



The LLV decomposition of hyper-Kähler cohomology (the known cases and the general conjectural behavior)

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Abstract

Looijenga–Lunts and Verbitsky showed that the cohomology of a compact hyper-Kähler manifold X admits a natural action by the Lie algebra $\mathfrak{so}(4, b_2(X) - 2)$, generalizing the Hard Lefschetz decomposition for compact Kähler manifolds. In this paper, we determine the Looijenga–Lunts–Verbitsky (LLV) decomposition for all known examples of compact hyper-Kähler manifolds, and propose a general conjecture on the weights occurring in the LLV decomposition, which in particular determines strong bounds on the second Betti number $b_2(X)$ of hyper-Kähler manifolds (see Kim and Laza in Bull Soc Math Fr 148(3):467–480, 2020). Specifically, in the $K3^{[n]}$ and Kum_n cases, we give generating series for the formal characters of the associated LLV representations, which generalize the well-known Göttsche formulas for the Euler numbers, Betti numbers, and Hodge numbers for these series of hyper-Kähler manifolds. For the two exceptional cases of O’Grady (OG6 and OG10) we refine the known results on their cohomology. In particular, we note that the LLV decomposition leads

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to a simple proof for the Hodge numbers of hyper-Kähler manifolds of OG10 type. In a different direction, for all known examples of hyper-Kähler manifolds, we establish the so-called Nagai's conjecture on the monodromy of degenerations of hyper-Kähler manifolds. More consequentially, we note that Nagai's conjecture is a first step towards a more general and more natural conjecture, that we state here. Finally, we prove that this new conjecture is satisfied by the known types of hyper-Kähler manifolds.

1 Introduction

The compact hyper-Kähler manifolds are one of the most interesting building blocks in algebraic and complex geometry, as they are the most likely case to admit a good general classification. Indeed, they are K -trivial varieties, and among the three possible irreducible pieces for K -trivial varieties, they occupy the middle ground between complex tori (trivial classification) and Calabi–Yau manifolds (already too varied in dimension 3). Unfortunately, all that is known so far is a small list of examples of hyper-Kähler manifolds: two infinite series, $K3^{[n]}$ and Kum_n in dimension $2n$, due to Beauville [1], and two exotic examples, OG10 and OG6 in dimension 10 and 6 respectively, due to O'Grady [40,41]. Not only it is not known if this list is essentially complete, but even the finiteness of the deformation types in any dimension $2n(> 2)$ is a wide open question.

Verbitsky's Global Torelli Theorem [20,55] says that a hyper-Kähler manifold X is essentially determined by the Hodge structure on the second cohomology $H^2(X)$. While this is similar to saying that a complex torus A is determined by $H^1(A)$, in contrast to the case of tori, the reconstruction of X from its second cohomology $H^2(X)$ is very mysterious. In this paper, as a more tangible goal, we focus on reconstructing the entire cohomology $H^*(X)$ from the second cohomology $H^2(X)$ (at least for the known examples mentioned above). Our starting point is the work of Verbitsky [52–54] and Looijenga–Lunts [26] who have noted that, for hyper-Kähler manifolds, $H^*(X)$ admits a natural representation by the Lie algebra $\mathfrak{g} = \mathfrak{so}(4, b_2(X) - 2)$, generalizing the usual $\mathfrak{sl}(2)$ representation that occurs in the Hard Lefschetz Theorem. We call this Lie algebra \mathfrak{g} the *Looijenga–Lunts–Verbitsky (LLV) algebra* of X . The LLV algebra \mathfrak{g} is determined by the second cohomology. Namely, \mathfrak{g} is the special orthogonal algebra associated to the quadratic space $V := (H^2(X, \mathbb{R}), q_X) \oplus U$, where q_X is the Beauville–Bogomolov–Fujiki quadratic form on $H^2(X)$, and U is the standard hyperbolic plane (V is the *Mukai completion* of $H^2(X)$). By construction, the resulting decomposition, referred throughout as the *LLV decomposition*, of $H^*(X)$ into irreducible \mathfrak{g} representations is a diffeomorphism invariant of X , and thus only depends on the deformation class of X . Furthermore, all natural decompositions of the cohomology $H^*(X)$ factor through the LLV decomposition. Here, examples of such decompositions include the Hodge decomposition once a complex structure is fixed, the usual $\mathfrak{sl}(2)$ -Lefschetz decomposition once a Kähler form is fixed, and Verbitsky's $\mathfrak{so}(4, 1)$ -decomposition once a hyper-Kähler metric is fixed.

Notation 1.1 The irreducible \mathfrak{g} -representations are indexed by their highest weights, and the latter are (non-negative) integral linear combinations of the fundamental weights $\{\varpi_j\}$. An irreducible \mathfrak{g} -module with highest weight μ will be denoted by V_μ . Sometimes we can describe V_μ explicitly. For example, an irreducible \mathfrak{g} -module V_{ϖ_1} with highest weight ϖ_1 is the standard representation $V = H^2(X) \oplus \mathbb{R}^2$. Similarly, an irreducible \mathfrak{g} -module $V_{k\varpi_1}$ with highest weight $k\varpi_1$ is the largest irreducible \mathfrak{g} -submodule of $\text{Sym}^k V$ (more precisely, $V_{k\varpi_1}$ is the kernel of the contraction map $\text{Sym}^k V \rightarrow \text{Sym}^{k-2} V$ with respect to the quadratic form, an element of $\text{Sym}^2 V^*$, defining \mathfrak{g}).

The existence of the LLV decomposition has strong consequences on the cohomology of hyper-Kähler manifolds. For instance, Verbitsky and Bogomolov described explicitly the subalgebra of $H^*(X)$ generated by $H^2(X)$ (see [54] and [4]), and for many questions this knowledge suffices. From our perspective, we interpret this result as saying that for a $2n$ -dimensional hyper-Kähler manifold X , the irreducible \mathfrak{g} -submodule of $H^*(X)$ containing the second cohomology is the unique irreducible component isomorphic to $V_{n\varpi_1} (\subset \text{Sym}^n V)$. We call it the *Verbitsky component*. The Verbitsky component $V_{n\varpi_1}$ is always present (with multiplicity 1) in the LLV decomposition of $H^*(X)$. The remaining question is what other representations occur in the LLV decomposition of a hyper-Kähler manifold X , and what restrictions do they satisfy. While some general results are established, our paper is primarily concerned with the study of the *known* cases of hyper-Kähler manifolds X , by which we mean X is of $\text{K3}^{[n]}$, Kum_n , OG6, or OG10 type. By extrapolating from these known cases, we arrive to a general Conjecture 1.19 on the structure of LLV decomposition for general hyper-Kähler manifolds (see [21] for some important consequences).

1.1 The LLV decomposition for the known cases

The Betti and Hodge numbers of all the known cases of hyper-Kähler manifolds were previously worked out by other authors. Specifically, Götsche and Soergel [12, 14] have studied the Hodge structure $H^*(X)$ for the two infinite series $\text{K3}^{[n]}$ and Kum_n . More recently, the two exceptional O’Grady cases were settled by Mongardi–Rapagnetta–Saccà [35] for OG6 type, and by de Cataldo–Rapagnetta–Saccà [6] for OG10 type. While these previous results are closely related to the LLV decomposition, surprisingly the question of actually describing the LLV decomposition does not seem to have been addressed previously (except some low dimensional [≤ 6] cases). Our first result does exactly this.

Theorem 1.2 *The LLV decompositions of the known classes of hyper-Kähler manifolds are as follows:*

(i) *The generating series of the formal characters of the $\mathfrak{so}(4, 21)$ -modules $H^*(\text{K3}^{[n]})$ is*

$$1 + \left(\sum_{i=0}^{11} (x_i + x_i^{-1}) \right) q + \sum_{n=2}^{\infty} \text{ch} \left(H^*(\text{K3}^{[n]}) \right) q^n$$

$$= \prod_{m=1}^{\infty} \prod_{i=0}^{11} \frac{1}{(1 - x_i q^m)(1 - x_i^{-1} q^m)}. \quad (1.3)$$

(The identity should be understood in the formal power series ring $A[[q]]$, where

$$A = \mathbb{Z}[x_0^{\pm 1}, \dots, x_{11}^{\pm 1}, (x_0 \cdots x_{11})^{\pm \frac{1}{2}}]^{\mathfrak{W}}$$

is the complex representation ring of $\mathfrak{so}(4, 21)$, and \mathfrak{W} indicates the Weyl group of B_{12} .)

(ii) Define the formal power series

$$B(q) = \prod_{m=1}^{\infty} \left[\prod_{i=0}^3 \frac{1}{(1 - x_i q^m)(1 - x_i^{-1} q^m)} \prod_j (1 + x_0^{j_0} x_1^{j_1} x_2^{j_2} x_3^{j_3} q^m) \right],$$

with $j = (j_0, \dots, j_3) \in \{-\frac{1}{2}, \frac{1}{2}\}^{\times 4}$ and $j_0 + \dots + j_3 \in 2\mathbb{Z}$. Let b_1 be the degree 1 coefficient of $B(q) = 1 + b_1 \cdot q + b_2 \cdot q^2 + \dots$, and $J_4(d) = d^4 \cdot \prod_{p|d} (1 - \frac{1}{p^4})$ be the fourth Jordan totient function. With these notations, the generating series of the formal characters of the $\mathfrak{so}(4, 5)$ -modules $H^*(\text{Kum}_n)$ is

$$1 + \left(\sum_{i=0}^3 (x_i + x_i^{-1}) + 16 \right) q + \sum_{n=2}^{\infty} \text{ch}(H^*(\text{Kum}_n)) q^n = \sum_{d=1}^{\infty} J_4(d) \frac{B(q^d) - 1}{b_1 \cdot q}. \quad (1.4)$$

(Again, the identity holds in $A[[q]]$ where $A = \mathbb{Z}[x_0^{\pm 1}, \dots, x_3^{\pm 1}, (x_0 \cdots x_3)^{\pm \frac{1}{2}}]^{\mathfrak{W}}$ is the complex representation ring of $\mathfrak{so}(4, 5)$.)

(iii) As a $\mathfrak{so}(4, 6)$ -module,

$$H^*(\text{OG6}) = V_{3\varpi_1} \oplus V_{\varpi_3} \oplus V^{\oplus 135} \oplus \mathbb{R}^{\oplus 240}, \quad (1.5)$$

where V is the standard representation, V_{ϖ_3} is isomorphic to $\wedge^3 V$, and $\mathbb{R}^{\oplus 240}$ stands for 240 copies of the trivial representation.

(iv) As a $\mathfrak{so}(4, 22)$ -module,

$$H^*(\text{OG10}) = V_{5\varpi_1} \oplus V_{2\varpi_2}, \quad (1.6)$$

where $V_{2\varpi_2}$ is the largest irreducible submodule of $\text{Sym}^2(\wedge^2 V)$.

While our result relies heavily on the previous work on the cohomology of hyper-Kähler manifolds (in particular, [5, 12, 14, 35], and [6]), the structure of $H^*(X)$ described in Theorem 1.2 is more refined. This is especially clear in the case of hyper-Kähler manifolds of exceptional types OG6 and OG10 (see however also Remark 1.9 below for $K3^{[n]}$ and Kum_n). In particular, we note that the arguments of Sect. 3.4 together with [8] give an independent and conceptually easier proof of the main result of [6] (see Remark 3.30). In a different direction, we note that the “functorial” nature

of the LLV decomposition can be used to extract geometric information. Specifically, frequently there are natural subalgebras $\mathfrak{g}' \subset \mathfrak{g}$ (for instance $\mathfrak{g}' = \mathfrak{g}_{NS}$, the Neron–Severi algebra; see [26, (1.11)]) and one might be interested in the \mathfrak{g}' -module structure of $H^*(X)$; this is easily determined from the LLV decomposition by applying the restriction functor. Concretely, this idea is used in a forthcoming paper of the third author with G. Pearlstein and Z. Zhang to study some special class of OG10 manifolds with a symplectic involution (see [24] and [25] for some related work).

Remark 1.7 A more compact version of (1.3) is

$$\sum_{n=0}^{\infty} \text{ch}(H^*(\text{K3}^{[n]}))q^n = \prod_{m=1}^{\infty} \prod_{i=0}^{11} \frac{1}{(1-x_i q^m)(1-x_i^{-1} q^m)}, \quad (1.8)$$

by noting that formally

$$\text{ch}(H^*(\text{K3}^{[0]})) = 1, \quad \text{ch}(H^*(\text{K3}^{[1]})) = \sum_{i=0}^{11} (x_i + x_i^{-1}).$$

The reason for using (1.3) above is that $H^*(\text{K3}^{[1]})$ does not have a structure of $\mathfrak{so}(4, 21)$ -module. Similar discussion applies also to the second identity (1.4) for Kum_n hyper-Kähler manifolds.

Remark 1.9 As is often the case for infinite families, it is more convenient to work with the generating series (1.3) and (1.4) to encode the LLV module structure of the cohomology of $\text{K3}^{[n]}$ and Kum_n types. However, one can also determine their explicit LLV decompositions. We refer to Corollaries 3.2 and 3.6 for these explicit descriptions when $\dim X \leq 10$. Furthermore, the two generating series can be easily specialized to the generating series for the Hodge–Deligne polynomials, Poincaré polynomials, signatures of the middle cohomology, or the Euler numbers. We recover this way some well known formulas of Götsche [12, 13] (see Corollaries 3.8 and 3.9 for the $\text{K3}^{[n]}$ and Kum_n case respectively). In particular, as a specialization of (1.3), one gets

$$\sum_{n=0}^{\infty} e(H^*(\text{K3}^{[n]}))q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{24}} = \frac{q}{\Delta(q)} \quad (1.10)$$

(with $\Delta(q)$ the weight 12 modular form), which is equivalent to the Yau–Zaslow formula on the number of rational curves on a $K3$ (see [2]). On the other hand, note that since the Kum_n construction involves both the Hilbert scheme of $(n+1)$ points on an abelian surface A , and taking the fiber of the sum map $A^{[n+1]} \rightarrow A$, the associated formulas are automatically more involved. Nonetheless, we believe that our formula (1.4) and its specializations are improvements over the existing literature. In particular, the role played by $J_4(n+1)$ in controlling the trivial representations in $H^*(\text{Kum}_n)$ (and thus universal Hodge cycles of middle dimension) seems new.

Remark 1.11 After the completion of our manuscript, we have learned that Letao Zhang [57] computed the generating series for the characters of $H^*(K3^{[n]})$, viewed as modules with respect to the generic Mumford–Tate algebra, a subalgebra of the LLV algebra \mathfrak{g} (see Sect. 2.3). The results and methods involved are similar to those of Theorem 1.1 (1).

Remark 1.12 While the work of Verbitsky [52–54] and Looijenga–Lunts [26] is now more than two decade old, we are not aware of a serious exploration of the full power of the LLV decomposition for hyper-Kähler manifolds until recently (see however Moonen [34] for the case of abelian varieties). For instance, to our knowledge, the only cases where the LLV decomposition was previously described were $K3^{[n]}$ for $n \leq 3$ ($n = 3$ due to Markman [28]) and Kum_2 (cf. [26]). In contrast, in the past year there seem to have been a flurry of applications related to the LLV decomposition. Perhaps the most spectacular application is Oberdieck’s simplification [39] (see also [38]) of Maulik–Negut [29] proof of Beauville’s conjecture [3] for Hilbert scheme of points of $K3$ surfaces. Essentially, by lifting the action of the Neron–Severi algebra \mathfrak{g}_{NS} from $H^*(X)$ to the Chow groups $\text{CH}^*(X)$, one gets Beauville’s conjecture as a corollary of Schur’s lemma (N.B. the same idea was used by Moonen [34] for abelian varieties). Some other recent applications (in various directions) of the LLV decomposition include [17, 47, 50], and [8].

1.2 Nagai’s conjecture

The original motivation for our paper was the seemingly unrelated study of degenerations of hyper-Kähler manifolds and specifically the so-called Nagai conjecture [37]. Let \mathfrak{X}/Δ be a one-parameter projective degeneration of hyper-Kähler manifolds. Similar to the K3 case, it is natural to define the *Type* of the degeneration to be *I*, *II*, or *III*, in accordance to the index of nilpotence v_2 of the log monodromy operator $N_2 = \log(T_2)_u$ on $H^2(X)$. However, in contrast to the case of K3 surfaces, the hyper-Kähler manifolds X of dimension $2n > 2$ have interesting higher cohomology $H^k(X)$, and thus, it is natural to investigate the behavior of $H^k(X)$ under degenerations. In particular, it is natural to ask how the monodromies on various cohomologies are related to each other. As the hyper-Kähler manifolds are controlled by their second cohomology, one might expect some tight connection between the second monodromy and higher monodromies. For instance, as a consequence of the fact that the Verbitsky component $V_{n\varpi_1} \subset H^*(X)$ controls the holomorphic part of the cohomology, one sees (e.g. [22, §6.2]) that Type III degenerations of hyper-Kähler manifolds (defined in terms of H^2) are equivalent to maximal unipotent monodromy (MUM) degenerations (defined in terms of the middle cohomology H^{2n}). More generally, it is natural to expect that the index of nilpotency v_{2k} of log monodromy N_{2k} on $H^{2k}(X_t)$ satisfies

$$v_{2k} = k \cdot v_2 \quad \text{for } k = 1, \dots, n. \quad (1.13)$$

We refer to (1.13) as *Nagai’s conjecture*, as Nagai [37] was the first to investigate this question. In particular, he established (1.13) for $K3^{[n]}$ type degenerations and partially for Kum_n type degenerations. Nagai’s conjecture was also verified for Type

I and III degenerations of any hyper-Kähler manifolds in [22], leaving only the Type II case open. Here we establish Nagai's conjecture in full for all known examples of hyper-Kähler manifolds.

Theorem 1.14 *Let \mathfrak{X}/Δ be a one-parameter degeneration of hyper-Kähler manifolds such that the general fiber X_t is of $\mathrm{K3}^{[n]}$, Kum_n , OG6, or OG10 type. Then Nagai's conjecture (1.13) holds (with $\dim X_t = 2n$).*

The approach in [22] to Nagai's conjecture is based on studying Kulikov type normalizations of the degeneration \mathfrak{X}/Δ , using both general results from the minimal model program and specific results on hyper-Kähler geometry. The approach here is essentially orthogonal, focusing exclusively on the cohomological behavior (in particular, our results will say little about the geometric shape of the degeneration).

To start, we consider the interplay between the LLV decomposition of the cohomology $H^*(X)$ and the period map. First, it is known that $H^2(X)$ determines the Hodge structure on $H^*(X)$ by means of LLV \mathfrak{g} -representation (see Sect. 2 below; cf. also [49]). We realize this fact again, using the language of period maps and period domains. Specifically, we prove that for families of hyper-Kähler manifolds, the period map on the second cohomology $H^2(X)$ determines the period map on the entire cohomology $H^*(X)$ (Theorem 4.1). It then follows that the log monodromy N_k on the k^{th} cohomology is determined by N_2 via the LLV representation, a result previously noticed by Soldatenkov [48, Proposition 3.4] (by a different method). In conclusion, Nagai's conjecture reduces to a representation theoretic question. Namely, the given data is the nilpotency index of $N_2 \in \bar{\mathfrak{g}}$ acting on the standard $\bar{\mathfrak{g}}$ -module $\bar{V} = H^2(X)$. We are interested in the nilpotency index of $N_{2k} = \rho_k(N_2)$ where $\rho_{2k} : \bar{\mathfrak{g}} \rightarrow \mathrm{End}(H^{2k}(X))$ is the degree $2k$ restriction of the LLV representation. Using a representation theoretic computation we conclude that Nagai's condition (1.13) is equivalent to the following condition on the dominant \mathfrak{g} -weights occurring in the LLV decomposition. For the purpose of stating the precise result, it is convenient to break with convention (see Notation 1.1) and use instead the following notation.¹

Notation 1.15 The highest weight μ of an irreducible \mathfrak{g} -representation V_μ can be written as a linear combination

$$\mu = (\mu_0, \mu_1, \dots, \mu_r) = \sum_{i=0}^r \mu_i \varepsilon_i,$$

with $\{\pm \varepsilon_i\}$ the nonzero weights of the standard representation V . We refer the reader to (A.1) and (A.5) of the appendix for the precise relationship between $\{\varpi_i\}$ and $\{\varepsilon_i\}$. Here we note that the Verbitsky component $V_{n\varpi_1}$ is denoted $V_{(n)}$ in this notation.

Proposition 1.16 *Let X be a compact hyper-Kähler manifold of dimension $2n$ with $b_2(X) \geq 5$. Let $\mathfrak{g} = \mathfrak{so}(4, b_2 - 2)$ be the associated LLV algebra, and*

$$H^*(X) \cong \bigoplus_{\mu \in S} V_\mu^{\oplus m_\mu}, \quad (1.17)$$

¹ The reason for preferring the notation $\mu = (\mu_0, \mu_1, \dots, \mu_r)$ for indexing the representations is Hodge theoretic. Namely, each V_μ carries its own Hodge structure, and this can be more naturally captured in terms of the notation μ_i (e.g., see (2.31)).

be the decomposition of the cohomology of X into irreducible \mathfrak{g} -representations (where $\mu = (\mu_0, \dots, \mu_r)$ indicates a dominant integral weight of \mathfrak{g} , $r = \lfloor b_2(X)/2 \rfloor$, and V_μ the irreducible \mathfrak{g} -module of highest weight μ). Then Nagai's condition (1.13) holds if all the highest weights $\mu \in S$ in (1.17) contributing to even cohomology satisfy

$$\mu_0 + \mu_1 + \mu_2 \leq n. \quad (1.18)$$

We note that a converse statement holds under mild conditions (likely to be true in general). For a more detailed discussion, see Theorem 5.2 and the remark following it. Using this criterion and our computation of the LLV decomposition for the known cases (Theorem 1.2), we can conclude that Nagai's conjecture holds in all known examples of hyper-Kähler manifolds (Theorem 1.14).

1.3 A conjecture on the cohomology of hyper-Kähler manifolds

We now take a closer look at the representation theoretic formulation (1.18) of Nagai's conjecture. While we are able to check (1.18) holds under some strong assumptions (e.g. $\dim(X) \leq 8$), we were not able to verify (1.18) in general. We expect (1.18) to be a new condition on the cohomology of hyper-Kähler manifolds (it holds for the known cases, but we believe it to be an open question in general). In fact, trying to prove (1.18) in general, we have arrived to a heuristic of motivic nature which gives the stronger and natural condition (1.20) below. Informally, condition (1.20) says that *the Verbitsky's component $V_{(n)}$ is the dominant component of the LLV representation of $H^*(X)$* (see also Remark 6.3).

Conjecture 1.19 *Let X be a compact hyper-Kähler manifold of dimension $2n$. Then the weights $\mu = (\mu_0, \dots, \mu_r)$ occurring in the LLV decomposition (1.17) of $H^*(X)$ satisfy*

$$\mu_0 + \dots + \mu_{r-1} + |\mu_r| \leq n. \quad (1.20)$$

Remark 1.21 Since the Hodge decomposition on $H^*(X)$ factors through the LLV decomposition (see the discussion of Sect. 2.2), the LLV decomposition (1.17) can be also interpreted as a Hodge structure decomposition. More precisely, the (Mukai) Hodge structure on the standard \mathfrak{g} -representation V induces a Hodge structure on any representation V_μ . Taking into account the Hodge weights, the LLV decomposition (1.17) reads

$$H^*(X) \cong \bigoplus_{\mu \in S} V_\mu (\mu_0 + \dots + \mu_{r-1} + |\mu_r| - n)^{\oplus m_\mu}, \quad (1.22)$$

where as usual, the notation $W(k)$ means the Tate twist by $\mathbb{Q}(k)$ of the Hodge structure W (N.B. the Tate twist lowers the weights by $2k$, or more precisely $W(k)^{p,q} = W^{p+k, q+k}$). In other words, Conjecture 1.19 means that $H^*(X)$ is obtained from the Mukai completion of a weight 2 Hodge structure of K3 type in an effective way, allowing only positive twists by the Lefschetz motives \mathbb{L}^l (with $l = n - (\mu_0 + \dots + \mu_{r-1} + |\mu_r|) \geq 0$). For instance, the Hodge structure on the Hilbert scheme $S^{[n]}$ of

points on a $K3$ surface is obtained by considering various symmetric powers $S^{(a)}$ of S and blow-ups with center $S^{(a)}$ (see (3.15) for a precise formula).

As evidence for this conjecture, we note that it holds for all known types of hyper-Kähler manifolds.

Theorem 1.23 *Conjecture 1.19 holds for hyper-Kähler manifolds of $K3^{[n]}$, Kum_n , $\text{OG}6$, or $\text{OG}10$ type.*

Conjecture 1.19 has immediate consequences on the boundedness of the second Betti number for hyper-Kähler manifolds (a first step towards the boundedness of hyper-Kähler manifolds). This is discussed in detail in [21], which generalizes the Beauville–Guan [16] bound: $b_2(X) \leq 23$ for hyper-Kähler fourfolds (N.B. Conjecture 1.19 holds for fourfolds). Here, we only note the following consequence of the conjecture.

Corollary 1.24 *Let X be a hyper-Kähler manifold of dimension $2n$ for which Conjecture 1.19 holds. Assume that $b_2(X) \geq 4n$, then X has no odd cohomology.*

In particular, *a posteriori*, we see that Theorem 1.23 implies the vanishing of odd cohomology for $\text{OG}10$, which then determines the Hodge diamond for $\text{OG}10$ from little geometric data (see Theorem 3.26).

1.4 Structure of the paper

We start in Sect. 2 with a discussion of the LLV algebra \mathfrak{g} and its action on the cohomology $H^*(X)$. While this is mostly standard material, we make some small but important observations. For instance, we note the LLV algebra \mathfrak{g} is defined over \mathbb{Q} and describe its \mathbb{Q} -algebra structure. This allows us to relate the LLV algebra \mathfrak{g} to the special Mumford–Tate (MT) algebra $\bar{\mathfrak{m}}$. Note that \mathfrak{g} is a diffeomorphism invariant, while on the other hand $\bar{\mathfrak{m}}$ varies in moduli (it depends on the Hodge structure). For hyper-Kähler manifolds, one can enlarge the Lie algebra $\bar{\mathfrak{m}}$ to its Mukai completion \mathfrak{m} . We note that $\mathfrak{m} \subset \mathfrak{g}$, and thus it acts on the cohomology of X . Typically, by construction, one understands the Hodge structure on $H^*(X)$, or equivalently the decomposition of $H^*(X)$ with respect to \mathfrak{m} . One of our main tools for the proof of Theorem 1.2 (in Sect. 3) is to use representation theory to lift this \mathfrak{m} -representation to a \mathfrak{g} -representation.

For concreteness, let us briefly discuss this procedure for a $K3^{[n]}$ type hyper-Kähler manifold X . By definition, we can specialize X to the one isomorphic to $S^{[n]}$ for a $K3$ surface S . Then the formula of Götsche–Soergel expresses the cohomology of $H^*(X)$ in terms of the cohomology of $H^*(S)$ as Hodge structures. From our perspective, these results express the cohomology of $H^*(X)$ as a representation of the Mumford–Tate algebra $\bar{\mathfrak{m}}$. Moreover, taking S as a very general non-projective $K3$ surface (which is allowed by [5]), we can further assume $\bar{\mathfrak{m}} \cong \mathfrak{so}(3, 19)$. Then, one can easily construct the Mukai completion, which means, that, using the natural degree grading on cohomology, one can lift the $\bar{\mathfrak{m}} (\cong \mathfrak{so}(3, 19))$ -module structure of $H^*(X)$ to a $\mathfrak{m} (\cong \mathfrak{so}(4, 20))$ -module structure. However, we are still not done, as this Lie algebra

is still slightly smaller than the LLV algebra $\mathfrak{g} \cong \mathfrak{so}(4, 21)$ for $K3^{[n]}$. We now use a representation theory fact on restriction representations. Namely, the Lie algebras $\mathfrak{so}(4, 20)$ and $\mathfrak{so}(4, 21)$ are of type D_{12} and B_{12} respectively, in particular are of the same rank. It follows that the restriction representation functor $\text{Rep}(B_{12}) \rightarrow \text{Rep}(D_{12})$ is injective on the level of objects. Thus, there is a unique lift of the \mathfrak{m} -module structure on $H^*(X)$ to a \mathfrak{g} -module structure. In other words, we have lifted the Götsche–Soergel presentation of $H^*(X)$ as a Hodge structure to the LLV decomposition of $H^*(X)$.

The Kum_n case is similar but more complicated as it contains nonvanishing odd cohomology and many trivial representations. For example, to get a flavor of this phenomenon of an excessive amount of trivial representations, the reader can consider the Kummer surface S ; it has 16 independent Hodge cycles in $H^2(S)$, which in turn will lead to trivial representations. More generally, we notice that the Jordan totient function $J_4(n+1) = O(n^4)$ governs the number of trivial representations for Kum_n . The exceptional O’Grady’s 10-dimensional example is surprisingly easier to handle. The reason for this is that the LLV algebra $\mathfrak{so}(4, 22)$ is large compared to the Euler number $e(X)$. Once one knows that the odd cohomology vanishes (cf. [6, 8]), there is not much space remaining for the complement of the Verbitsky component in the LLV decomposition of $H^*(X)$. On the other hand, O’Grady’s 6-dimensional example is harder, as there are two combinatorial solutions matching the Hodge diamond of [35]. In order to find the right choice for the LLV representation in the OG6 case, we need to revisit the geometric construction of OG6 used in [43] and [35].

The second part of the paper is concerned with Nagai’s conjecture (1.13). First, in Sect. 4, we discuss the relationship between higher period maps and the LLV algebra. We note that, except some overlap with the work of Soldatenkov [48, 49], the material here is new and possibly of independent interest².

Our main representation theoretic criterion (Proposition 1.16) is established in Sect. 5. Using our computation of the LLV decomposition for the known cases (Theorem 1.2), we prove Theorem 1.23 in Sect. 6. As noted, this is a stronger version of Theorem 1.14, concluding the proof of Nagai’s conjecture for all known examples of hyper-Kähler manifolds.

For reader’s convenience, we briefly review some relevant representation theory facts in the Appendix.

2 The Looijenga–Lunts–Verbitsky algebra for hyper-Kähler manifolds

A compact *hyper-Kähler manifold* (aka irreducible holomorphic symplectic manifold) X is a simply connected compact Kähler manifold such that $H^0(X, \Omega_X^2)$ is generated by a global holomorphic symplectic 2-form σ . To fix the notation, X will denote a compact hyper-Kähler manifold (not necessarily projective) of dimension $2n$. Throughout the paper, we will use

$$\bar{V} = H^2(X, \mathbb{Q}), \quad \bar{q} : \bar{V} \rightarrow \mathbb{Q}$$

² (Note added in proof) We refer also to Looijenga [27, Sect. 4] (which appeared after a first version of our manuscript) for some further discussion of higher period maps.

for the second cohomology endowed with the (rational) Beauville–Bogomolov quadratic form \bar{q} of X (i.e. $\bar{q}(x)^n = c \cdot x^{2n}$, for some constant c). Note that the quadratic space (\bar{V}, \bar{q}) (and the associated Lie algebra $\mathfrak{so}(\bar{V}, \bar{q})$) are diffeomorphism invariants of X . The purpose of this section is to introduce, following Verbitsky [53] and Looijenga–Lunts [26], the *Looijenga–Lunts–Verbitsky (LLV) algebra* \mathfrak{g} , which enhances $\mathfrak{so}(\bar{V}, \bar{q})$. The LLV algebra \mathfrak{g} acts naturally on the cohomology algebra $H^*(X)$, giving rise to a more refined diffeomorphism invariant of X , the *LLV decomposition* of the cohomology. After reviewing the basic structure and properties of \mathfrak{g} and its action on cohomology, we discuss the interplay between \mathfrak{g} and the natural Hodge structure on $H^*(X)$ (assuming a complex structure on X was fixed) and the associated Mumford–Tate algebra \bar{m} (an analytic invariant).

2.1 Looijenga–Lunts–Verbitsky algebra

The LLV algebra \mathfrak{g} and the associated LLV decomposition of $H^*(X)$ generalize the usual hard Lefschetz $\mathfrak{sl}(2)$ -decomposition of the cohomology in the presence of a Kähler class ω on X (for the moment X can be any compact Kähler manifold). Specifically, recall that ω defines two operators, the Lefschetz operator $L_\omega = \omega \cup \underline{}$, and the inverse Lefschetz operator $\Lambda_\omega (= \star^{-1} L_\omega \star)$. Then, L_ω and Λ_ω generate an $\mathfrak{sl}(2) \subset \mathfrak{gl}(H^*(X))$ acting on $H^*(X)$. Hard Lefschetz is equivalent to the resulting $\mathfrak{sl}(2)$ -decomposition of the cohomology. Looijenga–Lunts [26] have formalized this process and avoided the use of the Hodge star operator \star . To start, note that

$$[L_\omega, \Lambda_\omega] = h,$$

where h is the degree operator

$$\begin{aligned} h : H^*(X, \mathbb{Q}) &\rightarrow H^*(Y, \mathbb{Q}), \\ x &\mapsto (k - \dim X)x \quad \text{for } x \in H^k(X, \mathbb{Q}). \end{aligned} \tag{2.1}$$

It follows that $\{L_\omega, h, \Lambda_\omega\}$ is an $\mathfrak{sl}(2)$ -triple. The operator $L_x = x \cup \underline{}$ is well defined for any cohomology class $x \in H^2(X)$, while h is independent of any choice. The key observation now is that, due to the Jacobson–Morozov Theorem, the existence of an operator (automatically unique) Λ_x completing $\{L_x, h\}$ to an $\mathfrak{sl}(2)$ -triple is an open algebraic condition on the classes $x \in H^2(X)$. Thus, the dual Lefschetz operator Λ_x can be defined for almost all classes x , independent of being a Kähler class, or even of the complex structure of X . This allows to define a Lie algebra \mathfrak{g} (clearly definable over \mathbb{Q}) containing all these operators. By construction, \mathfrak{g} is a diffeomorphism invariant of X , \mathfrak{g} acts on $H^*(X)$, and any $\mathfrak{sl}(2)$ Lefschetz decomposition factors through \mathfrak{g} . The Kähler assumption is needed only to conclude that the set of $x \in H^2(X)$ for which Λ_x is defined is a *non-empty* (and thus dense) open Zariski set.

Definition 2.2 [26] Let X be a compact Kähler manifold (not necessarily hyper-Kähler). The *Looijenga–Lunts–Verbitsky (LLV) algebra*³ $\mathfrak{g}(X)$ of X is the Lie

³ $\mathfrak{g}(X)$ is called the *total Lie algebra* of X in [26], and denoted by $\mathfrak{g}_{tot}(X)$. Another natural algebra considered by [26] is the LLV Neron–Severi algebra $\mathfrak{g}_{NS}(X)$ which is generated by Lefschetz L_x and

subalgebra of $\mathfrak{gl}(H^*(X, \mathbb{Q}))$ generated by all formal Lefschetz and dual Lefschetz operators $L_x, \Lambda_x \in \mathfrak{gl}(H^*(X, \mathbb{Q}))$ associated to almost all elements $x \in H^2(X, \mathbb{Q})$.

The LLV algebra $\mathfrak{g}(X)$ is a semisimple Lie algebra defined over \mathbb{Q} (cf. [26, (1.9)]). We are interested in its structure and action on cohomology when X is a compact hyper-Kähler manifold (which we assume from now on). For notational simplicity, we write

$$\mathfrak{g} = \mathfrak{g}(X)$$

if no confusion is likely occur.

2.1.1 The structure of the LLV algebra (over \mathbb{Q}) for hyper-Kähler manifolds

The semisimple degree operator $h \in \mathfrak{g}$ induces an eigenspace decomposition of \mathfrak{g} . In the case of hyper-Kähler manifolds, only degrees 2, 0, and -2 occur⁴ and thus we have an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-2} \quad (2.3)$$

with respect to h acting on \mathfrak{g} by the adjoint action. The 0-eigenspace \mathfrak{g}_0 is a reductive subalgebra of \mathfrak{g} , which can be then decomposed as

$$\mathfrak{g}_0 = \bar{\mathfrak{g}} \oplus \mathbb{Q} \cdot h, \quad (2.4)$$

where $\bar{\mathfrak{g}}$ is the semisimple part ($\bar{\mathfrak{g}} = [\mathfrak{g}_0, \mathfrak{g}_0]$), and the 1-dimensional center $\mathfrak{z}(\mathfrak{g}_0)$ is spanned by the degree operator h . We refer to $\bar{\mathfrak{g}}$ as the *reduced LLV algebra* of X . Since $\bar{\mathfrak{g}} \subset \mathfrak{g}_0$ consists of degree 0 operators, the induced $\bar{\mathfrak{g}}$ -action on $H^*(X)$ preserves the degree. That is, we have a representation

$$\rho_k : \bar{\mathfrak{g}} \rightarrow \text{End}(H^k(X, \mathbb{Q})). \quad (2.5)$$

In particular, $\bar{\mathfrak{g}}$ acts on $H^2(X)$. On the other hand, it preserves the cup product as a derivation:

$$e.(x \cup y) = (e.x) \cup y + x \cup (e.y) \quad \text{for } e \in \bar{\mathfrak{g}}, \quad x, y \in H^*(X, \mathbb{Q}). \quad (2.6)$$

Together with the Fujiki relation $\bar{q}(x)^n = cx^{2n}$, one concludes that it also respects the Beauville–Bogomolov form \bar{q} on $H^2(X)$. That is, we have

$$\bar{\mathfrak{g}} \subset \mathfrak{so}(\bar{V}, \bar{q}),$$

inverse Lefschetz Λ_x operators for $x \in \text{NS}(X)$. Obviously, $\mathfrak{g}_{\text{NS}} \subset \mathfrak{g} (= \mathfrak{g}_{\text{tot}})$. We do not discuss \mathfrak{g}_{NS} in this paper.

⁴ In general, the Lefschetz operators L_η commute, but the dual Lefschetz operators Λ_η do not. For hyper-Kähler manifolds and abelian varieties, Λ_η do commute, resulting in the restricted range of weights.

where \bar{q} is the Beauville–Bogomolov form on the second cohomology $\bar{V} = H^2(X, \mathbb{Q})$ as before. In fact, the equality holds, and \bar{V} is the standard representation of $\bar{\mathfrak{g}}$. More precisely, \mathfrak{g} and $\bar{\mathfrak{g}}$ can be described as follows:

Theorem 2.7 (Looijenga–Lunts, Verbitsky). *Let X be a compact hyper-Kähler manifold. Then the LLV and reduced LLV algebras of X are described by*

$$\bar{\mathfrak{g}} \cong \mathfrak{so}(\bar{V}, \bar{q}), \quad (2.8)$$

$$\mathfrak{g} \cong \mathfrak{so}\left(\bar{V} \oplus \mathbb{Q}^2, \bar{q} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \quad (2.9)$$

In particular,

$$\bar{\mathfrak{g}}_{\mathbb{R}} \cong \mathfrak{so}(3, b_2(X) - 3) \quad \text{and} \quad \mathfrak{g}_{\mathbb{R}} \cong \mathfrak{so}(4, b_2(X) - 2). \quad (2.10)$$

Proof The isomorphism over \mathbb{R} is [26, Proposition 4.5] (also [53]). The isomorphism over \mathbb{Q} is not clearly addressed in the literature, so we provide the details here. For the reduced LLV algebra $\bar{\mathfrak{g}}$, since $\mathfrak{g} \subset \mathfrak{so}(\bar{V}, \bar{q})$ (both defined over \mathbb{Q}), the equality follows by dimension reasons.

For the identification of \mathfrak{g} we use the description from [23, Lemma 3.9] of \mathfrak{g} in terms of the subalgebra $\bar{\mathfrak{g}}$. Starting from the decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus (\bar{\mathfrak{g}} \oplus \mathbb{Q}h) \oplus \mathfrak{g}_2,$$

one sees that $\bar{\mathfrak{g}} \cong \mathfrak{so}(\bar{V}, \bar{q}) \cong \wedge^2 \bar{V}$, and $\mathfrak{g}_{\pm 2} \cong \bar{V}$ as $\bar{\mathfrak{g}}$ -representations (e.g. \mathfrak{g}_2 is generated by commuting Lefschetz operators L_x for $x \in H^2(X) = \bar{V}$). Identifying $\mathfrak{g}_{\pm 2}$ with \bar{V} , and $\bar{\mathfrak{g}}$ with $\wedge^2 \bar{V}$ by the rule

$$a \wedge b \mapsto \frac{1}{2}(\bar{q}(a, -) \otimes b - \bar{q}(b, -) \otimes a)$$

(recall $a, b \in \bar{V} = H^2(X, \mathbb{Q})$), we have the following bracket rules (which determine \mathfrak{g} starting from $\bar{\mathfrak{g}}$):

(1) The obvious grading relations:

- $[h, a] = -2a$, $[h, b] = 2b$, $[h, e] = 0$ for $a \in \mathfrak{g}_{-2}$, $b \in \mathfrak{g}_2$, $e \in \bar{\mathfrak{g}}$;
- $[a, a'] = 0$ for $a, a' \in \mathfrak{g}_{-2}$. $[b, b'] = 0$ for $b, b' \in \mathfrak{g}_2$.

(2) The identifications $\mathfrak{g} = \wedge^2 \bar{V}$ and $\mathfrak{g}_{\pm 2} = \bar{V}$ are as $\bar{\mathfrak{g}}$ -representations, i.e.

- $[e, e']$ for $e, e' \in \bar{\mathfrak{g}}$ is defined by the Lie bracket operation on $\bar{\mathfrak{g}}$;
- $[e, a] = e.a \in \mathfrak{g}_{-2}$, $[e, b] = e.b \in \mathfrak{g}_2$ for $a \in \mathfrak{g}_{-2}$, $b \in \mathfrak{g}_2$, $e \in \bar{\mathfrak{g}}$.

(3) Finally, the interesting cross-term relation:

- $[a, b] = a \wedge b + \bar{q}(a, b)h \in \mathfrak{g}_0$ for $a \in \mathfrak{g}_{-2}$, $b \in \mathfrak{g}_2$.

All of these bracket relations are defined over \mathbb{Q} . On the other hand, we note that the bracket relations above are exactly the same as those for $\mathfrak{so}(\bar{V} \oplus \mathbb{Q}^2, \bar{q} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$

described in terms of $\mathfrak{so}(\bar{V}, \bar{q})$. (Recall, in particular, that as $\mathfrak{so}(\bar{V}, \bar{q})$ -representation it holds $\wedge^2(\bar{V} \oplus \mathbb{Q}^2) = \bar{V} \oplus (\wedge^2 \bar{V} \oplus \mathbb{Q}) \oplus \bar{V}$.) \square

Example 2.11 If X is a K3 surface, $H^*(X)$ is naturally endowed with the Mukai pairing $H^*(X, \mathbb{Q}) \cong H^2(X, \mathbb{Q}) \oplus U$, even defined over \mathbb{Z} . In terms of representations, $H^2(X, \mathbb{Q})$ is the standard representation of $\bar{\mathfrak{g}}$ (whose real form is $\mathfrak{so}(3, 19)$), and $H^*(X, \mathbb{Q})$ is the standard representation of \mathfrak{g} (whose real form is $\mathfrak{so}(4, 20)$). Note that $\bar{\mathfrak{g}}$ is achieved by the generic special Mumford–Tate algebra of a K3 surface.

Interesting things happen for the case of Kummer surfaces, which can be considered as the case $n = 1$ in the series Kum_n . A Kummer surface is a K3 surface, so it has $\bar{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{so}(3, 19)$ as above. However, by construction it always contains 16 independent (-2) -curves. Due to this fact, its generic special Mumford–Tate algebra has a real form $\mathfrak{so}(3, 3)$. As a result, letting \bar{V} be its standard representation, we have $H^*(X, \mathbb{Q}) = H^0 \oplus (\bar{V} \oplus \mathbb{Q}^{16}) \oplus H^4$. This explains the meaning of the degree 1 term in Theorem 1.2(2).

Motivated by Theorem 2.7 above, we would rather like to consider the *Mukai completion*

$$V = \bar{V} \oplus \mathbb{Q}^2, \quad q = \bar{q} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of $\bar{V} (= H^2(X, \mathbb{Q}))$ as a more natural object associated to the cohomology of X . Theorem 2.7 says that \bar{V} is the standard representation of the reduced LLV algebra $\bar{\mathfrak{g}}$, while the Mukai completion V is the standard representation of \mathfrak{g} (N.B. only for K3 surfaces, $H^*(X, \mathbb{Q}) \cong V$).

Corollary 2.12 *Let X be a hyper-Kähler manifold, and $r = \lfloor b_2(X)/2 \rfloor$. The LLV algebra \mathfrak{g} is a simple Lie algebra of type B_{r+1} or D_{r+1} , depending on the parity of $b_2(X)$. Its reduced form $\bar{\mathfrak{g}}$ is a simple Lie algebra of type B_r or D_r .* \square

2.1.2 The LLV decomposition

The Looijenga–Lunts–Verbitsky algebra \mathfrak{g} is by definition a subalgebra of $\mathfrak{gl}(H^*(X, \mathbb{Q}))$ – it acts on the full cohomology $H^*(X, \mathbb{Q})$. Since \mathfrak{g} consists of only even degree operators (2.3), this action preserves the even and odd cohomology; that is, the action of \mathfrak{g} preserves the direct sum

$$H^*(X, \mathbb{Q}) = H_{\text{even}}^*(X, \mathbb{Q}) \oplus H_{\text{odd}}^*(X, \mathbb{Q}). \quad (2.13)$$

Since \mathfrak{g} is semisimple, the decomposition (2.13) may be further refined. We have

$$H^*(X, \mathbb{Q}) = \bigoplus_{\mu} V_{\mu}^{\oplus m_{\mu}}, \quad (2.14)$$

with V_{μ} the irreducible \mathfrak{g} module of highest weight μ . We call (2.14) the *LLV decomposition*; it is a basic diffeomorphism invariant of X .

With the notation of Appendix A, we write $\mu = (\mu_0, \dots, \mu_r)$ to indicate that $\mu = \sum_i \mu_i \varepsilon_i$. (Here ε are weights of the standard representation V .) For example, $V_{(n)}$ is the “largest” irreducible subrepresentation of $\text{Sym}^n V$.

Theorem 2.15 (Verbitsky). *Let X be a compact hyper-Kähler manifold X of dimension $2n$. Then the subalgebra $SH^2(X, \mathbb{Q}) \subset H^*(X, \mathbb{Q})$ generated by $H^2(X, \mathbb{Q})$ is an irreducible \mathfrak{g} -module $V_{(n)} \subset \text{Sym}^n V$ of highest weight $\mu = (n)$.*

Proof By definition $SH^2(X, \mathbb{Q})$ is the subalgebra generated by the second cohomology. Hence, every element in this subalgebra can be expressed by a linear combination of the product $x_1 \cdots x_k$ of elements in the second cohomology $x_i \in H^2(X, \mathbb{Q})$. Now from (2.6), one directly sees that $SH^2(X, \mathbb{Q})$ is $\bar{\mathfrak{g}}$ -invariant. Let us further show that it is in fact invariant under the full \mathfrak{g} -action.

Recall the decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus (\bar{\mathfrak{g}} \oplus \mathbb{Q}h) \oplus \mathfrak{g}_2$ in (2.3). The algebra $SH^2(X, \mathbb{Q})$ is clearly h -invariant. Thus it is enough to show $SH^2(X, \mathbb{Q})$ is closed under the \mathfrak{g}_2 -action and \mathfrak{g}_{-2} -action. Any element in \mathfrak{g}_2 is of the form L_x for $x \in H^2(X, \mathbb{Q})$, the multiplication operator by x . Hence $SH^2(X, \mathbb{Q})$ is closed under L_x by definition. The vector space \mathfrak{g}_{-2} is generated by the operators Λ_x for $x \in H^2(X, \mathbb{Q})$. To prove $SH^2(X, \mathbb{Q})$ is closed under Λ_x , we need the following standard trick in representation theory. For any $x_1, \dots, x_k \in H^2(X, \mathbb{Q})$, we have

$$\Lambda_x(x_1 x_2 \cdots x_k) = \Lambda_x(L_{x_1}(x_2 \cdots x_k)) = [L_{x_1}, \Lambda_x](x_2 \cdots x_k) - L_{x_1}(\Lambda_x(x_2 \cdots x_k)).$$

Since $[L_{x_1}, \Lambda_x] \in \mathfrak{g}_0 = \bar{\mathfrak{g}} \oplus \mathbb{Q}h$, we know the first component is contained in $SH^2(X, \mathbb{Q})$. Hence to prove $\Lambda_x(x_1 x_2 \cdots x_k) \in SH^2(X, \mathbb{Q})$, it is enough to prove $\Lambda_x(x_2 \cdots x_k)$ is contained in $SH^2(X, \mathbb{Q})$. Now use the induction on k . This proves $SH^2(X, \mathbb{Q})$ is closed under Λ_x , and hence closed under the full \mathfrak{g} -action.

By restricting to $\bar{\mathfrak{g}} \subset \mathfrak{g}$, we may regard $SH^2(X, \mathbb{Q})$ as a $\bar{\mathfrak{g}}$ -representation. From that perspective, Verbitsky [54] showed that

$$SH^2(X, \mathbb{Q})_{2k} = \begin{cases} \text{Sym}^k H^2(X, \mathbb{Q}) & \text{if } 0 \leq k \leq n \\ \text{Sym}^{2n-k} H^2(X, \mathbb{Q}) & \text{if } n < k \leq 2n \end{cases}. \quad (2.16)$$

In particular,

$$SH^2(X, \mathbb{Q}) = \text{Sym}^n \bar{V} \oplus (\text{Sym}^{n-1} \bar{V})^{\oplus 2} \oplus \cdots \oplus \mathbb{Q}^{\oplus 2}$$

as a $\bar{\mathfrak{g}}$ -module. Then the branching rules for $\bar{\mathfrak{g}} \subset \mathfrak{g}$ (§B.2) force $SH^2(X, \mathbb{Q}) = V_{(n)}$ as a \mathfrak{g} -representation. \square

Remark 2.17 Bogomolov [4] showed that

$$SH^2(X, \mathbb{Q}) \cong \text{Sym}^*(H^2(X, \mathbb{Q}))/\langle x^{n+1} : x \in H^2(X, \mathbb{Q}), \bar{q}(x) = 0 \rangle \quad (2.18)$$

as algebras.

Definition 2.19 We call $SH^2(X, \mathbb{Q}) \cong V_{(n)}$ the *Verbitsky component* of $H(X, \mathbb{Q})$.

Since \mathfrak{g} is semisimple, the cohomology admits a \mathfrak{g} -module decomposition

$$H^*(X, \mathbb{Q}) = V_{(n)} \oplus V'. \quad (2.20)$$

One of our goals in the paper is to describe the complement V' for the known cases of compact hyper-Kähler manifolds X (cf. Sect. 3). For arbitrary hyper-Kähler manifolds X we will see that the multiplicity of $V_{(n)}$ in $H^*(X, \mathbb{Q})$ is one (Proposition 2.32); equivalently, V' does not contain an irreducible \mathfrak{g} -module of highest weight $\mu = (n)$.

Remark 2.21 As the proof of Theorem 2.15 indicates it is sometimes convenient to restrict the \mathfrak{g} -action on $H^*(X)$ to a $\bar{\mathfrak{g}}$ -action (2.5), and apply branching rules. This argument will reappear again throughout the paper. Often, this restricted action is easier to understand. However, one of our main conclusions here is that it is better to consider the action of the larger algebra $\mathfrak{g} \supset \bar{\mathfrak{g}}$. This is essentially because the larger algebra encodes more symmetries; as a \mathfrak{g} -module, the cohomology admits fewer irreducible subrepresentations.

2.2 Further decompositions of the cohomology

We now discuss some finer decompositions of the cohomology which are obtained once certain choices have been made. For instance, the choice of complex structure determines a Hodge structure on $H^*(X)$ (which can be regarded as a decomposition with respect to the Deligne torus $\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$). Similarly, the choice of a twistor family (or equivalently a hyper-Kähler metric) determines an $\mathfrak{so}(4, 1)$ -decomposition of the cohomology $H^*(X)$, originally discovered by Verbitsky [52, 53]. Either of these finer decompositions factor through the LLV algebra, and in a certain sense the LLV algebra is the smallest subalgebra of $\mathfrak{gl}(H^*(X, \mathbb{R}))$ containing all these decompositions. More precisely, the LLV algebra \mathfrak{g} is generated by the (generic) Mumford–Tate algebra $\bar{\mathfrak{m}}$ (see Sect. 2.3) and Verbitsky’s algebra $\mathfrak{so}(4, 1)$ (see Sect. 2.2.2). Furthermore, the LLV algebra has the advantage of being defined over \mathbb{Q} .

2.2.1 Complex structures and Looijenga–Lunts–Verbitsky algebra

So far, the complex structure on X was not used in our discussion. In this subsection, we would like to take the complex structure into account and understand how it interacts with the LLV algebra \mathfrak{g} . Verbitsky’s Global Torelli for compact hyper-Kähler manifolds implies that the complex structure on X is captured by the Hodge structure on the cohomology $H^*(X, \mathbb{Q})$, and in fact $H^2(X, \mathbb{Q})$, up to some finite ambiguity.

Given a $2n$ -dimensional hyper-Kähler manifold X , we have a degree operator $h \in \mathfrak{g}_0 \subset \mathfrak{g}$

$$h : H^*(X, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q}), \quad x \mapsto (k - 2n)x \quad \text{for } x \in H^k(X, \mathbb{Q}). \quad (2.22)$$

Assuming a complex structure on X was fixed, we obtain a second operator $f \in \mathfrak{gl}(H^*(X, \mathbb{R}))$ defined by

$$\begin{aligned} f : H^*(X, \mathbb{R}) &\rightarrow H^*(X, \mathbb{R}), \\ x &\mapsto (q - p)\sqrt{-1}x \quad \text{for } x \in H^{p,q}(X), \end{aligned} \quad (2.23)$$

capturing the Hodge structure of the cohomology. While h and \mathfrak{g} are defined over \mathbb{Q} , f is in general only defined over \mathbb{R} . One sees that $f \in \mathfrak{g}_{\mathbb{R}} \subset \mathfrak{gl}(H^*(X, \mathbb{R}))$, and, in fact, a stronger statement holds.

Proposition 2.24 *The operator $f \in \mathfrak{gl}(H^*(X, \mathbb{R}))$ in (2.23) is contained in $\bar{\mathfrak{g}}_{\mathbb{R}}$ as a semisimple element.*

Proof Fix a hyper-Kähler metric g on X inducing the twistor complex structures I, J, K with I being the original complex structure of X . We can associate to the complex structures I, J, K the Kähler classes $\omega_I = g(I-, -)$, $\omega_J = g(J-, -)$ and $\omega_K = g(K-, -)$ (with $\omega_I, \omega_J, \omega_K \in H^2(X, \mathbb{R})$). Let L_I, L_J, L_K be the Lefschetz operators and $\Lambda_I, \Lambda_J, \Lambda_K$ the dual Lefschetz operators associated to them.

Let $f \in \mathfrak{gl}(H^*(X, \mathbb{R}))$ be the Hodge operator as in (2.23). Verbitsky [52] showed that

$$f = -[L_J, \Lambda_K] = -[L_K, \Lambda_J] \quad \text{on } H^*(X, \mathbb{R}).$$

Thus, f is contained in $\mathfrak{g}_{\mathbb{R}}$ by Definition 2.2. Since f is a degree 0 operator, we have in fact $f \in \mathfrak{g}_{0, \mathbb{R}}$. One can similarly define the operators $f_J, f_K \in \mathfrak{g}_{0, \mathbb{R}}$ for the Hodge structures of other complex structures J and K , with Verbitsky's relations $f_J = -[L_K, \Lambda_I]$ and $f_K = -[L_I, \Lambda_J]$. By symmetry, we get $f_J, f_K \in \mathfrak{g}_{0, \mathbb{R}}$. Now using the Jacobi identities and the relations above, we get

$$\begin{aligned} [f_J, f_K] &= [[L_K, \Lambda_I], [L_I, \Lambda_J]] \\ &= [L_K, [\Lambda_I, [L_I, \Lambda_J]]] - [\Lambda_I, [L_K, [L_I, \Lambda_J]]] \\ &= [L_K, [[\Lambda_I, L_I], \Lambda_J]] + [L_K, [L_I, [\Lambda_I, \Lambda_J]]] - [\Lambda_I, [[L_K, L_I], \Lambda_J]] \\ &\quad - [\Lambda_I, [L_I, [L_K, \Lambda_J]]] \\ &= [L_K, [-h, \Lambda_J]] + [L_K, [L_I, 0]] - [\Lambda_I, [0, \Lambda_J]] - [\Lambda_I, [L_I, -f]] \\ &= 2[L_K, \Lambda_J] - [\Lambda_I, 0] = -2f. \end{aligned}$$

We conclude $f \in [\mathfrak{g}_{0, \mathbb{R}}, \mathfrak{g}_{0, \mathbb{R}}] = \bar{\mathfrak{g}}_{\mathbb{R}}$. Finally, f is a semisimple element of $\bar{\mathfrak{g}}_{\mathbb{R}}$ since f acts diagonalizably on the faithful $\bar{\mathfrak{g}}_{\mathbb{R}}$ -representation $H^2(X, \mathbb{R})$. \square

Now we have two operators $h \in \mathfrak{g}$ and $f \in \mathfrak{g}_{\mathbb{R}}$. The action $h \in \mathfrak{g}$ on the standard \mathfrak{g} -module V induces an h -eigenspace decomposition

$$V = V_{-2} \oplus V_0 \oplus V_2, \quad \dim V_{\pm 2} = 1, \quad V_0 = \bar{V}. \quad (2.25)$$

Here the lower indexes indicate the eigenvalues of h . Similarly, the action $f \in \mathfrak{g}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ induces a f -eigenspace decomposition

$$V_{\mathbb{C}} = V_{\mathbb{C}, -2\sqrt{-1}} \oplus V_{\mathbb{C}, 0} \oplus V_{\mathbb{C}, 2\sqrt{-1}}, \quad \dim V_{\mathbb{C}, \pm 2\sqrt{-1}} = 1. \quad (2.26)$$

Since $h, f \in \mathfrak{g}_{\mathbb{C}}$ are commuting semisimple elements, there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ containing both h and f . Recall that \mathfrak{g} is a simple Lie algebra of rank $r + 1$,

so its Cartan subalgebra \mathfrak{h} has dimension $r + 1$. We will use the notation $\varepsilon_0, \dots, \varepsilon_r$ to denote our preferred choice of a basis of \mathfrak{h} in Appendix A. Note that we start the index from 0. Now the h and f -eigenspace decompositions above have the following interpretation. This was already appeared in the discussion of [48, §3.4].

Lemma 2.27 *Let $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ be a Cartan subalgebra containing both h and f . Then we must have*

$$h = \pm \varepsilon_i^{\vee}, \quad \sqrt{-1}f = \pm \varepsilon_j^{\vee} \quad \text{for some } i \neq j.$$

Proof The idea here is essentially the same as in Deligne's approach to the classification of Hermitian symmetric domains (see, e.g., Milne's note [31, p.12]). By definition, the weights $\varepsilon_0, \dots, \varepsilon_r$ are obtained by the weight decomposition of the standard \mathfrak{g} -module V (see Appendix A). More specifically, we have a weight decomposition with respect to the chosen Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$

$$\begin{aligned} V_{\mathbb{C}} &= V(\pm \varepsilon_0) \oplus \cdots \oplus V(\pm \varepsilon_r) \quad \text{or} \quad V(0) \\ &\oplus V(\pm \varepsilon_0) \oplus \cdots \oplus V(\pm \varepsilon_r), \quad \text{depending on the parity of } \dim V, \end{aligned}$$

where $V(\theta)$ denotes the weight θ subspace of $V_{\mathbb{C}}$. As an element in \mathfrak{h} , h acts on the weight space $V(\theta)$ by multiplication $\langle \theta, h \rangle$. Now by (2.25), this implies $\langle \theta, h \rangle = 0, -2, 2$ for $\theta = \pm \varepsilon_0, \dots, \pm \varepsilon_r$ and there is only one ε_i with $\langle \varepsilon_i, h \rangle = \pm 2$. This forces $h = \pm \varepsilon_i^{\vee}$ for some $i = 0, \dots, r$.

Same idea applies to f , but this time we need a coefficient $\sqrt{-1}$ as the eigenvalues of f are $0, \pm 2\sqrt{-1}$ in (2.26). Hence we deduce $\sqrt{-1}f = \pm \varepsilon_j^{\vee}$ for some $j = 0, \dots, r$. Here i and j cannot be the same, as certainly h and f are linearly independent. \square

Thanks to this lemma, after choosing an appropriate positive Weyl chamber, we may assume

$$h = \varepsilon_0^{\vee}, \quad \sqrt{-1}f = \varepsilon_1^{\vee}. \quad (2.28)$$

From now on, we fix an appropriate positive Weyl chamber so that we can use this condition freely.

Having discussed the complex structure, we may now consider the Hodge diamond of $H^*(X)$. Note that the Hodge diamond is in fact derived from a Hodge structure, which is again captured by the action of the operators $h = \varepsilon_0^{\vee}$ and $f = \frac{1}{\sqrt{-1}}\varepsilon_1^{\vee}$. An interesting conclusion is that *any* \mathfrak{g} -module, or $\bar{\mathfrak{g}}$ -module if we ignore the weight, possesses its own Hodge structure and hence its own Hodge diamond. Let us elaborate this fact a bit more precisely.

Consider the weight decomposition of a \mathfrak{g} -module V_{μ} . It is of the form

$$V_{\mu, \mathbb{C}} = \bigoplus_{\theta} V_{\mu}(\theta),$$

where $V_{\mu}(\theta)$ denotes the weight θ vector subspace of $V_{\mu, \mathbb{C}}$. The Hodge decomposition is obtained by the (h, f) -eigenspace decomposition. Namely, the Hodge decomposition of V_{μ} is

$$V_{\mu, \mathbb{C}} = \bigoplus_{p,q} V_{\mu}^{p,q}, \quad (2.29)$$

where $V_\mu^{p,q}$ is the (h, f) -eigenspace on which h acts by multiplication $p + q - 2n$ and f acts by multiplication $\sqrt{-1}(q - p)$. This Hodge decomposition of $V_{\mu, \mathbb{C}}$ can be easily deduced from the weight decomposition above. The operators h and f act on the weight subspace $V_\mu(\theta)$ by the multiplication $\langle \theta, h \rangle$ and $\langle \theta, f \rangle$, respectively. Hence the Hodge (p, q) -component $V_\mu^{p,q}$ is just a direct sum of weight subspaces $V_\mu(\theta)$ with $\langle \theta, h \rangle = p + q - 2n$ and $\langle \theta, f \rangle = \sqrt{-1}(q - p)$.

Recalling (2.28), if we denote the weight by $\theta = \theta_0 \varepsilon_0 + \dots + \theta_r \varepsilon_r$, then we have

$$\langle \theta, h \rangle = \langle \theta, \varepsilon_0^\vee \rangle = 2\theta_0, \quad \langle \theta, f \rangle = \frac{1}{\sqrt{-1}} \langle \theta, \varepsilon_1^\vee \rangle = -2\sqrt{-1}\theta_1. \quad (2.30)$$

This expresses p and q in terms of θ_0 and θ_1 :

$$p = \theta_0 + \theta_1 + n, \quad q = \theta_0 - \theta_1 + n. \quad (2.31)$$

Since θ_i are always mutually integers or half-integers (see (A.2) and (A.6)), both p and q are integers as we expect. There are several direct consequences of this simple observation.

Proposition 2.32 *The Hodge numbers $h_{(n)}^{p,q} = \dim V_{(n)}^{p,q}$ of the Verbitsky component $V_{(n)} \subset H^*(X, \mathbb{Q})$ satisfy $h^{2p,0} = 1$ and $h^{2p+1,0} = 0$ for all $0 \leq p \leq n$. In particular, the Verbitsky component occurs with multiplicity one ($m_{(n)} = 1$) in $H^*(X, \mathbb{Q})$.*

Proof Since the Verbitsky component $V_{(n)} \subset H^*(X, \mathbb{Q})$ is a \mathfrak{g} -submodule, it is also a sub-Hodge structure. Ignoring the notion of weight for simplicity, the Hodge decomposition of $\bar{V} = H^2(X, \mathbb{Q})$ is simply the f -eigenspace decomposition

$$\bar{V}_{\mathbb{C}} = \bar{V}^{1,-1} \oplus \bar{V}^{0,0} \oplus \bar{V}^{-1,1},$$

where $\dim \bar{V}^{1,-1} = \dim \bar{V}^{-1,1} = 1$. Now using the description (2.16) of the Verbitsky component, the $\bar{\mathfrak{g}}$ -module structures of each degrees of $V_{(n)}$ are $\text{Sym}^k \bar{V}$, which from the above Hodge structure on \bar{V} has the outermost Hodge numbers 1. Since the boundary Hodge numbers are $h^{2k,0} = h^{0,2k} = h^{2k,2n} = h^{2n,2k} = 1$ for compact hyper-Kähler manifolds X , the Verbitsky component already exhausts the boundary Hodge numbers 1. \square

The existence of the Hodge structure on \mathfrak{g} -modules also allows us to put more restrictions on the LLV components arising on the cohomology of X . Note that, even without the complex structure, the fact that $h = \varepsilon_0^\vee$ captures the degree of the cohomology implies every irreducible component $V_\mu \subset H^*(X, \mathbb{Q})$ satisfies

$$\langle \mu, h \rangle = \langle \mu, \varepsilon_0^\vee \rangle = 2\mu_0 \leq 2n. \quad (2.33)$$

Thus, we obtain $\mu_0 \leq n$. Taking into account also the Hodge structure, or equivalently f , we get a stronger inequality.

Proposition 2.34 *Every irreducible \mathfrak{g} -module V_μ contained in the full cohomology $H^*(X)$ satisfies either $\mu = (n)$ or $\mu_0 + \mu_1 \leq n - 1$.*

Proof By Proposition 2.32, the Verbitsky component $V_{(n)}$ always exhausts all the boundary Hodge numbers of X . Thus, if $\mu \neq (n)$ occurs as a highest weight in the LLV decomposition, then all the nonzero (p, q) -component arising in V_μ satisfy $1 \leq p \leq 2n - 1$. The highest \mathfrak{g} -module V_μ always contains the weight μ . By (2.31), all nonzero (p, q) -component in V_μ satisfy $p = \mu_0 + \mu_1 + n$ and $q = \mu_0 - \mu_1 + n$. Hence $\mu_0 + \mu_1 = p - n \leq n - 1$, as needed. \square

Similarly, we obtain the following easy restriction on the possible irreducible components of the LLV decomposition on the even and odd cohomology respectively.

Proposition 2.35 *Let X be a hyper-Kähler manifold, and \mathfrak{g} its LLV algebra.*

- (i) *Every irreducible \mathfrak{g} -module component $V_\mu \subset H_{\text{even}}^*(X)$ has integer coefficients $\mu_i \in \mathbb{Z}$. Similarly, every irreducible \mathfrak{g} -module component $V_\mu \subset H_{\text{odd}}^*(X)$ has half-integer coefficients $\mu_i \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.*
- (ii) *Every irreducible $\bar{\mathfrak{g}}$ -module component $\bar{V}_\lambda \subset H^{2k}(X)$ has integer coefficients $\lambda_i \in \mathbb{Z}$, while every irreducible $\bar{V}_\lambda \subset H^{2k+1}(X)$ has half-integer coefficients $\lambda_i \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.*

Proof Applying (2.31) to the highest weight μ of V_μ , we have $p = \mu_0 + \mu_1 + n$ and $q = \mu_0 - \mu_1 + n$. If we assume $V_\mu \in H_{\text{even}}^*(X)$, then we have an even $p+q = 2\mu_0 + 2n$. This proves $\mu_0 \in \mathbb{Z}$, and hence by (A.2) and (A.6) all the μ_i are integers. If we assume $V_\mu \in H_{\text{odd}}^*(X)$, then similar argument implies μ_0 is a half-integer and hence all μ_i are half-integers.

For the second statement, we cannot use the the operator $h \notin \bar{\mathfrak{g}}$, so we need to go back to (2.30). From it, we have $p - q = 2\lambda_1$. If $\bar{V}_\lambda \subset H^{2k}(X)$ lives in an even cohomology, then $p - q$ is even so λ_1 is integer. Hence all λ_i are integers. Similar argument proves the case $\bar{V}_\lambda \subset H^{2k+1}(X)$. \square

As an immediate corollary, we see that all reduced LLV modules $H^k(X)$ are faithful $\bar{\mathfrak{g}}$ -modules.

Corollary 2.36 *If $0 < k < 4n$ and $H^k(X) \neq 0$, then the map $\rho_k : \bar{\mathfrak{g}} \rightarrow \text{End}(H^k(X))$ is injective.*

Proof Since $\bar{\mathfrak{g}}$ is simple by Proposition 2.12, $\rho_k : \bar{\mathfrak{g}} \rightarrow \mathfrak{gl}(H^k(X))$ is injective unless $H^k(X)$ is a trivial $\bar{\mathfrak{g}}$ -module. For hyper-Kähler manifolds this cannot happen, since if k is odd then we can use Proposition 2.35, and if k is even then $\text{Sym}^{k/2} \bar{V} \subset H^k(X)$ because of the Verbitsky component $V_{(n)}$. \square

2.2.2 Hyper-Kähler metrics and twistor families

Hyper-Kähler manifolds admit *twistor families*. Let $X = (M, I)$ be a hyper-Kähler manifold, where M is the underlying real manifold and I a complex structure on M . A twistor family is a pencil of hyper-Kähler manifolds $(M, aI + bJ + cK)$ parameterized by $\{ai + bj + ck \in \mathbb{H} : a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1\} \cong \mathbb{P}^1$.

More precisely, let M be a compact real manifold of real dimension $4n$, admitting at least one hyper-Kähler metric g . That is, g is a Riemannian metric with the Holonomy

group isomorphic to $\mathrm{Sp}(n) \subset \mathrm{SO}(\mathbb{R}^{4n}, g)$. Then there exists a family of complex structures on M , the twistor family, $\{aI + bJ + cK : a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1\}$ such that any of these complex structure with (M, g) consists of a Kähler structure. Moreover, if we had two hyper-Kähler metrics g and g' inducing the same twistor family, then $g = g'$ by the uniqueness part of Calabi–Yau theorem. This means the choice of a hyper-Kähler metric g on M induces a twistor family and vice versa. For more details, see, e.g., [19, §24.2].

Now suppose we have a twistor family (M, I, J, K) (with $X = (M, I)$) corresponding to a hyper-Kähler metric g on M . There exist three distinguished Kähler forms $\omega_I = g(I-, -)$, $\omega_J = g(J-, -)$ and $\omega_K = g(K-, -)$ associated to this situation. These give us a distinguished choice of Kähler classes $\omega_I, \omega_J, \omega_K \in H^2(X, \mathbb{R})$. Hence, we have three Lefschetz and three dual Lefschetz operators associated to them

$$L_I, L_J, L_K, \Lambda_I, \Lambda_J, \Lambda_K \in \mathfrak{gl}(H^*(X, \mathbb{R})).$$

Now Verbitsky's Lie algebra \mathfrak{g}_g in [52, 53] is a real Lie subalgebra of $\mathfrak{gl}(H^*(X, \mathbb{R}))$ generated by these six operators. It is shown in loc. cit. that $\mathfrak{g}_g \cong \mathfrak{so}(4, 1)$. By definition of Looijenga–Lunts–Verbitsky algebra, we have an inclusion

$$\mathfrak{so}(4, 1) \cong \mathfrak{g}_g \subset \mathfrak{g}_{\mathbb{R}}.$$

Hence, the choice of a hyper-Kähler metric g on M induces the Verbitsky algebra \mathfrak{g}_g . If we vary a hyper-Kähler metric g on M , then \mathfrak{g}_g moves inside of the Looijenga–Lunts–Verbitsky algebra $\mathfrak{g}_{\mathbb{R}}$.

Finally, let us discuss the relationship between the Verbitsky algebra $\mathfrak{so}(4, 1)$ and Fujiki's work [10]. Once a hyper-Kähler metric g was fixed, Fujiki constructed an $\mathrm{Sp}(1)$ -action on each cohomology $H^k(X, \mathbb{R})$ by purely differential geometric methods. He studied the $\mathrm{Sp}(1)$ -representation theory on the cohomology $H^*(X, \mathbb{R})$ and as a result, obtained Hodge decomposition-type and hard Lefschetz-type theorems. In fact, the associated $\mathrm{Sp}(1)$ -decomposition essentially coincides with the decomposition associated to Verbitsky's $\mathfrak{g}_g \cong \mathfrak{so}(4, 1)$ -decomposition (and in particular, factors through the LLV decomposition). The decomposition (2.3) of the LLV algebra \mathfrak{g} induces a degree decomposition for Verbitsky's algebra $\mathfrak{g}_g \cong \mathfrak{so}(4, 1)$:

$$\mathfrak{g}_g = \mathfrak{g}_{g, -2} \oplus \mathfrak{g}_{g, 0} \oplus \mathfrak{g}_{g, 2}, \quad \mathfrak{g}_{g, 0} = \bar{\mathfrak{g}}_g \oplus \mathbb{R}h,$$

with $\bar{\mathfrak{g}}_g \cong \mathfrak{so}(3, \mathbb{R})$. Lifting the Lie algebra $\mathfrak{so}(3, \mathbb{R})$ to the level of Lie group gives us a simply connected real Lie group $\mathrm{Spin}(3, \mathbb{R})$, which is isomorphic to $\mathrm{Sp}(1)$ by an exceptional isomorphism (corresponding to $B_1 \equiv C_1$).

2.3 The Mumford–Tate algebra

The Looijenga–Lunts–Verbitsky algebra is a diffeomorphism invariant of a compact hyper-Kähler manifold X . A complex structure on X is encoded by the Hodge structure on the cohomology $H^*(X, \mathbb{Q})$. This Hodge structure is in turn given by a semisimple

element $f \in \bar{\mathfrak{g}}_{\mathbb{R}}$ (Proposition 2.24). To the Hodge structure is associated a (special) Mumford–Tate group. Here, we discuss the relationship between the Mumford–Tate algebra and the Looijenga–Lunts–Verbitsky algebra.

Definition 2.37 Let W be a \mathbb{Q} -Hodge structure. Define the operators $h \in \mathfrak{gl}(W)$ and $f \in \mathfrak{gl}(W)_{\mathbb{R}}$ by

$$\begin{aligned} h : W &\rightarrow W, & x \mapsto (p+q)x & \text{for } x \in W^{p,q} \\ f : W_{\mathbb{R}} &\rightarrow W_{\mathbb{R}}, & x \mapsto (q-p)\sqrt{-1}x & \text{for } x \in W^{p,q}, \end{aligned}$$

as in our previous notation (2.22) and (2.23). The *special Mumford–Tate algebra* of W is the smallest \mathbb{Q} -algebraic Lie subalgebra $\overline{\mathfrak{mt}}(W)$ of $\mathfrak{gl}(W)$ such that $f \in \overline{\mathfrak{mt}}(W)_{\mathbb{R}}$. The *Mumford–Tate algebra* of W is $\mathfrak{mt}_0(W) = \overline{\mathfrak{mt}}(W) \oplus \mathbb{Q}h$.

The Mumford–Tate algebra of W is usually defined as the associated Lie algebra of the Mumford–Tate group of W . Our definition coincides with this definition by the discussion in [56, §0.3.3]. The correspondence is as follows. Let \mathbb{S} be the Deligne torus, a nonsplit \mathbb{R} -algebraic torus of rank 2. According to Deligne, a \mathbb{Q} -Hodge structure W is a finite dimensional \mathbb{Q} -vector space W equipped with an appropriate \mathbb{S} -module structure on $W_{\mathbb{R}}$ (e.g. see [33]). That is, we have a morphism of \mathbb{R} -algebraic groups

$$\varphi : \mathbb{S} \rightarrow \mathrm{GL}(W)_{\mathbb{R}},$$

with $\mathbb{G}_{m,\mathbb{R}} \subset \mathbb{S} \rightarrow \mathrm{GL}(W)_{\mathbb{R}}$ defined over \mathbb{Q} . By definition, the Mumford–Tate group $\mathrm{MT}(W)$ of W is the smallest \mathbb{Q} -algebraic subgroup of $\mathrm{GL}(W)$ such that $\varphi(\mathbb{S}) \subset \mathrm{MT}(W)_{\mathbb{R}}$. Now take the differential of φ . We obtain a homomorphism of \mathbb{R} -Lie algebras

$$\varphi_* : \mathfrak{u}(1) \oplus \mathbb{R} \rightarrow \mathfrak{gl}(W)_{\mathbb{R}}.$$

The images the generators of $\mathfrak{u}(1)$ and \mathbb{R} are precisely f and h as above, giving the equivalence of the two definitions.

Returning to the hyper-Kähler geometry, we can consider the Hodge structures of degree k on $H^k(X, \mathbb{Q})$, and also of the full cohomology $H^*(X, \mathbb{Q})$. We will simply write

$$\bar{\mathfrak{m}} = \overline{\mathfrak{mt}}(H^*(X, \mathbb{Q}))$$

for the special Mumford–Tate algebra associated to the full cohomology of X . It is the \mathbb{Q} -algebraic Lie algebra closure of the one-dimensional real Lie algebra $\mathbb{R}f \subset \mathfrak{gl}(H^*(X, \mathbb{R}))$. There is a close relationship between $\bar{\mathfrak{m}}$, the Mumford–Tate algebras of the individual cohomologies $H^k(X, \mathbb{Q})$, and the Looijenga–Lunts–Verbitsky algebra \mathfrak{g} of X .

Proposition 2.38 Let X be a compact hyper-Kähler manifold of dimension $2n$.

(i) There exists an inclusion $\bar{\mathfrak{m}} \subset \bar{\mathfrak{g}}$. Equality holds for very general X .

(ii) If $0 < k < 4n$ and $H^k(X, \mathbb{Q}) \neq 0$, then $\overline{\text{mt}}(H^k(X, \mathbb{Q})) = \bar{m}$.

In particular, while Mumford–Tate algebras vary with the choice of complex structure on X , they remain subalgebras of the LLV algebra. In other words, all the Mumford–Tate algebras factor through \bar{g} (which is a diffeomorphism invariant).

Proof The first statement of (i) is a direct consequence of Proposition 2.24. By definition, the special Mumford–Tate algebra $\bar{m} \subset \mathfrak{gl}(H^*(X, \mathbb{Q}))$ is the smallest Lie algebra with $f \in \bar{m}_{\mathbb{R}}$. By Proposition 2.24, $f \in \bar{g}_{\mathbb{R}}$. Hence $\bar{m} \subset \bar{g}$.

Before proving the equality assertion of (i), let us first prove (ii). The \bar{g} -module structure (2.5) on $H^k(X, \mathbb{Q})$ is the composition

$$\rho_k : \bar{g} \subset \mathfrak{gl}(H^*(X, \mathbb{Q})) \xrightarrow{\pi_k} \mathfrak{gl}(H^k(X, \mathbb{Q})).$$

This map ρ_k is injective (Corollary 2.36). The paragraph above shows that \bar{m} is a posteriori the \mathbb{Q} -algebraic Lie algebra closure of f in $\bar{g}_{\mathbb{R}}$. Note that $\pi_k(f) \in \mathfrak{gl}(H^k(X, \mathbb{R}))$ is the operator encoding the Hodge structure of $H^k(X, \mathbb{Q})$. Thus $\overline{\text{mt}}(H^k(X, \mathbb{Q}))$ is the \mathbb{Q} -algebraic Lie algebra closure of $\pi_k(f)$ in $\mathfrak{gl}(H^k(X, \mathbb{R}))$. But we already have $\pi_k(f) \in \pi_k(\bar{g})_{\mathbb{R}}$, so by the same reason, $\overline{\text{mt}}(H^k(X, \mathbb{Q}))$ is the \mathbb{Q} -algebraic Lie algebra closure of $\pi_k(f)$ in $\pi_k(\bar{g})_{\mathbb{R}}$. But since ρ_k is injective, π_k induces an isomorphism between \bar{g} and $\pi_k(\bar{g})$. Thus $\pi_k : \bar{m} \rightarrow \overline{\text{mt}}(H^k(X, \mathbb{Q}))$ is an isomorphism.

It remains to prove the equality assertion of (i). It is a general fact in the theory of Mumford–Tate groups that the special Mumford–Tate group of the Hodge structure $\bar{V} = H^2(X, \mathbb{Q})$ of K3 type is $\text{SO}(\bar{V}, \bar{q})$ outside of the Noether–Lefschetz locus in the period domain of \bar{V} (see [15]). Since compact hyper-Kähler manifolds satisfy local Torelli theorem on second cohomology (or, even more, global Torelli theorem [20, 55]), this means $\overline{\text{mt}}(H^2(X, \mathbb{Q})) \cong \mathfrak{so}(\bar{V}, \bar{q})$ for very general X . Since $\overline{\text{mt}}(H^2(X, \mathbb{Q})) = \bar{m}$, the equality assertion follows. \square

By Proposition 2.38, the full Hodge structure on $H^*(X, \mathbb{Q})$ has the same degree of transcendence as the Hodge structure on $H^2(X, \mathbb{Q})$ over the special Mumford–Tate algebra $\overline{\text{mt}}(H^2(X, \mathbb{Q})) = \bar{m}$. This was anticipated by the Torelli principle for the second cohomology of hyper-Kähler manifolds (e.g., [19, Cor. 24.5] or [49]). As a byproduct of this proposition, following Zarhin [56], one can classify the special Mumford–Tate algebra of projective hyper-Kähler manifolds.

3 The LLV decomposition for the known examples of hyper-Kähler manifolds

In this section, we determine the LLV decomposition for all known examples of hyper-Kähler manifolds (Theorem 1.2). We begin with a review of what is known about these cohomology groups. From our perspective these results are equivalent to describing the structure of $H^*(X)$ as $\bar{m} = \mathfrak{so}(3, b_2 - 4)$ module (Sect. 2.3). This structure is the restriction of an $\bar{m} \cong \mathfrak{so}(4, b_2 - 3)$ module structure (as in the proof of Theorem 2.7). We then show that this second structure is in turn the restriction to \bar{m} of a $\bar{g} \cong \mathfrak{so}(4, b_2 - 2)$ –representation. This argument works for both the $K3^{[n]}$

types and the Kum_n types as the necessarily initial module structure is known [14], even in the non-projective cases [5]. For the OG6 and OG10 types, we do not have the full \bar{m} -module structures, only the Hodge numbers (cf. [35] and [6]). For OG10 this suffices, as we are dealing with a big algebra $\mathfrak{g} \cong \mathfrak{so}(4, 22)$, and a relatively small Euler number $e(X) = 176, 904$. For OG6, these considerations reduce us to two possible LLV module structures. To identify the correct \mathfrak{g} -representation, we need to delve deeper into the geometric construction of [35]. The proof of Theorem 1.2 is presented case by case in Sects. 3.2–3.5.

Remark 3.1 Since our arguments make use of the special Mumford–Tate algebra \bar{m} , it is important that the LLV algebra \mathfrak{g} is defined over \mathbb{Q} . However, it is more convenient to work over \mathbb{R} ; we will do so throughout (unless \mathbb{Q} coefficients are needed). For instance, this allows us to write

$$\mathfrak{g} = \mathfrak{so}(4, b_2(X) - 2), \quad \bar{\mathfrak{g}} = \mathfrak{so}(3, b_2(X) - 3)$$

ignoring the rational quadratic structure. Similarly, we write $H^*(X) = H^*(X, \mathbb{R})$. Everything is however defined over \mathbb{Q} , and the discussion can be easily adapted to \mathbb{Q} coefficients.

To our knowledge, very little was previously known on the LLV decomposition for the known cases. The $K3$ surface and the Kummer surface are clear (Ex. 2.11). Recall that, for any hyper-Kähler manifold X , the Verbitsky component $V_{(n)}$ occurs with multiplicity 1 in the LLV decomposition (Proposition 2.32). Dimension counts force $H^*(K3^{[2]}) = V_{(2)}$. The only other LLV decompositions that we are aware of are the next simplest cases, Kum_2 (see [26, Ex 4.6]) and $K3^{[3]}$ (see [28, Example 14]). The main result of the section is stated as Theorem 1.2 in the introduction. For reader’s convenience we state two corollaries of this result, namely the explicit LLV decompositions for hyper-Kähler manifolds of type $K3^{[n]}$ and Kum_n respectively for small values of n .

Corollary 3.2 *Let $\mathfrak{g} \cong \mathfrak{so}(4, 21)$ be the LLV algebra for hyper-Kähler manifolds of $K3^{[n]}$ type (with $n \geq 2$). Then, for $n \in \{2, \dots, 7\}$, the associated LLV decomposition of the cohomology is as follows:*

$$\begin{aligned} H^*(K3^{[2]}) &= V_{(2)} \\ H^*(K3^{[3]}) &= V_{(3)} \oplus V_{(1,1)} \\ H^*(K3^{[4]}) &= V_{(4)} \oplus V_{(2,1)} \oplus V_{(2)} \oplus \mathbb{R} \\ H^*(K3^{[5]}) &= V_{(5)} \oplus V_{(3,1)} \oplus V_{(3)} \oplus V_{(2,1)} \oplus V_{(1,1)} \oplus V \\ H^*(K3^{[6]}) &= V_{(6)} \oplus V_{(4,1)} \oplus V_{(4)} \oplus V_{(3,1)} \oplus V_{(3)} \oplus V_{(2,2)} \\ &\quad \oplus V_{(2,1)} \oplus V_{(2)}^{\oplus 2} \oplus V_{(1,1,1)} \oplus V \oplus \mathbb{R} \\ H^*(K3^{[7]}) &= V_{(7)} \oplus V_{(5,1)} \oplus V_{(5)} \oplus V_{(4,1)} \oplus V_{(4)} \oplus V_{(3,2)} \\ &\quad \oplus V_{(3,1)}^{\oplus 2} \oplus V_{(3)}^{\oplus 2} \oplus V_{(2,1,1)} \\ &\quad \oplus V_{(2,1)}^{\oplus 2} \oplus V_{(2)} \oplus V_{(1,1)}^{\oplus 2} \oplus V^{\oplus 2} \end{aligned}$$

Remark 3.3 As an illustration, let us discuss the case of hyper-Kähler manifold X of $K3^{[3]}$ type. The LLV algebra of X is $\mathfrak{g} \cong \mathfrak{so}(4, 21)$, and the above result says $H^*(X) = V_{(3)} \oplus V_{(1,1)}$ as \mathfrak{g} -modules, or equivalently

$$H^*(X) = V_{3\varpi_1} \oplus V_{\varpi_2} \quad (3.4)$$

in terms of the fundamental weights. Further decomposing (3.4) as a module of the reduced LLV algebra $\bar{\mathfrak{g}} = \mathfrak{so}(3, 20)$ accounts for disassembling the Mukai completion. By definition, the standard \mathfrak{g} -module V decomposes as $V = \mathbb{R}(1) \oplus \bar{V} \oplus \mathbb{R}(-1)$ when viewed as $\bar{\mathfrak{g}}$ -module. Here $\mathbb{R}(\pm 1)$ indicates the degree ∓ 2 parts of V ; \bar{V} has degree 0. The branching rules (see Appendix B) give

$$\begin{aligned} H^*(X) = & \mathbb{R}(3) \oplus \bar{V}(2) \oplus (\text{Sym}^2 \bar{V} \oplus \bar{V})(1) \\ & \oplus (\text{Sym}^3 \bar{V} \oplus \wedge^2 \bar{V} \oplus \mathbb{R}) \oplus (\text{Sym}^2 \bar{V} \oplus \bar{V})(-1) \\ & \oplus \bar{V}(-2) \oplus \mathbb{R}(-3), \end{aligned} \quad (3.5)$$

which the reader will notice is much more involved than (3.4). The decomposition (3.5) yields $H^0(X) = \mathbb{R}$, $H^2(X) = \bar{V}$, $H^4(X) = \text{Sym}^2 \bar{V} \oplus \bar{V}$, and so on, recovering Markman's computation [28, Ex. 14]. Finally, specializing X to $X = S^{[3]}$ for some $K3$ surface S , the generic Mumford–Tate algebra $\bar{\mathfrak{m}}$ of X in this locus becomes slightly smaller than $\bar{\mathfrak{g}}$ (see Proposition 2.38). More specifically, we have $\bar{\mathfrak{m}} = \mathfrak{so}(3, 19)$ contained in $\bar{\mathfrak{g}} = \mathfrak{so}(3, 20)$. Restricting the above identity further to $\bar{\mathfrak{m}}$, we recover the Götsche–Soergel's formula on Hodge structures [14], which is equivalent to the $\bar{\mathfrak{m}}$ -module structure.

Similarly, we have the following formulas for the low dimensional Kum_n cases.

Corollary 3.6 *Let $\mathfrak{g} \cong \mathfrak{so}(4, 5)$ be the LLV algebra for hyper-Kähler manifolds of Kum_n type (with $n \geq 2$). Then, for $n \in \{2, \dots, 5\}$, the associated LLV decomposition of the cohomology is as follows:*

$$\begin{aligned} H^*(\text{Kum}_2) = & V_{(2)} \oplus \mathbb{R}^{\oplus 80} \oplus V_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \\ H^*(\text{Kum}_3) = & V_{(3)} \oplus V_{(1,1)} \oplus V^{\oplus 16} \oplus \mathbb{R}^{\oplus 240} \oplus V_{(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \\ H^*(\text{Kum}_4) = & V_{(4)} \oplus V_{(2,1)} \oplus V_{(2)} \oplus V_{(1,1,1)} \oplus V_{(1,1)} \oplus \mathbb{R}^{\oplus 625} \oplus V_{(\frac{5}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \\ & \oplus V_{(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \oplus V_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \\ H^*(\text{Kum}_5) = & V_{(5)} \oplus V_{(3,1)} \oplus V_{(3)} \oplus V_{(2,1,1)} \oplus V_{(2,1)}^{\oplus 2} \\ & \oplus V_{(2)}^{\oplus 16} \oplus V_{(1,1,1,1)} \oplus V_{(1,1)} \\ & \oplus V^{\oplus 82} \oplus \mathbb{R}^{\oplus 1200} \oplus V_{(\frac{7}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \oplus V_{(\frac{5}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \oplus V_{(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})} \\ & \oplus V_{(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})}^{\oplus 2} \oplus V_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})}^{\oplus 17} \end{aligned}$$

Remark 3.7 We do not have closed formulas for the irreducible LLV decompositions of the general case $K3^{[n]}$ and Kum_n , but as one can see, the cohomology of Kum_n is

fairly complicated. Note in particular, the presence of several spin type representations, and the large number of trivial representations. The number of trivial representations is controlled by the fourth Jordan totient function $J_4(n+1) \sim (n+1)^4$. Specifically note that in the range that we have worked out the representations explicitly ($n \in \{1, \dots, 5\}$), the values of $J_4(n+1)$ are 15, 80, 240, 624, and 1, 200, while the number of trivial representations is 16, 80, 240, 625, and 1, 200 respectively. Geometrically, this means that a variety of Kum_n type contains many Hodge cycles (of order n^4) even if it is non-projective.

Other consequences of Theorem 1.2 are formulas for the generating series for the Euler numbers, the Poincaré polynomials, and Hodge–Deligne polynomials for the two series $\text{K3}^{[n]}$ and Kum_n . In the case of $\text{K3}^{[n]}$, we recover the formulas of Göttsche (see esp. [13, Thm 2.3.10] and [13, Rem 2.3.12]).

Corollary 3.8 *The generating series for $\text{K3}^{[n]}$ are as follows.*

(i) *The generating series for the Euler numbers of $\text{K3}^{[n]}$ is*

$$\sum_{n=0}^{\infty} e(\text{K3}^{[n]}) q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{24}} = \frac{q}{\Delta(q)},$$

where $\Delta(q)$ is the weight 12 modular form.

(ii) *The generating series for the Poincaré polynomials of $\text{K3}^{[n]}$ is*

$$\begin{aligned} \sum_{n=0}^{\infty} b(\text{K3}^{[n]}, t) q^n \\ = \prod_{m=1}^{\infty} \frac{1}{(1-t^2 q^m)(1-t^{-2} q^m)(1-q^m)^{22}}, \end{aligned}$$

where $b(\text{K3}^{[n]}, t)$ indicates the Poincaré polynomial with the Betti numbers $b_k(\text{K3}^{[n]})$ encoded in the coefficient of t^{k-2n} .

(iii) *The generating series for the Hodge–Deligne polynomials of $\text{K3}^{[n]}$ is*

$$\begin{aligned} \sum_{n=0}^{\infty} h(\text{K3}^{[n]}, s, t) q^n \\ = \prod_{m=1}^{\infty} \frac{1}{(1-st q^m)(1-st^{-1} q^m)(1-s^{-1} t q^m)(1-s^{-1} t^{-1} q^m)(1-q^m)^{20}}, \end{aligned}$$

where $h(\text{K3}^{[n]}, s, t)$ indicates the Hodge–Deligne polynomial with the Hodge numbers $h^{p,q}(\text{K3}^{[n]})$ encoded in the coefficient of $s^{p-n} t^{q-n}$.

Proof Recall the discussion in §2.2 that the Hodge structure is captured by the \mathfrak{g} -module structure. Specifically, the Hodge component $W^{p,q}$ of a \mathfrak{g} -module W is the

direct sum of the weight spaces $W(\theta)$, for the weights $\theta = \theta_0 \varepsilon_0 + \cdots + \theta_{11} \varepsilon_{11}$ such that

$$p = \theta_0 + \theta_1 + n, \quad q = \theta_0 - \theta_1 + n,$$

cf. (2.29) and (2.31). Note that the dimension of $W(\theta)$ is captured by the coefficient of $x_0^{\theta_0} \cdots x_{11}^{\theta_{11}}$ in the formal character. Setting $x_0 = st$, $x_1 = st^{-1}$, $x_2 = \cdots = x_{11} = 1$ gives us the transformation

$$x_0^{\theta_0} \cdots x_{11}^{\theta_{11}} = s^{\theta_0 + \theta_1} t^{\theta_0 - \theta_1} = s^{p-n} t^{q-n},$$

whose coefficient contributes to the Hodge number $h^{p,q}$ of W . This means setting $x_0 = st$, $x_1 = st^{-1}$, $x_2 = \cdots = x_{11} = 1$ in (1.3) of Theorem 1.2 gives us the generating series of the Hodge–Deligne polynomial of them. This proves (iii).

A similar argument implies that setting $x_0 = t^2$, $x_1 = x_2 = \cdots = x_{11} = 1$ yields (ii). Finally, for (i), note that the Euler number is an alternating sum of the Betti numbers. This amounts to setting $t = -1$. \square

For the Kum_n case, specializing the generating series of Theorem 1.2(2), we obtain the following formula for the Hodge–Deligne polynomials. This formula seems new and slightly simpler than those existing in the literature, but still not as neat as in the $\text{K3}^{[n]}$ case.

Corollary 3.9 *The generating series of Hodge–Deligne polynomials of Kum_n is*

$$\sum_{n=0}^{\infty} h(\text{Kum}_n, s, t) q^n = \sum_{d=1}^{\infty} J_4(d) \frac{st(B(q^d) - 1)}{(s+1)^2(t+1)^2 q}$$

as in (1.4), but with the formal power series $B(q)$ in this case defined by

$$B(q) = \prod_{m=1}^{\infty} \frac{(1+sq^m)^2(1+s^{-1}q^m)^2(1+tq^m)^2(1+t^{-1}q^m)^2}{(1-stq^m)(1-st^{-1}q^m)(1-s^{-1}tq^m)(1-s^{-1}t^{-1}q^m)(1-q^m)^4}.$$

Proof The proof is the same as that of Corollary 3.8. Setting $x_0 = st$, $x_1 = st^{-1}$ and $x_2 = x_3 = 1$ gives us the desired result. One can also observe the first coefficient b_1 of $B(q)$ is $b_1 = \frac{1}{st}(s+1)^2(t+1)^2$. \square

3.1 The Mukai completion

In this subsection, we assume X to be an arbitrary compact hyper-Kähler manifold. Let $\bar{\mathfrak{m}}$ and $\mathfrak{m}_0 = \bar{\mathfrak{m}} \oplus \mathbb{Q}h$ be the special Mumford–Tate algebra and Mumford–Tate algebra of X respectively (see Sect. 2). By Proposition 2.38, the Mumford–Tate algebra $\bar{\mathfrak{m}}$ is contained in $\bar{\mathfrak{g}}$. If we assume S is projective, then we further have a classification of the special Mumford–Tate algebra $\bar{\mathfrak{m}}$ by Zarhin [56]; either $\bar{\mathfrak{m}} \cong \mathfrak{so}_E(\bar{T}, \bar{q})$ or $\mathfrak{u}_{E_0}(\bar{T}, \bar{q})$ for a totally real or CM number field E (where E is determined by the endomorphisms

of the Hodge structure). In particular, if $E = \mathbb{Q}$ then we have $\bar{\mathfrak{m}} = \mathfrak{so}(\bar{T}, \bar{q})$. The assumption $E = \mathbb{Q}$ holds when X is a very general projective hyper-Kähler manifold.

Now, assume we had $\bar{\mathfrak{m}} \cong \mathfrak{so}(\bar{T}, \bar{q})$ for some sub-Hodge structure $\bar{T} \subset \bar{V}$. This assumption is satisfied in the following two cases:

- (A) If X is a very general projective hyper-Kähler manifold with a fixed polarization, then the assumption is satisfied with \bar{T} the transcendental Hodge structure of \bar{V} with $\dim \bar{T} = \dim \bar{V} - 1$, by the above discussion.
- (B) If X is a very general non-projective hyper-Kähler manifold, then the assumption is again satisfied with $\bar{T} = \bar{V}$, by Proposition 2.38(1).

Recall the relation between the two Lie algebras $\bar{\mathfrak{g}}$ and \mathfrak{g} in Theorem 2.7. In these cases, we can formally imitate this relation to enlarge the Lie algebra $\bar{\mathfrak{m}}$ to a new Lie algebra \mathfrak{m} . This process is often used in the theory of moduli of sheaves on K3 surfaces, and called Mukai extension or Mukai completion of the second cohomology.

Definition 3.10 Let (\bar{T}, \bar{q}) be a quadratic space over \mathbb{Q} and $\bar{\mathfrak{m}} = \mathfrak{so}(\bar{T}, \bar{q})$ a \mathbb{Q} -Lie algebra. We call $(T, q) = (\bar{T} \oplus \mathbb{Q}^2, \bar{q} \oplus (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$ the *Mukai completion* of (\bar{T}, \bar{q}) , and $\mathfrak{m} = \mathfrak{so}(T, q)$ the *Mukai completion* of $\bar{\mathfrak{m}}$.

The proof of Theorem 2.7 can be interpreted as saying that one can recover the Lie algebra \mathfrak{g} as the Mukai completion of the smaller Lie algebra $\bar{\mathfrak{g}}$. Now consider the special Mumford–Tate algebra $\bar{\mathfrak{m}}$ of X . It is contained in $\bar{\mathfrak{g}}$. If we apply the Mukai completion to $\bar{\mathfrak{m}}$, then get an abstract Lie algebra \mathfrak{m} . Since \mathfrak{g} is also the Mukai completion of $\bar{\mathfrak{g}}$, one can easily conclude

$$\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_2 \subset \mathfrak{g}, \quad \mathfrak{m}_0 = \bar{\mathfrak{m}} \oplus \mathbb{Q}h, \quad \mathfrak{m}_{\pm 2} = \mathfrak{m} \cap \mathfrak{g}_{\pm 2}. \quad (3.11)$$

Lemma 3.12 Assume the special Mumford–Tate algebra $\bar{\mathfrak{m}}$ of X is isomorphic to $\mathfrak{so}(\bar{T}, \bar{q})$, e.g., assume X satisfies either (A) or (B) above. Then its formal Mukai completion \mathfrak{m} is contained in \mathfrak{g} , and respects the degree of \mathfrak{g} in the sense of (3.11). \square

3.2 Cohomology of Hilbert schemes of K3 surfaces

The main result of this subsection is the proof of Theorem 1.2(1) concerning the generating series for $K3^{[n]}$. Specifically, we establish:

Theorem 3.13 Let \mathfrak{g} be the Looijenga–Lunts–Verbitksy algebra of a hyper-Kähler manifold of $K3^{[n]}$ type with $n \geq 2$. Then the generating series of the formal characters of \mathfrak{g} -modules $H^*(K3^{[n]})$ is

$$1 + \left(\sum_{i=0}^{11} (x_i + x_i^{-1}) \right) q + \sum_{n=2}^{\infty} \text{ch}(H^*(K3^{[n]})) q^n = \prod_{m=1}^{\infty} \prod_{i=0}^{11} \frac{1}{(1 - x_i q^m)(1 - x_i^{-1} q^m)}.$$

Let X be a $K3^{[n]}$ type hyper-Kähler manifold. Since the \mathfrak{g} -module structure on $H^*(X)$ is a diffeomorphism invariant, we may specialize X to $S^{[n]}$ with S a complex K3 surface. Since the statement is a diffeomorphism invariant, we may also vary the

complex structure of S at our convenience. The Looijenga–Lunts–Verbitsky algebras for S and X are different, and we indicate them by $\mathfrak{g}(S)$ and $\mathfrak{g}(X)$ respectively. As discussed, $\mathfrak{g}(S) = \mathfrak{so}(H^*(S, \mathbb{Q}), q_S)$ where q_S is the Mukai completion of the intersection pairing on the second cohomology of S . On the other hand, $\mathfrak{g}(X) = \mathfrak{so}(V, q)$ where (V, q) is the Mukai completion of the second cohomology $(\bar{V} = H^2(X), \bar{q})$ of X endowed with the Beauville–Bogomolov form. The relationship between (\bar{V}, \bar{q}) and $H^2(S)$ is well understood. Specifically,

$$(\bar{V}, \bar{q}) = (H^2(S, \mathbb{Q}), \bar{q}_S) \oplus \langle -2(n-1) \rangle.$$

This implies the inclusion $\bar{\mathfrak{g}}(S) \subset \bar{\mathfrak{g}}(X)$, whence the inclusion of Looijenga–Lunts–Verbitsky algebras

$$\mathfrak{g}(S) \subset \mathfrak{g}(X).$$

The Hodge structure of the hyper-Kähler manifold $S^{[n]}$ was determined by Götsche–Soergel [14]. We interpret this as giving the decomposition of $S^{[n]}$ as a representation of the Mumford–Tate algebra $\bar{\mathfrak{m}} \cong \mathfrak{so}(3, 19) (= \bar{\mathfrak{g}}(S))$. By considering the grading operator h , we can lift this decomposition of $H^*(S^{[n]})$ to a decomposition as a $\mathfrak{g}(S) \cong \mathfrak{so}(4, 20)$ -module. Since $\mathfrak{g}(S) \cong \mathfrak{so}(4, 20)$ and $\mathfrak{g}(X) \cong \mathfrak{so}(4, 21)$ have the same rank (type D_{12} and B_{12} respectively), there exists a unique $\mathfrak{g}(X)$ -module structure compatible (by restriction) to the $\mathfrak{g}(S)$ -module structure that we have determined. We conclude that essentially formally starting from Götsche–Soergel results, we recover the LLV decomposition for $\mathrm{K3}^{[n]}$.

Theorem 3.14 *Let S be a K3 surface and $X = S^{[n]}$. Denote $W = H^*(S)$ by the standard $\mathfrak{g}(S)$ -module. Then the $\mathfrak{g}(X)$ -module structure on $H^*(X)$ is uniquely determined by the isomorphism of $\mathfrak{g}(S)$ -modules*

$$H^*(X) \cong \bigoplus_{\alpha \vdash n} \left(\bigotimes_{i=1}^n \mathrm{Sym}^{a_i} W \right).$$

Here $\alpha = (1^{a_1}, \dots, n^{a_n})$ runs through all the partitions of $n = \sum_{i=1}^n i a_i$.

Proof The main result of Götsche–Soergel [14] is the existence of a canonical isomorphism of \mathbb{Q} -Hodge structures:

$$H^*(X)(n) = \bigoplus_{\alpha \vdash n} H^* \left(S^{(a_1)} \times \dots \times S^{(a_n)} \right) (a_1 + \dots + a_n). \quad (3.15)$$

Here $S^{(a)} = S^a / \mathfrak{S}_a$ denotes the a -th symmetric power of S , and the additional parentheses indicate Tate twistings by $\mathbb{Q}(n)$ and $\mathbb{Q}(a_1 + \dots + a_n)$ respectively. For notational simplicity, we omit the Tate twists⁵ henceforth. Now using the Hodge structure iso-

⁵ The Tate twisting has the effect of centering the Hodge weights at 0, instead instead of the natural $2n$ center for $H^*(X)$. This type of shifting is customary in Hodge theory, and the reason for it is to align the weights arising geometrically (centered at $2n$) to the weight arising from representation theory (centered at 0).

morphism $H^*(S^{(a_i)}) = \text{Sym}^{a_i} W$, we see that the desired identity holds at the level of Hodge structures.

In [14], S is assumed to be a polarized $K3$ surface. The algebraicity on S is not necessary as shown by de Cataldo–Migliorini [5]. This allows us to assume that the Mumford–Tate algebra of S is as big as possible, i.e., $\bar{\mathfrak{m}}(S) \cong \mathfrak{so}(3, 19)$.

This isomorphism (3.15) gives that the Mumford–Tate algebras $\bar{\mathfrak{m}}(S) = \bar{\mathfrak{m}}(X)$ coincide. Indeed, the Hodge structure of $H^*(X)$ is obtained from a suitable tensor construction applied to the Hodge structure $W = H^*(S)$. By [33, Rem 1.8], the special Mumford–Tate algebra of a tensor construction of W is an image of the special Mumford–Tate algebra of W . This means we have a surjection $\bar{\mathfrak{m}}(S) \twoheadrightarrow \bar{\mathfrak{m}}(X)$. On the other hand, choosing $\alpha = (a_1 = 0, \dots, a_{n-1} = 0, a_n = 1)$, we have a component L on the right hand side. This means $\bar{\mathfrak{m}}(S) \subset \bar{\mathfrak{m}}(X)$. Thus we must have $\bar{\mathfrak{m}}(X) = \bar{\mathfrak{m}}(S)$ by dimension reasons. We write $\bar{\mathfrak{m}}$ for both $\bar{\mathfrak{m}}(S)$ and $\bar{\mathfrak{m}}(X)$, and we understand them as identified via (3.15).

It follows that the identity [14] can be interpreted as an $\bar{\mathfrak{m}}$ -module isomorphism. As discussed, we can assume $\bar{\mathfrak{m}}$ is as large as possible, i.e. $\bar{\mathfrak{m}} = \bar{\mathfrak{g}}(S) \cong \mathfrak{so}(3, 19)$. Recall that it holds

$$\mathfrak{g}(S)_0 = \bar{\mathfrak{g}}(S) \oplus \mathbb{R}h$$

where h is the grading operator. Since the isomorphism (3.15) respects the natural grading (when the Tate twists are taken into account), we lift (3.15) to an isomorphism of $\mathfrak{g}(S)_0$ -modules. Since the weight lattices of $\mathfrak{g}(S)_0$ and $\mathfrak{g}(S)$ are the same, this is enough to conclude the both hand sides are isomorphic as $\mathfrak{g}(S)$ -modules. (Here the left hand side has a structure of $\mathfrak{g}(X)$ -module, which by restriction gives the structure of a $\mathfrak{g}(S)$ -module. While the right hand side only has a natural structure of $\mathfrak{g}(S)$ -module.)

Finally, the $\mathfrak{g}(X)$ -module structure on $H^*(X)$ is in fact uniquely determined by its $\mathfrak{g}(S)$ -module structure. Note that $\mathfrak{g}(S) = \mathfrak{so}(W, q_S)$ and $\mathfrak{g}(X) = \mathfrak{so}(V, q)$ are type D_{12} and B_{12} simple Lie algebras. Hence we can apply Proposition B.6. \square

The above Theorem 3.14 gives us a tool to compute the $\mathfrak{g}(X)$ -module structure of $H^*(X)$, because we can determine its formal character by computing the right hand side of the equality. This method is already very useful to compute the formal character and hence the $\mathfrak{g}(X)$ -module structure of the cohomology of $K3^{[n]}$ type hyper-Kähler manifold. We can make the formula even better by taking care of them all; we consider the generating function of the formal characters of $K3^{[n]}$ hyper-Kähler manifolds. The advantage of this is that we can get rid of the delicate part of partitions in the formula.

Proof of Theorem 3.13 For simplicity, let us write $s_i = \text{ch}(\text{Sym}^i W)$ for the formal character of the symmetric power of the standard $\mathfrak{g}(S)$ -module W . By Theorem 3.14, we can write down the generating function by

$$\begin{aligned} \sum_{n=0}^{\infty} \text{ch}(K3^{[n]}) q^n &= \sum_{n=0}^{\infty} \sum_{\alpha \vdash n} s_{a_1} s_{a_2} \cdots s_{a_n} q^n \\ &= \sum_{\alpha} s_{a_1} q^{a_1} \cdot s_{a_2} q^{2a_2} \cdots \cdots \cdot s_{a_n} q^{na_n}, \end{aligned}$$

where $\alpha = (1^{a_1}, \dots, n^{a_n})$ runs through all the partitions of $n = a_1 + 2a_2 + \dots + na_n$ for all nonnegative integers n . Hence, forgetting about the partition α and just thinking about a_i , we can simply rewrite the last expression by

$$\begin{aligned} & \sum_{\alpha} s_{a_1} q^{a_1} \cdot s_{a_2} q^{2a_2} \cdot \dots \cdot s_{a_n} q^{na_n} \\ &= \left(\sum_{a_1=0}^{\infty} s_{a_1} q^{a_1} \right) \left(\sum_{a_2=0}^{\infty} s_{a_2} q^{2a_2} \right) \left(\sum_{a_3=0}^{\infty} s_{a_3} q^{3a_3} \right) \dots. \end{aligned} \quad (3.16)$$

Now setting $A(q) = \sum_{i=0}^{\infty} s_i q^i$, this value is just $A(q)A(q^2)A(q^3)\dots = \prod_{m=1}^{\infty} A(q^m)$. Moreover, the expression $A(q)$ can be further simplified into

$$\begin{aligned} A(q) &= \sum_{i=0}^{\infty} s_i q^i \\ &= 1 + \text{ch } Wq + \text{ch}(\text{Sym}^2 W)q^2 + \dots \\ &= 1 + (x_0 + \dots + x_{11} + x_0^{-1} + \dots + x_{11}^{-1})q + (x_0^2 + x_0 x_1 + \dots + x_{11}^{-2})q^2 + \dots \\ &= \prod_{i=0}^{11} (1 + x_i q + x_i^2 q^2 + \dots) (1 + x_i^{-1} q + x_i^{-2} q^2 + \dots) \\ &= \prod_{i=0}^{11} \frac{1}{(1 - x_i q)(1 - x_i^{-1} q)}. \end{aligned}$$

The theorem follows. \square

3.3 Cohomology of generalized Kummer varieties

In this subsection, we prove Theorem 1.2(2).

Theorem 3.17 *Let \mathfrak{g} be the Looijenga–Lunts–Verbitsky algebra of a hyper-Kähler manifold of Kum_n type. Let us define the formal power series*

$$B(q) = \prod_{m=1}^{\infty} \left[\prod_{i=0}^3 \frac{1}{(1 - x_i q^m)(1 - x_i^{-1} q^m)} \prod_j (1 + x_0^{j_0} x_1^{j_1} x_2^{j_2} x_3^{j_3} q^m) \right], \quad (3.18)$$

with $j = (j_0, \dots, j_3) \in \{-\frac{1}{2}, \frac{1}{2}\}^{\times 4}$ and $j_0 + \dots + j_3 \in 2\mathbb{Z}$. Let b_1 be the degree 1 coefficient of $B(q) = 1 + b_1 \cdot q + b_2 \cdot q^2 + \dots$, and $J_4(d)$ be the fourth Jordan totient function. With these notations, the generating series of the formal characters of the \mathfrak{g} -modules $H^*(\text{Kum}_n)$ is

$$1 + \left(\sum_{i=0}^3 (x_i + x_i^{-1}) + 16 \right) q + \sum_{n=2}^{\infty} \text{ch}(H^*(\text{Kum}_n)) q^n = \sum_{d=1}^{\infty} J_4(d) \frac{B(q^d) - 1}{b_1 \cdot q}. \quad (3.19)$$

Remark 3.20 Considering the degree n terms of the identity (3.19), we obtain

$$\mathrm{ch}(H^*(\mathrm{Kum}_n)) = \frac{1}{b_1} \sum_{d|n+1} J_4\left(\frac{n+1}{d}\right) \cdot b_d,$$

where b_d are the coefficients of $B(q) = 1 + b_1q + b_2q^2 + \dots$ given by (3.18). In particular, if $n+1 = p$ is prime, then

$$\mathrm{ch}(H^*(\mathrm{Kum}_{p-1})) = \frac{b_p}{b_1} + J_4(p) = \frac{b_p}{b_1} + (p^4 - 1)$$

has a simple expression. As previously mentioned, the constant term $p^4 - 1$ is an indicator of the trivial representations in $H^*(\mathrm{Kum}_{p-1})$.

For the proof of Theorem 3.17, we follow the same strategy as for $K3^{[n]}$. The only difference here is that the Hodge structure of the generalized Kummer varieties is much more complicated than that of the Hilbert scheme of K3 surfaces, essentially because of the existence of the odd cohomology. Fortunately, the first step, interpreting the Götsche–Soergel [14] result in our language, is fairly straightforward.

Theorem 3.21 *Let A be a complex torus of dimension 2 and X be the generalized Kummer variety associated to A . Write $W = H_{\mathrm{even}}^*(A)$ and $U = H_{\mathrm{odd}}^*(A)$ as the standard and semi-spin $\mathfrak{g}(A)$ -modules. Then the $\mathfrak{g}(X)$ -module structure on $H^*(X)$ is uniquely determined by the $\mathfrak{g}(A)$ -module isomorphism*

$$H^*(X) \otimes (W \oplus U) = \bigoplus_{\alpha \vdash n+1} \left[\bigotimes_{i=1}^{n+1} \left(\bigoplus_{j=0}^{a_i} \mathrm{Sym}^{a_i-j} W \otimes \wedge^j U \right) \right]^{\oplus g(\alpha)^4},$$

where $\alpha = (1^{a_1}, \dots, (n+1)^{a_{n+1}})$ runs through all the partitions of $n+1 = \sum_{i=1}^{n+1} i a_i$, and $g(\alpha)$ is defined by

$$g(\alpha) = \gcd\{k : 1 \leq k \leq n+1, a_k \neq 0\}.$$

Proof Again, the following isomorphism is proved in [14] on the level of \mathbb{Q} -Hodge structures (ignoring Tate twists as in Theorem 3.14).

$$H^*(X \times A) = \bigoplus_{\alpha \vdash n+1} H^*(A^{(a_1)} \times \dots \times A^{(a_{n+1})})^{\oplus g(\alpha)^4}.$$

Here $A^{(a)} = A^a / \mathfrak{S}_a$ indicates the symmetric power of A . Since A in this case has an odd cohomology, the Hodge structure of $A^{(a)}$ has a more complicated form

$$H^*(A^{(a)}) = \bigoplus_{j=0}^a \mathrm{Sym}^{a-j} W \otimes \wedge^j U.$$

The proof of it can be found, for example, in [30].

In Theorem 3.14, as K3 surfaces are also hyper-Kähler manifolds, we could avoid the discussion of Looijenga–Lunts–Verbitsky algebra and Mumford–Tate algebra of them. In this case, since A is not a hyper-Kähler manifold, we first need to (1) compute the Looijenga–Lunts–Verbitsky algebra of A and (2) compute the special Mumford–Tate algebra of A . Fortunately, both issues can be handled without much difficulty. First, Looijenga and Lunts [26] already computed the Looijenga–Lunts–Verbitsky algebra for an arbitrary complex torus:

$$\mathfrak{g}(A) \cong \mathfrak{so}(H^1(A, \mathbb{Q}) \oplus (H^1(A, \mathbb{Q}))^\vee, (,))$$

where $(,)$ is the canonical pairing. For dimension 2, this coincides with $\mathfrak{so}(U^{\oplus 4}) = \mathfrak{so}(H^2(A, \mathbb{Q}) \oplus \mathbb{Q}^2, q_A \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$. Hence, the theory of Looijenga–Lunts–Verbitsky algebra of complex tori of dimension 2 coincides with that of hyper-Kähler manifolds (with $b_2 = 6$). Second, the special Mumford–Tate Lie algebra of $H^*(A, \mathbb{Q})$ is that of $H^1(A, \mathbb{Q})$ because $H^*(A, \mathbb{Q}) = \wedge^* H^1(A, \mathbb{Q})$. This also coincides with the special Mumford–Tate algebra of $H^2(X, \mathbb{Q}) = \wedge^2 H^1(A, \mathbb{Q})$, which is a Hodge structure of K3 type, so we can also apply the same argument for complex tori of dimension 2.

Now lifting the Hodge structure isomorphism to an \mathfrak{m} -module isomorphism can be done as in the proof of Theorem 3.14. Also, using the Torelli theorem for complex tori, we can vary the complex structure of A to enhance this isomorphism to a $\mathfrak{g}(A)$ -module isomorphism. (Again, we have made use of [5] to be able to work with non-projective complex tori.) \square

The second part of the theorem requires a new idea. This is because of the additional wedge product terms appearing in Theorem 3.21, and also because of the delicate term $g(\alpha)^4$.

Proof of Theorem 3.17 Write $s_i = \text{ch}(\text{Sym}^i W)$ and $w_i = \text{ch}(\wedge^i U)$. Observe that U is of dimension 8, so we have only w_1, \dots, w_8 . By Theorem 3.21, we can directly compute

$$\begin{aligned} & \sum_{n=0}^{\infty} \text{ch}(H^*(\text{Kum}_n))(s_1 + w_1)q^{n+1} \\ &= \sum_{n=0}^{\infty} \sum_{\alpha \vdash n+1} g(\alpha)^4 (s_{a_1} + s_{a_1-1}w_1 + s_{a_1-2}w_2 + \dots) \\ & \quad \dots (s_{a_{n+1}} + s_{a_{n+1}-1}w_1 + s_{a_{n+1}-2}w_2 + \dots)q^{n+1} \\ &= \sum_{\substack{\alpha \neq 0}} g(\alpha)^4 (s_{a_1} + s_{a_1-1}w_1 + \dots)q^{a_1}(s_{a_2} \\ & \quad + s_{a_2-1}w_1 + \dots)q^{2a_2}(s_{a_3} + s_{a_3-1}w_1 + \dots)q^{3a_3} \dots \end{aligned}$$

Here in the last expression, $\alpha = (1^{a_1}, 2^{a_2}, \dots)$ runs through all *nonempty* partition and we used $n+1 = a_1 + 2a_2 + \dots$. Now, as we did in the proof of Theorem 3.13, we would like to transform the expression without involving the partition α . This cannot

be in the same way because of the problematic term $g(\alpha)^4$. Let us introduce a set $K = \{k : a_k \neq 0\}$ associated to the partition α . This set is nonempty because α cannot be an empty partition. Now running α through all nonempty partition corresponds to running K for all *nonempty finite* subset of \mathbb{Z} , and varying the multiplicity $a_k \in \mathbb{Z}_{\geq 1}$ for all elements $k \in K$. Moreover, the notation $g(\alpha)$ is converted simply to $\gcd(K)$. Hence, we can convert the last expression by

$$\begin{aligned} & \sum_{\alpha \neq 0} g(\alpha)^4 (s_{a_1} + s_{a_1-1}w_1 + \cdots) q^{a_1} (s_{a_2} \\ & \quad + s_{a_2-1}w_1 + \cdots) q^{2a_2} (s_{a_3} + s_{a_3-1}w_1 + \cdots) q^{3a_3} \cdots \\ & = \sum_K \sum_{(a_k)_{k \in K}} \prod_{k \in K} \gcd(K)^4 (s_{a_k} + s_{a_k-1}w_1 + \cdots) q^{ka_k} \\ & = \sum_K \gcd(K)^4 \sum_{(a_k)} \prod_{k \in K} (s_{a_k} + s_{a_k-1}w_1 + \cdots) q^{ka_k}, \end{aligned} \quad (3.22)$$

where the tuple $(a_k)_{k \in K}$ runs through all the possible functions $K \rightarrow \mathbb{Z}_{\geq 1}$. Now one can apply the same factorization technique we used in (3.16) to simplify the last expression into

$$\begin{aligned} & \sum_{(a_k)} \prod_{k \in K} (s_{a_k} + s_{a_k-1}w_1 + \cdots) q^{ka_k} \\ & = \prod_{k \in K} \left((s_1 + w_1)q^k + (s_2 + s_1w_1 + w_2)q^{2k} + \cdots \right). \end{aligned}$$

For simplicity, let us define $A(q) = (s_1 + w_1)q + (s_2 + s_1w_1 + w_2)q^2 + \cdots$. Then we can write down the last expression in (3.22) simply by

$$\sum_K \gcd(K)^4 \prod_{k \in K} A(q^k).$$

It is surprising to observe this expression admits a further simplification. \square

Lemma 3.23 *Let $A(q)$ be an arbitrary formal power series on q . Then we have an identity*

$$\sum_K \gcd(K)^4 \prod_{k \in K} A(q^k) = \sum_{d=1}^{\infty} J_4(d)(B(q^d) - 1),$$

where K runs through all the nonempty finite subset of \mathbb{Z} , $J_4(d)$ denotes the fourth Jordan totient function and $B(q) = (1 + A(q))(1 + A(q^2))(1 + A(q^3)) \cdots$.

Proof of Lemma 3.23 Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle group and $T^4 = (S^1)^4$ the 4-torus. We define the following character function $\delta_k : T^4 \rightarrow \{0, 1\}$ of the k^4 -lattice on T^4

$$\delta_k(x, y, z, w) = \begin{cases} 1 & \text{if } x, y, z, w \in \left(\frac{1}{k}\mathbb{Z}\right)/\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

Let $d\mu$ be the counting measure on T^4 . The idea is to capture the nuisance term $\gcd(K)^4$ by the integration of a multiplication of the character functions

$$\gcd(K)^4 = \int_{T^4} \left(\prod_{k \in K} \delta_k \right) d\mu.$$

Using this, the left hand side of the identity can be transformed in the following way.

$$\begin{aligned} \sum_{K \neq \emptyset} \gcd(K)^4 \prod_{k \in K} A(q^k) &= \int_{T^4} \left[\sum_{K \neq \emptyset} \left(\prod_{k \in K} A(q^k) \delta_k \right) \right] d\mu \\ &= \int_{T^4} [-1 + (1 + A(q)\delta_1)(1 + A(q^2)\delta_2)(1 + A(q^3)\delta_3) \cdots] d\mu. \end{aligned}$$

Let us compute this last expression by evaluating the integral at the d^4 -lattice points $((\frac{1}{d}\mathbb{Z})/\mathbb{Z})^4 \subset T^4$, inductively starting from the lower values of d . At the point $(\frac{c_1}{d}, \dots, \frac{c_4}{d}) \in T^4$, if $\gcd(c_1, c_2, c_3, c_4, d) > 1$, then this point was already counted when we considered the $(d')^4$ -lattice points with $d' < d$. When $\gcd(c_1, c_2, c_3, c_4, d) = 1$, the evaluation of the integral at this point gives precisely the formal power series

$$-1 + (1 + A(q^d))(1 + A(q^{2d}))(1 + A(q^{3d})) \cdots = B(q^d) - 1.$$

The number of points $(\frac{c_1}{d}, \dots, \frac{c_4}{d})$ with $\gcd(c_1, c_2, c_3, c_4, d) = 1$ is by definition the fourth Jordan totient function value $J_4(d)$. This proves the lemma. \square

Returning to the proof of Theorem 3.17, we now have

$$\sum_{n=0}^{\infty} \text{ch}(H^*(\text{Kum}_n))(s_1 + w_1)q^{n+1} = \sum_{d=1}^{\infty} J_4(d)(B(q^d) - 1).$$

In our case, we have the further identities

$$\begin{aligned} 1 + A(q) &= 1 + (s_1 + w_1)q + (s_2 + s_1w_1 + w_2)q^2 + (s_3 + s_2w_1 + s_1w_2 + w_3)q^3 + \cdots \\ &= (1 + s_1q + s_2q^2 + \cdots)(1 + w_1q + \cdots + w_8q^8) \\ &= \prod_{i=0}^3 \frac{1}{(1 - x_i q)(q - x_i^{-1} q)} \cdot \prod_j (1 + x_0^{j_0} x_1^{j_1} x_2^{j_2} x_3^{j_3} q). \end{aligned}$$

We finally note $A(q) = 1 + (s_1 + w_1)q + \dots$, so that $B(q) = (1 + A(q))(1 + A(q^2)) \dots = 1 + (s_1 + w_1)q + \dots$. Hence we can set $s_1 + w_1 = b_1$, where b_1 denotes the first q -coefficient of $B(q)$. This completes the proof of the theorem. \square

3.4 Cohomology of O'Grady's 10-dimensional example

The case of hyper-Kähler manifolds of OG10 type is in some sense the most interesting case, as it shows the power of the LLV decomposition of the cohomology. To start, we recall the very recent result of de Cataldo–Rapagnetta–Saccà [6].

Theorem 3.24 (de Cataldo–Rapagnetta–Saccà [6]) *Let X be a hyper-Kähler manifold of OG10 type. Then*

- (i) *There is no odd cohomology ($H_{odd}^*(X) = 0$).*
- (ii) *The Hodge numbers of $H_{even}^*(X)$ are as follows (we list only the first quadrant):*

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & 22 & & 1 & & & \\
 254 & & 22 & & 1 & & \\
 2,299 & & 276 & & 23 & & 1 \\
 16,490 & & 2,531 & & 276 & & 22 \\
 88,024 & & 16,490 & & 2,299 & & 254 & & 22 & 1
 \end{array} \tag{3.25}$$

While in the other cases we have made heavy use of the knowledge of the Hodge numbers, it turns out that in the OG10 case the existence of the LLV decomposition with respect to $\mathfrak{so}(4, 22)$ is a very constraining condition. In fact, all that we need to prove Theorem 1.2(4) is the vanishing of the odd cohomology. In particular, we obtain that item (ii) of Theorem 3.24 is a corollary of item (i). To emphasize this fact, we state the following somewhat artificial result:

Theorem 3.26 *Let X be a 10-dimensional hyper-Kähler manifold. Assume the following*

- (1) $b_2(X) = 24$.
- (2) $e(X) = 176, 904$.
- (3) *There is no odd cohomology ($H_{odd}^*(X) = 0$).*

Then X has the following LLV decomposition as a $\mathfrak{g} = \mathfrak{so}(4, 22)$ -module:

$$H^*(X) = V_{(5)} \oplus V_{(2,2)}. \tag{3.27}$$

In particular, the Hodge numbers are as in (3.25).

Remark 3.28 An alternative notation for this result, the one written in the introduction, is $H^*(X) = V_{5\varpi_1} \oplus V_{2\varpi_2}$. Here ϖ_1 is the fundamental weight associated to the standard representation V (and thus $V_{5\varpi_1}$ is the leading representation in $\text{Sym}^5 V$), and ϖ_2 is the fundamental weight associated to the irreducible \mathfrak{g} -module $\wedge^2 V$.

Corollary 3.29 *If X is a hyper-Kähler manifold of OG10 type, then its LLV decomposition is given by (3.27) and the Hodge numbers are as in (3.25).*

Proof The three conditions of Theorem 3.26 were established by Rapagnetta [44], Mozgovoy [36] (see also [18]), and de Cataldo–Rapagnetta–Saccá [6] (Theorem 3.24(i)) respectively. \square

Remark 3.30 Lie Fu noted that the arguments of [8] (especially Theorem 1.3 in loc. cit.) directly imply the vanishing of the odd cohomology for hyper-Kähler manifolds of OG10 type. Essentially, if one follows O’Grady’s original geometric construction [40] for an OG10 hyper-Kähler manifold X starting from a projective K3 surface S , it can be shown that the Hodge structure $H^*(X)$ can be realized by a tensor construction starting from the Hodge structure $H^2(S)$. It follows that X has no odd cohomology. Combined with Theorem 3.26, one obtains an independent proof of Theorem 3.24.

Remark 3.31 It is interesting to note that in the OG10 case, the vanishing of the odd cohomology is equivalent to Theorem 1.23. Specifically, assuming no odd cohomology, we obtain the LLV decomposition (3.27), which obviously satisfies the condition (1.20) of Conjecture 1.19. Conversely, assuming (1.20), we conclude that there is no odd cohomology. Namely, for OG10, the rank of the LLV algebra \mathfrak{g} is 13. Any irreducible \mathfrak{g} -module occurring in the odd cohomology $V_\mu \subset H_{\text{odd}}^*(X, \mathbb{Q})$ has all the coefficients of $\mu = (\mu_0, \dots, \mu_{12})$ half-integers. But then,

$$\mu_0 + \dots + \mu_{11} + |\mu_{12}| \geq \frac{13}{2} > 5,$$

violating the the inequality (1.20). The same argument applies more generally. Namely, the condition (1.20) forces the vanishing of odd cohomology for $2n$ -dimensional hyper-Kähler manifolds satisfying

$$b_2(X) \geq 4n. \quad (3.32)$$

The rest of the section is concerned with the proof of Theorem 3.26. In addition to the numerical assumptions of the theorem, we are using the following three general facts about the cohomology of hyper-Kähler manifolds.

- (A) *$H^*(X)$ admits an action by the LLV algebra $\mathfrak{g} \cong \mathfrak{so}(4, b_2 - 2)$.* In this situation, the assumptions of Theorem 3.26 give $\mathfrak{g} \cong \mathfrak{so}(4, 22)$ and $H^*(X) = H_{\text{even}}^*(X)$ has dimension 176, 904. The main point here is that this dimension is relatively small with respect to \mathfrak{g} .
- (B) *The Verbitsky component $V_{(5)}$ occurs in $H^*(X)$.* Since $\dim V_{(5)} = 139, 230$, we obtain that the other irreducible \mathfrak{g} -representations occurring in $H^*(X)$ have total dimension 37, 674.
- (C) *A $2n$ -dimensional hyper-Kähler manifold satisfies Salamon’s relation:*

$$2 \sum_{i=1}^{2n} (-1)^i (3i^2 - n) b_{2n-i} = nb_{2n},$$

which we find convenient to rewrite as $\sum_{i=1}^{2n} (-1)^i i^2 b_{2n-i} = \frac{n}{6} \cdot e(X)$. Assuming no odd cohomology, this gives us

$$\sum_{k=0}^n (n-k)^2 b_{2k} = \frac{n}{24} \cdot e(X) \quad (3.33)$$

(e.g. in dimension $2 = 2n$, this reads $b_0 = \frac{e(X)}{24}$, which is equivalent to Noether's formula for hyper-Kähler [$\equiv K3$] surfaces).

In the particular case considered here, we obtain the following four equations for the six even Betti numbers:

$$\begin{aligned} b_0 &= 1, & b_2 &= 24 \\ 25b_0 + 16b_2 + 9b_4 + 4b_6 + b_8 &= 36,855 \\ 2b_0 + 2b_2 + 2b_4 + 2b_6 + 2b_8 + b_{10} &= 176,904 \end{aligned}$$

There are finitely many non-negative integer solutions b_{2i} to the above equations. It turns out that there is a unique solution compatible with the LLV structure. Specifically, we have

$$H_{\text{even}}^*(X) = V_{(5)} \oplus V', \quad (3.34)$$

for some $\mathfrak{so}(4, 22)$ -module V' . The dimension bound discussed above greatly limits the possibilities for the irreducible summands of V' .

Lemma 3.35 *The possible dominant integral $\mathfrak{so}(4, 22)$ -weights μ such that V_μ can be contained in V' are*

$$S = \{(4), (3), (2, 2), (2, 1), (2), (1, 1, 1, 1), (1, 1, 1), (1, 1), (1), (0)\}.$$

Proof As discussed, $\dim V' = 37,674$. On the other hand, by Proposition 2.35, $\mu = (\mu_0, \dots, \mu_{12})$ has integer coefficients μ_i . Using Weyl dimension formula and Lemma A.9, one can check that S is the complete list of dominant integral weights for the $\mathfrak{so}(4, 22)$ -modules satisfying these constraints. \square

As discussed in Sect. 2, each of the $\mathfrak{so}(4, 22)$ -modules V_μ carry a Hodge structure (induced by $h, f \in \mathfrak{g} = \mathfrak{so}(4, 22)$), and hence each V_μ admits its own Betti numbers. We list the relevant Betti numbers in Table 1. Writing

$$V' = \bigoplus_{\mu \in S} V_\mu^{\oplus m_\mu}$$

for the irreducible decomposition of V' , and using Table 1, we obtain the following constraints:

- (i) The betti number $b_2 = 24$ forces $m_{(4)} = 0$.

Table 1 The relevant irreducible $\mathfrak{so}(4, 22)$ -modules for OG10

	b_0	b_2	b_4	b_6	b_8	b_{10}	Dimension	$\sum(5-i)^2 b_{2i}$
$V_{(5)}$	1	24	300	2600	17,550	98,280	139,230	31,059
$V_{(4)}$		1	24	300	2600	17,550	23,400	4032
$V_{(3)}$			1	24	300	2600	3250	405
$V_{(2,2)}$				299	4600	27,876	37,674	5796
$V_{(2,1)}$					24	576	4624	672
$V_{(2)}$					1	24	300	28
$V_{(1,1,1,1)}$						2024	10,902	14,950
$V_{(1,1,1)}$						276	2048	2600
$V_{(1,1)}$						24	277	325
$V_{(1)}$						1	24	26
$V_{(0)}$							1	0

(ii) The Euler characteristic yields

$$3, 250m_{(3)} + 37, 674m_{(2,2)} + 5, 824m_{(2,1)} + 350m_{(2)} + 14, 950m_{(1,1,1,1)} + 2, 600m_{(1,1,1)} + 325m_{(1,1)} + 26m_{(1)} + m_{(0)} = 37, 674. \quad (3.36)$$

iii) Salamon's relation gives us

$$405m_{(3)} + 5, 796m_{(2,2)} + 672m_{(2,1)} + 28m_{(2)} + 2, 024m_{(1,1,1,1)} + 276m_{(1,1,1)} + 24m_{(1,1)} + m_{(1)} = 5, 796. \quad (3.37)$$

It turns out that this system of equations has a unique (obvious) solution.

Lemma 3.38 *The above equations admit a unique nonnegative integer solution*

$$m_{(2,2)} = 1, \quad m_{(4)} = m_{(3)} = m_{(2,1)} = \cdots = m_{(0)} = 0.$$

Proof By dimension reasons, $m_{(2,2)} \geq 1$ forces the solution listed in the lemma. Thus, we can assume $m_{(2,2)} = 0$. Rescaling the Euler characteristic equation (3.36) by $\frac{2}{13}$, we get

$$500m_{(3)} + 896m_{(2,1)} + \left(53 + \frac{11}{13}\right)m_{(2)} + 2, 300m_{(1,1,1,1)} + 400m_{(1,1,1)} + 50m_{(1,1)} + 4m_{(1)} + \frac{2}{13}m_{(0)} = 5, 796.$$

Notice that the coefficients of this equation are all larger than the corresponding ones in the Salamon's relation (3.37) (while the value on the right-hand-side stays the same). We conclude that there is no non-negative solution with $m_{(2,2)} = 0$. The lemma follows. \square

This concludes the proof of Theorem 3.26 (and thus Theorem 1.2(4)).

Table 2 The Hodge diamond of OG6, the component $V_{(3)}$, and the residual component V'

1		—	1		=	0		
6	1		6	1		0	0	
173	12	1	22	6	1	151	6	0
1,144	173	6	1	62	22	6	1	1,082
						151	0	0

3.5 Cohomology of O'Grady's 6-dimensional example

We now prove the OG6 case of Theorem 1.2. Specifically, we prove:

Theorem 3.39 *Let X be a hyper-Kähler manifold of OG6 type and $\mathfrak{g} \cong \mathfrak{so}(4, 6)$ its Looijenga–Lunts–Verbitsky algebra. Then the \mathfrak{g} -module irreducible decomposition of the cohomology of X is*

$$H^*(X) = V_{(3)} \oplus V_{(1,1,1)} \oplus V^{\oplus 135} \oplus \mathbb{R}^{\oplus 240}.$$

Remark 3.40 An alternative notation for this result, the one written in the introduction, is $H^*(X) = V_{3\varpi_1} \oplus V_{\varpi_3} \oplus V^{\oplus 135} \oplus \mathbb{R}^{\oplus 240}$. Here ϖ_1 is the fundamental weight associated to the standard representation V and ϖ_3 is the fundamental weight associated to the irreducible \mathfrak{g} -module $\wedge^3 V$.

The starting point of our result is the Hodge numbers computed by Mongardi–Rapagnetta–Saccà [35]. Again, there is no odd cohomology, and the relevant Hodge numbers are given in Table 2.

Splitting off the Verbitsky component from the cohomology of OG6 hyper-Kähler manifold X

$$H^*(X) = V_{(3)} \oplus V',$$

it remains to understand the residual component V' (see Table 2 for the numerics). We proceed as for the OG10 case. Unfortunately, it turns out that there are two possible solutions to the numerical constraints satisfied by OG6 (even when the Hodge numbers are taken into account).

Proposition 3.41 *The LLV decomposition of $H^*(\text{OG6})$ as a $\mathfrak{so}(4, 6)$ -module is either*

$$H^*(X) = V_{(3)} \oplus V_{(1,1,1)} \oplus V^{\oplus 135} \oplus \mathbb{R}^{\oplus 240} \quad \text{or} \quad V_{(3)} \oplus V_{(1,1)}^{\oplus 6} \oplus V^{\oplus 115} \oplus \mathbb{R}^{\oplus 290}. \quad (3.42)$$

Proof Straight-forward manipulations of the Hodge numbers, similar to the OG10 case (see Sect. 3.4). We omit the details. \square

In order to decide which of the two possibilities of (3.42) actually occurs in the LLV decomposition of the OG6 example, we need to investigate further the geometric construction of [35]. First, it is not hard to lift the computations of Hodge numbers in loc. cit. to a statement about Hodge structures (Proposition 3.45). This allows us to

understand the decomposition with respect to the Mumford–Tate algebra $\bar{\mathfrak{m}}$ of $H^4(X)$ (Proposition 3.47). Finally, we complete the proof in Sect. 3.5.3, by considering the possible restriction representations of the two cases in (3.42) from \mathfrak{g} -representations to $\bar{\mathfrak{m}}$ -representations (recall $\bar{\mathfrak{m}} \subset \bar{\mathfrak{g}} \subset \mathfrak{g}$).

Remark 3.43 There are two heuristic reasons why the situation is more complicated in the OG6 case versus the OG10 case. First the LLV algebra is much smaller in this case $\mathfrak{so}(4, 6)$ (vs. $\mathfrak{so}(4, 22)$). Secondly, OG6 is an exceptional case of the Kum_n series, meaning that multiple trivial representations will occur, which in turn means less rigidity for the numerical constraints.

3.5.1 Review of [35] construction

Let X be a hyper-Kähler manifold of OG6 type. The basic topological invariants of X were found by Rapagnetta [43] by realizing X as the resolution of the quotient of some companion $K3^{[3]}$ hyper-Kähler manifold Y by a birational involution ι . This model was then used by Mongardi–Rapagnetta–Sacca [35] for the computation of Hodge numbers. We review their construction, and extract some further consequences.

Let $X = \widetilde{Y}/\iota$ as above (N.B. since the involution is only birational, the equality should be understood as contacting Y to a singular model on which the involution is regular, followed by a symplectic resolution of the quotient). To avoid working with birational involutions and singular models, one considers a blow-up \hat{Y} of Y on which the involution lifts to a regular involution $\hat{\iota}$. The quotient $\tilde{X} = \hat{Y}/\hat{\iota}$ is a blow-up of the OG6 manifold X . More specifically, one has the following diagram:

$$\begin{array}{ccc}
 \hat{Y} & & \\
 \text{blowup } \Delta \downarrow & \searrow \hat{\iota} & \\
 \tilde{Y} & & \tilde{X} \\
 \text{blowup } 256 \mathbb{P}^3 \downarrow & & \downarrow \text{blowdown } 256 \text{ quadrics} \\
 Y & \dashrightarrow \iota & X
 \end{array} \tag{3.44}$$

The following facts (cf. [35, 43]) will be needed in our arguments:

- (0) Let A be a *very general* principally polarized abelian surface. Let $S = S(A) = \widetilde{A/\pm 1}$ be the Kummer $K3$ surface associated to A (it contains 16 disjoint \mathbb{P}^1). The $K3^{[3]}$ hyper-Kähler manifold Y is birational to $S^{[3]}$, and it contains 256 disjoint \mathbb{P}^3 . The OG6 manifold X is obtained as a moduli of sheaves on A and resolving (as in [41]). By construction there is a birational involution ι on Y such that birationally $Y/\iota \cong_{bir} X$.
- (1) The blow-up $\hat{Y} \rightarrow Y$ is the composition of the blow-up \tilde{Y} of the 256 copies of \mathbb{P}^3 in Y , followed by the blow-up of the strict transform $\Delta \subset \tilde{Y}$ of a certain diagonal locus.
- (2) The center Δ of the blow-up $\hat{Y} \rightarrow \tilde{Y}$ is smooth and isomorphic to the blow-up of 256 nodes of $(A \times A^\vee)/\pm 1$. In particular, the exceptional divisor $\hat{\Delta} \subset \hat{Y}$ is a \mathbb{P}^1 bundle over Δ .

- (3) The involution ι lifts to a regular involution $\hat{\iota}$ on \hat{Y} . The quotient variety $\tilde{X} = \hat{Y}/\hat{\iota}$ is smooth, and the divisor $\tilde{\Delta} \subset \tilde{X}$ is $\hat{\iota}$ -invariant.
- (4) The OG6 manifold X is obtained from \tilde{X} by contracting 256 disjoint smooth threefolds, each isomorphic to a quadric threefold.

In [35], the Hodge numbers of X are obtained from the knowledge of the Hodge numbers of Y and tracing through the diagram (3.44) using the above mentioned facts. While not explicitly mentioned in loc. cit., a careful reading of [35, Sect. 6] gives the following statement about the relationship between the Hodge structures on X and Y . The key fact to notice here is the factor $A \times A^\vee$ arising from the blow-up of the diagonal divisor Δ . For the following discussion about the \mathbb{Q} -Hodge structures and Mumford–Tate algebras, it is necessary to work with the field \mathbb{Q} .

Proposition 3.45 *There exists an isomorphism of \mathbb{Q} -Hodge structures*

$$H^*(X, \mathbb{Q}) = H^*(Y, \mathbb{Q})^\sigma \oplus H_{\text{even}}^*(A \times A^\vee, \mathbb{Q})(-1) \oplus 256\mathbb{Q}(-3).$$

Here $H^*(Y, \mathbb{Q})^\sigma$ indicates the invariant cohomology of $H^*(Y, \mathbb{Q})$ with respect to an appropriate involution σ .

Proof This follows from Section 6 of [35]. Their statements are formulated in terms of Hodge numbers, but in fact all their proofs apply at the level of Hodge structures. First, by [35, Lemma 6.2(1)] we get

$$\begin{aligned} H^*(\tilde{X}, \mathbb{Q}) &= H^*(X, \mathbb{Q}) \oplus 256\mathbb{Q}(-1) \oplus 512\mathbb{Q}(-2) \\ &\oplus 512\mathbb{Q}(-3) \oplus 512\mathbb{Q}(-4) \oplus 256\mathbb{Q}(-5). \end{aligned}$$

Lemmas 6.2(2) and 6.3 in [35] give the Hodge structure isomorphism

$$\begin{aligned} H^*(\tilde{X}, \mathbb{Q}) &= H^*(\tilde{Y}, \mathbb{Q})^\iota \oplus H_{\text{even}}^*(A \times A^\vee, \mathbb{Q})(-1) \\ &\oplus 256\mathbb{Q}(-2) \oplus 256\mathbb{Q}(-3) \oplus 256\mathbb{Q}(-4). \end{aligned}$$

Finally, [35, Lemma 6.5(2)] states the Hodge structure isomorphism

$$\begin{aligned} H^*(\tilde{Y}, \mathbb{Q})^\iota &= H^*(Y, \mathbb{Q})^\sigma \oplus 256\mathbb{Q}(-1) \\ &\oplus 256\mathbb{Q}(-2) \oplus 512\mathbb{Q}(-3) \oplus 256\mathbb{Q}(-4) \oplus 256\mathbb{Q}(-5). \end{aligned}$$

Combining the three isomorphisms, we get the desired identification. \square

3.5.2 The Mumford–Tate decomposition

Recall that A was a very general principally polarized abelian surface. Hence, the Mumford–Tate algebra of A is

$$\bar{\mathfrak{m}} = \mathfrak{so}(\bar{T}, \bar{q}_A),$$

where $\bar{T} \subset H^2(A, \mathbb{Q})$ is the transcendental Hodge structure of the second cohomology of A and \bar{q}_A is the intersection form. Note that $\dim_{\mathbb{Q}} \bar{T} = 5$ and the signature of \bar{q}_A is $(2, 3)$. This implies the real form of $\bar{\mathfrak{m}}$ is isomorphic to $\mathfrak{so}(2, 3)$.

Lemma 3.46 *The special Mumford–Tate algebras of X and Y are both isomorphic to $\bar{\mathfrak{m}}$.*

Proof Since Y is obtained from the Kummer surface $S = A/\pm 1$, the statement is standard. Using Proposition 3.45, the statement follows also for X . We omit further details. \square

Using Lemma 3.46 and a more careful inspection of the involution σ from [35, Sect. 6], we obtain the decomposition of the cohomology of $H^4(X, \mathbb{Q})$ as a $\bar{\mathfrak{m}}$ -module.

Proposition 3.47 *Let \bar{W} be the standard $\bar{\mathfrak{m}}$ -module. Then the fourth cohomology of X has the $\bar{\mathfrak{m}}$ -module decomposition*

$$H^4(X, \mathbb{Q}) = \bar{W}_{(2)} \oplus \bar{W}_{(1,1)} \oplus 6\bar{W} \oplus 145\mathbb{Q}.$$

Proof Proposition 3.45 gives

$$H^4(X, \mathbb{Q}) = H^4(Y, \mathbb{Q})^\sigma \oplus H^2(A \times A^\vee, \mathbb{Q})(-1).$$

Let us first compute the second component. Applying Künneth and standard representation theory, we obtain

$$\begin{aligned} H^2(A \times A^\vee, \mathbb{Q}) &= H^2(A, \mathbb{Q}) \oplus H^2(A^\vee, \mathbb{Q}) \oplus \left[H^1(A, \mathbb{Q}) \otimes H^1(A, \mathbb{Q}) \right] \\ &= 2\bar{W} \oplus 2\mathbb{Q} \oplus (\bar{W}_{(\frac{1}{2}, \frac{1}{2})})^{\otimes 2} \\ &= \bar{W}_{(1,1)} \oplus 3\bar{W} \oplus 3\mathbb{Q}. \end{aligned} \tag{3.48}$$

Next, we need to compute $H^4(Y, \mathbb{Q})^\sigma$. To do so, we imitate the trick used in the proof of [35, Lem 6.6]. We first compare the second cohomology of the identification of Proposition 3.45. This gives us the Hodge structure isomorphism

$$H^2(X, \mathbb{Q}) = H^2(Y, \mathbb{Q})^\sigma \oplus \mathbb{Q}(-1).$$

But we already know what the Hodge structure of $H^2(X, \mathbb{Q})$ is by Lemma 3.46 with our old Proposition 2.38(2). Both $H^2(X, \mathbb{Q})$ and \bar{W} are $\bar{\mathfrak{m}}$ -modules and their Hodge numbers are $(1, 6, 1)$ and $(1, 3, 1)$, respectively. This forces an $\bar{\mathfrak{m}}$ -module isomorphism $H^2(X, \mathbb{Q}) = \bar{W} \oplus 3\mathbb{Q}$. Hence, we get

$$H^2(Y, \mathbb{Q})^\sigma = \bar{W} \oplus 2\mathbb{Q}$$

as $\bar{\mathfrak{m}}$ -modules. By similar Hodge number argument, we have $H^2(Y, \mathbb{Q}) = \bar{W} \oplus 18\mathbb{Q}$. Writing $H_+^2(Y, \mathbb{Q}) = H^2(Y, \mathbb{Q})^\sigma$ and $H_-^2(Y, \mathbb{Q})$ by the ± 1 eigenspaces of the

involution σ on $H^2(Y, \mathbb{Q})$, we have $\bar{\mathfrak{m}}$ -module isomorphisms

$$H_+^2(Y, \mathbb{Q}) = H^2(Y, \mathbb{Q})^\sigma = \bar{W} \oplus 2\mathbb{Q}, \quad H_-^2(Y, \mathbb{Q}) = 16\mathbb{Q}.$$

The involution σ was constructed as a monodromy operator on the space Y (see [35, §6]). Since the monodromy action respects the (reduced) Looijenga–Lunts–Verbitsky algebra $\bar{\mathfrak{g}}$ -structure on each cohomology (cf. [28]), it follows that

$$H^4(Y, \mathbb{Q}) = \text{Sym}^2 H^2(Y, \mathbb{Q}) \oplus H^2(Y, \mathbb{Q})$$

as $\bar{\mathfrak{g}}(Y)$ -modules by the computation in Remark 3.3. This means $H^4(Y, \mathbb{Q})^\sigma$ is precisely

$$\begin{aligned} H^4(Y, \mathbb{Q})^\sigma &= \text{Sym}^2 H_+^2(Y, \mathbb{Q}) \oplus \text{Sym}^2 H_-^2(Y, \mathbb{Q}) \oplus H_+^2(Y, \mathbb{Q}) \\ &= \text{Sym}^2(\bar{W} \oplus 2\mathbb{Q}) \oplus \text{Sym}^2(16\mathbb{Q}) \oplus \bar{W} \oplus 2\mathbb{Q} = \bar{W}_{(2)} \oplus 3\bar{W} \oplus 142\mathbb{Q}. \end{aligned} \quad (3.49)$$

Combining (3.48) and (3.49), we deduce the result. \square

3.5.3 Completion of the proof of Theorem 3.5

We complete the computations of the LLV decomposition in the OG6 case by studying the possible restrictions of the \mathfrak{g} -representations occurring in (3.42) to $\bar{\mathfrak{m}}$ -representations. For reader's convenience let's recall the inclusions of algebras $\bar{\mathfrak{m}} \subset \bar{\mathfrak{g}} \subset \mathfrak{g}$, with $\mathfrak{g} \cong \mathfrak{so}(4, 6)$ the LLV algebra, $\bar{\mathfrak{g}} \cong \mathfrak{so}(3, 5)$ the reduced LLV algebra, and finally $\bar{\mathfrak{m}} \cong \mathfrak{so}(2, 3)$ the Mumford–Tate algebra.

$$\begin{array}{ccccc} \bar{\mathfrak{m}} & \subset & \bar{\mathfrak{g}} & \subset & \mathfrak{g} \\ \parallel & & \parallel & & \parallel \\ \mathfrak{so}(2, 3) & \subset & \mathfrak{so}(3, 5) & \subset & \mathfrak{so}(4, 6) \end{array}$$

We also recall that $\bar{\mathfrak{g}}$ (and thus also $\bar{\mathfrak{m}}$) respect the cohomological degree. We focus on degree 4 cohomology $H^4(X)$ as the first non-obvious piece for the $\bar{\mathfrak{g}}$ -action. First we investigate the restriction of the two cases of (3.42) from \mathfrak{g} to $\bar{\mathfrak{g}}$ -modules.

Lemma 3.50 *Let X be a hyper-Kähler 6-fold with $b_2(X) = 8$.*

(i) *Assume that the LLV decomposition of $H^*(X)$ is*

$$H^*(X) = V_{(3)} \oplus V_{(1,1,1)} \oplus 135V \oplus 240\mathbb{R}$$

as $\mathfrak{g} \cong \mathfrak{so}(4, 6)$ -modules. Then

$$H^4(X) = \bar{V}_{(2)} \oplus \bar{V}_{(1,1)} \oplus 136\mathbb{R}. \quad (3.51)$$

as $\bar{\mathfrak{g}} \cong \mathfrak{so}(3, 5)$ -modules.

(ii) Assume that the LLV decomposition of $H^*(X)$ is

$$H^*(X) = V_{(3)} \oplus 6V_{(1,1)} \oplus 115V \oplus 290\mathbb{R}$$

as $\mathfrak{g} \cong \mathfrak{so}(4, 6)$ -modules. Then

$$H^4(X) = \bar{V}_{(2)} \oplus 6\bar{V} \oplus 116\mathbb{R}. \quad (3.52)$$

as $\bar{\mathfrak{g}} \cong \mathfrak{so}(3, 5)$ -modules.

Proof We proceed as in Remark 3.3 (see also Appendix B). Recall that the standard representation V of \mathfrak{g} is the Mukai completion of the standard representation \bar{V} of $\bar{\mathfrak{g}}$. Regarding V as a $\bar{\mathfrak{g}}$ module gives

$$V = \mathbb{R}(1) \oplus \bar{V} \oplus \mathbb{R}(-1),$$

where we indicate the twist to keep track of the cohomological degree. It is immediate to see

$$\begin{aligned} V_{(3)} &= \mathbb{R}(3) \oplus \bar{V}(2) \oplus \text{Sym}^2 \bar{V}(1) \oplus \text{Sym}^3 \bar{V} \oplus \text{Sym}^2 \bar{V}(-1) \oplus \bar{V}(-2) \oplus \mathbb{R}(-3), \\ V_{(1,1,1)} &= \wedge^2 \bar{V}(1) \oplus [\wedge^3 \bar{V} \oplus \bar{V}] \oplus \wedge^2 \bar{V}(-1), \\ V_{(1,1)} &= \bar{V}(1) \oplus [\wedge^2 \bar{V} \oplus \mathbb{R}] \oplus \bar{V}(-1). \end{aligned}$$

It follows that

$$\begin{aligned} H^*(X) &= \mathbb{R}(3) \oplus \bar{V}(2) \oplus \left[\text{Sym}^2 \bar{V} \oplus \wedge^2 \bar{V} \oplus 135\mathbb{R} \right] (1) \\ &\quad \oplus \left[\text{Sym}^3 \bar{V} \oplus \wedge^3 \bar{V} \oplus 136\bar{V} \oplus 240\mathbb{R} \right] \oplus \dots, \end{aligned}$$

for the first case, and

$$\begin{aligned} H^*(X) &= \mathbb{R}(3) \oplus \bar{V}(2) \oplus \left[\text{Sym}^2 \bar{V} \oplus 6\bar{V} \oplus 115\mathbb{R} \right] (1) \\ &\quad \oplus \left[\text{Sym}^3 \bar{V} \oplus 6\wedge^2 \bar{V} \oplus 116\bar{V} \oplus 296\mathbb{R} \right] \oplus \dots, \end{aligned}$$

for the second case. The lemma follows. \square

Finally, we restrict from $\bar{\mathfrak{g}}$ -representation on $H^4(X)$ to a $\bar{\mathfrak{m}}$ -representation.

Proposition 3.53 *With notations and assumptions as in Lemma 3.50*

(i) *If (3.51) holds, then*

$$H^4(X) = \bar{W}_{(2)} \oplus \bar{W}_{(1,1)} \oplus 6\bar{W} \oplus 145\mathbb{R}$$

as $\bar{\mathfrak{m}} \cong \mathfrak{so}(2, 3)$ -modules.

(ii) *If (3.52) holds, then*

$$H^4(X) = \bar{W}_{(2)} \oplus 9\bar{W} \oplus 140\mathbb{R}.$$

as $\bar{\mathfrak{m}} \cong \mathfrak{so}(2, 3)$ -modules.

Proof This follows directly from the result of Lemma 3.50 with the decomposition $\bar{V} = \bar{W} \oplus 3\mathbb{R}$. The latter fact follows from the comparison of the Hodge diamond of \bar{V} and \bar{W} , which are $(1, 6, 1)$ and $(1, 3, 1)$ respectively. \square

Proof Theorem 3.39 Using the numerical restrictions on OG6 type, we have determined two compatible LLV $\mathfrak{g} = \mathfrak{so}(4, 6)$ -decompositions of the cohomology (Proposition 3.41). In Proposition 3.53, we have determined the restrictions of these two cases as representations of the Mumford–Tate algebra $\bar{\mathfrak{m}} = \mathfrak{so}(2, 3)$. Only one of them matches the geometric possibility identified in Proposition 3.47. The claim follows. \square

4 Period maps, monodromy, and the LLV algebra

As discussed in Sect. 2.2.1, the Hodge structure of the cohomology of X is determined by two operators h, f contained in the LLV algebra \mathfrak{g} . The purpose of this section is to discuss the behavior of the Hodge structure of hyper-Kähler manifolds in families, and more precisely to discuss the higher degree period maps. Given that the Torelli theorem holds for the second cohomology of hyper-Kähler manifolds, it is no surprise that the higher degree period maps and monodromy operators are determined by those for $H^2(X)$. Our main results (Theorems 4.1 and 4.9) of this section achieve precisely these. We note the recent papers of Soldatenkov [48, 49] cover similar ground. Theorem 4.9 is already proved in [48, Prop. 3.4], and here we provide its alternative Hodge theoretic proof, more related to the spirit of Torelli theorem and our discussion on LLV decomposition. We also note that many of the discussions here are already conceptually treated in [15, Ch. III–IV], though our two main results about higher period maps of hyper-Kähler manifolds are not discussed there.

4.1 Higher degree period maps

Throughout this subsection⁶, we fix a compact hyper-Kähler manifold X . Consider any smooth proper family \mathfrak{X}/S of hyper-Kähler manifolds over a complex manifold S , whose fiber at $0 \in S$ is isomorphic to X . For each degree $0 \leq k \leq 4n$, one associates the period map

$$\Phi_k : \tilde{S} \rightarrow D_k$$

from the universal cover \tilde{S} of S to the classifying space D_k of Hodge structures with specified Hodge numbers, matching those of $H^k(X)$. Verbitsky’s global Torelli

⁶ The arguments in this section follow a suggestion by an anonymous referee. They replace our original infinitesimal approach.

theorem [20,55] says that a compact hyper-Kähler manifold is essentially recovered from its second Hodge structure. Thus, one expects that the k -th period map Φ_k is recovered from the second period map Φ_2 . Here, we make this more precise.

To start, we define the *k -th period variety* (for $k > 2$) to be the symmetric space parameterizing Hodge flags with specified Hodge numbers

$$\check{D}_k = \text{Flag}(H^k(X, \mathbb{C}), (f^\bullet)),$$

where (f^\bullet) indicates the dimensions of the Hodge filtration of the k -th cohomology of X . It is a smooth projective variety, on which the general linear group $\text{GL}(H^k(X, \mathbb{C}))$ acts transitively. Let us fix a reference point $o_k \in \check{D}_k$ corresponding to the Hodge structure of the original hyper-Kähler manifold X . Then we have an identification

$$\check{D}_k = \text{GL}(H^k(X, \mathbb{C}))/P_k,$$

where $P_k \subset \text{GL}(H^k(X, \mathbb{C}))$ is the stabilizer at o_k . The case $k = 2$ is special as we take into account the Beauville–Bogomolov form (giving a polarization). The structure group reduces to the special orthogonal group, and we can define the *second period variety*, or the *second compact dual* (of the *period domain*) by $\check{D}_2 = \text{SO}(\bar{V}, \bar{q})_{\mathbb{C}}/P_2$. For our purpose, it will be convenient to replace the group $\text{SO}(\bar{V}, \bar{q})$ by its degree 2 universal cover $\text{Spin}(\bar{V}, \bar{q})$ and represent the compact dual period domain by the quotient of a Spin group

$$\check{D}_2 = \text{Spin}(\bar{V}, \bar{q})_{\mathbb{C}}/P_2.$$

The *second period domain* D_2 can be realized by an open subdomain in \check{D}_2 as a $\text{Spin}(\bar{V}, \bar{q})_{\mathbb{R}}$ -orbit. In the absence of a polarization on $H^k(X)$, $k \geq 3$, there is no naturally defined period domain $D_k \subset \check{D}_k$. Nevertheless, we still have the holomorphic period map $\Phi_k : \tilde{S} \rightarrow \check{D}_k$.

With these preliminaries, we can state our first main result of this section:

Theorem 4.1 *For each k , there exists a morphism $\psi_k : \check{D}_2 \rightarrow \check{D}_k$ with the following property. Let \mathfrak{X}/S be a smooth proper family of hyper-Kähler manifolds whose fiber at $0 \in S$ is isomorphic to X . Let \tilde{S} be the universal covering of S , and $\Phi_2 : \tilde{S} \rightarrow \check{D}_2$ and $\Phi_k : \tilde{S} \rightarrow \check{D}_k$ the second and k -th period maps associated to the family. Then we have a factorization $\Phi_k = \psi_k \circ \Phi_2$.*

$$\begin{array}{ccccc} \tilde{S} & \xrightarrow{\Phi_2} & \check{D}_2 & \xrightarrow{\psi_k} & \check{D}_k \\ & & \underbrace{\hspace{10em}}_{\Phi_k} & & \end{array} \tag{4.2}$$

The only nontrivial part of the theorem is the construction of such a morphism ψ_k . After it, the proof will be a formal consequence of our discussions in Sect. 2. As one can expect, the essential point is the existence of the reduced LLV representation $\rho_k : \bar{\mathfrak{g}} \rightarrow \mathfrak{gl}(H^k(X, \mathbb{Q}))$ in (2.5). Throughout, we will abuse the notation and denote

its algebraic group version by the same symbol

$$\rho_k : \mathrm{Spin}(\bar{V}, \bar{q}) \rightarrow \mathrm{GL}(H^k(X, \mathbb{Q})).$$

Recall that we have fixed the base points o_k of the period varieties \check{D}_k that correspond to the original hyper-Kähler manifold X . Note that the theorem claims (when S is a point) $\psi_k(o_2) = o_k$.

Proposition 4.3 *There exists a unique equivariant morphism $\psi_k : \check{D}_2 \rightarrow \check{D}_k$ with $\psi_k(o_2) = o_k$.*

Proof Recall from above that the period varieties can be described as $\check{D}_2 = \mathrm{Spin}(\bar{V}, \bar{q})_{\mathbb{C}}/P_2$ and $\check{D}_k = \mathrm{GL}(H^k(X, \mathbb{C}))/P_k$. We would like to construct a mor-

$$\begin{array}{ccc} \mathrm{Spin}(\bar{V}, \bar{q})_{\mathbb{C}} & \xrightarrow{\rho_k} & \mathrm{GL}(H^k(X, \mathbb{C})) \\ \downarrow & & \downarrow \\ \check{D}_2 & \xrightarrow{\psi_k} & \check{D}_k \end{array}.$$

phism ψ_k by a quotient of ρ_k :

Here the identity elements in the first row are sent to o_2 and o_k respectively, giving $\psi_k(o_2) = o_k$. To descend to a quotient map as desired, we need to prove $\rho_k(P_2) \subset P_k$. By the Lie algebra–algebraic group correspondence, it suffices to prove its Lie algebra version $\rho_k(\mathfrak{p}_2) \subset \mathfrak{p}_k$.

Let $f \in \bar{\mathfrak{g}}_{\mathbb{R}}$ be the Hodge operator associated to the complex structure of X . The Lie algebra homomorphism $\rho_k : \bar{\mathfrak{g}}_{\mathbb{R}} \rightarrow \mathfrak{gl}(H^k(X, \mathbb{R}))$ sends the operator f to $\rho_k(f)$, the operator defining the Hodge structure of $H^k(X)$. Since ρ_k is a Lie algebra homomorphism, it in particular respects the (adjoint) f -action. In other words, ρ_k is a Hodge structure homomorphism. By standard Hodge theory, the Lie algebras of the stabilizers \mathfrak{p}_2 and \mathfrak{p}_k admit their own Hodge decompositions

$$\mathfrak{p}_2 = \bar{\mathfrak{g}}^{1,-1} \oplus \bar{\mathfrak{g}}^{0,0}, \quad \mathfrak{p}_k = \mathfrak{gl}(H^k(X))^{k,-k} \oplus \cdots \oplus \mathfrak{gl}(H^k(X))^{0,0}.$$

Hence ρ_k sends \mathfrak{p}_2 into \mathfrak{p}_k and the claim follows. Here one can further prove ψ_k is a horizontal map, but we omit its proof as we will not need this fact in our future discussion. \square

Our definition of the morphism ψ_k is formal. In order to prove the more geometric Theorem 4.1, we need to understand it more concretely. Let $\sigma \in \check{D}_2 \subset \check{D}_2$. This corresponds to a Hodge structure on $H^2(X, \mathbb{Q})$ with $H^{2,0} = \mathbb{C}\sigma$, $H^{0,2} = \mathbb{C}\bar{\sigma}$ and $H^{1,1} = \mathbb{C}\{\sigma, \bar{\sigma}\}^{\perp}$. This gives rise to a Hodge operator $f_{\sigma} \in \bar{\mathfrak{g}}_{\mathbb{R}}$ by our discussion in Sect. 2. Now consider the element $\rho_k(f_{\sigma}) \in \mathfrak{gl}(H^k(X, \mathbb{R}))$ given by the reduced LLV representation. Its eigenspaces define a Hodge structure on $H^k(X, \mathbb{Q})$.

Lemma 4.4 *Let $\sigma \in \check{D}_2$, and $f_{\sigma} \in \bar{\mathfrak{g}}_{\mathbb{R}}$ be as above. Then $\psi_k(\sigma) \in \check{D}_k$ defines a Hodge structure corresponding to the Hodge operator $\rho_k(f_{\sigma}) \in \mathfrak{gl}(H^k(X, \mathbb{R}))$.*

Proof The idea is to use the equivariance of the morphism ψ_k in Proposition 4.3. Note that $\mathrm{Spin}(\bar{V}, \bar{q})_{\mathbb{R}}$ acts on the period domain \check{D}_2 transitively, so we may assume $\sigma = g.o_2$ for some $g \in \mathrm{Spin}(\bar{V}, \bar{q})_{\mathbb{R}}$. Now use the equivariance of ψ_k to write $\psi_k(\sigma) = \rho_k(g).\psi_k(o_2) = \rho_k(g).o_k$.

Let F_2^\bullet and F_k^\bullet be the reference Hodge filtrations associated to the base points o_2 and o_k . The Hodge operators corresponding to them are $f \in \bar{\mathfrak{g}}_{\mathbb{R}}$ and $\rho_k(f) \in \mathfrak{gl}(H^k(X, \mathbb{R}))$. Hence the Hodge filtrations associated to $g.o_2$ and $\rho_k(g).o_k$ are $g.F_2^\bullet$ and $\rho_k(g).F_k^\bullet$. Their associated Hodge operators are $f_\sigma = g.f$ and $\rho_k(g).\rho_k(f)$, where the actions here are the adjoint actions (conjugations). The claim follows from $\rho_k(g).\rho_k(f) = \rho_k(g.f) = \rho_k(f_\sigma)$. \square

The remaining step is to relate our formal construction to the definition of period maps.

Proof of Theorem 4.1 Let $s \in \tilde{S}$ be any point. Let us first review the definition of the period mappings $\Phi_2(s)$ and $\Phi_k(s)$. First, fix any path connecting 0 and s . Diffeomorphically trivialize the given family \mathfrak{X}/S along the path, so that we have a “parallel transport” identification between the fibers $\text{PT} : X = X_0 \rightarrow X_s$. The map PT is a diffeomorphism. Hence, (1) it induces an isomorphism of LLV algebras $\mathfrak{g}(X_s)$ and $\mathfrak{g}(X)$; and (2) the pullback $\text{PT}^* : H^*(X_s, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ becomes an LLV module isomorphism. However, PT^* is not a Hodge structure isomorphism as it is not biholomorphic. We define $\Phi_k(s) \in \check{D}_k$ by the Hodge structure of $H^k(X_s, \mathbb{Q})$ transported into the vector space $H^k(X, \mathbb{Q})$ under the identification PT^* . For hyper-Kähler manifolds, the Hodge structure on $H^k(X_s, \mathbb{Q})$ is captured by the Hodge operator $\rho_k(f_s) \in \bar{\mathfrak{g}}(X_s)_{\mathbb{R}}$ by discussions in Sect. 2. Hence $\Phi_k(s)$ is captured by $\text{PT}^*(\rho_k(f_s)) \in \bar{\mathfrak{g}}(X)_{\mathbb{R}}$.

Note that PT^* is an LLV module isomorphism. Hence $\text{PT}^*(\rho_k(f_s)) = \rho_k(\text{PT}^*(f_s))$. By definition of Φ_2 (in the previous paragraph), $\text{PT}^*(f_s) \in \bar{\mathfrak{g}}_{\mathbb{R}}$ is the Hodge operator associated to $\Phi_2(s)$. Hence by Lemma 4.4, the Hodge operator $\rho_k(\text{PT}^*(f_s))$ is associated to $\psi_k(\Phi_2(s))$. This proves $\Phi_k(s) = \psi_k(\Phi_2(s))$. \square

So far, we did not assume the family \mathfrak{X}/S is projective and everything was proved without the projectivity assumption. Now if we further assume the family is projective, then we can restrict our commutative diagram (4.2) further to the period domains. This can be done with the aid of the theory of Mumford–Tate subdomains of period varieties [15]. Let us assume X is a projective hyper-Kähler manifold and fix a polarization

$$l \in H^2(X, \mathbb{Z}). \quad (4.5)$$

Set $D_2^l \subset \check{D}_2$ and $D_k^l \subset \check{D}_k$ by the period domain associated to the primitive Hodge structures $H_{\text{prim}}^2(X, \mathbb{Q})$ and $H_{\text{prim}}^k(X, \mathbb{Q})$ with respect to the polarization l .

Corollary 4.6 *Assume further in Theorem 4.1 that \mathfrak{X}/S is a smooth projective family of hyper-Kähler manifolds with a polarization l . Then the following diagram commutes.*

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\Phi_2} & D_2^l \xrightarrow{\psi_k} D_k^l \\ & \underbrace{\curvearrowright}_{\Phi_k} & \end{array} \quad (4.7)$$

Proof It is enough to show $\psi_k(D_2^l) \subset D_k^l$. Let $\overline{\text{MT}}_2 = \text{SO}(H_{\text{prim}}^2(X, \mathbb{Q}), \bar{q})$ be the generic special Mumford–Tate group of the Hodge structures $H^2(X, \mathbb{Q})$ with $l \in H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$. The period domain D_2^l can be obtained as a Mumford–Tate

subdomain of \check{D}_2 , the $\overline{\text{MT}}_{2,\mathbb{R}}$ -orbit of a point $\Phi_2(s_0) \in \check{D}_2$. By Proposition 2.38, the k -th generic special Mumford–Tate group is

$$\overline{\text{MT}}_k = \rho_k(\overline{\text{MT}}_2^\sim),$$

where $\overline{\text{MT}}_2^\sim$ is the inverse image of $\overline{\text{MT}}_2$ by the degree 2 isogeny $\text{Spin}(\bar{V}, \bar{q}) \rightarrow \text{SO}(\bar{V}, \bar{q})$. Hence the k -th Mumford–Tate subdomain in \check{D}_k is the orbit of $\rho_k(\overline{\text{MT}}_2)_{\mathbb{R}}$, which is just $\psi_k(D_2^l)$ as ψ_k is an equivariant map by construction. This proves $\psi_k(D_2^l) \subset D_k^l$. \square

4.2 Higher degree monodromy operators

Let us now consider a one-parameter projective degeneration \mathfrak{X}/Δ of a hyper-Kähler manifold X . By this, we mean a flat projective morphism $\mathfrak{X} \rightarrow \Delta$ over the unit disk Δ such that its restriction $\mathfrak{X}^* \rightarrow \Delta^*$ to the punctured disk Δ^* is smooth. In this situation, one gets a monodromy operator $T_k \in \text{GL}(H^k(X))$ associated to the smooth family \mathfrak{X}^*/Δ^* for each cohomological degree $0 \leq k \leq 4n$. Since each T_k is a quasi-unipotent operator by the monodromy theorem (e.g. [45]), we can define the k -th log monodromy operator by its logarithm

$$N_k = \frac{1}{m} \log((T_k)^m) \in \text{End}(H^k(X)), \quad (4.8)$$

where $m \in \mathbb{Z}_{>0}$ is such that $(T_k)^m$ is unipotent.

Any diffeomorphism of X respects the Beauville–Bogomolov form on $H^2(X)$. Thus, for the second monodromy we can further say $T_2 \in \text{SO}(\bar{V}, \bar{q})$ and hence $N_2 \in \mathfrak{so}(\bar{V}, \bar{q}) = \bar{\mathfrak{g}}$. Recall we had the reduced LLV representation $\rho_k : \bar{\mathfrak{g}} \rightarrow \text{End}(H^k(X))$, so we have two elements N_k and $\rho_k(N_2)$ in $\text{End}(H^k(X))$. It was already observed by Soldatenkov that these two elements coincide. Here we recover his result using our results on the relation between the second and higher degree period maps (Theorem 4.1 and Corollary 4.6); this is the second main theorem of this section.

Theorem 4.9 (Soldatenkov [48]). *For any one-parameter projective degeneration \mathfrak{X}/Δ of hyper-Kähler manifolds, the k -th log monodromy is determined by the second log monodromy by the relation*

$$N_k = \rho_k(N_2) \quad \text{for all } 0 \leq k \leq 4n.$$

In fact, we will prove a slightly stronger result over an arbitrary complex manifold S . For its statement, consider the degree 2 universal covering $\text{Spin}(\bar{V}, \bar{q}) \rightarrow \text{SO}(\bar{V}, \bar{q})$. Among the two preimages of $T_2 \in \text{SO}(\bar{V}, \bar{q})$, let us make a choice and denote it by

$$\tilde{T}_2 \in \text{Spin}(\bar{V}, \bar{q}). \quad (4.10)$$

The choice will be not a big issue because the square of the two preimages are the same; our result is about sufficient powers of the monodromies.

Theorem 4.11 *Let \mathfrak{X}/S be a smooth projective family of hyper-Kähler manifolds with a fixed polarization l . Let $\gamma \in \pi_1(S, s_0)$ and T_k the image of γ by the monodromy representation $\pi_1(S, s_0) \rightarrow \mathrm{GL}(H^k(X))$. Let $\tilde{T}_2 \in \mathrm{Spin}(\tilde{V}, \tilde{q})$ be the lifting of T_2 as in (4.10). Then there exists a positive integer $d = d(l) \in \mathbb{Z}_{>0}$, solely depending on l , such that*

$$(T_k)^d = \rho_k(\tilde{T}_2)^d \quad \text{for all } 0 \leq k \leq 4n.$$

Theorem 4.9 follows directly from Theorem 4.11 by letting $S = \Delta^*$ and taking the logarithm of the equality. For the proof of Theorem 4.11, the following simple lemma is crucial.

Lemma 4.12 *Let D be a real homogeneous space on which a semisimple real Lie group $G_{\mathbb{R}}$ is acting transitively. Fix any point $o \in D$ and assume its stabilizer $K \subset G_{\mathbb{R}}$ is a compact subgroup. Let $G_{\mathbb{Z}} \subset G_{\mathbb{R}}$ be an arithmetic subgroup, and suppose we have two elements $T, T' \in G_{\mathbb{Z}}$ with the property*

$$T^m \cdot o = (T')^m \cdot o \quad \text{for all } m \in \mathbb{Z}.$$

Then there exists a positive integer $0 < d \leq |G_{\mathbb{Z}} \cap K|$ such that $T^d = (T')^d$.

Proof Since $K \subset G_{\mathbb{R}}$ is compact and $G_{\mathbb{Z}} \subset G_{\mathbb{R}}$ is discrete, $G_{\mathbb{Z}} \cap K$ is a finite group. From the given assumption, we obtain $(T^{-m}(T')^m) \cdot o = o$ and hence $T^{-m}(T')^m \in G_{\mathbb{Z}} \cap K$ for all m . Since $G_{\mathbb{Z}} \cap K$ is finite, by pigeonhole principle there exist two distinct integers $0 \leq m_1 < m_2 \leq |G_{\mathbb{Z}} \cap K|$ so that $T^{-m_1}(T')^{m_1} = T^{-m_2}(T')^{m_2}$. This implies $T^{m_2-m_1} = (T')^{m_2-m_1}$. \square

Proof of Theorem 4.11 Consider the diagram (4.7) in Corollary 4.6. By construction, the period maps certainly satisfy the following relations for all $s \in \tilde{S}$:

$$\Phi_k(\gamma \cdot s) = T_k \cdot \Phi_k(s), \quad \Phi_2(\gamma \cdot s) = T_2 \cdot \Phi_2(s) = \tilde{T}_2 \cdot \Phi_2(s).$$

Now from the commutativity $\Phi_k = \psi_k \circ \Phi_2$ of the diagram, we have a sequence of identities

$$\Phi_k(\gamma \cdot s) = \psi_k(\Phi_2(\gamma \cdot s)) = \psi_k(\tilde{T}_2 \cdot \Phi_2(s)) = \rho_k(\tilde{T}_2) \cdot \psi_k(\Phi_2(s)) = \rho_k(\tilde{T}_2) \cdot \Phi_k(s).$$

This gives us the condition

$$(T_k)^m \cdot \Phi_k(s) = \Phi_k(\gamma^m \cdot s) = \rho_2(\tilde{T}_2)^m \cdot \Phi_k(s) \quad \text{for all } m \in \mathbb{Z}.$$

Now we can apply Lemma 4.12 to T_k and $\rho_k(\tilde{T}_2)$. It follows $(T_k)^d = \rho_k(\tilde{T}_2)^d$ for some $0 < d \leq A$, where A denotes the cardinality of the finite set $\mathrm{GL}(H_{\mathrm{prim}}^k(X, \mathbb{Z})) \cap K_k$. Hence $(T_k)^{A!} = \rho_k(\tilde{T}_2)^{A!}$ and $A!$ does not depend on the degeneration and γ .

Refining the argument gives us a smaller constant. We can replace the period domain D_k^l in Corollary 4.6 by the Mumford–Tate subdomain $\psi_k(D_2^l)$. Note that we can still

apply Lemma 4.12 in this situation, because the monodromy T_k is contained in the Mumford–Tate group $\overline{\text{MT}}_k = \rho_k(\overline{\text{MT}}_2)$. Hence we can replace the constant A by the cardinality of $\text{GL}(H^2(X, \mathbb{Z})) \cap K_2$, which is much smaller than the previous one. Also, one can replace $A!$ by $\text{lcm}\{d : 1 \leq d \leq A\}$. \square

5 Nagai’s conjecture and the Looijenga–Lunts–Verbitsky decomposition

Let \mathcal{X}/Δ be a projective one-parameter degeneration of hyper-Kähler manifolds. Consider the associated monodromy operators N_k on the degree k -cohomology (cf. (4.8)), and let ν_k be their nilpotency indices. Nagai [37] conjectured that ν_2 determines the higher nilpotency indices by

$$\nu_{2k} = k \cdot \nu_2 \quad \text{for } k = 1, \dots, n \quad (1.13 \text{ (restated)})$$

(see [48] for a partial discussion of the odd cohomology case). The purpose of this section is to show that the equation above is in fact equivalent to a non-trivial condition on the LLV decomposition of hyper-Kähler manifolds. Specifically, let us write the LLV decomposition on the even cohomology part:

$$H_{\text{even}}^*(X) \cong \bigoplus_{\mu \in S} V_\mu^{\oplus m_\mu}, \quad (5.1)$$

where $\mu = (\mu_0, \dots, \mu_r)$ indicates a dominant integral weight of \mathfrak{g} and V_μ its associated highest weight module.

Also, recall that the Type of the degeneration is I, II and III, depending on the nilpotency index $\nu_2 \in \{0, 1, 2\}$ of the second log monodromy. Nagai’s conjecture was already established for the Type I and III cases by geometric methods in [22]. Hence, the Type II degeneration is the remaining interest. With these preliminaries, we can state the main result of the section.

Theorem 5.2 *Let X be a projective hyper-Kähler manifold with $b_2(X) \geq 5$. Suppose that every irreducible \mathfrak{g} -module component V_μ appearing in (5.1) satisfies the inequality*

$$\mu_0 + \mu_1 + \mu_2 \leq n. \quad (5.3)$$

Then Nagai’s conjecture (1.13) holds for any one-parameter projective degeneration of X . Conversely, if Nagai’s conjecture holds for a single Type II degeneration of X then the inequality (5.3) holds.

Remark 5.4 In other words, Nagai’s conjecture is essentially equivalent to the condition (5.3), except for the hypothetical situation when there is no Type II degeneration (in which case, Nagai’s conjecture would be trivially true). To understand this case, let us recall that the moduli space of polarized hyper-Kähler manifolds is a locally symmetric variety D/Γ , with D a Type IV Hermitian symmetric domain and Γ an arithmetic group. In this set-up, the existence of a Type II degeneration is equivalent to the existence of a rank 2 totally isotropic sublattice in the Beauville–Bogomolov

lattice $H^2(X, \mathbb{Z})$. For signature reasons, such an isotropic sublattice can not exist if $b_2(X) \leq 4$. On the other hand, for $b_2(X) \geq 7$, general lattice theory guarantees the existence of rank 2 totally isotropic sublattices, and thus the equivalence of Nagai's conjecture to condition (5.3). Hence, the only ambiguous cases (where the existence of Type II degenerations is unclear) are $b_2(X) = 5$ or 6. Of course, at this point, no such examples of hyper-Kähler manifolds are known. Note also that the condition $b_2(X) \geq 7$ occurs naturally in [51].

The rest of this section will be devoted to the proof of Theorem 5.2. We divide the proof into three cases depending on the Type of the degeneration. As mentioned, Nagai's conjecture for the Type I and III cases were already established in [22]. From our perspective, Type I is trivial as N_{2k} is determined (see Sect. 4) via the LLV decomposition from $N_2 (= 0$ for Type I). The argument for Type III is similar to that in [22] (essentially the Verbitsky component $V_{(n)}$ is always present in the LLV decomposition). Finally, the Type II case requires a more delicate representation theoretic argument.

5.1 Type I and III degenerations

In Sect. 4 we have discussed the interplay between the LLV decomposition and the period map. In particular, we have seen in Theorem 4.9 that in the case of one-parameter degenerations, the second monodromy operator $N = N_2 \in \bar{\mathfrak{g}}$ determines (for simplicity, we write N instead of N_2 from now on) all the monodromy operators N_k by

$$N_k = \rho_k(N)$$

where $\rho_k : \bar{\mathfrak{g}} \rightarrow \text{End}(H^k(X))$ is the representation of the reduced LLV algebra. As an immediate consequence we obtain the Type I and III cases of Nagai's conjecture without any further restrictions.

Proposition 5.5 *Nagai's conjecture (1.13) holds for type I and III degenerations of projective hyper-Kähler manifolds.*

Proof Type I degeneration is equivalent to $v_2 = 0$, i.e. $N = 0$. Since $N_k = \rho_k(N)$, we conclude $N_k = 0$ for all k (compare [22, Cor. 3.2]). In particular, $v_{2k} = 0$ as needed.

For Type III degenerations ($v_2 = 2$), on one hand, we have the general bound on the index on nilpotency on H^{2n} by the monodromy theorem, i.e. $v_{2k} \leq 2k$. Conversely, we recall that the LLV decomposition of $H^*(X)$ always contains the Verbitsky component $V_{(n)}$. From Appendix B, Verbitsky component splits as a direct sum of $\text{Sym}^k \bar{V} \subset H^{2k}(X)$ as a $\bar{\mathfrak{g}}$ -module. Hence we have a $\bar{\mathfrak{g}}$ -module decomposition

$$H^{2k}(X) = \text{Sym}^k \bar{V} \oplus \bar{V}'_{2k},$$

or equivalently ρ_{2k} splits as $\text{Sym}^k \rho_2 \oplus \rho'_{2k}$, where $\text{Sym}^k \rho_2 : \bar{\mathfrak{g}} \rightarrow \text{End}(\text{Sym}^k \bar{V})$ and ρ'_{2k} is some residual representation. Since $N_{2k} = \rho_{2k}(N)$, and the nilpotency index on $\text{Sym}^k \rho_2$ is $2k$ (cf. Lemma 5.6), the claim follows. \square

The following lemma is standard. For completeness, we include the proof.

Lemma 5.6 *Let $\text{Sym}^k \rho_2 : \bar{\mathfrak{g}} \rightarrow \text{End}(\text{Sym}^k \bar{V})$ be the $\bar{\mathfrak{g}}$ -module structure on $\text{Sym}^k \bar{V}$. Then $\text{Sym}^k \rho_2(N)$ has nilpotency index $k \cdot v_2$.*

Proof The operator $\text{Sym}^k \rho_2(N)$ acts on $\text{Sym}^k \bar{V}$ as follows:

$$(\text{Sym}^k \rho_2(N))(x_1 \cdots x_k) = \sum_{i=1}^k x_1 \cdots x_{i-1} N(x_i) x_{i+1} \cdots x_k.$$

Here we considered $N \in \text{End}(\bar{V})$ as a linear operator on \bar{V} . Recall by definition of the nilpotency index v_2 , we have $N^{v_2+1} = 0$ but $N^{v_2} \neq 0$. One computes

$$(\text{Sym}^k \rho_2(N))^{k v_2}(x^k) = (\text{const.})(N^{v_2}(x))^k, \quad (\text{Sym}^k \rho_2(N))^{k v_2+1} = 0,$$

which establishes the claim. \square

5.2 Type II degeneration

Consider the reduced LLV decomposition the $2k$ -th cohomology

$$H^{2k}(X) \cong \bigoplus_{\lambda} \bar{V}_{\lambda}^{\oplus n_{\lambda}}. \quad (5.7)$$

Here $\lambda = (\lambda_1, \dots, \lambda_r)$ denotes a dominant integral weight of $\bar{\mathfrak{g}}$ and \bar{V}_{λ} denotes a highest $\bar{\mathfrak{g}}$ -module of weight λ . Proposition 2.35 tells us that every λ in this decomposition has integer coefficients λ_i . For each of such components \bar{V}_{λ} 's, we can in fact compute the nilpotency index of the log monodromy N_{2k} on this component. This is the content of the next lemma, which is the core computation used in the proof of Theorem 5.2.

Lemma 5.8 *Assume $b_2(X) \geq 5$. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a dominant integral weight of $\bar{\mathfrak{g}}$ with $\lambda_i \in \mathbb{Z}$ and $\rho_{\lambda} : \bar{\mathfrak{g}} \rightarrow \text{End}(\bar{V}_{\lambda})$ the highest $\bar{\mathfrak{g}}$ -module associated to it.*

- (i) *If $v_2 = 1$, then $\rho_{\lambda}(N)$ has nilpotency index $\lambda_1 + \lambda_2$.*
- (ii) *If $v_2 = 2$, then $\rho_{\lambda}(N)$ has nilpotency index $2\lambda_1$.*

We note that the proof of this lemma is *not* purely representation theoretic. The fact that N is obtained from a degeneration of Hodge structure, and hence associated to a limit mixed Hodge structure of the Hodge structure \bar{V} of K3 type, will be crucially used. For an arbitrary choice of an element $N \in \bar{\mathfrak{g}}$, the lemma would not hold.

The proof of Lemma 5.8 is quite lengthy, so we would like to devote the rest of this subsection for its proof. The proof of Theorem 5.2 is then completed in Sect. 5.3. Before getting into the proof, note that the statement of the lemma does not depend on the base field. Hence, it is enough to prove the lemma over \mathbb{C} . For simplicity, let us omit the base change index \mathbb{C} and assume everything is complexified from now on.

The first step is to give a normalization of the monodromy action on $\bar{V} = H^2(X)$. Since we are working over \mathbb{C} , we can assume that the quadratic space (\bar{V}, \bar{q}) has one

of the following standard form:

$$\begin{pmatrix} 0 & \text{id}_{r \times r} & 0 \\ \text{id}_{r \times r} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \text{id}_{r \times r} \\ \text{id}_{r \times r} & 0 \end{pmatrix} \quad (5.9)$$

depending on the parity of the dimension of \bar{V} . The content of the following proposition is to say that both N and \bar{q} can be suitably normalized.

Lemma 5.10 *Assume $b_2(X) = \dim \bar{V} \geq 5$. Let $N = N_2 \in \text{End}(\bar{V})$ be the second log monodromy and v_2 its nilpotency index.*

(i) *If $v_2 = 1$, then $\dim(\text{im } N) = 2$. Moreover, there exists a basis*

$$\{e_1, \dots, e_r, e'_1, \dots, e'_r, (e_{r+1})\}$$

of \bar{V} such that \bar{q} with respect to it has a matrix form (5.9), and

$$N \left(\sum_{i=1}^r a_i e_i + \sum_{i=1}^r a'_i e'_i + (a_{r+1} e_{r+1}) \right) = -a_2 e'_1 + a_1 e'_2. \quad (5.11)$$

(ii) *If $v_2 = 2$, then $\dim(\text{im } N) = 2$ and $\dim(\text{im } N^2) = 1$. Moreover, there exists a basis*

$$\{e_1, \dots, e_r, e'_1, \dots, e'_r, (e_{r+1})\}$$

of \bar{V} such that \bar{q} with respect to it has a matrix form (5.9), and

$$N \left(\sum_{i=1}^r a_i e_i + \sum_{i=1}^r a'_i e'_i + (a_{r+1} e_{r+1}) \right) = a_1 e_2 - (a_2 + a'_2) e'_1 + a_1 e'_2. \quad (5.12)$$

Proof Our arguments follow closely [46, Prop 4.1] (they go back to the study of degenerations of $K3$ surface, e.g. in [9] even a normalization over \mathbb{Z} is given). For completeness and notational consistency, we give a proof here.

Assume first that we have a Type II degeneration. The one-parameter degeneration produces a limit mixed Hodge structure \bar{V}_{\lim} . It is a degeneration of the second cohomology \bar{V} of $K3$ type. The nilpotency index is $v_2 = 1$, so we have the monodromy weight filtration

$$0 \subset W_1 \subset W_2 \subset W_3 = \bar{V}_{\lim}$$

with $W_1 = \text{im } N$ and $W_2 = \ker N$. Since it is a degeneration of a $K3$ type Hodge structure, there is only one possibility of the Hodge diamond of \bar{V}_{\lim} as in Table 3. From it, we deduce $\dim W_1 = 2$ and $\dim W_2 = b_2(X) - 2$. This proves $\dim(\text{im } N) = 2$. Next, we choose two elements $x, y \in \bar{V}$ in as follows:

Table 3 The limit mixed Hodge structure \tilde{V}_{\lim} for $v_2 = 1$ and 2 respectively

	0		1
	1	1	0 0
0	$b_2 - 4$	0	0 $b_2 - 2$ 0
	1	1	0 0
	0		1

- (i) Choose any $x \notin \ker N$. Choose any $y \in x^\perp \setminus (Nx)^\perp$. Since $Nx \in \ker N$, x and Nx are linearly independent.
- (ii) Adjust y so that $\bar{q}|_{\mathbb{C}\{y, Nx\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is possible because $\bar{q}(Nx) = -(x, N^2x) = 0$.
- (iii) Adjust x so that $\bar{q}|_{\mathbb{C}\{x, -Ny\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is possible because $\bar{q}(Ny) = -(y, N^2y) = 0$, $(x, -Ny) = (Nx, y) = 1$, and we can replace the original x by $x - \frac{q(x)}{2}Ny$ (this doesn't change the value of Nx , preserving (ii) above).

We define $e_1 = x$, $e_2 = y$, $e'_1 = -Ny$, and $e'_2 = Nx$. By construction, the intersection pairing on the subspace spanned by $\{e_1, e_2, e'_1, e'_2\}$ is $U^{\oplus 2}$ as needed. For the orthogonal space $\langle e_1, e_2, e'_1, e'_2 \rangle^\perp$ we choose a basis $\{e_3, \dots, e_r, e'_3, \dots, e'_r, e_{r+1}\}$ as needed in the normal form (5.9) (since working over \mathbb{C} , this can be accomplished). Finally note that since $e_3, \dots, e_r, e'_3, \dots, e'_r, e_{r+1}$ are all perpendicular to e'_1 and e'_2 , they are perpendicular to $\text{im } N$, and thus contained in $\ker N = (\text{im } N)^\perp$. The formula (5.11) for N follows.

The Type III case is similar (and standard). We omit the details. \square

Once a choice of basis as in the lemma above has been made, we can further adjust the Cartan subalgebra and simple roots of $\bar{\mathfrak{g}}$ so that it becomes compatible with this choice of basis. (N.B. this choice of a Cartan subalgebra is only for this situation, and unrelated to the previous one containing the operator f in (2.23).)

Lemma 5.13 *We can choose a Cartan subalgebra $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$ and simple roots of $\bar{\mathfrak{g}}$ so that the basis elements $e_1, \dots, e_r, e'_1, \dots, e'_r$ (and e_{r+1}) in Lemma 5.10 are the weight vectors associated to the weights $\varepsilon_1, \dots, \varepsilon_r, -\varepsilon_1, \dots, -\varepsilon_r$ (and 0) of the standard $\bar{\mathfrak{g}}$ -module \tilde{V} .* \square

The normalizations above allow us to compute explicitly the nilpotency index as stated in Lemma 5.8. First, we have the following special case of Lemma 5.8.

Lemma 5.14 *Assume $b_2(X) = \dim \tilde{V} \geq 5$. Consider the $\bar{\mathfrak{g}}$ -module $\rho : \bar{\mathfrak{g}} \rightarrow \text{End}(\wedge^i \tilde{V})$ with $2 \leq i \leq r$.*

- (i) *If $v_2 = 1$, then $\rho(N)$ has nilpotency index 2.*
- (ii) *If $v_2 = 2$, then $\rho(N)$ also has nilpotency index 2.*

Proof Note that

$$(\rho(N))(x_1 \wedge \cdots \wedge x_i) = \sum_{j=1}^i x_1 \wedge \cdots \wedge N(x_j) \wedge \cdots \wedge x_i.$$

Assume $v_2 = 1$. From Lemma 5.10, we have $\dim(\text{im } N) = 2$ and $N^2 = 0$. Hence

$$\begin{aligned} (\rho(N))^2(x_1 \wedge \cdots \wedge x_i) &= 2 \sum N(x_1) \wedge N(x_2) \wedge x_3 \wedge \cdots \wedge x_i, \\ (\rho(N))^3(x_1 \wedge \cdots \wedge x_i) &= 0. \end{aligned}$$

This proves the nilpotency index of $\rho(N)$ is at most 2. Using the basis of Lemma 5.10, we get

$$(\rho(N))^2(e_1 \wedge \cdots \wedge e_i) = 2e'_1 \wedge e'_2 \wedge e_3 \wedge \cdots \wedge e_i \neq 0.$$

This proves the nilpotency index of $\rho(N)$ is precisely 2.

Assume $v_2 = 2$. Again from Lemma 5.10, we have $\dim(\text{im } N) = 2$, $\dim(\text{im } N^2) = 1$ and $N^3 = 0$. Thus

$$\begin{aligned} (\rho(N))^2(x_1 \wedge \cdots \wedge x_i) &= \sum N^2(x_1) \wedge x_2 \wedge \cdots \wedge x_i \\ &\quad + 2 \sum N(x_1) \wedge N(x_2) \wedge x_3 \wedge \cdots \wedge x_i, \\ (\rho(N))^3(x_1 \wedge \cdots \wedge x_i) &= 3 \sum N^2(x_1) \wedge N(x_2) \wedge x_3 \wedge \cdots \wedge x_i. \end{aligned}$$

Now using the preferred basis, we can further compute

$$\begin{aligned} (\rho(N))^2(e_1 \wedge \cdots \wedge e_i) &= 2e'_1 \wedge e'_2 \wedge e_3 \wedge \cdots \wedge e_i \neq 0, \\ \rho(N)^3 &= 0. \end{aligned}$$

(We have been unable to see the last identity without working with a suitable basis). The lemma follows. \square

One subtlety that needs an attention is that $\wedge^r \bar{V}$ is *not* an irreducible $\bar{\mathfrak{g}}$ -module for $b_2(X) = \dim \bar{V}$ even. In fact, in that case, it holds $\wedge^r \bar{V} = \bar{V}_{2\omega_{r-1}} \oplus \bar{V}_{2\omega_r}$. See Appendix A. With this in mind, we complete the proof of Lemma 5.8.

Proof of Lemma 5.8 Set

$$a_i = \lambda_i - \lambda_{i+1} \quad \text{for } 1 \leq i \leq r-1, \quad a_r = \lambda_r.$$

Consider a $\bar{\mathfrak{g}}$ -module

$$W = \text{Sym}^{a_1} \bar{V} \otimes \text{Sym}^{a_2}(\wedge^2 \bar{V}) \otimes \cdots \otimes \text{Sym}^{a_r}(\wedge^r \bar{V}).$$

The highest weight of this module becomes exactly λ (there are two highest weights when $b_2(X)$ is even and $\lambda_r \neq 0$, the other one being $(\lambda_1, \dots, \lambda_{r-1}, -\lambda_r)$). Hence \bar{V}_λ should be contained in W . Using Lemma 5.14, This proves the nilpotency index of $\rho_\lambda(N)$ is at most $a_1 + 2a_2 + \cdots + 2a_r = \lambda_1 + \lambda_2$ for $v_2 = 1$, and $2a_1 + \cdots + 2a_r = 2\lambda_1$ for $v_2 = 2$.

To prove the nilpotency index is precisely the desired value, we need Lemma 5.10 with Lemma 5.13. Using these results, one can see that $\wedge^i V$ has a unique highest weight vector $e_1 \wedge \cdots \wedge e_i$ up to scalar (there are two when $b_2(X)$ is even and $i = r$, the other one being $e_1 \wedge \cdots \wedge e_{r-1} \wedge e'_r$). Hence W has a unique (two) highest weight vector, up to scalar,

$$x := e_1^{a_1} \otimes (e_1 \wedge e_2)^{a_2} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_r)^{a_r},$$

and this x is contained in \bar{V}_λ .

Assume $v_2 = 1$. Then the computations in the proof of Lemma 5.14 shows

$$\begin{aligned} (\rho(N))^{a_1+2a_2+\cdots+2a_r}(x) &= (e'_1)^{a_1} \otimes (2e'_1 \wedge e'_2)^{a_2} \\ &\otimes \cdots \otimes (2e'_1 \wedge e'_2 \wedge e_3 \wedge \cdots \wedge e_r)^{a_r} \neq 0. \end{aligned}$$

Assume $v_2 = 2$. Then again, computations in Lemma 5.14 shows

$$\begin{aligned} (\rho(N))^{2a_1+2a_2+\cdots+2a_r}(x) &= (-2e'_1)^{a_1} \otimes (2e'_1 \wedge e'_2)^{a_2} \\ &\otimes \cdots \otimes (2e'_1 \wedge e'_2 \wedge e_3 \wedge \cdots \wedge e_r)^{a_r} \neq 0. \end{aligned}$$

This proves the nilpotency indexes are precisely as stated. \square

5.3 Completion of the proof of Theorem 5.2

Now that we know Lemma 5.8 holds, we can compute the nilpotency index v_{2k} explicitly.

Proposition 5.15 *Assume $b_2(X) \geq 5$ and $v_2 = 1$. Let $H^{2k}(X) = \bigoplus_{\lambda \in S} \bar{V}_\lambda^{\oplus n_\lambda}$ be a $\bar{\mathfrak{g}}$ -module irreducible decomposition of the $2k$ -th cohomology. Then*

$$v_{2k} = \max\{\lambda_1 + \lambda_2 : \lambda = (\lambda_1, \dots, \lambda_r) \in S\}.$$

Proof The representation $\rho_{2k} : \bar{\mathfrak{g}} \rightarrow \text{End}(H^{2k}(X))$ decomposes into

$$\rho_{2k} : \bar{\mathfrak{g}} \rightarrow \bigoplus_{\lambda \in S} \text{End}(\bar{V}_\lambda)^{\oplus n_\lambda} \subset \text{End}(H^{2k}(X)).$$

Hence $\rho_{2k}(N)$ is the direct sum of each $\rho_\lambda(N)$, and its nilpotency index is the maximum of those of $\rho_\lambda(N)$. Thus the statement follows from Lemma 5.8. \square

By Lemma 5.6, if $v_2 = 1$ then we always have $v_{2k} \geq k$. Thus, it is enough to show every irreducible $\bar{\mathfrak{g}}$ -module component \bar{V}_λ of $H^{2k}(X)$ satisfies $\lambda_1 + \lambda_2 \leq k$.

Corollary 5.16 *Assume $b_2(X) \geq 5$ and $v_2 = 1$. Then $v_{2k} = k$ for all $0 \leq k \leq n$ if and only if every highest $\bar{\mathfrak{g}}$ -weight λ appearing in (5.7) satisfies the inequality $\lambda_1 + \lambda_2 \leq k$ for all $0 \leq k \leq n$.* \square

The final step now is to lift the condition $\lambda_1 + \lambda_2 \leq k$ (in terms of the $\bar{\mathfrak{g}}$ -module structure on $H^k(X)$) to a condition in terms of the \mathfrak{g} -module structure on $H_{\text{even}}^*(X)$. Recall that the $\bar{\mathfrak{g}}$ -module structure on $H^{2k}(X)$ was induced from a more rigid \mathfrak{g} -module structure on $H_{\text{even}}^*(X)$. The \mathfrak{g} -module irreducible decomposition of the full even cohomology was

$$H_{\text{even}}^*(X) \cong \bigoplus_{\mu \in S} V_\mu^{\oplus m_\mu}, \quad (5.1 \text{ (restated)})$$

where $\mu = (\mu_0, \dots, \mu_r)$ indicates a dominant integral weight of \mathfrak{g} and V_μ indicates the associated \mathfrak{g} -module.

Let us start from the lifting of $\bar{\mathfrak{g}}$ -module structure to the \mathfrak{g}_0 -module structure. Recall the definition $\mathfrak{g}_0 = \bar{\mathfrak{g}} \oplus \mathbb{R}h$ in (2.4). Assume $0 \leq k \leq n$. The $\bar{\mathfrak{g}}$ -module \bar{V}_λ contained in $H^{2k}(X)$ can be thought of an irreducible \mathfrak{g}_0 -module of highest weight $(k-n)\varepsilon_0 + \lambda$ contained in $H_{\text{even}}^*(X)$. This is because the operator $h = \varepsilon_0^\vee$ acts on $H^{2k}(X)$ by the multiplication $2k - 2n$, whence giving us the coefficient $(k-n)\varepsilon_0$. We abuse our notation and write this \mathfrak{g}_0 -module as

$$\bar{V}_{(k-n)\varepsilon_0 + \lambda}.$$

Note that the Cartan subalgebra and weight lattices of \mathfrak{g} and \mathfrak{g}_0 are exactly the same. The difference between their representation theory comes from their Weyl group. The Weyl group \mathfrak{W}_0 of \mathfrak{g}_0 is strictly smaller than the Weyl group \mathfrak{W} of \mathfrak{g} ; the Weyl group \mathfrak{W}_0 loses all the symmetries coming from the weight ε_0 . This explains why V_μ decomposes further as a \mathfrak{g}_0 -module.

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ of \mathfrak{g} . The weights of \mathfrak{g} live in the space $\mathfrak{h}_\mathbb{R}^\vee$. We define the weight polytope $\text{WP}(V_\mu)$ of V_μ as the smallest convex hull in $\mathfrak{h}_\mathbb{R}^\vee$ containing all the weights of V_μ . The following simple lemma about the weight polytope will be useful.

Lemma 5.17 *Let us define a subset of $\mathfrak{h}_\mathbb{R}^\vee$ by*

$$K = \left\{ \sum_{i=0}^r t_i \varepsilon_i \in \mathfrak{h}_\mathbb{R}^\vee : t_i \in \mathbb{R} \text{ for } 0 \leq i \leq r, \quad |t_0| + |t_1| + |t_2| \leq n \right\}.$$

If a dominant integral weight μ of \mathfrak{g} is contained in K , then the whole weight polytope $\text{WP}(V_\mu)$ is contained in K .

Proof Note that a dominant integral weight $\mu = \sum_{i=0}^r \mu_i \varepsilon_i$ satisfies $\mu_0 \geq \dots \geq \mu_{r-1} \geq |\mu_r| \geq 0$. Thus, $|\mu_0| + |\mu_1| + |\mu_2| \leq n$ implies $|\mu_i| + |\mu_j| + |\mu_k| \leq n$ for all different i, j, k . Now, a Weyl group action $w \in \mathfrak{W}$ of type BD acts on μ by permutation of coefficients μ_i and changing their signs. Hence the sum of absolute value of the first three coefficients of $w.\mu$ is always $|\mu_i| + |\mu_j| + |\mu_k| \leq n$. This proves all the vertices of $\text{WP}(V_\mu)$ is contained in P . Since the weight polytope $\text{WP}(V_\mu)$ is a convex hull generated by its vertices and K is a convex set, we conclude the statement. \square

We now conclude the proof of Theorem 5.2.

Proof of Theorem 5.2 Assume $\mu_0 + \mu_1 + \mu_2 \leq n$ for all $\mu \in S$. Consider any $\bar{V}_\lambda \subset H^{2k}(X)$. We lift it to a \mathfrak{g}_0 -module $\bar{V}_{(k-n)\varepsilon_0+\lambda} \subset H_{\text{even}}^*(X)$. Then there exists a unique irreducible \mathfrak{g} -submodule $V_\mu \subset H_{\text{even}}^*(X)$ containing $\bar{V}_{(k-n)\varepsilon_0+\lambda}$. For such μ , we certainly have $(k-n)\varepsilon_0 + \lambda \in \text{WP}(V_\mu)$. Now Lemma 5.17 says $(n-k) + \lambda_1 + \lambda_2 \leq n$. Hence $\lambda_1 + \lambda_2 \leq k$, and now Nagai's conjecture follows from Corollary 5.16.

Conversely, assume there exists $\mu \in S$ with $\mu_0 + \mu_1 + \mu_2 > n$. Define a dominant integral weight λ of $\bar{\mathfrak{g}}$ by

$$\lambda = \begin{cases} \mu_1\varepsilon_1 + \cdots + \mu_r\varepsilon_r & \text{if } b_2(X) \text{ is odd,} \\ \mu_1\varepsilon_1 + \cdots + \mu_{r-1}\varepsilon_{r-1} - \mu_r\varepsilon_r & \text{if } b_2(X) \text{ is even.} \end{cases}$$

Consider a dominant integral \mathfrak{g}_0 -weight $-\mu_0\varepsilon_0 + \lambda$. It is also a \mathfrak{g} -weight since the weight lattices of \mathfrak{g}_0 and \mathfrak{g} are the same. Let us define the Weyl group action $w \in \mathfrak{W}$ as follows: if $b_2(X)$ is odd then w changes the sign of ε_0 , and if $b_2(X)$ is even then w changes the sign of both ε_0 and ε_r . Regardless of the parity of $b_2(X)$, we always have $-\mu_0\varepsilon_0 + \lambda = w.\mu$. Since $\mu \in \text{WP}(V_\mu)$, we have $-\mu_0\varepsilon_0 + \lambda \in \text{WP}(V_\mu)$ as one of the vertices. This forces $\bar{V}_{-\mu_0\varepsilon_0+\lambda} \subset V_\mu \subset H_{\text{even}}^*(X)$ as \mathfrak{g}_0 -submodules. But then by the discussion above, we have $\bar{V}_\lambda \subset H^{2k}(X)$ for $k = -\mu_0 + n$ with the property $\lambda_1 + \lambda_2 = \mu_1 + \mu_2 > -\mu_0 + n = k$. Again by Corollary 5.16, this proves Nagai's conjecture fails for any Type II degeneration. \square

6 Nagai's conjecture for the known examples of hyper-Kähler manifolds

At this point, we conclude with the proof of Nagai's conjecture (Theorem 1.14) for the known cases of hyper-Kähler manifolds. In fact, as announced in the introduction, a stronger representation theoretic condition holds for all known cases. Specifically, the following holds (also stated as Theorem 1.23 in the introduction):

Theorem 6.1 *Let X be a $2n$ -dimensional hyper-Kähler manifold of $\text{K3}^{[n]}$, Kum_n , OG6 , or OG10 type. Then any irreducible \mathfrak{g} -module component V_μ occurring in the LLV decomposition of $H^*(X)$ satisfies*

$$\mu_0 + \cdots + \mu_{r-1} + |\mu_r| \leq n. \quad (6.2)$$

Remark 6.3 There are at least two other equivalent ways to state the condition (6.2). The first one is in terms of weight polytopes. The condition (6.2) is equivalent to the weight polytope of V_μ being contained in that of the Verbitsky component $V_{(n)}$:

$$\text{WP}(V_\mu) \subset \text{WP}(V_{(n)}).$$

This can be easily seen as follows. The weight polytope of the Verbitsky component $V_{(n)}$ has vertices $\pm n\varepsilon_0, \dots, \pm n\varepsilon_r$, obtained by applying the Weyl group actions to the

highest weight $n\varepsilon_0$. From it, one shows its weight polytope is

$$\text{WP}(V_{(n)}) = \{\theta = \theta_0\varepsilon_0 + \cdots + \theta_r\varepsilon_r \in \mathfrak{h}_{\mathbb{R}}^* : |\theta_0| + \cdots + |\theta_r| \leq n\}.$$

In our case, μ is a dominant integral weight so we have an assumption $\mu_0 \geq \cdots \geq \mu_{r-1} \geq |\mu_r| \geq 0$. Hence (6.2) is equivalent to $\mu \in \text{WP}(V_{(n)})$, which is again equivalent to our condition on the weight polytopes. In this sense, the condition (6.2) in some sense means that the Verbitsky component is the *dominant component* among the LLV components arising in $H^*(X)$.

The second equivalent way to state the condition (6.2) is to use the notion of a cocharacter. Assume $\dim V = 2r + 3$ is odd. Let us denote ϖ_r the fundamental weight associated to the spin representation V_{ϖ_r} . Then one can consider the cocharacter $\varpi_r^\vee := \frac{2(\varpi_r, -)}{(\varpi_r, \varpi_r)}$ associated to it. Then the inequality (6.2) is equivalent to the inequality

$$\langle \mu, \varpi_r^\vee \rangle \leq \langle n\varepsilon_0, \varpi_r^\vee \rangle.$$

That is, the highest weight μ is again *dominated* in terms of the pairing with the cocharacter ϖ_r^\vee associated to the spin representation. If $\dim V = 2r + 2$ is even, then we have to take care of the case $\mu_r < 0$, so (6.2) is in fact equivalent to two inequalities $\mu_0 + \cdots + \mu_{r-1} - \mu_r \leq n$ and $\mu_0 + \cdots + \mu_{r-1} + \mu_r \leq n$. This case, we have two half-spin representations associated to the fundamental weights $\varpi_{r-1} = \frac{1}{2}(\varepsilon_0 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$ and $\varpi_r = \frac{1}{2}(\varepsilon_0 + \cdots + \varepsilon_r)$. Hence the condition (6.2) is equivalent to

$$\langle \mu, \varpi_{r-1}^\vee \rangle \leq \langle n\varepsilon_0, \varpi_{r-1}^\vee \rangle \quad \text{and} \quad \langle \mu, \varpi_r^\vee \rangle \leq \langle n\varepsilon_0, \varpi_r^\vee \rangle.$$

This can be again interpreted as the highest weight μ is dominated in terms of the pairing with the cocharacters associated to the two half-spin representations.

Proof of Theorem 6.1 The inequality for OG6 and OG10 follows directly from the irreducible LLV decomposition in Theorem 1.2 items (3) and (4) respectively. Assume now that X is a hyper-Kähler manifold of $\text{K3}^{[n]}$ type. We will in fact prove that *every* weight μ associated to $H^*(X)$ satisfies the desired inequality above. In this situation, we have the following generating series for the formal character (cf. Theorem 1.2(1)):

$$\sum_{n=0}^{\infty} \text{ch}(H^*(\text{K3}^{[n]}, \mathbb{R}))q^n = \prod_{m=1}^{\infty} \prod_{i=0}^{11} \frac{1}{(1 - x_i q^m)(1 - x_i^{-1} q^m)}.$$

By definition, the coefficient of q^n gives all the \mathfrak{g} -weights of $H^*(X)$. The weight $\mu = \mu_0\varepsilon_0 + \cdots + \mu_{11}\varepsilon_{11}$ corresponds to the monomial $x_0^{\mu_0} x_1^{\mu_1} \cdots x_{11}^{\mu_{11}}$ in the representation ring. Hence, the desired inequality $\mu_0 + \cdots + \mu_{11} \leq n$ is equivalent to saying that the x_i -degree of the coefficient of q^n is $\leq n$. This is obvious from the form of the right hand side; whenever we increase the degree of q by $m > 0$, then the degree of x_i increases at most by 1 ($\leq m$). Thus, for every monomial in the generating series, the x_i -degree is at most the q -degree. The claim follows.

For a Kum_n type hyper-Kähler manifold X , the same argument applies. In this case, the generating series of the formal character of $H^*(X)$ is (cf. Theorem 1.2(2)):

$$\sum_{n=0}^{\infty} \text{ch}(H^*(\text{Kum}_n, \mathbb{R}))q^n = \sum_{d=1}^{\infty} J_4(d) \frac{B(q^d) - 1}{b_1 q},$$

where $B(q)$ is defined by

$$B(q) = \prod_{m=1}^{\infty} \left[\prod_{i=0}^3 \frac{1}{(1 - x_i q^m)(1 - x_i^{-1} q^m)} \prod_j (1 + x_0^{j_0} x_1^{j_1} x_2^{j_2} x_3^{j_3} q^m) \right].$$

Note that in the denominator we have

$$b_1 = x_0 + \cdots + x_3 + x_0^{-1} + \cdots + x_3^{-1} + \sqrt{x_0 x_1 x_2 x_3} + \cdots + \sqrt{x_0 x_1 x_2 x_3}^{-1}.$$

Assume on the contrary that there exists some monomial $x_0^{\mu_0} \cdots x_3^{\mu_3} q^n$ in the generating series such that $\mu_0 + \cdots + \mu_3 \geq n + \frac{1}{2}$. After multiplying $b_1 q$, it follows that some $B(q^d)$ contains a monomial $x_0^{\mu_0 + \frac{1}{2}} \cdots x_3^{\mu_3 + \frac{1}{2}} q^{n+1}$. This means that $B(q)$ contains a monomial with x_i -degree at least $\frac{3}{2}$ bigger than the q -degree. One can see without difficulty this cannot happen in $B(q)$ defined as above. \square

Combining Theorem 6.1 with the representation theoretic formulation of Nagai's conjecture (Theorem 5.2), we conclude Nagai's conjecture holds for all currently known examples of hyper-Kähler manifolds.

Corollary 6.4 *Nagai's conjecture (1.13) holds for all one-parameter degenerations of projective hyper-Kähler manifolds of $\text{K3}^{[n]}$, Kum_n , OG6, or OG10 type.* \square

It is natural to speculate that Nagai's conjecture (or even the stronger inequality (6.2)) holds for any hyper-Kähler manifold. We do not have much to say in this direction. However, for completeness, we note Nagai's conjecture holds in general for low (≤ 8) dimensional cases.

Proposition 6.5 *Nagai's conjecture (1.13) holds when $\dim X \leq 8$.*

Proof If $b_2(X) \leq 4$ then Remark 5.4 shows Nagai's conjecture is always true. If $b_2(X) \geq 5$ then we may apply Theorem 5.2. From Proposition 2.34, every highest weight μ in the LLV decomposition of the even cohomology $H_{\text{even}}^*(X) = \bigoplus_{\mu} V_{\mu}^{\oplus m_{\mu}}$ satisfies either $\mu_0 + \mu_1 \leq n - 1$ or $\mu = (n)$. The case $\mu = (n)$ clearly satisfies $\mu_0 + \mu_1 + \mu_2 \leq n$. If $n \leq 4$, we get $\mu_0 + \mu_1 \leq 3$ and hence $\mu_2 \leq \mu_1 \leq 1$. This proves $\mu_0 + \mu_1 + \mu_2 \leq n$. \square

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While preparing this manuscript, the papers [48] and [49] appeared. We acknowledge their influence; in particular, occasionally they helped streamline some of our arguments.

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Appendix A: Representation theory of simple Lie algebras of type BD

We present a short review and fix notation for finite dimensional representation theory of simple Lie algebras of type BD. Throughout this section, we fix the notation $\mathfrak{g} = \mathfrak{so}(V, q)$ for a special orthogonal Lie algebra associated to an arbitrary *nondegenerate* quadratic space (V, q) over \mathbb{Q} . Over the complex numbers \mathbb{C} , there is only one quadratic space of dimension n up to isomorphism, so every type BD simple Lie algebra over \mathbb{C} is isomorphic to $\mathfrak{so}(n, \mathbb{C})$. Over the real numbers \mathbb{R} , by Sylvester’s classification, quadratic forms on \mathbb{R}^n is classified by its signature (a, b) , so every Lie algebra $\mathfrak{so}(V, q)$ over \mathbb{R} is isomorphic to $\mathfrak{so}(a, b)$. (N.B. not every type BD simple Lie algebra over \mathbb{R} is of the form $\mathfrak{so}(V, q)$.) Over the rational numbers \mathbb{Q} , the case of interest here, the classification of quadratic forms on \mathbb{Q}^n is well understood, but more subtle (e.g., [42]).

The LLV algebra of a hyper-Kähler manifold is of the form $\mathfrak{g} = \mathfrak{so}(V, q)$ (see Theorem 2.7) for a rational quadratic space (V, q) . We will review some representation theory facts in this simplest case of type BD Lie algebra. We will do the representation theory over \mathbb{Q} as much as possible. By definition, a finite dimensional \mathbb{Q} -vector space W is called a *\mathfrak{g} -module*, or a *\mathfrak{g} -representation*, if it is equipped with a Lie algebra homomorphism $\mathfrak{g} \rightarrow \mathfrak{gl}(W)$. Our main references for this appendix are [11] for representation theory over \mathbb{C} , and Milne [32] for that over \mathbb{Q} .

A.1 Type B

Assume (V, q) is a rational quadratic space of odd dimension $2r + 1 \geq 3$.

Fix any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$. Let $0, \pm \varepsilon_1, \dots, \pm \varepsilon_r$ be the associated weights of the standard representation $V_{\mathbb{C}}$ with respect to \mathfrak{h} . We can choose a positive Weyl chamber appropriately so that it is generated by the fundamental weights

$$\varpi_i = \varepsilon_1 + \dots + \varepsilon_i \quad \text{for } 1 \leq i \leq r-1, \quad \varpi_r = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_r). \quad (\text{A.1})$$

Let $\Lambda \subset \mathfrak{h}_{\mathbb{R}}^{\vee}$ be the weight lattice of \mathfrak{g} . It is a rank r lattice in the Euclidean space $\mathfrak{h}_{\mathbb{R}}^{\vee}$ generated by the above fundamental weights. The intersection of the positive Weyl

chamber and Λ is the monoid of dominant integral weights

$$\Lambda^+ = \{\lambda = \sum_{i=1}^r \lambda_i \varepsilon_i : \lambda_1 \geq \dots \geq \lambda_r \geq 0, \lambda_i \in \frac{1}{2}\mathbb{Z}, \lambda_i - \lambda_j \in \mathbb{Z}\}. \quad (\text{A.2})$$

We will often denote a dominant integral weight $\lambda = \sum_{i=1}^r \lambda_i \varepsilon_i$ simply as $\lambda = (\lambda_1, \dots, \lambda_r)$, and omit the zeros in the end for simplicity. Whenever we use this notation, we assume λ is a dominant integral weight and the conditions on λ_i above (A.2) are tacitly assumed.

Let λ be a dominant integral weight of \mathfrak{g} . Over \mathbb{C} , we always have a unique irreducible $\mathfrak{g}_{\mathbb{C}}$ -module $V_{\lambda, \mathbb{C}}$ with highest weight λ . We call this a highest $\mathfrak{g}_{\mathbb{C}}$ -module of weight λ . Over \mathbb{Q} , this is not always possible. However, in our case of $\mathfrak{g} = \mathfrak{so}(V, q)$, we have a strong condition that the standard \mathfrak{g} -module V is defined over \mathbb{Q} . This implies that many of the modules relevant to us are in fact defined over \mathbb{Q} .

Proposition A.3 *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a dominant integral weight of \mathfrak{g} . If λ_i are integers, then there exists a unique irreducible \mathfrak{g} -module V_{λ} such that its complexification $(V_{\lambda})_{\mathbb{C}}$ is isomorphic to the highest $\mathfrak{g}_{\mathbb{C}}$ -module $V_{\lambda, \mathbb{C}}$. We call this V_{λ} the highest \mathfrak{g} -module of weight λ .*

Proof The orthogonal Schur-Weyl construction [11, Thm 19.22] realizes the highest $\mathfrak{g}_{\mathbb{C}}$ -module $V_{\lambda, \mathbb{C}}$ as an explicit tensor construction starting from the standard module $V_{\mathbb{C}}$. This construction works over \mathbb{Q} , and hence one can apply it to the rational \mathfrak{g} -module V and end up with a rational \mathfrak{g} -module V_{λ} . This proves $V_{\lambda, \mathbb{C}}$ is in fact defined over \mathbb{Q} . Uniqueness is a general fact in representation theory over an arbitrary field (see, e.g., [32, Thm 25.34]). \square

The highest \mathfrak{g} -modules associated to the fundamental weights (A.1) are most easily described. For $1 \leq i \leq r-1$, the highest \mathfrak{g} -module of weight ϖ_i is isomorphic to $\wedge^i V$, the i -th wedge power of the standard module V . The highest module associated to the weight ϖ_r is exceptional; it is the spin \mathfrak{g} -module $V_{\varpi_r, \mathbb{C}}$. Note that Proposition A.3 does not guarantee $V_{\varpi_r, \mathbb{C}}$ is defined over \mathbb{Q} . Indeed, it is completely possible that the spin module is not even defined over \mathbb{R} (see, e.g., [7]). However, one should be aware that $V_{2\varpi_r}$ is defined over \mathbb{Q} and isomorphic to $\wedge^r V$. It is an irreducible \mathfrak{g} -module.

Any \mathfrak{g} -module W over \mathbb{Q} admits an associated $\text{Spin}(V, q)$ -module structure and vice versa [32, Thm 22.53]. The existence of a degree 2 isogeny $\text{Spin}(V, q) \rightarrow \text{SO}(V, q)$ says there are exactly half the irreducible $\text{SO}(V, q)$ -modules than that of $\text{Spin}(V, q)$ -modules. More specifically, V_{λ} admits an associated $\text{SO}(V, q)$ -module structure if and only if λ is contained in the following submonoid of Λ^+ :

$$\Lambda_{\text{SO}}^+ = \left\{ \lambda = \sum_{i=1}^r \lambda_i \varepsilon_i : \lambda_1 \geq \dots \geq \lambda_r \geq 0, \lambda_i \in \mathbb{Z} \right\}.$$

Note that this consists of precisely the dominant integral weights stated in Proposition A.3. Hence, every $\text{SO}(V, q)$ -module is defined over \mathbb{Q} .

The Weyl group \mathfrak{W} of \mathfrak{g} is the symmetry group of its root system consisting of permutations of the weights $\varepsilon_1, \dots, \varepsilon_r$ and their sign changes. More specifically, $\mathfrak{W} \cong \mathfrak{S}_r \ltimes (\mathbb{Z}/2)^{\times r}$ where \mathfrak{S}_r is the symmetric group of order r and the semidirect product is defined in terms of the group \mathfrak{S}_r acting on $(\mathbb{Z}/2)^{\times r}$ by permuting factors. For every highest weight λ , the set of weights of $V_{\lambda, \mathbb{C}}$ is \mathfrak{W} -invariant as a subset in the Euclidean space $\mathfrak{h}_{\mathbb{R}}^{\vee}$. Moreover, if we consider the convex hull in $\mathfrak{h}_{\mathbb{R}}^{\vee}$ generated by all of the weights of $V_{\lambda, \mathbb{C}}$, then we have a weight polytope $WP(V_{\lambda})$. The vertices of this polytope are exactly the points $w \cdot \lambda$ where $w \in \mathfrak{W}$ varies through all the Weyl group actions. Some of them can coincide.

The Weyl dimension formula provides a convenient way to compute the dimension of the highest weight modules $V_{\lambda, \mathbb{C}}$. The formula is as follows.

$$\dim V_{\lambda, \mathbb{C}} = \prod_{1 \leq i < j \leq r} \frac{(\lambda + \rho, \varepsilon_i + \varepsilon_j) \cdot (\lambda + \rho, \varepsilon_i - \varepsilon_j)}{(\rho, \varepsilon_i + \varepsilon_j) \cdot (\rho, \varepsilon_i - \varepsilon_j)} \cdot \prod_{i=1}^r \frac{(\rho + \lambda, \varepsilon_i)}{(\rho, \varepsilon_i)}. \quad (\text{A.4})$$

Here $\rho = \sum_{i=1}^r (r - i + \frac{1}{2}) \varepsilon_i$ is half the sum of the positive roots and $(,)$ is the standard Euclidean inner product on $\mathfrak{h}_{\mathbb{R}}^{\vee}$ with respect to the basis ε_i , the Killing form. Of course, if $V_{\lambda, \mathbb{C}}$ is defined over \mathbb{Q} with its \mathbb{Q} -form V_{λ} , then the \mathbb{Q} -dimension of V_{λ} can be computed by exactly the same formula.

A.2 Type D

Assume (V, q) is a rational quadratic space of even dimension $2r \geq 4$. There is an analogue but slightly different story in this case. Again, start with fixing any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$. We have $\pm \varepsilon_1, \dots, \pm \varepsilon_r$ as the weights associated to the standard $\mathfrak{g}_{\mathbb{C}}$ -module $V_{\mathbb{C}}$ with respect to \mathfrak{h} . Taking an appropriate positive Weyl chamber, we can choose the fundamental weights by

$$\begin{aligned} \varpi_i &= \varepsilon_1 + \dots + \varepsilon_i \quad \text{for } 1 \leq i \leq r-2, & \varpi_{r-1} &= \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{r-1} - \varepsilon_r), \\ \varpi_r &= \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_r). \end{aligned} \quad (\text{A.5})$$

The monoid of dominant integral weights is

$$\Lambda^+ = \left\{ \lambda = \sum_{i=1}^r \lambda_i \varepsilon_i : \lambda_1 \geq \dots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0, \quad \lambda_i \in \frac{1}{2}\mathbb{Z}, \quad \lambda_i - \lambda_j \in \mathbb{Z} \right\}. \quad (\text{A.6})$$

We often denote $\lambda = (\lambda_1, \dots, \lambda_r)$ for a dominant integral weight $\lambda = \sum_{i=1}^r \lambda_i \varepsilon_i$, satisfying the condition (A.6).

We denote $V_{\lambda, \mathbb{C}}$ the highest $\mathfrak{g}_{\mathbb{C}}$ -module of weight λ . A similar proposition on their field of definition holds for $V_{\lambda, \mathbb{C}}$, but this time a bit more complicated than the previous one.

Proposition A.7 *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a dominant integral weight of \mathfrak{g} . If $\lambda_r = 0$ then there exists a unique irreducible \mathfrak{g} -module V_{λ} such that its base change over \mathbb{C} is the highest module $V_{\lambda, \mathbb{C}}$.*

Proof The proof is the same as above. Notice the conditions on λ are different for type B and type D. \square

Note that the orthogonal Schur–Weyl construction argument [11, Thm 19.22] still says $V_{\lambda, \mathbb{C}} \oplus V_{\lambda', \mathbb{C}}$ is defined over \mathbb{Q} for $\lambda_r \in \mathbb{Z} \setminus \{0\}$, where $\lambda' = (\lambda_1, \dots, \lambda_{r-1}, -\lambda_r)$. Hence one cannot say the highest $\mathfrak{g}_{\mathbb{C}}$ -module $V_{\lambda, \mathbb{C}}$ is always defined over \mathbb{Q} when $\lambda_r \in \mathbb{Z} \setminus \{0\}$. However, only two cases can possibly arise:

- (i) There do exist irreducible \mathfrak{g} -modules V_{λ} and $V_{\lambda'}$, whose complexification are the highest $\mathfrak{g}_{\mathbb{C}}$ -modules $V_{\lambda, \mathbb{C}}$ and $V_{\lambda', \mathbb{C}}$.
- (ii) There does not exist any irreducible \mathfrak{g} -module whose complexification is the highest $\mathfrak{g}_{\mathbb{C}}$ -module $V_{\lambda, \mathbb{C}}$ (resp. $V_{\lambda', \mathbb{C}}$). Nonetheless, there exists a unique irreducible \mathfrak{g} -module $V_{\lambda} = V_{\lambda'}$ whose complexification is $V_{\lambda, \mathbb{C}} \oplus V_{\lambda', \mathbb{C}}$.

The highest \mathfrak{g} -modules associated to the fundamental weights (A.5) are $V_{\varpi_i} = \wedge^i V$ for $1 \leq i \leq r-2$. For $i = r-1, r$, we get the two half-spin representations $V_{\varpi_{r-1}, \mathbb{C}}$ and $V_{\varpi_r, \mathbb{C}}$. Again, it is totally possible these half-spin representations are not defined over \mathbb{Q} [7]. We also note the isomorphisms $\wedge^{r-1} V = V_{\varpi_{r-1} + \varpi_r}$, and $\wedge^r V = V_{2\varpi_{r-1}} \oplus V_{2\varpi_r}$. In particular, $\wedge^{r-1} V$ is an irreducible \mathfrak{g} -module whereas $\wedge^r V$ is not.

Any \mathfrak{g} -module W over \mathbb{Q} admits an associated $\text{Spin}(V, q)$ -module structure and vice versa. There exists a degree 2 isogeny $\text{Spin}(V, q) \rightarrow \text{SO}(V, q)$. This says V_{λ} admits an associated $\text{SO}(V, q)$ -module structure if and only if λ is contained in

$$\Lambda_{\text{SO}}^+ = \left\{ \lambda = \sum_{i=1}^r \lambda_i \varepsilon_i : \lambda_1 \geq \dots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0, \lambda_i \in \mathbb{Z} \right\}.$$

In this case, the center of $\text{Spin}(V, q)$ is isomorphic to $(\mathbb{Z}/2)^{\times 2}$ and hence there exists a further degree 2 isogeny $\text{SO}(V, q) \rightarrow \text{PSO}(V, q)$. Therefore, there are more possibility of \mathbb{Q} -algebraic groups with the associated Lie algebra \mathfrak{g} .

The Weyl group \mathfrak{W} of \mathfrak{g} consists of permutations of the weights $\varepsilon_1, \dots, \varepsilon_r$ and even number of their sign changes. The group \mathfrak{W} is an index 2 subgroup of $\mathfrak{S}_r \ltimes (\mathbb{Z}/2)^{\times r}$, consisting of elements of even number of 1's in $(\mathbb{Z}/2)^{\times r}$. The set of weights of V_{λ} is \mathfrak{W} -invariant, and generates a convex hull $\text{WP}(V_{\lambda})$, the weight polytope of V_{λ} . The vertices of $\text{WP}(V_{\lambda})$ are exactly the points $w \cdot \lambda$ where $w \in \mathfrak{W}$ varies through all the Weyl group actions.

For any dominant integral weight λ , the Weyl dimension formula for this case has the following form:

$$\dim V_{\lambda, \mathbb{C}} = \prod_{1 \leq i < j \leq r} \frac{(\lambda + \rho, \varepsilon_i + \varepsilon_j) \cdot (\lambda + \rho, \varepsilon_i - \varepsilon_j)}{(\rho, \varepsilon_i + \varepsilon_j) \cdot (\rho, \varepsilon_i - \varepsilon_j)}. \quad (\text{A.8})$$

Here $\rho = \sum_{i=1}^r (r-i) \varepsilon_i$ denotes again half the sum of the positive roots and (\cdot, \cdot) is the standard Euclidean inner product on $\mathfrak{h}_{\mathbb{R}}^{\vee}$ with respect to the basis ε_i . As an example, in this paper, we have used the Weyl dimension formula to generate Table 1 for hyper-Kähler manifolds of OG10 type. Since, in this case, the rank of \mathfrak{g} is $r = 13$, our computations were computer-aided. Finally, we provide the following lemma about

the dimension comparison of V_λ (cf. [11, Ex. 24.9]). It is used in Sect. 3.4 for the study of the LLV decomposition of OG10 hyper-Kähler manifolds.

Lemma A.9 *Let λ, μ be dominant integral weights of \mathfrak{g} . Then $\dim V_{\lambda+\mu, \mathbb{C}} \geq \dim V_{\lambda, \mathbb{C}}$.*

Appendix B: Representation ring and restriction representations

Since many of our results involve several different Lie algebras and heavily depends on the relation between their representation theory, we provide a separate section to discuss this topic.

B.1 Representation ring and restriction representations

Let \mathfrak{g} be a reductive Lie algebra over \mathbb{Q} . Recall that a (rational) \mathfrak{g} -module is a finite dimensional rational vector space V equipped with a Lie algebra homomorphism $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$. We define a complex \mathfrak{g} -module by a finite dimensional complex vector space V equipped with a Lie algebra homomorphism $\mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{gl}(V)$. Notice that the notion of a complex \mathfrak{g} -module is nothing but just a $\mathfrak{g}_{\mathbb{C}}$ -module. If we have a rational \mathfrak{g} -module $V_{\mathbb{Q}}$, then its complexification $(V_{\mathbb{Q}})_{\mathbb{C}}$ is clearly a complex \mathfrak{g} -module. On the other hand, not every complex \mathfrak{g} -module can be obtained by the complexification of a rational \mathfrak{g} -module.

Let $\text{Rep}(\mathfrak{g})$ and $\text{Rep}_{\mathbb{C}}(\mathfrak{g})$ ($= \text{Rep}(\mathfrak{g}_{\mathbb{C}})$) be the categories of finite dimensional rational \mathfrak{g} -modules and complex \mathfrak{g} -modules, respectively. Since we have assumed \mathfrak{g} is reductive, both categories are semisimple, i.e., every object in the category is completely reducible. The discussion in the previous paragraph implies there exists a complexification functor

$$\text{Rep}(\mathfrak{g}) \rightarrow \text{Rep}_{\mathbb{C}}(\mathfrak{g}).$$

Consider the Grothendieck ring $K(\mathfrak{g})$ and $K_{\mathbb{C}}(\mathfrak{g})$ of the categories $\text{Rep}(\mathfrak{g})$ and $\text{Rep}_{\mathbb{C}}(\mathfrak{g})$, respectively. These rings are called the *representation ring* (resp. complex representation ring) of \mathfrak{g} . Since $\text{Rep}(\mathfrak{g})$ (resp. $\text{Rep}_{\mathbb{C}}(\mathfrak{g})$) is semisimple, the representation ring $K(\mathfrak{g})$ (resp. $K_{\mathbb{C}}(\mathfrak{g})$) coincides with the abelianization of the monoid of isomorphism classes of \mathfrak{g} -modules (resp. complex \mathfrak{g} -modules). Moreover, the above complexification functor induces an injective ring homomorphism (see [32, §25.d])

$$K(\mathfrak{g}) \hookrightarrow K_{\mathbb{C}}(\mathfrak{g}). \quad (\text{B.1})$$

Thus, to describe the structure of (rational or complex) \mathfrak{g} -modules up to isomorphism, it is enough to describe them as elements in $K_{\mathbb{C}}(\mathfrak{g})$.

Proposition B.2 *Let V be a \mathfrak{g} -module. Then the \mathfrak{g} -module structure of V is completely determined by an element $[V_{\mathbb{C}}] \in K_{\mathbb{C}}(\mathfrak{g})$ in the complex representation ring.*

The structure of the representation ring $K_{\mathbb{C}}(\mathfrak{g})$ for simple Lie algebras \mathfrak{g} is completely understood. It is related to the character theory and weights of \mathfrak{g} -modules. Fix

a Cartan subalgebra of \mathfrak{g} and let Λ be the weight lattice of \mathfrak{g} . Consider its group ring $\mathbb{Z}[\Lambda]$. To use a multiplicative notation for the multiplication operation in $\mathbb{Z}[\Lambda]$, we use a notation $e^\mu \in \mathbb{Z}[\Lambda]$ to represent $\mu \in \Lambda$ as an element in $\mathbb{Z}[\Lambda]$.

Definition B.3 Let V be any complex \mathfrak{g} -module. Consider its weight decomposition $V = \bigoplus_\mu V(\mu)$, where $V(\mu)$ indicates the weight μ subvector space of V . We define the *formal character map* of \mathfrak{g} by a ring homomorphism

$$\text{ch} : K_{\mathbb{C}}(\mathfrak{g}) \rightarrow \mathbb{Z}[\Lambda], \quad [V] \mapsto \sum_\mu \dim V(\mu) e^\mu.$$

We recall the following well known result (e.g. [11, §23]).

Theorem B.4 *The formal character map ch is injective, and the image of it is the Weyl group invariant ring $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$. That is, $\text{ch} : K_{\mathbb{C}}(\mathfrak{g}) \rightarrow \mathbb{Z}[\Lambda]^{\mathfrak{W}}$ is a ring isomorphism.*

B.1.1 Representation ring of type BD simple Lie algebras

Now let us specialize our discussion to the case of our primary interest, $\mathfrak{g} = \mathfrak{so}(V, q)$ for a rational quadratic space (V, q) .

Assume $\dim V = 2r + 1$ is odd for $r \geq 1$ (Case B_r). The complexification of \mathfrak{g} is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2r + 1, \mathbb{C})$. Recall from Sect. A that the weight lattice Λ of it is generated by the fundamental weights

$$\begin{aligned} \varpi_1 &= \varepsilon_1, & \varpi_2 &= \varepsilon_1 + \varepsilon_2, & \dots, & \varpi_{r-1} &= \varepsilon_1 + \dots + \varepsilon_{r-1}, \\ \varpi_r &= \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_r). \end{aligned}$$

Let us simply write $x_i = e^{\varepsilon_i}$ for $i = 1, \dots, r$. Then we can describe the group ring $\mathbb{Z}[\Lambda]$ explicitly as

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]. \quad (\text{B.5})$$

Recall the Weyl group of \mathfrak{g} is isomorphic to $\mathfrak{W}_{2r+1} \cong \mathfrak{S}_r \ltimes (\mathbb{Z}/2)^{\times r}$. It acts on $\mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]$ as follows: $\sigma \in \mathfrak{S}_r$ acts as a permutation on x_1, \dots, x_r , and $1 \in \mathbb{Z}/2$ in the i -th factor $\mathbb{Z}/2$ acts as $x_i \mapsto x_i^{-1}$. Finally, Theorem B.4 completes the explicit description of $K_{\mathbb{C}}(\mathfrak{g})$ by

$$\text{ch} : K_{\mathbb{C}}(\mathfrak{g}) \xrightarrow{\cong} \mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]^{\mathfrak{W}_{2r+1}}.$$

Now assume $\dim V = 2r$ is even for $r \geq 2$ (Case D_r). The complexification of \mathfrak{g} is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2r, \mathbb{C})$. The weight lattice Λ of it is generated by the fundamental weights

$$\begin{aligned} \varpi_1 &= \varepsilon_1, & \dots, & \varpi_{r-2} &= \varepsilon_1 + \dots + \varepsilon_{r-2}, & \varpi_{r-1} &= \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{r-1} - \varepsilon_r), \\ \varpi_r &= \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_r). \end{aligned}$$

Let us also write $x_i = e^{e_i}$ for $i = 1, \dots, r$. Then the group ring $\mathbb{Z}[\Lambda]$ becomes the same as above:

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}].$$

However, the Weyl group becomes smaller. The Weyl group \mathfrak{W}_{2r} in this case is an order 2 subgroup of $\mathfrak{S}_r \times (\mathbb{Z}/2)^{\times r}$, consisting of elements of even number of 1's in $(\mathbb{Z}/2)^{\times r}$. It acts on $\mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]$ in the same way as above. Theorem B.4 gives us the isomorphism

$$\text{ch} : K_{\mathbb{C}}(\mathfrak{g}) \xrightarrow{\cong} \mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]^{\mathfrak{W}_{2r}}.$$

B.1.2 Restriction representations

A direct but interesting consequence of the above discussions is the following.

Proposition B.6 *Let (V, q) be a rational quadratic space and $T \subset V$ a nondegenerate quadratic subspace with $\dim V = 2r + 1$ and $\dim T = 2r$. Set $\mathfrak{g} = \mathfrak{so}(V, q)$ and $\mathfrak{m} = \mathfrak{so}(T, q)$. Then the restriction representation functor $\text{Res} : \text{Rep}(\mathfrak{g}) \rightarrow \text{Rep}(\mathfrak{m})$ induces an injective ring homomorphism on the level of representation rings. That is, the following diagram commutes with all the horizontal arrows injective.*

$$\begin{array}{ccc} K(\mathfrak{g}) & \xrightarrow{\text{Res}} & K(\mathfrak{m}) \\ \downarrow & & \downarrow \\ K_{\mathbb{C}}(\mathfrak{g}) & \xrightarrow{\text{Res}} & K_{\mathbb{C}}(\mathfrak{m}) \\ \text{ch} \downarrow \cong & & \text{ch} \downarrow \cong \\ \mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]^{\mathfrak{W}_{2r+1}} & \hookrightarrow & \mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]^{\mathfrak{W}_{2r}} \end{array}$$

Proof The statement follows almost directly from the previous discussions. Observe that $\mathfrak{W}_{2r} \subsetneq \mathfrak{W}_{2r+1}$. This implies the bottom map is injective. Since the character homomorphisms are isomorphisms by Theorem B.4, the middle restriction map on the complex representation ring is also injective. It follows the restriction map on the first row is also injective, because the two vertical maps $K(\mathfrak{g}) \rightarrow K_{\mathbb{C}}(\mathfrak{g})$ and $K(\mathfrak{m}) \rightarrow K_{\mathbb{C}}(\mathfrak{m})$ are both injective by (B.1). \square

That is, in the set-up of the proposition, if W is a \mathfrak{g} -module then the \mathfrak{m} -module structure on W by restriction representation determines its \mathfrak{g} -module structure. In particular, since b_2 is even for $K3$ surfaces, notice that this applies to case of the (Mukai completed) MT algebra $\mathfrak{m}_{\mathbb{R}} = \mathfrak{so}(4, 20)$ for $K3$ surfaces and the LLV algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(4, 21)$ for $K3^{[n]}$ (and similarly, for the Kum_n series). This fact plays a key role in Sect. 3.

B.2 Some explicit examples of branching rules

A branching rule is simply a combinatorial rule describing how the restriction representation of the two Lie algebras $\mathfrak{m} \subset \mathfrak{g}$ behave. Although we discussed above the theoretical framework of restriction representations, more explicit combinatorial descriptions are often easier to deal with. We collect a few branching rules for type BD Lie algebras, which are useful for us.

B.2.1 The branching rule of $\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{so}(n+1, \mathbb{C})$

Let us consider the branching rule of $\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{so}(n+1, \mathbb{C})$. We temporarily assume everything is over \mathbb{C} for this discussion. However, applying (B.1), one also concludes exactly the same branching rule for rational Lie algebras. Denote $V = \mathbb{C}^{n+1}$ and $W = \mathbb{C}^n$ for the standard representations of $\mathfrak{so}(n+1)$ and $\mathfrak{so}(n)$. In [11, §25.3], there is an explicit branching rule describing how the highest $\mathfrak{so}(n+1)$ -module V_λ of weight λ splits as a direct sum of irreducible $\mathfrak{so}(n)$ -modules. The description is as follows.

Assume $n = 2r$ and let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a dominant integral weight of $\mathfrak{so}(2r+1)$. Then we have an $\mathfrak{so}(2r)$ -module irreducible decomposition of the highest $\mathfrak{so}(2r+1)$ -module

$$V_\lambda = \bigoplus_{\lambda'} W_{\lambda'},$$

where $\lambda' = (\lambda'_1, \dots, \lambda'_r)$ runs through all the $\mathfrak{so}(2r)$ -dominant integral weights with

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \dots \geq \lambda'_{r-1} \geq \lambda_r \geq |\lambda'_r| \geq 0,$$

and λ_i and λ'_i are simultaneously all integers or half-integers.

Assume $n = 2r-1$ and let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a dominant integral weight of $\mathfrak{so}(2r)$. Then we have an $\mathfrak{so}(2r-1)$ -module irreducible decomposition of the highest $\mathfrak{so}(2r)$ -module

$$V_\lambda = \bigoplus_{\lambda'} W_{\lambda'},$$

where $\lambda' = (\lambda'_1, \dots, \lambda'_{r-1})$ runs through all the $\mathfrak{so}(2r-1)$ -dominant integral weights with

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \dots \geq \lambda'_{r-1} \geq |\lambda_r| \geq 0,$$

and λ_i and λ'_i are simultaneously all integers or half-integers.

B.2.2 Some special branching rules for $\mathfrak{m} \subset \mathfrak{g}$

Let (V, q) be a rational quadratic space and $W \subset V$ be a nondegenerate quadratic subspace. Set $\mathfrak{g} = \mathfrak{so}(V, q)$ and $\mathfrak{m} = \mathfrak{so}(W, q)$ be rational Lie algebras. Since any nondegenerate subspace $W \subset V$ has its orthogonal complement, evidently we have an

inclusion $\mathfrak{m} \subset \mathfrak{g}$. Applying the above discussion on the branching rule of $\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{so}(n+1, \mathbb{C})$ several times (with the aid of (B.1)), we can get an explicit branching rule for $\mathfrak{m} \subset \mathfrak{g}$.

However, in the two special cases $V_{(k)}$ and $V_{(1, \dots, 1)}$, there is another easier way to obtain a branching rule. Let us first consider the case of the \mathfrak{g} -module $V_{(k)}$. This is precisely the case for the Verbitsky component in the cohomology of compact hyper-Kähler manifold. Classically, this component is viewed as a “symmetric power” of the second cohomology as its $2k$ -th degree part is isomorphic to the k -th symmetric power of the second degree part. We can recover this fact in the following way. Assume that we are in the standard set-up of the Mukai completion, i.e.

$$(V, q) = (\bar{V}, \bar{q}) \oplus U,$$

where U is the 2-dimensional hyperbolic quadratic space, and denote $\bar{\mathfrak{g}} = \mathfrak{so}(\bar{V}, \bar{q})$. Then we have an equality $V = \bar{V} \oplus \mathbb{Q}^2$ (since we are interested in the $\mathfrak{so}(\bar{V}, \bar{q})$ -structure, the precise structure on the second component \mathbb{Q}^2 does not matter). One can compute

$$\begin{aligned} \text{Sym}^k V &= \text{Sym}^k(\bar{V} \oplus \mathbb{Q}^2) = \text{Sym}^k \bar{V} \oplus 2 \text{Sym}^{k-1} \bar{V} \\ &\oplus 3 \text{Sym}^{k-2} \bar{V} \oplus \dots \oplus k \bar{V} \oplus (k+1)\mathbb{Q} \end{aligned}$$

as a $\bar{\mathfrak{g}}$ -module decomposition. Now it is well known $\text{Sym}^k V = \text{Sym}^{k-2} V \oplus V_{(k)}$ as \mathfrak{g} -modules, so this leads us to the identity

$$V_{(k)} = \text{Sym}^k \bar{V} \oplus 2 \text{Sym}^{k-1} \bar{V} \oplus 2 \text{Sym}^{k-2} \bar{V} \oplus \dots \oplus 2 \bar{V} \oplus 2\mathbb{Q}.$$

In particular, this recovers the symmetric power description of the Verbitsky component $V_{(k)}$. If one also wants to capture the degree of the components, then one can consider the decomposition $V = \mathbb{Q}(-1) \oplus V \oplus \mathbb{Q}(1)$ instead, where $\mathbb{Q}(-1)$ and $\mathbb{Q}(1)$ denote the ± 2 eigenspaces of the “grading operator” h (see Sect. 2, esp. (2.1)).

The branching rule for the \mathfrak{g} -module $V_{(1, \dots, 1)} = V_{(1^k)}$ (k times of 1's) will be used when we discuss the LLV decomposition of hyper-Kähler manifolds of OG6 type. Here we assume $\mathfrak{m} = \mathfrak{so}(W, q)$ with $\dim V - \dim W = m$. Thus, we can write $V = W \oplus \mathbb{Q}^m$ and get

$$\begin{aligned} V_{(1^k)} &= \wedge^k V = \wedge^k(W \oplus \mathbb{Q}^m) \\ &= \wedge^k W \oplus m \wedge^{k-1} W \oplus \binom{m}{2} \wedge^{k-2} W \oplus \dots \oplus \binom{m}{k-1} W \oplus \binom{m}{k} \mathbb{Q} \\ &= W_{(1^k)} \oplus m W_{(1^{k-1})} \oplus \dots \oplus \binom{m}{k-1} W \oplus \binom{m}{k} \mathbb{Q}. \end{aligned}$$

This gives us the decomposition of $V_{(1^k)}$ into a direct sum of irreducible \mathfrak{m} -modules.

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