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On the Normalized Ground States of Second Order PDE's with Mixed Power Non-linearities

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Abstract: For each $\lambda > 0$ and under necessary conditions on the parameters, we construct normalized waves for second order PDE's with mixed power non-linearities, with $\|u\|^2_{L^2(\mathbf{R}^n)} = \lambda, n \geq 1$. We show that these are bell-shaped smooth and localized functions, and we compute their precise asymptotics. We study the question for the smoothness of the Lagrange multiplier with respect to the L^2 norm of the waves, namely the map $\lambda \to \omega_\lambda$, a classical problem related to its stability. We show that this is intimately related to the question for the non-degeneracy of the said solitons. We provide a wide class of non-linearities, for which the waves are non-degenerate. Under some minimal extra assumptions, we show that a.e. in λ, the map $\lambda \to f_{\omega_\lambda}$ is differentiable and the waves $e^{i\omega_\lambda t} f_{\omega_\lambda}$ are spectrally (and in some cases orbitally) stable as solutions to the NLS equation. Similar results are obtained for the same waves, as traveling waves of the Zakharov–Kuznetsov system.

1. Introduction

We consider the Schrödinger equation with general Hamiltonian non-linearity

$$iu_t + \Delta u + F(|u|^2)u = 0, \quad u : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{C}, \tag{1.1}$$

where $F : \mathbf{R}_+ \to \mathbf{R}$ will be henceforth assumed to be $C^1(\mathbf{R}_+)$ function. These type of models are ubiquitous in current applications (especially in quantum mechanical context, such as nonlinear optics and additionally in the theory of water waves). Of particular importance of the theory and applications to physics and technology, is the study of the existence and properties of ground states, that is, standing wave solutions in the form $e^{i\omega t} f_\omega$, where $f_\omega > 0$. Clearly, they satisfy the elliptic profile equation

$$-\Delta f_\omega + \omega f_\omega - F(f_\omega^2) f_\omega = 0. \tag{1.2}$$

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The existence of solutions of (1.2), together with their properties, including their uniqueness has been the subject of hundreds of papers in the literature, we refer the reader to the landmark papers, [2,20,34] and for some recent developments to the review paper [36].

In addition, and somewhat in parallel of the study of the solitary waves, various mathematical aspects of the theory have been rigorously established in the literature–such as conditions on the parameters guaranteeing local and global well-posedness, asymptotic properties of the solutions etc. We do not even attempt to review these here, instead we refer to the excellent (and by now classical) books, [5] and [35].

More recently, more advanced topics of investigations have been concerned with the questions of the global dynamics of these models. In these studies (and in many previous works), it became clear that the behavior near solitary waves is of utmost importance. In particular, we should mention the soliton resolution conjecture (SRC), which predicts that if the system does not support unstable solitons, any sufficiently smooth and localized data, produces a global solution which resolves, as $t \to \infty$, into a solitonic part plus a radiation term. This has been established in a variety of NLS models, in different dimensions and specific non-linearities in the form $F(z) = z^p$. The SRC is otherwise widely believed to hold true, at least in very generic circumstances. Important advances were made towards that goal in that various dispersive estimates for the Schrödinger evolution, [9], we also refer to [6,32,33] for further related issues and discussions.

As one can see from the recent developments–the existence, functional and most importantly stability properties of the solitons are really a starting point towards an attempt at understanding the global dynamic of a model like (1.1). It should be mentioned though, as this will be the focus of this paper, that *the cases outside of the simple power non-linearity, that is $F(z) = z^p$, have not been well-understood at all–at least from point of view of existence and stability of the corresponding solitary waves*. Clearly, this is an important question, both from a theoretical and practical point of view.

As an example of a model of this type, which naturally appears in the shallow water waves approximation models is the Gardner equation, which features cubic and quintic terms, or in terms of F, $F(z) = az \pm bz^2$, $a, b > 0$. One should note that this is a model in one spatial dimension, where the profile equation (1.2) allows reduction of order. In fact, it should be pointed out that matters in this particular case, that is $n = 1$ are more or less fully understood.[1] For the one dimensional case, in the paper [14], under pretty general conditions on the non-linearity F, the authors have established the existence of ground state waves. In addition, the stability of such waves was reduced to a sign of an explicit quadrature involving the nonlinearity F. As this condition is very non-explicit (even for simple combinations of two powers), Ohta, [29], followed by Maeda, [25], have further studied the conditions for power nonlinearities of the form $F(z) = az^p \pm bz^q$. They discovered an interesting new paradigm, namely that even for fixed a, b, p, q, the stability of the waves f_ω, may change with ω. This is a complete departure from the case of a single power non-linearity, $F(z) = z^p$, since the stability in such a case happens exactly for $p : 0 < p < 2$, and then *for all values of ω*. In fact, we provide a quick and self-contained introduction to the existence and stability of the waves in one spatial dimension–see Appendix C.

[1] In the periodic case, the theory is slightly more technical, due to the appearance of an additional integration constant, but the theory goes through.

The purpose of this work is to examine these questions for general power non-linearities, in high dimensions, $n \geq 2$. We work with power functions of the form

$$F(r) = \sum_{k=1}^{K} a_k r^{p_k} - \sum_{l=1}^{L} b_l r^{q_l}, \quad 0 < p_1 < \cdots < p_K, a_1, \ldots, a_K > 0; q_1 < \cdots < q_L; b_1, \ldots, b_L > 0.$$

Within this class, we require that we work with nonlinearities with at least one focusing term, that is $K \geq 1$. There are several reasons in favor of working with explicit power functions. One reason is to avoid imposing hard to verify conditions on F. A second one is to be able to illustrate the results better - including how they stack up against the standard threshold results for stability, non-degeneracy[2] among others.

In closing of the introductory remarks, let us point out that our results for NLS will transfer nicely to the Zakharov–Kuznetsov equation. This is a higher dimensional version of the KdV equation, and more precisely,

$$u_t + \partial_{x_1}(\Delta u + F(u^2)u) = 0, \quad u : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R} \tag{1.3}$$

The problem was initially derived in three spatial dimensions (and quadratic nonlinearity) by Zakharov and Kuznetsov, [37] to describe weakly magnetized ion-acoustic waves in a strongly magnetized plasma, but later found applications in two spatial dimensions as well, [26,27]. Finally, in [21], the equation was derived from the Euler-Poisson system with magnetic field in the long wave limit approximation.

Here, we consider waves, traveling in the direction of x_1, with a speed of ω. In other words, we impose the traveling wave ansatz, $u(x,t) = f_\omega(x_1 - \omega t, x_2, \ldots, x_n)$. After plugging in (1.3) and taking into account that f_ω is vanishing at infinity, we obtain the same profile Eq. (1.2).

1.1. The linearized problem. In this section, we consider the linearized problems and introduce the relevant notions of stability. Taking the ansatz $u = e^{i\omega t}[f_\omega + v(t, \cdot)]$ into the NLS problem (1.1), we obtain, after ignoring $O(v^2)$ terms,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: \mathcal{J}\mathcal{L}\vec{v} \tag{1.4}$$

where $v = v_1 + iv_2$, and the self-adjoint operators, $\mathcal{L}_\pm$ (with $D(\mathcal{L}_\pm) = H^2(\mathbf{R}^n)$) are given by

$$\mathcal{L}_- := -\Delta + \omega - F(f_\omega^2)$$
$$\mathcal{L}_+ := -\Delta + \omega - F(f_\omega^2) - 2F'(f_\omega^2) f_\omega^2.$$

Applying the ansatz $u = f_\omega(x_1 - \omega t, x') + v(t, x_1 - \omega t, x')$ in the Zakharov–Kuznetsov model, (1.3), we arrive at the linearized problem

$$v_t = \partial_{x_1}\mathcal{L}_+ v. \tag{1.5}$$

It is immediate that by (1.2), $\mathcal{L}_-[f_\omega] = 0$, while taking a derivative in any x_j, $j = 1, \ldots, n$ results[3] in $\mathcal{L}_+[\partial_j f_\omega] = 0$, $j = 1, \ldots, n$. Actually, from Nöther's principle, all

[2] To be defined shortly.

[3] This is all formal for now, but it will turn out to be justified, once we review the relevant properties of f_ω.

elements of[4] $Ker[\mathscr{L}]$ arising out of the known symmetries of the system–translational and modulational, are accounted for. Still, it is unclear whether these are all elements of $Ker[\mathscr{L}]$. While it is usually pretty easy to establish that zero is the bottom of the spectrum for $\mathscr{L}_-$, whence zero is a simple eigenvalue spanned by f_ω, (see Theorem 3 below), the fact that $Ker[\mathscr{L}_+]$ is spanned by ∇f_ω is not straightforward and it is an open question in a surprising number of applications. In fact, we shall introduce an intermediate property.

Definition 1. We say that the wave f_ω is non-degenerate, if

$$Ker[\mathscr{L}_+] = span[\partial_j f_\omega, j = 1, \dots, n].$$

We say that f_ω is weakly non-degenerate, if $f_\omega \perp Ker[\mathscr{L}_+]$.

The weak non-degeneracy of course easily follows from the non-degeneracy. While it does not seem to be a standard notion in the literature, we introduce it herein since it turns out it plays an important role in stability considerations and it is also closely related to the differentiability of the map $\omega \to f_\omega$. This brings us to the second main objective of this paper–beside the construction of the waves, it is a common assumption in the literature that "the map $\omega \to f_\omega$ is a C^1 in some interval Ω". This is of course easily verifiable in the case of a single power non-linearity, $F(z) = z^p$, but it is a non-trivial fact for just about any other non-linearity. We address this issue, in the framework of normalized waves, in Theorems 1 and 4 below.

Finally, we formally introduce the different notions of stability.

Definition 2. We say that the wave f_ω, as a solution to the NLS problem (1.1), is spectrally stable, if the equation

$$\mathscr{J}\mathscr{L}\vec{v} = \lambda\vec{v},$$

does not have solutions, with $\vec{v} \in H^2(\mathbf{R}^n)$, $\vec{v} \neq 0$, $\lambda : \Re\lambda > 0$. Similarly, f_ω is stable as a solution to (1.3), if $\partial_{x_1}\mathscr{L}_+ v = \lambda v$ does not have solutions $v \in H^2(\mathbf{R}^n)$, $\vec{v} \neq 0$, $\lambda : \Re\lambda > 0$.

We say that the wave f_ω is orbitally stable solution of (1.1), if for any $\epsilon > 0$, there exists $\delta > 0$, so that whenever the initial data is picked so that $\|u_0 - f_\omega\|_{H^1(\mathbf{R}^n)} < \delta$, then the corresponding solution u satisfies

$$\sup_{t>0} \inf_{\theta\in[0,2\pi], y\in\mathbf{R}^n} \|u(t, \cdot - y) - e^{i\theta} f_\omega\|_{H^1(\mathbf{R}^n)} < \epsilon.$$

For traveling wave solutions of (1.3), orbital stability means that for every $\epsilon > 0$, there is $\delta > 0$, so that for all $\|u_0 - f_\omega\|_{H^1(\mathbf{R}^n)} < \delta$, one has $\sup_{t>0} \inf_{y\in\mathbf{R}^n} \|u(t, x - y) - f_\omega(x_1 - \omega t, x')\|_{H^1_x(\mathbf{R}^n)} < \epsilon$.

There is of course the notion of asymptotic stability, but since we claim no results in this direction, we do not introduce it here.

[4] And here, it is important to note that we are interested in a description of all elements of $Ker[\mathscr{L}\pm] \subset D(\mathscr{L}_\pm) = H^2(\mathbf{R}^n)$.

1.2. Variational setup: normalized waves. Of specific interests are the properties of the so-called normalized ground states. More specifically, these are solutions (if they exist!) of the following constrained minimization problem

$$\begin{cases} I[u] := \int_{\mathbf{R}^n} |\nabla u(x)|^2 - \int_{\mathbf{R}^n} G(|u(x)|^2)dx \to min \\ \int_{\mathbf{R}^n} |u(x)|^2 dx = \lambda, \lambda > 0 \end{cases} \tag{1.6}$$

where $G(0) = 0$, $G'(r) = F(r)$, or equivalently,

$$G(r) = \sum_{k=1}^{K} \frac{a_k}{p_k + 1} r^{p_k+1} - \sum_{l=1}^{L} \frac{b_l}{q_l + 1} r^{q_l+1}. \tag{1.7}$$

The question for existence of solutions to (1.6) is in fact a hard one to analyze, despite many recent advances. In fact, this is one of the central issues that we would like to address in this paper. To that end, introduce the following function $m : [0, \infty) \to \mathbf{R} \cup \{-\infty\}$,

$$m(\lambda) := \inf_{\int_{\mathbf{R}^n} |u(x)|^2 dx = \lambda} I[u].$$

Note that $m = m_{\vec{a},\vec{b},\vec{p},\vec{q}}(\lambda)$ and it is possible that $m(\lambda) = -\infty$ for a substantial portion of the domain. Clearly, $m(\lambda) > -\infty$ is a necessary condition for (1.6) to have a solution, in which case we refer to (1.6) as well-posed. *In addition, it turns out that the requirement that m is a non-increasing function in λ is a sufficient*[5] *condition for the existence of solution to the constrained minimization problem* (1.6).

More precisely, we have the following existence results.

1.3. Existence results. The standard notion of bell-shapedness will appear frequently, so we introduce it formally here–namely, we say that a function $f : \mathbf{R}^n \to \mathbf{R}$ is bell shaped, if there exists a decreasing function $\rho : (0, \infty) \to \mathbf{R}_+$, so that $f(x) = \rho(|x|)$.

Theorem 1. *If*

$$p_K < \max\left(\frac{2}{n}, q_L\right) \tag{1.8}$$

then[6] *the constrained minimization problem* (1.6) *is well-posed, that is* $m(\lambda) > -\infty$. *In such case,* $m(0) = 0$. *If in addition,* $p_K < \frac{2}{n-2}$ *and the parameters* $\vec{p} = (p_1, \dots, p_K)$, $\vec{q} = (q_1, \dots, q_L)$, $\vec{a} = (a_1, \dots, a_K)$, $\vec{b} = (b_1, \dots, b_L)$ *are so that*

$$m \text{ is non-increasing on the interval } \Omega, \tag{1.9}$$

then, the problem (1.6) *has a solution* $\varphi_\lambda : \lambda \in \Omega$, *which is smooth and bell-shaped. It also satisfies the Euler-Lagrange equation* (1.2), *that is there is a Lagrange multiplier* $\omega = \omega_\lambda$, *so that* (1.2) *holds in a distributions sense. There are the following properties*

(1) the linearized operator $\mathscr{L}_+$ *satisfies* $\mathscr{L}_+|_{\{\varphi_\lambda\}^\perp} \geq 0$. *In fact, it has exactly one negative eigenvalue.*

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[5] And as we will show, in the most important cases, it is necessary as well.

[6] With the understanding that in the absence of de-focusing terms, that is $b_1 = \dots = b_L = 0$, $\max\left(\frac{2}{n}, q_L\right) = \frac{2}{n}$.

(2) The function $m(\lambda)$ is locally Lipschitz, that is for each interval $(a, b) \subset (0, \infty)$, there is $C_{a,b}$ so that $\sup_{x,y\in(a,b)} |m(x) - m(y)| \leq C_{a,b}|x - y|$. As such, it is differentiable a.e. and its derivative is $m'(\lambda) = -\frac{\omega_\lambda}{2}$. In addition, there is the representation formula

$$m(\lambda_2) - m(\lambda_1) = -\frac{1}{2}\int_{\lambda_1}^{\lambda_2} \omega_\lambda d\lambda. \tag{1.10}$$

for each $\lambda_1, \lambda_2 > 0$.

Remarks:

- The condition (1.8) is necessary for the existence of the waves, otherwise $m(\lambda) = -\infty$, see Proposition 1 below.
- The condition (1.9) is also necessary, see Proposition 1.
- Implicitly in the statement, we have that the Lagrange multiplier ω_λ also depends on the particular minimizer φ_λ. That is, we cannot rule out the possibility that for the same $\lambda > 0$, there are two minimizers $\varphi_\lambda, \tilde{\varphi}_\lambda : \|\varphi_\lambda\|^2 = \|\tilde{\varphi}_\lambda\|^2_{L^2} = \lambda$, with $\omega_{\lambda,\varphi} \neq \omega_{\lambda,\tilde{\varphi}}$. On the other hand, on the set where m' exists (which is a.e.), we have that $\omega_\lambda = -2m'(\lambda)$, which is independent on the minimizers.

Next, we turn to the necessity of the assumptions made in Theorem 1.

Proposition 1. *For the constrained minimization problem (1.6), we have the following*

- *(necessity of (1.8)) If $p_K > \max\left(\frac{2}{n}, q_L\right)$, then $m(\lambda) = -\infty$.*
- *(normalized waves exist only for $\omega > 0$) If $f_\omega \in H^1(\mathbf{R}^n) \cap L^{2p_K+1}(\mathbf{R}^n) \cap L^{2q_L+1}(\mathbf{R}^n)$ is a minimizer of (1.6), then $\omega > 0$.*
- *Suppose that (1.8) holds and the constrained minimization problem (1.6) has a solution for each $\lambda > 0$. Then, $\lambda \to m(\lambda)$ is a non-increasing function.*

As an easy and useful corollary of Theorem 1, we have the following

Proposition 2. *Let φ be one of the constrained minimizers described in Theorem 1. If in addition, $\langle \mathscr{L}_+\varphi, \varphi\rangle = -2\int_{\mathbf{R}^n} F'(\varphi^2)\varphi^4 dx < 0$, then the wave φ is weakly non-degenerate, i.e. $\varphi \perp Ker[\mathscr{L}_+]$.*

In particular, if F has only focusing terms, the corresponding wave is always weakly non-degenerate. We now discuss the properties of the solutions to (1.2). In doing so, one has to keep in mind that in general, we do not know uniqueness for (1.2), while on the other hand, some solutions are generated by the constrained minimization procedure, as described in Theorem 1.

Theorem 2. *Assume $\omega > 0$, (1.8) holds, and f is a bell-shaped function, with $f \in L^2(\mathbf{R}^n)$, so that f is a strong solution of (1.2), that is*

$$f = (-\Delta + \omega)^{-1}[F(f^2)f]. \tag{1.11}$$

Then, $f \in L^\infty(\mathbf{R}^n)$ and moreover, f has exponential decay rate at ∞. More precisely, $f(x) \leq C(1 + |x|)^{-\frac{n-1}{2}} e^{-\sqrt{\omega}|x|}$ and in fact, there is $c > 0$, so that for all large $|x|$,

$$f(x) = c\frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}} + o\left(\frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}}\right).$$

Next, we have a general result about $\mathscr{L}_-, \mathscr{L}_+$.

1.4. Spectral results about $\mathcal{L}_-$, $\mathcal{L}_+$.

Theorem 3. *Suppose $\omega > 0$, $f_\omega > 0$, $f_\omega \in H^2(\mathbf{R}^n)$ solves (1.2) and it has exponential decay. Then, the operators $\mathcal{L}_-$, $\mathcal{L}_+$ enjoy the following spectral properties:*

(1) $\mathcal{L}_- \geq 0$, so that 0 is a simple eigenvalue, with an eigenspace spanned by f_ω.
(2) $\mathcal{L}_+$ has at least one negative eigenvalue.
(3) Assume in addition that $n \neq 2$, $n(\mathcal{L}_+) = 1$. Then, $Ker[\mathcal{L}_+] \supseteq \{\partial_1 f_\omega, \dots, \partial_n f_\omega\}$ is either n or $n+1$ dimensional. In the former case, $Ker[\mathcal{L}_+] = span\{\partial_1 f_\omega, \dots, \partial_n f_\omega\}$, while in the latter $Ker[\mathcal{L}_+] = span\{\partial_1 f_\omega, \dots, \partial_n f_\omega, \Psi_0\}$, where Ψ_0 is a function, depending on the radial variable only, with exactly one zero in $(0, \infty)$. In addition, Ψ_0 is a bounded function and there is the exponential bound $|\Psi_0(x)| \leq C(1 + |x|)^{-\frac{n-1}{2}} e^{-\sqrt{\omega}|x|}$. In fact, there is $c > 0$, so that for all large $|x|$,

$$\Psi_0(x) = c\frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}} + o\left(\frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}}\right).$$

Remark: The requirement for exponential decay of f_ω could be weakened significantly. However, in view of the result listed in Theorem 2, the minimizers of (1.6) do have exponential decay. Thus, generalizing Theorem 3 to cover f_ω with less than exponential decay seems like a mute point.

1.5. Smoothness of $\lambda \to m(\lambda)$ and the non-degeneracy of the constrained minimizers. We start with a lemma that is interesting in its own right, but it will turn out to be relevant for the smoothness $\lambda \to m(\lambda)$.

Proposition 3. *Assume that (1.8) and (1.9) holds on an interval $\Omega = (a, b)$. Let $\lambda \in (a, b)$ be a point of differentiability for $\omega(\lambda)$. Then, for each sequence $\delta_j \to 0$, there exists a subsequence δ_{j_k} and Φ_λ, so that*

- $\lim_{k\to\infty} \|\varphi_{\lambda+\delta_{j_k}} - \Phi_\lambda\|_{H^1} = 0$,
- *Φ_λ is a constrained minimizer for (1.6), in particular it satisfies $\Phi_\lambda \in H^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ and the Euler-Lagrange equation (1.2), hence Theorem 2 applies to it.*

Remark: For the purposes of the presentation below, we shall call Φ_λ obtained according to the procedure described in Proposition 3 *a limit wave*.[7]

For the next theorem, we make some remarks concerning the Lagrange multipliers ω. As we have alluded to above, in general, one cannot claim, without any additional arguments, the continuity of the map $\lambda \to \omega_\lambda$ and even the independence of ω_λ on the particular minimizer φ_λ. Some of the smoothness issues were touched upon by Maris, [24].

Theorem 4. *Let $p_K < \frac{2}{n-2}$. Assume that for a fixed interval (a, b), $0 < a < b \leq \infty$, and for each $\lambda \in (a, b)$, φ_λ is a minimizer for (1.6) and*

$$\lim_{\delta\to 0} \|\varphi_{\lambda+\delta} - \varphi_\lambda\|_{L^2} = 0 \tag{1.12}$$

then

[7] This is not a standard notion by any means, but it arises naturally in our considerations, we name it.

(1) $\lambda \to \omega(\lambda)$ *is a continuous function on* (a, b) *and* $\lambda \to m(\lambda)$ *is a* $C^1(a, b)$ *function, given by* (1.10).
(2) The function $\lambda \to m(\lambda)$ *is a strictly concave function on* (a, b). *In particular,* m *is twice differentiable almost everywhere,* $\omega'(\lambda) = -2m''(\lambda) > 0$, *whenever* $\omega'(\lambda)$ *exists.*
(3) Assuming that $\omega'(\lambda)$ *exists, then the waves* φ_λ *are weakly non-degenerate, that is* $\varphi_\lambda \perp Ker[\mathscr{L}_+^{\varphi_\lambda}]$.

For the rest, assume $n \geq 3$ *and* $\omega'(\lambda)$ *exists.*

(1) If φ_λ *is non-degenerate, that is* $Ker[\mathscr{L}_+] = span[\partial_1\varphi_\lambda, \ldots, \partial_n\varphi_\lambda]$, *then the function* $\lambda \to \varphi_\lambda$ *is differentiable as an* $L^2(\mathbf{R}^n)$*-valued mapping, at all points of differentiability of* ω. *Also, we have the formula*

$$\partial_\lambda\varphi_\lambda = -\omega'(\lambda)\mathscr{L}_+^{-1}\varphi_\lambda.$$

In particular, $\langle \mathscr{L}_+^{-1}\varphi_\lambda, \varphi_\lambda\rangle = -\frac{1}{2\omega'(\lambda)} < 0$.
(2) If φ_λ *is degenerate, i.e.* $Ker[\mathscr{L}_+] = span[\partial_1\varphi_\lambda, \ldots, \partial_n\varphi_\lambda, \Psi_0]$, *but we assume the stronger condition*

$$\lim_{\delta\to 0} \frac{\|\varphi_{\lambda+\delta} - \varphi_\lambda\|_{L^2}}{\sqrt{|\delta|}} = 0, \tag{1.13}$$

then again, the function $\lambda \to \varphi_\lambda$ *is differentiable as an* $L^2(\mathbf{R}^n)$*-valued mapping, at all points of differentiability*[8] *of* ω *and*

$$\partial_\lambda\varphi_\lambda = -\omega'(\lambda)\mathscr{L}_+^{-1}\varphi_\lambda,$$

and consequently $\langle \mathscr{L}_+^{-1}\varphi_\lambda, \varphi_\lambda\rangle = -\frac{1}{2\omega'(\lambda)} < 0$.

Remarks:

- The assumptions (1.12), in the non-degenerate case, is very weak, compared to the conclusions. Note that it is claimed that $\lambda \to \varphi_\lambda$ is differentiable,[9] which implies

$$\lim_{\delta\to 0} \frac{\|\varphi_{\lambda+\delta} - \varphi_\lambda\|_{L^2}}{|\delta|} = \|\partial_\lambda\varphi_\lambda\| = |\omega'(\lambda)|\|\mathscr{L}_+^{-1}\varphi_\lambda\|.$$

  Clearly, this last identity implies (5.3) and it is indeed stronger.
- Even in the weakly non-degenerate case, the stronger assumption (1.13) is much weaker than the subsequent claim. In the same fashion, it is claimed that in particular $\lim_{\delta\to 0} \frac{\|\varphi_{\lambda+\delta} - \varphi_\lambda\|_{L^2}}{|\delta|}$ exists, which implies, and it is in fact stronger than, (1.13).

Our next result concerns some cases in which we can assert the non-degeneracy of φ_λ.

Proposition 4. *Assume that* φ *is a bell-shaped wave, which is weakly non-degenerate, that is* $\varphi_\lambda \perp Ker[\mathscr{L}_+]$. *Assume in addition that* $n \geq 3$, $n(\mathscr{L}_+) = 1$ *and one of the following holds*

$$F(r) = \sum_{k=1}^{K} a_k r^{p_k}, \tag{1.14}$$

[8] Which is at least almost everywhere.
[9] At the points of differentiability of ω.

or

$$F(r) = \sum_{k=1}^{K} a_k r^{p_k} - br^q, 0 < q < p_1 \tag{1.15}$$

or

$$F(r) = \sum_{k=1}^{K} a_k r^{p_k} - br^q, 0 < p_K < q. \tag{1.16}$$

Then, the corresponding constrained minimizer φ_λ is non-degenerate, i.e. $Ker[\mathscr{L}_+] = span[\nabla\varphi]$.

Clearly, in order to ensure that a wave φ like that exists, we need further assumptions in Proposition 4, like $p_K < \frac{2}{n}$ in (1.14) and (1.15), and $p_K < \frac{2}{n-2}$ in (1.16).

1.6. Applications to the stability of normalized waves for Schrödinger and Zakharov–Kuznetsov equation. We finally state our results concerning the stability of the waves constructed in Theorem 1.

Theorem 5 (Focusing nonlinearity). *Let* (1.14) *holds and $n \geq 3$. Then, for every $\lambda > 0$, there exists an a.e. differentiable function $\omega = \omega(\lambda) > 0$ and a bell-shaped constrained minimizer $f_\omega \in H^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ for the problem* (1.6) *with*

$$f_\omega(x) = c_\omega \frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}} + o\left(\frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}}\right), \quad |x| \to \infty.$$

In addition, for every point of differentiability of $\lambda \to \omega(\lambda)$, let f_{ω_λ} be the limit wave, in the sense of Proposition 3. Then, f_ω is non-degenerate, in the sense of Definition 1. Finally, $e^{i\omega_\lambda t} f_{\omega_\lambda}$ is orbitally stable solution of the NLS and the Zakharov–Kuznetsov system.

Remark: We show that the assumption (1.14) implies (1.9). The rest of the statement is a combination of Theorems 1, 2, 3, Proposition 4 and Proposition 3.

Our next result concerns mixed nonlinearities–some focusing and one defocussing, as in (1.15) or (1.16). The only difference with Theorem 5 is that we now need to explicitly assume that (1.9) holds. Note that such assumption is necessary, by Proposition 1, if we were to expect normalized waves.

Theorem 6 (nonlinearity with at most one defocussing term). *Let the nonlinearity be in the form* (1.15) *or* (1.16) *holds. Assume that* (1.9) *holds. Then, for every $\lambda > 0$, there exists an a.e. differentiable function $\omega = \omega(\lambda) > 0$ and a bell-shaped constrained minimizer $f_\omega \in H^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ for the problem* (1.6) *with*

$$f_\omega(x) = c_\omega \frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}} + o\left(\frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}}\right), \quad |x| \to \infty.$$

In addition, assuming that $n \geq 3$ and for every point of differentiability of $\lambda \to \omega(\lambda)$, let f_{ω_λ} be a limit wave, in the sense of Proposition 3. Then, f_ω is non-degenerate, in the sense of Definition 1. Finally, $e^{i\omega_\lambda t} f_{\omega_\lambda}$ is orbitally stable for the NLS Eq. (1.1)*, and $f_\omega(x_1 - \omega_\lambda t, x')$ is spectrally stable solution to the Zakharov–Kuznetsov model* (1.3).

Our most general result, applies to general mixed power non-linearities, satisfying (1.8). Unfortunately, in this case, in order to obtain any stability result, we need to require (1.13).

Theorem 7. *Assume* (1.8)*,* (1.9)*. Then, for every* $\lambda > 0$*, there exists an a.e. differentiable function* $\omega = \omega(\lambda) > 0$ *and a bell-shaped constrained minimizer* $f_\omega \in H^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ *for the problem* (1.6) *with* $f_\omega(x) = c_\omega \frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}} + o\left(\frac{e^{-\sqrt{\omega}|x|}}{|x|^{\frac{n-1}{2}}}\right)$ *as* $|x| \to \infty$*. If in addition* $n \geq 3$*,* λ *is a point of differentiability for* $\omega(\lambda)$ *and*

$$\lim_{\delta \to 0} |\delta|^{-1/2} \| f_{\omega_{\lambda+\delta}} - f_{\omega_\lambda} \|_{L^2(\mathbf{R}^n)} = 0,$$

then the wave $e^{i\omega_\lambda t} f_{\omega_\lambda}$ *is a spectrally stable solution of NLS* (1.1)*, while* $f_\omega(x_1 - \omega_\lambda t, x')$ *is spectrally stable solution to the Zakharov–Kuznetsov equation,* (1.3).

Let us finish this introduction with an outline of the paper. In Sect. 2, we introduce some basic notions and standard results, in particular we present the basics of the Hamilton instability index count in Sect. 2.3. In Sect. 3, we give the variational construction of the waves, including the Euler-Lagrange equations, some initial smoothness results about the important function m as well as the necessity of the assumptions of Theorem 1, formulated in Proposition 1 above. Section 3 finishes with the simple proof of Proposition 2. In Sect. 4, we discuss the general functional properties of the waves, beyond the basics established in Sect. 3. In fact, for most of this section, we take (the more general) viewpoint of the waves as solutions to PDE, rather than constrained minimizers. We establish L^∞ bounds at zero as well as precise asymptotic behavior at ∞. In Sect. 5, we start with an in depth analysis of the spectral properties of the linearized operators $\mathscr{L}_-, \mathscr{L}_+$. In it, we need to resort to the spherical harmonic decomposition, thanks to the radiality of the potential. In Sect. 6, we show smoothness and non-degeneracy properties of the normalized waves. In particular, we prove Proposition 3. We also discuss the subtle issues of the dependence of the Lagrange multiplier ω_λ on the particular minimizer φ_λ, its continuity and concavity of $\lambda \to m(\lambda)$. In Sect. 6.4, we establish the weak non-degeneracy of the waves, under the assumptions in Theorem 4. In Sects. 6.5 and 6.6 , we explore the differentiability of the (Banach space valued) mapping $\lambda \to \varphi_\lambda$, under weak non-degeneracy and non-degeneracy assumptions. This allows us to compute the sign of Vakhitov-Kolokolov index, which in turn implies spectral stability. As it turns out, this is intimately related to the concavity properties of m. In Sect. 6.7, we establish the non-degeneracy of the wave in the cases considered in Theorems 5, 6, 7. This is the key remaining ingredient of the orbital stability of the corresponding waves for the NLS models, as stated in the aforementioned theorems. This is done via an abstract result yielding orbital stability from spectral stability and non-degeneracy. Finally, due to the failure of the abstract theory to cover the Zakharov–Kuzntesov case, we provide a direct proof of the orbital stability for the Zakharov–Kuznetsov model in Sect. 7.2, see Proposition 8.

2. Preliminaries

We use standard notations for L^p spaces, $W^{s,p}$ for Sobolev spaces etc. We use the following definition of Fourier transform and its inverse

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx, \; f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

In this setting, the Laplacian is given by the symbol $-4\pi^2|\xi|^2$. A decreasing rearrangement for a function $f : \mathbf{R}^n \to \mathbf{C}$ is the radially decreasing function $f^* : \mathbf{R}^n \to \mathbf{R}_+$, which has the same distribution function as f. It is standard that for all lattice norms (i.e. those that depend only on the distribution function $d_f(\alpha) = |\{x \in \mathbf{R}^n : |f(x)| \geq \alpha\}|$), $\|f\|_X = \|f^*\|_X$. In addition, there is the Polya-Szegö inequality

$$\|\nabla f\|_{L^2} \geq \|\nabla f^*\|_{L^2}, \tag{2.1}$$

where in addition, equality is achieved only if $f = f^*$, modulo the usual invariance group. This is then a good place to introduce bell-shaped functions.

Definition 3. We say that a function $f : \mathbf{R}^n \to \mathbf{R}$ is bell-shaped, if $f = f^*$.

The bell-shaped functions will have the following point-wise decay property that will be used throughout in the sequel. Let $x : |x| = R$, then a bell-shaped function f satisfies, for all $q > 0$,

$$\|f\|_{L^q}^q \geq \int_{|y|<R} |f(y)|^q dy \geq c_n R^n |f(x)|^q,$$

whence

$$0 \leq f(x) \leq C_n \|f\|_{L^q} |x|^{-\frac{n}{q}}.$$

The uniform convexity property of the L^r, $r > 1$ norms will be useful in the variational arguments in the sequel.

Proposition 5. *Let $r > 1$, $\{f_n\}$ be a bounded sequence in L^r, with a weak limit f, $f_n \rightharpoonup f$. Then,*

$$\liminf_n \|f_n\|_{L^r} \geq \|f\|_{L^r}.$$

If in addition $\lim_n \|f_n\|_{L^r} = \|f\|_{L^r}$, *then $f_n \to f$ in L^r norm, that is* $\lim_n \|f_n - f\|_{L^r} = 0$.

2.1. Precise asymptotic of the Green's function of $(-\Delta + 1)^{-1}$. We record the formula for the Green function of $(-\Delta+1)^{-1}$, that is $\hat{Q}(\xi) = (1+4\pi^2|\xi|^2)^{-1}$ (see [10], p. 418)

$$Q(x) = (2\sqrt{\pi})^{-n} \int_0^\infty e^{-(t+\frac{|x|^2}{4t})} \frac{dt}{t^{n/2}}. \tag{2.2}$$

Note that $Q > 0$, radial and radially decreasing. Also, $\|Q\|_{L^1(\mathbf{R}^n)} = \int_{\mathbf{R}^n} Q(x)dx = \hat{Q}(0) = 1$, but note that $Q(0) = +\infty$ for $n \geq 2$. In fact, we have the following lemma about Q.

Lemma 1. *The Green's function Q introduced in (2.2) satisfies, for all $|x| > 1$,*

$$|Q(x)| \leq Ce^{-|x|}.$$

For $|x| \leq 1$, we have the asymptotic formula

$$Q(x) \sim \begin{cases} |x|^{2-n} + O(1) & n \geq 3 \\ \ln(\frac{1}{|x|}) + O(1) & n = 2 \end{cases}.$$

In particular, $Q \in L^q(\mathbf{R}^n)$, whenever $q < \frac{n}{n-2}$ (or $q < \infty$, when $n = 2$).

Remark: More precise asymptotics will give the optimal decay rate for large $|x|$, which is $|Q(x)| \leq C|x|^{-\frac{n-1}{2}} e^{-|x|}$.

Proof. The asymptotics near zero are well-known, see Proposition 6.1.5, p. 418, [10]. Regarding $|x| >> 1$, we start by rewriting Q in (2.2). We obtain

$$Q(x) = \frac{e^{-|x|}}{|x|^{\frac{n}{2}-1}} \int_0^\infty e^{-|x|(u-\frac{1}{2u})^2} \frac{du}{u^{n-1}}.$$

It remains to check that

$$\sup_{\mu>1} \int_0^\infty e^{-\mu(u-\frac{1}{2u})^2} \frac{du}{u^{n-1}} < \infty.$$

This follows easily, once we split the integration in (0, 1/2), (1/2, 1) and (1, ∞). □

2.2. Eigenspaces of spherical Laplacians and applications to Schrödinger operators with radial potentials. The Laplacian operator can be written in its radial and angular components as follows

$$\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \frac{\Delta_{\mathbf{S}^{n-1}}}{r^2}.$$

Let $\mathscr{X}_0 = L^2_{rad}(\mathbf{R}^n)$ be the radial subspace of $L^2(\mathbf{R}^n)$, defined by

$$\mathscr{X}_0 = L^2_{rad}(\mathbf{R}^n) = \{f(|\cdot|) : \int_0^\infty |f(r)|^2 r^{n-1} dr < \infty\}.$$

It is well-known that for each $k = 1, 2, \dots$, the eigenvalues of $\Delta_{\mathbf{S}^{n-1}}$ are given by $-k(k+n-2)$, with the spherical harmonics Y_k as eigenfunctions[10] $\Delta_{\mathbf{S}^{n-1}} Y_k = -k(k+n-2)Y_k$. In fact, it is easy to identify the eigenfunctions corresponding to $k = 1$, as these are exactly $\frac{x_j}{r}$, $j = 1, \dots, n$,

$$-\Delta_{\mathbf{S}^{n-1}} \frac{x_j}{r} = (n-1)\frac{x_j}{r}, \quad j = 1, \dots, n. \tag{2.3}$$

Accordingly, introduce the invariant for Δ subspaces

$$\mathscr{X}_k := span\{f(r)Y_k : f_k \in L^2_{rad}(\mathbf{R}^n), -\Delta_{\mathbf{S}^{n-1}} Y_k = k(k+n-2)Y_k\}, k = 1, 2, \dots$$

so that there is a orthogonal decomposition

$$L^2(\mathbf{R}^n) = \oplus_{k=0}^\infty \mathscr{X}_k.$$

Next, consider a Schrödinger operator in the form $\mathscr{H} := -\Delta + \omega - V(|x|)$, where the potential is a radial function. Clearly, $\mathscr{H}$ acts invariantly on $\mathscr{X}_k$, $k = 0, 1, \dots$ as well.

[10] It is also well-known that the multiplicity of the eigenvalue $k(k+n-2)$ is exactly $\binom{n+k-1}{k} - \binom{n+k-3}{k-2}$, but this fact will not be used later on.

Denoting $\mathcal{H}_k := \mathcal{H}|_{\mathcal{X}_k}$, we observe that $\mathcal{H}_k$ can be viewed as an operator acting on the subspace of radial functions $L^2_{rad}(\mathbf{R}^n)$, through the formula

$$\mathcal{H}_k = -\partial_{rr} - \frac{n-1}{r}\partial_r + \omega + \frac{k(k+n-2)}{r^2} - V(r), k = 0, 1, 2, \dots. \tag{2.4}$$

In addition, $\mathcal{H}_0 < \mathcal{H}_1 < \cdots < \mathcal{H}_k < \cdots$, as operators acting on $L^2_{rad}(\mathbf{R}^n)$, in particular,

$$\sigma_{L^2(\mathbf{R}^n)}(\mathcal{H}) = \cup_{k=0}^{\infty}\sigma_{L^2_{rad}(\mathbf{R}^n)}(\mathcal{H}_k).$$

It is now easy to apply these ideas to the operator $\mathcal{L}_+$. Suppose that f_ω is radial and sufficiently smooth and decaying. Since $\mathcal{L}_+[\nabla f_\omega] = 0$ and $\partial_j f_\omega = \frac{x_j}{r} f'_\omega(r)$, whence (recall that by (2.3), x_j/r is an eigenfunction corresponding to $k = 1$)

$$0 = \mathcal{L}_+[\partial_j f_\omega] = \mathcal{L}_{+,1}[f'].$$

That is, the function f'_ω is an eigenfunction, corresponding to zero eigenvalue for $\mathcal{L}_{+,1}$. Recalling that $\mathcal{L}_{+,0} = \mathcal{L}_{+,1} - \frac{n-1}{r^2}$, we conclude

$$\begin{aligned}\langle \mathcal{L}_{+,0}[f'], f'\rangle &= \langle \mathcal{L}_{+,1}[f'], f'\rangle - (n-1)\int_0^\infty (f'(r))^2 r^{n-3} dr \\ &= -(n-1)\int_0^\infty (f'(r))^2 r^{n-3} dr < 0.\end{aligned}$$

Applying the Ritz-Rayleigh principle implies the following lemma.

Lemma 2. *$\mathcal{L}_+$ always has at least one negative eigenvalue.*

2.3. Index theory and spectral stability. In this section, we introduce some basic consequences of the index theory, as developed over the last thirty years. In its most basic form, it was put forward by Grillakis, Shatah and Strauss in a series of seminal papers, [11,12]. Their theory applies to the eigenvalue problem of the type (1.4), where the skew symmetric operator $\mathcal{J}$ is invertible. For eigenvalue problem (1.5), $\mathcal{J} = \partial_x$ in particular fails to be invertible, the GSS theory does not apply to it. This case is covered in more recent works, [18,30] and more recently [23]. For the results that we quote below, we follow the book [17] for (1.4) and the recent paper [23] for the eigenvalue problem (1.5).

For (1.4), we have the following setup. The eigenvalue problem is in the form

$$\mathcal{J}\mathcal{L}f = \lambda f, \tag{2.5}$$

where $\mathcal{J}$ is assumed to be bounded, invertible and skew-symmetric ($\mathcal{J}^* = -\mathcal{J}$), while $(\mathcal{L}, D(\mathcal{L}))$ is self-adjoint($\mathcal{L}^* = \mathcal{L}$) and not necessarily bounded, with finite dimensional kernel $Ker[\mathcal{L}]$. Assume in addition that $\mathcal{L}$ has a finite number of negative eigenvalues, $n(\mathcal{L})$ and $\mathcal{J}^{-1} : Ker[\mathcal{L}] \to Ker[\mathcal{L}]^\perp$. Let k_r denote the number of positive eigenvalues of (2.5), k_c be the number of quadruplets of eigenvalues with non-zero real and imaginary parts, and k_i^-, the number of pairs of purely imaginary eigenvalues with negative Krein-signature.[11] Let $Ker[\mathcal{L}] = \{\phi_1, \dots, \phi_m\}$, , then introduce a matrix $D = (D_{ij})_{i,j=1}^m$

$$D_{ij} := \langle \mathcal{L}^{-1}[\mathcal{J}^{-1}\phi_i], \mathcal{J}^{-1}\phi_j\rangle, \quad i, j = 1, \dots, m \tag{2.6}$$

[11] The precise definition of those is provided in [15]. For us, $k_i^- = 0$, so this will be irrelevant.

where the formula is meaningful, since $\mathscr{J}^{-1}\phi_i \in Ker[\mathscr{L}]^{\perp}$. The index counting theorem, see Theorem 1, [16] states that if $det(D) \neq 0$, then

$$k_r + 2k_c + 2k_i^- = n(\mathscr{L}) - n(D). \tag{2.7}$$

The most common corollary, which we use, is that $n(\mathscr{L}) = 1$, whence stability follows once we establish $n(D) \geq 1$. In the case of the eigenvalue problem (1.4), this is simply a consequence of $\langle \mathscr{L}_+^{-1} f_\omega, f_\omega\rangle < 0$. The case of the eigenvalue problem (1.5) is slightly more involved, as is was alluded to above. Nevertheless, as shown in [23], spectral stability follows in the same way (formula similar to (2.7) holds true), provided $\langle \mathscr{L}_+^{-1} f_\omega, f_\omega\rangle < 0$. Thus, in all our spectral stability calculations, we have reduced matters to the computation of the scalar index $\langle \mathscr{L}_+^{-1} f_\omega, f_\omega\rangle < 0$, sometimes referred to as Vakhitov-Kolokolov criteria for stability. In short, we have shown the following

Proposition 6. *For the eigenvalue problem* (1.4)*, assume that*

- $n(\mathscr{L}_+) = 1$, $n(\mathscr{L}_-) = 0$,
- $f_\omega \perp Ker[\mathscr{L}_+]$ *and* $\langle \mathscr{L}_+^{-1} f_\omega, f_\omega\rangle < 0$.

Then, (1.4) *is spectrally stable, in the sense of Definition* 2*. For the eigenvalue problem* (1.5)*, assume*

- $n(\mathscr{L}_+) = 1$
- $f_\omega \perp Ker[\mathscr{L}_+]$ *and* $\langle \mathscr{L}_+^{-1} f_\omega, f_\omega\rangle < 0$.

Then, (1.5) *is spectrally stable, in the sense of Definition* 2*.*

Regarding orbital stability for the NLS problem, it follows from spectral stability, the non-degeneracy of f_ω (i.e. $Ker[\mathscr{L}_+] = span[\partial_1 f_\omega, \ldots, \partial_n f_\omega]$), in addition to the smoothness of the map $\omega \to f_\omega$ as an H^1 valued mapping.

Remark: The last statement about orbital stability is a corollary of a very general result, namely Theorem 5.2.11, [17]. Note the requirement about smoothness $\omega \to f_\omega$ as an H^1 valued mapping, right under (5.2.47) on p. 139.

3. Existence of the Waves

3.1. Proof of Theorem 1. We first show that the problem is well-posed, i.e. $m(\lambda) > -\infty$ for each $\lambda > 0$, if (1.8) holds. If $p_K < \frac{2}{n}$, we have by Sobolev embedding, for each $p \in (0, \frac{2}{n})$,

$$\|u\|_{L^{2p+2}} \leq C\|u\|_{\dot{H}^{n(\frac{1}{2}-\frac{1}{2p+2})}} \leq C\|\nabla u\|^{n(\frac{1}{2}-\frac{1}{2p+2})}\|u\|_{L^2}^{1-n(\frac{1}{2}-\frac{1}{2p+2})}.$$

Noting that $2(p+1)n(\frac{1}{2} - \frac{1}{2p+2}) < 2$, we conclude that for each $u : \|u\|_{L^2}^2 = \lambda$, we have the estimate

$$\sum_{k=1}^{K} \frac{a_k}{p_k+1}\|u\|_{L^{2p_k+2}}^{2p_k+2} \leq \epsilon\|\nabla u\|_{L^2}^2 + C_{\epsilon,\lambda} \tag{3.1}$$

for each $\epsilon > 0$. Choosing $\epsilon = \frac{1}{2}$, it follows that

$$I[u] \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 + \sum_{l=1}^{L} \frac{b_l}{q_l+1}\|u\|_{L^{2q_l+2}}^{2q_l+2} - C_{\frac{1}{2},\lambda} \geq -C_{\frac{1}{2},\lambda}.$$

If on the other hand, $p_K < q_L$, we have by Gagliardo-Nirenberg inequality for all $p \in (0, q_L)$, $\|u\|_{L^{2p+2}} \leq \|u\|_{2q_L+2}^{\theta} \|u\|_{L^2}^{1-\theta}$, where $\theta \in (0,1) : \frac{1}{2p+2} = \frac{\theta}{2q_l+2} + \frac{1-\theta}{2}$. Thus,

$$\sum_{k=1}^{K} \frac{a_k}{p_k+1} \|u\|_{L^{2p_k+2}}^{2p_k+2} \leq \epsilon \|u\|_{L^{2q_L+2}}^{2q_L+2} + C_{\epsilon,\lambda}. \tag{3.2}$$

Once again,

$$I[u] \geq \|\nabla u\|_{L^2}^2 + \sum_{l=1}^{L} \frac{b_l}{q_l+1} \|u\|_{L^{2q_l+2}}^{2q_l+2} - \epsilon \|u\|_{L^{2q_L+2}}^{2q_L+2} - C_{\epsilon,\lambda} \geq -C_{\epsilon,\lambda},$$

for appropriate choice of ϵ.

Next, we take on the existence of a minimizer, now that we know that $m(\lambda) > -\infty$. First, observe that when minimizing $I[u]$, it is always better to take u^* instead of u. Indeed, by Polya-Szegö inequality and $\|u\|_{L^r} = \|u^*\|_{L^r}$, we conclude that $I[u] \geq I[u^*]$, while $\|u\|_{L^2}^2 = \lambda = \|u^*\|_{L^2}^2$. Furthermore, by the conditions for equality in Ploya-Szegö, the minimizer, if it exists is necessarily a bell-shaped function,[12] i.e. $u = u^*$. So, it suffices to focus our attention to bell-shaped functions.

Take a minimizing sequence, say u_j, of bell-shaped functions, which satisfy $\|u_j\|_{L^2}^2 = \lambda$ and

$$I[u_j] \leq m(\lambda) + \frac{1}{j}.$$

We have shown that due to the assumption (1.8), we have either (3.1)or (3.2). In either case, we have

$$\|\nabla u_j\|_{L^2}^2 + \sum_{l=1}^{L} \frac{b_l}{q_l+1} \|u_j\|_{L^{2q_l+2}}^{2q_l+2} \leq m(\lambda) + \frac{1}{j} + \sum_{k=1}^{K} \frac{a_k}{p_k+1} \|u_j\|_{L^{2p_k+2}}^{2p_k+2}$$
$$\leq m(\lambda) + \frac{1}{j} + \epsilon(\|\nabla u_j\|_{L^2}^2 + \sum_{l=1}^{L} \|u_j\|_{L^{2q_l+2}}^{2q_l+2}) + C_{\epsilon,\lambda}.$$

Thus, for appropriate choice of ϵ, we conclude

$$\|\nabla u_j\|_{L^2}^2 + \sum_{l=1}^{L} \|u_j\|_{L^{2q_l+2}}^{2q_l+2} \leq C_\lambda, \tag{3.3}$$

where C_λ is an explicit and continuous function of λ, depending only on n and the parameters $\vec{p}, \vec{q}$. Since this last quantity controls $\|u_j\|_{L^{p_j}}$, $\{u_j\}$ is a bounded sequence in all these spaces. By taking a subsequence, we can without loss of generality assume that u_j converges to $u_j \rightharpoonup \varphi$, in all of these weak topologies. Recall now that $u_j = u_j^* :$ $\|u_j\|^2 = \lambda$, whence

$$\lambda = \int_{\mathbf{R}^n} |u_j(x)|^2 dx \geq \int_{|x| \leq R} |u_j(x)|^2 dx \geq c_n R^n |u_j(x)|^2,$$

12 After accounting for translations.

for every $x : |x| = R$. We have $|u_j(x)| \leq c_n |x|^{-n/2}$, whence

$$\int_{|x|>R} |u_j(x)|^{2+2p} dx \leq c_n R^{-np}.$$

In addition, we have that for $p < \frac{2}{n-2}$, $\alpha = 1 - \frac{np}{2(p+1)} > 0$ and hence by Sobolev embedding, $\|u_j\|_{W^{\alpha,p}(\mathbf{R}^n)} \leq C\|u_j\|_{H^1(\mathbf{R}^n)} \leq C_\lambda$. Thus, by the Riesz-Relich compactness criteria, u_n is a compact subsequence in the strong topology of all L^{2p_k+2}, whence (after eventual taking a subsequence), $\lim_j \|u_j - u\|_{L^{2p_k+2}} = 0$, $k = 1, \ldots, K$. Using the lower semi-continuity of the weak norm, with respect to the strong norm in L^r, $r > 1$, we have $\|\varphi\|_{L^2}^2 \leq \liminf_j \|u_j\|_{L^2}^2 = \lambda$ and

$$\begin{aligned} m(\lambda) = \lim_{j\to\infty} I[u_j] &\geq \liminf_{j\to\infty} [\|\nabla u_j\|_{L^2}^2 + \sum_{l=1}^{L} \frac{b_l}{q_l+1} \|u_j\|_{L^{2q_l+2}}^{2q_l+2}] \\ &\quad - \lim_j \sum_{k=1}^{K} \frac{a_k}{p_k+1} \|u_j\|_{L^{2p_k+2}}^{2p_k+2} \geq \\ &\geq \|\nabla\varphi\|_{L^2}^2 + \sum_{l=1}^{L} \frac{b_l}{q_l+1} \|\varphi\|_{L^{2q_l+2}}^{2q_l+2} \\ &\quad - \sum_{k=1}^{K} \frac{a_k}{p_k+1} \|\varphi\|_{L^{2p_k+2}}^{2p_k+2} = I[\varphi] \geq m(\|\varphi\|^2) \geq m(\lambda), \end{aligned}$$

where in the last step, we have used the fact that m is non-increasing. Clearly, in all the above chain of inequalities we have equalities. In particular, $\|\varphi\|_{L^2}^2 = \lambda$, $I[\varphi] = m(\lambda)$, whence φ is a minimizer of (1.6). In addition, observe that $\liminf_j \|\nabla u_j\|_{L^2} = \|\nabla\varphi\|_{L^2}$ and $\liminf_j \|u_j\|_{L^{2q_l+2}} = \|\varphi\|_{L^{2q_l+2}}$, $l = 1, \ldots, L$. By Proposition 5, u_j tends to φ in the norm of $H^1(\mathbf{R}^n) \cap L^{2q_L+2}(\mathbf{R}^n)$.

3.2. Euler-Lagrange equations. We now derive the Euler-Lagrange equation, which are satisfied by the minimizers φ_λ. The starting point is that for arbitrary test function h and a real parameter ϵ, there is the inequality

$$I\left[\sqrt{\lambda}\frac{\varphi_\lambda + \epsilon h}{\|\varphi_\lambda + \epsilon h\|_{L^2}}\right] \geq I[\varphi_\lambda], \tag{3.4}$$

which exploits the fact that φ_λ is a minimizer. For simplicity, take h real-valued so that $h : \langle h, \varphi_\lambda\rangle = 0$, $\|\varphi_\lambda + \epsilon h\|_{L^2}^2 = \lambda + \epsilon^2\|h\|^2$. Expanding in orders of ϵ, we find

$$\begin{aligned} I\left[\sqrt{\lambda}\frac{\varphi_\lambda + \epsilon h}{\|\varphi_\lambda + \epsilon h\|_{L^2}}\right] &= \int_{\mathbf{R}^n} |\nabla[\varphi_\lambda + \epsilon h]|^2 - \int_{\mathbf{R}^n} G(\varphi_\lambda^2 + 2\epsilon\varphi_\lambda h) + O(\epsilon^2) \\ &= I[\varphi_\lambda] + 2\epsilon(\langle -\Delta\varphi_\lambda - G'(\varphi_\lambda^2)\varphi_\lambda, h\rangle) + O(\epsilon^2). \end{aligned}$$

It follows that $\langle -\Delta\varphi_\lambda - G'(\varphi_\lambda^2)\varphi_\lambda, h\rangle = 0$, whenever $h \perp \varphi_\lambda$. Equivalently, there is a Lagrange multiplier ω_λ, so that

$$-\Delta\varphi_\lambda - F(\varphi_\lambda^2)\varphi_\lambda = -\Delta\varphi_\lambda - G'(\varphi_\lambda^2)\varphi_\lambda = -\omega_\lambda\varphi_\lambda. \tag{3.5}$$

Note that so far, this equation is only satisfied in weak sense, since we only know $\varphi_\lambda \in H^1(\mathbf{R}^n)$! This is of course nothing but the elliptic profile Eq. (1.2). Taking dot product with φ_λ (which is justified even for weak solutions φ_λ) gives the useful relation

$$\omega_\lambda = \frac{\int_{\mathbf{R}^n} F(\varphi_\lambda^2)\varphi_\lambda^2 - \|\nabla\varphi_\lambda\|^2}{\lambda}. \tag{3.6}$$

Taking into account (3.6) and expanding up to second order in ϵ in (3.4) (keeping in mind that $h \perp \varphi_\lambda$), we obtain

$$\begin{aligned} I\left[\sqrt{\lambda}\frac{\varphi_\lambda + \epsilon h}{\|\varphi_\lambda + \epsilon h\|_{L^2}}\right] &= \\ &= (1 - \frac{\epsilon^2}{\lambda}\|h\|^2)\int_{\mathbf{R}^n} |\nabla[\varphi_\lambda + \epsilon h]|^2 \\ &\quad - \int_{\mathbf{R}^n} G\left((\varphi_\lambda^2 + 2\epsilon\varphi_\lambda h + \epsilon^2 h^2)(1 - \frac{\epsilon^2}{\lambda}\|h\|^2)\right) \\ &= I[\varphi_\lambda] + \epsilon^2\Bigg(\langle(-\Delta - F(\varphi^2) - 2F'(\varphi^2)\varphi^2)h, h\rangle \\ &\quad + \frac{1}{\lambda}(\int_{\mathbf{R}^n} F(\varphi_\lambda^2)\varphi_\lambda^2 - \|\nabla\varphi_\lambda\|^2)\|h\|^2\Bigg) + O(\epsilon^3) \\ &= I[\varphi_\lambda] + \epsilon^2\langle\mathscr{L}_+ h, h\rangle + O(\epsilon^3). \end{aligned}$$

It follows that $\langle\mathscr{L}_+ h, h\rangle \geq 0$, whenever $h \perp \varphi_\lambda$. It follows that $\mathscr{L}_+$ has at most one negative eigenvalue. In Morse index notations, $n(\mathscr{L}_+) \leq 1$. It follows from Lemma 2 that $n(\mathscr{L}_+) \geq 1$, so we conclude that $n(\mathscr{L}_+) = 1$.

3.3. Properties of the function $m(\lambda)$.

Lemma 3. *The function $m : (0, \infty) \to \mathbf{R}\cup\{-\infty\}$ is a non-positive function. In addition, supposing that the requirement (1.8) of Theorem 1 is met, then m is a Lipschitz function, with a locally bounded Lipschitz constant.*

Proof. Fix $\lambda > 0$ and a Schwartz function $\chi : \|\chi\|_{L^2}^2 = \lambda$, so that $\chi_\mu(x) := \mu^{n/2}\chi(\mu x)$, has the property $\|\chi_\mu\|^2 = \lambda$. Then, $m(\lambda) \leq I[\chi_\mu]$ for each $\mu > 0$, whence

$$m(\lambda) \leq \liminf_{\mu\to 0+} I[\chi_\mu] = \liminf_{\mu\to 0+}[\mu^2\|\nabla\chi\|_{L^2}^2 - \mu^{-n}\int_{\mathbf{R}^n} G(\mu^n\chi^2(x))dx] = 0.$$

Suppose now that (1.8) is satisfied. According to[13] (3.3), we may define

$$m(\lambda) = \inf_{\|u\|^2=\lambda, \|u\|_{H^1\cap L^{2q_l+2}} \leq 2C_\lambda} I[u] = \inf_{\|u\|^2=\lambda, \|u\|_{H^1\cap L^{2q_l+2}} \leq 1.5C_\lambda} I[u].$$

Upon introducing a new variable, $U : u = \sqrt{\lambda}U$, we can write

$$\begin{aligned} k(\lambda) := \frac{m(\lambda)}{\lambda} = \inf_{\|U\|^2=1, \|U\|_{H^1\cap L^{2q_l+2}} \leq 2C_\lambda} &[\|\nabla U\|^2 - \sum_{k=1}^{K}\frac{a_k\lambda^{\frac{p_k-1}{2}}}{p_k+1}\int_{\mathbf{R}^n}|U|^{2+2p_k} \\ &+ \sum_{l=1}^{L}\frac{b_l\lambda^{\frac{q_l-1}{2}}}{q_l+1}\int_{\mathbf{R}^n}|U|^{2+2q_l}]. \end{aligned}$$

[13] Which holds whenever $m(\lambda) > -\infty$, hence it is enough to assume only (1.8).

Clearly, it suffices to check that k is Lipschitz. Fix an U that satisfies the constraint for $\lambda+\delta$, that is $\|U\|=1$, $\|U\|_{H^1\cap L^{2q_l+2}} \leq 2C_{\lambda+\delta}$. For each such U we have

$$\begin{aligned}
&\|\nabla U\|^2 - \sum_{k=1}^{K} \frac{a_k(\lambda+\delta)^{\frac{p_k-1}{2}}}{p_k+1}\int_{\mathbf{R}^n}|U|^{2+2p_k} + \sum_{l=1}^{L}\frac{b_l(\lambda+\delta)^{\frac{q_l-1}{2}}}{q_l+1}\int_{\mathbf{R}^n}|U|^{2+2q_l}\\
&= \|\nabla U\|^2 - \sum_{k=1}^{K} \frac{a_k\lambda^{\frac{p_k-1}{2}}}{p_k+1}\int_{\mathbf{R}^n}|U|^{2+2p_k} + \sum_{l=1}^{L}\frac{b_l\lambda^{\frac{q_l-1}{2}}}{q_l+1}\int_{\mathbf{R}^n}|U|^{2+2q_l} + E_{\delta,\lambda},
\end{aligned}$$

where the error term $E_{\delta,\lambda}$ clearly can be estimated as follows

$$|E_{\delta,\lambda}| \leq C|\delta|(\frac{1}{\lambda}+\lambda^{q_L}+\lambda^{p_K})(1+\|U\|^{2+2q_L}_{L^{2+2q_L}}+\|U\|^{2+2p_K}_{L^{2+2p_K}}) \leq |\delta|D_\lambda,$$

where again D_λ is an explicit, continuous (and computable in terms of C_λ, $\vec{p},\vec{q}$ etc.) function of λ. It follows that, by taking $|\delta|$ small enough so that $2C_{\lambda+\delta} > 1.5C_\lambda$ and consequently taking infimum over all U satisfying the constraints for $\lambda+\delta$ (and hence, by the choice of δ for λ as well)

$$k(\lambda) - D_\lambda|\delta| \leq k(\lambda+\delta) \leq k(\lambda) + D_\lambda|\delta|. \tag{3.7}$$

This is the desired Lipschitzness for k, with a constant D_λ. Due to the fact that m is Lipschitz, we have that it is differentiable a.e. We show now that $\omega_\lambda \geq 0$ and whenever $m'(\lambda)$ exists, we have the formula $m'(\lambda) = -\frac{\omega_\lambda}{2}$. Indeed, start with the inequality

$$I[\varphi_\lambda+\epsilon h] \geq m(\|\varphi_\lambda+\epsilon h\|^2) = m(\lambda+2\epsilon\langle\varphi_\lambda,h\rangle+\epsilon^2\|h\|^2), \tag{3.8}$$

valid for all ϵ and all test functions h. On the other hand, there is

$$\begin{aligned}
I[\varphi_\lambda+\epsilon h] &= I[\varphi_\lambda] - \epsilon\omega_\lambda\langle\varphi_\lambda,h\rangle + \frac{\epsilon^2}{2}\langle(\mathcal{L}_+-\omega_\lambda)h,h\rangle + O(\epsilon^3)\\
&= m(\lambda) - \epsilon\omega_\lambda\langle\varphi_\lambda,h\rangle + O(\epsilon^2).
\end{aligned} \tag{3.9}$$

Taking $h=\varphi_\lambda$ yields

$$m(\lambda+2\lambda\epsilon+\epsilon^2\lambda) \leq m(\lambda) - \epsilon\lambda\omega_\lambda + O(\epsilon^2). \tag{3.10}$$

For $\epsilon < 0$, we have $-\omega_\lambda \leq \frac{m(\lambda+2\lambda\epsilon+\epsilon^2\lambda)-m(\lambda)}{\epsilon}$, whence by taking $\liminf_{\epsilon\to 0-}$ and taking into account that m is decreasing (and since $2\lambda\epsilon+\lambda\epsilon^2 < 0$ for all small enough $\epsilon < 0$),

$$-\omega_\lambda \leq \liminf_{\epsilon\to 0-}\frac{m(\lambda+2\lambda\epsilon+\epsilon^2\lambda)-m(\lambda)}{\lambda\epsilon} \leq 0,$$

so, $\omega_\lambda \geq 0$. If $m'(\lambda)$ exists, we can compute it from (3.10). Indeed, taking $\epsilon \to 0+$ yields

$$m'(\lambda) = \lim_{\epsilon\to 0+}\frac{m(\lambda+2\lambda\epsilon+\epsilon^2\lambda)-m(\lambda)}{2\lambda\epsilon} \leq -\frac{\omega_\lambda}{2}, \tag{3.11}$$

whereas taking $\lim_{\epsilon\to 0-}$ yields

$$m'(\lambda) = \lim_{\epsilon\to 0-} \frac{m(\lambda + 2\lambda\epsilon + \epsilon^2\lambda) - m(\lambda)}{2\lambda\epsilon} \geq -\frac{\omega_\lambda}{2}.$$

Combining the last two inequalities gives the desired formula $m'(\lambda) = -\frac{\omega_\lambda}{2}$, whenever m' exists. Since m is Lipschitz and hence absolutely continuous, there is the formula (1.10). □

In the case $\omega_\lambda > 0$, we can actually say that φ_λ is a classical solution of (3.5). Indeed, for φ_λ (which is initially known to be only in $H^1(\mathbf{R}^n) \cap L^{2q_L+2}(\mathbf{R}^n)$), we can write

$$\varphi_\lambda = (-\Delta + \omega_\lambda)^{-1}[F[\varphi_\lambda^2]\varphi_\lambda]. \tag{3.12}$$

Since the radial potential $V := F[\varphi_\lambda^2]$ has some decay at ∞, we conclude from Theorem 2 that in fact $|\varphi_\lambda(x)| \leq Ce^{-\sqrt{\omega_\lambda}|x|}$. Going back to (3.12), it is clear that the bell-shaped function φ_λ is in fact $H^2(\mathbf{R}^n)$. This can clearly be bootstrapped further, we will not need to do so here.

3.4. Necessity of the assumptions: Proof of Proposition 1.

3.4.1. (1.8) is necessary Assuming that (1.8) fails, let $\lambda > 0$ and fix a Schwartz function $\chi : \|\chi\|_{L^2}^2 = \lambda$. Consider testing (1.6) with the sequence $\chi_N = N^{n/2}\chi(Nx) : \|\chi_N\|_{L^2}^2 = \lambda$, for $N >> 1$. We obtain

$$I[\chi_N] = N^2\|\nabla\chi\|_{L^2}^2 - N^{np_K}\|\chi\|_{L^{2p_K+2}}^{2+2p_K} + N^{nq_L}\|\chi\|_{L^{2q_L+2}}^{2+2q_L} + o(N^{np_K}).$$

Clearly, N^{np_K} is the dominant term, whence $m(\lambda) \leq \liminf_N I[\chi_N] = -\infty$.

3.4.2. Waves exist only for $\omega_\lambda > 0$ One can directly use the Pohozaev's identities (A.2). From it, and assuming that f_ω is a minimizer, we have

$$\omega_\lambda n\|f_\omega\|^2 = n\int_{\mathbf{R}^n} G(f_\omega^2(x))dx - (n-2)\|\nabla f_\omega\|^2 = 2\|\nabla f_\omega\|^2 - nm(\lambda) > 0,$$

taking into account that $m(\lambda) \leq 0$, as established earlier. It follows that $\omega_\lambda > 0$. Note for future reference that if f_ω is a constrained minimizer, then $\omega_\lambda \geq -\frac{m(\lambda)}{\lambda}$. In particular, on an interval (λ_1, λ_2), since m is non-increasing, we obtain

$$\inf_{\lambda\in(\lambda_1,\lambda_2)} \omega_\lambda \geq -\frac{m(\lambda_1)}{\lambda_1}. \tag{3.13}$$

3.4.3. $\lambda \to m(\lambda)$ must be non-increasing We have essentially showed this already. Indeed, recall that $\lambda \to m(\lambda)$ was shown to be Lipschitz, only under the assumption (1.8) (see Lemma 3). As such, it is absolutely continuous function, with a derivative a.e. Finally, assuming that a minimizer for (1.6) exists, we have (3.8) and subsequently (3.10), whence we compute the derivative to be $m'(\lambda) = -\frac{\omega_\lambda}{2}$. Since a.c. functions are integrals of their derivatives, we have for each $0 < \lambda_1 < \lambda_2$,

$$m(\lambda_2) - m(\lambda_1) = \int_{\lambda_1}^{\lambda_2} m'(\lambda)d\lambda = -\frac{1}{2}\int_{\lambda_1}^{\lambda_2} \omega_\lambda d\lambda \leq 0,$$

since $\omega_\lambda \geq 0$. Thus, m is non-increasing.

3.5. Proof of Proposition 2. Recall that by Theorem 1, $\mathcal{L}_+|\{\varphi\}^\perp \geq 0$. Take any element $\Psi \in Ker[\mathcal{L}_+]$. Then, $\Psi - \|\varphi\|^{-2}\langle\Psi, \varphi\rangle\varphi \in \{\varphi\}^\perp$. Thus, it follows that

$$0 \leq \langle\mathcal{L}_+(\Psi - \|\varphi\|^{-2}\langle\Psi, \varphi\rangle\varphi), \Psi - \|\varphi\|^{-2}\langle\Psi, \varphi\rangle\varphi\rangle = \|\varphi\|^{-4}\langle\Psi, \varphi\rangle^2\langle\mathcal{L}_+\varphi, \varphi\rangle.$$

But since $\langle\mathcal{L}_+\varphi, \varphi\rangle < 0$, we conclude that $\langle\Psi, \varphi\rangle = 0$, otherwise a contradiction with the previous inequality. This establishes Proposition 2.

4. Proof of Theorem 2

In the next lemma, we show that the solutions to (1.2) are bounded at zero, provided (1.8) is assumed. Recall the notation $V(x) = F(f^2(x))$.

4.1. Bounds at zero.

Lemma 4. *Assume $\omega > 0$, (1.8) holds, and f is a bell-shaped function, with $f \in L^2(\mathbf{R}^n)$, so that f is a strong solution of (1.2), that is*

$$f(x) = (-\Delta + \omega)^{-1}[Vf] = \omega^{\frac{n}{2}-1}\int_{\mathbf{R}^n} Q(\sqrt{\omega}(x-y))F(f^2(y))f(y)dy. \quad (4.1)$$

Then, $f \in L^\infty(\mathbf{R}^n)$.

Proof. Since f is bell-shaped, clearly $f(0) = \sup_{x\in\mathbf{R}^n}|f(x)|$, so we need to show that $f(0) < \infty$. Since $Q > 0$, $f > 0$ and after ignoring the negative part of the function F, we obtain

$$\begin{aligned} 0 < f(x) &< \omega^{\frac{n}{2}-1}\int_{\mathbf{R}^n} Q(\sqrt{\omega}(x-y))(\sum_{k=1}^K a_k f^{2p_k+1}(y))dy \\ &= \omega^{\frac{n}{2}-1}\sum_{k=1}^K a_k \int_{\mathbf{R}^n} Q(\sqrt{\omega}(x-y))f^{2p_k+1}(y)dy. \end{aligned}$$

By the exponential decay of Q, the integral clearly converges for large y, so the issue is controlling the integration, say over $|y| < 1$.

Assume $n \geq 3$, the case $n = 2$ is treated similarly. As we saw in our earlier arguments for bell-shaped functions in L^q spaces, we have that $f(x) \leq c_n\|f\|_{L^q}|x|^{-n/q}$, for all $x \neq 0$. Consider

$$s_0 = \inf\{s > 0 : |f(x)| \leq C_s|x|^{-s}, \text{ for } |x| < 1\}.$$

Clearly, since $f \in L^2$, we have that $0 \leq s_0 \leq \frac{n}{2}$. We will actually show $s_0 = 0$. Assume not, so $s_0 > 0$ and take any $s > s_0$. Take $\delta \in (0, 1)$ and $x : |x| = \delta$. Then,

$$\begin{aligned} \int_{|y|<2\delta} Q(\sqrt{\omega}(x-y))f^{2p+1}(y)dy &= \int_{|y|<\frac{\delta}{2}} Q(\sqrt{\omega}(x-y))f^{2p+1}(y)dy + \\ &+ \int_{\frac{\delta}{2}<|y|<2\delta} Q(\sqrt{\omega}(x-y))f^{2p+1}(y)dy \lesssim \delta^{-(n-2)}\int_{|y|<\frac{\delta}{2}} |y|^{-s(2p+1)}dy + \\ &+ \delta^{-(2p+1)s}\int_{\frac{\delta}{2}<|y|<2\delta} |x-y|^{-(n-2)}dy \lesssim \delta^{2-(2p+1)s}. \end{aligned}$$

We have also good bounds for $\int_{2\delta<|y|<1} Q(\sqrt{\omega}(x-y)) f^{2p+1}(y)dy$. Indeed, for k_0 : $2^{k_0-1}\delta < 1 \leq 2^{k_0}\delta$, we have

$$\begin{aligned}\int_{2\delta<|y|<1} & Q(\sqrt{\omega}(x-y)) f^{2p+1}(y)dy \\ &\leq \sum_{k=1}^{k_0} \int_{2^k\delta<|y|<2^{(k+1)\delta}} Q(\sqrt{\omega}(x-y)) f^{2p+1}(y)dy \lesssim \\ &\lesssim \sum_{k=1}^{k_0} (2^k\delta)^{-(n-2)} \int_{2^k\delta<|y|<2^{(k+1)\delta}} |y|^{-s(2p+1)}dy \\ &\lesssim \sum_{k=1}^{k_0} (2^k\delta)^{2-(2p+1)s} \lesssim \max(1, \delta^{2-(2p+1)s})\end{aligned}$$

and also, a bound by a constant for $\int_{|y|>1} Q(\sqrt{\omega}(x-y)) f^{2p+1}(y)dy$, due to the exponential bounds for Q. The least favorable bounds occur of course for $p = p_K$, so this shows that $|f(x)| \leq C \max(|x|^{-((2p_K+1)s-2)}, 1)$. If $(2p_K+1)s-2 \leq 0$, we have $s_0 = 0$ and we are done. Otherwise, if $(2p_K+1)s-2 > 0$,

$$s_0 \leq (2p_K+1)s - 2$$

for all $s > s_0$. This leads to the inequality $s_0 \geq \frac{1}{p_K}$. But, $p_K < \frac{2}{n}$, whence $s_0 > \frac{n}{2}$. But, we already know that $s_0 \leq \frac{n}{2}$, a contradiction. So, $s_0 = 0$.

This means that for all $\epsilon > 0$, there is C_ϵ, so that $f(x) \leq C_\epsilon |x|^{-\epsilon}$. Clearly, by our argument above, with $s = \epsilon$,

$$\int_{|y|<2\delta} Q(\sqrt{\omega}(x-y)) f^{2p+1}(y)dy \lesssim \delta^{2-(2p_K+1)\epsilon} \leq \delta$$

for small enough ϵ. Similar to the previous estimate, now with $s = \epsilon$, $\int_{2\delta<|y|} Q(\sqrt{\omega}(x-y)) f^{2p+1}(y)dy \lesssim 1$. Thus, the boundedness of $f(0)$ is established. □

4.2. Asymptotics at infinity for eigenfunctions and waves. The next lemma is about the existence and properties of Jost solutions, with the expected prescribed behavior at ∞. This result will be an important ingredient in two related, but overall different situations. First, to establish that the (radial portion of the) eigenfunctions for Schrödinger operators with radial potentials have exponential asymptotics at ∞, and second, to actually show that the waves (which are solutions to a non-linear problem!) actually do decay exponentially at ∞.

Lemma 5. *Let $k > 0$, $\alpha \in \mathbf{R}$, $A >> 1$ and let V be a smooth potential, with $V \in L^1(A, \infty)$. Let Φ be a non-trivial C^∞ decaying solution in (A, ∞) of the problem*

$$-\Phi''(r) + k^2\Phi(r) + \frac{\alpha}{r^2}\Phi - V(r)\Phi(r) = 0. \tag{4.2}$$

Then, as $r \to \infty$,

$$\Phi(r) = c_0 e^{-kr} + o(e^{-kr}),$$

with some $c_0 \neq 0$.

Remark: Assuming faster decay for V, say $V = o(r^{-3})$, we can explicitly calculate the next asymptotic term, namely

$$\Phi(r) = c_0[e^{-kr} + \frac{\alpha}{2k} r^{-1} e^{-kr}] + o(r^{-1} e^{-kr}).$$

Proof. At issue here is the construction of a complete system of linear independent Jost solutions for (4.2). We will show that there are solutions $g_1 = e^{-kr} + o(e^{-kr})$, $g_2 = e^{kr} + o(e^{kr})$ of (4.2). Once this is done, the proof follows easily, since this is a complete system and any solution, and in particular Φ, is a linear combination of g_1, g_2. Since Φ, g_1 are localized, whereas g_2 is growing at ∞, it follows that $\Phi = c_0 g_1$, which is the claim.

In order to construct g_1 (construction of g_2 is identical), recall the formula for solutions of the inhomogeneous problem $-h'' + k^2 h = G$, which takes the form

$$h(r) = -\frac{1}{2k} \int_r^\infty e^{k(s-r)} G(s) ds + \frac{1}{2k} \int_r^\infty e^{-k(s-r)} G(s) ds. \tag{4.3}$$

We will show that the ansatz $g_1(r) = e^{-kr} + \Psi_1(r)$ produces a solution with the required bounds. Note that Ψ_1 solves

$$-\Psi_1''(r) + k^2 \Psi_1(r) + \left(\frac{\alpha}{r^2} - V(r)\right)(e^{-kr} + \Psi_1(r)) = 0.$$

Thus, we need to solve the integral equation

$$\Psi_1(r) = -\frac{1}{2k} \int_r^\infty e^{k(s-r)} \left(V(s) - \frac{\alpha}{s^2}\right)(e^{-ks} + \Psi_1(s)) ds$$
$$+ \frac{1}{2k} \int_r^\infty e^{-k(s-r)} \left(V(s) - \frac{\alpha}{s^2}\right)(e^{-ks} + \Psi_1(s)) ds.$$

Introduce $\psi_1(r) := e^{kr} \Psi_1(r)$, so

$$\psi_1(r) = -\frac{1}{2k} \int_r^\infty \left(V(s) - \frac{\alpha}{s^2}\right)(1 + \psi_1(s)) ds$$
$$+ \frac{e^{2kr}}{2k} \int_r^\infty e^{-2ks} \left(V(s) - \frac{\alpha}{s^2}\right)(1 + \psi_1(s)) ds. \tag{4.4}$$

The linear operator

$$\Lambda f(r) = -\frac{1}{2k} \int_r^\infty \left(V(s) - \frac{\alpha}{s^2}\right) f(s) ds + \frac{e^{2kr}}{2k} \int_r^\infty e^{-2ks} \left(V(s) - \frac{\alpha}{s^2}\right) f(s) ds$$

clearly has small norm, when acting on the space $L^\infty(A, \infty)$ for $A >> 1$, say

$$\|\Lambda\|_{L^\infty(A,\infty) \to L^\infty(A,\infty)} \leq \frac{1}{2}.$$

Thus, we can resolve (4.4) as follows

$$\psi = (Id - \Lambda)^{-1}[-\frac{1}{2k} \int_r^\infty \left(V(s) - \frac{\alpha}{s^2}\right) ds + \frac{e^{2kr}}{2k} \int_r^\infty e^{-2ks} \left(V(s) - \frac{\alpha}{s^2}\right) ds].$$

It follows that

$$\|\psi\|_{L^\infty(A,\infty)} \leq 2\| - \frac{1}{2k}\int_r^\infty \left(V(s) - \frac{\alpha}{s^2}\right) ds + \frac{e^{2kr}}{2k}\int_r^\infty e^{-2ks}\left(V(s) - \frac{\alpha}{s^2}\right) ds\|_{L_r^\infty(A,\infty)}.$$

Since $V \in L^1(A,\infty)$, for large enough A, it follows that

$$\|\psi\|_{L^\infty(A,\infty)} \leq C_{k,\alpha}\int_A^\infty [|V(s)| + \frac{|\alpha|}{s^2}]ds \leq C_{k,\alpha}(A^{-1} + \int_A^\infty |V(s)|)$$

so the result is established. If V has even faster decay, say $o(s^{-3})$, we can compute explicitly the next order term for ψ to be

$$-\frac{1}{2k}\int_r^\infty \left(V(s) - \frac{\alpha}{s^2}\right) ds = \frac{\alpha}{2k} r^{-1} + o(r^{-1}).$$

Thus, we have the asymptotic formula (for large r)

$$\Phi(r) = e^{-kr} + \frac{\alpha}{2k} r^{-1} e^{-kr} + o(r^{-1}e^{-kr}).$$

□

Next, we deal with the question of the asymptotic behavior at ∞ of bell-shaped solutions of (1.2). Clearly, Lemma 5 will be helpful in this regard. Indeed, a solution of (1.2) satisfies the ODE

$$-f''(r) - \frac{n-1}{r} f'(r) + \omega f(r) - V(r)f(r) = 0, r \in (1,\infty)$$

where recall $V = F(f^2)$. We make the transformation $g(r) := r^{\frac{n-1}{2}} f(r)$, so that g satisfies

$$-g''(r) + \frac{(n-1)(n-3)}{4r^2} g(r) + \omega g(r) - V(r)g(r) = 0. \tag{4.5}$$

Note that by the bell-shapedness of f, we have $0 < f(x) < c_n\|f\|_{L^2}|x|^{-n/2}$, whence $0 < g(r) < c_n r^{-1/2}$. Clearly, (4.5) is in the form (4.2), with $\omega = k^2$ and $\alpha = \frac{(n-1)(n-3)}{4}$.

The only missing piece is that the potential $V(r) = F(f^2) = F(r^{-(n-1)}g^2(r))$, does not satisfy *a priori* the required integrability condition $V \in L^1(1,\infty)$. Indeed, since we only assume $f \in L^2(\mathbf{R}^n)$, we can only infer a decay $f(r) \sim |r|^{-\frac{n}{2}}$, whence $V(r) \sim \min(|r|^{-p_1 n}, r^{-nq_1})$. This does not satisfy the condition only when $\min(p_1, q_1) \leq \frac{1}{n}$, but it turns out that one can address this issue, even for small $\min(p_1, q_1)$.

We set up a bootstrap argument as follows. Let

$$\sigma_0 := \sup\{\sigma > 0 : f(r) < C_\sigma r^{-\sigma}, \text{ for } r > 1\}.$$

We already know that $\sigma_0 \geq \frac{n}{2}$. It remains to show that $\sigma_0 = \infty$, whence the result will follow, since $V(r) = F(f^2(r)) \lesssim (f^2(r))^{\min(p_1,q_1)} \leq C_N r^{-N}$ for any N and $r > 1$.

Assume that $\sigma_0 < \infty$ and let $0 < \sigma < \sigma_0$. Use the representation of f (as a function on $\mathbf{R}^n$)

$$f(x) = \omega^{\frac{n}{2}-1} \int_{\mathbf{R}^n} Q(\sqrt{\omega}(x-y))F(f^2(y))f(y)dy.$$

Let $x : |x| > 1$, so we estimate (by using the boundedness of f),

$$\begin{aligned} f(x) &\leq \int_{|y|<\frac{|x|}{2}} Q(\sqrt{\omega}(x-y))|F(f^2(y))f(y)|dy \\ &\quad + \int_{\frac{|x|}{2}<|y|} Q(\sqrt{\omega}(x-y))|y|^{-\sigma(2\min(p_1,q_1)+1)}dy \\ &\lesssim e^{-\frac{\sqrt{\omega}}{2}|x|}|x|^n + |x|^{-\sigma(2\min(p_1,q_1)+1)} \int Q(\sqrt{\omega}(x-y))dy \\ &\lesssim e^{-\frac{\sqrt{\omega}}{2}|x|}|x|^n + |x|^{-\sigma(2\min(p_1,q_1)+1)}. \end{aligned}$$

It follows that $\sigma_0 \geq \sigma(2\min(p_1,q_1)+1)$ for all $\sigma < \sigma_0$, a contradiction.

5. Proof of Theorem 3

We start with the spectral analysis of $\mathscr{L}_-$.

5.1. The operator $\mathscr{L}_-$. Denote $V(r) := F(f_\omega^2(r))$, so that $\mathscr{L}_- = -\Delta + w - V(|\cdot|)$. Clearly $\mathscr{L}_-[f_\omega] = 0$. We apply the spectral decomposition of Sect. 2.2. We obtain a sequence of operators $\mathscr{L}_{-,k}$, $k = 0, 1, \ldots$ acting on L^2_{rad}, so that $\mathscr{L}_{-,0} < \mathscr{L}_{-,1} < \ldots$. In order to show that $\mathscr{L}_- \geq 0$, with a simple eigenvalue at zero, it clearly suffices to show $\mathscr{L}_{-,0} \geq 0$, with a simple eigenvalue at zero. Set the eigenvalue problem $\mathscr{L}_{-,0}$ for radial valued functions f

$$-\partial_{rr} f - \frac{n-1}{r}\partial_r f + \omega f - V(r)f = \mu f. \tag{5.1}$$

Introduce a change of variables, $g : g(r) = r^{\frac{n-1}{2}} f(r)$. Note that $\|g\|_{L^2(0,\infty)} = \|f\|_{L^2_{rad}(\mathbf{R}^n)}$ and so, (5.1) becomes, in terms of g

$$-g'' + \frac{(n-1)(n-3)}{4r^2} g + \omega g - V(r)g = \mu g. \tag{5.2}$$

Denoting $V_1(r) := V(r) - \frac{(n-1)(n-3)}{4r^2}$, we recast the eigenvalue problem in the form $-g'' + \omega g - V_1(r)g = \mu g$, where $g \in L^2(0,\infty)$. This is slightly unusual eigenvalue problem, but observe that the operator $L_{-,0} := -\frac{d^2}{dr^2} + \omega - V_1$ is essentially self-adjoint on the Hilbert space $L^2(0,\infty)$, when considered over the domain $\{u : u \in C_0^\infty(0,\infty)\}$. See laso [31], p. 91, where similar eigenvalue problems arise.

Clearly, $L_{-,0}[\tilde{f}_\omega] = 0$, $\tilde{f}_\omega(r) := r^{\frac{n-1}{2}} f_\omega(r)$. We will show that this is the bottom of the spectrum. This is essentially contained in the Sturm oscillation theorem (Lemma 2, p. 92, [31]), but we shall give a direct proof, as the result in [31] is stated with boundary conditions at zero, which are not relevant for us.

So, assume for a contradiction, that there is a negative eigenvalue for $\mathscr{L}_{-,0}$. That is, a function Ψ and $\sigma_0 > 0$, so that $L_{-,0}[\Psi] = -\sigma_0^2\Psi$. Following the proof of Lemma 2, p. 92, [31], let $(r_0, r_1) : 0 \le r_0 < r_1 \le \infty$, is an interval in which Ψ does not change sign, but $\Psi(r_0) = \Psi(r_1) = 0$. Without loss of generality, $\Psi|_{(r_0,r_1)} > 0$, otherwise take $-\Psi$. Note that $\Psi'(r_0) \ge 0$ and $\Psi'(r_1) \le 0$ (in fact $\Psi'(r_1) < 0$, if $r_1 < \infty$). Consider

$$
\begin{aligned}
I &= \int_{r_0}^{r_1} (\Psi' \tilde{f}_\omega - \Psi \tilde{f}_\omega{}')' dr = (\Psi' \tilde{f}_\omega - \Psi \tilde{f}_\omega{}')|_{r_0}^{r_1} \\
&= (\Psi'(r_1)\tilde{f}_\omega(r_1) - \Psi(r_1)\tilde{f}_\omega{}'(r_1)) - (\Psi'(r_0)\tilde{f}_\omega(r_0) - \Psi(r_0)\tilde{f}_\omega{}'(r_0)) \le 0,
\end{aligned}
$$

since $\Psi(r_0) = 0$, $\Psi(r_1) = 0$, $\Psi'(r_1) \le 0$, $\Psi'(r_0) \ge 0$ and $\tilde{f}_\omega > 0$. On the other hand, using the fact that $L_{-,0}[\Psi] = -\sigma_0^2\Psi$ and $L_{-,0}[\tilde{f}_\omega] = 0$, we have

$$
I = \int_{r_0}^{r_1} (\Psi'' \tilde{f}_\omega - \Psi \tilde{f}_\omega{}'') dr = \sigma_0^2 \int_{r_0}^{r_1} \Psi \tilde{f}_\omega dr > 0.
$$

This is of course a contradiction, whence Ψ has only one zero, at $r_1 = +\infty$. This means that the function $\Psi \ge 0$, in particular $\langle \Psi, \tilde{f}_\omega \rangle > 0$. This is a contradiction again, since eigenfunctions corresponding to different eigenvalues are orthogonal.

So, $\mathscr{L}_{-,0}$ does not have a negative eigenvalue and zero is at the bottom of $\sigma(\mathscr{L}_{-,0})$. Similar argument produces a contradiction, if one assumes that there is a second, independent from $\tilde{f}_\omega$ eigenfunction, corresponding to the zero eigenvalue. Thus, zero is a simple eigenvalue for $\mathscr{L}_{-,0}$ and hence for $\mathscr{L}_{-}$.

5.2. The operator $\mathscr{L}_+$. We apply the decomposition in eigenspaces of the spherical Laplacian as described in Sect. 2.2. More specifically, the operators $\mathscr{L}_{+,k}$, $k = 0, 1, \ldots$ act on the space $\mathscr{L}^2_{rad.}$ as follows

$$
\begin{aligned}
\mathscr{L}_{+,0} &= -\partial_{rr} - \frac{n-1}{r}\partial_r + \omega - W_1(r), \\
\mathscr{L}_{+,k} &= -\partial_{rr} - \frac{n-1}{r}\partial_r + \frac{k(k+n-2)}{r^2} + \omega - V_1(r), k = 1, 2, \ldots
\end{aligned}
$$

where $W_1(r) := F(f_\omega^2(r)) + 2F'(f_\omega^2(r))f_\omega^2(r), r > 0$. Note that $0 = \mathscr{L}_+[\nabla f_\omega] = \mathscr{L}_+[\frac{x}{r} f'_\omega]$ is equivalent to $\mathscr{L}_{+,1}[f'_\omega] = 0$, since $\frac{x_j}{r}$, $j = 1, \ldots, n$ are the first non-trivial harmonics, corresponding to the eigenvalue $(n-1)$.

Since $\mathscr{L}_{+,0} < \mathscr{L}_{+,1} < \ldots$, and by the assumption $n(\mathscr{L}_+) = 1$, we clearly must have that $n(\mathscr{L}_{+,0}) = 1$, while $\mathscr{L}_{+,k} \ge 0, k = 1, 2, \ldots$. The remaining statements about $\mathscr{L}_+$ in Theorem 3 amount to establishing the following

$$
Ker[\mathscr{L}_{+,0}] = \{0\} \text{ or } Ker[\mathscr{L}_{+,0}] = span\{\Psi_0\}, \tag{5.3}
$$

$$
\mathscr{L}_{+,1} = span\{f'_\omega\}, \quad \mathscr{L}_{+,1}|_{\{f'_\omega\}^\perp} > 0, \tag{5.4}
$$

where $\{\cdot\}^\perp$ is in the sense of the Hilbert space $L^2_{rad.}$, equipped with its dot product $\langle f, g\rangle = \int_0^\infty f(r)\bar{g}(r)r^{n-1}dr$.

5.2.1. Proof of (5.4) We apply the transformation $g(r) = r^{\frac{n-1}{2}} f(r)$. Thus, the eigenvalue problem, for the zero eigenvalue of $\mathscr{L}_{+,1}$ becomes

$$- g'' + \frac{(n-1)(n+1)}{4r^2} g + \omega g - W_1(r) g = 0 \tag{5.5}$$

where $g \in L^2(0, \infty) : \int_0^\infty |g(r)|^2 dr < \infty$. It suffices to show that there is no second localized eigenfunction for (5.5), other than $g_0(r) = r^{\frac{n-1}{2}} f'(r)$. To look for a second eigenfunction, we set the usual ansatz $g_0(r) G(r)$, which leads us to the ODE $G'' g_0 = -2 g_0' G'$. Solving it, we obtain a solution $G(r) = \int_1^r \frac{1}{g_0^2(\tau)} d\tau$. Note that the function g_0 does not vanish in $(0, \infty)$, whence this formula makes sense for all $r \in (0, \infty)$ and a second eigenfunction is in the form

$$g(r) = g_0(r) \int_1^r \frac{1}{g_0^2(\tau)} d\tau.$$

The function g is linearly independent from g_0, because $g(1) = 0$, while $g'(1) = \frac{1}{g_0(1)}$ and so the Wronskian is non-trivial, since $\det \begin{pmatrix} g(1) & g'(1) \\ g_0(1) & g_0'(1) \end{pmatrix} = -1$.

We now argue that g is not localized at $r = \infty$, hence precluding the possibility for a second eigenfunction, corresponding to eigenvalue zero. To this end, note that for $r > 2$,

$$g(r) = g_0(r) \int_1^r \frac{1}{g_0^2(\tau)} d\tau > g_0(r) \int_{r-1}^r \frac{1}{g_0^2(\tau)} d\tau = \frac{g_0(r)}{g_0^2(\tilde{r})},$$

for some $\tilde{r} \in (r-1, r)$. We now show that $\lim_{r \to \infty} g(r) = \infty$. Recall that the function g_0 solves (5.5). By Lemma 5, $g_0(r) = c_0 e^{-\sqrt{\omega} r} + o(e^{-\sqrt{\omega} r})$ as $r \to \infty$. Thus,

$$\lim_{r \to \infty} g(r) \geq \lim_{r \to \infty} \frac{g_0(r)}{g_0^2(r-1)} = \lim_{r \to \infty} \frac{e^{-\sqrt{\omega} r}}{c_0 e^{-2\sqrt{\omega}(r-1)}} = \infty.$$

It follows that g_0 is the only localized eigenfunction (and hence zero is simple eigenvalue for $\mathscr{L}_{+,0}$), since every other eigenfunction must be a non-trivial linear combination of g_0, g, and as such it will not be localized at ∞.

5.2.2. Proof of (5.3) As before, with the change of variables $g(r) = r^{\frac{n-1}{2}} f(r)$, we consider the operator

$$L_{+,0} := -\partial_r^2 + \frac{(n-1)(n-3)}{4r^2} + \omega - W_1(r).$$

More specifically, we consider $L_{+,0}$ as given by the Friedrich's extension for the form domain[14] $\{g \in L^2(0, \infty) : g(0) = 0, \int_0^\infty |g'(r)|^2 dr < \infty, \int_0^1 \frac{g^2(r)}{r^2} dr < \infty\}$. Note that in order to satisfy the integrability condition at zero (that is $\int_0^1 \frac{g^2(r)}{r^2} dr < \infty$), for functions in the form $g = r^{\frac{n-1}{2}} f(r)$, $f \in L^2_{rad.}(\mathbf{R}^n)$, we need $n \geq 3$. Thus, for $n \geq 3$,

[14] For the case $n = 3$, the integrability at zero condition is clearly not necessary.

the spectral problem of interest, that is $\mathcal{L}_{+,0}$ on $L^2_{rad.}(\mathbf{R}^n)$, becomes equivalent to the spectral problem for $L_{+,0}$ (as a Friedrich's extension).

Per our assumptions, $n(\mathcal{L}_+) = 1$, whence $n(\mathcal{L}_{+,0}) = 1$, whence $n(L_{+,0}) = 1$. That is, $L_{+,0}$ has a negative eigenvalue, say $-\sigma_0^2$. Similar to the arguments for $\mathcal{L}_-$, the next eigenfunction, say Ψ_0 (if there is one at all!) must change sign at least once in $(0, \infty)$. Clearly, this eigenfunction cannot correspond to a negative eigenvalue, as this would contradict $n(L_{+,0}) = 1$. Therefore, it may correspond to a positive eigenvalue, in which case we are done–this implies $Ker[L_{+,0}] = \{0\}$. Finally, there is the possibility that the eigenfunction Ψ_0, corresponds to a zero eigenvalue, that is $L_{+,0}\Psi_0 = 0$.

We will now show that Ψ_0 cannot change sign twice. Suppose that it does changes signs twice, say at $r_1, r_2 : 0 < r_1 < r_2 < \infty$. Following the argument in Lemma 1, p. 91, [31], we set $\Psi_1(r) = \Psi_0\chi_{(0,r_1)}$, $\Psi_2(r) = \Psi_0\chi_{(r_1,r_2)}$, $\Psi_3(r) = \Psi_0\chi_{(r_2,\infty)}$. Note $\Psi_0(0) = 0$ and $\int_0^1 \frac{|\Psi_0(r)|^2}{r^2} dr < \infty$.

Clearly, Ψ_j, $j = 1, 2, 3$ are continuous and piecewise C^1, but they do not belong to $H^2(0, \infty) = D(-\frac{d^2}{dr^2})$. On the other hand, they do belong to the form domain. For arbitrary coefficients a_j, $j = 1, 2, 3$, we compute

$$
\begin{aligned}
&\langle \sum_{j=1}^3 a_j \Psi_j, L_{+,0}[\sum_{j=1}^3 a_j \Psi_j]\rangle \\
&= \int_0^\infty |\sum_{j=1}^3 a_j \Psi_j'(r)|^2 dr + \frac{(n-1)(n-3)}{4} \int_0^\infty \frac{|\sum_{j=1}^3 a_j \Psi_j|^2}{r^2} dr \\
&\quad + \int_0^\infty (\omega - W_1(r))|\sum_{j=1}^3 a_j \Psi_j|^2 dr \\
&= \sum_{j=1}^3 |a_j|^2 \int_{r_{j-1}}^{r_j} \left[|\Psi_j'(r)|^2 + \frac{(n-1)(n-3)}{4} \frac{|\Psi_j(r)|^2}{r^2} + (\omega - V_1(r))|\Psi_j(r)|^2 \right] dr \\
&= \sum_{j=1}^3 |a_j|^2 \int_{r_{j-1}}^{r_j} \overline{\Psi}_j \left[-\Psi_j'' + \frac{(n-1)(n-3)}{4r^2}\Psi_j + (\omega - W_1(r))\Psi_j \right] dr = 0.
\end{aligned}
$$

It follows that on a three dimensional subspace X, $\sup_{f\in X}\langle f, L_{+,0} f\rangle \le 0$. Hence, $L_{+,0}$ has either two negative eigenvalues (a contradiction with $n(L_{+,0}) = 1$), or zero is a double eigenvalue. We can rule out the second eigenfunction at zero (and hence contradiction with the two zero of the function Ψ_0) in a similar manner as in Sect. 5.2.1. Clearly Ψ satisfies $L_{+,0}[\Psi] = 0$ in (r_*, ∞), whence by Lemma 5, $\Psi_0(r) = c_0 e^{-\sqrt{\omega}r} + o(e^{-\sqrt{\omega}r})$ as $r \to \infty$. In particular, $\Psi_0(r) \neq 0$ for all large enough r. Take r_* to be the largest zero of Ψ_0 and define a second eigenfunction, in (r_*, ∞) via the formula

$$\Psi(r) = \Psi_0(r) \int_{r_*}^r \frac{1}{\Psi_0^2(y)} dy.$$

Similar to our arguments before, there is $\tilde{r} \in (r-1, r)$, $\Psi(r) \ge \frac{\Psi_0(r)}{\Psi_0^2(\tilde{r})}$. By the asymptotics for Ψ_0, it follows that

$$\lim_{r\to\infty} \Psi(r) \ge \lim_{r\to\infty} \frac{\Psi_0(r)}{\Psi_0^2(\tilde{r})} = \infty.$$

Thus Ψ is not localized and so no eigenfunction, other than Ψ_0, is localized. Thus, we have reached a contradiction again, which was due to our previous assumption that Ψ_0 has two zeros in $(0, \infty)$. Thus, Ψ_0 has exactly one zero in $(0, \infty)$.

6. Smoothness and Non-degeneracy Properties of the Normalized Waves

We start with the proof of Proposition 3.

6.1. Proof of Proposition 3. Let us first show that there is a convergent subsequence of $\varphi_{\lambda+\delta_j}$. Recall that in the course of the proof of Theorem 1, we have shown there that *each* minimizing sequence has a convergent subsequence (denoted the same), in H^1 sense, to a constrained minimizer, Φ_λ. It remains to show that $\sqrt{\frac{\lambda}{\lambda+\delta_j}}\varphi_{\lambda+\delta_j}$ is minimizing. Clearly,

$$\|\sqrt{\frac{\lambda}{\lambda+\delta_j}}\varphi_{\lambda+\delta_j}\|^2 = \lambda.$$

Also,

$$I[\sqrt{\frac{\lambda}{\lambda+\delta_j}}\varphi_{\lambda+\delta_j}] = I[\varphi_{\lambda+\delta_j}] + O[\delta_j] = m(\lambda+\delta_j) + O(\delta_j) \to m(\lambda),$$

since the function m is continuous. It follows that $\sqrt{\frac{\lambda}{\lambda+\delta_j}}\varphi_{\lambda+\delta_j}$ is minimizing and hence converges to what we call Φ_λ. Clearly,

$$\lim_j \varphi_{\lambda+\delta_j} = \lim_j \sqrt{\frac{\lambda}{\lambda+\delta_j}}\varphi_{\lambda+\delta_j} = \Phi_\lambda,$$

in H^1 sense. From here on, the proof of Proposition 3 follows the scheme of the proof of Theorem 4, except we have a discrete sequence δ_j, instead of a continuous variable δ, as it approaches zero.

6.2. On the independence of ω_λ on the minimizer, its continuity and $m \in C^1(a, b)$. First, we note that while ω_λ might potentially depend on the minimizer, $m(\lambda)$ certainly does not. On the other hand, it was already established that $m'(\lambda) = -\frac{\omega_\lambda}{2}$, whenever the derivative exists. Thus, on the full measure subset of $\mathbf{R}_+$, $\mathscr{A} := \{\lambda > 0 : m'(\lambda) \text{ exists}\}$, ω_λ is independent on the minimizers, in the sense described in the statement of Theorem 4. Clearly $\mathscr{A}$ is a dense set as well. Recall the formula (1.10), where we can think of the integrand ω_λ as being only defined over $\mathscr{A}$, and hence independent on the minimizers. *If we are able to show now that the function $\omega|_{\mathscr{A}\cap(a,b)}$ has a continuous extension over (a, b)*, then we can use (1.10) to conclude that the derivative of m is a continuous function and hence in class $C^1(a, b)$. Hence, we will have a legitimate formula $\omega_\lambda = -2m'(\lambda)$ for all $\lambda \in (a, b)$. In particular, ω_λ would be independent of minimizers as a derivative of m, which is naturally independent on the minimizers, due to its definition. Thus, it remains to establish the continuity of $\omega|_{\mathscr{A}}$. In fact, it is enough to establish the following proposition.

Proposition 7. *Suppose that* $\lim_{\delta\to 0+}\|\varphi_{\lambda+\delta}-\Phi\|_{L^2}=0$. *Then, for each* $r: 0<r<\infty$, $n=2$ *or* $0<r<\frac{2}{n-2}$, $n\geq 3$, *we have that*

$$\lim_{\delta\to 0+}\int_{\mathbf{R}^n}\varphi_{\lambda+\delta}^{2+2r}dx=\int_{\mathbf{R}^n}\Phi^{2+2r}dx. \tag{6.1}$$

Let us show how to obtain the continuity of $\lambda\to\omega(\lambda)$ under the assumption (1.12). Supposing that (1.12) holds, we have from the proof of Proposition 3 $\lim_{\delta\to 0}\|\varphi_{\lambda+\delta}-\Phi\|_{H^1\cap L^{2q_L+2}}=0$–indeed, for every sequence $\delta_j\to 0+$, we will be able to take a subsequence δ_{j_k}, so that $\varphi_{\lambda+\delta_{j_k}}$ converges to Φ in $H^1\cap L^{2q_L+2}$, which implies exactly that $\lim_{\delta\to 0}\|\varphi_{\lambda+\delta}-\Phi\|_{H^1\cap L^{2q_L+2}}=0$.

Now, the formula (3.6) for $\omega_{\lambda,\Phi}$ represents it as a linear combination of $\|\nabla\Phi\|$ and various L^{p_k}, L^{q_j} norms. The convergence of the L^q norms is guaranteed already, whereas Proposition 7 (more specifically (6.1)) provides the convergence of the L^p norms. With that, supposing that for any $\{\lambda+\delta_j\}_j\subset(a,b)$, we will have proven $\lim_j\omega_{\lambda+\delta_j}=\lim_j\omega_{\lambda+\delta_j,\varphi_{\lambda+\delta_j}}=\omega_{\lambda,\Phi}$. Thus, the function $\omega|_{\mathscr{A}\cap(a,b)}$ can be extended as a continuous function on (a,b).

Proof. We have already shown that $\lim_{\delta\to 0+}\|\varphi_{\lambda+\delta}-\Phi\|_{L^2}=0$ implies $\lim_{\delta\to 0}\|\varphi_{\lambda+\delta}-\Phi\|_{H^1\cap L^{2q_L+2}}=0$. The formula (6.1) is a consequence of the Sobolev embedding $H^1(\mathbf{R}^n)\hookrightarrow L^{2+2r}(\mathbf{R}^n)$, valid for exactly the range of r specified in the statement. □

The next order of business is the concavity of m. Note that the concavity property is independent on the assumption (1.12).

6.3. The function $\lambda\to m(\lambda)$ *is concave down.* Our starting point is the inequality (3.8) established earlier. Taking into account (3.9), it reads

$$m(\lambda+2\epsilon\langle\varphi_\lambda,h\rangle+\epsilon^2\|h\|^2)\leq m(\lambda)-\epsilon\omega_\lambda\langle\varphi_\lambda,h\rangle+\frac{\epsilon^2}{2}\langle(\mathscr{L}_+-\omega_\lambda)h,h\rangle+O(\epsilon^3). \tag{6.2}$$

Writing the same inequality with $\epsilon\to-\epsilon$ and adding the two yields

$$\begin{aligned}&m(\lambda+2\epsilon\langle\varphi_\lambda,h\rangle+\epsilon^2\|h\|^2)+m(\lambda-2\epsilon\langle\varphi_\lambda,h\rangle+\epsilon^2\|h\|^2)\\&\leq 2m(\lambda)+\epsilon^2\langle(\mathscr{L}_+-\omega_\lambda)h,h\rangle+O(\epsilon^3).\end{aligned} \tag{6.3}$$

This inequality is valid for all h, but we wish to apply it for the eigenfunction, corresponding to the negative eigenvalue for $\mathscr{L}_+$. Recall that according to Theorem 1 and Theorem 3, $\mathscr{L}_+$ has exactly one (simple) negative eigenvalue, say $-\sigma_\lambda^2$, with a normalized eigenfunction $\chi_\lambda:\|\chi_\lambda\|=1$.

We note that $\langle\chi_\lambda,\varphi_\lambda\rangle\neq 0$, since otherwise, we will get a contradiction with the property $\mathscr{L}_+|_{\{\varphi_\lambda\}^\perp}\geq 0$. Take $h:=\frac{\chi_\lambda}{2\langle\chi_\lambda,\varphi_\lambda\rangle}$. Applying (6.3), we obtain

$$\begin{aligned}&m(\lambda+\epsilon+\epsilon^2\|h\|^2)+m(\lambda-\epsilon+\epsilon^2\|h\|^2)-2m(\lambda)\\&\leq-\epsilon^2\omega(\lambda)\|h\|^2-\frac{\epsilon^2\sigma_\lambda^2}{4\langle\chi_\lambda,\varphi_\lambda\rangle^2}+O(\epsilon^3).\end{aligned} \tag{6.4}$$

We have by (1.10)

$$m(\lambda \pm \epsilon + \epsilon^2\|h\|^2) - m(\lambda \pm \epsilon) = \int_{\lambda\pm\epsilon}^{\lambda\pm\epsilon+\epsilon^2\|h\|^2} m'(z)dz = -\frac{1}{2}\int_{\lambda\pm\epsilon}^{\lambda\pm\epsilon+\epsilon^2\|h\|^2} \omega(z)dz$$

whence by the uniform continuity of $\omega(\lambda)$ on (a, b),

$$m(\lambda \pm \epsilon + \epsilon^2\|h\|^2) - m(\lambda \pm \epsilon) + \frac{\epsilon^2\|h\|^2}{2}\omega(\lambda)$$
$$= -\frac{1}{2}\int_{\lambda\pm\epsilon}^{\lambda\pm\epsilon+\epsilon^2\|h\|^2} [\omega(z) - \omega(\lambda)]dz = o_{uniform}(\lambda, \epsilon^2),$$

meaning $\lim_{\epsilon\to 0}\sup_{\lambda\in(a,b)} \frac{o_{uniform}(\lambda,\epsilon^2)}{\epsilon^2} = 0$. Thus, applying this in (6.4), we obtain for all $\epsilon \neq 0$,

$$m(\lambda + \epsilon) + m(\lambda - \epsilon) - 2m(\lambda) \leq -\frac{\epsilon^2\sigma_\lambda^2}{4\langle\chi_\lambda, \varphi_\lambda\rangle^2} + o_{uniform}(\lambda, \epsilon^2). \tag{6.5}$$

It follows that

$$\limsup_{\epsilon\to 0}\sup_{\lambda\in(a,b)} \frac{m(\lambda + \epsilon) + m(\lambda - \epsilon) - 2m(\lambda)}{\epsilon^2} \leq -\inf_{\lambda\in(a,b)} \frac{\sigma_\lambda^2}{4\langle\chi_\lambda, \varphi_\lambda\rangle^2} \leq 0. \tag{6.6}$$

We now finish with the following Lemma.

Lemma 6. *Let $f : (a, b) \to \mathbf{R}$ be a continuous function that satisfies*

$$\limsup_{\epsilon\to 0}\sup_{\lambda\in(a,b)} \frac{f(\lambda + \epsilon) + f(\lambda - \epsilon) - 2f(\lambda)}{\epsilon^2} \leq 0.$$

Then, f is concave down on (a, b).

We postpone the proof of Lemma 6 to the Appendix. Based on the lemma, we conclude that the function m is concave down. As such, m is twice differentiable a.e.. In fact, based on (6.5), we have that $m''(\lambda) \leq -\frac{\sigma_\lambda^2}{4\langle\chi_\lambda,\varphi_\lambda\rangle^2}$ for almost all λ. Thus, for all points of differentiability of ω (which is a.e. in λ),

$$\omega'(\lambda) = -2m''(\lambda) \geq \frac{\sigma_\lambda^2}{2\langle\chi_\lambda, \varphi_\lambda\rangle^2} > 0. \tag{6.7}$$

6.4. The weak non-degeneracy for φ_λ. In this section, we establish that under the assumptions in Theorem 4, we have that $\varphi_\lambda \perp Ker[\mathscr{L}_+]$. In view of Theorem 3, this is something to worry about only in case where the (strong) non-degeneracy does not hold, namely when $Ker[\mathscr{L}_+] = span\{\partial_1\varphi_\lambda, \ldots, \partial_n\varphi_\lambda, \Psi_0\}$. Indeed, we trivially have that $\varphi_\lambda \perp span\{\partial_1\varphi_\lambda, \ldots, \partial_n\varphi_\lambda\}$. Thus, we only need to show that $\varphi_\lambda \perp \Psi_0$, (if such a function exists in the first place!).

To this end, starting with the elliptic problem (1.2), which φ_λ satisfies, with $\omega = \omega_\lambda$. Let $\lambda \in (a, b)$ be a point of differentiability [15] for $\omega(\lambda)$. Let $\delta > 0$ be so small that

[15] Which applies to a.e. point.

$\lambda + \delta \in (a, b)$. We will write the equations for φ_λ and $\varphi_{\lambda+\delta}$ respectively and eventually, we will take their difference. In doing so, it is convenient to introduce the notation

$$\varphi_{\lambda+\delta} = \varphi_\lambda + \delta \frac{\varphi_{\lambda+\delta} - \varphi_\lambda}{\delta} =: \varphi_\lambda + \delta Z_\delta, \tag{6.8}$$

and to prepare a few calculations. First, the key assumption (1.12) reads $\lim_{\delta\to 0} \delta \|Z_\delta\|_{L^2} = 0$. Since the functions $\varphi_\lambda, \varphi_{\lambda+\delta}$ are bounded, we have that for each $r \in (2, \infty)$,

$$\delta \|Z_\delta\|_{L^r(\mathbf{R}^n)} \le (\delta \|Z_\delta\|_{L^\infty})^{1-\frac{2}{r}} (\delta \|Z_\delta\|_{L^2})^{\frac{2}{r}} \to 0,$$

as $\delta \to 0+$. Next, for each power $p > 0$, we use the first order expansion

$$\varphi_{\lambda+\delta}^{2p+1} = \varphi_\lambda^{2p+1} + \delta Z_\delta (2p+1) \varphi_\lambda^{2p} + E_{\delta,\lambda;p},$$

where the error term satisfies

$$|E_{\delta,\lambda,p}(x)| \le C \begin{cases} \varphi_\lambda^{2p-1} (\delta Z_\delta)^2 & \delta |Z_\delta(x)| \le \frac{\varphi_\lambda(x)}{10} \\ (\delta Z_\delta)^{2p+1} & \delta |Z_\delta(x)| \ge \frac{\varphi_\lambda(x)}{10} \end{cases}. \tag{6.9}$$

It may appear that there are terms with *exponential growth* in the spatial variables, such as φ_λ^{2p-1}, when $p < \frac{1}{2}$ (recall that p is generally small in our assumptions). This turns out not to be the case. As we know while $\varphi_\lambda \sim e^{-\sqrt{\omega_\lambda}|x|}$, $|Z_\delta| \le e^{-\min(\sqrt{\omega_\lambda}, \sqrt{\omega_{\lambda+\delta}})|x|}$. So, for example $E_{\delta,\lambda,p}$ has decay rate

$$|E_{\delta,\lambda,p}| \le C e^{-[2(\min \sqrt{\omega_\lambda}, \sqrt{\omega_{\lambda+\delta}}) + (2p-1)\sqrt{\omega_\lambda}]|x|},$$

or about $e^{-(2p+1)\sqrt{\omega_\lambda}|x|}$, since $\delta << 1$ and λ is a point of continuity for ω_λ.

Plugging in the formula (6.8) in (1.2) and taking differences and dividing by δ, we obtain

$$\mathscr{L}_+[Z_\delta] + \frac{\omega_{\lambda+\delta} - \omega_\lambda}{\delta} (\varphi_\lambda + \delta Z_\delta) - \delta^{-1} E_{\delta,\lambda} = 0, \tag{6.10}$$

where $E_{\delta,\lambda} = \sum_{k=1}^K a_k E_{\delta,\lambda,p_k} - \sum_{l=1}^L b_l E_{\delta,\lambda,q_l}$.

We now take a decomposition of Z_δ across the spectrum of $\mathscr{L}_+$. Since Z_δ is radial, the only non-trivial projection onto $Ker[\mathscr{L}_+]$ is potentially only over Ψ_0, so we have

$$Z_\delta = \langle Z_\delta, \Psi_0 \rangle \Psi_0 + z_\delta =: a(\delta) \Psi_0 + z_\delta,$$

where $z_\delta \perp Ker \mathscr{L}_+$, so in particular $z_\delta \perp \Psi_0$. Note that since $\delta^2 a^2(\delta) + \delta^2 \|z_\delta\|^2 = \delta^2 \|Z_\delta\|_{L^2}^2 \to 0$, it follows that $\lim_{\delta\to 0+} \delta a(\delta) = 0$ and $\lim_{\delta\to 0+} \delta \|z_\delta\| = 0$. In view of that and our earlier arguments, it follows that for each $r \in (2, \infty)$, $\lim_{\delta\to 0+} \delta \|z_\delta\|_{L^r} = 0$ as well. In addition, the exponential bounds for Z_δ and Ψ_0 carry over to z_δ. We collect the estimates for $a(\delta)$, z_δ in the following

$$\delta |z_\delta(x)| \le C e^{-\sqrt{\min(\omega_\lambda, \omega_{\lambda+\delta})}|x|}; \ \lim_{\delta\to 0+} \delta \|z_\delta\|_{L^r} = 0, 2 \le r < \infty; \ \lim_{\delta\to 0+} \delta a(\delta) = 0, \tag{6.11}$$

where C is independent on $\delta > 0$.

Since $\mathscr{L}_+[Z_\delta] = \mathscr{L}_+[z_\delta]$, we have from (6.10)

$$z_\delta = \mathscr{L}_+^{-1} P_{\{\Psi_0\}^\perp}[-\frac{\omega_{\lambda+\delta} - \omega_\lambda}{\delta}(\varphi_\lambda + \delta Z_\delta) + \delta^{-1} E_{\delta,\lambda}]. \tag{6.12}$$

Since $D(\mathscr{L}_+) = H^2(\mathbf{R}^n)$, $\mathscr{L}_+^{-1} P_{\{\Psi_0\}^\perp} : L^2 \to H^2(\mathbf{R}^n)$ and we obtain the bound

$$\|z_\delta\|_{H^2} \leq C \left(\frac{|\omega_{\lambda+\delta} - \omega_\lambda|}{\delta}(\|\varphi_\lambda\| + \delta\|Z_\delta\|) + \delta^{-1}\|E_{\delta,\lambda}\|_{L^2} \right). \tag{6.13}$$

We now need appropriate estimate for $\delta^{-1}\|E_{\delta,\lambda}\|_{L^2}$.

Lemma 7.

$$\delta^{-1}\|E_{\delta,\lambda}\|_{L^2} \leq o(1)\|z_\delta\|_{H^2} + C\delta a^2(\delta).$$

Proof. In the regime $\delta|Z_\delta| \leq \frac{\varphi_\lambda}{10}$, the function $\delta^{-1}|E_{\delta,\lambda}|$ is estimated as follows

$$\begin{aligned}
\delta^{-1}\|E_{\delta,\lambda}\|_{L^2(\delta|Z_\delta|\leq\frac{\varphi_\lambda}{10})} &\leq \sum_{k=1}^K |a_k| \|\varphi^{2p_k-1}\delta[a^2(\delta)\Psi_0^2 + 2a(\delta)\Psi_0 z_\delta + z_\delta^2]\|_{L^2} \\
&+ \sum_{l=1}^L |b_l| \|\varphi^{2q_l-1}\delta[a^2(\delta)\Psi_0^2 + 2a(\delta)\Psi_0 z_\delta + z_\delta^2]\|_{L^2} \\
&\leq C\delta a^2(\delta) \sup_{r\in p_1,\ldots p_K; q_1,\ldots q_L} \|\Psi_0^2\varphi_\lambda^{2r-1}\|_{L^2} \\
&+ C\delta|a(\delta)| \sup_{r\in p_1,\ldots p_K; q_1,\ldots q_L} \|\Psi_0 z_\delta \varphi_\lambda^{2r-1}\|_{L^2} + C\delta \sup_{r\in p_1,\ldots p_K; q_1,\ldots q_L} \|\varphi_\lambda^{2r-1} z_\delta^2\|_{L^2}.
\end{aligned}$$

Clearly, since $\Psi_0, \varphi_\lambda \sim e^{-\sqrt{\omega_\lambda}|x|} + o(e^{-\sqrt{\omega_\lambda}|x|})$, we have that $\varphi_\lambda^{-1}\Psi_0$ is a bounded function. Hence $\|\Psi_0^2\varphi_\lambda^{2r-1}\|_{L^2} < \infty$, $\|\Psi_0\varphi_\lambda^{2r-1}\|_{L^\infty} < \infty$. For the last term, choose any $\sigma : 2 < \sigma < \infty$, so that there is the Sobolev embedding $H^2(\mathbf{R}^n) \hookrightarrow L^\sigma(\mathbf{R}^n)$, i.e. $\sigma < \frac{2n}{n-2}$. We have

$$\delta\|\varphi_\lambda^{2r-1} z_\delta^2\|_{L^2} \leq \|z_\delta\|_{L^\sigma} \|\varphi_\lambda^{2r-1}\delta z_\delta\|_{L^{\sigma_1}} \leq C\|z_\delta\|_{H^2}\|\varphi_\lambda^{2r-1}\delta z_\delta\|_{L^{\sigma_1}},$$

where $\sigma_1 : \frac{1}{\sigma} + \frac{1}{\sigma_1} = \frac{1}{2}$. Thus,

$$\|\varphi_\lambda^{2r-1}\delta z_\delta\|_{L^{\sigma_1}} \leq \||\delta z_\delta|^{(1-\theta)}\|_{L^{\sigma_1}} \|\varphi_\lambda^{2r-1}|\delta z_\delta|^\theta\|_{L^\infty} \leq C\|\delta z_\delta\|_{L^{\sigma_1(1-\theta)}}^{\frac{1}{1-\theta}} = o(1)$$

where $\theta \in (0, 1)$ is designed so that the L^∞ term is bounded. This is possible, since there is the estimate

$$\varphi_\lambda^{2r-1}|\delta z_\delta|^\theta \leq Ce^{-(2r-1)\sqrt{\omega_\lambda}|x|}e^{-\theta\sqrt{\min(\omega_\lambda,\omega_{\lambda+\delta})}|x|}$$

which can be made exponentially decaying at ∞ (and hence bounded), provided $1 > \theta > 1 - 2r$ and δ is sufficiently small. Combining it all together, in view of (6.11), we have shown the required estimate for $\delta^{-1}\|E_{\delta,\lambda}\|_{L^2(\delta|Z_\delta|\leq\frac{\varphi_\lambda}{10})}$.

The estimate for $\delta^{-1}\|E_{\delta,\lambda}\|_{L^2(\delta|Z_\delta|>\frac{\varphi_\lambda}{10})}$ is in fact simpler. More specifically, note that since $\lim_{\delta\to 0+}\delta a(\delta) = 0$, we have that for all small enough $\delta > 0$, $\delta a(\delta)\Psi_0 << \varphi_\lambda$, whence $\delta|z_\delta| \geq \frac{1}{2}\delta|Z_\delta| \geq \frac{\varphi_\lambda}{10}$.

For every $r \in \{p_1, \dots p_K; q_1, \dots q_L\}$, we estimate by Hölder's with $\sigma : 0 < \sigma - 2 << 1$ and $r_1 : \frac{1}{2} = \frac{1}{\sigma} + \frac{1}{\sigma_1}$

$$\|z_\delta (\delta z_\delta)^{2r}\|_{L^2} \le \|z_\delta\|_{L^\sigma} (\delta \|z_\delta\|_{L^{2\sigma_1 r}})^{2r}.$$

We now select σ so close to 2 (and consequently σ_1 can be made as big as we wish), so that $2\sigma_1 r > 2$. As a consequence, $\|z_\delta\|_{L^\sigma} (\delta \|z_\delta\|_{L^{2\sigma_1 r}})^{2r} = o(1)\|z_\delta\|_{H^2(\mathbf{R}^n)}$, according to (6.11). $\square$

Going back to (6.13), we see that for all small enough δ, and taking into account that λ is a point of differentiability for ω_λ (and hence $\lim_{\delta \to 0+} \frac{\omega_{\lambda+\delta} - \omega_\lambda}{\delta} = \omega'_\lambda$) there is the bound

$$\|z_\delta\|_{H^2} \le C_\lambda (|\omega'(\lambda)| + o(1)\|z_\delta\|_{H^2} + \delta a^2(\delta)).$$

Since we can hide $o(1)\|z_\delta\|_{H^2}$ behind the left hand side, we arrive at the bound, in a schematic form,

$$\|z_\delta\|_{H^2} \le C\delta a^2(\delta) + O(1) = o(1)|a(\delta)| + O(1), \tag{6.14}$$

in view of $\lim_{\delta \to 0+} \delta a(\delta) = 0$. We now show that this by itself implies the weak non-degeneracy of φ_λ. Compute

$$\delta^2 \|Z_\delta\|^2 = \langle \varphi_{\lambda+\delta} - \varphi_\lambda, \varphi_{\lambda+\delta} - \varphi_\lambda \rangle = 2\lambda + \delta - 2\langle \varphi_{\lambda+\delta}, \varphi_\lambda \rangle = \delta(1 - 2\langle Z_\delta, \varphi_\lambda \rangle). \tag{6.15}$$

On the other hand,

$$\delta \|Z_\delta\|^2 = \delta a^2(\delta) + \delta \|z_\delta\|^2,$$

while

$$1 - 2\langle Z_\delta, \varphi_\lambda \rangle = 1 - 2a(\delta)\langle \Psi_0, \varphi_\lambda \rangle - 2\langle z_\delta, \varphi_\lambda \rangle.$$

It follows that

$$2a(\delta)\langle \Psi_0, \varphi_\lambda \rangle = 1 - 2\langle z_\delta, \varphi_\lambda \rangle - \delta a^2(\delta) - \delta \|z_\delta\|^2. \tag{6.16}$$

From (6.14), $|\langle z_\delta, \varphi_\lambda \rangle| \le C_\lambda \|z_\delta\| \le C_\lambda (o(\delta)|a(\delta)| + O(1))$ and $\delta \|z_\delta\|^2 \le o(\delta)|a(\delta)| + O(\delta)$. Hence,

$$2a(\delta)\langle \Psi_0, \varphi_\lambda \rangle = o(\delta)|a(\delta)| + O(1).$$

So, if it happens that $\langle \Psi_0, \varphi_\lambda \rangle \ne 0$ (i.e. we assume weak degeneracy for a contradiction), we must have $a(\delta) = O(1)$. In that case, take a dot product of (6.10)with Ψ_0, so that $\langle \mathscr{L}_+ z_\delta, \Psi_0 \rangle = 0$. We have

$$\left| \frac{\omega_{\lambda+\delta} - \omega_\lambda}{\delta} \right| |\langle \Psi_0, \varphi_\lambda \rangle| \le \left| \frac{\omega_{\lambda+\delta} - \omega_\lambda}{\delta} \right| \delta |\langle Z_\delta, \Psi_0 \rangle| + \delta^{-1} |\langle E_{\delta,\lambda}, \Psi_0 \rangle|.$$

Note that the right hand side is $o(1)$, if $a(\delta) = O(1)$. On the other hand, this is contradiction, since

$$\lim_{\delta \to 0+} \left| \frac{\omega_{\lambda+\delta} - \omega_\lambda}{\delta} \right| |\langle \Psi_0, \varphi_\lambda \rangle| = |\omega'(\lambda)||\langle \Psi_0, \varphi_\lambda \rangle| > 0,$$

according to (6.7). This leads us to the conclusion that $\langle \Psi_0, \varphi_\lambda \rangle = 0$, which is the weak non-degeneracy of φ_λ. Let us record, for future reference, the identity that follows from (6.15), in view of the fact that $\langle \Psi_0, \varphi_\lambda \rangle = 0$,

$$2\langle z_\delta, \varphi_\lambda \rangle = 1 - \delta a^2(\delta) - \delta \|z_\delta\|^2. \tag{6.17}$$

6.5. On the differentiability of the map $\lambda \to \varphi_\lambda$ in the non-degenerate case. In this section, we assume that the wave φ_λ is non-degenerate, that is $Ker[\mathscr{L}_+] = span\{\partial_1\varphi_\lambda, \ldots, \partial_n\varphi_\lambda\}$. Under these assumptions for the kernel, we can essentially run the same argument as in the previous section, by assuming $a(\delta) = 0$ or equivalently $Z_\delta = z_\delta$. In particular, Lemma 7 applies to yield

$$\delta^{-1}\|E_{\delta,\lambda}\|_{L^2} \leq o(1)\|Z_\delta\|_{H^2}. \tag{6.18}$$

From (6.13), combined with (6.18), we obtain

$$\|Z_\delta\|_{H^2} \leq C|\omega'(\lambda)|\|\varphi_\lambda\| + o(1)\|Z_\delta\|_{H^2} + o(1).$$

All in all, it follows that $\limsup_{\delta\to 0}\|Z_\delta\|_{H^2} \leq C|\omega'(\lambda)|\|\varphi_\lambda\| < \infty$. Using this information, we can actually take limits as $\delta \to 0$ in (6.10). Indeed, applying $\mathscr{L}_+^{-1}$ to it[16]

$$Z_\delta + \omega'(\lambda)\mathscr{L}_+^{-1}\varphi_\lambda + o_{L^2}(1) + O(\delta\|Z_\delta\|_{H^2}) = 0.$$

Thus,

$$\lim_{\delta\to 0}\|Z_\delta + \omega'(\lambda)\mathscr{L}_+^{-1}\varphi_\lambda\|_{L^2} = 0.$$

This means that the function $\varphi : (a, b) \to L^2(\mathbf{R}^n)$ is differentiable, at least at the points of differentiability of ω. In fact,

$$\partial_\lambda\varphi_\lambda = -\omega'(\lambda)\mathscr{L}_+^{-1}\varphi_\lambda.$$

Finally,

$$\langle\mathscr{L}_+^{-1}\varphi_\lambda, \varphi_\lambda\rangle = -\frac{1}{\omega'(\lambda)}\langle\partial_\lambda\varphi_\lambda, \varphi_\lambda\rangle = -\frac{1}{2\omega'(\lambda)}\partial_\lambda\|\varphi_\lambda\|^2 = -\frac{1}{2\omega'(\lambda)} < 0.$$

6.6. Differentiability of $\lambda \to \varphi_\lambda$ in the weakly non-degenerate case. We have already established the weak non-degeneracy of φ_λ, when $\omega'(\lambda)$ exists. For $\delta > 0$, we have the identity

$$\delta^{-1}\|\varphi_{\lambda+\delta} - \varphi_\lambda\|^2 = \delta\|Z_\delta\|^2 = \delta a^2(\delta) + \delta\|z_\delta\|^2,$$

whence the assumption (1.13) implies that $\lim_{\delta\to 0}\delta a^2(\delta) = 0 = \lim_{\delta\to 0}\delta\|z_\delta\|^2$. This simplifies matters quite a bit–by combining (6.13) and the estimate in Lemma 7, we obtain

$$\|z_\delta\|_{H^2} \leq C|\omega'(\lambda)| + o(1) + C\delta(|a(\delta)| + \|z_\delta\|_{L^2}) + C\delta a^2(\delta) + o(1)\|z_\delta\|_{L^2}.$$

Thus, $\|z_\delta\|_{H^2} = O(1)$. In particular by Lemma 7, $\delta^{-1}E_{\delta,\lambda} = o(1)$. We now easily obtain, by taking limit as $\delta \to 0$ in (6.12),

$$\lim_{\delta\to 0}\|z_\delta + \omega'(\lambda)\mathscr{L}_+^{-1}[\varphi_\lambda]\| = 0.$$

[16] This is justified since all the terms appearing in (6.10) are radial and hence orthogonal to $Ker[\mathscr{L}_+]$ by the non-degeneracy assumption.

In particular, the ($L^2(\mathbf{R}^n)$ valued) function $\lambda \to \varphi_\lambda$ is differentiable, and

$$\partial_\lambda \varphi_\lambda = -\omega'(\lambda)\mathscr{L}_+^{-1}[\varphi_\lambda].$$

Also,

$$\lim_{\delta\to 0}\langle z_\delta, \varphi_\lambda\rangle = -\omega'(\lambda)\langle \mathscr{L}_+^{-1}[\varphi_\lambda], \varphi_\lambda\rangle$$

while by virtue of (6.16) (recall $\langle \Psi_0, \varphi_\lambda\rangle = 0$), we have $\lim_{\delta\to 0}\langle z_\delta, \varphi_\lambda\rangle = \frac{1}{2}$.

6.7. Non-degeneracy of φ_λ: Proof of Proposition 4. According to Theorem 3, we only need to rule out the existence of a radial eigenfunction Ψ_0 in $Ker[\mathscr{L}_+]$, which vanishes at exactly one point, say $r_* \in (0, \infty)$.

Recall $\varphi \perp Ker[\mathscr{L}_+]$. A direct inspection establishes the well-known identity

$$\mathscr{L}_+[\sum_{j=1}^n x_j \partial_j \varphi] = -2\Delta\varphi. \tag{6.19}$$

This shows that $\Delta\varphi \perp Ker[\mathscr{L}_+]$ as well. In addition,

$$\mathscr{L}_+[\varphi] = -2F'(\varphi^2)\varphi^3,$$

while from the profile equation $F(\varphi^2)\varphi = -\Delta\varphi + \omega\varphi \perp Ker[\mathscr{L}_+]$. It follows that $F'(\varphi^2)\varphi^3, F(\varphi^2)\varphi \perp Ker[\mathscr{L}_+]$.

We will show that in the three examples, (1.14), (1.15) and (1.16), listed in Proposition 4, this allows us to rule out Ψ_0. Recall that $\Psi_0(r) > 0, r \in (0, r_*)$, $\Psi_0(r) < 0, r \in (r_*, \infty)$. Assume (1.14). Choose $c_0 > 0$, so that $c_0 \sum_{k=1}^K a_k \varphi_\lambda^{2p_k}(r_*) = 1$. Consider the function

$$h(r) := c_0 F(\varphi^2)\varphi - \varphi.$$

On one hand, $h \perp Ker[\mathscr{L}_+]$, as linear combination of two functions in $Ker[\mathscr{L}_+]^\perp$. On the other hand, since φ is bell-shaped[17] for $r \in (0, r_*)$,

$$h(r) = \varphi(r)(c_0 \sum_{k=1}^K a_k \varphi^{2p_k}(r) - 1) > \varphi(r)(c_0 \sum_{k=1}^K a_k \varphi^{2p_k}(r_*) - 1) = 0.$$

For $r \in (r_*, \infty)$, we have the opposite inequality, since

$$h(r) = \varphi(r)(c_0 \sum_{k=1}^K a_k \varphi^{2p_k}(r) - 1) < \varphi(r)(c_0 \sum_{k=1}^K a_k \varphi^{2p_k}(r_*) - 1) = 0.$$

Clearly, $\langle h, \Psi_0\rangle = \int_0^{r_*} h(r)\Psi_0(r) r^{n-1} dr + \int_{r_*}^\infty h(r)\Psi_0(r) r^{n-1} dr > 0$, in contradiction with $h \perp Ker[\mathscr{L}_+]$ and in particular $h \perp \Psi_0$.

[17] And so strictly decreasing in $(0, \infty)$).

The proof in the cases of (1.15) and (1.16) follows the same logic, but it is slightly more involved. The conditions $F'(\varphi^2)\varphi^3$, $F(\varphi^2)\varphi \perp Ker[\mathscr{L}_+]$ read

$$F(\varphi^2)\varphi = \sum_{k=1}^{K} a_k \varphi^{2p_k+1} - b\varphi^{2q+1} \perp Ker[\mathscr{L}_+]$$

$$F'(\varphi^2)\varphi^3 = \sum_{k=1}^{K} (2p_k+1) a_k \varphi^{2p_k+1} - b(2q+1)\varphi^{2q+1} \perp Ker[\mathscr{L}_+].$$

Taking a linear combination $(2q+1)F(\varphi^2)\varphi - F'(\varphi^2)\varphi^3$, we eliminate the term φ^{2q+1} and we obtain yet another element of $Ker[\mathscr{L}_+]^\perp$, namely $\sum_{k=1}^K 2(q-p_k)a_k\varphi^{2p_k+1}$. Clearly, in the cases when $q > p_K$ or $q < p_1$, we have an element of $Ker[\mathscr{L}_+]^\perp$ in the form

$$\sum_{k=1}^{K} \tilde{a}_k \varphi_\lambda^{2p_k+1}, \tilde{a}_k > 0,$$

which can be used to produce a contradiction with the existence of Ψ_0, the same way as we did under the assumption (1.14).

7. Proof of Theorems 5, 6, 7

We first check (1.9) for the case of a purely focusing nonlinearity (1.14).

7.1. Verification of (1.9) for focussing non-linearities. Write as before

$$m(\lambda) = \inf_{\|u\|^2=\lambda} I[u] = \lambda \inf_{\|v\|^2=1} \left[\int |\nabla v|^2 - \lambda^{-1} \int G(\lambda |v(x)|^2) dx \right]$$
$$= \lambda \inf_{\|v\|^2=1} \left[\int |\nabla v|^2 - \sum_k \frac{a_k \lambda^{p_k}}{p_k+1} \int |v|^{2+2p_k} dx \right] =: \lambda M(\lambda).$$

Clearly, the function $\lambda \to M(\lambda)$ is decreasing. In addition $M(\lambda) = \frac{m(\lambda)}{\lambda} < 0$, since $m(\lambda) < 0$. So, for $0 < \lambda_1 < \lambda_2$, we have

$$m(\lambda_1) = \lambda_1 M(\lambda_1) > \lambda_2 M(\lambda_1) > \lambda_2 M(\lambda_2) = m(\lambda_2).$$

Thus, (1.9) holds true.

We now turn our attention to the stability claims in Theorems 5, 6, 7 as the others were explained in details immediately after the statements. The spectral stability of the waves is a consequence of the formulas $\langle \mathscr{L}_+^{-1}\varphi_\lambda, \varphi_\lambda \rangle = -\frac{1}{2\omega'(\lambda)} < 0$, the fact that $n(\mathscr{L}_+) = 1$, $n(\mathscr{L}_-) = 0$ and the index theory, introduced in Sect. 2.3, more specifically Proposition 6.

Orbital stability follows from the end of the same proposition, once we take into account the non-degeneracy of the waves φ_λ, the local invertibility of the map $\lambda \to \omega_\lambda$ and the smoothness of $\lambda \to \varphi_\lambda$, stated in Theorem 4. Unfortunately, there is no abstract result providing orbital stability for the Zakharov–Kuznetsov model, due to the failure of a key assumption in Theorem 5.2.11 in [17], namely the invertibility of $\mathcal{J} = \partial_{x_1}$ does not hold.

In the section below however, we provide a direct proof of this fact, by adapting slightly the Benjamin's method, [1]. Similar, albeit slightly more elaborated method can be applied to produce a direct proof of the orbital stability of the NLS equation, instead of referring to Theorem 5.2.11, [17], but we will not do so here.

7.2. Orbital stability for the Zakharov–Kuznetsov models. The local well-posedness theory for the ZK, (1.3) follows by classical semigroup theory in the energy space $H^1(\mathbf{R}^n)$, under the assumptions for L^2 sub-critical powers, as considered herein. This is then upgraded to global well-posedness theory in $H^1(\mathbf{R}^n)$, thanks to the conservation laws

$$\mathcal{H}[u] = \int_{\mathbf{R}^n} |\nabla u(x)|^2 - \int_{\mathbf{R}^n} G(|u(x)|^2)dx, \mathcal{P}(u) = \int_{\mathbf{R}^n} |u(x)|^2 dx.$$

Thus, we are reduced to showing the following proposition.

Proposition 8. *Let φ be a smooth wave, satisfying*

$$-\Delta\varphi + \omega\varphi - F(\varphi^2)\varphi = 0 \tag{7.1}$$

and the following assumptions:

- *The operator $\mathcal{L}_+ = -\Delta + \omega - F(\varphi^2) - 2F'(\varphi^2)\varphi^2$ satisfies $\mathcal{L}_+|_{\{\varphi\}^\perp} \geq 0$.*
- *φ is non-degenerate, i.e. $Ker[\mathcal{L}_+] = span[\partial_1\varphi, \dots, \partial_n\varphi]$.*

Then, φ is orbitally stable in the sense of Definition 2.

Remark: Clearly, the proposition above applies to the limit waves $\varphi = f_{\omega_\lambda}$ described in Theorems 5, 6, 7 as they were established to enjoy the desired properties described above. Note also that the method that we present does not require the differentiability[18] of ω.

Proof. The proof proceeds by a contradiction argument. Assuming that orbital stability does not hold, there is a $\epsilon_0 > 0$ and a sequence $u_l \to \varphi$ in H^1, so that the corresponding solutions

$$\sup_{0\leq t<\infty} \inf_{r\in\mathbf{R}^n} \|u_l(t,\cdot) - \varphi(\cdot - r)\|_{H^1} \geq \epsilon_0. \tag{7.2}$$

For $0 < \epsilon << 1$, consider a neighborhood $\mathcal{U}_\varepsilon$ in the set of all real-valued functions, which are closed to translations of φ_λ

$$\mathcal{U}_\varepsilon = \{u \in H^1_{real}(\mathbf{R}^n) \quad : \quad \inf_{r\in\mathbf{R}^n} ||u - \varphi(\cdot - r)||_{H^1} < \varepsilon\}.$$

By Lemma 3.2, [11], there exists $\epsilon_0(\varphi) > 0$, so that for all $0 < \epsilon < \epsilon_0(\varphi)$, there is a unique C^1 map $\beta : \mathcal{U}_\varepsilon \mapsto \mathbb{R}$, such that

$$\langle u(\cdot + \beta(u)), \partial_j\varphi\rangle = 0, j = 1, \dots, n. \tag{7.3}$$

[18] Which is on the other hand used already in the proof of the non-degeneracy of the waves.

Note that $\beta(\varphi) = 0$. Since we need $\epsilon < \min(\epsilon_0(\varphi), \epsilon_0)$, take the new ϵ_0 to be the minimum of the $\epsilon_0, \epsilon_0(\varphi)$. Introduce the total energy functional $\mathscr{E}(u) = \mathscr{H}(u)+\omega\mathscr{P}(u)$. In terms of $\mathscr{H}$, $\mathscr{P}$ the profile equation (7.1) reads

$$\mathscr{E}'[\varphi] = \mathscr{H}'(\varphi) + \omega\mathscr{P}'(\varphi) = 0.$$

Let

$$\epsilon_l := |\mathscr{E}(u_l(t)) - \mathscr{E}(\varphi)| + |\mathscr{P}(u_l(t)) - \mathscr{P}(\varphi)|,$$

which is conserved in time. Note that $\lim_l \epsilon_l = 0$, since $\lim_l \|u_l - \varphi\|_{H^1} = 0$.

By the continuity of the solution map and the map β, we have that there exists $t_l > 0$, so that for $t \in (0, t_l)$, $\|u_l(t, \cdot) - \varphi\|_{H^1} < \frac{\epsilon}{2}$ and $\beta(u_l(t))$ is so close to $\beta(\varphi) = 0$, that

$$\|\varphi - \varphi(\cdot - \beta(u_l(t)))\|_{H^1} < \frac{\epsilon}{2}.$$

Consequently,

$$\begin{aligned}\|u_l(t, \cdot + \beta(u_l(t))) - \varphi\|_{H^1} &= \|u_l(t, \cdot) - \varphi(\cdot - \beta(u_l(t)))\|_{H^1}\\ &\le \|u_l(t, \cdot) - \varphi\|_{H^1} + \|\varphi - \varphi(\cdot - \beta(u_l(t)))\|_{H^1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

With that in mind, take

$$T_l^* = \sup\{\tau_0 : \sup_{0<\tau<\tau_0} \|u_l(\tau, \cdot + \beta(u_l)) - \varphi(\cdot)\|_{H^1} < \epsilon\}.$$

The previous calculation shows $T_l^* \ge t_l > 0$. We aim at showing that for all sufficiently small ϵ and for all large enough l, $T_l^* = \infty$, which will provide the sought contradiction with (7.2). We henceforth work with $t \in (0, T_l^*)$. Denote

$$\psi_l(t, \cdot) = u_l(t, \cdot + \beta(u_l)) - \varphi(\cdot) = \mu_l(t)\varphi + \eta_l(t, \cdot), \quad \eta_l \perp \varphi.$$

We have that

$$\mathscr{P}(u_l(t)) = \mathscr{P}(\varphi) + 2\langle\varphi, \mu_l\varphi + \eta_l\rangle + \|\psi_l\|_{L^2}^2 = \mathscr{P}(\varphi) + 2\mu_l\|\varphi\|^2 + \|\psi_l\|_{L^2}^2.$$

It follows that $2\mu_l\|\varphi\|^2 = \mathscr{P}(u_l) - \mathscr{P}(\varphi) - \|\psi_l\|_{L^2}^2$, whence

$$|\mu_l(t)| \le \frac{|\mathscr{P}(u_l) - \mathscr{P}(\varphi)| + \|\psi_l\|_{L^2}^2}{2\|\varphi\|^2} \le C(\epsilon_l + \|\psi_l\|_{L^2}^2). \tag{7.4}$$

But $\mathscr{E}'(\varphi) = 0$. So expansion in Taylor's and various Sobolev embedding estimates yield the formula

$$\begin{aligned}\mathscr{E}(u_l(t)) - \mathscr{E}(\varphi) &= \mathscr{E}(u_l(t, \cdot + \beta(u_l(t)))) - \mathscr{E}(\varphi) = \mathscr{E}(\varphi + \psi_l) - \mathscr{E}(\varphi)\\ &= \frac{1}{2}\langle\mathscr{L}_+\psi_l, \psi_n\rangle + O(\|\psi_l\|_{H^1}^3)\\ &= \frac{1}{2}\langle\mathscr{L}_+\eta_l, \eta_n\rangle + \frac{1}{2}(\mu_l^2\langle\mathscr{L}_+\varphi, \varphi\rangle + 2\mu_l\langle\mathscr{L}_+\varphi, \eta_l\rangle) + O(\|\psi_l\|_{H^1}^3).\end{aligned}$$

By construction, $\eta_l \perp \varphi$. In addition, from (7.3), we have for all $j = 1, 2, \ldots n$,

$$\langle\eta_l, \partial_j\varphi\rangle = \langle u_l(t, \cdot + \beta(u_l(t))) - \varphi - \mu_l\varphi, \partial_j\varphi\rangle = 0.$$

So, it turns out that $\eta_l \perp span\{\varphi, \nabla\varphi\}$. But recall that we have assumed $\mathscr{L}_+|_{\{\varphi\}^\perp} \geq 0$. In addition, by the non-degeneracy assumption, $Ker[\mathscr{L}_+] = span[\nabla\varphi]$. Thus,

$$\mathscr{L}_+|_{span\{\varphi, \nabla\varphi\}^\perp} \geq \kappa > 0.$$

In particular,

$$\langle \mathscr{L}_+\eta_l, \eta_l\rangle \geq \kappa\|\eta_l\|^2_{H^1}. \tag{7.5}$$

Plugging this information into the expression for $\mathscr{E}(u_l) - \mathscr{E}(\varphi) = \mathscr{E}(u_l(t)) - \mathscr{E}(\varphi)$, we arrive at

$$\frac{\kappa}{2}\|\eta_l\|^2_{H^1} \leq C\epsilon_l + C\|\psi_l\|^3_{H^1}. \tag{7.6}$$

By the definition of η_l and (7.4), we have however

$$\|\eta_l\|_{H^1} \geq \|\psi_l - \mu_l\varphi\|_{H^1} \geq \|\psi_l\|_{H^1} - |\mu_l|\|\varphi\|_{H^1} \geq \|\psi_l\|_{H^1} - C(\epsilon_l + \|\psi_l\|^2_{H^1}). \tag{7.7}$$

We now select ϵ so small that $C\epsilon < \min(\frac{1}{100}, \frac{\kappa}{64})$, for any C that appears in the argument.

We claim that for all large enough n, $\|\psi_l(t)\|_{H^1} < \epsilon_l^{\frac{1}{4}}$, for $t \in (0, T_l^*)$. Suppose not–this will then yield a subsequence, denoted the same, so that $\|\psi_l(\tau_l)\|_{H^1} \geq \epsilon_l^{\frac{1}{4}}$ for some $\tau_l \in (0, T_l^*)$. Note that by the definition of T_l^*, we still have $\|\psi_l(\tau_l)\|_{H^1} \leq \epsilon$. From (7.7), we have now, for large enough n,

$$\|\eta_l(\tau_l)\|_{H^1} \geq \frac{1}{2}\|\psi_l(\tau_l)\|_{H^1} - C\epsilon_l \geq \frac{1}{4}\|\psi_l(\tau_l)\|_{H^1} \tag{7.8}$$

since by the choice of ϵ, we have $\|\psi_l(\tau_l)\|_{H^1} - C\|\psi_l(\tau_l)\|^2_{H^1} \geq \frac{1}{2}\|\psi_l(\tau_l)\|_{H^1}$ (since $C\epsilon < \frac{1}{100}$). In addition, $\frac{1}{2}\|\psi_l(\tau_l)\|_{H^1} - C\epsilon_l \geq \frac{1}{4}\|\psi_l(\tau_l)\|_{H^{\frac{\alpha}{2}}}$ since $\epsilon_l^{\frac{1}{4}} >> \epsilon_l$. Using this in (7.6) yields

$$\frac{\kappa}{32}\|\psi_l(\tau_l)\|^2_{H^1} \leq C\epsilon_l + C\|\psi_l(\tau_l)\|^3_{H^1} \leq C\epsilon_l + \frac{\kappa}{64}\|\psi_l(\tau_l)\|^2_{H^1}. \tag{7.9}$$

It follows that

$$\frac{\kappa}{64}\|\psi_l(\tau_l)\|^2_{H^1} \leq C\epsilon_l,$$

which is a contradiction with $\|\psi_l(\tau_l)\|_{H^1} \geq \epsilon_l^{\frac{1}{4}}$ for large l. Thus, for large l, $\|\psi_l(t)\|_{H^1} < \epsilon_l^{\frac{1}{4}}$ for $t \in (0, T_l^*)$. But this exactly means that for all large l, $T_l^* = \infty$, whence we arrive at a contradiction with (7.2). □

Appendix A. Pohozaev

Proposition 9. *Any weak solution* $f \in H^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ *of* (1.2) *satisfy*

$$\|\nabla f_\omega\|^2 + \omega\|f_\omega\|^2 - \int_{\mathbf{R}^n} F(f_\omega^2(x))f_\omega^2(x)dx = 0, \tag{A.1}$$

$$(n-2)\|\nabla f_\omega\|^2 + \omega n\|f_\omega\|^2 - n\int_{\mathbf{R}^n} G(f_\omega^2(x))dx = 0. \tag{A.2}$$

Proof. We first verify it for classical solutions. The relation (A.1) follows by taking dot product of (1.2) with f_ω. For (A.2), take dot product with $\sum_{j=1}^n x_j\partial_j f$. Since,

$$\begin{aligned}\langle -\Delta f, \sum_{j=1}^n x_j\partial_j f\rangle &= \sum_{j=1}^n [\sum_{k\neq j}\int_{\mathbf{R}^n} x_j\partial_k f\partial_{jk} f dx \\ &\quad - \int_{\mathbf{R}^n}\partial_j f\partial_{jj} f x_j dx] = -\frac{n-2}{2}\|\nabla f\|^2. \\ &\quad -\sum_{j=1}^n \int F(f^2(x))f(x)x_j\partial_j f(x)dx \\ &= \frac{n}{2}\int G(f^2)dx, \quad \sum_{j=1}^n\int x_j f(x)\partial_j f(x)dx = -\frac{n}{2}\|f\|^2\end{aligned}$$

we conclude (A.2).

For weak solutions, take dot products with $f\chi(x/N)$ and $\sum_{j=1}^n x_j\partial_j f\chi(|x|/N)$ respectively, where χ is a C_0^∞ function, supported in $(-2, 2)$, so that $\chi(r) = 1 : |r| < 1$. After integration by parts and taking $\lim_{N\to\infty}$ we get again (A.1) and (A.2). □

Appendix B. Concavity Lemma

We prove Lemma 6. Assume that f is not concave. Then, since it is continuous, it is not "concave" with $\theta = \frac{1}{2}$. That is, there is $\lambda_0 \in (a, b)$ and $\epsilon_0, \delta_0 > 0$, so that $\lambda_0 \pm \epsilon_0 \in (a, b)$,

$$f(\lambda_0 + \epsilon_0) + f(\lambda_0 - \epsilon_0) \geq 2f(\lambda_0) + \delta_0.$$

We claim that at least one of the following three inequalities will hold true

$$\begin{aligned} f(\lambda_0 + \epsilon_0) + f(\lambda_0) - 2f(\lambda_0 + \frac{\epsilon_0}{2}) &\geq \frac{\delta_0}{4}, \\ f(\lambda_0 - \epsilon_0) + f(\lambda_0) - 2f(\lambda_0 - \frac{\epsilon_0}{2}) &\geq \frac{\delta_0}{4}, \\ f(\lambda_0 + \frac{\epsilon_0}{2}) + f(\lambda_0 - \frac{\epsilon_0}{2}) - 2f(\lambda_0) &\geq \frac{\delta_0}{4}. \end{aligned}$$

Indeed, assuming all three are false, add the first two to twice the third one. We obtain

$$f(\lambda_0 + \epsilon_0) + f(\lambda_0 - \epsilon_0) - 2f(\lambda_0) < \delta_0,$$

which is a contradiction. Thus, we have shown that inside $(\lambda_0 - \epsilon_0, \lambda_0 + \epsilon_0)$ there is an interval, with half the length, on which f is not concave with $\theta = \frac{1}{2}$. Continuing in this fashion, we construct a sequence of nested intervals $I_j = [\lambda_j - \frac{\epsilon_0}{2^j}, \lambda_j + \frac{\epsilon_0}{2^j}]$, on which there is the inequality

$$f(\lambda_j + \frac{\epsilon_0}{2^j}) + f(\lambda_j - \frac{\epsilon_0}{2^j}) - 2f(\lambda_j) \geq \frac{\delta_0}{2^{2j}}.$$

Since $\lambda_j \to \tilde{\lambda} = \cap_{j=0}^{\infty} I_j \subset (a, b)$, we obtain as a consequence

$$\limsup_j \frac{f(\lambda_j + \frac{\epsilon_0}{2^j}) + f(\lambda_j - \frac{\epsilon_0}{2^j}) - 2f(\lambda_j)}{\left(\frac{\epsilon_0}{2^j}\right)^2} \geq \frac{\delta_0}{\epsilon_0^2} > 0.$$

This is however in contradiction with the assumption in Lemma 6.

Appendix C. The one dimensional case

In this section, we provide an alternative approach to the existence and stability of solitary waves for NLS with general non-linearity, in one spatial dimension. Note that here, we do not necessarily restrict ourselves to normalized waves, but we in fact consider all waves.

The existence and stability of these waves is a known result, but we wanted to present a version here, with explicit assumptions, in order to be able to directly compare with the higher dimensional case, considered herein.

Theorem 8. *Let $F : (0, \infty) \to \mathbf{R}$ be a C^1 function, so that the function $H(z) := \frac{G(z)}{z}$, where $G : G' = F, G(0) = 0$ satisfies*

- *There exist $z_0 = z_0(\omega) : H(z_0) = \omega$, so that $H(z) < \omega$, for $z \in (0, z_0(\omega))$. In addition, z_0 is non-degenerate zero of $H(z) - \omega = 0$, i.e. $H'(z_0(\omega)) \neq 0$.*
- *$|H(z)| \leq C|z|^\delta$ for $z \in (0, z_0(\omega))$ and some $\delta > 0$.*

Then, there exists a bell-shaped solution f_ω, with $f_\omega(0) = \sqrt{z_0(\omega)}$. In addition, the function $\omega \to \int_0^{z_0(\omega)} \frac{1}{\sqrt{\omega - H(z)}} dz > 0$ is differentiable for all $\omega > 0$ and the wave f_ω is orbitally stable[19] *if and only if*

$$\partial_\omega \int_0^{z_0(\omega)} \frac{1}{\sqrt{\omega - H(z)}} dz > 0.$$

Proof. The profile equation is

$$-f'' + \omega f - F(f^2) f = 0.$$

This can be of course integrated once to

$$f'(x) = -\sqrt{\omega f^2(x) - G(f^2(x))}, \quad x \in \mathbf{R}. \tag{C.1}$$

An easy analysis shows that a bell-shaped solution of (C.1) exists with $f(0) = \sqrt{z_0(\omega)}$, $f'(0) = 0$. Note that f_ω must be strictly decreasing. It is easier to work with the new variable $z(x) := f^2(x)$. In it, the Eq. (C.1) becomes

$$z'(x) = -2z(x)\sqrt{\omega - H(z(x))}, \quad x \in \mathbf{R}. \tag{C.2}$$

[19] Outside of the points, where this quantity is zero, which is known to be a delicate issue. However, in all cases where this has been studied in detail, nonlinear instability has been established.

The non-degeneracy condition $H'(z_0(\omega))$ ensures that a solution with $z(0) = z_0(\omega)$ exists, since we have from (C.2) that for every $x > 0$,

$$x = \frac{1}{2}\int_{z(x)}^{z_0(\omega)} \frac{dz}{z\sqrt{\omega - H(z)}},$$

so the last integral needs to be convergent close to $z_0(\omega)$. This is of course not the case, unless $H'(z_0(\omega)) \neq 0$, which we have assumed to be true.

The linearized operators $\mathscr{L}_\pm$, as previously defined are now second order self-adjoint operators, with domain $D(\mathscr{L}_\pm) = H^2(\mathbf{R})$. In addition, $\mathscr{L}_-[f] = 0$ and $f > 0$, whence $\mathscr{L}_- \geq 0$, the zero is a simple eigenvalue and $\mathscr{L}_-|_{\{f\}^\perp} > 0$. By direct differentiation of the profile equation, $\mathscr{L}_+[f'] = 0$. Since f' has an unique zero, at zero, Sturm-Liouville theory applies to imply that the zero is the second smallest (simple) eigenvalue, the smallest one being strictly negative. So, $n(\mathscr{L}_+) = 1$.

The classical stability theory, say Grillakis-Shatah-Strauss applies to imply that the stability of such waves is dictated by the sign of the quantity $\partial_\omega \|f_\omega\|_{L^2}^2$, namely the stability occurs exactly when $\partial_\omega \|f_\omega\|_{L^2}^2 > 0$. Before we proceed with this, let us explicitly compute $\|f_\omega\|_{L^2}^2$. We use the z variable again. We have, by (C.2)

$$\|f_\omega\|^2 = 2\int_0^\infty f^2(x)dx = 2\int_{z_0(\omega)}^0 z\frac{dx}{dz}dz = \int_0^{z_0(\omega)} \frac{1}{\sqrt{\omega - H(z)}}dz.$$

In the last formula, it is not even clear that this is differentiable in ω, due to the (mild) singularity at $z_0(\omega)$. It turns out, after some elementary calculations that this is not an issue and $\omega \to \int_0^{z_0(\omega)} \frac{1}{\sqrt{\omega - H(z)}}dz$ is indeed differentiable in ω. The precise stability condition is exactly $\partial_\omega \int_0^{z_0(\omega)} \frac{1}{\sqrt{\omega - H(z)}}dz > 0$.

For the particular case of a single power non-linearity, $F(z) = z^p$, we have $H(z) = (p+1)^{-1}z^p$ and we obtain $\int_0^{z_0(\omega)} \frac{1}{\sqrt{\omega - H(z)}}dz = const.\omega^{\frac{1}{p}-\frac{1}{2}}$. The stability is then equivalent to $\partial_\omega[\omega^{\frac{1}{p}-\frac{1}{2}}] > 0$ or the familiar $p < 2$. $\square$

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