Journal of Dynamics and Differential Equations (2020) 32:85–99
https://doi.org/10.1007/s10884-018-9713-8

Small Amplitude Traveling Waves in the Full-Dispersion Whitham Equation

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Received: 27 February 2018 / Revised: 3 October 2018 / Published online: 27 October 2018

Abstract
In this article, we provide an alternative way to construct small amplitude traveling waves for general Whitham type equations, in both periodic and whole line contexts. More specifically, Fourier analysis techniques allow us to reformulate the problem to the study of waves that are small and regular perturbations of well-understood ODE's. In addition, rigorous spectral stability of these waves is established.

Keywords Whitham equation · Full-dispersion models · Solitary waves · Nonlocal systems · Water waves · Orbital stability

1 Introduction

The equation,

$$u_t + \mathscr{W} u_x + \partial_x (u^2) = 0, \quad \widehat{\mathscr{W} u}(k) = \sqrt{\frac{\tanh(k)}{k}}\widehat{u}(k) \tag{1}$$

was proposed by Whitham [17] as an alternative model to the ubiquitous Korteweg–de Vries (KdV) approximation ($u_t + u_{xxx} + 2uu_x = 0$) for water waves. In particular, (1) is driven by the non-local operator $\mathscr{W}$, which (modulo some rescalings) gives the "full-dispersion" relation for the corresponding water waves equation. At the rigorous level, (1) is asymptotically as accurate a model for the full water-wave problem as KdV [13], and in fact performs better than KdV when its solutions are compared directly with experimental data [1] or with numerically computed solutions of the water wave problem [15]. It also allows for wave breaking [18], a desirable realistic feature for such models.

Stefanov is partially supported by NSF-DMS under Grant # 1614734. Wright is partially supported by NSF-DMS under Grant # 1511488.

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Let us mention some well-known facts about the dynamics of (1), which will be helpful in the sequel. There are three known *formally* conserved quantities, namely the Hamiltonian, the energy P and the average

$$\mathscr{H}[u] := \frac{1}{2}\langle \mathscr{W} * u, u\rangle + \frac{1}{3}\int u^3(x)dx, \quad \mathscr{P}[u] := \int u^2(x)dx \quad \text{and} \quad I[u] = \int u(x)dx.$$

The question of when this formally conserved quantities are actually conserved for particular solutions is a delicate one, as it is for the Euler equation. It is in fact related to the well-posedness theory for (1), which is unfortunately rather incomplete. The main reason is that wave breaking actually does occur and one needs to account for that aspect of the theory. Nevertheless, in [2], the authors show local well-posedness in $H^{3/2+}$, both in the periodic and whole line contetxts. In [3], it is shown that H^1 solutions conserve $\mathscr{H}[u]$ and $\mathscr{P}[u]$.

In this article we study a generalization of (1). More specifically, we allow for the following sort of "pseudo-differential equations of Whitham type":

$$u_t + (Lu + n(u))_x = 0, \quad u = u(x,t) \in \mathbf{R}, \quad x \in \mathbf{R} \quad \text{and} \quad t \in \mathbf{R}, \tag{2}$$

where $n : \mathbf{R} \to \mathbf{R}$ is purely nonlinear. The operator L is a Fourier multiplier operator with symbol m. That is

$$\widehat{Lf}(k) = m(k)\widehat{f}(k)$$

where $\widehat{f}(k)$ is the Fourier transform of $f(x)$. Precise conditions on m and n will be set forth below, but the prototypical choices will be of course

$$m(k) = \sqrt{\tanh(k)/k} \quad \text{and} \quad n(u) = u^2,$$

which then leads us to the original model (1). The main focus of the current work lies in the existence and the properties of a class of special solutions, namely traveling waves.

More specifically, we make the traveling wave ansatz $u(x,t) = w(x - \nu t)$, where $\nu \in \mathbf{R}$ is an as yet undetermined wave speed. After one integration we arrive at:

$$(\nu - L)w = n(w). \tag{3}$$

The question of existence and the corresponding properties of traveling waves, that is solutions of (3), in either the whole line or periodic context, has been the subject of numerous papers over the last ten years. We mention the papers [4–6], where the question for existence periodic waves is investigated, both rigorously and numerically. Finally, in the *tour de force*, [3], the authors have constructed (through an involved constrained variational with penalization construction), traveling waves for the whole line problem, with speeds slightly bigger than the sonic speed $\nu = 1$. The question of stability of these waves, mostly in the periodic context, was considered recently in [16]. It should be noted that both in the analytical and numerical results discussed herein and elsewhere, it appears that there is some natural barrier for the the wave speeds, $1 < \nu < 1.141\ldots$, which is still not fully understood. Thus, the "slightly supersonic" assumption in these papers appears to be well-warranted. The methods in these papers are varied and rather technical. In some cases, the analysis is supplemented by numerical simulations, which is justified given the lack of precise formulas, even in the classical case (1).

In this article, we take a slightly different point of view. A rescaling of the problem, together with some Fourier analysis reformulates the problem in such a way that the governing equations for the traveling waves are small and regular perturbations of well-understood ordinary differential equations. Then we use an implicit function theorem to prove the existence of solutions when the scaling parameter is small. The main ideas of the method are inspired by the work of Friesecke and Pego [7] and Friesecke and Mikikits-Leitner [8] on traveling waves in Fermi–Pasta–Ulam–Tsingou lattices, whose governing equations are nonlocal in a way similar to those we study here.

1.1 Assumptions and Main Results

We make the following assumption regarding $n(u)$.

Assumption 1 There exists $\delta_* > 0$ such that the nonlinearity $n : (-\delta_*, \delta_*) \to \mathbf{R}$ is $C^{2,1}$ (that is, its second derivative exists and is uniformly Lipschitz continuous) and satisfies

$$n(0) = n'(0) = 0 \quad \text{and} \quad n''(0) > 0.$$

And here is our assumption on the multiplier m, which is a sort of combination of convexity near zero with boundededness for large k:

Assumption 2 The multiplier $m : \mathbf{R} \to \mathbf{R}$ is even and there exists $k_* > 0$ which has the following properties:

- m is $C^{3,1}$ (that is, its third derivative exists and is uniformly Lipschitz continuous) on $[-k_*, k_*]$ and

$$m_2 := \max_{|k| \le k_*} m''(k) < 0. \tag{4}$$

In particular $m''(0) < 0$.

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$$m_1 := \sup_{k \ge k_*} m(k) < m(0). \tag{5}$$

Both Assumptions 1 and 2 are easily verified for the choices which give the full-dispersion Whitham equation (1). An important quantity that will arise in the analysis is

$$\gamma := -\frac{n''(0)}{m''(0)}, \tag{6}$$

which, by Assumptions 1 and 2, is positive.

Here are our main results. Note that our construction provides explicit leading order terms for both for the wave speed and the traveling wave profile.[1]

Theorem 1 *The following hold when Assumptions 1 and 2 are met. There exists $\epsilon_0 > 0$, so that for every $\epsilon \in (0, \epsilon_0)$, there is a traveling wave solution $u(x,t) = \epsilon^2 W_\epsilon(\epsilon(x - \nu_\epsilon t))$ of (2). Moreover, $W_\epsilon \in H^1_{even}(\mathbf{R})$,*

$$\nu_\epsilon = m(0) - \frac{1}{2} m''(0)\epsilon^2, \tag{7}$$

$$W_\epsilon(x) = \frac{3}{2\gamma} \operatorname{sech}^2\left(\frac{x}{2}\right) + O_{H^1(\mathbf{R})}(\epsilon^2). \tag{8}$$

In addition, assume the boundedness of m. Then, the waves $\epsilon^2 W_\epsilon(\epsilon(x - \nu_\epsilon t))$ are in fact spectrally stable, for all small enough values of ϵ.

[1] In principle, one could compute explicitly the next terms, up to any degree of accuracy.

Remark (1) Assuming higher regularity of n, say $C^{l+2,1}(\mathbf{R})$, we have that $W_\epsilon \in H^l(\mathbf{R})$.
(2) In the proof, we can actually verify the non-degeneracy of the solution $\epsilon^2 W_\epsilon(\epsilon x)$ in the sense that the linearized operator has kernel spanned exactly by the group of symmetries.[2] It is expected that under these circumstances, one can in fact show orbital stability. The main obstacle towards that is a good local well-posedness result.

We also prove the existence of periodic "cnoidal" solutions of (2).

Theorem 2 *The following hold when Assumptions 1 and 2 are met. There exists $P_0 > 0$ such that the following holds for all $P > P_0$. There exists $\epsilon_P > 0$, so that for every $\epsilon \in (0, \epsilon_P)$ there is a $2P/\epsilon$-periodic, even, non-zero traveling wave solution $u(x,t) = \epsilon^2 W_{P,\epsilon}(\epsilon(x - \nu_\epsilon t))$ of (2). Moreover, $W_{P,\epsilon} \in H^1_{even}(\mathbf{T}_P)$,*

$$\nu_\epsilon = m(0) - \frac{1}{2} m''(0)\epsilon^2, \tag{9}$$

$$W_{P,\epsilon}(x) = \phi_P(x) + O_{H^1(\mathbf{T}_P)}(\epsilon^2), \tag{10}$$

where ϕ_P is the unique even, non-zero $2P$-periodic solution of $-\phi_P'' + \phi_P - \gamma\phi_P^2 = 0$.

In addition, assume the boundedness of m. For $0 < \epsilon \ll 1$, the waves $W_{P,\epsilon}$ are spectrally stable, with respect to co-periodic perturbations (that is perturbations of the same period $2P/\epsilon$).

Finally, regarding orbital stability - as we have discussed above, such result should follow by the general theory, see for example Theorem 5.2.11 in [11]. The issue here is again that we are lacking good well-posedness theory. In addition, the Hamiltonian, having $\mathscr{W}$ as a leading order differential operator term, does not control any L^p norms by virtue of Sobolev embedding. As is well-known, this is problematic with any sort of general approach. We leave this technical issue aside for the moment, we plan on addressing it in a future publication.

1.2 Conventions

By $H^s(\mathbf{R})$ we mean the usual L^2-based order s Sobolev space defined on $\mathbf{R}$. By $H^s(\mathbf{T}_P)$ we mean the usual L^2-based order s Sobolev space of periodic functions with period $2P$. Restricting attention only to even functions in the above results in the spaces $H^s_{even}(\mathbf{R})$ and $H^s_{even}(\mathbf{T}_P)$. If X is a Banach space then $B(X)$ is the space of bounded linear maps from X to itself, endowed with the usual norm.

For a function $f \in H^s(\mathbf{R})$ we use the following normalizations for the Fourier transform and its inverse:

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{\mathbf{R}} f(x)e^{-ixk}dx \quad \text{and} \quad f(x) = \int_{\mathbf{R}} \widehat{f}(k)e^{ixk}dk.$$

For a function $f \in H^s(\mathbf{T}_P)$ we use the following normalizations for the Fourier series and its inverse:

$$\widehat{f}(k) := \frac{1}{2P}\int_{-P}^{P} f(x)e^{-ik\pi x/P}dx \quad \text{and} \quad f(x) = \sum_{k\in\mathbf{Z}} \widehat{f}(k)e^{ik\pi x/P}.$$

If X is a Banach space and q_ϵ is an ϵ dependent quantity in X, we write

$$q_\epsilon = O_X(\epsilon^p)$$

[2] In this case, the only symmetry is the translation in the x variable.

if there exists ϵ_0 and $C > 0$ such that

$$\|q_\epsilon\|_X \le C\epsilon^p$$

for $0 < \epsilon \le \epsilon_0$.

2 Existence of Small Solutions

We present a detailed proof for the whole line case. The result for the periodic waves, which proceeds in an almost identical fashion, is proved in Sect. 2.3.

Our approach consists of introducing and analyzing a rescaled system, which is then showed to approximate the standard equation which gives the traveling wave solutions for KdV.

2.1 The Rescaled System

We make the "long wave/small amplitude/slightly supersonic" scalings

$$w(y) = \epsilon^2 W(\epsilon y) \quad \text{and} \quad \nu = m(0) - \frac{1}{2}m''(0)\epsilon^2$$

where $0 < \epsilon \ll 1$.[3] With this, (3) becomes

$$\left(m(0) - \frac{1}{2}m''(0)\epsilon^2 - L_\epsilon\right) W = \epsilon^{-2} n(\epsilon^2 W) \tag{11}$$

where L_ϵ is the Fourier multiplier operator with symbol

$$m_\epsilon(k) = m(\epsilon k). \tag{12}$$

Since $n(u)$ is $C^{2,1}$ by assumption, Taylor's theorem tells us that

$$\epsilon^{-2} n(\epsilon^2 W) = \frac{\epsilon^2}{2} n''(0) W^2 + \epsilon^4 \rho_\epsilon(W)$$

with

$$|\rho_\epsilon(W)| \le C|W|^3 \quad \text{and} \quad |\partial_x[\rho_\epsilon(W(x))]| \le C|W'(x)||W^2(x)| \tag{13}$$

when $|W| \le \delta_*/\epsilon^2$. Thus (11) becomes:

$$\left(m(0) - \frac{1}{2}m''(0)\epsilon^2 - L_\epsilon\right) W = \frac{\epsilon^2}{2} n''(0) W^2 + \epsilon^4 \rho_\epsilon(W). \tag{14}$$

Assumption 2 implies the following result.

Lemma 3 *Given Assumption 2, there exists $C > 0$ such that*

$$\sup_{K \in \mathbf{R}} \left| \frac{\epsilon^2}{m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K)} + \frac{1}{\frac{1}{2}m''(0)(1 + K^2)} \right| \le C\epsilon^2 \tag{15}$$

when ϵ is sufficiently close to zero.

[3] Note that the precise form for ν is primarily for convenience. We could let $\nu = m(0) + \kappa\epsilon^2$ for any $\kappa > 0$ and all that would really change are some nonessential coefficients.

We postpone the technical proof until "Appendix", below. Note however that quite a bit of information is packed into this Lemma. The first piece is that it guarantees that the symbol $(m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K))^{-1}$ is bounded. And so we can rewrite (14) as:

$$\underbrace{W - \epsilon^2 \left(m(0) - \frac{1}{2}m''(0)\epsilon^2 - L_\epsilon\right)^{-1} \left(\frac{1}{2}n''(0)W^2 + \epsilon^2 \rho_\epsilon(W)\right)}_{\Phi(W,\epsilon)} = 0. \tag{16}$$

Our goal is to resolve (16), at least for $0 < \epsilon \ll 1$. To do so, we will rely on the implicit function theorem and as such we need the behavior of the limiting system at $\epsilon = 0$. To be clear, in (16) by $\left(m(0) - \frac{1}{2}m''(0)\epsilon^2 - L_\epsilon\right)^{-1}$ we mean the operator with symbol $(m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K))^{-1}$.

Moreover, Lemma 3 implies that

$$\epsilon^2 \left(m(0) - \frac{1}{2}m''(0)\epsilon^2 - L_\epsilon\right)^{-1} = -\frac{2}{m''(0)}(1 - \partial_x^2)^{-1} + O_{B(X)}(\epsilon^2) \tag{17}$$

where X is either $H^s(\mathbf{R})$ or $H^s(\mathbf{T}_P)$. Thus, if we set $\epsilon = 0$ in (16), we get:

$$W - \gamma(1 - \partial_X^2)^{-1} W^2 = 0 \tag{18}$$

or rather

$$-W'' + W - \gamma W^2 = 0. \tag{19}$$

Here $\gamma > 0$ is given above in (6).

2.2 Existence of Localized Traveling Waves

Observe that (19), and so (18), has (a unique!) non-zero even localized solution, namely

$$W(X) = \sigma(x) := \frac{3}{2\gamma} \operatorname{sech}^2\left(\frac{x}{2}\right). \tag{20}$$

In other words, we have $\Phi(\sigma, 0) = 0$. The linearization of (19) about $\sigma(x)$ results in the self-adjoint operator

$$\mathscr{L} := -\partial_x^2 + 1 - 2\gamma\sigma,$$

which is well-studied in the literature.

Remark 1 It is known that $\mathscr{L}$ has exactly one simple negative eigenvalue, $-\zeta_0^2$, a simple eigenvalue at zero, spanned by σ', and outside of these two directions, the operator $\mathscr{L}$ is strictly positive.

2.2.1 Solvability of (16)

If we compute $\mathscr{K} := D_W \Phi(\sigma, 0)$ we get

$$\mathscr{K} = Id - 2\gamma(1 - \partial_x^2)^{-1}(\sigma \cdot).$$

The following lemma is proved in [7]:

Lemma 4 *For all $l \geq 0$, $l \in \mathbf{Z}$, $\mathscr{K} : H^l_{even}(\mathbf{R}) \to H^l_{even}(\mathbf{R})$ is bounded and has a bounded inverse.*

Here is a brief explanation of the proof. It is by now a classical result that $(1-\partial_x^2)^{-1}(\sigma\cdot) : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ and indeed $(1-\partial_x^2)^{-1}(\sigma\cdot) : H^1(\mathbf{R}) \to H^1(\mathbf{R})$ is a compact operator. Thus, the set $\sigma(\mathscr{K}) \setminus \{1\}$ has only eigenvalues of finite multiplicity. Note that when restricted to the even (and also odd subspaces), $\mathscr{K}$ acts invariantly, that is $\mathscr{K} : H^1_{even}(\mathbf{R}) \to H^1_{even}(\mathbf{R})$. We claim that $\mathscr{K}$ is invertible on $H^1_{even}(\mathbf{R})$. Indeed, assuming otherwise, it must be, by the Fredholm alternative, that there is an eigenfunction $f_0 \in H^1_{even} : \mathscr{K} f_0 = 0$. One quickly realizes that this implies $f_0 \in H^2(\mathbf{R})$ and $\mathscr{L} f_0 = 0$. This is a contradiction, since $f_0 \in Ker[\mathscr{L}] = span[\sigma']$, which then implies that f_0 is an odd function.

We use the following version of the implicit function theorem:

Theorem 5 *Let X be a Banach space and suppose that $\Phi : X \times \mathbf{R} \to X$ has the following properties: (a) Φ is continuously differentiable (b) $\Phi(x_*, \mu_*) = 0$ and (c) $D_x\Phi(x_*, \mu_*)$ has bounded inverse from X to X. Then there exists a neighborhoods U of x_* and M of μ_* and a continuously differentiable function $\chi : M \to U$ such that $\Phi(\chi(\mu), \mu) = 0$ and $\Phi(x, \mu) = 0$ iff $x = \chi(\mu)$ for all $(x, \mu) \in U \times M$.*

According to Theorem 5, the solvability of (16), that is $\Phi(W, \epsilon) = 0$, holds. Indeed, by our previous considerations, $\Phi(\sigma, 0) = 0$, the functional $\Phi : H^1_{even}(\mathbf{R}) \times \mathbf{R} \to H^1_{even}(\mathbf{R})$, $s > \frac{1}{2}$ is continuously differentiable. In addition, $\mathscr{K} = D_W\Phi(\sigma, 0) : H^1_{even}(\mathbf{R}) \to H^1_{even}(\mathbf{R})$ is invertible, according to Lemma 4. This gives a family of solutions, say $W_\epsilon \in H^1_{even}(\mathbf{R})$, at least in a small neighborhood of $\epsilon \in (0, \epsilon_0)$, $\epsilon_0 < 1$. That $W_\epsilon - \sigma$ is $O_{H^1(\mathbf{R})}(\epsilon^2)$ follows in routine way from (16), (17) and (13). This finishes the proof of the existence part of Theorem 1.

Remark Note that with the current assumptions on m, one cannot obtain a higher regularity results on W_ϵ, since the operator $\left(m(0) - \frac{1}{2}m''(0)\epsilon^2 - L_\epsilon\right)^{-1}$ cannot be guaranteed to be smoothing.[4] We can however claim higher regularity, by essentially the same arguments as above, once we know a higher regularity of the remainder term $\rho_\epsilon(z) = \frac{n(\epsilon^2 z) - \frac{n''(0)}{2}\epsilon^4 z^2}{\epsilon^3}$ or, what is the same, a higher regularity of the nonlinearity n. Indeed, assuming $n \in C^{l+2,1}(\mathbf{R})$, we obtain $\rho \in C^{l,1}(\mathbf{R})$ and then, we can claim that the map $\Phi : H^l_{even}(\mathbf{R}) \times \mathbf{R} \to H^l_{even}(\mathbf{R})$ is continuously differentiable. Since $\mathscr{K}$ will also be invertible on $H^l_{even}(\mathbf{R})$, an application of the implicit function theorem will produce a solution $W_\epsilon \in H^l_{even}(\mathbf{R})$.

2.3 Existence of Periodic Traveling Waves

Return attention to (19). In addition to the solitary wave solution $\sigma(X)$, this equation has a one-parameter family of even periodic solutions. While there are explicit formulas available for these solutions [8] in terms of the elliptic functions "cn" (hence the nomenclature "cnoidal" waves) we do not need these formulas here. Instead, we summarize the properties of such waves.

Theorem 6 *For all $\gamma > 0$ there exists $P_0 > 0$ and a family functions $\{\phi_P(x)\}_{P>P_0}$ with the following properties*

(1) *$\phi_P(x)$ is C^∞, non-constant and even.*

4 And in fact, for the Whitham example, where $m(k) = \sqrt{\frac{\tanh(k)}{k}}$ it is not smoothing.

(2) *$\phi_P(x)$ is periodic with principal period $2P$.*
(3) *$W(x) = \phi_P(x)$ solves (19) (and thus (18))*
(4) *The kernel of*

$$\mathscr{L}_P := -\partial_x^2 + 1 - 2\gamma\phi_P$$

(as an operator in $H^s(\mathbf{T}_P)$) is exactly span $\{\phi'_P(x)\}$.

This theorem tells us that $\Phi(\phi_P, 0) = 0$. Our strategy for continuing such solutions to $\epsilon > 0$ via the implicit function theorem is not that different than the one used for the localized waves above. If we compute $\mathscr{K}_P := D_W\Phi(\phi_P, 0)$ we get

$$\mathscr{K}_P = Id - 2\gamma(1 - \partial_x^2)^{-1}(\phi_P \cdot).$$

In [8] (their Lemma 5.1) the following is shown:

Lemma 7 *$\mathscr{K}_P : L^2_{even}(\mathbf{T}_P) \to L^2_{even}(\mathbf{T}_P)$ is bounded and has a bounded inverse. Also, for any $l \geq 1$, we have $\mathscr{K}_P : H^l_{even}(\mathbf{T}_P) \to H^l_{even}(\mathbf{T}_P)$ is bounded and invertible.*

This follows from part (3) of Theorem 6 and the argument is very much the same as the proof of Lemma 4. Indeed, the operator $\mathscr{K}_p$ is easily seen to map H^l_{even} into itself for any $l \geq 1$, due to the smoothness of ϕ_P. Finally, the spectrum of $\mathscr{K}_P$ is independent on the space H^l, so $0 \notin \sigma_{L^2}(\mathscr{K}_P) = \sigma_{H^l}(\mathscr{K}_P)$, hence $\mathscr{K}_P$ is invertible on those as well.

At this stage we appeal to the implicit function theorem as above and arrive at the conclusions of Theorem 2.

3 Proof of Theorem 1: The Stability of the Small Whitham Waves

Now that we have constructed the solutions W_ϵ for $0 < \epsilon \ll 1$, let us address the question for their stability. We first linearize around the traveling wave solution.

3.1 The Linearized Problem and Stability

We take the perturbation of the solution $\epsilon^2 W_\epsilon(\epsilon(x - \nu t))$ in the form $u = \epsilon^2(W_\epsilon(\epsilon(x - \nu t)) + v(\epsilon t, \epsilon(x - \nu t)))$. Plugging in this ansatz in the Eq. (2) and ignoring terms of order $O(v^2)$ and transforming $x - \nu t \to x$, we obtain the following linearized system

$$v_t + \partial_x[L_\epsilon v - \nu v + n'(\epsilon^2 W_\epsilon)v] = 0. \tag{21}$$

Introduce the linearized operator

$$\mathscr{L}_\epsilon := -L_\epsilon + \nu - n'(\epsilon^2 W_\epsilon).$$

Passing to the time independent problem via the map $v(t, x) \to e^{\lambda t} z(x)$, we arrive at the eigenvalue problem

$$\partial_x \mathscr{L}_\epsilon z = \lambda z \tag{22}$$

It is then time to introduce the notion of stability.

Definition 1 We say that the traveling wave $\epsilon^2 W_\epsilon(\epsilon(x - \nu t))$ is spectrally stable, if the eigenvalue problem (22) does not have non-trivial solutions $(\lambda, z) : \Re\lambda > 0, z \in L^2(\mathbf{R})$.

Next, we discuss the instability index count theory, which gives sufficient (and in many cases necessary) conditions for stability/instability. We mostly follow the general theory, as developed in [14], although earlier relevant results are available, see [9–12].

3.2 Instability Index Theory

For the eigenvalue problem

$$\mathscr{J}\mathscr{L}f = \lambda f \tag{23}$$

make the following assumptions regarding $\mathscr{L}$, $\mathscr{J}$. There exists a real Hilbert space X, so that[5]

(1) The operator $\mathscr{L} : X \to X^*$ is bounded and symmetric, in the sense that $(u, v) \to \langle \mathscr{L}u, v\rangle$, understood as a pairing of an element of X^* with an element of X, is a symmetric bilinear form on X.
(2) $dim(Ker[\mathscr{L}]) < \infty$ and there is the $\mathscr{L}$ invariant decomposition of the space X,

$$X = X_- \oplus Ker[\mathscr{L}] \oplus X_+,$$

where $dim(X_-) < \infty$, and for some $\delta > 0$, $\mathscr{L}|_{X_-} \leq -\delta$ (i.e. $\langle \mathscr{L}u_-, u_-\rangle \leq -\delta\|u_-\|_X^2$ for all $u_- \in X_-$), $\mathscr{L}|_{X_+} \geq \delta > 0$.
(3) $\mathscr{J} : D(\mathscr{J}) \subset X^* \to X$ is anti-symmetric, in the sense that for every $u, v \in D(\mathscr{J})$, one has $\langle \mathscr{J}u, v\rangle = -\langle u, \mathscr{J}v\rangle$.

Moreover, introduce the Morse index $n^-(\mathscr{L}) = dim(X_-)$. Consider the generalized eigenspace at zero for the operator $\mathscr{J}\mathscr{L}$, that is $E_0 = \{u \in X : (\mathscr{J}\mathscr{L})^k u = 0, k = 1, 2, \dots\}$. Clearly, $Ker[\mathscr{L}]$ is a (finite dimensional) subspace of E_0 and so, one can complement it as follows $E_0 = Ker[\mathscr{L}] \oplus \widetilde{E}_0$. Then,

$$k_0^{\leq 0} := \max\{dim(Z) : Z \text{ subspace of } \widetilde{E}_0 : \langle \mathscr{L}z, z\rangle < 0, z \in Z\}.$$

Under these assumptions, it was proved (see Theorem 2.3, [14]) that[6]

$$k_{unstable} \leq n^-(\mathscr{L}) - k_0^{\leq 0}(\mathscr{L}). \tag{24}$$

where $k_{unstable}$ is the number of non-trivial (that is pairs (λ, z) with $\Re\lambda > 0, z \neq 0, z \in X$) and linearly independent unstable solutions to (23). In the next section, we apply this theory to the linearized problem (22).

3.3 Spectral Stability Analysis for the Small Whitham Waves

For the eigenvalue problem (22), we have $\mathscr{J} = \partial_x$, which is anti self-adjoint, while clearly $\mathscr{L}_\epsilon : \mathscr{L}_\epsilon^* = \mathscr{L}_\epsilon$ is a bounded symmetric operator, if we assume the boundedness of its symbol m, which we do in this section. In the setup of the instability index theory, we take $X = L^2(\mathbf{R})$, while $D(\partial_x) = H^1(\mathbf{R})$.

We will establish below that $\mathscr{L}_\epsilon$ has, at least for small enough values of ϵ, a single and simple negative eigenvalue (i.e. $n^-(\mathscr{L}_\epsilon) = 1$), while its kernel is one dimensional and it is in fact spanned by W_ϵ'. Assuming that for the moment, let us proceed to establish a sufficient

[5] The most common example one should think is $X = H^s$, a Sobolev space, with dual $X^* = H^{-s}$.

[6] A much more precise result is contained in Theorem 2.3, [14], but we state this corollary, as it is enough for our purposes.

condition for the stability. According to (24), $k_{unstable} \leq 1 - k_0^{\leq 0}$. Thus, the stability of the solitary waves $\epsilon^2 W_\epsilon(\epsilon x)$, will be established, once we show that[7] $k_0^{\leq 0}(\mathscr{L}_\epsilon) \geq 1$.

To this end, we can identify an element in $gKer(\partial_x \mathscr{L}_\epsilon) \setminus Ker[\partial_x \mathscr{L}_\epsilon]$. Note that $Ker[\partial_x \mathscr{L}_\epsilon] = Ker[\mathscr{L}_\epsilon] = span\{W_\epsilon'\}$. In addition, $W_\epsilon \perp W_\epsilon'$, whence $W_\epsilon \perp Ker[\mathscr{L}_\epsilon]$. Thus, $\Psi_\epsilon := \mathscr{L}_\epsilon^{-1}[W_\epsilon]$ is well-defined. Since,

$$(\partial_x \mathscr{L}_\epsilon)^2[\Psi_\epsilon] = \partial_x \mathscr{L}_\epsilon \partial_x [W_\epsilon] = \partial_x \mathscr{L}_\epsilon [W_\epsilon'] = 0,$$

we have that $\Psi_\epsilon \in gKer(\partial_x \mathscr{L}_\epsilon) \setminus Ker[\partial_x \mathscr{L}_\epsilon]$. According to the definition of $k_0^{\leq 0}(\mathscr{L}_\epsilon)$, we will have established $k_0^{\leq 0}(\mathscr{L}_\epsilon) \geq 1$, once we verify that

$$0 > \langle \mathscr{L}_\epsilon \Psi_\epsilon, \Psi_\epsilon \rangle = \langle \mathscr{L}_\epsilon^{-1}[W_\epsilon], \epsilon^2 W_\epsilon \rangle.$$

Thus, we will need to verify the negativity of the Vakhitov–Kolokolov type quantity

$$\langle \mathscr{L}_\epsilon^{-1}[W_\epsilon], W_\epsilon \rangle < 0, \tag{25}$$

once we check that for all small enough ϵ, $n^-(\mathscr{L}_\epsilon) = 1$, $Ker[\mathscr{L}_\epsilon] = span\{W_\epsilon'\}$. We do this in the next Lemma.

Lemma 8 *There exists $\epsilon_0 > 0$ so that for all $\epsilon \in (0, \epsilon_0)$, $n^-(\mathscr{L}_\epsilon) = 1$, $Ker[\mathscr{L}_\epsilon] = span\{W_\epsilon'\}$.*

Proof Start by taking a sufficiently large $\mu > 0$, to be specified later. We will construct the operator $\left(\epsilon^{-2}\mathscr{L}_\epsilon + \mu\right)^{-1}$ for all small enough ϵ. Indeed, since

$$n'(\epsilon^2 W_\epsilon) = n''(0)\epsilon^2 W_\epsilon + O_{H^1}(\epsilon^4) = n''(0)\epsilon^2 \sigma + O_{H^1}(\epsilon^4),$$

where σ is the explicit $sech^2$ function, see (20). We have

$$\begin{aligned} \epsilon^{-2}\mathscr{L}_\epsilon + \mu &= \epsilon^{-2}[\mathscr{L}_\epsilon + \mu\epsilon^2] = \epsilon^{-2}[-L_\epsilon + \nu - \epsilon^2 n''(0)\sigma + \mu\epsilon^2 + O_{H^1}(\epsilon^4)] \\ &= [Id - [n''(0)\sigma - \mu + O_{H^1}(\epsilon^2)]\epsilon^2(\nu - L_\epsilon)^{-1}]\epsilon^{-2}(\nu - L_\epsilon). \end{aligned}$$

Recall now that the operator $\epsilon^2(\nu – L_\epsilon)^{-1}$ is associated with the multiplier $\frac{\epsilon^2}{m(0) – \frac{1}{2}m''(0)\epsilon^2 – m(\epsilon k)}$. So, according to Lemma 3 [and more precisely (15)],

$$\epsilon^2(\nu - L_\epsilon)^{-1} = -\frac{2}{m''(0)}(1 - \partial_x^2)^{-1} + O_{B(L^2)}(\epsilon^2). \tag{26}$$

Thus,

$$\begin{aligned} \epsilon^{-2}\mathscr{L}_\epsilon + \mu &= \left(Id + \frac{2}{m''(0)}[n''(0)\sigma - \mu + O_{H^1}(\epsilon^2)](1 - \partial_x^2)^{-1} \right) \epsilon^{-2}(\nu - L_\epsilon) \\ &= \left(Id + 2\left[-\gamma\sigma - \frac{\mu}{m''(0)} + O_{H^1}(\epsilon^2) \right](1 - \partial_x^2)^{-1} \right) \epsilon^{-2}(\nu - L_\epsilon). \end{aligned}$$

Recall now that $\mathscr{L} := -\partial_x^2 + 1 - 2\gamma\sigma$. So,

$$\begin{aligned} \mathscr{L} - \frac{2\mu}{m''(0)} + O_{H^1}(\epsilon^2) &= 1 - \partial_x^2 - 2\gamma\sigma - \frac{2\mu}{m''(0)} + O_{H^1}(\epsilon^2) \\ &= \left[Id + 2\left[-\gamma\sigma - \frac{\mu}{m''(0)} + O_{H^1}(\epsilon^2) \right](1 - \partial_x^2)^{-1} \right](1 - \partial_x^2). \end{aligned}$$

[7] And hence $k_0^{\leq 0}(\mathscr{L}_\epsilon) = 1$, since the left hand side of (24) is non-negative.

Now, we select $\mu > 0$ large and $\epsilon \ll 1$, so that $\mathscr{L} - \frac{2\mu}{m''(0)} + O_{H^1}(\epsilon^2)$ is invertible. This is possible, since $-\frac{2\mu}{m''(0)} > 0$ and $\mathscr{L}$ is bounded from below.[8] Moreover, $(\mathscr{L} - \frac{2\mu}{m''(0)} + O_{H^1}(\epsilon^2))^{-1} : L^2 \to H^2$. Thus, we can write

$$\left[Id + 2[-\gamma\sigma - \frac{\mu}{m''(0)} + O_{H^1}(\epsilon^2)](1-\partial_x^2)^{-1}\right]^{-1}$$
$$= (1-\partial_x^2)\left(\mathscr{L} - \frac{2\mu}{m''(0)} + O_{H^1}(\epsilon^2)\right)^{-1} : L^2 \to L^2.$$

Hence, we can invert [by means of the previous formula and (26)]

$$\begin{aligned}(\epsilon^{-2}\mathscr{L}_\epsilon + \mu)^{-1} &= \epsilon^2(\nu - L_\epsilon)^{-1}\left[Id + 2\left[-\gamma\sigma - \frac{\mu}{m''(0)} + O_{H^1}(\epsilon^2)\right](1-\partial_x^2)^{-1}\right]^{-1} \\ &= \left(-\frac{2}{m''(0)}(1-\partial_x^2)^{-1} + O_{B(L^2)}(\epsilon^2)\right)(1-\partial_x^2)\left(\mathscr{L} - \frac{2\mu}{m''(0)} + O_{H^1}(\epsilon^2)\right)^{-1} \\ &= -\frac{2}{m''(0)}\left(\mathscr{L} - \frac{2\mu}{m''(0)}\right)^{-1} + O_{B(L^2)}(\epsilon^2).\end{aligned}$$

That is,

$$(\epsilon^{-2}\mathscr{L}_\epsilon + \mu)^{-1} = \left(-\frac{m''(0)}{2}\mathscr{L} + \mu\right)^{-1} + O_{B(L^2)}(\epsilon^2). \tag{27}$$

We can now use this formula to study the spectrum of $\mathscr{L}_\epsilon$. To this end, recall the negative eigenvalue of $\mathscr{L}$ is $-\zeta_0^2$, so that for some ψ_0, $\|\psi_0\| = 1$, $\mathscr{L}\psi_0 = -\zeta_0^2\psi_0$. Then,

$$\sup_{f:\|f\|=1}\left\langle\left(-\frac{m''(0)}{2}\mathscr{L} + \mu\right)^{-1} f, f\right\rangle \geq \left\langle\left(-\frac{m''(0)}{2}\mathscr{L} + \mu\right)^{-1}\psi_0, \psi_0\right\rangle = \frac{1}{-\frac{m''(0)}{2}(-\zeta_0^2) + \mu} > \frac{1}{\mu}.$$

Thus, by (27) and for all $\epsilon << 1$,

$$\sup_{f:\|f\|=1}\langle(\epsilon^{-2}\mathscr{L}_\epsilon + \mu)^{-1} f, f\rangle > \frac{1}{\mu}. \tag{28}$$

Next, take $f : f \perp \psi_0$, $\|f\| = 1$. Since we have $\mathscr{L}|_{\{\psi_0\}^\perp} \geq 0$ and $\mathscr{L}[\sigma'] = 0$,

$$\frac{1}{\mu} = \left\langle\left(-\frac{m''(0)}{2}\mathscr{L} + \mu\right)^{-1}\frac{\sigma'}{\|\sigma'\|}, \frac{\sigma'}{\|\sigma'\|}\right\rangle \leq \sup_{f\perp\psi_0, \|f\|=1}\left\langle\left(-\frac{m''(0)}{2}\mathscr{L} + \mu\right)^{-1} f, f\right\rangle \leq \frac{1}{\mu},$$

so $\sup_g \sup_{f\perp g, \|f\|=1}\langle(-\frac{m''(0)}{2}\mathscr{L} + \mu)^{-1} f, f\rangle = \sup_{f\perp\psi_0, \|f\|=1}\langle(-\frac{m''(0)}{2}\mathscr{L} + \mu)^{-1} f, f\rangle$ $= \frac{1}{\mu}$, whence

$$\sup_g \sup_{f\perp g, \|f\|=1}\langle(\epsilon^{-2}\mathscr{L}_\epsilon + \mu)^{-1} f, f\rangle = \frac{1}{\mu} + O(\epsilon^2). \tag{29}$$

Further, according to the spectral information for $\mathscr{L}$, see Remark 1, its second smallest eigenvalue - zero, is also simple, in particular, $\mathscr{L}|_{span\{\psi_0, \sigma'\}^\perp} \geq \delta Id > 0$. Therefore,

$$\sup_{f\perp\psi_0, f\perp\sigma', \|f\|=1}\left\langle\left(-\frac{m''(0)}{2}\mathscr{L} + \mu\right)^{-1} f, f\right\rangle \leq \frac{1}{-\delta\frac{m''(0)}{2} + \mu} < \frac{1}{\mu}. \tag{30}$$

[8] And in fact it has a single negative eigenvalue.

Again,

$$\sup_{g_1,g_2}\sup_{f\perp g_1, f\perp g_2, \|f\|=1}\left\langle\left(-\frac{m''(0)}{2}\mathscr{L}+\mu\right)^{-1}f,f\right\rangle=\sup_{f\perp\psi_0, f\perp\sigma', \|f\|=1}\left\langle\left(-\frac{m''(0)}{2}\mathscr{L}+\mu\right)^{-1}f,f\right\rangle$$

so from (27) and for all $\epsilon << 1$,

$$\sup_{g_1,g_2}\sup_{f\perp g_1, f\perp g_2, \|f\|=1}\langle(\epsilon^{-2}\mathscr{L}_\epsilon+\mu)^{-1}f,f\rangle<\frac{1}{\mu}. \tag{31}$$

Using min–max formulas for the eigenvalues of self-adjoint operators, the relations (28), (29) and (31) imply that the spectrum of the bounded operator $(\epsilon^{-2}\mathscr{L}_\epsilon+\mu)^{-1}$ is bounded from above by two simple eigenvalues, $\kappa_0>\kappa_1$, so that $\kappa_0>\frac{1}{\mu}$ and $\kappa_1=\frac{1}{\mu}+O(\epsilon^2)$. In addition, the rest of the spectrum is smaller than κ_1, that is $\sigma((\epsilon^{-2}\mathscr{L}_\epsilon+\mu)^{-1})\subset(-\infty,\kappa_1)$. Note that since $\mathscr{L}_\epsilon[W'_\epsilon]=0$, we know that $\frac{1}{\mu}$ is an eigenvalue for $(\epsilon^{-2}\mathscr{L}_\epsilon+\mu)^{-1}$, so $\kappa_1=\frac{1}{\mu}$.

Equivalently, $\epsilon^{-2}\mathscr{L}_\epsilon$ has the smallest eigenvalue in the form $\lambda_0(\epsilon^{-2}\mathscr{L}_\epsilon)=\frac{1}{\kappa_0}-\mu+O(\epsilon^2)<0$, while the second smallest eigenvalue is $\lambda_1(\epsilon^{-2}\mathscr{L}_\epsilon)=0$, while the rest of the spectrum of $\epsilon^{-2}\mathscr{L}_\epsilon$ satisfies

$$\sigma(\epsilon^{-2}\mathscr{L}_\epsilon)\setminus\{\lambda_0,\lambda_1\}\subset\left(-\delta\frac{m''(0)}{2}+O(\epsilon^2),\infty\right)\subset(0,\infty),$$

according to (30). This finishes the proof of Lemma 8. □

It remains to finally verify (25). Now that we know that $Ker[\mathscr{L}_\epsilon]=span\{W'_\epsilon\}$, we conclude that $\mathscr{L}_\epsilon$ is invertible on the even subspace L^2_{even}. In fact, we may use the formula (27) with $\mu=0$. In addition, from (8), we have

$$\begin{aligned}\langle\mathscr{L}_\epsilon^{-1}[W_\epsilon],W_\epsilon\rangle&=-\frac{2}{m''(0)}\epsilon^{-2}\langle(\mathscr{L}^{-1}+O_{B(L^2)}(\epsilon^2)[\sigma+O_{H^1}(\epsilon^2),\sigma+O_{H^1}(\epsilon^2)\rangle\\&=-\frac{2}{m''(0)}\epsilon^{-2}[\langle\mathscr{L}^{-1}\sigma,\sigma\rangle+O(\epsilon^2)].\end{aligned}$$

The quantity $\langle\mathscr{L}^{-1}\sigma,\sigma\rangle$ is well-known in the theory of stability for the corresponding KdV/NLS models. Its negativity is exactly in the same way equivalent to the (well-known) stability of the corresponding traveling/standing waves. It actually may be computed explicitly as follows.

Consider (19) and a function $W_\lambda:=\lambda^2\sigma(\lambda\cdot)$, $\lambda>0$. This solves

$$-W''_\lambda+\lambda^2W_\lambda-\gamma W_\lambda^2=0.$$

Taking a derivative in λ and evaluating at $\lambda=1$ yields

$$\mathscr{L}\left[\frac{d}{d\lambda}W_\lambda|_{\lambda=1}\right]=-2\sigma$$

Thus, $\mathscr{L}^{-1}\sigma=-\frac{1}{2}\frac{d}{d\lambda}W_\lambda|_{\lambda=1}=-\frac{1}{2}(2\sigma+x\sigma')$. It follows that

$$\langle\mathscr{L}^{-1}\sigma,\sigma\rangle=-\frac{1}{2}\langle2\sigma+x\sigma',\sigma\rangle=-\frac{3}{4}\|\sigma\|^2<0.$$

Thus, the Vakhitov–Kolokolov condition (25) is verified and the proof of Theorem 1 is complete.

3.4 Spectral Stability of the Periodic Waves

The stability calculation for the periodic waves proceed in an identical fashion. The eigenvalue problem is in the form (22), where now the operators are acting on the corresponding periodic spaces $H^s(\mathbf{T}_P)$. In fact, noting that for $\lambda \neq 0$, the right hand side z is an exact derivative, allows us to restrict the consideration of (22) to the space $L^2_0(\mathbf{T}_P) = \{f \in L^2(\mathbf{T}_P) : \int_{-P}^{P} f(x)dx = 0\}$. The advantage of this is that now $\mathscr{J} = \partial_x$ is boundedly invertible, hence allowing for the results of [11] to kick in.

The instability index theory outlined in Sect. 3.2 applies. According to (24) and the analysis in Sect. 3.3—(25) implies the spectral stability. Moreover, Lemma 8 applies as well to the periodic waves. That is, the Morse index of $\mathscr{L}_\epsilon$ is one and the wave is non-degenerate, in the sense that $Ker[\mathscr{L}_\epsilon] = span[W'_{P,\epsilon}]$. The verification of (25) is reduced, in the same way, to the verification of the inequality $\langle \mathscr{L}_P^{-1}\phi_P, \phi_P\rangle < 0$. This quantity can be computed fairly precisely, in terms of elliptic functions, but we will not do so here.

Appendix: Assorted Proofs

Proof (Lemma 3) Take $K > k_*/\epsilon$. Then we clearly have

$$\left|\frac{1}{\frac{1}{2}m''(0)(1+K^2)}\right| \leq \frac{2\epsilon^2}{|m''(0)|k_*} \leq C\epsilon^2.$$

Since $K > k_*/\epsilon$, we know that $m(\epsilon K) \leq m_1$ by (5). Thus we have

$$m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K) > m(0) - m_1 > 0.$$

Here we have used that fact that $m''(0) < 0$, which is implied by (4). This in turn implies:

$$\left|\frac{\epsilon^2}{m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K)}\right| \leq \frac{\epsilon^2}{m(0)-m_1} \leq C\epsilon^2.$$

The triangle inequality gives:

$$\sup_{|K|\geq k_*/\epsilon}\left|\frac{\epsilon^2}{m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K)} + \frac{1}{\frac{1}{2}m''(0)(1+K^2)}\right| \leq C\epsilon^2. \tag{32}$$

Now suppose that $|K| \leq k_*/\epsilon$. We have

$$\begin{aligned} &\frac{\epsilon^2}{m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K)} + \frac{1}{\frac{1}{2}m''(0)(1+K^2)} \\ &\quad = \frac{m(0) + \frac{1}{2}m''(0)\epsilon^2K^2 - m(\epsilon K)}{[m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K)][\frac{1}{2}m''(0)(1+K^2)]} \end{aligned} \tag{33}$$

The fact that m is even and $C^{3,1}$ implies, by way of Taylor's theorem, that there exists $C > 0$ such that

$$\left|m(0) + \frac{1}{2}m''(0)k^2 - m(k)\right| \leq Ck^4$$

when $|k| \leq k_*$. This implies that

$$\left|m(0) + \frac{1}{2}m''(0)\epsilon^2 K^2 - m(\epsilon K)\right| \leq C\epsilon^4 K^4 \tag{34}$$

when $|K| \leq k_*/\epsilon$.

The fundamental theorem of calculus implies that

$$m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(k) = -\frac{1}{2}m''(0)\epsilon^2 - \int_0^k \int_0^s m''(\sigma)d\sigma ds.$$

Here we used the fact that $m(k)$ is even. Then we use (4) to see that

$$m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(k) \geq -\frac{1}{2}m''(0)\epsilon^2 - \frac{1}{2}m_2 k^2$$

so long as $|k| \leq k_*$. Thus, for $|K| \leq k_*/\epsilon$ we have

$$m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K) \geq \epsilon^2 \left(-\frac{1}{2}m''(0) - \frac{1}{2}m_2 K^2\right)$$

Since $m''(0)$ and m_2 are both negative this implies:

$$\left|m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K)\right| \geq C\epsilon^2\left(1 + K^2\right) \tag{35}$$

when $|K| \leq k_*/\epsilon$.

Thus we can control the left hand side of (33) using (34) and (35) as:

$$\left|\frac{\epsilon^2}{m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K)} + \frac{1}{\frac{1}{2}m''(0)(1 + K^2)}\right| \leq \frac{C\epsilon^2 K^4}{(1 + K^2)^2}$$

when $K \leq k_*/\epsilon$. Since $K^4/(1 + K^2)^2 \leq 1$ we have

$$\sup_{|K| \leq k_*/\epsilon} \left|\frac{\epsilon^2}{m(0) - \frac{1}{2}m''(0)\epsilon^2 - m(\epsilon K)} + \frac{1}{\frac{1}{2}m''(0)(1 + K^2)}\right| \leq C\epsilon^2$$

□

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