Adaptive NN-Based Boundary Control for Output Tracking of A Wave Equation with Matched and Unmatched Boundary Uncertainties

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Abstract— **This paper is focused on the output tracking control problem of a wave equation with both matched and unmatched boundary uncertainties. An adaptive boundary feedback control scheme is proposed by utilizing radial basis function neural networks (RBF NNs) to deal with the effect of system uncertainties. Specifically, two RBF NN models are first developed to approximate the matched and unmatched system uncertain dynamics respectively. Based on this, an adaptive NN control scheme is derived, which consists of: (i) an adaptive boundary feedback controller embedded by the NN model approximating the matched uncertainty, for rendering stable and accurate tracking control; and (ii) a reference model embedded by the NN model approximating the unmatched uncertainty, for generating a prescribed reference trajectory. Rigorous analysis is performed using the Lyapunov theory and the C_0-semigroup theory to prove that our proposed control scheme can guarantee closed-loop stability and well-posedness. Simulation study has been conducted to demonstrate effectiveness of the proposed approach.**

I. INTRODUCTION

Flexible distributed parameter systems (DPSs) represent a class of very important modern engineering systems, such as spacecraft with flexible attachments, flexible link robot arm, and flexible marine riser [1]. Such systems are usually described by wave equation and/or beam equation—a class of second-order partial differential equations. The fundamental research on control of wave equations and beam equations has been attracting ever-increasing attention in recent years, see, e.g., [2], [3], [4] and the references therein.

Boundary control of DPSs has been of particular interests, due to its practical advantage of demanding fewer sensors and actuators in control design and implementation. Some research efforts have been devoted to the development of boundary feedback control of DPSs modeled by wave equations. For example, [5] proposed a backstepping technique-based boundary control scheme for the stabilization problem of an unstable wave equation. [6] considered the stabilization problem of a wave equation that has anti-damping on the uncontrolled boundary. All these schemes are developed for the stabilization problem of wave equations. In addition to the stabilization problem, many practical applications, e.g., flexible structures in robotics or manufacturing [7], [8], require the operating system's states/output to accurately track certain prescribed trajectories or to realize high-performance point-to-point motion. In these situations, the tracking-control design will be of theoretical and practical importance. A few research efforts have been dedicated to the boundary tracking control design of wave equations, e.g., [9], [10], [11], which however still leave some challenging issues yet to be adequately addressed. For example, [9] proposed an adaptive servomechanism output feedback control scheme for a wave equation with boundary disturbance, which is limited to the case of harmonic disturbances. [10] developed a disturbance estimator for general disturbance, including internal uncertainty and external disturbance, which is applicable only to unmatched uncertainties, i.e., system uncertainties and control input are at different boundaries. [11] considered the control problem of a wave equation that has in-domain and boundary uncertainties, which however required system uncertainties to be structured by a finite-dimensional exosystem. Despite rich literature, most of these schemes cannot be applied to wave equations that have both matched and unmatched boundary uncertainties, especially to the ones where system uncertainties are presented with unstructured nonlinear dynamics. In this case, associated boundary control design problem is still under-explored.

For the control design of systems with uncertain nonlinearities, neural networks (NNs) together with adaptive control techniques provide useful tools due to their inherent approximation and online learning capabilities. Some attempts have been made for the development of adaptive NN-based control design of uncertain nonlinear DPSs, e.g., existing research works in [12], [13], [14]. These schemes utilized adaptive NN to deal with the effect of system uncertainties, such that system stability, control accuracy and robustness property of the resulting control system can be guaranteed based on the Lyapunov stability theory. However, most of these schemes are interior or distributed control approaches, and their implementation requires the system states to be measurable for all spatial locations and all time instants, which may not be feasible in many application scenarios. For extending NN technique to the boundary feedback control design, which is a more effective control strategy developing better practical applicability for DPSs, only a few research results have been obtained in [15], [1]. In [15], radial basis function neural network (RBF NN) was used to deal with unknown input saturations, dead zones and model uncertainties occurring in a flexible manipulator. [1] developed an adaptive neural boundary feedback control scheme for nonlinear flexible

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DPSs modeled by a nonuniform wave equation. However, these schemes are applicable only to the stabilization problem of DPSs, and required the system uncertainty to be matched with control input. It is therefore of interest to develop a new adaptive NN-based boundary control scheme for the tracking problem of flexible DPSs with both matched and unmatched system uncertain nonlinearities.

In this paper, we focus on the output tracking control problem of a wave equation that has both matched and unmatched system uncertain nonlinearities, i.e., the system uncertainties that respectively lie at the controlled boundary and the uncontrolled boundary. An adaptive NN-based boundary feedback control scheme will be proposed by employing RBF NN to deal with the effect of system uncertainties. Specifically, we first develop two RBF NN models to respectively approximate the matched and unmatched system uncertain nonlinear dynamics. An adaptive NN-based control framework will be proposed with these RBF NN models, which consists of: (i) an adaptive boundary feedback controller embedded with the NN model approximating the matched uncertainty, to render stable and accurate tracking control; and (ii) a reference model with the NN model approximating the unmatched uncertainty, to generate a prescribed reference trajectory. Rigorous analysis is performed based on the Lyapunov stability theory and the C_0-semigroup theory to prove that our proposed control scheme can guarantee overall system stability and closed-loop well-posedness. Simulation study of a numerical example is conducted to justify the effectiveness of the proposed approach.

The main contributions of this paper are summarized as follows. (i) we investigate the reference-tracking problem of a wave equation with both matched and unmatched uncertainties; (ii) we propose an adaptive NN-based boundary control scheme by extending the adaptive RBF NN technique to an uncertain wave equation; and (iii) we provide rigorous analysis for the performances of our proposed approaches, including the overall system stability and closed-loop well-posedness.

The rest of this paper is organized as follows. Section II provides preliminary results and problem formulation. Section III presents the proposed adaptive NN-based boundary feedback tracking control scheme. Simulation results are in Section IV. Finally, the conclusion is in Section V.

Notation. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}_+$ denote, respectively, the set of complex numbers, the set of real numbers, the set of positive real numbers and the set of positive integers; $\mathbb{R}^{m\times n}$ denotes the set of $m\times n$ real matrices; $\mathbb{R}^n$ denotes the set of $n\times 1$ real column vectors; $|\cdot|$ is the absolute value of a real number; $\|\cdot\|$ is the 2-norm of a vector or a matrix; $C^1(\Omega)$ denotes the class of all differentiable functions whose derivative is continuous on a measure space Ω; $L^2(\Omega)$ denotes the set of square integrable L_2-functions on a measure space Ω, i.e., $L^2(\Omega) = \{\phi(x) : \Omega \to \mathbb{C}, \int_\Omega |\phi(x)|^2 \, dx < \infty\}$; $L^2_{loc}(\Omega)$ denotes the set of locally square integrable L_2-functions on a measure space Ω, i.e., $L^2_{loc}(\Omega) = \{\phi(x) : \Omega \to \mathbb{C}, |\phi|_{\Omega_0} \in L^2(\Omega_0), \forall \Omega_0 \subset \Omega, \Omega_0 \text{ is compact}\}$; $L^\infty(\Omega)$ denotes the set of functions that are almost everywhere (a.e.) bounded on a measure space Ω, i.e., $L^\infty(\Omega) = \{\phi(x) : \Omega \to \mathbb{C}, \phi \text{ is a.e. bounded}\}$; $H^1(\Omega)$ denotes Sobolev space of order 1 on the space Ω, i.e., $H^1(\Omega) = \{\phi(x) \in L^2(\Omega), \frac{\partial \phi}{\partial x} \in L^2(\Omega)\}$; $(\cdot)_x$ and $(\cdot)'$ denote $\frac{\partial(\cdot)}{\partial x}$; $(\cdot)_{xx}$ and $(\cdot)''$ denote $\frac{\partial^2(\cdot)}{\partial x^2}$; $(\cdot)_t$ and $\dot{(\cdot)}$ denote $\frac{\partial(\cdot)}{\partial t}$; $(\cdot)_{tt}$ and $\ddot{(\cdot)}$ denote $\frac{\partial^2(\cdot)}{\partial t^2}$.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Preliminaries

1) RBF NNs: The RBF networks can be described by $f_{nn}(Z) = \sum_{i=1}^{N_n} w_i s_i(Z) = W^\top S(Z)$ [16], where $Z \in \Omega_Z \subset \mathbb{R}^q$ is the input vector, $W = [w_1, \cdots, w_{N_n}]^\top \in \mathbb{R}^{N_n}$ is the weight vector, N_n is the NN node number, and $S(Z) = [s_1(\|Z - \varsigma_1\|), \cdots, s_{N_n}(\|Z - \varsigma_{N_n}\|)]^\top$, with $s_i(\cdot)$ being a radial basis function, and ς_i $(i = 1, 2, \cdots, N_n)$ being distinct points in state space. The Gaussian function $s_i(\|Z - \varsigma_i\|) = \exp[\frac{-(Z-\varsigma_i)^\top (Z-\varsigma_i)}{\eta_i^2}]$ is one of the most commonly used radial basis functions, where $\varsigma_i = [\varsigma_{i1}, \varsigma_{i2}, \cdots, \varsigma_{iq}]^\top$ is the center of the receptive field and η_i is the width of the receptive field. The Gaussian function belongs to the class of localized RBFs in the sense that $s_i(\|Z - \varsigma_i\|) \to 0$ as $\|Z\| \to \infty$. It is noted that $S(Z)$ is bounded. And, there exists a real constant $S_M \in \mathbb{R}_+$ such that $\|S(Z)\| \leqslant S_M$ [17, Lemma 2.1]. It has been shown in [16] that for any continuous function $f(Z) : \Omega_Z \to \mathbb{R}$ where $\Omega_Z \subset \mathbb{R}^q$ is a compact set, and for the NN approximator, where the node number N_n is sufficiently large, there exists an ideal constant weight vector W^*, such that for any $\epsilon^* > 0$, $f(Z) = W^{*\top} S(Z) + \epsilon$, $\forall Z \in \Omega_Z$, where $|\epsilon| < \epsilon^*$ is the ideal approximation error. The ideal weight vector W^* is an "artificial" quantity required for analysis, and is defined as the value of W that minimizes $|\epsilon|$ for all $Z \in \Omega_Z \subset \mathbb{R}^q$, i.e., $W^* \triangleq argmin_{W\in\mathbb{R}^{N_n}} \{\sup_{Z\in\Omega_Z} |f(Z) - W^\top S(Z)|\}$.

2) C_0-semigroup: Let Ω be a (real or complex) Banach space. A C_0-semigroup on Ω is defined as follows:

Definition 1: A C_0-semigroup on Ω (also a strongly continuous semigroup) is a function $T : [0, +\infty) \to \mathcal{L}(\Omega)$, where $\mathcal{L}(\Omega)$ denotes the space of bounded linear operators in Ω with domain all of Ω, satisfying

(i) $T(t + s) = T(t)T(s)$, for all $t, s \geq 0$;
(ii) $\lim_{t\to 0+} T(t)x = x$ for all $x \in \Omega$.

Particularly, the C_0-semigroup $T(t)$ on Ω will be associated with the solutions of the initial value problem for a linear autonomous differential equation on $[0, +\infty)$, i.e.,

$$\frac{d}{dt} y(t) = Ay(t), \quad y(0) = y_0, \tag{1}$$

where A is a linear operator with dense domain $D(A)$ in Ω. If $y_0 \in D(A)$, then, the function $y : [0, +\infty) \to \Omega$ given by $y(t) = T(t)y_0$ should be the unique solution of the initial value problem given in (1).

B. Problem Formulation

Consider a one-dimensional wave equation with matched and unmatched boundary uncertainties as follows:

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t), \quad x \in (0,1),\, t>0 \\ y_x(0,t) = c_1 y_t(0,t) + f_1(y_m(0,t)) \\ y_x(1,t) = u(t) + f_2(y_m(1,t)) \\ y(x,0) = y_0(x),\, y_t(x,0) = y_1(x) \\ y_m(x,t) = [y(x,t), y_t(x,t)]^\top \\ y_{out}(t) = y(1,t), \end{cases} \tag{2}$$

where $y(x,t) \in \mathbb{R}$ is the system state variable at the position $x \in [0,1]$ for time $t \geq 0$; $u \in \mathbb{R}$ is the boundary system control input; $c_1 > 0$ is a known constant; $f_1(y_m(0,t))$ and $f_2(y_m(1,t))$ are unknown nonlinear functions satisfying locally Lipschitz continuous, which represent the system uncertainties that are unmatched and matched with control input, respectively; $y_0(x)$ and $y_1(x)$ are unknown functions representing initial conditions; $y_{out}(t)$ is the performance output signal to be regulated/controlled; $y_m(0,t) = [y(0,t), y_t(0,t)]^\top$ and $y_m(1,t) = [y(1,t), y_t(1,t)]^\top$ are boundary state signals that are assumed to be measurable.

In this paper, our objective is to design a boundary feedback control scheme for the system (2), aiming to drive the system output $y_{out}(t) = y(1,t)$ to track a prescribed reference signal $y_{ref}(t)$ with guaranteed system well-posedness and stability. To this end, adaptive NNs will be used to deal with the system uncertainties $f_1(y_m(0,t))$ and $f_2(y_m(1,t))$.

Assumption 1: The reference signal satisfies $y_{ref}(t) \in W^{2,\infty}(0,\infty) := \{y \,|\, y \in L^\infty(0,\infty),\, \dot{y} \in L^\infty(0,\infty),\, \ddot{y} \in L^\infty(0,\infty)\}$.

Before proceeding, we first study the well-posedness of the system (2). Define a Hilbert space $\mathcal{H} = H^1(0,1) \times L^2(0,1)$ with inner product:

$$\begin{aligned} &\left\langle (\phi_1,\psi_1)^\top, (\phi_2,\psi_2)^\top \right\rangle_{\mathcal{H}} \\ =& c_2\phi_1(1)\overline{\phi_2(1)} + \int_0^1 \left(\phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)}\right) dx \\ &+ \mu \int_0^1 (-2+x)\left(\phi_1'(x)\overline{\psi_2(x)} + \psi_1(x)\overline{\phi_2'(x)}\right) dx, \\ &\qquad\qquad \forall (\phi_1,\psi_1)^\top, (\phi_2,\psi_2)^\top \in \mathcal{H}, \end{aligned} \tag{3}$$

where $\overline{(\cdot)}$ represents the conjugate of $(\cdot)$, $c_2 > 0$, $0 < \mu < \min\{\frac{c_1}{1+c_1^2}, \frac{1}{2}\}$ with $c_1 > 0$ given in (2). This inner product is well-defined and positive-definite because $\forall (\phi,\psi)^\top \in \mathcal{H}$,

$$\begin{aligned} \|(\phi,\psi)\|_{\mathcal{H}}^2 &= \left\langle (\phi,\psi)^\top, (\phi,\psi)^\top \right\rangle_{\mathcal{H}} \\ &\geq (1-2\mu)\int_0^1 \left(|\phi'(x)|^2 + |\psi(x)|^2\right) dx. \end{aligned} \tag{4}$$

Lemma 1: The system (2) is well-posed, i.e., for any initial value $[y_0(\cdot), y_1(\cdot)]^\top \in \mathcal{H}$, and $u(t) \in L^2_{loc}(0,\infty)$, there exists a unique solution to (2) such that $[y(\cdot,t), y_t(\cdot,t)]^\top \in C^1([0,\infty);\mathcal{H})$.

Proof: To show the well-posedness of the system (2), we can rewrite this system into the following form:

$$\begin{aligned} \frac{d}{dt}y_m(\cdot,t) =& \mathcal{A}y_m(\cdot,t) + \mathcal{B}_1 f_1(y_m(0,t)) \\ &+ \mathcal{B}_2(c_2 y(1,t) + u(t) + f_2(y_m(1,t))), \end{aligned} \tag{5}$$

for $y_m(\cdot,0) \in \mathcal{H}$, where $y_m(\cdot,t) = [y(\cdot,t), y_t(\cdot,t)]^\top$, $y_m(\cdot,0) = [y_0(\cdot), y_1(\cdot)]^\top$, $\mathcal{B}_1 = [0, -\delta(x)]^\top$, $\mathcal{B}_2 = [0, \delta(x-1)]^\top$ with $\delta(\cdot)$ being a Dirac delta distribution, and the operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is defined as:

$$\begin{aligned} &\mathcal{A}(\phi,\psi)^\top = (\psi, \phi'')^\top, \\ &D(\mathcal{A}) = \{(\phi,\psi)^\top \in \mathcal{H} \,|\, \mathcal{A}(\phi,\psi)^\top \in \mathcal{H}, \phi'(0) = c_1\psi(0), \\ &\qquad\qquad \phi'(1) = -c_2\phi(1)\}. \end{aligned} \tag{6}$$

We first verify that the operator $\mathcal{A}$ of (6) can generate an exponentially stable C_0-semigroup on $\mathcal{H}$. To this end, we consider the following system:

$$\frac{d}{dt}y_m(\cdot,t) = \mathcal{A}y_m(\cdot,t). \tag{7}$$

Define a positive-definite Lyapunov function as:

$$\begin{aligned} V_y(t) =& c_2 y^2(1,t) + \int_0^1 (y_x^2(x,t) + y_t^2(x,t))dx \\ &+ 2\mu \int_0^1 (-2+x) y_x(x,t) y_t(x,t) dx. \end{aligned} \tag{8}$$

From (6)–(7), noting that $0 < \mu < \frac{c_1}{1+c_1^2}$, the derivative of $V_y(t)$ can be derived as follows:

$$\begin{aligned} &\dot{V}_y(t) = 2c_2 y_t(1,t) y(1,t) \\ &\quad + 2\int_0^1 \left(y_x(x,t)y_{xt}(x,t) + y_t(x,t)y_{tt}(x,t)\right) dx \\ &\quad + 2\mu \int_0^1 (-2+x)\left(y_x(x,t)y_{tt}(x,t) + y_{xt}(x,t)y_t(x,t)\right) dx \\ &= 2y_x(x,t)y_t(x,t)|_0^1 + \mu(-2+x)\left(y_x^2(x,t) + y_t^2(x,t)\right)|_0^1 \\ &\quad - \mu \int_0^1 \left(y_x^2(x,t) + y_t^2(x,t)\right) dx + 2c_2 y_t(1,t) y(1,t) \\ &= -2(c_1 - \mu(1+c_1^2)) y_t^2(0,t) - \mu(c_2^2 y^2(1,t) + y_t^2(1,t)) \\ &\quad - \mu \int_0^1 \left(y_x^2(x,t) + y_t^2(x,t)\right) dx \\ &\leq -\mu c_2^2 y^2(1,t) - \mu \int_0^1 \left(y_x^2(x,t) + y_t^2(x,t)\right) dx. \end{aligned} \tag{9}$$

From (8)–(9), it can be deduced that there exists a positive constant κ such that $\dot{V}_y(t) \leq -\kappa V_y(t)$. This implies that the operator $\mathcal{A}$ of (7) is dissipative [18]. Furthermore, it is easily seen that $\mathcal{A}^{-1}$ exists and is bounded in $\mathcal{H}$. A simple computation shows:

$$\mathcal{A}^{-1}(\phi,\psi)^\top = (\phi^*, \psi^*)^\top,\ \forall (\phi,\psi)^\top \in \mathcal{H}, \tag{10}$$

with

$$\begin{cases} \phi^*(x) = & (x-1)\int_0^x \psi(s)ds + \int_x^1 (s-1)\psi(s)ds \\ & -\frac{1}{c_2}\int_0^1 \psi(s)ds - c_1\phi(0)(1-x+\frac{1}{c_2}), \\ \psi^*(x) = & \phi(x). \end{cases} \tag{11}$$

Thus, according to the Lumer-Phillips theorem [18], it can be deduced that the operator $\mathcal{A}$ of (6) generates an exponentially stable C_0-semigroup on $\mathcal{H}$.

Then, consider the system (5), following a similar line of the analysis in [10], it can be verified that $\mathcal{B}_1$, $\mathcal{B}_2$ are admissible to $e^{\mathcal{A}t}$. Furthermore, note that the functions $f_1(y_m(0,t))$ and $f_2(y_m(1,t))$ in (2) are both locally Lipschiz continuous. According to [19, Proposition 4.2.5], the system (5) will be well-posed, i.e., for any $t_f > 0$, $y_m(\cdot,0) \in \mathcal{H}$ and $u(t) \in L^2_{loc}(0,t_f)$, there exists a unique solution in $\mathcal{H}$ to system (2) with the form of $y_m(\cdot,t) = e^{\mathcal{A}t}y_m(\cdot,0) + \int_0^t e^{\mathcal{A}(t-s)}(\mathcal{B}_1 f_1(y_m(0,s)) + \mathcal{B}_2(c_2 y(1,s) + u(s) + f_2(y_m(1,s))))ds$, $t \in [0,t_f]$. ■

III. MAIN RESULTS

In this section, an adaptive NN-based boundary feedback control scheme will be presented. First of all, according to Section II-A, we know that there exist constant NN weights $W_1^* \in \mathbb{R}^{N_{n_1}}$ and $W_2^* \in \mathbb{R}^{N_{n_2}}$ with N_{n_i} $(i = 1,2)$ denoting the number of NN nodes, such that:

$$\begin{aligned} f_1(y_m(0,t)) &= W_1^{*\top} S_1(y_m(0,t)) + \epsilon_1, \\ f_2(y_m(1,t)) &= W_2^{*\top} S_2(y_m(1,t)) + \epsilon_2, \end{aligned} \tag{12}$$

where $S_1(\cdot) : \mathbb{R}^2 \to \mathbb{R}^{N_{n_1}}$ and $S_2(\cdot) : \mathbb{R}^2 \to \mathbb{R}^{N_{n_2}}$ are smooth RBF vectors, $\epsilon_i \in \mathbb{R}$ $(i = 1,2)$ is the estimation error satisfying $|\epsilon_i| < \epsilon^*$ with ϵ^* being a positive constant that can be made arbitrarily small given a sufficiently large number of neurons.

Based on (12), we first design a reference model to generate a prescribed reference trajectory $y_{ref}(t)$ as follows:

$$\begin{cases} \hat{y}_{tt}(x,t) = \hat{y}_{xx}(x,t) \\ \hat{y}_x(0,t) = c_1\hat{y}_t(0,t) + \hat{W}_1^\top S_1(y_m(0,t)) \\ \hat{y}(1,t) = y_{ref}(t) \\ \hat{y}(x,0) = \hat{y}_0(x),\ \hat{y}_t(x,0) = \hat{y}_1(x), \end{cases} \tag{13}$$

where $\hat{y}(x,t) \in \mathbb{R}$ is the state of reference model, $\hat{W}_1 \in \mathbb{R}^{N_{n_1}}$ is the estimate of W_1^* in (12), $\hat{y}_0(x)$ and $\hat{y}_1(x)$ are initial conditions, and c_1 is a known constant given in (2). Then, we propose to develop an adaptive NN-based boundary feedback controller as:

$$u(t) = -c_2\left(y(1,t) - y_{ref}(t)\right) + \hat{y}_x(1,t) - \hat{W}_2^\top S_2(y_m(1,t)), \tag{14}$$

where $\hat{W}_2 \in \mathbb{R}^{N_{n_2}}$ is the estimate of W_2^* in (12), and $c_2 > 0$ is a design constant. The adaptation law of NN weights $\hat{W}_1$ and $\hat{W}_2$ for (13) and (14) are given as:

$$\begin{aligned} \dot{\hat{W}}_1 &= \Gamma_1(2\mu c_1 - 1)\tilde{y}_t(0,t)S_1 - \Gamma_1\gamma_1\hat{W}_1, \\ \dot{\hat{W}}_2 &= \Gamma_2(\tilde{y}_t(1,t) + \mu c_2\tilde{y}(1,t))S_2 - \Gamma_2\gamma_2\hat{W}_1, \end{aligned} \tag{15}$$

where $\tilde{y}(x,t) = y(x,t) - \hat{y}(x,t)$, $\Gamma_i = \Gamma_i^\top > 0$, $\gamma_i > 0$ $(i = 1,2)$ and $0 < \mu < \min\{\frac{c_1}{1+c_1^2}, \frac{1}{2}\}$ are design constants.

Lemma 2: The reference system (13) is well-posed. That is, for any initial value $[\hat{y}_0(\cdot), \hat{y}_1(\cdot)]^\top \in \mathcal{H}$, $\hat{W}_1^\top S_1 \in L^2_{loc}(0,\infty)$, and $y_{ref}(t) \in W^{2,\infty}(0,\infty)$ (with $W^{2,\infty}(0,\infty)$ defined in Assumption 1), there exists a unique solution to (13) such that $[\hat{y}(\cdot,t), \hat{y}_t(\cdot,t)]^\top \in C^1([0,\infty);\mathcal{H})$.

Proof: We first transform the model (13) into the following system by using the transformation $\nu(x,t) = \hat{y}(x,t) - xy_{ref}(t)$:

$$\begin{cases} \nu_{tt}(x,t) = \nu_{xx}(x,t) - x\ddot{y}_{ref}(t) \\ \nu_x(0,t) = c_1\nu_t(0,t) + \hat{W}_1^\top S_1 - y_{ref}(t) \\ \nu(1,t) = 0 \\ \nu(x,0) = \nu_0(x),\ \nu_t(x,0) = \nu_1(x), \end{cases} \tag{16}$$

where $\nu_0(x) = \hat{y}_0(x) - xy_{ref}(0)$, and $\nu_1(x) = \hat{y}_1(x) - x\dot{y}_{ref}(0)$. Then, this system can be rewritten as:

$$\frac{d}{dt}\upsilon(t) = \mathcal{A}_0\upsilon(t) + F(\cdot,t) + \mathcal{B}_1(\hat{W}_1^\top S_1 - y_{ref}(t)), \tag{17}$$

for $\upsilon(0) \in \mathcal{H}$, where $\upsilon(t) = [\nu(\cdot,t), \nu_t(\cdot,t)]^\top$, $\upsilon(0) = [\nu_0(\cdot), \nu_1(\cdot)]^\top$, $F(x,t) = [0, -x\ddot{y}_{ref}(t)]^\top$, $\mathcal{B}_1 = [0, -\delta(x)]^\top$ and the operator $\mathcal{A}_0 : D_0(\mathcal{A}_0) \to \mathcal{H}$ is defined as:

$$\begin{aligned} &\mathcal{A}_0(\phi,\psi)^\top = (\psi, \phi'')^\top, \\ &D_0(\mathcal{A}_0) = \{(\phi,\psi)^\top \in \mathcal{H} \,|\, \mathcal{A}_0(\phi,\psi)^\top \in \mathcal{H}, \phi'(0) = c_1\psi(0), \\ &\qquad\qquad \phi(1) = 0\}. \end{aligned} \tag{18}$$

Following a similar line of the proof in Lemma 1, it can be verified that the operator $\mathcal{A}_0$ is dissipative, and $\mathcal{A}_0^{-1}$ exists and is bounded, which can be seen using a simple example:

$$\mathcal{A}_0^{-1}(\phi,\psi)^\top = (\phi^*,\psi^*)^\top, \quad \forall(\phi,\psi)^\top \in \mathcal{H}, \tag{19}$$

with

$$\begin{cases} \phi^*(x) = & c_1\phi(0)(x-1) - \int_x^1\int_0^z \psi(s)dsdz, \\ \psi^*(x) = & \phi(x). \end{cases} \tag{20}$$

Thus, it is seen that the operator $\mathcal{A}_0$ can generate an exponentially stable C_0-semigroup on $\mathcal{H}$. Furthermore, $\mathcal{B}_1$ is admissible to $e^{\mathcal{A}_0 t}$ following the analysis of [10]. Then, note that for any $y_{ref}(t) \in W^{2,\infty}(0,\infty)$, we have $y_{ref}(t) \in L^2_{loc}(0,\infty)$ and $F(\cdot,t) = [0, -x\ddot{y}_{ref}(t)]^\top \in L^2_{loc}(0,\infty)$. It can be deduced that for any $t_f > 0$, $y_{ref}(t) \in W^{2,\infty}(0,\infty)$, $\hat{W}_1^\top S_1 \in L^2_{loc}(0,t_f)$, and $\upsilon(0) \in \mathcal{H}$, there exists a unique solution in $\mathcal{H}$ to system (17) with the form of $\upsilon(t) = e^{\mathcal{A}_0 t}\upsilon(0) + \int_0^t e^{\mathcal{A}_0(t-s)}(F(\cdot,s) + \mathcal{B}_1(\hat{W}_1^\top S_1 - y_{ref}(s)))ds$, $t \in [t,t_f]$. This ends the proof of well-posedness of (13). ■

Remark 1: The above analysis implies that the signals $\upsilon(t) = [\nu(\cdot,t), \nu_t(\cdot,t)]^\top$ will be bounded as long as $\hat{W}_1^\top S_1$, $y_{ref}(t)$ and $\ddot{y}_{ref}(t)$ are bounded. Based on this, noting that $\hat{y}(x,t) = \nu(x,t) + xy_{ref}(t)$ and $y_{ref}(t)$, $\dot{y}_{ref}(t)$, $\ddot{y}_{ref}(t)$ are all bounded under Assumption 1, it can be deduced that the state signals $[\hat{y}(x,t), \hat{y}_t(x,t)]^\top$ of the designed reference model (13) can be bounded as long as $\hat{W}_1^\top S_1$ is bounded.

Based on Lemma 1 and Lemma 2, well-posedness and overall stability of the closed-loop system consisting of (2) and (13)–(15) are established in the following theorem.

Theorem 1: Consider the closed-loop system consisting of the plant (2), the reference model (13), the controller (14) and the weight adaptation law (15). Under Assumption 1, if the design constants $\gamma_1 > 0$ and $\mu > 0$ satisfy: $\gamma_1 - \mu S_M^2 > 0$, with S_M being the upper bound of $S_1(\cdot)$, then, we have: (i) the closed-loop system is well-posed; and (ii) all signals of the closed-loop system are bounded.

Proof: To prove the first part, from (2), (12), (13) and (14), we derive the following error system:

$$\begin{cases} \tilde{y}_{tt}(x,t) = \tilde{y}_{xx}(x,t) \\ \tilde{y}_x(0,t) = c_1\tilde{y}_t(0,t) - \tilde{W}_1^\top S_1 + \epsilon_1 \\ \tilde{y}_x(1,t) = -c_2\tilde{y}(1,t) - \tilde{W}_2^\top S_2 + \epsilon_2 \\ \tilde{y}(x,0) = \tilde{y}_0(x),\ \tilde{y}_t(x,0) = \tilde{y}_1(x), \end{cases} \tag{21}$$

where $\tilde{y}(x,t) = y(x,t) - \hat{y}(x,t)$, $\tilde{y}_0(x) = y_0(x) - \hat{y}_0(x)$, $\tilde{y}_1(x) = y_1(x) - \hat{y}_1(x)$, and $\tilde{W}_i = \hat{W}_i - W_i^*$ $(i = 1, 2)$. This system can be rewritten as:

$$\begin{aligned} \frac{d}{dt}\tilde{y}_m(\cdot,t) =& \mathcal{A}\tilde{y}_m(\cdot,t) + \mathcal{B}_1(-\tilde{W}_1^\top S_1 + \epsilon_1) \\ &+ \mathcal{B}_2(-\tilde{W}_2^\top S_2 + \epsilon_2), \end{aligned} \tag{22}$$

for $\tilde{y}_m(\cdot,0) \in \mathcal{H}$, where $\tilde{y}_m(\cdot,t) = [\tilde{y}(\cdot,t), \tilde{y}_t(\cdot,t)]^\top$, $\tilde{y}_m(\cdot,0) = [\tilde{y}_0(\cdot), \tilde{y}_1(\cdot)]^\top$, the operator $\mathcal{A}$ is defined in (6), $\mathcal{B}_1 = [0, -\delta(x)]^\top$ and $\mathcal{B}_2 = [0, \delta(x-1)]^\top$. Based on (22), the error system (21) can be verified to be well-posed by following a similar line of the proof in Lemma 1. Furthermore, it is known from Lemma 2 that the reference system (13) is also well-posed. Thus, it can be deduced that the closed-loop system consisting of (2), (13), (14) and (15) is well-posed.

We further study the overall stability of the closed-loop system. From (2), (12), (13), (14) and (15), we can obtain the closed-loop error system as follows:

$$\begin{cases} \tilde{y}_{tt}(x,t) = \tilde{y}_{xx}(x,t) \\ \tilde{y}_x(0,t) = c_1\tilde{y}_t(0,t) - \tilde{W}_1^\top S_1 + \epsilon_1 \\ \tilde{y}_x(1,t) = -c_2\tilde{y}(1,t) - \tilde{W}_2^\top S_2 + \epsilon_2 \\ \dot{\hat{W}}_1 = \Gamma_1(2\mu c_1 - 1)\tilde{y}_t(0,t)S_1 - \Gamma_1\gamma_1\hat{W}_1 \\ \dot{\hat{W}}_2 = \Gamma_2(\tilde{y}_t(1,t) + \mu c_2\tilde{y}(1,t))S_2 - \Gamma_2\gamma_2\hat{W}_1. \end{cases} \tag{23}$$

For this system, we define a positive-definite Lyapunov function candidate as follows:

$$\begin{aligned} V(t) =& c_2\tilde{y}^2(1,t) + \int_0^1 (\tilde{y}_x^2(x,t) + \tilde{y}_t^2(x,t))dx \\ &+ 2\mu\int_0^1 (-2+x)\tilde{y}_x(x,t)\tilde{y}_t(x,t)dx \\ &+ \tilde{W}_1^\top\Gamma_1^{-1}\tilde{W}_1 + \tilde{W}_2^\top\Gamma_2^{-1}\tilde{W}_2. \end{aligned} \tag{24}$$

The derivative of $V(t)$ can be derived as follows:

$$\begin{aligned} \dot{V}(t) =& 2c_2\tilde{y}_t(1,t)\tilde{y}(1,t) \\ &+ 2(-c_2\tilde{y}(1,t) - \tilde{W}_2^\top S_2 + \epsilon_2)\tilde{y}_t(1,t) \\ &- 2(c_1\tilde{y}_t(0,t) - \tilde{W}_1^\top S_1 + \epsilon_1)\tilde{y}_t(0,t) \\ &- \mu((-c_2\tilde{y}(1,t) - \tilde{W}_2^\top S_2 + \epsilon_2)^2 + \tilde{y}_t^2(1,t)) \\ &+ 2\mu((c_1\tilde{y}_t(0,t) - \tilde{W}_1^\top S_1 + \epsilon_1)^2 + \tilde{y}_t^2(0,t)) \\ &- \mu\int_0^1 \left(\tilde{y}_x^2(x,t) + \tilde{y}_t^2(x,t)\right)dx \\ &+ 2\tilde{W}_1^\top((2\mu c_1 - 1)\tilde{y}_t(0,t)S_1 - \gamma_1\hat{W}_1) \\ &+ 2\tilde{W}_2^\top((\tilde{y}_t(1,t) + \mu c_2\tilde{y}(1,t))S_2 - \gamma_2\hat{W}_2) \\ =& -2(c_1 - \mu(1+c_1^2))\tilde{y}_t^2(0,t) - \mu c_2^2\tilde{y}^2(1,t) - \mu\tilde{y}_t^2(1,t) \\ &- \mu\int_0^1 (\tilde{y}_x^2(x,t) + \tilde{y}_t^2(x,t))dx - \mu(\tilde{W}_2^\top S_2 - \epsilon_2)^2 \\ &+ 2(2\mu c_1 - 1)\tilde{y}_t(0,t)\epsilon_1 - 4\mu\tilde{W}_1^\top S_1\epsilon_1 + 2\mu\epsilon_1^2 \\ &+ 2(\tilde{y}_t(1,t) + \mu c_2\tilde{y}(1,t))\epsilon_2 + 2\mu\tilde{W}_1^\top S_1\tilde{W}_1^\top S_1 \\ &- 2\gamma_1\tilde{W}_1^\top\tilde{W}_1 - 2\gamma_2\tilde{W}_2^\top\tilde{W}_2 - 2\gamma_1\tilde{W}_1^\top W_1^* - 2\gamma_2\tilde{W}_2^\top W_2^* \\ \leq& -2(c_1 - \mu(1+c_1^2))\tilde{y}_t^2(0,t) - \mu c_2^2\tilde{y}^2(1,t) - \mu\tilde{y}_t^2(1,t) \\ &- \mu\int_0^1 (\tilde{y}_x^2(x,t) + \tilde{y}_t^2(x,t))dx - 2(\gamma_1 - \mu S_M^2)\left\|\tilde{W}_1\right\|^2 \\ &- 2\gamma_2\left\|\tilde{W}_2\right\|^2 + 2(2\mu c_1 - 1)\tilde{y}_t(0,t)\epsilon_1 + 2\mu\epsilon_1^2 \\ &+ 2\tilde{y}_t(1,t)\epsilon_2 + 2\mu c_2\tilde{y}(1,t)\epsilon_2 + 4\mu S_M\left\|\tilde{W}_1\right\||\epsilon_1| \\ &+ 2\gamma_1\left\|\tilde{W}_1\right\|\|W_1^*\| + 2\gamma_2\left\|\tilde{W}_2\right\|\|W_2^*\|, \end{aligned} \tag{25}$$

where $\hat{W}_i = \tilde{W}_i + W_i^*$ $(i = 1, 2)$ and $\|S_1\| < S_M$. By completing the associated squares of (25), and noting that $|\epsilon_i| < \epsilon^*$ $(i = 1, 2)$, $0 < \mu < \frac{c_1}{1+c_1^2}$ and $\gamma_1 - \mu S_M^2 > 0$, we have:

$$\begin{aligned} \dot{V}(t) \leq& -\frac{\mu}{2}c_2^2\tilde{y}^2(1,t) - \mu\int_0^1 \left(\tilde{y}_x^2(x,t) + \tilde{y}_t^2(x,t)\right)dx \\ &- (\gamma_1 - \mu S_M^2)\left\|\tilde{W}_1\right\|^2 - \gamma_2\left\|\tilde{W}_2\right\|^2 \\ &+ \rho\epsilon^{*2} + \frac{2\gamma_1^2}{\gamma_1 - \mu S_M^2}\|W_1^*\|^2 + \gamma_2\|W_2^*\|^2, \end{aligned} \tag{26}$$

where $\rho := \frac{(2\mu c_1 - 1)^2}{c_1 - \mu(1+c_1^2)} + \frac{8\mu^2 S_M^2}{\gamma_1 - \mu S_M^2} + \frac{2}{\mu} + 2\mu$. Based on this, we have $\dot{V}(t) < 0$ whenever:

$$\begin{aligned} &\tilde{y}^2(1,t) \geq \frac{2\rho}{\mu c_2^2}\epsilon^{*2} + \frac{4\gamma_1^2}{(\gamma_1 - \mu S_M^2)\mu c_2^2}\|W_1^*\|^2 + \frac{2\gamma_2}{\mu c_2^2}\|W_2^*\|^2; \\ &\int_0^1 \tilde{y}_x^2 dx \geq \frac{\rho}{\mu}\epsilon^{*2} + \frac{2\gamma_1^2}{(\gamma_1 - \mu S_M^2)\mu}\|W_1^*\|^2 + \frac{\gamma_2}{\mu}\|W_2^*\|^2; \\ &\int_0^1 \tilde{y}_t^2 dx \geq \frac{\rho}{\mu}\epsilon^{*2} + \frac{2\gamma_1^2}{(\gamma_1 - \mu S_M^2)\mu}\|W_1^*\|^2 + \frac{\gamma_2}{\mu}\|W_2^*\|^2; \\ &\left\|\tilde{W}_1\right\|^2 \geq \frac{\rho}{\gamma_1 - \mu S_M^2}\epsilon^{*2} + \frac{2\gamma_1^2}{(\gamma_1 - \mu S_M^2)^2}\|W_1^*\|^2 \\ &\qquad + \frac{\gamma_2}{\gamma_1 - \mu S_M^2}\|W_2^*\|^2; \\ &\left\|\tilde{W}_2\right\|^2 \geq \frac{\rho}{\gamma_2}\epsilon^{*2} + \frac{2\gamma_1^2}{(\gamma_1 - \mu S_M^2)\gamma_2}\|W_1^*\|^2 + \|W_2^*\|^2. \end{aligned} \tag{27}$$

This guarantees that the signals of $\tilde{y}(1,t)$, $\tilde{y}_x(x,t)$, $\tilde{y}_t(x,t)$, $\tilde{W}_1$ and $\tilde{W}_2$ of (23) are all bounded. Based on this and from the Poincare inequality of $\int_0^1 \tilde{y}^2(x,t)dx \leq 2\tilde{y}^2(1,t) + 4\int_0^1 \tilde{y}_x^2(x,t)dx$, we have: $\tilde{y}(x,t)$ is bounded. Then, since $\hat{W}_i = \tilde{W}_i + W_i^*$ $(i = 1, 2)$, we have: $\hat{W}_i$ is also bounded. Noting that S_i is bounded such that $\hat{W}_i^\top S_i$ is bounded, the control signal of (14) can be guaranteed bounded, and the signals of reference model (13) will also be bounded, as argued in Remark 1. Consequently, it is verified that all signals in the closed-loop system consisting of (2), (13), (14) and (15) are bounded. This ends the proof. ∎

IV. SIMULATION STUDIES

In this section, we will use a numerical example to demonstrate the effectiveness of our scheme. Specifically, consider the system of (2) with $c_1 = 1$, $f_1(y(0,t), y_t(0,t)) = 0.2\sin^2(y(0,t)) + 0.5y^3(0,t)$, and $f_2(y(1,t), y_t(1,t)) = 0.9y(1,t)) + 0.2\cos(y_t^2(1,t))$. The reference model (13) is designed with $y_{ref}(t) = 0.5\sin(t) + 0.5\cos(\frac{\pi}{2}t + 2) + 1$. The RBF NN $\hat{W}_1^\top S_1$ is constructed in a regular lattice with the number of nodes $N_n = 11 \times 17$, the centers evenly spaced on $[-0.5, 1] \times [-1, 1.5]$, and the widths $\eta_i = 0.15$ $(i = 1, 2, \cdots, 187)$; and $\hat{W}_2^\top S_2$ is with the number of nodes $N_n = 11 \times 19$, the centers evenly spaced on $[0, 2] \times [-1.8, 1.8]$, and the widths $\eta_i = 0.2$ $(i = 1, 2, \cdots, 209)$. The controller (14) and the NN weight adaptation law (15) are implemented with $c_2 = 3$, $\Gamma_1 = \Gamma_2 = 5$, $\mu = 0.45$, $\gamma_1 = 0.5$ and $\gamma_2 = 0.02$. The initial conditions are given as: $y_0(x) = 0.2 - 0.2x$, $y_1(x) = 0.1 - 0.2x$, $\hat{y}_0(x) = 0.15 - 0.15x$, $\hat{y}_1(x) = 0.075 - 0.15x$, $\hat{W}_1(0) = 0$ and $\hat{W}_2(0) = 0$.

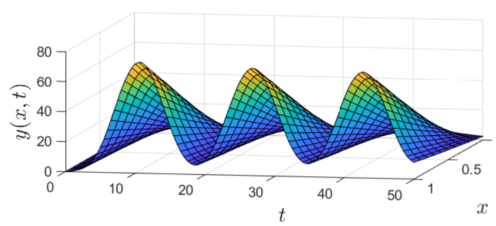

Fig. 1: Open-loop system state $y(x,t)$ of (2) with control input $u(t) = 0$.

Consider the system (2), the system state $y(x,t)$ with control signal $u(t) = 0$ (i.e., the system (2) is open-loop) is shown in Fig. 1. For this system, with the control scheme (13)–(15), associated tracking control performances are plotted in Figs. 2–3. Specifically, Fig. 2 shows that all the signals, including the real-time system state $y(x,t)$, the reference signal $\hat{y}(x,t)$, the control signal $u(t)$, and the NN weights $\hat{W}(t) = [\hat{W}_1(t); \hat{W}_2(t)]$, are stable. Fig. 3 illustrates that the system output $y_{out}(t) = y(1,t)$ can accurately track the given reference trajectory $y_{ref}(t)$. These results justify the effectiveness of our scheme.

V. CONCLUSIONS

In this paper, an adaptive RBF NN-based boundary feedback control scheme has been proposed for output tracking control of a wave equation with both matched and unmatched boundary uncertainties. Two RBF NN models were developed to respectively approximate matched and unmatched system uncertain nonlinear dynamics, based on which an adaptive NN-based controller was proposed for stable tracking control. Rigorous analysis has been performed based on the Lyapunov stability theory and C_0-semigroup theory, which demonstrates that our proposed approach can guarantee overall system stability and closed-loop well-posedness.

Based on the scheme proposed in this paper, there are several promising directions worth for future investigations, including (i) to extend the proposed tracking control scheme to a more general problem of driving the holistic system state to track a prescribed infinite-dimensional reference trajectory; (ii) to investigate a new adaptive NN-based learning control scheme to enable not only stable tracking control but also accurate dynamics learning for PDE systems.

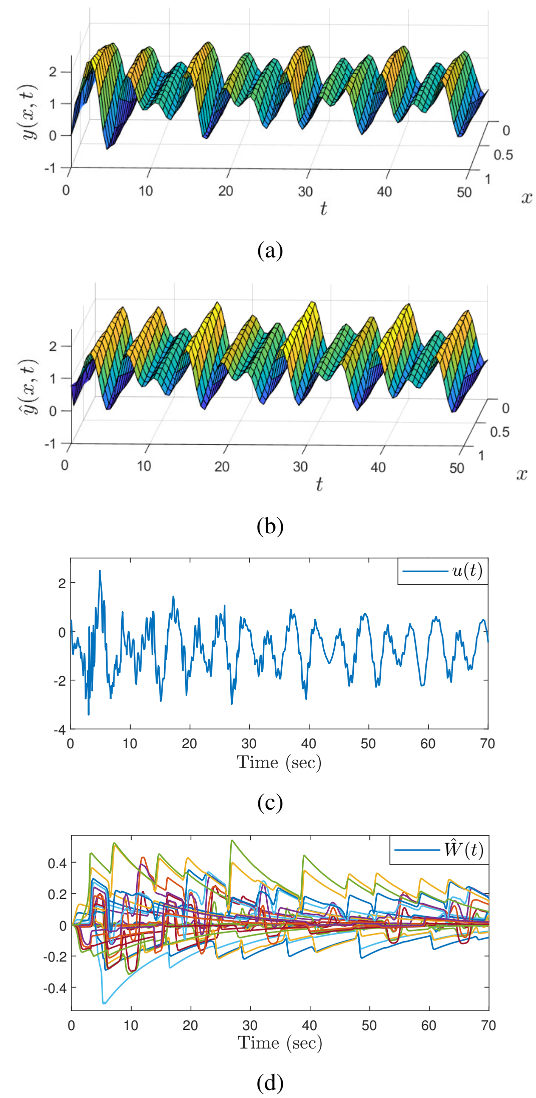

Fig. 2: Closed-loop system response of (2) with control scheme (13)–(15): (a) system state $y(x,t)$; (b) reference state $\hat{y}(x,t)$; (c) control signal $u(t)$; and (d) NN weights $\hat{W}(t) = [\hat{W}_1(t); \hat{W}_2(t)]$.

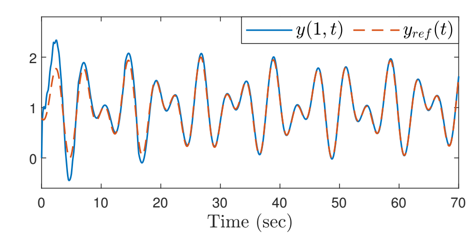

Fig. 3: Tracking performance for the output $y_{out}(t) = y(1,t)$ of system (2) to a prescribed reference trajectory $y_{ref}(t)$.

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