach space that satisfies Pełczyński's accordion property, i.e., for some  $1 \leq p \leq \infty$  we have that  $X \simeq \ell_p(X)$ . Assume that there exist  $C \geq 1$  and  $0 < \varepsilon < 1/2$  so that every bounded linear operator  $T : X \rightarrow X$  is a  $C$ -projectional factor with error  $\varepsilon$  of a scalar operator, i.e., a scalar multiple of the identity. Then, for every bounded linear operator  $T : X \rightarrow X$  the identity  $2C/(1 - 2\varepsilon)$  factors through either  $T$  or  $I - T$ . In particular,  $X$  is primary.

*Proof.* Let  $Y$  be a subspace of  $X$  that is isomorphic to  $X$  and complemented in  $X$ , with associated projection and isomorphism  $P, A : X \rightarrow Y$ , so that  $\|A^{-1}P\|\|A\| \leq C$  and so that there exists a scalar  $\lambda$  with  $\|(A^{-1}P)TA - \lambda I\| \leq \varepsilon$ . If  $|\lambda| \geq 1/2$  then

$$\left\| \underbrace{\lambda^{-1}A^{-1}PTA}_{=:B} - I \right\| \leq 2\varepsilon < 1$$

and thus  $B^{-1}$  exists with  $\|B^{-1}\| \leq 1/(1 - 2\varepsilon)$ . We obtain that if  $S = B^{-1}\lambda^{-1}A^{-1}P$  then  $STA = I$  and  $\|S\|\|A\| \leq 2C/(1 - 2\varepsilon)$ . If, on the other hand  $|\lambda| < 1/2$  then, because  $\|A^{-1}P(I - T)A - (1 - \lambda)I\| \leq \varepsilon$ , we achieve the same conclusion for  $I - T$  instead of  $T$ .

If  $X = Y \oplus Z$  and  $Q : X \rightarrow Y$  is a projection then we deduce that either  $Y$  or  $Z$  contains a complemented subspace isomorphic to  $X$ . To see that we can assume that for some scalar  $\lambda$ , with  $|\lambda| \geq 1/2$ ,  $Q$  is a  $C$ -projectional factor with error  $\varepsilon \in (0, 1/2)$  of  $\lambda I$ . Otherwise we replace  $Q$  by  $I - Q$ . From what we proved so far we deduce that there are operators  $S, A : X \rightarrow X$  so that  $SQA = I$ . Then  $W = QA(X)$  is a subspace of  $Y$  that is isomorphic to  $X$ . It is also complemented via the projection  $R = (S|_W)^{-1}S : X \rightarrow W$ . So we obtain that  $Y$  is a complemented subspace of  $X$  and  $X$  is isomorphic to complemented subspace of  $Y$ . Since in addition  $X$  satisfies the accordion property it follows from Pełczyński's famous classical argument from [25] that  $X \simeq Y$ . Similarly, if  $(I - Q)$  is a factor of the identity we deduce  $X \simeq Z$ .  $\square$

**2.2. The Haar system in  $L_1$ .** We denote by  $L_1$  the space of all (equivalence classes of) integrable scalar functions  $f$  with domain  $[0, 1)$  endowed with the norm  $\|f\|_1 = \int_0^1 |f(s)|ds$ . We will denote the Lebesgue measure of a measurable subset  $A$  of  $[0, 1)$  by  $|A|$ .

We denote by  $\mathcal{D}$  the collection of all dyadic intervals in  $[0, 1)$ , namely

$$\mathcal{D} = \left\{ \left[ \frac{i-1}{2^j}, \frac{i}{2^j} \right) : j \in \mathbb{N} \cup \{0\}, 1 \leq i \leq 2^j \right\}.$$

We define the bijective function  $\iota : \mathcal{D} \rightarrow \{2, 3, \dots\}$  by

$$\left[ \frac{i-1}{2^j}, \frac{i}{2^j} \right) \mapsto 2^j + i.$$

The function  $\iota$  defines a linear order on  $\mathcal{D}$ . We recall the definition of the Haar system  $(h_I)_{I \in \mathcal{D}}$ . For  $I = [(i-1)/2^j, i/2^j) \in \mathcal{D}$  we define  $I^+, I^- \in \mathcal{D}$  as follows:  $I^+ = [(i-1)/2^j, (2i-1)/2^{j+1})$ ,  $I^- = [(2i-1)/2^{j+1}, i/2^j)$ , and

$$h_I = \chi_{I^+} - \chi_{I^-}.$$

We additionally define  $h_\emptyset = \chi_{[0,1)}$  and  $\mathcal{D}^+ = \mathcal{D} \cup \{\emptyset\}$ . We also define  $\iota(\emptyset) = 1$ . Then,  $(h_I)_{I \in \mathcal{D}^+}$  is a monotone Schauder basis of  $L_1$ , with the linear order induced by  $\iota$ . Henceforth, whenever we write  $\sum_{I \in \mathcal{D}^+}$  we will always mean that the sum is taken with this linear order  $\iota$ .

For each  $n \in \mathbb{N} \cup \{0\}$  we define

$$\mathcal{D}_n = \{I \in \mathcal{D} : |I| = 2^{-n}\} \text{ and } \mathcal{D}^n = \{\emptyset\} \cup (\cup_{k=0}^n \mathcal{D}_k).$$

An important realization, that will be used multiple times in the sequel is the following. Let  $I \in \mathcal{D}$ . Then there exists a unique  $k_0 \in \mathbb{N}$  and a unique decreasing sequence of intervals  $(I_k)_{k=0}^{k_0}$  in  $(\mathcal{D}^+)$ , so that  $I_0 = \emptyset$ ,  $I_1 = [0, 1)$ , and  $I_{k_0} = I$ , and for  $k = 1, 2, \dots, k_0-1$ ,  $I_{k+1} = I_k^+$ , or  $I_{k+1} = I_k^-$ . In other words  $(I_k)_{k=1}^{k_0}$  consists of all elements of  $\mathcal{D}^+$  which contain  $I$ , decreasingly ordered. For  $k = 1, 2, \dots, k_0-1$  put  $\theta_k = 1$ , if  $I_{k+1} = I_k^+$  and  $\theta_k = -1$  if

$I_{k+1} = I_k^-$ . We then have the following formula, already discovered by Haar,

$$(19) \quad |I|^{-1}\chi_I = |I_{k_0}|^{-1}\chi_{I_{k_0}} = h_{I_0} + \sum_{k=1}^{k_0-1} \theta_k |I_k|^{-1} h_{I_k}.$$

Note that in the above representation, if we define  $I_{k_0} = I$ , then  $I_k = I_{k-1}^-$  or  $I_k = I_{k-1}^+$  for  $k = 2, \dots, k_0$ . To simplify notation, we will henceforth make the convention  $\theta_0 = 1$  and  $|I_0|^{-1} = |\emptyset|^{-1} = 1$  to be able to write

$$(20) \quad |I_{k_0}|^{-1}\chi_{I_{k_0}} = \sum_{k=0}^{k_0-1} \theta_k |I_k|^{-1} h_{I_k}.$$

This representation will be used multiple times in this paper.

A relevant definition is that of  $[\mathcal{D}^+]$ , the collection of all sequences  $(I_k)_{k=0}^\infty$  in  $\mathcal{D}^+$  so that  $I_0 = \emptyset$ ,  $I_1 = [0, 1)$ , and for each  $k \in \mathbb{N}$ ,  $I_{k+1} = I_k^+$  or  $I_{k+1} = I_k^-$ . Note that for  $(I_k)_{k=0}^\infty \in [\mathcal{D}^+]$  and  $k \in \mathbb{N}$ ,  $I_k \in \mathcal{D}_{k-1}$ . Each  $(I_k)_{k=0}^\infty$  defines a sequence  $(\theta_k)_{k=1}^\infty$  as described in the paragraph above. This yields a bijection between  $[\mathcal{D}^+]$  and  $\{-1, 1\}^\mathbb{N}$ . This fact will be used more than once. On  $\{-1, 1\}^\mathbb{N}$  we will consider the product of the uniform distribution on  $\{-1, 1\}$ , which via this bijection generates a probability on  $[\mathcal{D}^+]$ , which we will also denote by  $|\cdot|$ . Also, we consider on  $[\mathcal{D}^+]$  the image topology of the product of the discrete topology on  $\{-1, 1\}$  via that bijection.

**2.3. Haar multipliers on  $L_1$ .** A Haar multiplier is a linear map  $D$ , defined on the linear span of the Haar system, for which every Haar vector  $h_I$  is an eigenvector with eigenvalue  $a_I$ . We denote the space of bounded Haar multipliers  $D : L_1 \rightarrow L_1$  by  $\mathcal{L}_{HM}(L_1)$ . In this subsection we recall a formula for the norm of a Haar multiplier that was observed by Semenov and Uksusov in [29]. We then use Haar multipliers to sketch a proof of the fact that every bounded linear operator on  $L_1$  is an approximate 1-projectional factor of a scalar operator.

**Proposition 2.5.** Let  $(I_k)_{k=0}^\infty \in [\mathcal{D}^+]$  associated to  $(\theta_k)_{k=1}^\infty \in \{-1, 1\}^\mathbb{N}$ . For  $k \in \mathbb{N}$  define  $B_k = I_k \setminus I_{k+1}$  and let  $(a_k)_{k=0}^\infty$  be a sequence of scalars.

Then we have

$$(21) \quad \left\| \sum_{k=0}^n a_k \theta_k |I_k|^{-1} h_{I_k} \right\|_{L_1} \leq \sum_{k=1}^n |a_k - a_{k-1}| + |a_n|.$$

and for any  $1 \leq m < n$

$$(22) \quad \left\| \left( \sum_{k=0}^n a_k \theta_k |I_k|^{-1} h_{I_k} \right) \Big|_{\bigcup_{j=m}^n B_j} \right\|_{L_1} \geq \frac{1}{3} \sum_{k=m+1}^n |a_k - a_{k-1}|.$$

*Proof.* Note that the sequence  $(B_k)_{k=1}^\infty$  is a partition of  $[0, 1)$  and for  $k \in \mathbb{N}$   $B_k$  is the set in  $[0, 1]$  of measure  $2^{-k}$ , on which  $\theta_k h_{I_k}$  takes the value  $-1$ .

Let  $f = a_0 h_\emptyset + \sum_{k=1}^n \theta_k a_k |I_k|^{-1} h_{I_k}$ . For  $k \in \mathbb{N}$  put  $b_k = a_k$  if  $k \leq n$  and  $b_k = 0$  otherwise. For each  $k \in \mathbb{N}$  the function  $f$  is constant on  $B_k$  and in fact for  $s \in B_k$  we have

$$f(s) = b_0 + \sum_{j=1}^{k-1} |I_j|^{-1} b_j - |I_k|^{-1} b_k = b_0 + \sum_{j=1}^{k-1} 2^{j-1} b_j - 2^{k-1} b_k =: c_k.$$

Therefore, for any  $m = 1, 2, \dots, n$

$$(23) \quad \left\| f \chi_{\bigcup_{j=m}^n B_j} \right\|_{L_1} = \sum_{k=m}^n |X_k| \text{ and } \left\| f \chi_{\bigcup_{j=m}^\infty B_j} \right\|_{L_1} = \sum_{k=m}^\infty |X_k|$$

where for each  $k \in \mathbb{N}$ ,

$$X_k = \frac{c_k}{2^k} = \frac{1}{2^k} b_0 + \sum_{j=1}^{k-1} \frac{2^{j-1}}{2^k} b_j - \frac{1}{2} b_k.$$

Putting  $X_0 = 0$ , a calculation yields that for all  $k \in \mathbb{N}$

$$(24) \quad X_k = \frac{1}{2} X_{k-1} + \frac{1}{2} (b_{k-1} - b_k).$$

Applying the triangle inequality to (23) and (24) we conclude

$$\begin{aligned} \|f\|_{L_1} &= \sum_{k=1}^\infty |X_k| = \sum_{k=1}^\infty 2|X_k| - |X_{k-1}| \\ &\leq \sum_{k=1}^\infty |2X_k - X_{k-1}| = \sum_{k=1}^\infty |b_k - b_{k-1}| \end{aligned}$$

which yields (21). In order to obtain (22), we deduce from (24)

$$\sum_{k=m+1}^n |X_k| \geq \frac{1}{2} \sum_{k=m+1}^n |b_k - b_{k-1}| - \frac{1}{2} \sum_{k=m}^{n-1} |X_k|$$

and therefore

$$\frac{3}{2} \sum_{k=m+1}^n |X_k| + \frac{1}{2} |X_m| \geq \frac{1}{2} \sum_{k=m+1}^n |b_k - b_{k-1}|$$

which yields

$$\left\| f \chi_{\bigcup_{j=m}^n B_j} \right\|_{L_1} = \sum_{k=m}^n |X_k| \geq \sum_{k=m+1}^n |X_k| + \frac{1}{3} |X_m| \geq \frac{1}{3} \sum_{k=m+1}^n |b_k - b_{k-1}|$$

and proves (22).  $\square$

**Theorem 2.6** (Semenov-Uksusov, [29]). Let  $(a_I)_{I \in \mathcal{D}^+}$  be a collection of scalars, and  $D$  be the associated Haar multiplier. Define

$$(25) \quad |||D||| = \sup \left( \sum_{k=1}^{\infty} |a_{I_k} - a_{I_{k-1}}| + \lim_k |a_{I_k}| \right)$$

where the supremum is taken over all  $(I_k)_{k=0}^{\infty} \in [\mathcal{D}^+]$ . Then,  $D$  is bounded (and thus extends to a bounded linear operator on  $L_1(X)$ ) if and only if  $|||D||| < \infty$ . More precisely,

$$(26) \quad \|D\| \leq |||D||| \leq 3\|D\|.$$

*Proof.* By (19),  $D$  is always well defined on the linear span of the set  $\mathcal{X} = \{|I|^{-1}\chi_I : I \in \mathcal{D}\}$ . In fact, the closed convex symmetric hull of  $\mathcal{X}$  is the unit ball of  $L_1$ . We deduce that  $\|D\| = \sup\{\|Df\| : f \in \mathcal{X}\}$ , under the convention that  $\|D\| = \infty$  if and only if  $D$  is unbounded. Fix  $f = |I|^{-1}\chi_I \in \mathcal{X}$ . Use (19) to write

$$f = |I_{k_0}|^{-1}\chi_{I_{k_0}} = \sum_{k=0}^{k_0-1} \theta_k |I_k|^{-1} h_{I_k}, \text{ i.e., } Df = \sum_{k=0}^{k_0-1} a_k \theta_k |I_k|^{-1} h_{I_k}.$$

Extend  $(I_k)_{k=0}^{k_0}$  to a branch  $(I_k)_{k=0}^{\infty}$ . By (21) we have

$$(27) \quad \frac{1}{3} \left( \sum_{k=1}^{k_0-1} |a_{I_k} - a_{I_{k-1}}| + |a_{I_{k_0-1}}| \right) \leq \|Df\|_{L_1} \leq \sum_{k=1}^{k_0-1} |a_{I_k} - a_{I_{k-1}}| + |a_{I_{k_0-1}}|.$$

By the triangle inequality,  $\|Df\|_{L_1} \leq \sum_{k=1}^{\infty} |a_{I_k} - a_{I_{k-1}}| + \lim_k |a_{I_k}| \leq |||D|||$ . The lower bound is achieved by taking in (27) all  $f \in \mathcal{X}$ .  $\square$

The following special type of Haar multiplier will appear in the sequel.

**Example 2.7.** Let  $\mathcal{A} \subset [\mathcal{D}^+]$  be a non-empty set and define the set  $\mathcal{A} = \cup_{k_0=0}^{\infty} \{I_{k_0} : (I_k)_{k=0}^{\infty} \in \mathcal{A}\} \subset \mathcal{D}^+$ . Let  $P_{\mathcal{A}}$  denote the Haar multiplier that has entries  $a_I = 1$  for  $I \in \mathcal{A}$  and  $a_I = 0$  otherwise. Then, by Theorem 2.6,  $\|P_{\mathcal{A}}\| \leq |||P_{\mathcal{A}}||| = 1$  and therefore  $P_{\mathcal{A}}$  defines a norm-one projection onto  $Y_{\mathcal{A}} = \langle \{h_I : I \in \mathcal{A}\} \rangle$ .

The following elementary remark will be useful eventually.

**Remark 2.8.** Let  $\mathcal{A}$  be a non-empty closed subset of  $[\mathcal{D}^+]$  and  $\mathcal{A} = \cup_{k_0=0}^{\infty} \{I_{k_0} : (I_k)_{k=0}^{\infty} \in \mathcal{A}\}$ . Let  $D$  be a Haar multiplier with entries that are zero outside  $\mathcal{A}$ . Then,  $|||D||| = \sup_{(I_k)_{k=0}^{\infty} \in \mathcal{A}} \left( \sum_{k=1}^{\infty} |a_{I_k} - a_{I_{k-1}}| + \lim_k |a_{I_k}| \right)$ .

Haar multipliers provide a short path to a proof of the fact that every operator on  $L_1$  is an approximate 1-projectional factor of a scalar operator, which in turn yields Enflo's theorem [22] that  $L_1$  is primary.

**Theorem 2.9.** The following are true in the space  $L_1$ .

- (i) Let  $D : L_1 \rightarrow L_1$  be a bounded Haar multiplier. For every  $\varepsilon > 0$ ,  $D$  is a 1-projectional factor with error  $\varepsilon$  of a scalar operator.

- (ii) Let  $T : L_1 \rightarrow L_1$  be a bounded linear operator. For every  $\varepsilon > 0$ ,  $T$  is a 1-projectional factor with error  $\varepsilon$  of a bounded Haar multiplier  $D : L_1 \rightarrow L_1$ .

In particular, for every  $\varepsilon > 0$ , every bounded linear operator  $T : L_1 \rightarrow L_1$  is a 1-projectional factor with error  $\varepsilon$  of a scalar operator.

We wish to provide a sketch of the proof of the above. Firstly, we will use it at the end of the paper and secondly it provides an introduction to the basis of the methods used in the paper. Now, and numerous times in the sequel, we require the following notation and definition.

**Notation.** For every disjoint collection  $\Delta$  of  $\mathcal{D}^+$  and  $\theta \in \{-1, 1\}^\Delta$  we denote  $h_\Delta^\theta = \sum_{J \in \Delta} \theta_J h_J$ . If  $\theta_J = 1$  for all  $J \in \Delta$  we write  $h_\Delta = \sum_{J \in \Delta} h_J$ . For a finite disjoint collection  $\Delta$  of  $\mathcal{D}$  we denote  $\Delta^* = \cup\{I : I \in \Delta\}$ .

**Definition 2.10.** A *faithful Haar system* is a collection  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$  so that for each  $I \in \mathcal{D}^+$  the function  $\tilde{h}_I$  is of the form  $\tilde{h}_I = h_{\Delta_I}^{\theta_I}$ , for some finite disjoint collection  $\Delta_I$  of  $\mathcal{D}$ , and so that

- (i)  $\Delta_\emptyset^* = \Delta_{[0,1)}^* = [0, 1)$  and for each  $I \in \mathcal{D}$  we have  $|\Delta_I| = |I|$ ,
- (ii) for every  $I \in \mathcal{D}$  we have that  $\Delta_{I^+}^* = [\tilde{h}_\emptyset \tilde{h}_I = 1]$  and  $\Delta_{I^-}^* = [\tilde{h}_\emptyset \tilde{h}_I = -1]$ .

**Remark 2.11.** It is immediate that  $(\tilde{h}_\emptyset \tilde{h}_I)_{I \in \mathcal{D}^+}$  is distributionally equivalent to  $(h_I)_{I \in \mathcal{D}^+}$ . Therefore,  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$  is isometrically equivalent to  $(h_I)_{I \in \mathcal{D}^+}$ , both in  $L_1$  and in  $L_\infty$ . In particular,

$$Pf = \sum_{I \in \mathcal{D}^+} \langle \tilde{h}_I, f \rangle |I|^{-1} \tilde{h}_I$$

defines a norm-one projection onto a subspace  $Z$  of  $L_1$  that is isometrically isomorphic to  $L_1$ . Note that, unless  $h_\emptyset = 1$ ,  $P$  is not a conditional expectation as  $P\chi_{[0,1)} = 0$ . Instead, it is of the form  $Pf = \tilde{h}_\emptyset E(\tilde{h}_\emptyset f | \Sigma)$ , where  $\Sigma = \sigma(\tilde{h}_\emptyset \tilde{h}_I)_{I \in \mathcal{D}^+}$ . Since  $\tilde{h}_\emptyset$  is not  $\Sigma$ -measurable it cannot be eliminated. The advantage of the notion of a faithful Haar system is that one can be constructed in every tail of the Haar system. The drawback is that it causes a slight notational burden when having to adjust for the initial function  $\tilde{h}_\emptyset$  in several situations.

We will several times recursively construct faithful Haar systems  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$ , which means that we first choose  $\tilde{h}_\emptyset$ , secondly  $\tilde{h}_{[0,1)}$ , and then  $\tilde{h}_I$ ,  $I \in \mathcal{D}$ , assuming that  $\tilde{h}_J$  was chosen for all  $J \in \mathcal{D}^+$  with  $\iota(J) < \iota(I)$ .

*Proof of Theorem 2.9.* Let us sketch the proof of the first statement. Let  $(a_I)_{I \in \mathcal{D}^+}$  be the entries of  $D$ . For every  $I \in \mathcal{D}$  denote by  $Q_I$  the Haar multiplier that has entries 1 for all  $J \subset I$  and zero all others. Then,  $\|Q_I\| = 1$ . First note that, for every  $\varepsilon > 0$ , there exists  $I_0 \in \mathcal{D}^+$  so that  $\|DQ_{I_0} - a_{I_0}Q_{I_0}\| \leq \varepsilon$ . Otherwise, we could easily deduce  $\|D\| = \infty$ . Construct a dilated and renormalized faithful Haar system  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$  with closed

linear span  $Z$  in the range of  $Q_{I_0}$  and let  $P : L_1 \rightarrow Z$  be the corresponding norm-one projection and  $A : L_1 \rightarrow Z$  be an onto isometry. Then,  $\|A^{-1}PDA - a_{I_0}I\| \leq \varepsilon$ .

For the second part we will use that the Rademacher sequence  $(r_n)_n$  (i.e.,  $r_n = \sum_{L \in \mathcal{D}_n} h_L$ , for  $n \in \mathbb{N}$ ) is weakly null in  $L_1$  and  $w^*$ -null in  $(L_1)^* \equiv L_\infty$ . Using this fact, we inductively construct a faithful Haar system  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$  so that for each  $I \neq J$  we have

$$|\langle \tilde{h}_I, T(|J|^{-1}\tilde{h}_J) \rangle| \leq \varepsilon_{(I,J)},$$

where  $(\varepsilon_{(I,J)})_{(I,J) \in \mathcal{D}^+}$  is a pre-chosen collection of positive real numbers with  $\sum \varepsilon_{(I,J)} \leq \varepsilon$ . This is done as follows. If we have chosen  $\tilde{h}_I$  for  $\iota(I) = 1, \dots, k-1$ . Let  $I \in \mathcal{D}^+$  with  $\iota(I) = k$  and let  $I_0$  be the predecessor of  $I$ , i.e., either  $I = I_0^+$  or  $I = I_0^-$ . Let us assume  $I = I_0^+$ . We then choose the next function  $\tilde{h}_I$  among the terms of a Rademacher sequence with support  $[\tilde{h}_\emptyset \tilde{h}_{I_0} = 1]$ . Denote by  $Z$  the closed linear span of  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$  and take the canonical projection  $P : L_1 \rightarrow Z$  as well as the onto isometry  $A : L_1 \rightarrow Z$  given by  $Ah_I = \tilde{h}_I$ . Consider the operator  $S = A^{-1}PTA : L_1 \rightarrow L_1$  and note that for all  $I \neq J$  we have  $|\langle h_I, S(|J|^{-1}h_J) \rangle| = |\langle \tilde{h}_I, T(|J|^{-1}\tilde{h}_J) \rangle| \leq \varepsilon_{(I,J)}$ . It follows that the entries  $a_I = \langle h_I, S(|I|^{-1}h_I) \rangle$  define a bounded Haar multiplier  $D$  and  $\|S - D\| \leq \varepsilon$ , i.e.,  $T$  is a 1-projectional factor with error  $\varepsilon$  of  $D$ .  $\square$

**2.4. Haar system spaces.** We define Haar system spaces. These are Banach spaces of scalar function generated by the Haar system in which two functions with the same distribution have the same norm. This abstraction does not impose any notational burden to the proof of the main result. The only difference to the case  $X = L_p$  is the normalization of the Haar basis. Properties such as unconditionality of the Haar system or reflexivity of  $L_p$  are never deployed.

**Definition 2.12.** A *Haar system space*  $X$  is the completion of  $Z = \langle \{h_L : L \in \mathcal{D}^+\} \rangle = \langle \{\chi_I : I \in \mathcal{D}\} \rangle$  under a norm  $\|\cdot\|$  that satisfies the following properties.

- (i) If  $f, g$  are in  $Z$  and  $|f|, |g|$  have the same distribution then  $\|f\| = \|g\|$ .
- (ii)  $\|\chi_{[0,1]}\| = 1$ .

We denote the class of Haar system spaces by  $\mathcal{H}$ .

Obviously, property (ii) may be achieved by scaling the norm of a space that satisfies (i). We include it anyway for notational convenience.

An important class of spaces which satisfy Definition 2.12, according to [21, Proposition 2.c.1], are separable rearrangement invariant function spaces on  $[0, 1]$ . Recall that a (non-zero) Banach space  $Y$  of measurable scalar functions on  $[0, 1]$  is called *rearrangement invariant* (or as in [26] *symmetric*) if the following conditions hold true: First, whenever  $f \in Y$  and  $g$  is a measurable function with  $|g| \leq |f|$  a.e. then  $g \in Y$  and  $\|g\|_Y \leq \|f\|_Y$ . Second, if  $u, v$  are in  $Y$  and they have the same distribution then  $\|u\|_Y = \|v\|_Y$ .

The following properties of a Haar system space  $X$  follow from elementary arguments. For completeness, we provide the proofs.

**Proposition 2.13.** Let  $X$  be a Haar system space.

- (a) For every  $f \in Z = \langle \{\chi_I : I \in \mathcal{D}\} \rangle$  we have  $\|f\|_{L_1} \leq \|f\| \leq \|f\|_{L_\infty}$ . Therefore,  $X$  can be naturally identified with a space of measurable scalar functions on  $[0, 1)$  and  $\overline{Z}^{\|\cdot\|_{L_\infty}} \subset X \subset L_1$ .
- (b)  $Z = \langle \{\chi_I : I \in \mathcal{D}\} \rangle$  naturally coincides with a subspace of  $X^*$  and its closure  $\overline{Z}$  in  $X^*$  is also a Haar system space.
- (c) The Haar system, in the usual linear order, is a monotone Schauder basis of  $X$ .
- (d) For a finite union  $A$  of elements of  $\mathcal{D}$  we put  $\mu_A = \|\chi_A\|_X^{-1}$  and  $\nu_A = \|\chi_A\|_{X^*}^{-1}$ . Then,  $\mu_A \nu_A = |A|^{-1}$ . In particular,  $(\nu_L h_L, \mu_L h_L)_{L \in \mathcal{D}^+}$  is a biorthogonal system in  $X^* \times X$ .
- (e) A faithful Haar system  $(\hat{h}_L)_{L \in \mathcal{D}^+}$  is isometrically equivalent to  $(h_L)_{L \in \mathcal{D}^+}$ . In particular,  $Pf = \sum_{L \in \mathcal{D}^+} \langle \nu_L \hat{h}_L, f \rangle \mu_L \hat{h}_L$  defines a norm-one projection onto a subspace of  $X$  that is isometrically isomorphic to  $X$ .

*Proof.* By the first condition in Definition 2.12, we have

$$\left\| \sum_{I \in \mathcal{D}_n} a_I \chi_{\pi(I)} \right\| = \left\| \sum_{I \in \mathcal{D}_n} a_I \chi_I \right\|,$$

for all  $n \in \mathbb{N}$ , all permutations  $\pi$  on  $\mathcal{D}_n$ , and all scalar families  $(a_I : I \in \mathcal{D}_n)$ .

To show the first inequality in (a) let  $n \in \mathbb{N}$ ,  $f = \sum_{I \in \mathcal{D}_n} a_I \chi_I \in Z$  and let  $\pi : \mathcal{D}_n \rightarrow \mathcal{D}_n$ , be cyclic (i.e.,  $\{\pi^r(I) : r = 1, 2, \dots, 2^n\} = \mathcal{D}_n$  for  $I \in \mathcal{D}_n$ ). Then

$$\|f\| = \left\| \sum_{I \in \mathcal{D}_n} a_I \chi_I \right\| \geq \frac{1}{2^n} \left\| \sum_{r=1}^{2^n} \sum_{I \in \mathcal{D}_n} |a_I| \chi_{\pi^r(I)} \right\| = \frac{1}{2^n} \left| \sum_{I \in \mathcal{D}_n} |a_I| \right| = \|f\|_{L_1}.$$

The second inequality in (a) follows from the observation that for each  $n \in \mathbb{N}$  the family  $(\chi_I : I \in \mathcal{D}_n)$  is 1-unconditional.

We identify each  $g \in Z$  with the bounded functional  $x_g^*$ , defined by  $x_g^*(f) = \int_0^1 f g$ , and we denote the dual norm by  $\|\cdot\|_*$ . From this representation it is clear that  $\|\cdot\|_*$  also satisfies the first condition in Definition 2.12. Since  $\|1_{[0,1]}\| = 1$  and since for all  $f \in Z$ ,  $\int f \leq \|f\|_1 \leq \|f\|$ , we deduce that the second condition in Definition 2.12 holds true for the norm  $\|\cdot\|_*$ .

Let  $(h_n)$  be the Haar basis linearly ordered in the usual way, meaning that if  $m < n$ , then either  $\text{supp}(h_n) \subset \text{supp}(h_m)$  or  $\text{supp}(h_n) \cap \text{supp}(h_m) = \emptyset$ . The claim of condition (c) follows from the fact that if  $f = \sum_{j=1}^n a_j h_j \in Z$ , then for any scalar  $a_{n+1}$  the absolute values of the functions  $f + a_{n+1} h_{n+1}$  and  $f - a_{n+1} h_{n+1}$  have the same distribution and their average is  $f$ .

Let  $n \in \mathbb{N}$  and  $I \in \mathcal{D}_n$  using for  $k > n$  cyclic permutations on  $\{J \in \mathcal{D}_k, J \subset I\}$  we deduce that  $\sup_{f \in Z, \|f\| \leq 1} \int_I f$  is attained for  $f = \chi_I / \|\chi_I\|$  and thus  $\|\chi_I\| \cdot \|\chi_I\|_* = 2^{-n}$ . Since secondly, for each  $n$ ,  $(\chi_I : I \in \mathcal{D}_n)$  is an orthogonal family, we deduce (d).

Since faithful Haar systems have the same joint distribution we deduce the first part of (e). Since by (b), this is also true with respect to the dual norm we deduce the second part of (e).  $\square$

In different parts of proof we will require additional properties of Haar system spaces. The following class of Haar system spaces is the one for which we prove our main theorem.

**Definition 2.14.**  $\mathcal{H}^*$  is the class of all Banach spaces  $X$  in  $\mathcal{H}$  satisfying

- ( $\star$ ) the Rademacher sequence  $(r_n)_n$  is not equivalent to the  $\ell_1$ -unit vector basis.

$\mathcal{H}^{**}$  is the class of all Banach spaces  $X$  in  $\mathcal{H}$  satisfying

- ( $\star\star$ ) no subsequence of the  $X$ -normalized Haar system  $(\mu_L h_L)_{L \in \mathcal{D}^+}$  is equivalent to the  $\ell_1$ -unit vector basis.

**Remark 2.15.** Examples of Haar system spaces which satisfy ( $\star$ ) and ( $\star\star$ ) are separable reflexive r.i. spaces.

We note and will use several times that ( $\star$ ) for Haar system spaces, is equivalent with the condition that the Rademacher sequence  $(r_n)$  is weakly null. To see this, first note that for any  $(a_n) \in c_{00}$ , any  $\sigma = (\sigma_n) \subset \{\pm 1\}$ , and permutation  $\pi$  on  $\mathbb{N}$  the distribution of  $\sum_{n \in \mathbb{N}} a_n \sigma_n r_{\pi(n)}$ , does not depend on  $\sigma$  on  $\pi$ . It follows that  $(r_n)$  is a symmetric basic sequence in  $X$ . This implies that either  $r_n$  is equivalent to the  $\ell_1$  unit vector basis or it is weakly null in  $X$ . Indeed, if it is not equivalent to the unit vector basis of  $\ell_1$ , and by symmetry no subsequence, is equivalent to the  $\ell_1$  unit vector basis, it must by Rosenthal's  $\ell_1$  Theorem have weakly Cauchy subsequence and thus for some subsequence  $(n_k) \subset \mathbb{N}$  the sequence  $(r_{n_{2k}} - r_{n_{2k-1}} : k \in \mathbb{N})$  is weakly null. But then also the sequence  $(r_{n_{2k}} + r_{n_{2k-1}} : k \in \mathbb{N})$  is weakly null, and thus  $r_{n_{2k}}$  is weakly null and by symmetry  $(r_n)$  is weakly null.

**2.5. Complemented subspaces of  $L_1(X)$  isomorphic to  $L_1(X)$ .** Let  $E, F$  be Banach spaces. The projective tensor product of  $E$  and  $F$  is the completion of the algebraic tensor product  $E \otimes F$  under the norm

$$(28) \quad \|u\| = \inf \left\{ \sum_{n=1}^N \|e_n\| \|f_n\| : u = \sum_{n=1}^N e_n \otimes f_n \right\}.$$

It is well known and follows from the definition of Bochner-Lebesgue spaces that for any Banach space  $X$ ,  $L_1 \otimes_\pi X \equiv L_1(X)$  via the identification  $(f \otimes x)(s) = f(s)x$ . Then,  $L_\infty(X^*)$  canonically embeds into  $(L_1(X))^*$  via the identification  $\langle u, v \rangle = \int_0^1 \langle u(s), v(s) \rangle ds$ . Recall that by the definition of tensor norms the projective tensor norm satisfies the following property we will use.

- ( $\bullet$ ) For any pair of bounded linear operators  $T : E \rightarrow E$  and  $S : F \rightarrow F$  there exists a unique bounded linear operator  $T \otimes S : E \otimes_\pi F \rightarrow E \otimes_\pi F$  with  $(T \otimes S)(e \otimes f) = (Te) \otimes (Sf)$  and  $\|T \otimes S\| = \|T\| \|S\|$ .

The next standard statement explains one of the main features of the projective tensor product. For the sake of completeness, and because it is essential in this paper, we include the proof.

**Proposition 2.16.** Let  $Z$  be a subspace of  $L_1$  that is isometrically isomorphic to  $L_1$  via  $A : L_1 \rightarrow Z$  and 1-complemented in  $L_1$  via  $P : L_1 \rightarrow Z$ . Let  $X$  be a Banach space and let  $W$  be a subspace of  $X$  that is isometrically isomorphic to  $X$  via  $B : X \rightarrow W$  and 1-complemented in  $X$  via  $Q : X \rightarrow W$ .

Then the space  $Z(W) = \overline{Z \otimes W}^{L_1(X)}$  coincides with  $Z \otimes_\pi W$  and is isometrically isomorphic to  $L_1(X)$  via  $A \otimes B : L_1(X) \rightarrow Z(W)$  and 1-complemented in  $L_1(X)$  via  $P \otimes Q : L_1(X) \rightarrow Z(W)$ .

*Proof.* It is immediate that  $P \otimes Q$  is a norm-one projection onto  $Z(W)$  and that  $A \otimes B$  is a norm-one map with dense image. It also follows that  $A \otimes B$  is 1-1 on  $L_1 \otimes X$ . One way to see this is to identify  $L_1 \otimes X$  and  $Z \otimes W$  with spaces of bilinear forms on  $(L_1)^* \times X^*$  and  $Z^* \times W^*$  respectively. To conclude that  $A \otimes B$  is an isometry and that  $Z(W) = Z \otimes_\pi W$  take  $u$  in  $L_1 \otimes X$ . Note that  $v := (A \otimes B)(u)$  is in  $Z \otimes W \subset L_1 \otimes X$  and write  $v = \sum_{i=1}^n f_i \otimes x_i$ , where  $f_1, \dots, f_n \in L_1$  and  $x_1, \dots, x_n \in X$ . We will see that  $\sum_{i=1}^n \|f_i\| \|x_i\| \geq \|u\|$ , which will imply the conclusion, by the definition of  $\|v\|$ . Indeed,  $v = (P \otimes Q)(v) = \sum_{i=1}^n (Pf_i) \otimes (Qx_i)$  and

$$\begin{aligned} \|v\| &\geq \sum_{i=1}^n \|Pf_i\| \|Qx_i\| = \sum_{i=1}^n \|A^{-1}Pf_i\| \|B^{-1}Qx_i\| \\ &\geq \left\| \underbrace{\sum_{i=1}^n (A^{-1}Pf_i) \otimes (B^{-1}Qx_i)}_{=:y} \right\|. \end{aligned}$$

It is immediate that  $(A \otimes B)(y) = v$  and thus  $y = u$ .  $\square$

The following standard example will be used often to define projectional factors of an operator  $T : L_1(X) \rightarrow L_1(X)$ .

**Example 2.17.** Let  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$ ,  $(\hat{h}_L)_{L \in \mathcal{D}^+}$  be a faithful Haar systems and let  $X$  be a Haar system space. Take

$$Z = \overline{\langle \tilde{h}_I : I \in \mathcal{D}^+ \rangle} \subset L_1 \text{ and } W = \overline{\langle \hat{h}_L : L \in \mathcal{D}^+ \rangle} \subset X.$$

Then the map  $P : L_1(X) \rightarrow L_1(X)$  given by

$$Pu = \sum_{I \in \mathcal{D}^+} \sum_{L \in \mathcal{D}^+} \langle \tilde{h}_I \otimes \nu_L \hat{h}_L, u \rangle |I|^{-1} \tilde{h}_I \otimes \mu_L \hat{h}_L$$

(recall that  $\mu_I = \|\chi_I\|_X^{-1}$  and  $\nu_L = \|\chi_L\|_{X^*}^{-1}$ ) is a norm-one projection onto  $Z(X) = \overline{\langle \tilde{h}_I \otimes \hat{h}_L : I, L \in \mathcal{D}^+ \rangle}$  and the map

$$A : L_1(X) \rightarrow L_1(X) \text{ given by } A(h_I \otimes h_L) = \tilde{h}_I \otimes \hat{h}_L$$

is a linear isometry onto  $Z(X)$ . Then, any bounded linear operator  $T : L_1(X) \rightarrow L_1(X)$  is a 1-projectional factor of  $S = A^{-1}PTA : L_1(X) \rightarrow L_1(X)$ , so that for all  $I, J, L, M \in \mathcal{D}^+$  we have

$$\langle h_I \otimes h_L, S(h_J \otimes h_M) \rangle = \langle \tilde{h}_I \otimes \hat{h}_L, T(\tilde{h}_J \otimes \hat{h}_M) \rangle.$$

**Proposition 2.18.** Let  $\mathcal{A} \subset [\mathcal{D}^+]$  be a subset that has positive measure. Denote by  $\mathcal{A} = \bigcup_{k_0=0}^{\infty} \{I_{k_0} : (I_k)_{k=0}^{\infty} \in \mathcal{A}\}$  and  $Y_{\mathcal{A}} = \overline{\langle \{h_I : I \in \mathcal{A}\} \rangle}$ . Then, there exists a subspace  $Z$  of  $Y_{\mathcal{A}}$  which is isometrically isomorphic to  $L_1$  and 1-complemented in  $L_1$ .

*Proof.* By approximating  $\mathcal{A}$  in measure by closed sets from the inside, we can assume that  $\mathcal{A}$  is closed. For  $k \in \mathbb{N}$  let  $A_k = \bigcup \{I : I \in \mathcal{A} \cap \mathcal{D}_k\}$ , and  $\mathcal{A} = \{(I_n) \in [\mathcal{D}^+] : I_k \in \mathcal{D}_k, I_k \subset A_k\}$ . Then it follows that  $\mathcal{A} = \bigcap_k \mathcal{A}_k$  and letting  $A = \bigcap_k A_k$ , we deduce that

$$|A| = \lim_{k \rightarrow \infty} |A_k| = \lim_{k \rightarrow \infty} |\mathcal{A}_k| = |\mathcal{A}|.$$

But also, for any  $J \notin \mathcal{A}$ , we have  $J \cap A = \emptyset$ . It follows that for any  $f \in L_1$  with  $f|_{A^c} = 0$  and  $J \notin \mathcal{A}$  we have  $\langle h_J, f \rangle = 0$  and thus  $f \in Y_{\mathcal{A}}$ . In particular, the restriction operator  $R_A : L_1 \rightarrow L_1$  is a 1-projection onto a subspace that is isometrically isomorphic to  $L_1$ .  $\square$

The above proposition leads to the following example, which will be useful in the sequel.

**Example 2.19.** Let  $\mathcal{A} \subset [\mathcal{D}^+]$  be a subset that has positive measure and let  $X$  be a Banach space. Then, there exists a subspace  $Z$  of  $Y_{\mathcal{A}}$  that is isometrically isomorphic to  $L_1$  via  $A : L_1 \rightarrow Z$  and 1-complemented in  $L_1$  via  $P : L_1 \rightarrow Z$ . In particular, for any Banach space  $X$  the space

$$Z(X) = \overline{Z \otimes X} \subset L_1(X)$$

is isometrically isomorphic to  $L_1(X)$  via  $A \otimes I : L_1(X) \rightarrow Z(X)$  and 1-complemented in  $L_1(X)$  via  $P \otimes I$ .

**2.6. Decompositions of operators on  $L_1(X)$ .** We begin by listing further standard facts about projective tensor products. We then use these facts to associate to each bounded linear operator  $T : L_1(X) \rightarrow L_1(X)$  a family of bounded linear operators on  $L_1$ . In the next section we will study compactness properties of this family. In later sections we use these properties to extract information about projectional factors of the operator  $T$ .

Let  $E, F$  be Banach spaces.

- (a) For every  $e_0^* \in E^*$  and  $f_0^* \in F^*$  we may define the bounded linear maps  $q_{(e_0^*)} : E \otimes_{\pi} F \rightarrow F$  and  $q^{(f_0^*)} : E \otimes_{\pi} F \rightarrow E$  given by  $q_{(e_0^*)}(e \otimes f) = e_0^*(e)f$  and  $q^{(f_0^*)}(e \otimes f) = f_0^*(f)e$ . Then,  $\|q_{(e_0^*)}\| = \|e_0^*\|$  and  $\|q^{(f_0^*)}\| = \|f_0^*\|$ .
- (b) For every  $e_0 \in E$  and  $f_0 \in F$  we may define the maps  $j_{(e_0)} : F \rightarrow E \otimes_{\pi} F$  and  $j^{(f_0)} : E \rightarrow E \otimes_{\pi} F$  given by  $j_{(e_0)}f = e_0 \otimes f$  and  $j^{(f_0)}e = e \otimes f_0$ . Then,  $\|j_{(e_0)}\| = \|e_0\|$  and  $\|j^{(f_0)}\| = \|f_0\|$ .

- (c) For every bounded linear operator  $T : E \otimes_\pi F \rightarrow E \otimes_\pi F$ ,  $f_0^* \in F^*$ , and  $f_0 \in F$  the map  $T^{(f_0^*, f_0)} := q(f_0^*)Tj(f_0) : E \rightarrow E$  is the unique bounded linear map so that for all  $e^* \in E^*$  and  $e \in E$  we have  $\langle e^*, T^{(f_0^*, f_0)}e \rangle = \langle e^* \otimes f_0^*, T(e \otimes f_0) \rangle$ .
- (d) For every bounded linear operator  $T : E \otimes_\pi F \rightarrow E \otimes_\pi F$ ,  $e_0^* \in E^*$ , and  $e_0 \in E$  the map  $T_{(e_0^*, e_0)} := q(e_0^*)Tj(e_0) : F \rightarrow F$  is the unique bounded linear map so that for all  $f^* \in F^*$  and  $f \in F$  we have  $\langle f^*, T_{(e_0^*, e_0)}f \rangle = \langle e_0^* \otimes f^*, T(e_0 \otimes f) \rangle$ .

**Notation.** Let  $X$  be a Haar system space. For  $L \in \mathcal{D}^+$  we denote

- (i)  $q^L = q^{(\nu_L h_L)} : L_1(X) \rightarrow L_1$ ,
- (ii)  $j^L = j^{(\mu_L h_L)} : L_1 \rightarrow L_1(X)$ , and
- (iii)  $P^L = j^L q^L : L_1(X) \rightarrow L_1(X)$ .

Note that, for any  $k \in \mathbb{N}$ ,  $\|\sum_{\{L: \iota(L) \leq k\}} P^L\| = 1$ . This is because this operator coincides with  $I \otimes P^{[\iota \leq k]}$ , where  $P^{[\iota \leq k]} : X \rightarrow X$  is the basis projection onto  $(\mu_L h_L)_{\iota(L) \leq k}$  (this is easy to verify on vectors of the form  $u = h_I \otimes h_L$  whose linear span is dense in  $L_1(X)$ ). We may therefore state the following.

**Remark 2.20.** Let  $X$  be a Haar system space.

- (i) For each  $L \in \mathcal{D}^+$ ,  $P^L$  is a projection with image

$$Y^L = \{f \otimes (\mu_L h_L) : f \in L_1\}$$

that is isometrically isomorphic to  $L_1$ .

- (ii)  $(Y^L)_{L \in \mathcal{D}^+}$  forms a monotone Schauder decomposition of  $L_1(X)$ . In particular, for every  $u \in L_1(X)$

$$u = \sum_{L \in \mathcal{D}^+} P^L u = \sum_{L \in \mathcal{D}^+} (q^L u) \otimes (\mu_L h_L).$$

Thus,  $u$  admits a unique representation  $u = \sum_{L \in \mathcal{D}^+} f_L \otimes (\mu_L h_L)$ .

**2.7. Operators on  $L_1$  associated to an operator on  $L_1(X)$ .** For a Haar system space  $X$ , we represent every bounded linear operator  $T : L_1(X) \rightarrow L_1(X)$  as a matrix of operators  $(T^{(L,M)})_{(L,M) \in \mathcal{D}^+}$ , each of which is defined on  $L_1$ .

**Notation.** Let  $X$  be a Haar system space and let  $T : L_1(X) \rightarrow L_1(X)$  be a bounded linear operator. For  $L, M \in \mathcal{D}^+$  we denote  $T^{(L,M)} = T^{(\nu_L h_L, \mu_M h_M)}$  (recall from Proposition 2.13 that scalars  $\mu_M$  and  $\nu_L$  positive, and chosen so that  $\mu_M h_M$  is normalized in  $X^*$  and  $\nu_L h_L$  is normalized in  $X$ ), so that for every  $u \in L_1(X)$  we have

$$\begin{aligned}
 (29) \quad Tu &= \sum_{L \in \mathcal{D}^+} P^L T \left( \sum_{M \in \mathcal{D}^+} P^M u \right) = \sum_{L \in \mathcal{D}^+} \sum_{M \in \mathcal{D}^+} j^L T^{(L,M)} q^M u \\
 &= \sum_{L \in \mathcal{D}^+} \sum_{M \in \mathcal{D}^+} \left( T^{(L,M)}(q^M u) \right) \otimes (\mu_L h_L).
 \end{aligned}$$

For  $L \in \mathcal{D}^+$  we denote  $T^L = T^{(L,L)}$ .

The following type of operator is essential as it is easier to work with. A big part of the paper is to show that, within the constraints of the problem under consideration, every operator  $T : L_1(X) \rightarrow L_1(X)$  is a 1-projectional factor with error  $\varepsilon$  of an  $X$ -diagonal operator.

**Definition 2.21.** Let  $X$  be a Haar system space. A bounded linear operator  $T : L_1(X) \rightarrow L_1(X)$  is called  $X$ -diagonal if for all  $L \neq M \in \mathcal{D}^+$ ,  $T^{(L,M)} = 0$ . We then call  $(T^L)_{L \in \mathcal{D}^+}$  the entries of  $T$ .

Note that  $T$  is  $X$ -diagonal if and only if for all  $f \in L_1$  and  $L \in \mathcal{D}^+$  we have  $T(f \otimes (\mu_L h_L)) = (T^L f) \otimes (\mu_L h_L)$  if and only if for all  $L \in \mathcal{D}^+$  the space  $Y^L$  is  $T$ -invariant.

**Remark 2.22.** If  $X$  is a Haar system space and  $T : L_1(X) \rightarrow L_1(X)$  is a bounded linear operator so that  $\sum_{L \neq M} \|T^{(L,M)}\| = \varepsilon < \infty$ , then (29) yields that there exists an  $X$ -diagonal operator  $\bar{T} : L_1(X) \rightarrow L_1(X)$  with entries  $(T^L)_{L \in \mathcal{D}^+}$  so that  $\|T - \bar{T}\| \leq \varepsilon$ .

### 3. COMPACTNESS PROPERTIES OF FAMILIES OF OPERATORS

In this section we extract compactness properties of families of operators associated to a  $T : L_1(X) \rightarrow L_1(X)$ . These results will be eventually applied to families that resemble ones of the form  $(T^{(L,M)})_{(L,M) \in \mathcal{D}^+}$ . The achieved compactness will later be used in a regularization process that will allow us to extract “nicer” operators that projectionally factor through  $T$ . We have chosen to present this section in a more abstract setting that permits more elegant statements and proofs.

**3.1. WOT-sequentially compact families.** Taking WOT-limits of certain sequences of operators of the form  $T^{(x^*, x)}$  is an important component of the proof. This element was already present in the approach of Capon [8, 9].

This essential Lemma due to Rosenthal is necessary in this subsection as well as the next one. A proof can be given, e.g., by induction on  $j$  for  $\varepsilon = 2^{-j} \sup_n \|\xi_n\|_1$ .

**Lemma 3.1.** ([27, Lemma 1.1]) Let  $(\xi_n)_n$  be a bounded sequence of elements of  $\ell_1$  and  $\varepsilon > 0$ . Then, there exists an infinite set  $N = \{n_j : j \in \mathbb{N}\} \in [\mathbb{N}]^\infty$  so that for every  $j_0 \in \mathbb{N}$  we have  $\sum_{j \neq j_0} |\xi_{n_{j_0}}(n_j)| \leq \varepsilon$ .

Here, WOT stands for the weak operator topology in  $L_1(X)$ .

**Theorem 3.2.** Let  $X$  be a Banach space,  $T : L_1(X) \rightarrow L_1(X)$  be a bounded linear operator, and  $A, B$  be bounded subsets of  $X^*$  and  $X$ , respectively. Assume that  $B$  contains no sequence that is equivalent to the unit vector basis of  $\ell_1$ . Then, for every  $f \in L_1$  the set

$$\{T^{(x^*, x)} f : (x^*, x) \in A \times B\}$$

is a uniformly integrable (and thus weakly relatively compact) subset of  $L_1$ . In particular, every sequence in  $\{T^{(x^*, x)} : (x^*, x) \in A \times B\}$  has a WOT-convergent subsequence.

*Proof.* The “in particular” part follows from the separability of  $L_1$  and the fact that the set in question is bounded by  $\|T\| \sup_{(x^*, x) \in A \times B} \|x^*\| \|x\|$ .

Fix a sequence  $(x_n^*, x_n) \in A \times B$ . Assume that  $(T^{(x_n^*, x_n)} f)_n$  is not uniformly integrable. Then, after passing to a subsequence, there exist  $\delta > 0$  and a sequence of disjoint measurable subsets  $(A_n)_n$  of  $[0, 1]$  so that for all  $n \in \mathbb{N}$  we have

$$\delta \leq \left| \int_{A_n} (T^{(x_n^*, x_n)} f)(s) ds \right| = |\langle \chi_{A_n}, T^{(x_n^*, x_n)} f \rangle| = |\langle \chi_{A_n} \otimes x_n^*, T(f \otimes x_n) \rangle|.$$

For every  $n \in \mathbb{N}$  define the scalar sequence  $\xi_n = (\xi_n(m))_m$  given by  $\xi_n(m) = \langle \chi_{A_m} \otimes x_m^*, T(f \otimes x_n) \rangle$ . Then for every  $m_0 \in \mathbb{N}$  we have that for appropriate scalars  $(\zeta_m)_{m=1}^{m_0}$  of modulus one

$$\begin{aligned} (30) \quad \sum_{m=1}^{m_0} |\xi_n(m)| &= \left| \left\langle \sum_{m=1}^{m_0} \chi_{A_m} \otimes \zeta_m x_m^*, T(f \otimes x_n) \right\rangle \right| \\ &\leq \underbrace{\left\| \sum_{m=1}^{m_0} \chi_{A_m} \otimes (\zeta_m x_m^*) \right\|}_{=\max_{1 \leq m \leq m_0} \|x_m^*\|} \|T\| \|f\| \|x_n\| \\ &\leq \|T\| \|f\| \sup_{(x^*, x) \in A \times B} \|x^*\| \|x\|. \end{aligned}$$

By Rosenthal’s Lemma 3.1, there exists an infinite subset  $N = \{n_j : j \in \mathbb{N}\}$  of  $\mathbb{N}$  so that for all  $i_0 \in \mathbb{N}$  we have  $\sum_{j \neq i_0} |\xi_{n_{i_0}}(n_j)| \leq \delta/2$ . After relabelling, for all  $n_0 \in \mathbb{N}$  we have

$$\sum_{m \neq n_0} |\xi_{n_0}(m)| \leq \delta/2.$$

We now show that  $(x_n)_n$  is equivalent to the unit vector basis of  $\ell_1$ . Fix scalars  $a_1, \dots, a_N$ . For appropriate scalars  $\theta_1, \dots, \theta_N$  of modulus 1 we have

$$(31) \quad \sum_{n=1}^N a_n \theta_n \langle \chi_{A_n} \otimes x_n^*, T(f \otimes x_n) \rangle \geq \delta \sum_{n=1}^N |a_n|.$$

Put

$$\begin{aligned} \Lambda &= \left| \left\langle \sum_{m=1}^N \chi_{A_m} \otimes (\theta_m x_m^*), T\left(\sum_{n=1}^N f \otimes a_n x_n\right) \right\rangle \right| \\ &\leq \underbrace{\left\| \sum_{m=1}^N \chi_{A_m} \otimes (\theta_m x_m^*) \right\|}_{=\max_{1 \leq m \leq N} \|x_m^*\|} \|T\| \|f\| \left\| \sum_{n=1}^N a_n x_n \right\| \end{aligned}$$

$$\leq \|T\| \|f\| \sup_{x^* \in A} \|x^*\| \left\| \sum_{n=1}^N a_n x_n \right\|.$$

Also,

$$\begin{aligned} \Lambda &= \left| \sum_{n=1}^N a_n \theta_n \langle \chi_{A_n} \otimes x_n^*, T(f \otimes x_n) \rangle + \sum_{n=1}^N a_n \sum_{m \neq n} \theta_m \langle \chi_{A_m} \otimes x_m^*, T(f \otimes x_n) \rangle \right| \\ &\geq \delta \sum_{n=1}^N |a_n| - \sum_{n=1}^N |a_n| \sum_{m \neq n} |\xi_n(m)| \geq \delta/2 \sum_{n=1}^N |a_n|. \end{aligned}$$

Thus,  $\left\| \sum_{n=1}^N a_n x_n \right\| \geq c \sum_{n=1}^N |a_n|$ , where  $c = \delta/(2\|T\|\|f\| \sup_{x^* \in A} \|x^*\|)$ .  $\square$

**3.2. Compactness in operator norm.** We discuss families that are uniformly eventually close to multipliers and how to obtain compact sets from them. This is particularly important in the sequel because compactness will be essential in achieving strong stabilization properties of operators  $T : L_1(X) \rightarrow L_1(X)$ .

**Notation.** For  $n \in \mathbb{N}$  we denote by  $P_{(\leq n)} : L_1 \rightarrow L_1$  the norm-one canonical basis projection onto  $\langle \{h_I : I \in \mathcal{D}^n\} \rangle$ . We also denote  $P_{(>n)} = I - P_{(\leq n)}$ .

**Definition 3.3.** A set  $\mathcal{T}$  of bounded linear operators on  $L_1$  is called *uniformly eventually close to Haar multipliers* if there exists a collection  $(D_T)_{T \in \mathcal{T}}$  in  $\mathcal{L}_{HM}(L_1)$  so that

$$\limsup_n \sup_{T \in \mathcal{T}} \left( \|(T - D_T)P_{(>n)}\| + \|P_{(>n)}(T - D_T)\| \right) = 0.$$

The main result of this subsection is the first one in the paper that requires a certain amount of legwork.

**Theorem 3.4** (Fundamental Lemma). Let  $X$  be a Banach space,  $A, B$  be bounded subsets of  $X^*$  and  $X$ , respectively, and  $C \subset A \times B$ . Let  $T : L_1(X) \rightarrow L_1(X)$  be a bounded linear operator and assume the following.

- (i) The set  $B$  contains no sequence that is equivalent to the unit vector basis of  $\ell_1$ .
- (ii) The set  $\{T^{(x^*, x)} : (x^*, x) \in C\}$  is uniformly eventually close to Haar multipliers.

Then, for every  $\eta > 0$ , there exists a closed subset  $\mathcal{A}$  of  $[\mathcal{D}^+]$  with  $|\mathcal{A}| > 1 - \eta$  so that the set  $\{T^{(x^*, x)} P_{\mathcal{A}} : (x^*, x) \in C\}$  is relatively compact in the operator norm topology.

**Remark 3.5.** It is not hard to see that the unit ball of  $\mathcal{L}_{HM}(L_1)$  is a compact set in the strong operator topology of  $L_1$ . In fact, this is the  $w^*$ -topology inherited by a predual of  $\mathcal{L}_{HM}(L_1)$ , namely Rosenthal's Stopping Time space studied by Bang and Odell in [4, 5], by Dew in [11], and by Apatsidis in [2]. The Fundamental Lemma (Theorem 3.4) states that

under the right conditions, strong operator convergence yields convergence in operator norm on a big subspace of  $L_1$ . Therefore, this is a type of Egorov Theorem. We point out that some restriction to the family of operators is necessary for the conclusion to hold. If one takes for example  $D_n = P_{(\leq n)}$  then this converges to  $I$  in the strong operator topology. Yet, for no non-empty set of branches  $\mathcal{A}$  the set  $\{D_n P_{\mathcal{A}} : n \in \mathbb{N}\}$  is relatively compact in the operator norm topology.

**Lemma 3.6.** Let  $r > 0$ ,  $(I_k)_{k=0}^\infty \in [\mathcal{D}^+]$  associated to  $(\theta_k)_{k=1}^\infty \in \{-1, 1\}^\mathbb{N}$ , and  $(a_k^n)_{(k,n) \in (\{0\} \cup \mathbb{N}) \times \mathbb{N}}$  be a collection of scalars. Assume that there exist  $k_1 < \ell_1 < k_2 < \ell_2 < \dots$  so that for each  $n \in \mathbb{N}$  we have

$$\sum_{k=k_n+1}^{\ell_n} |a_k^n - a_{k-1}^n| \geq r.$$

For every  $\ell, n \in \mathbb{N}$  define  $f_n^\ell = \sum_{k=0}^\ell a_k^n \theta_k |I_k|^{-1} h_{I_k}$ . Then, there exists a strictly increasing sequence of disjoint measurable subsets  $(A_n)_n$  of  $[0, 1]$  so that for all  $n \in \mathbb{N}$  and  $\ell \geq \ell_n$  we have

$$f_n^\ell(s) = f_n^{\ell_n}(s) \text{ on } A_n \text{ and } \int_{A_n} |f_n^{\ell_n}(s)| ds \geq r/3.$$

*Proof.* Let  $(B_k)$  be the partition of  $[0, 1]$ , defined by  $B_k = I_k \setminus I_{k+1}$ ,  $k \in \mathbb{N}$ . We conclude from the inequality (22) in Proposition 2.5 that:

- (i) for every  $k \leq \ell_n \leq \ell \in \mathbb{N}$  and  $s \in B_k$  we have  $f_n^\ell(s) = f_n^{\ell_n}(s)$  and
- (ii) for every  $m \leq \ell_n \in \mathbb{N}$  we have

$$\int_{\cup_{k=m}^{\ell_n} B_k} |f_n^{\ell_n}(s)| ds \geq \frac{1}{3} \sum_{k=m+1}^{\ell_n} |a_k - a_{k-1}|.$$

Put  $A_n = \cup_{i=k_n}^{\ell_n} B_i$ . The conclusion follows directly from (i) and (ii).  $\square$

*Proof of Theorem 3.4.* Put  $\mathcal{T} = \{T^{(x^*, x)} : (x^*, x) \in A \times B\}$ . Take a family  $(D_T)_T$  that witnesses Definition 3.3. For each  $T \in \mathcal{T}$  we have

$$(32) \quad \|(T - D_T)P_{(>k)}\| \leq \sup_{S \in \mathcal{T}} \left( \|(S - D_S)P_{(>k)}\| \right) = \varepsilon_k.$$

$$(33) \quad \begin{aligned} \|P_{(>k)} T P_{(\leq k)}\| &\leq \underbrace{\|P_{(>k)} D_T P_{(\leq k)}\|}_{=0} + \|P_{(>k)} (T - D_T) P_{(\leq k)}\| \\ &\leq \|P_{(>k)} (T - D_T)\| \\ &\leq \sup_{S \in \mathcal{T}} \left( \|P_{(>k)} (S - D_S)\| \right) = \delta_k. \end{aligned}$$

Both  $(\varepsilon_k)_k$  and  $(\delta_k)_k$  tend to zero. For each  $T \in \mathcal{T}$  denote by  $(a_I^T)_{I \in \mathcal{D}^+}$  the entries of  $D_T$ .

*Claim:* Fix  $\sigma = (I_k)_{k=1}^\infty \in [\mathcal{D}^+]$  and  $r > 0$ . Then, there exists  $k_0 \in \mathbb{N}$  so that for all  $T \in \mathcal{T}$  we have  $\sum_{k=k_0}^\infty |a_{I_k}^T - a_{I_{k-1}}^T| \leq r$ .

We will assume that the claim is true and proceed with the rest of the proof. For every  $N, k_0 \in \mathbb{N}$  let

$$\mathcal{A}_{N,k_0} = \left\{ \sigma = (I_k)_{k=1}^\infty \in [\mathcal{D}^+] : \sup_{T \in \mathcal{T}} \sum_{k=k_0}^\infty |a_{I_k}^T - a_{I_{k-1}}^T| \leq 2^{-N} \right\},$$

which is a closed subset of  $[\mathcal{D}^+]$  and by the claim we have  $\cup_{k_0} \mathcal{A}_{N,k_0} = [\mathcal{D}^+]$ , for all  $N \in \mathbb{N}$ . We may therefore pick a strictly increasing sequence of natural numbers  $(k_N)$  so that for each  $N$  we have  $|\mathcal{A}_{N,k_N}| \geq 1 - \eta/2^N$ . We put  $\mathcal{A} = \cap_N \mathcal{A}_{N,k_N}$  and we demonstrate that this is the desired set.

To show that  $\{TP_{\mathcal{A}} : T \in \mathcal{T}\}$  is relatively compact with respect to the operator norm we fix  $\varepsilon > 0$  and  $(T_n)_n$  in  $\mathcal{T}$ . For each  $n \in \mathbb{N}$  denote  $D_n = D_{T_n}$ . We will find  $M \in [\mathbb{N}]^\infty$  so that for all  $n, m \in M$  we have  $\|T_n P_{\mathcal{A}} - T_m P_{\mathcal{A}}\| \leq 11\varepsilon$ . Fix  $N \in \mathbb{N}$  so that  $2^{-N} \leq \varepsilon$ ,  $\varepsilon_{k_N} \leq \varepsilon$ , and  $\delta_{k_N} \leq \varepsilon$ . For each  $n \in \mathbb{N}$  write

$$T_n = D_n P_{(>k_N)} + \underbrace{(T_n - D_n) P_{(>k_N)}}_{=: A_n} + \underbrace{P_{(>k_N)} T_n P_{(\leq k_N)}}_{=: B_n} + \underbrace{P_{(\leq k_N)} T_n P_{(\leq k_N)}}_{=: C_n}.$$

Then we have  $\|A_n\| \leq \varepsilon_{k_N} \leq \varepsilon$  and  $\|B_n\| \leq \delta_{k_N} \leq \varepsilon$ . By passing to a subsequence of  $(T_n)$  we may assume that for all  $n, m \in \mathbb{N}$  we have (letting  $a_I^n = a_I^{T_n}$ )

$$(34) \quad \sum_{\left\{ \substack{I \in \mathcal{D}^+ \\ |I| \geq 1/2^{k_N+1}} \right\}} |a_I^n - a_I^m| \leq \varepsilon.$$

Since the  $C_n$  are bounded elements of a finite dimensional space, we can also assume that  $\|C_n - C_m\| \leq \varepsilon$ , for  $m, n \in \mathbb{N}$ . Therefore, for  $n, m \in \mathbb{N}$  we have

$$\|T_n P_{\mathcal{A}} - T_m P_{\mathcal{A}}\| \leq \underbrace{\|D_n P_{(>k_N)} P_{\mathcal{A}} - D_m P_{(>k_N)} P_{\mathcal{A}}\|}_{=: \Lambda} + 5\varepsilon.$$

Luckily, the remaining quantity  $\Lambda$  is the norm of a Haar multiplier on  $L_1$  and we know how to compute this. If for  $\sigma = (I_k)_{k=0}^\infty \in \mathcal{A}$  we put

$$\begin{aligned} \Lambda_\sigma &= \sum_{k=k_N+1}^\infty |(a_{I_k}^n - a_{I_k}^m) - (a_{I_{k-1}}^n - a_{I_{k-1}}^m)| + |a_{I_{k_N}}^n - a_{I_{k_N}}^m| + \lim_k |a_{I_k}^n - a_{I_k}^m| \\ &\leq 2 \underbrace{\sum_{k=k_N+1}^\infty |(a_{I_k}^n - a_{I_k}^m) - (a_{I_{k-1}}^n - a_{I_{k-1}}^m)|}_{\leq 2/2^N \leq 2\varepsilon} + 2 \underbrace{|a_{I_{k_N}}^n - a_{I_{k_N}}^m|}_{\leq \varepsilon} \leq 6\varepsilon. \end{aligned}$$

Then, by Remark 2.8,  $\Lambda = \sup_{\sigma \in \mathcal{A}} \Lambda_\sigma$  and thus  $\|T_n P_{\mathcal{A}} - T_m P_{\mathcal{A}}\| \leq 11\varepsilon$ .

We now provide the owed proof of the claim. We fix  $\sigma = (I_k)_{k=0}^\infty$ , with associated signs  $(\theta_k)_{k=0}^\infty$ . Let us assume that the conclusion fails. Then, we may find  $(T_n)_n = (T^{(x_n^*, x_n)})_n$  in  $\mathcal{T}$ , each  $T_n$  is associated with a  $D_n$  (each

$D_n$  has entries  $(a_I^n)_{I \in \mathcal{D}^+}$ , and  $k_1 < \ell_1 < k_2 < \ell_2 < \dots$  so that for all  $n \in \mathbb{N}$

$$\sum_{k=k_n+1}^{\ell_n} |a_{I_k}^n - a_{I_{k-1}}^n| \geq r.$$

Pick  $k_0 \in \mathbb{N}$  so that  $\varepsilon_{k_0} \leq r/12$ . For  $k, n \in \mathbb{N}$  define  $b_k^n = 0$  if  $k \leq k_0$  and  $b_k^n = a_{I_k}^n$  if  $k > k_0$ . If we additionally assume that  $k_1 > k_0$  then for all  $n \in \mathbb{N}$  we have

$$(35) \quad \sum_{k=k_n+1}^{\ell_n} |b_k^n - b_{k-1}^n| \geq r.$$

For each  $n, \ell \in \mathbb{N}$  put

$$f_n^\ell = \sum_{k=0}^{\ell} b_k^n \theta_k |I_k|^{-1} h_{I_k} = D_n P_{(>k_0)}(|I_{\ell+1}|^{-1} \chi_{I_{\ell+1}}).$$

$\underbrace{\hspace{10em}}_{=: \psi_\ell}$

By Lemma 3.6 we may find a sequence of  $(A_n)_n$  of disjoint measurable sets so that for each  $n \in \mathbb{N}$  the sequence  $(f_n^\ell(s))_{\ell \geq \ell_n}$  is constant for all  $s \in A_n$  and  $\|f_n^{\ell_n}|_{A_n}\|_{L_1} \geq r/3$ . For each  $n \in \mathbb{N}$  fix  $g_n$  in the unit sphere of  $L_\infty$  with support in  $A_n$  so that for all  $\ell \geq \ell_n$

$$\begin{aligned} r/3 &\leq \|f_n^{\ell_n}|_{A_n}\|_{L_1} = |\langle g_n, f_n^\ell \rangle| = |\langle g_n, D_n P_{(>k_0)}(\psi_\ell) \rangle| \\ &\leq |\langle g_n, T_n P_{(>k_0)}(\psi_\ell) \rangle| + r/12 = |\langle g_n \otimes x_n^*, T(\underbrace{(P_{(>k_0)} \psi_\ell)}_{\phi_\ell} \otimes x_n) \rangle| + r/12. \end{aligned}$$

Note that for all  $\ell \in \mathbb{N}$ ,  $\|\phi_\ell\|_{L_1} \leq 2$ . Then, for all  $n \in \mathbb{N}$  and  $\ell \geq \ell_n$

$$|\langle g_n \otimes x_n^*, T(\phi_\ell \otimes x_n) \rangle| \geq r/4.$$

Pick an  $L \in [\mathbb{N}]^\infty$  so that for each  $m, n \in \mathbb{N}$  the limit

$$\xi_n(m) := \lim_{\ell \in L} \langle g_m \otimes x_m^*, T(\psi_\ell \otimes x_n) \rangle \text{ exists.}$$

Because the sequence  $(g_m)_m$  is disjointly supported, an identical calculation as in (30) yields that for all  $n \in \mathbb{N}$  we have

$$\sum_m |\xi_n(m)| \leq 2\|T\| \sup_{(x^*, x) \in C} \|x^*\| \|x\|.$$

Thus, by Rosenthal's Lemma 3.1 we may pass to a subsequence and relabel so that for all  $n_0 \in \mathbb{N}$  we have  $\sum_{m \neq n_0} |\xi_{n_0}(m)| \leq r/8$ .

We will show that  $(x_n)_n$  must be equivalent to the unit vector basis of  $\ell_1$ , which would contradict our assumption and thus finish the proof. Fix scalars  $a_1, \dots, a_N$  and for  $\ell \in L$  with  $\ell \geq \ell_N$  pick appropriate scalars  $\zeta_1^\ell, \dots, \zeta_N^\ell$  of modulus one so that we have

$$\frac{r}{4} \sum_{n=1}^N |a_n| \leq \sum_{n=1}^N \langle g_n \otimes (\zeta_n^\ell x_n^*), T(\phi_\ell \otimes (a_n x_n)) \rangle$$

and put

$$\begin{aligned}\Lambda_\ell &= \left| \left\langle \sum_{m=1}^N g_n \otimes (\zeta_m^\ell x_m^*), \sum_{n=1}^N T(\phi_\ell \otimes (\sum_{n=1}^N a_n x_n)) \right\rangle \right| \\ &\leq \left( 2\|T\| \sup_{x^* \in A} \|x^*\| \right) \left\| \sum_{n=1}^N a_n x_n \right\|.\end{aligned}$$

But also,

$$\begin{aligned}\lim_{\ell \in L} \Lambda_\ell &= \lim_{\ell \in L} \left| \sum_{n=1}^N \langle g_n \otimes (\zeta_n^\ell x_n^*), T(\phi_\ell \otimes (a_n x_n)) \rangle \right. \\ &\quad \left. + \sum_{n=1}^N a_n \sum_{m \neq n} \zeta_m^\ell \langle g_m \otimes x_m^*, T(\psi_\ell \otimes x_n) \rangle \right| \\ &\geq \frac{r}{4} \sum_{n=1}^N |a_n| - \sum_{n=1}^N |a_n| \sum_{m \neq n} |\xi_n(m)| \geq \frac{r}{8} \sum_{n=1}^N |a_n|.\end{aligned}$$

Therefore,  $\left\| \sum_{n=1}^N a_n x_n \right\| \geq r/(16\|T\| \sup_{x^* \in A} \|x^*\|) \sum_{n=1}^N |a_n|$ .  $\square$

#### 4. PROJECTIONAL FACTORS OF $X$ -DIAGONAL OPERATORS

The main purpose of the section is to prove the following first step towards the final result. The Fundamental Lemma (Theorem 3.4) is a necessary part of the proof.

**Theorem 4.1.** Let  $X$  be in  $\mathcal{H}^*$  and let  $T : L_1(X) \rightarrow L_1(X)$  be a bounded linear operator. Then, for every  $\varepsilon > 0$ ,  $T$  is a 1-projectional factor with error  $\varepsilon$  of an  $X$ -diagonal operator  $S : L_1(X) \rightarrow L_1(X)$ .

The strategy is to first pass to an operator  $S$  with the family  $(S^{(L,M)})_{L \neq M}$  uniformly eventually close to Haar multipliers (in reality,  $S$  satisfies something slightly stronger). We will then use the Fundamental Lemma to eliminate these entries altogether. The following result states how uniform eventual proximity to Haar multipliers is achieved in practice.

**Lemma 4.2.** Let  $\mathcal{T}$  be a subset of  $\mathcal{L}(L_1)$  and  $(\varepsilon_{(I,J)})_{(I,J) \in \mathcal{D}^+ \times \mathcal{D}^+}$  be a summable collection of positive real numbers. If for every  $I \neq J \in \mathcal{D}^+$  and  $T \in \mathcal{T}$  we have  $|\langle h_I, T(|J|^{-1} h_J) \rangle| \leq \varepsilon_{(I,J)}$  then  $\mathcal{T}$  is uniformly eventually close to Haar multipliers.

*Proof.* For fixed  $T \in \mathcal{T}$  put  $a_I = \langle h_I, T(|I|^{-1} h_I) \rangle$ . This collection defines a bounded Haar multiplier  $D_T$  because for all  $f$  in the unit ball of  $L_1$ ,  $\|(T - D_T)f\| \leq \sum_{I \in \mathcal{D}^+} \sum_{\{J \in \mathcal{D}^+ : J \neq I\}} |\langle h_I, T(|J|^{-1} h_J) \rangle| < \infty$ . Also, for all  $n \in \mathbb{N}$ ,

$$\left\| TP_{(>n)} - D_T P_{(>n)} \right\| \leq \sum_{I \in \mathcal{D}^+} \sum_{J \in \mathcal{D}^+ \setminus \mathcal{D}^n} \varepsilon_{(I,J)} =: \varepsilon_n \text{ and}$$

$$\left\| P_{(>n)}T - P_{(>n)}D_T \right\| \leq \sum_{I \in \mathcal{D}^+ \setminus \mathcal{D}^n} \sum_{J \in \mathcal{D}^+} \varepsilon_{(I,J)} =: \delta_n.$$

Both  $(\varepsilon_n)_n$  and  $(\delta_n)_n$  tend to zero.  $\square$

The next lemma is the basic tool used to achieve the first step.

**Lemma 4.3.** Let  $X$  be in  $\mathcal{H}^*$  and  $\mathcal{T} \subset \mathcal{L}(X)$ ,  $G \subset X^*$ , and  $F \subset X$  be finite sets. Then, for any  $\varepsilon > 0$ , there exists  $i_0 \in \mathbb{N}$  so that for any disjoint collection  $\Delta$  of  $\mathcal{D}^+$  with  $\min \iota(\Delta) \geq i_0$  and any  $\theta \in \{-1, 1\}^\Delta$  we have

$$\max_{g \in G, T \in \mathcal{T}} |\langle g, T(h_\Delta^\theta) \rangle| \leq \varepsilon \text{ and } \max_{f \in F, T \in \mathcal{T}} |\langle h_\Delta^\theta, T(f) \rangle| \leq \varepsilon$$

(recall that  $h_\Delta^\theta$  was introduced before Definition 2.10).

*Proof.* The result is an immediate consequence of the following fact: let  $(\Delta_k)$  be a sequence of finite disjoint collections of  $\mathcal{D}^+$  with  $\lim_k \min \iota(\Delta_k) = \infty$  and for every  $k \in \mathbb{N}$  let  $\theta_k \in \{-1, 1\}^{\Delta_k}$ .

- (a) The sequence  $(h_{\Delta_k}^{\theta_k})_k$  is weakly null.
- (b) The sequence  $(h_{\Delta_k}^{\theta_k})_k$  is a bounded block sequence in  $X^*$  and thus it is  $w^*$ -null.

There is nothing further to say about statement (b). We now explain how statement (a) is achieved. Note that any sequence of independent  $\{-1, 1\}$ -valued random variables of mean 0 is distributionally equivalent to  $(r_n)_n$  and thus weakly null. Any sequence as in statement (a) has a subsequence which is of the form  $(\frac{r_n + r'_n}{2})$ , where  $(r_n)$  and  $(r'_n)$  are both sequences of independent  $\{-1, 1\}$ -valued random variables of mean 0. Thus, it is weakly null as well.  $\square$

We carry out the first step towards the proof of Theorem 4.1

**Proposition 4.4.** Let  $X$  be in  $\mathcal{H}^*$  and denote by  $C$  the set of all pairs  $(g, f)$  in  $B_{X^*} \times B_X$  so that  $g$  and  $f$  have finite and disjoint supports with respect to the Haar system. Then, every bounded linear operator  $T : L_1(X) \rightarrow L_1(X)$  is a 1-projectional factor of a bounded linear operator  $S : L_1(X) \rightarrow L_1(X)$  so that the family  $\{S^{(f,g)} : (f, g) \in C\}$  is uniformly eventually close to Haar multipliers.

*Proof.* We will inductively construct faithful Haar systems  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$  and  $(\hat{h}_L)_{L \in \mathcal{D}^+}$ . In each step  $k$  of the induction we will define  $\tilde{h}_I$  and then  $\hat{h}_L$  with  $k = \iota(I) = \iota(L)$  (i.e.,  $I = L$  but we separate the notation for clarity). These vectors are of the form  $\tilde{h}_I = \sum_{J \in \Delta_I} h_J$  and  $\hat{h}_L = \sum_{M \in \Gamma_L} h_M$ . The inductive assumption is the following.

For every  $J, J', M, M' \in \mathcal{D}^+$  with  $\iota(J') \neq \iota(J) \leq k$  and  $\iota(M') \neq \iota(M) \leq k$  we have

$$(36) \quad |\langle \tilde{h}_J \otimes \nu_M \hat{h}_M, T(|J'|^{-1} \tilde{h}_{J'} \otimes \mu_{M'} \hat{h}_{M'}) \rangle| \leq 2^{-(\iota(J) + \iota(J') + \iota(M) + \iota(M'))}.$$

We may start by picking  $\tilde{h}_\emptyset = \hat{h}_\emptyset = h_\emptyset$ . We now carry out the  $k$ 'th inductive step. Let  $I = L \in \mathcal{D}^+$  with  $\iota(I) = \iota(L) = k$ . We will apply Lemma 4.3 twice, once for  $L_1$  and once for  $X$ . First, define the following finite sets.

$$\begin{aligned}\mathcal{T}_1 &= \{T^{(\nu_M \hat{h}_M, \mu_{M'} \hat{h}_{M'})} : \iota(M), \iota(M') < k\} \subset \mathcal{L}(L_1) \\ G_1 &= \{\tilde{h}_J : \iota(J) < k\} \subset (L_1)^* \text{ and } F_1 = \{|J| \tilde{h}_J : \iota(J) < k\} \subset L_1.\end{aligned}$$

Use Lemma 4.3 to pick  $\tilde{h}_I$  so that

$$\max_{g \in G_1, T \in \mathcal{T}_1} |\langle g, T(\tilde{h}_I) \rangle| \leq 2^{-4k} \text{ and } \max_{f \in F_1, T \in \mathcal{T}_1} |\langle \tilde{h}_I, T(f) \rangle| \leq |I| 2^{-4k}.$$

Next, we take the finite sets

$$\begin{aligned}\mathcal{T}_2 &= \{T_{(\tilde{h}_J, |J|^{-1} \tilde{h}_{J'})} : \iota(J), \iota(J') \leq k\} \subset \mathcal{L}(X) \\ G_2 &= \{\nu_M \hat{h}_M : \iota(M) < k\} \subset (L_1)^* \text{ and } F_2 = \{\mu_M \hat{h}_M : \iota(M) < k\} \subset X.\end{aligned}$$

Use Lemma 4.3 to pick  $\hat{h}_M$  so that

$$\max_{g \in G_2, T \in \mathcal{T}_2} |\langle g, T(\hat{h}_L) \rangle| \leq \mu_L^{-1} 2^{-4k} \text{ and } \max_{f \in F_2, T \in \mathcal{T}_2} |\langle \hat{h}_L, T(f) \rangle| \leq |I| \nu_L^{-1} 2^{-4k}.$$

The inductive step is complete and it is straightforward to check that the inductive hypothesis is preserved.

Take the operator  $S$  given in Example 2.17. We will show that it has the desired property. Fix  $g \in B_{X^*}$ ,  $f \in B_X$  with  $g = \sum_{M \in E} b_M \nu_M h_M$  and  $f = \sum_{M \in F} a_M \mu_M h_M$  so that  $F, G$  are finite and disjoint. Then,

$$S^{(g,f)} = \sum_{M \in E} \sum_{M' \in F} b_M a_{M'} S^{(M,M')}$$

and for  $I \neq J \in \mathcal{D}^+$  we have

$$\begin{aligned}|\langle h_I, S^{(g,f)}(|J|^{-1} h_J) \rangle| &\leq \sum_{M \in E} \sum_{M' \in F} |\langle h_I, S^{(M,M')}(|J|^{-1} h_J) \rangle| \\ &= \sum_{M \in E} \sum_{M' \in F} |\langle h_I \otimes (\nu_M h_M), S((|J|^{-1} h_J) \otimes (\mu_{M'} h_{M'})) \rangle| \\ &= \sum_{M \in E} \sum_{M' \in F} |\langle \tilde{h}_I \otimes (\nu_M \hat{h}_M), T((|J|^{-1} \tilde{h}_J) \otimes (\mu_{M'} \hat{h}_{M'})) \rangle| \\ &\leq \sum_{M \in E} \sum_{M' \in F} 2^{-(\iota(J) + \iota(J') + \iota(M) + \iota(M'))} \leq 2^{-(\iota(I) + \iota(J))} =: \varepsilon_{(I,J)}.\end{aligned}$$

By Lemma 4.2, the family under consideration is uniformly eventually close to Haar multipliers.  $\square$

**Remark 4.5.** Proposition 4.4 can be achieved if we merely assume that  $X$  is a Haar system space as condition  $(\star)$  of Definition 2.14 can be replaced with a probabilistic argument. We presented the slightly simpler proof that assumes  $(\star)$ .

We now eliminate the off-diagonal entries to obtain an  $X$ -diagonal operator that projectionally factors through  $T$ .

*Proof of Theorem 4.1.* By Proposition 4.4,  $T$  is a 1-projectional factor of an  $S : L_1(X) \rightarrow L_1(X)$  that satisfies condition (ii) of Theorem 3.4.

For a finite pairwise disjoint collection  $\Gamma \subset \mathcal{D}^+$  we define  $\Gamma(n) = \{D \in \mathcal{D}_n : D \subset \Gamma^*\}$ . Note that  $\Gamma(n)$  is a partition of  $\Gamma^*$  for large enough  $n$ . Also note that from our condition (\*) it follows that for two finite subsets  $\Gamma, \Gamma'$  of  $\mathcal{D}$ , with  $\Gamma^* \cap (\Gamma')^* = \emptyset$ , the set

$$C^{(\Gamma, \Gamma')} = \{(\nu_{(\Gamma')^*} h_{\Gamma'(n)}, \mu_{\Gamma^*} h_{\Gamma}) : n \in \mathbb{N}\} \cup \{(\nu_{\Gamma^*} h_{\Gamma}, \mu_{(\Gamma')^*} h_{\Gamma'(n)}) : n \in \mathbb{N}\}$$

satisfies condition (i) of Theorem 3.4. The following claim will be the main step towards recursively defining an appropriate faithful Haar system  $(\tilde{h}_L)$ .

*Claim.* There are  $\mathcal{A} \subset [\mathcal{D}]$ , with  $|\mathcal{A}| > 1 - \eta$ , and  $\mathcal{U} \in [\mathbb{N}]^\infty$ , so that

$$\lim_{n \in \mathcal{U}} S^{(\nu_{(\Gamma')^*} h_{\Gamma'(n)}, \mu_{\Gamma^*} h_{\Gamma})} P_{\mathcal{A}} = 0 \text{ and } \lim_{n \in \mathcal{U}} S^{(\nu_{\Gamma^*} h_{\Gamma}, \mu_{(\Gamma')^*} h_{\Gamma'(n)})} P_{\mathcal{A}} = 0$$

with respect to the operator norm, for all  $\Gamma, \Gamma' \subset \mathcal{D}$ , with  $\Gamma^* \cap (\Gamma')^* = \emptyset$ .

In order to show the claim we choose for each pair  $(\Gamma, \Gamma')$ , with  $\Gamma, \Gamma' \subset \mathcal{D}$  being finite,  $\eta_{(\Gamma, \Gamma')} > 0$ , with  $\sum \eta_{(\Gamma, \Gamma')} < \eta$ . Then, using Theorem 3.4, we choose a closed set  $\mathcal{A}_{(\Gamma, \Gamma')}$  in  $[\mathcal{D}^+]$  with  $|\mathcal{A}_{(\Gamma, \Gamma')}| > 1 - \eta_{(\Gamma, \Gamma')}$ , so that  $\{S^{(g, f)} P_{\mathcal{A}_{(\Gamma, \Gamma')}} : (g, f) \in C^{(\Gamma, \Gamma')}\}$  is relatively compact in the operator norm topology. Put  $\mathcal{A} = \bigcap \mathcal{A}_{(\Gamma, \Gamma')}$  and note that  $|\mathcal{A}| > 1 - \eta$  and that for each  $(\Gamma, \Gamma')$  we still have that  $\{S^{(g, f)} P_{\mathcal{A}} : (g, f) \in C^{(\Gamma, \Gamma')}\}$  is relatively compact.

Via a Cantor diagonalization find  $\mathcal{U} \in [\mathbb{N}]^\infty$  so that for every pair  $(\Gamma, \Gamma')$  both limits

$$S_1^{(\Gamma, \Gamma')} := \lim_{n \in \mathcal{U}} S^{(\nu_{(\Gamma')^*} h_{\Gamma'(n)}, \mu_{\Gamma^*} h_{\Gamma})} P_{\mathcal{A}} \text{ and } S_2^{(\Gamma, \Gamma')} := \lim_{n \in \mathcal{U}} S^{(\nu_{\Gamma^*} h_{\Gamma}, \mu_{(\Gamma')^*} h_{\Gamma'(n)})} P_{\mathcal{A}}$$

exist with respect to the operator norm. As we will now see right away,  $S_1^{(\Gamma, \Gamma')} = S_2^{(\Gamma, \Gamma')} = 0$ . Indeed, for any  $g \in L_\infty$  and  $f \in L_1$  we have

$$\begin{aligned} \langle g, S_2^{(\Gamma, \Gamma')} f \rangle &= \lim_{n \in \mathcal{U}} \langle g, S^{(\nu_{\Gamma^*} h_{\Gamma}, \mu_{(\Gamma')^*} h_{\Gamma'(n)})} (P_{\mathcal{A}} f) \rangle \\ &= \lim_{n \in \mathcal{U}} \langle g \otimes (\nu_{\Gamma^*} h_{\Gamma}), S((P_{\mathcal{A}} f) \otimes (\mu_{(\Gamma')^*} h_{\Gamma'(n)})) \rangle = 0 \end{aligned}$$

because  $(h_{\Gamma'(n)})_n$  is weakly null in  $X$ , by Lemma 4.3. With the same computation,  $S_2^{(\Gamma, \Gamma')} = 0$  because  $(h_{\Gamma'(n)})_n$  is  $w^*$ -null in  $X^*$ . This finishes the proof of the claim.

We now choose inductively a faithful Haar system  $(\tilde{h}_L)_{L \in \mathcal{D}^+}$  so that for every  $L \neq M \in \mathcal{D}^+$  we have

$$(37) \quad \|S^{(\nu_L \tilde{h}_L, \mu_M \tilde{h}_M)} P_{\mathcal{A}}\| \leq \varepsilon 2^{-(\iota(L) + \iota(M))}.$$

Assume  $M \in \mathcal{D}$  and  $\tilde{h}_L = h_{L_L}$ , has been chosen for all  $L \in \mathcal{D}^+$  with  $\iota(L) < \iota(M)$ , ( $\tilde{h}_\emptyset = h_\emptyset$  and  $\tilde{h}_{(0,1)} = h_{[0,1]}$  by definition). Without loss of generality we can assume that  $M = K^+$  for some  $K \in \mathcal{D}$  with  $\iota(K) < \iota(M)$ .

Thus we will choose  $\Gamma_M$  so that  $\Gamma_M^* = [\tilde{h}_K = 1]$ . For large enough  $n_0 \in \mathbb{N}$  it follows that  $[\tilde{h}_K = 1] = (\Gamma')^*$  for some  $\Gamma' \subset \mathcal{D}_{n_0}$ . Then we can use our claim that for large enough  $n > 0$ , we let  $\Gamma_M = \Gamma'(n)$  we deduce (37) for all  $L \in \mathcal{D}$ , with  $\iota(L) < \iota(M)$

Apply Proposition 2.18 to find a subspace  $Z$  of  $Y_{\mathcal{A}}$  (i.e., in the image of  $P_{\mathcal{A}}$ ) that is 1-complemented in  $L_1$  via  $P : L_1 \rightarrow Z$  and isometrically isomorphic to  $L_1$  via  $A : L_1 \rightarrow Z$ . Let also  $W$  be the closed linear span of  $(\tilde{h}_L)_{L \in \mathcal{D}^+}$  in  $X$ , let  $Q : X \rightarrow W$  be the canonical 1-projection, and  $B : X \rightarrow W$  be the canonical onto isometry.

By Proposition 2.16 the operator  $R = ((A^{-1}P) \otimes (B^{-1}Q))S(A \otimes B)$  is a 1-projectional factor of  $S$ , and thus also of  $T$ . It remains to see that  $R$  is  $\varepsilon$ -close to an  $X$ -diagonal operator. Fix  $L \neq M$ . To compute the norm of  $R^{(L,M)}$  we also fix  $g \in B_{L_\infty}$  and  $f \in B_{L_1}$ .

$$\begin{aligned} |\langle g, R^{(L,M)} f \rangle| &= |\langle g \otimes (\nu_L h_L), R(f \otimes \mu_M h_M) \rangle| \\ &= |\langle \underbrace{P^* A^{-1*} g}_{:= v \in B_{L_\infty}} \otimes \underbrace{(Q^* B^{-1*} \nu_L h_L)}_{:= \nu_L \tilde{h}_L}, S(\underbrace{A f}_{:= u \in B_{Y_{\mathcal{A}}}} \otimes \underbrace{B \mu_M h_M}_{:= \mu_M \tilde{h}_M}) \rangle| \\ &= |\langle v, S^{(\nu_L \tilde{h}_L, \mu_M \tilde{h}_M)}(P_{\mathcal{A}} u) \rangle| \leq \varepsilon 2^{-(\iota(L) + \iota(M))}. \end{aligned}$$

By Remark 2.22,  $R$  is  $\varepsilon$ -close to an  $X$ -diagonal operator.  $\square$

## 5. STABILIZING ENTRIES OF $X$ -DIAGONAL OPERATORS

Once we have an  $X$ -diagonal operator at hand we can pass to another  $X$ -diagonal operator whose entries are stable in an extremely strong sense.

**Theorem 5.1.** Let  $X$  be in  $\mathcal{H}^{**}$  and let  $T : L_1(X) \rightarrow L_1(X)$  be a bounded  $X$ -diagonal operator. Then, for any collection of positive real numbers  $(\varepsilon_L)_{L \in \mathcal{D}^+}$ ,  $T$  is a 1-projectional factor of an operator  $S : L_1(X) \rightarrow L_1(X)$  with the following properties:

- (a)  $S$  is  $X$ -diagonal with entries  $(S^L)_{L \in \mathcal{D}^+}$  and
- (b) for every  $L, M \in \mathcal{D}^+$  with  $L \subset M$  we have  $\|S^L - S^M\| \leq \varepsilon_M$ .

The above theorem is proved in two steps. The first one is to pass, from an arbitrary  $X$ -diagonal operator, to another one whose entries are uniformly eventually close to Haar multipliers. This is perhaps the most challenging part of the entire process. For presentation purposes we momentarily skip this. Instead, we describe the step that follows it, which is the strong stabilization of the entries, given the uniform eventual proximity to Haar multipliers. This is based on the Fundamental Lemma (Theorem 3.4) and a simple concentration inequality. This proof also serves as an icebreaker for the proof of the first step which is presented afterwards in this section.

**Proposition 5.2.** Let  $X$  be in  $\mathcal{H}^{**}$  and let  $T : L_1(X) \rightarrow L_1(X)$  be a bounded  $X$ -diagonal operator. Assume that the set of entries  $\{T^L : L \in \mathcal{D}^+\}$  of  $T$  is uniformly eventually close to Haar multipliers. Then, for any

collection of positive real numbers  $(\varepsilon_L)_{L \in \mathcal{D}^+}$ ,  $T$  is a 1-projectional factor of an  $X$ -diagonal operator  $S : L_1(X) \rightarrow L_1(X)$  so that for every  $L, M \in \mathcal{D}^+$  with  $L \subset M$  we have  $\|S^L - S^M\| \leq \varepsilon_M$ .

We start with the probabilistic component required in the proof.

**Lemma 5.3.** Let  $N \in \mathbb{N}$ ,  $M \geq 0$ ,  $\Omega$  be a uniform probability space with  $2N$  elements, and let  $\Omega = \sqcup_{n=1}^N \{\omega_n^{-1}, \omega_n^1\}$  be a partition of  $\Omega$  into doubletons. For a function  $G : \Omega \rightarrow [-M, M]$  define  $\Phi : \{-1, 1\}^N \rightarrow [-M, M]$  given by

$$(38) \quad \Phi(\varepsilon) = \frac{1}{N} \sum_{n=1}^N G(\omega_n^{\varepsilon_n}).$$

Then,  $\mathbb{E}(\Phi) = \mathbb{E}(G)$  and  $\text{Var}(\Phi) \leq M^2/N$ , where on  $\{-1, 1\}^N$  we also consider the uniform probability measure. In particular, for any  $\eta > 0$ ,

$$(39) \quad \mathbb{P}\left(\left|\Phi - \mathbb{E}(G)\right| \geq \eta\right) \leq \frac{M^2}{N\eta^2}.$$

*Proof.* For  $1 \leq n \leq N$  let  $\Phi_n : \{-1, 1\}^N \rightarrow [-M, M]$  given by  $\Phi_n(\varepsilon) = G(\omega_n^{\varepsilon_n})$ . This is an independent sequence of random variables and for each  $n \in \mathbb{N}$  we have

$$\mathbb{E}(\Phi_n) = \frac{1}{2}(G(\omega_n^{-1}) + G(\omega_n^1)) \text{ and } \text{Var}(\Phi_n) = \frac{1}{4}(G(\omega_n^{-1}) - G(\omega_n^1))^2.$$

Then,  $\mathbb{E}(\Phi) = (1/N) \sum_{n=1}^N \mathbb{E}(\Phi_n) = \mathbb{E}(G)$ . By independence we obtain

$$\text{Var}(\Phi) = \frac{1}{N^2} \sum_{n=1}^N \text{Var}(\Phi_n) \leq \frac{1}{N^2} N \frac{4M^2}{4} = \frac{M^2}{N}.$$

□

**Lemma 5.4.** Let  $K$  be a relatively compact subset of a Banach space. Then, for every  $\varepsilon > 0$  and  $\eta > 0$  there exists  $N(K, \varepsilon, \eta) \in \mathbb{N}$  so that for every  $N \geq N(K, \varepsilon, \eta)$  the following holds. For every uniform probability space  $\Omega$  with  $2N$  elements and partition  $\Omega = \sqcup_{n=1}^N \{\omega_n^{-1}, \omega_n^1\}$  into doubletons, for any function  $G : \Omega \rightarrow K$ , if we define  $\Phi : \{-1, 1\}^N \rightarrow \text{conv}(K)$  given by

$$\Phi(\varepsilon) = \frac{1}{N} \sum_{n=1}^N G(\omega_n^{\varepsilon_n})$$

then  $\mathbb{E}(\Phi) = \mathbb{E}(G)$  and

$$(40) \quad \mathbb{P}\left(\left\|\Phi - \mathbb{E}(G)\right\| \geq \eta\right) \leq \varepsilon.$$

*Proof.* The statement  $\mathbb{E}(\Phi) = \mathbb{E}(G)$  is proved exactly as in the scalar valued scenario and it is in fact independent of the choice of  $N(K, \varepsilon, \eta)$ . For the second part fix  $\varepsilon, \eta > 0$ , and take a finite  $\eta/3$ -net  $(k_i)_{i=1}^{d(K, \eta)}$  of the set  $\text{conv}(K \cup (-K))$ . Fix norm-one functionals  $(f_i)_{i=1}^{d(k, \eta)}$  so that for each  $1 \leq i \leq d(K, \eta)$  we have  $f_i(k_i) = \|k_i\|$ . In particular, for any  $k_1, k_2 \in \text{co}(K)$

with  $\|k_1 - k_2\| \geq \eta$  there exists  $1 \leq i \leq d(K, \eta)$  so that  $|f_i(k_1) - f_i(k_2)| \geq \eta/3$ . Also set  $M = \sup_{k \in K} \|k\|$ .

If we now fix  $N$ ,  $X$ ,  $G$ , and  $\Phi$  as in the statement. For  $1 \leq i \leq d(K, \eta)$  put  $G_i = f_i \circ G$  and  $\Phi_i = f_i \circ \Phi$  then

$$\{\omega : \|\Phi(\omega) - \mathbb{E}(G)\| > \eta\} \subset \bigcup_{1 \leq i \leq d(K, \eta)} \{\omega : |\Phi_i(\omega) - \mathbb{E}(G_i)| \geq \eta/3\}$$

and thus by Lemma 5.3 we have

$$\mathbb{P}(\|\Phi - \mathbb{E}(G)\| > \eta) \leq d(K, \eta) \frac{9M^2}{N\eta^2}.$$

Picking any  $N(K, \varepsilon, \eta) \geq 9d(K, \eta)M^2/(\varepsilon\eta^2)$  completes the proof.  $\square$

**Remark 5.5.** Let  $X$  be and Haar system space,  $T : L_1(X) \rightarrow L_1(X)$  be an  $X$ -diagonal operator, and  $\Gamma$  be a disjoint collection of  $\mathcal{D}^+$ . Then, for every  $g \in L_\infty$ ,  $f \in L_1$ , and  $\theta$  in  $\{-1, 1\}^\Gamma$  we have

$$\begin{aligned} \langle g, T(\nu_{\Gamma^*} h_\Gamma^\theta, \mu_{\Gamma^*} h_\Gamma^\theta) f \rangle &= \langle g \otimes \nu_{\Gamma^*} h_\Gamma^\theta, T(f \otimes \mu_{\Gamma^*} h_\Gamma^\theta) \rangle \\ &= |\Gamma^*|^{-1} \sum_{M \in \Gamma} \sum_{L \in \Gamma} \theta_M \theta_L \langle g \otimes h_M, T(f \otimes h_L) \rangle \\ &= |\Gamma^*|^{-1} \sum_{M \in \Gamma} \sum_{L \in \Gamma} \theta_M \theta_L \langle g \otimes h_M, (T^L f) \otimes h_L \rangle \\ &= |\Gamma^*|^{-1} \sum_{M \in \Gamma} \sum_{L \in \Gamma} \theta_M \theta_L \langle g, T^L f \rangle \langle h_M, h_L \rangle \\ &= \sum_{L \in \Gamma} (|L|/|\Gamma^*|) \langle g, T^L f \rangle. \end{aligned}$$

In particular, the above expression does not depend on the choice of signs  $\theta$ , i.e., we may write

$$T^\Gamma := T(\nu_{\Gamma^*} h_\Gamma^\theta, \mu_{\Gamma^*} h_\Gamma^\theta) = \sum_{L \in \Gamma} (|L|/|\Gamma^*|) T^L.$$

*Proof of Proposition 5.2.* Since  $X$  is in  $\mathcal{H}^{**}$ , the conditions of the Fundamental Lemma (Theorem 3.4) are satisfied for  $B = \{\mu_L h_L : L \in \mathcal{D}^+\}$ . Fix some  $\eta \in (0, 1)$ . We apply the Fundamental Lemma to find a closed subset  $\mathcal{A}$  of  $[\mathcal{D}^+]$  with  $|\mathcal{A}| > 1 - \eta$  and so that  $\{T^L P_{\mathcal{A}} : L \in \mathcal{D}^+\}$  is relatively compact. By Proposition 2.18 there exists a subspace  $Z$  of  $P_{\mathcal{A}}(L_1)$  that is isometrically isomorphic to  $L_1$  via  $A : L_1 \rightarrow Z$  and 1-complemented in  $L_1$  via  $P : L_1 \rightarrow Z$ . The operator  $T$  is a 1-projectional factor of  $S = ((A^{-1}P) \otimes I)T(A \otimes I)$ . and in fact for every  $L \in \mathcal{D}^+$  we have  $S^L = A^{-1}PT^L A = A^{-1}PT^L P_{\mathcal{A}} A$ . In particular, for every set  $\{S^L : L \in \mathcal{D}^+\}$  is relatively compact. Let  $K$  be the closed convex hull of  $\{S^L : L \in \mathcal{D}^+\}$ , with respect to the operator norm.

As in the proof of Theorem 4.1, for every finite disjoint collection  $\Gamma$  of  $\mathcal{D}^+$  and  $n \in \mathbb{N}$  define  $\Gamma(n) = \{L \in \mathcal{D}_n : L \subset \Gamma^*\}$ . For finitely many  $n \in \mathbb{N}$ ,  $\Gamma(n)$

may be empty however eventually  $\Gamma^* = \Gamma(n)^*$ . Note that for  $n$  sufficiently large so that  $\Gamma^* = \Gamma(n)^*$  we have

$$(41) \quad S_n^\Gamma := S^{\Gamma(n)} = \frac{1}{\#\Gamma(n)} \sum_{L \in \Gamma(n)} S^L \in K.$$

By the relative compactness of  $K$  pass to an infinite subset  $\mathcal{U}$  of  $\mathbb{N}$  so that for each disjoint collection  $\Gamma$  the limit  $S_\infty^\Gamma = \lim_{n \in \mathcal{U}} S_n^\Gamma$  exists. We point out for later that for any partition  $\Gamma = \Gamma_1 \sqcup \dots \sqcup \Gamma_k$  we have

$$(42) \quad S_\infty^\Gamma = (|\Gamma_1^*|/|\Gamma^*|)S_\infty^{\Gamma_1} + \dots + (|\Gamma_k^*|/|\Gamma^*|)S_\infty^{\Gamma_k}.$$

Pick  $(\delta_L)_{L \in \mathcal{D}^+}$  so that for all  $M \in \mathcal{D}^+$  we have  $\sum_{L \subset M} \delta_L \leq \varepsilon_M/3$ . We will recursively define a faithful Haar system  $(\hat{h}_L)_{L \in \mathcal{D}^+}$  so that each  $\hat{h}_L = \sum_{M \in \Gamma_L} \zeta_M h_M$  with  $\Gamma_L \subset \mathcal{D}_{n_L}$ , with  $n_L \in \mathcal{U}$ . We will require that additional conditions are satisfied.

For each  $L$  put  $\Gamma_L^+ = \{M \in \mathcal{D}_{n_L+1} : M \subset [\hat{h}_\emptyset \hat{h}_L = 1]\}$  and  $\Gamma_L^- = \{M \in \mathcal{D}_{n_L+1} : M \subset [\hat{h}_\emptyset \hat{h}_L = -1]\}$ . In the case  $L = \emptyset$  the set  $\Gamma_\emptyset^-$  is empty and we don't consider it, which is consistent with the fact that there is only one immediate successor of  $\emptyset$  in  $\mathcal{D}^+$ . For each  $L$  we define a disjoint collection  $E_L$  of  $\mathcal{D}^+$  with  $E_L^* = \Gamma_L^*$ . This auxiliary collection  $E_L$  will be chosen in the inductive step before  $\Gamma_L^*$  and in fact it will be used to choose the latter. If  $L = \emptyset$  put  $E_L = \{[0, 1]\}$ , if  $L = [0, 1]$  put  $E_L = \Gamma_\emptyset$ , if  $L = L_0^+$  put  $E_L = \Gamma_{L_0}^+$ , and if  $L = L_0^-$  put  $E_L = \Gamma_{L_0}^-$ . Below are the additional requirements for each  $L \in \mathcal{D}^+$ .

- (i) The set  $\Gamma_L$  is of the form  $E_L(n_L)$ .
- (ii)  $\|S_{n_L}^{E_L} - S_\infty^{E_L}\| \leq \delta_L$ .
- (iii)  $\|S_\infty^{E_L} - S_\infty^{\Gamma_L^+}\| \leq \delta_L$  and  $\|S_\infty^{E_L} - S_\infty^{\Gamma_L^-}\| \leq \delta_L$ .

If we have achieved this construction we define

$$Q : L_1(X) \rightarrow L_1(X), \text{ by } Q(f) = \sum_{J, M \in \mathcal{D}^+} \langle h_J \otimes \nu_M \hat{h}_M, f \rangle |J|^{-1} h_J \otimes \mu_M \hat{h}_M,$$

$$B : L_1(X) \rightarrow L_1(X), \text{ by } B(h_I \otimes h_L) = h_I \otimes \hat{h}_L.$$

Put  $R = B^{-1}QSB$ . It follows that  $R$  is  $X$ -diagonal and, by Remark 5.5, for each  $L \in \mathcal{D}^+$  we have  $R^L = S_{n_L}^{E_L}$ . Then, for each  $L$  we have

$$(43) \quad \begin{aligned} \|R^L - R^{L^+}\| &= \|S_{n_L}^{E_L} - S_{n_{L^+}}^{E_{L^+}}\| \leq \|S_\infty^{\Gamma_L^+} - S_{n_{L^+}}^{E_{L^+}}\| + \|S_{n_L}^{E_L} - S_\infty^{\Gamma_L^+}\| \\ &\leq \|S_\infty^{\Gamma_L^+} - S_{n_{L^+}}^{E_{L^+}}\| + 2\delta_L \text{ (by (ii) \& (iii))} \\ &\leq \|S_\infty^{\Gamma_L^+} - S_\infty^{E_{L^+}}\| + \|S_\infty^{E_{L^+}} - S_{n_{L^+}}^{E_{L^+}}\| + 2\delta_L \\ &\stackrel{(ii)}{\leq} \|S_\infty^{\Gamma_L^+} - S_\infty^{E_{L^+}}\| + \delta_{L^+} + 2\delta_L = 2\delta_L + \delta_{L^+}, \end{aligned}$$

because, by definition,  $\Gamma_L^+ = E_{L^+}$ . Similarly, we deduce  $\|R^L - R^{L^-}\| \leq 2\delta_L + \delta_{L^-}$ . Also, using  $S^{\Gamma_\emptyset} = S^{\Gamma_{[0,1]}}$  we deduce  $\|R^\emptyset - R^{[0,1]}\| \leq 2\delta_\emptyset$ . By

iterating this process, we may deduce that for every  $L \subset M$  we have that  $\|R^L - R^M\| \leq 3 \sum_{N \subset M} \delta_N \leq \varepsilon_M$ .

It remains to explain how we ensure that conditions (i), (ii), and (iii) are upheld. We start by putting  $E_\emptyset = \{[0, 1)\}$ , by picking  $n_\emptyset$  sufficiently large so that  $\|S_{n_\emptyset}^{E_\emptyset} - S_\infty^{E_\emptyset}\| \leq \delta_\emptyset$ , and by taking  $\zeta_M = 1$  for  $M \in E_\emptyset(n_\emptyset) = \Gamma_\emptyset$ . Assume that we have carried out the construction up to a certain point and the time has come to pick  $\hat{h}_L$ . Let  $L_0$  be the immediate predecessor of  $L$ . We will assume  $L = L_0^+$ . Similar arguments work if  $L = L_0^-$  or if  $L = [0, 1)$ . Put  $E_L = \Gamma_{L_0}^+$  and pick  $n_L \in \mathcal{U}$  so that

$$(44) \quad \|S_{n_L}^{E_L} - S_\infty^{E_L}\| \leq \delta_L \text{ and } \#E_L(n_L) \geq N(K, 1/2, \delta_L),$$

where  $N(K, 1/2, \delta_L)$  is given by Lemma 5.4 to the compact set  $K$ , defined in beginning of this proof. We now apply that Lemma to  $G : E_L(n_L + 1) \rightarrow K$  with  $G(M) = S_\infty^{\{M\}}$ . If we endow  $G$  with the uniform probability measure, by (42),  $\mathbb{E}(G) = S_\infty^{E_L(n_L + 1)}$ . Because  $E_L(n_L + 1)^* = E_L^*$ , we may instead write  $\mathbb{E}(G) = S_\infty^{E_L}$ . We partition  $E_L(n_L + 1)$  into doubletons by writing  $E_L(n_L + 1) = \sqcup_{M \in E_L(n_L)} \{M^+, M^-\}$ . For  $M \in E_L(n_L) = \Gamma_L$  define  $M^1$  and  $M^{-1}$  as follows.

$$M^1 = \begin{cases} M^+ & \text{if } M \subset [\hat{h}_\emptyset = 1] \\ M^- & \text{if } M \subset [\hat{h}_\emptyset = -1] \end{cases} \quad \text{and} \quad M^{-1} = \begin{cases} M^- & \text{if } M \subset [\hat{h}_\emptyset = 1] \\ M^+ & \text{if } M \subset [\hat{h}_\emptyset = -1] \end{cases}.$$

Take  $\Phi : \{-1, 1\}^{\Gamma_L} \rightarrow \text{conv}(K)$  given by

$$\Phi(\zeta) = \frac{1}{\#\Gamma_L} \sum_{M \in \Gamma_L} G(M^{\zeta(M)}) = \frac{1}{\#\Gamma_L} \sum_{M \in \Gamma_L} S_\infty^{\{M^{\zeta(M)}\}}.$$

By the choice of  $n_L$  so that  $\#E_L(n_L) \geq N(K, 1/2, \delta_L)$ , there exists a choice  $\zeta \in \{-1, 1\}^{\Gamma_L}$  so that

$$\|\Phi(\zeta) - \mathbb{E}(G)\| = \|\Phi(\zeta) - S_\infty^{E_L}\| \leq \delta_L.$$

By (42) and the definition of  $\Phi$  we deduce that  $(1/2)(\Phi(\zeta) + \Phi(-\zeta)) = S_\infty^{E_L}$  and therefore we also have that

$$\|G(\zeta) - S_\infty^{E_L}\| \leq \delta_L.$$

To finish the proof, it remains to observe that if we take  $\hat{h}_L = \sum_{M \in \Gamma_L} \zeta(M) h_M$  we have that  $S_{n_L}^{\Gamma_L^+} = \Phi(\zeta)$  and  $S_{n_L}^{\Gamma_L^-} = \Phi(-\zeta)$ . Indeed, taking a long and hard look at the definition of  $M^1$  and  $M^{-1}$  we eventually observe that for each  $M \in \Gamma_L$  we have  $(h_\emptyset \zeta(M) h_M)|_{M^{\zeta(M)}} = 1$  and  $(h_\emptyset \zeta(M) h_M)|_{M^{-\zeta(M)}} = -1$ . This can be seen, e.g., by examining all four possible combinations of values of  $h_\emptyset|_M$  and  $\zeta(M)$ . Therefore, it is now evident that

$$\Gamma_L^+ = \{M \in E_L(n_L + 1) : M \subset [h_\emptyset h_L = 1]\} = \cup \{M^{\zeta(M)} : M \in \Gamma_L\}$$

and therefore  $S_{n_L}^{\Gamma_L^+} = (\#\Gamma_L)^{-1} \sum_{M \in \Gamma_L} S_\infty^{\{M^{\zeta(M)}\}} = \Phi(\zeta)$ . Finally, by using

$$\frac{1}{2} (S_{n_L}^{\Gamma_L^+} + S_{n_L}^{\Gamma_L^-}) = S_{n_L}^{\Gamma_L} = S_\infty^{E_L} = \frac{1}{2} (\Phi(\zeta) + \Phi(-\zeta))$$

we see that  $S_{\infty}^{\Gamma_L^-} = \Phi(-\zeta)$ .

□

Now that we are warmed up by the proof of Proposition 5.2 we are ready to proceed to the slightly more challenging proof of the following. We point out at this point that Theorem 5.1 is an immediate consequence of Proposition 5.2 and the following Proposition 5.6

**Proposition 5.6.** Let  $X$  be in  $\mathcal{H}^{**}$  and  $T : L_1(X) \rightarrow L_1(X)$  be an  $X$ -diagonal operator and let  $(\varepsilon_{(I,J)})_{(I,J) \in (\mathcal{D}^+)^2}$  be a collection of positive real numbers. Then,  $T$  is a 1-projectional factor of an  $X$ -diagonal operator  $S$  with entries  $(S_L)_{L \in \mathcal{D}^+}$  and the property that for every  $L \in \mathcal{D}^+$  and  $I \neq J \in \mathcal{D}^+$  we have

$$(45) \quad |\langle h_I, S^L(|J|^{-1}h_J) \rangle| \leq \varepsilon_{(I,J)}.$$

In particular, the entries of  $S$  are uniformly eventually close to Haar multipliers.

*Proof.* The “in particular” part follows from Lemma 4.2 we therefore focus on achieving (45).

For each finite disjoint collection  $\Gamma$  of  $\Delta^+$  we define  $\Gamma(n)$ ,  $T^\Gamma$ , and  $T_n^\Gamma$  as in the proof of Proposition 5.2. Because  $X \in \mathcal{H}^{**}$ , by Theorem 3.2, applied to the set  $B = \{\mu_L h_L : L \in \mathcal{D}^+\}$ , for every  $f \in L_1$  the set  $\{T^L f : L \in \mathcal{D}^+\}$  is relatively compact and thus so is its convex hull. In particular, for every finite disjoint collection  $\Gamma$ ,  $\{T_n^\Gamma f : n \in \mathbb{N}\}$  is relatively compact by (41). By a Cantor diagonalization, we may find  $\mathcal{U} \in [\mathbb{N}]^\infty$  so that  $\text{WOT-lim}_{n \in \mathcal{U}} T_n^\Gamma = T_\infty^\Gamma$  exists for every finite disjoint collection  $\Gamma$ .

We will define inductively two faithful Haar systems  $(\tilde{h}_I)_{I \in \mathcal{D}^+}$ ,  $(\hat{h}_L)_{L \in \mathcal{D}^+}$ . In each step of the induction we will build a single vector  $\tilde{h}_I$  but we will build an entire level of vectors  $\hat{h}_L$ . For example, in each of the first four steps of the inductive process we will define respectively the collections of vectors

$$\{\tilde{h}_\emptyset; \hat{h}_\emptyset\}, \{\tilde{h}_{[0,1)}; \hat{h}_{[0,1)}\}, \{\tilde{h}_{[0,1/2)}; \hat{h}_{[0,1/2)}, \hat{h}_{[1/2,1)}\}, \text{ and} \\ \{\tilde{h}_{[1/2,1)}; \hat{h}_{[0,1/4)}, \hat{h}_{[1/4,1/2)}, \hat{h}_{[1/2,3/4)}, \hat{h}_{[3/4,1)}\}.$$

This asymmetric choice is necessary because whenever we pick a new vector  $\tilde{h}_I$  we have to stabilize its interaction with all  $\hat{h}_L$  that will be defined in the future. For each  $I, L \in \mathcal{D}^+$  we will have  $\tilde{h}_I = \sum_{J \in \Delta_I} h_J$  and  $\hat{h}_L = \sum_{M \in \Gamma_L} \zeta_M h_M$ , for some family  $(\zeta_M : M \in \Gamma_L) \subset \{\pm 1\}$ .

Let us set up the stage that will allow us to state the somewhat lengthy inductive hypothesis. For each  $I \in \mathcal{D}^+$  let

$$\varepsilon'_I = \min_{\substack{J, J' \in \mathcal{D}^+ : \\ \iota(J), \iota(J') \leq \iota(I)}} \varepsilon_{(J, J')}$$

and fix  $(\delta_L)_{L \in \mathcal{D}^+}$  so that for all  $M \in \mathcal{D}^+$  we have

$$\sum_{L \subset M} \delta_L \leq \varepsilon'_M/6.$$

Here,  $\mathcal{D}_{-1} = \{\emptyset\}$  and  $\mathcal{D}_0 = \{[0, 1)\}$ . For each  $k \in \mathbb{N}$  and for every  $L \in \mathcal{D}_{k-2}$  we have for some  $n_L \in \mathcal{U}$ ,  $\Gamma_L$  is a finite disjoint collection of  $\mathcal{D}_{n_L}$  and  $|\Gamma_L^*| = |L|$ . Additionally, if  $\iota(I) = k$  the following hold.

- (a) For some  $\alpha_k \in \mathbb{N}$ ,  $\Delta_I$  is a disjoint collection of  $\mathcal{D}^{\alpha_k} \setminus \mathcal{D}^{\alpha_{k-1}}$  and  $|\Delta_I| = |I|$ .  
If  $k > 1$ , then  $\alpha_k > \alpha_{k-1}$  and we put  $\mathcal{D}^{\alpha_0} = \emptyset$ .
- (b) For every  $J \in \mathcal{D}^+$  with  $\iota(J) < k$  and every  $M \in \mathcal{D}^{k-2}$  we have

$$|\langle \tilde{h}_I, T^{\Gamma_M}(|J|^{-1} \tilde{h}_J) \rangle| \leq \varepsilon'_I/2 \text{ and } |\langle \tilde{h}_J, T^{\Gamma_M}(|I|^{-1} \tilde{h}_I) \rangle| \leq \varepsilon'_I/2.$$

We will impose additional conditions. As in the proof of Proposition 5.2 we put  $\Gamma_L^+ = \{M \in \mathcal{D}_{n_L+1} : M \subset [\tilde{h}_\emptyset \tilde{h}_L = 1]\}$  and  $\Gamma_L^- = \{M \in \mathcal{D}_{n_L+1} : M \subset [\tilde{h}_\emptyset \tilde{h}_L = -1]\}$ , for each  $L$ . If  $L = \emptyset$  put  $E_L = \{[0, 1)\}$ , if  $L = [0, 1)$  put  $E_L = \Gamma_\emptyset$ , if  $L = L_0^+$  put  $E_L = \Gamma_{L_0}^+$ , and if  $L = L_0^-$  put  $E_L = \Gamma_{L_0}^-$ . Furthermore, for each  $\alpha \in \mathbb{N}$  let  $P_\alpha : L_1 \rightarrow L_1$  denote the canonical projection onto  $\langle \{h_I : I \in \mathcal{D}^\alpha\} \rangle$ . We require the following for each  $L \in \mathcal{D}_{k-2}$ .

- (i) The set  $\Gamma_L$  is of the form  $E_L(n_L)$ .
- (ii)  $\|P_{\alpha_k}(T_{n_L}^{E_L} - T_{\alpha_k}^{E_L})P_{\alpha_k}\| \leq \delta_L$ .
- (iii)  $\|P_{\alpha_k}(T_\infty^{E_L} - T_\infty^{\Gamma_L^+})P_{\alpha_k}\| \leq \delta_L$  and  $\|P_{\alpha_k}(T_\infty^{E_L} - T_\infty^{\Gamma_L^-})P_{\alpha_k}\| \leq \delta_L$ .

One might jump to the conclusion that the weaker property that, for each  $k \in \mathbb{N}$ ,  $\lim_{n \in \mathcal{U}} P_k T_n^\Gamma P_k$  exists is sufficient to yield the same result. This is in fact false. We would not know that  $T_\infty^\Gamma : L_1 \rightarrow L_1$  is well defined as the Haar system is not boundedly complete. In the inductive step, the operators  $T_\infty^{E_L}$ ,  $L \in \mathcal{D}_{k-2}$  are used in the choice of  $\tilde{h}_I$ ,  $\iota(I) = k$ . Therefore the fact that for each  $\Gamma$ ,  $\text{WOT-}\lim_{n \in \mathcal{U}} T_n^\Gamma = T_\infty^\Gamma$  is necessary.

We assume that we have completed the construction to finish the proof. Take the isometry  $A$  given by  $A(h_I \otimes h_L) = \tilde{h}_I \otimes \hat{h}_L$  and the norm-one projection  $P$  onto the image of  $A$  given by

$$P(u) = \sum_{I, L \in \mathcal{D}^+} \langle \tilde{h}_I \otimes |L|^{-1/q} \hat{h}_L, u \rangle |I|^{-1} \tilde{h}_I \otimes |L|^{-1/p} \hat{h}_L.$$

The operator  $T$  is a 1-projectional factor of  $S = A^{-1}PTA$  and  $S$  is  $X$ -diagonal with entries  $(S^L)_{L \in \mathcal{D}^+}$  so that for each  $L, I, J \in \mathcal{D}^+$  we have

$$\langle h_I, S^L(h_J) \rangle = \langle \tilde{h}_I, T^{\Gamma_L} \tilde{h}_J \rangle.$$

We fix  $I \neq J \in \mathcal{D}^+$  with  $\iota(J) < \iota(I) = k$  and  $L \in \mathcal{D}^+$ . If  $L \in \mathcal{D}^{k-2}$  then by (b) we have

$$(46) \quad \begin{aligned} |\langle h_I, S^L(|J|^{-1} h_J) \rangle| &\leq \varepsilon'_I/2 \leq \varepsilon_{(I, J)} \text{ and} \\ |\langle h_J, S^L(|I|^{-1} h_I) \rangle| &\leq \varepsilon'_I/2 \leq \varepsilon_{(J, I)}. \end{aligned}$$

Assume then that  $L \in \mathcal{D}_{k'-2}$  with  $k' > k$ . Let  $L_k, \dots, L_{k'-1}, L_{k'} = L$  be a sequence with  $L_j \in \mathcal{D}_{j-2}^+$  and each term is a direct successor of the one before it. Repeat the argument from (43) to deduce that for  $k \leq j < k'$

$$\begin{aligned} \|P_{\alpha_k}(T^{\Gamma_{L_j}} - T^{\Gamma_{L_{j+1}}})P_{\alpha_k}\| &\leq \|P_{\alpha_j}(T^{\Gamma_{L_j}} - T^{\Gamma_{L_{j+1}}})P_{\alpha_j}\| \leq 2\delta_{L_j} + \delta_{L_{j+1}}, \text{ i.e.,} \\ \|P_{\alpha_k}(T^{\Gamma_L} - T^{\Gamma_{L_k}})P_{\alpha_k}\| &\leq 3 \sum_{M \subset L_k} \delta_M \leq \varepsilon'_{L_k}/2 \leq \varepsilon'_I/2 \text{ (because } \iota(I) \leq \iota(L_k)). \end{aligned}$$

Therefore,

$$\begin{aligned} |\langle h_I, S^L(|J|^{-1}h_J) \rangle| &= |\langle \tilde{h}_I, T^{\Gamma_L}(|J|^{-1}\tilde{h}_J) \rangle| = |\langle \tilde{h}_I, P_{\alpha_k}T^{\Gamma_L}P_{\alpha_k}(|J|^{-1}\tilde{h}_J) \rangle| \\ &\leq |\langle \tilde{h}_I, P_{\alpha_k}T^{\Gamma_{L_k}}P_{\alpha_k}(|J|^{-1}\tilde{h}_J) \rangle| + \varepsilon'_I/2 \\ &= |\langle \tilde{h}_I, T^{\Gamma_{L_k}}(|J|^{-1}\tilde{h}_J) \rangle| + \varepsilon'_I/2 \\ &= |\langle h_I, S^{L_k}(|J|^{-1}h_J) \rangle| + \varepsilon'_I/2 \stackrel{(46)}{\leq} \varepsilon'_I/2 + \varepsilon'_I/2 \leq \varepsilon_{(I,J)}. \end{aligned}$$

Repeating the argument yields  $|\langle h_J, S^L(|I|^{-1}h_I) \rangle| \leq \varepsilon_{(J,I)}$ . To complete the proof we still need to carry out the inductive construction. In the first step we may take  $\tilde{h}_\emptyset = h_\emptyset$  (i.e.,  $\Delta_\emptyset = \{\emptyset\}$ ) and thus we may take, e.g.,  $\alpha_1 = 1$ . Next, we pick  $n_\emptyset \in \mathcal{U}$  sufficiently large so that we have  $\|P_1(T_{n_\emptyset}^{\{[0,1]\}} - T_\infty^{\{[0,1]\}})P_1\| \leq \delta_\emptyset$ . We put  $\hat{h}_\emptyset = \sum_{M \in E_\emptyset(n_\emptyset)} h_L$  (i.e.,  $\Gamma_\emptyset = E_\emptyset(n_\emptyset)$  with  $E_\emptyset = \{[0,1]\}$  and  $\zeta_M = 1$  for  $M \in \Gamma_\emptyset$ ). The only non-trivial condition to check is (iii), which follows from the fact that  $E_\emptyset^* = (\Gamma_\emptyset^+)^*$  and thus  $T_\infty^{E_\emptyset} = T_\infty^{\Gamma_\emptyset^+}$ . We do not consider the set  $\Gamma_\emptyset^-$ .

We now present the  $k$ 'th step for  $k \geq 2$ . Let  $I \in \mathcal{D}^+$  with  $\iota(I) = k$  and denote by  $I_0$  its immediate predecessor. We will assume that  $I = I_0^+$ . For each  $L \in \mathcal{D}_{k-2}$  we denote its immediate predecessor by  $L_0$ . Recall that for each such  $L$  the set  $E_L$  has been defined based on whether  $L = L_0^+$  or  $L = L_0^-$ . Consider the following finite sets.

$$\begin{aligned} \mathcal{T} &= \{T^{\Gamma_L} : L \in \mathcal{D}^{k-3}\} \cup \{T_\infty^{E_L} : L \in \mathcal{D}_{k-2}\} \subset \mathcal{L}(L_1), \\ G &= \{\tilde{h}_J : \iota(K) < k\} \subset L_\infty \text{ and } F = \{|J|^{-1}\tilde{h}_J : \iota(J) < k\}. \end{aligned}$$

By Lemma 4.3, there exists  $i_0 \in \mathbb{N}$  so that for any finite disjoint collection  $\Delta \subset \mathcal{D}^+$  with  $\min \iota(\Delta) \geq i_0$  and any  $\theta \in \{-1, 1\}^\Delta$  we have that for all  $T \in \mathcal{T}$ ,  $g \in G$ , and  $f \in F$

$$(47) \quad |\langle g, T(h_\Delta^\theta) \rangle| \leq |I|\varepsilon'_I/3 \text{ and } |\langle h_\Delta^\theta, T(f) \rangle| \leq \varepsilon'_I/3.$$

We pick  $\Delta_I$  with  $\min \iota(\Delta_I) \geq i_0$  and so that (a) is satisfied. The integer  $\alpha_k$  is simply chosen so that  $P_{\alpha_k}\tilde{h}_I = \tilde{h}_I$ . It is immediate that condition (b) is satisfied for all  $M \in \mathcal{D}^{k-3}$ . Later we will show that (b) also holds for  $M \in \mathcal{D}^{k-2}$ .

In the next step, for each  $L \in \mathcal{D}_{k-2}$  we need to pick  $n_L$  that defined  $\Gamma_L$  and  $\zeta_L \in \{-1, 1\}^{\Gamma_L}$ . The choice of  $n_L$  so that (i) and (ii) are satisfied is easy. However, we wish to ensure that we can additionally achieve condition

(iii) and for this we need Lemma 5.4. Consider the relatively compact set  $K = \{P_{\alpha_k} T_{\infty}^{\Gamma} P_{\alpha_k} : \Gamma \text{ is a finite disjoint collection of } \mathcal{D}^+\} \subset \mathcal{L}(L_1)$  and take  $N(K, 2^{-k}, \varepsilon'_I/6)$  given by Lemma 5.4. For each  $L \in \mathcal{D}^{k-2}$  pick  $n_L \in \mathcal{U}$  so that (ii) is satisfied as well as  $\#E_L(n_L) \geq N(K, 2^{-k}, \varepsilon'_I/6)$ . The objective is to pick, for each  $L \in \mathcal{D}_{k-2}$ , signs  $\zeta_L \in \{-1, 1\}^{\Gamma_L}$  so that (iii) is satisfied. Repeating, word for word, the argument from the last few paragraphs of the proof of Proposition 5.2 we can do exactly that.

The final touch that is required to complete the proof is to observe that (b) is now also satisfied for all  $L \in \mathcal{D}_{k-2}$ . Indeed, for  $J \in \mathcal{D}^+$  with  $\iota(J) < k$  we have

$$\begin{aligned} |\langle \tilde{h}_I, T^{\Gamma_L}(|J|^{-1} \tilde{h}_J) \rangle| &= |\langle P_{\alpha_k}^* \tilde{h}_I, T_{n_L}^{E_L} P_{\alpha_k}(|J|^{-1} \tilde{h}_J) \rangle| \\ &= |\langle \tilde{h}_I, P_{\alpha_k} T_{n_L}^{E_L} P_{\alpha_k}(|J|^{-1} \tilde{h}_J) \rangle| \\ &\stackrel{(ii)}{\leq} |\langle \tilde{h}_I, P_{\alpha_k} T_{\infty}^{E_L} P_{\alpha_k}(|J|^{-1} \tilde{h}_J) \rangle| + \delta_L \\ &\stackrel{(47)}{\leq} \varepsilon'_I/3 + \delta_L \leq \varepsilon'_I/3 + \varepsilon'_I/6 = \varepsilon'_I/2. \end{aligned}$$

The same argument yields  $|\langle \tilde{h}_J, T^{\Gamma_L}(|I|^{-1} \tilde{h}_I) \rangle| \leq \varepsilon'_I/2$ .  $\square$

## 6. PROJECTIONAL FACTORS OF SCALAR OPERATORS

In this section we put the finishing touches to prove our main result.

**Theorem 6.1.** Let  $X$  be in  $\mathcal{H}^*$  and  $\mathcal{H}^{**}$  and let  $T : L_1(X) \rightarrow L_1(X)$  be a bounded linear operator. Then, for every  $\varepsilon > 0$ ,  $T$  is a 1-projectional factor with error  $\varepsilon$  of a scalar operator. In particular,  $L_1(X)$  is primary.

We first need to prove a perturbation result that will allow us to pass from Theorem 5.1 to the conclusion.

**Proposition 6.2.** Let  $X$  be a Haar system space and  $T : L_1(X) \rightarrow L_1(X)$  be an  $X$ -diagonal operator with entries  $(T^L)_{L \in \mathcal{D}^+}$  and let  $\varepsilon > 0$ . Assume that for all  $L, M \in \mathcal{D}^+$  with  $L \subset M$  we have  $\|T^L - T^M\| \leq \varepsilon|M|^2$ . Then,  $\|T - T^{\emptyset} \otimes I\| \leq 7\varepsilon$ .

**Remark 6.3.** Let  $n_0 \in \mathbb{N}$  and for each  $L \in \mathcal{D}_{n_0}$  let  $(\theta_k^L)_{k=0}^{n_0}$  be the signs given by (20). Then, for scalars  $(a_L)_{L \in \mathcal{D}_{n_0}}$  we may write

$$\sum_{L \in \mathcal{D}_{n_0}} a_L |L|^{-1} \chi_L = \left( \sum_{L \in \mathcal{D}_{n_0}} a_L \right) h_{\emptyset} + \sum_{k=1}^{n_0} \sum_{M \in \mathcal{D}_{k-1}} \left( \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} \theta_k^L a_L \right) |M|^{-1} h_M.$$

We now take an RI space  $X$  and translate this into the  $X$  setting. For  $k = 0, \dots, n_0$  put  $\mu_k = \mu_L$  and  $\nu_k = \nu_L$  for  $L \in \mathcal{D}_k$ . Multiply both sides by  $\nu_{n_0}^{-1}$  so that for  $L \in \mathcal{D}_{n_0}$  we have  $|L|^{-1} \nu_{n_0}^{-1} = \mu_L$ .

$$(48) \quad \sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L =$$

$$\nu_{n_0}^{-1} \left( \sum_{L \in \mathcal{D}_{n_0}} a_L \right) h_\emptyset + \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} \left( \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} \theta_k^L a_L \right) \mu_M h_M.$$

Scalar multiplication may be replaced with tensor multiplication to obtain the same formula (i.e., consider  $a_L \otimes \chi_L$  where  $a_L$  is, e.g., in  $L_1$ ).

Let us additionally observe that for any  $1 \leq k \leq n_0$  and  $M \in \mathcal{D}_{k-1}$  we have

$$\begin{aligned} (49) \quad \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} |a_L| &= \left\langle \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} |a_L| \mu_L \chi_L, \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} \nu_L \chi_L \right\rangle \\ &\leq \left\| \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} |a_L| \mu_L \chi_L \right\| \left\| \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} \nu_L \chi_L \right\|_{X^*} \\ &\leq \left\| \sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L \right\| \nu_{n_0} \|\chi_M\|_{X^*} \\ &= \nu_{n_0} \nu_{k-1}^{-1} \left\| \sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L \right\|. \end{aligned}$$

*Proof of Proposition 6.2.* For  $n = 0, 1, \dots$  consider the auxiliary operator  $S_n = \sum_{L \in \mathcal{D}_n} T^L \otimes R^L$ , where  $R^L : X \rightarrow X$  denotes the restriction onto  $L$ , i.e.,  $R^L f = \chi_L f$ . We observe that

$$\begin{aligned} \|S_n - S_{n+1}\| &= \left\| \sum_{L \in \mathcal{D}_n} T^L \otimes (R^{L^+} + R^{L^-}) - \sum_{L \in \mathcal{D}_n} (T^{L^+} \otimes R^{L^+} + T^{L^-} \otimes R^{L^-}) \right\| \\ &\leq \sum_{L \in \mathcal{D}_n} (\|T^L - T^{L^+}\| \|R_{L^+}\| + \|T^L - T^{L^-}\| \|R^{L^-}\|) \\ &\leq 2\varepsilon \sum_{L \in \mathcal{D}_n} |L|^2 = \varepsilon 2^{-n+1}. \end{aligned}$$

In particular, for all  $n \in \mathbb{N}$  we have

$$(50) \quad \|T^\emptyset \otimes I - S_n\| = \|S_0 - S_n\| \leq 4\varepsilon.$$

By (28), to estimate  $\|T - T^\emptyset \otimes I\|$  it is sufficient to consider vectors of the form  $f \otimes g$ , with  $f \in B_{L_1}$ ,  $g = \sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L$ , and  $\|\sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L\| = 1$ . By (48)

$$\begin{aligned} (51) \quad f \otimes g &= \nu_{n_0}^{-1} \left( \sum_{L \in \mathcal{D}_{n_0}} a_L \right) f \otimes h_\emptyset \\ &\quad + \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} \left( \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} \theta_k^L a_L \right) f \otimes \mu_M h_M. \end{aligned}$$

From (50) it follows that

$$\|(T - T^\emptyset \otimes I)(f \otimes g)\| \leq 4\varepsilon + \|(T - S^{n_0})(f \otimes g)\|.$$

We next evaluate  $T$  and  $S_{n_0}$  on  $f \otimes g$ . Since  $T$  is  $X$ -diagonal we have

$$\begin{aligned} T(f \otimes g) &\stackrel{(51)}{=} \nu_{n_0}^{-1} \left( \sum_{L \in \mathcal{D}_{n_0}} a_L \right) (T^\emptyset f) \otimes h_\emptyset \\ &\quad + \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} \left( \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} \theta_k^L a_L \right) (T^M f) \otimes \mu_M h_M. \end{aligned}$$

For the other valuation note that for  $L \in \mathcal{D}_{n_0}$  we have  $S_{n_0}(f \otimes \mu_L \chi_L) = (T^L f) \otimes \mu_L \chi_L$ . Therefore,

$$\begin{aligned} S_{n_0}(f \otimes g) &\stackrel{(48)}{=} \nu_{n_0}^{-1} \left( \sum_{L \in \mathcal{D}_{n_0}} a_L (T^L f) \right) \otimes h_\emptyset \\ &\quad + \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} \left( \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} \theta_k^L a_L (T^L f) \right) \otimes \mu_M h_M. \end{aligned}$$

Therefore,

$$\begin{aligned} &\| (T - S_{n_0})(f \otimes g) \| \\ &\leq \nu_{n_0}^{-1} \sum_{L \in \mathcal{D}_{n_0}} |a_L| \underbrace{\|T^\emptyset - T^L\|}_{\leq \varepsilon} + \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} \sum_{\substack{L \in \mathcal{D}_{n_0} \\ L \subset M}} |a_L| \underbrace{\|T^M - T^L\|}_{\leq \varepsilon |M|^2} \\ &\stackrel{(49)}{\leq} \underbrace{\nu_{n_0}^{-1} \varepsilon \left\| \sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L \right\|}_{=1} \underbrace{\nu_{n_0}^{-1} \nu_0^{-1}}_{=1} \\ &\quad + \varepsilon \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} |M|^2 \left\| \sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L \right\| \nu_{n_0} \nu_{k-1}^{-1} \\ &= \varepsilon + \varepsilon \sum_{k=1}^{n_0} \sum_{M \in \mathcal{D}_{k-1}} |M|^2 = \varepsilon + \varepsilon \sum_{k=1}^{n_0} \frac{2^{k-1}}{2^{2k-2}} \leq 3\varepsilon. \end{aligned}$$

In conclusion,  $\|(T - T^\emptyset \otimes I)(f \otimes g)\| \leq 4\varepsilon + 3\varepsilon$ .  $\square$

We give the proof of the main result.

*Proof of Theorem 6.1.* Recall that, by virtue of Proposition 2.3, being an approximate 1-projectional factor is a transitive property, during which the compounded errors are under control. We successively apply Theorem 4.1, Theorem 5.1 and Proposition 6.2 to find a bounded linear operator  $S : L_1 \rightarrow L_1$  so that  $T$  is a 1-projectional factor with error  $\varepsilon$  of  $S \otimes I : L_1(X) \rightarrow L_1(X)$ . By Theorem 2.9,  $S$  is a 1-projectional factor with error  $\varepsilon$  of a scalar operator  $\lambda I : L_1 \rightarrow L_1$  and therefore  $S \otimes I : L_1(X) \rightarrow L_1(X)$  is a 1-projectional factor with error  $\varepsilon$  of  $\lambda I : L_1(X) \rightarrow L_1(X)$ . Finally,  $T$  is a 1-projectional

factor with error  $2\varepsilon$  of  $\lambda I : L_1(X) \rightarrow L_1(X)$ . Thus, our claim follows from Proposition 2.4.  $\square$

## 7. FINAL DISCUSSION

Characterizing the complemented subspaces of  $L_1$  and those of  $C(K)$  remain the most prominent problems in the study of decompositions of classical Banach spaces. This motivates in particular the study of biparameter spaces, especially those with an  $L_1$  or  $C(K)$  component. The proof, e.g., of primariness for each such type of space presents a different challenge and therefore an opportunity to extract new information on the structure of  $L_1$  or  $C(K)$  and their operators. Here is a list of classical biparameter spaces, for which primariness remains unresolved.

- (a)  $L_p(L_1)$  for  $1 < p \leq \infty$ .
- (b)  $L_p(L_\infty) \simeq L_p(\ell_\infty)$  for  $1 \leq p < \infty$ .
- (c)  $\ell_p(C(K))$  for a compact metric space  $K$  and  $1 \leq p \leq \infty$ .
- (d)  $L_p(C(K))$  for a compact metric space  $K$  and  $1 \leq p \leq \infty$ .
- (e)  $C(K, \ell_p)$  for a compact metric space  $K$  and  $1 \leq p \leq \infty$ .
- (f)  $C(K, L_p)$  for a compact metric space  $K$  and  $1 \leq p < \infty$ .

Noteworthily, all the other biparameter Lebesgue spaces  $\ell_p(\ell_q)$ [10],  $\ell_p(L_q)$ [7],  $L_p(L_q)$  [8],  $L_p(\ell_q)$ [9], and  $\ell_\infty(L_q)$ [31] ( $1 < p, q < \infty$ ) are known to be primary. The space  $L_1(C[0, 1])$  resists the approach of this paper but perhaps some of the tools developed here could be of some use. If this were to be resolved, it is conceivable, that techniques from [17] may be useful in transcending the separability barrier to show that  $L_1(L_\infty)$  is primary. Such methods may also be useful in the investigation of whether for non-separable RI space  $X \neq L_\infty$ ,  $L_1(X)$  is primary. In more generality, one may ask for what types of Banach spaces  $X$ , the spaces  $L_1(X)$ ,  $L_p(X)$ ,  $H_1(X)$  and  $H_p(X)$  are primary.

For any two rearrangement invariant Banach function spaces  $X$  and  $Y$  on  $[0, 1]$  one can define the biparameter space  $X(Y)$  as the space of all functions  $f : [0, 1]^2 \rightarrow \mathbb{C}$ , for which,  $f(s, \cdot) \in Y$  for all  $s \in [0, 1]$ , and  $g = g_f : [0, 1] \rightarrow \mathbb{R}$ ,  $s \mapsto \|f(s, \cdot)\|_Y$  is in  $X$ . The norm of  $f$  in  $X(Y)$  would then be  $\|f\|_{X(Y)} = \|g_f\|_X$ . It would be interesting to formulate general conditions on  $X$  and  $Y$ , which imply that  $X(Y)$  is primary, or has the factorization property (formulated below) with respect to some basis.

The above list may be expanded to the tri-parameter spaces, in which setting there has been little progress.

It is natural to study general conditions under which an operator  $T$  on a Banach space is a factor of the identity. A bounded linear operator  $T$  on a Banach space  $X$  with a Schauder basis  $(e_n)_n$  is said to have *large diagonal* if  $\inf_n |e_n^*(Te_n)| > 0$ . If every operator on  $X$  with large diagonal is a factor of the identity then we say that  $X$  has the *factorization property*. The study of the factorization property and that of primariness are closely related. Our proof does not directly show that the spaces under investigation have the

factorization property. We may therefore ask: for what Haar system spaces  $X$  and  $Y$  does the biparameter Haar system  $(h_I \otimes h_L)_{(I,L) \in \mathcal{D}^+ \times \mathcal{D}^+}$  have the factorization property in  $X(Y)$ ?

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