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On the standing waves of the Schrödinger equation with concentrated nonlinearity

Abba Ramadan[1] · Atanas G. Stefanov[1]

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Abstract

We study the concentrated NLS on $\mathbf{R}^n$, with power non-linearities, driven by the fractional Laplacian, $(-\Delta)^s, s > \frac{n}{2}$. We construct the solitary waves explicitly, in an optimal range of the parameters, so that they belong to the natural energy space $H^s(\mathbf{R}^n)$. Next, we provide a complete classification of their spectral stability. Finally, we show that the waves are non-degenerate and consequently orbitally stable, whenever they are spectrally stable. Incidentally, our construction shows that the soliton profiles for the concentrated NLS are in fact exact minimizers of the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$, which provides an alternative calculation and justification of the sharp constants in these inequalities.

Keywords Concentrated NLS · Solitons · Stability

Mathematics Subject Classification Primary 35Q55 · 35Q40

1 Introduction

The (focusing) nonlinear Schrödinger equation, with generalized power nonlinearity

$$iu_t + \Delta u + |u|^{2\sigma} u = 0, (t, x) \in \mathbf{R} \times \mathbf{R}^n \tag{1.1}$$

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✉ Atanas G. Stefanov
stefanov@ku.edu

Abba Ramadan
aramadan@ku.edu

1 Department of Mathematics, University of Kansas, 1460 Jayhawk Boulevard, Lawrence, KS 66045–7523, USA

is a basic model in theoretical physics and applied mathematics. Examples of such physical applications can be found in fractional quantum mechanics and Lévy path integrals [19]. Other applications arise in water waves theory and other engineering models.

Equation (1.1) has been studied extensively in the last fifty years, in particular with regards to the well-posedness of the Cauchy problem and the stability of its solitary waves. The well-posedness theory is classical by now, [11] states that local well-posedness holds for any $\sigma > 0$, whenever the data $u_0 \in H^s(\mathbf{R}^n)$, $s \geq 0$. The global well-posedness results rely upon the conservation law, which state that the following quantities, namely the mass $M(u)$ and the energy $E(u)$, are conserved

$$M(u) = \int_{\mathbf{R}^n} |u(t,x)|^2 dx = const$$
$$E(u) = \frac{1}{2}\int_{\mathbf{R}^n} |\nabla u(t,x)|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbf{R}^n} |u(t,x)|^{2\sigma+2} dx = const.$$

As such, initial data $u_0 \in H^1(\mathbf{R}^n)$ yields global solutions whenever the problem is L^2 sub-critical, i.e. for $\sigma < \frac{2}{n}$, while for $\sigma \geq \frac{2}{n}$, some initial data gives rise to finite time blow-ups. Interestingly, the ground states for (1.1) are stable exactly in the L^2 sub-critical range $\sigma < \frac{2}{n}$, while they are unstable in the supercritical regime $\sigma > \frac{2}{n}$. In the L^2 critical case, $\sigma = \frac{2}{n}$, the Eq. (1.1) exhibits an additional symmetry, the so-called quasi-conformal invariance, which allows one to find special self-similar type solutions. Thus blow-up also occurs in the critical case.

In this work, we analyze a related model, the focusing non-linear Schrödinger equation with concentrated non-linearity. As our dispersive models will be driven by fractional Laplacians, let us introduce the proper framework. We set the Fourier transform and its inverse by the formulas

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx; \quad f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

In that case, the Laplacian is given as a Fourier multiplier (on the space of Schwartz functions $\mathcal{S}$) via $\widehat{(-\Delta)f} = 4\pi^2|\xi|^2\hat{f}(\xi)$. More generally, for all $s > 0$

$$\widehat{(-\Delta)^s f} = (2\pi|\xi|)^{2s}\hat{f}(\xi).$$

Now, the focusing NLS with concentrated non-linearity is the following

$$\begin{cases} iu_t = ((-\Delta)^s - |u|^{2\sigma}\delta_0)u, & (t,x) \in \mathbf{R}\times\mathbf{R}^n \\ u(0,x) = u_0(x) \end{cases}. \tag{1.2}$$

Our definition of a solution is as follows: a continuous in x function u is a weak solution of (1.2), if it satisfies

$$i\left(\langle u(t,\cdot), \psi(t,\cdot)\rangle - \langle u_0, \psi(0,\cdot)\rangle - \int_0^t \langle u(\tau,\cdot), \psi_\tau(\tau,\cdot)\rangle d\tau\right)$$

$$= \int_0^t \langle (-\Delta)^{\frac{s}{2}} u(\tau, \cdot), (-\Delta)^{\frac{s}{2}} \psi(\tau, \cdot) \rangle d\tau - \int_0^t |u(\tau, 0)|^{2\sigma} u(\tau, 0) \psi(\tau, 0) d\tau$$

for all test functions ψ. For the case of the standard Laplacian, i.e. $s = 1$, the Eq. (1.2) has been used to model resonant tunneling, [14], the dynamics of mixed states, [20], quantum turbulence, [9] and the generation of weakly bounded states close to the instability, [23].

The fractional Laplacian, perturbed by a delta potential, together with its self-adjoint extensions and various applications, has recently been considered in [10]. In the case of one spatial dimension, $n = 1$ and $s > \frac{1}{2}$, the local well-posedness as well as the conservation of mass and energy

$$M(u) = \int_{\mathbf{R}^n} |u(t, x)|^2 dx = const. \tag{1.3}$$

$$E(u) = \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \frac{1}{2\sigma + 2} |u(t, 0)|^{2\sigma+2} = const. \tag{1.4}$$

was recently established in [10]. Even though the results in [10] are stated for the one dimensional case only, it seems plausible that they can be extended in any dimension n and $s > \frac{n}{2}$ using similar techniques. It is important to note that, since our interest is in continuous in x functions, the natural spaces for well-posedness, in the scale of the Sobolev spaces, should be $H^s(\mathbf{R}^n)$, $s > \frac{n}{2}$. Another reason why this is, in our opinion, a more natural class of problems to consider, is that we would like the waves to belong to the energy space $H^s(\mathbf{R}^n)$ as dictated by the conservation of $E(u)$. As we shall see below, the solitary waves are in this space only for $s > \frac{n}{2}$.

It has to be noted, however, that it is certainly possible (and it is in fact considerably more challenging, the further one is from the threshold $s = \frac{n}{2}$) to consider (1.2) in case $s < \frac{n}{2}$. This has been addressed, at least in low dimensional situations, in the recent papers, [3–7]. Regarding analysis of blow-up solutions for the concentrated NLS (although not necessarily in the case of interest $s > \frac{n}{2}$), this was carried out recently in [5].

Our main interest in the model (1.2) is to investigate its solitary waves and their stability. More specifically, we consider solutions in the form $u = e^{i\omega t}\phi$, where ϕ is real-valued. Such solutions satisfy the profile equation

$$(-\Delta)^s \phi + \omega\phi - |\phi(0)|^{2\sigma} \phi(0)\delta_0 = 0. \tag{1.5}$$

This is understood in the weak sense described above. We take the opportunity to note that in the cases considered herein, one cannot expect the positivity of ϕ, as in the classical case. This is why we keep the absolute value in (1.5).

Regarding the physical modeling which necessitates the fractional differential operators and the delta function in the non-linearity, we encourage interested reader to consult the appendix of [12] and also [13]. Note that both works deal with the case $s \in (0, 1)$. We believe that our results can motivate further investigation of such structure for $s > 1$.

The question for the stability of these waves, when $s = 1$, has been considered in several contexts recently, see [2], [6], [7] for the three dimensional case $n = 3$ and [1], for $n = 2$. Again, some of these works deal with cases mostly outside of the range of consideration herein, which is $s > \frac{n}{2}$.

Before we address the construction of the solitons (that is, of (1.5)), and since our situation is a bit non-standard, we would like to outline the framework for the stability of the waves.

1.1 Linearized problem for the concentrated NLS

As is customary, the spectral/linearized stability of the standing waves, i.e. the solutions of (1.5), guides us in the study of the actual non-linear dynamics, when one starts close to these solutions[1]. More specifically, if we linearize around the solitary waves and ignore quadratic and higher order contributions, we obtain a linear system, whose spectral picture plays a crucial part in the dynamics. To that end, we take $u = e^{i\omega t}(\phi + v)$ and plug it in (1.2). Ignoring any $O(v^2)$ term and utilizing (1.5), after setting $v := (\Re v, \Im v)$, we obtain

$$\begin{pmatrix} \Re v \\ \Im v \end{pmatrix}_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix} \begin{pmatrix} \Re v \\ \Im v \end{pmatrix}, \tag{1.6}$$

where the following fractional Schrödinger operators are introduced

$$\begin{aligned} \mathcal{L}_+ &= (-\Delta)^s + \omega - (2\sigma + 1)|\phi(0)|^{2\sigma} \delta_0, \\ \mathcal{L}_- &= (-\Delta)^s + \omega - |\phi(0)|^{2\sigma} \delta_0. \end{aligned}$$

This formulas are heuristic in the sense that the operators $\mathcal{L}_\pm$ are not yet properly defined, in terms of domains, etc. This is generally not an easy task,[2] and will be properly carried in Sect. 2.2. Introducing the operators

$$\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathcal{L} := \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix},$$

and the assignment $\begin{pmatrix} \Re v \\ \Im v \end{pmatrix} \to e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: e^{\lambda t}\mathbf{v}$, we obtain the following time-independent linearized eigenvalue problem

$$\mathcal{J}\mathcal{L}\mathbf{v} = \lambda \mathbf{v}. \tag{1.7}$$

Since we are interested in stability of waves, it will be appropriate to give a standard definition of stability as follows.

[1] And indeed in the understanding of the ranges of σ that give global existence viz. a viz blow up, as discussed above.

[2] Although, as it turns out, we shall need to restrict to the case $s > \frac{n}{2}$, which will make such definitions in a sense canonical.

Definition 1 The wave $e^{i\omega t}\phi$ is said to be **spectrally unstable** if the eigenvalue problem (1.7) has a solution $(\lambda, \mathbf{v})$ with $\Re\lambda > 0$ and $\mathbf{v} \neq 0$, $\mathbf{v} \in D(\mathcal{L})$. Otherwise, if (1.7) has no non-trivial solutions with $\Re\lambda > 0$, we say that the wave is **spectrally stable**.

We say that $e^{i\omega t}\phi$ is **orbitally stable** solution of (1.2), if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$, so that whenever $\|u_0 - \phi\|_{H^s(\mathbf{R}^n)} < \delta$, then the following statements hold.

- The solution u of (1.2), in appropriate sense, with initial data $u_0 \in H^s(\mathbf{R}^n)$ is global in $H^s(\mathbf{R}^n)$, i.e. $u(t, \cdot) \in H^s(\mathbf{R}^n)$.
- $\sup_{t>0} \inf_{\theta\in\mathbf{R}} \|u(t, \cdot) - e^{-i(\omega t+\theta)}\phi(\cdot)\|_{H^s(\mathbf{R}^n)} < \epsilon$.

The connection between the two main notions of stability, namely spectral and orbital stability, has been explored extensively in the literature—see for example the excellent book [15]. Generally speaking, spectral stability is a prerequisite for orbital stability, and in many cases of interest and under some natural, but not necessarily easy to check conditions (see Section 5.2.2 in [15]), spectral stability implies orbital stability. In the case under consideration, the Assumption 5.2.5 a) on p. 136, [15] does not apply. We provide a direct proof of orbital stability in the cases of spectral stability, via contradiction argument, by following the original idea of T.E. Benjamin.

We should also point out the reverse connection, namely spectral instability implies orbital instability. Basic heuristics (or even some more formal arguments) may suggest that this must be indeed the case. However, in terms of rigorous results, see for example [18], which states that if there is a positive instability mode present, via a direct ODE Lyapunov method spectral instability implies orbital instability. As in the stability case, there is no satisfactory general result that would cover our examples, so we leave our rigorous conclusions at the level of spectral instability of the waves and we do not comment further on orbital instability thereof.

1.2 Main results

Before we present our existence result for the singular elliptic problem (1.5), let us introduce a function $\mathcal{G}_s^\lambda$, which will be a basic building block in our analysis. Namely, for all $\lambda > 0$ and $s > 0$,

$$\widehat{\mathcal{G}_s^\lambda}(\xi) = \frac{1}{(2\pi|\xi|)^{2s} + \lambda}.$$

We first state a few results related to the existence of the waves ϕ_ω,[3] under some conditions on the parameters s, ω, n, which turn out to be necessary as well. Then, we discuss the fact that these waves are also minimizers of a Sobolev embedding inequality and we present its exact constant.

[3] Here the subscript ω is to emphasis the ω dependency of ϕ. Whenever such dependence is too important for the particular discussion, ϕ will be referred to as ϕ_ω.

1.2.1 Existence of the waves ϕ_ω

Theorem 1 (Existence standing waves of the concentrated NLS) *Let $\omega > 0$, $s > \frac{n}{2}$ and $\sigma > 0$. Then, the function ϕ, with*

$$\hat{\phi}_\omega(\xi) = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega} d\xi\right)^{-(1+\frac{1}{2\sigma})} \frac{1}{(2\pi|\xi|)^{2s} + \omega}.$$

is a solution of (1.5)*. Alternatively,*

$$\phi_\omega(x) = \frac{\mathcal{G}_s^\omega(x)}{(\mathcal{G}_s^\omega(0))^{1+\frac{1}{2\sigma}}}.$$

Interestingly, the conditions for ω and s in Theorem 1 are necessary for the existence of solutions $\phi \in H^s(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ of (1.5).

Proposition 1 *Let $\phi \in H^s(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ be a weak solution of* (1.5)*. Then, $\omega(s - \frac{n}{2}) > 0$. That is, either $\omega > 0, s > \frac{n}{2}$ or $\omega < 0, s < \frac{n}{2}$.*

The proof of Proposition 1 proceeds via the Pohozaev's identities, see Sect. 2.1 below.

In the process of the variational construction of the waves ϕ_ω, we establish a non-surprising connection to the problem for the optimal constant in the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$. More specifically, we establish that $\mathcal{G}_s = \mathcal{G}_s^1$ (and consequently ϕ_1) are H^s functions that saturate the Sobolev embedding, with the optimal Sobolev constant

$$s 2^n \pi^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right) \sin\left(\frac{n\pi}{2s}\right) \|u\|_{L^\infty}^2 \le \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \|u\|_{L^2}^2. \tag{1.8}$$

We formulate the result in the following proposition.

Proposition 2 *The function $\mathcal{G}_s$ is a solution to the Sobolev embedding minimization problem*

$$\inf_{u \in \mathcal{S}: u \neq 0} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \|u\|_{L^2}^2}{\|u\|_{L^\infty}^2} = s 2^n \pi^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right) \sin\left(\frac{n\pi}{2s}\right).$$

Next, we turn our attention towards the stability results. We first state spectral stability/instability result, followed by orbital stability statements.

1.2.2 Stability characterization of the waves ϕ_ω

Theorem 2 *Let $n \ge 1$, $s > \frac{n}{2}$ and $\omega > 0$. Then, the waves $e^{i\omega t}\phi_\omega$ are spectrally stable if and only if*

$$0 < \sigma < \frac{2s}{n} - 1.$$

That is, the waves are stable for all $0 < \sigma < \frac{2s}{n} - 1$ *and unstable, when* $\sigma > \frac{2s}{n} - 1$. *Moreover, the instability is due to a presence of a single and simple real mode in the eigenvalue problem* (1.7).

Finally, before we state our orbital stability results, we need to make some natural assumptions regarding the well-posedness of the Cauchy problem (1.2).

Clearly, the orbital stability is only expected to hold for the case $\sigma < \frac{2s}{n} - 1$, so we assume that henceforth. We make the following **key assumptions**:

(1) The solution map $g \to u_g$ has **continuous dependence on initial data property in a neighborhood of** ϕ. That is, there exists $T_0 > 0$, so that for all $\epsilon > 0$, there exists $\delta > 0$, so that whenever $g : \|g - \phi\|_{H^s} < \delta$, then $\sup_{0<t<T_0} \|u_g(t, \cdot) - e^{-i\omega t}\phi_\omega\|_{H^s} < \epsilon$.

(2) **All initial data, sufficiently close to** ϕ_ω**in** H^s**norm, generates a global in time solution** u_g of (1.2). In addition, the L^2-norm and the Hamiltonian for these solutions are conserved. That is

$$M[u_g(t)] = M[g], E[u_g(t)] = E[g].$$

First, let us mention that this exact result is already available in the one dimensional case $n = 1$, [10]. For dimensions, $n \geq 2$, we conjecture that this is also the case. That is, in parallel with the results for the standard semi-linear Schrödinger equation, we make the following conjecture—please refer to the definitions of the operator $\mathcal{L}_c$ and $D(\mathcal{L}_c)$ in (2.8) and (2.9) below.

Conjecture 1 *For* $n \geq 2$, $s > \frac{n}{2}$, $u_0 \in D(\mathcal{L}_c)$, (1.2) *is locally well-posed and the quantities* (1.3), (1.4) *are conserved. Moreover, if* $0 < \sigma < \frac{2s}{n} - 1$, *the solutions are global, whereas for* $\sigma \geq \frac{2s}{n} - 1$, *finite time blow-up is possible, at least for some initial data.*

Remark For the case $n = 1$, $s > \frac{1}{2}$, this is exactly the statement in [10].

We are now ready to state our orbital stability results.

Theorem 3 *Let* $n \geq 1$, $\omega > 0$, $s > \frac{n}{2}$, $0 < \sigma < \frac{2s}{n} - 1$. *In addition, assume continuous dependence on initial data and globality of the solutions close to* ϕ_ω, *as outlined above. Then, the solitons* $e^{i\omega t}\phi_\omega$ *are orbitally stable.*

We plan our paper as follows. In Sect. 2, we prove the Pohozaev's identities, which in turn imply the necessary conditions for existence of the waves. Then, we discuss a self-adjoint realization of the operators $(-\Delta)^s + \lambda - c\delta_0$ for $\lambda > 0, c > 0$.

In Sect. 3, we first provide a variational construction of the waves ϕ_ω. The special relation to the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$, $s > \frac{n}{2}$ is highlighted. The precise results are stated in the explicit formulas in Proposition 2. Finally, in Sect. 3.4, we discuss the lower part of the spectrum for operators in the form $(-\Delta)^s + \lambda - \mu\delta_0$. In the particular case of the linearized operator $\mathcal{L}_+$, this yields the non-degeneracy of the waves. In our specific case, this takes the form $Ker(\mathcal{L}_+) = \{0\}$, due to the broken translational symmetry.

In Sect. 4, we start with a short introduction to the instability index count theory in general, and then we apply it to the spectral stability of the waves ϕ_ω. We explicitly calculate the relevant Vakhitov-Kolokolov quantity $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle$, which provides the stability characterization of the waves described in Theorem 2. Finally, under the necessary and sufficient condition for spectral stability, $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle < 0$, we derive the coercivity of $\mathcal{L}_+$ on $\{\phi_\omega\}^\perp$, which is of course crucial in the proof of the orbital stability.

2 Preliminaries

We use the standard notations for the L^p, $1 \le p \le \infty$ spaces. The Sobolev norms $\|\cdot\|_{W^{s,p}}$ are given by

$$\|f\|_{\dot{W}^{s,p}} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p}; \quad \|f\|_{W^{s,p}} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} + \|f\|_{L^p}, 1 < p < \infty,$$

while the corresponding spaces are the completions of Schwartz functions $\mathcal{S}$ in these norms.

Of particular importance will be the Sobolev embedding, $\dot{W}^{s,p}(\mathbf{R}^n) \hookrightarrow L^q(\mathbf{R}^n)$, for $1 < p < q < \infty : s \ge n\left(\frac{1}{p} - \frac{1}{q}\right)$. Also, recall that for $s > \frac{n}{p}$, there is the embedding[4] $W^{s,p} \hookrightarrow C^{[s-\frac{n}{p}],\gamma}(\mathbf{R}^n) : 0 < \gamma < s - \frac{n}{p}$. As is well-known, the embedding $H^{\frac{n}{2}}(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$ fails, but sometimes an useful replacement estimate is the following: for all $\delta \in (0, \frac{n}{2})$, there is C_δ, so that

$$\|f\|_{L^\infty} \le C_\delta(\|f\|_{\dot{H}^{\frac{n}{2}-\delta}} + \|f\|_{\dot{H}^{\frac{n}{2}+\delta}}). \tag{2.1}$$

2.1 Pohozaev's identities and consequences

We would like to address the question for existence of solutions for the profile Eq. (1.5). Eventually, we will write them down explicitly, but first, we need to identify some necessary conditions on the parameters, which turn out to be sufficient as well. The approach here is classical, even though our problem is certainly not. We build some Pohozaev's identities, which proceeds by establishing relations between various norms of the eventual solution ϕ, which are *a priori* assumed finite. As a consequence, we find that the parameters must obey certain constraints.

Proposition 3 *Let $s > 0$ and $\phi \in H^s(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ be a weak solution of* (1.5). *Then,*

$$\|\phi\|_{L^2}^2 = \frac{2s-n}{2s\omega}|\phi(0)|^{2\sigma+2} \tag{2.2}$$

$$\|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2}^2 = \frac{n}{2s}|\phi(0)|^{2\sigma+2}. \tag{2.3}$$

Proof Testing (1.5) with ϕ itself results in

$$\|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 - |\phi(0)|^{2\sigma+2} = 0. \tag{2.4}$$

[4] Here $\{x\} = x - [x]$, where $[x] = \max\{n : n \le x\}$.

Next, we test (1.5) against $x \cdot \nabla\Psi$, for a test function Ψ. We obtain, by taking into account the commutation relation $[(-\Delta)^s, x\cdot\nabla] = 2s(-\Delta)^s$, that

$$\begin{aligned}\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}[x\cdot\nabla\Psi]\rangle &= \langle\phi, x\cdot\nabla(-\Delta)^s\Psi\rangle + 2s\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}\Psi\rangle = \\ &= -\langle x\cdot\nabla\phi, (-\Delta)^s\Psi\rangle + (2s-n)\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}\Psi\rangle = \\ &= -\langle(-\Delta)^{\frac{s}{2}}[x\cdot\nabla\phi], (-\Delta)^{\frac{s}{2}}\Psi\rangle + (2s-n)\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}\Psi\rangle.\end{aligned}$$

This implies the equality

$$\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}[x\cdot\nabla\Psi]\rangle + \langle(-\Delta)^{\frac{s}{2}}[x\cdot\nabla\phi], (-\Delta)^{\frac{s}{2}}\Psi\rangle = (2s-n)\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}\Psi\rangle.$$

Note that the right-hand side of this expression makes sense for[5], $\Psi = \phi$ whence

$$\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}[x\cdot\nabla\Psi]\rangle = (s-\frac{n}{2})\|(-\Delta)^{\frac{s}{2}}\phi\|^2. \tag{2.5}$$

Also[6]

$$\langle\phi, x\cdot\nabla\Psi\rangle = -n\langle\phi,\Psi\rangle - \langle x\cdot\nabla\phi, \Psi\rangle,$$

which also makes sense for $\Psi = \phi$, whence

$$\langle\phi, x\cdot\nabla\Psi\rangle = -\frac{n}{2}\|\phi\|^2_{L^2}. \tag{2.6}$$

Finally, we claim that $\langle\delta_0, x\cdot\nabla\Psi\rangle = 0$ for each test function Ψ. Indeed, let us introduce a smooth radial function $V : \mathbf{R}^n \to \mathbf{R}$, which is non-negative, supported on $\mathbf{B} := \{x \in \mathbf{R}^n : \|x\| < 1\}$ and normalized so that $\int_{\mathbf{R}^n} V(x)dx = 1$. It is well-known that, in a distribution sense, one can use the approximation $N^n V(Nx) \to \delta_0$. That is, $\lim_{N\to\infty}\langle N^n V(N\cdot), f\rangle = f(0)$. So,

$$\begin{aligned}\langle\delta_0, x\cdot\nabla\Psi\rangle &= \lim_{N\to\infty} N^n \sum_{j=1}^n \int_{\mathbf{R}^n} V(Nx)x_j\partial_j\Psi(x)dx \\ &= \lim_{N\to\infty}\left[-nN^n\int_{\mathbf{R}^n} V(Nx)\Psi(x)dx - N^{n+1}\int_{\mathbf{R}^n}|x|V'(Nx)\Psi(x)dx\right] = 0,\end{aligned}$$

since

$$N^{n+1}\int_{\mathbf{R}^n}|x|V'(Nx)dx = \int_{\mathbf{R}^n}|y|V'(y)dy = |\mathbb{S}^{n-1}|\int_0^\infty V'(\rho)\rho^n d\rho = -n\int_0^\infty V(\rho)\rho^{n-1}d\rho = -n.$$

[5] One can formally take limits of $\Psi_n : \|\Psi_n - \phi\|_{H^s} \to 0$.

[6] Note that $\phi \in H^1(\mathbf{R}^n)$ makes this well-defined.

Putting $\langle \delta_0, x \cdot \nabla \Psi \rangle = 0$ together with (2.5), (2.6), implies

$$(s - \frac{n}{2}) \|(-\Delta)^{\frac{s}{2}} \phi\|_{L^2}^2 - \frac{\omega n}{2} \|\phi\|_{L^2}^2 = 0. \tag{2.7}$$

Solving the system of Eqs. (2.4) and (2.7) results in the relations (2.2) and (2.3). □

An immediate corollary of these results follows from the positivity of the norms in both (2.2) and (2.3). This is given by Proposition 1. Namely, either $\omega > 0, s > \frac{n}{2}$ or $\omega < 0, s < \frac{n}{2}$. Clearly, the case $\omega > 0, s > \frac{n}{2}$ is a more physical situation to consider—after all, one has the embedding $H^s(\mathbf{R}^n) \hookrightarrow C(\mathbf{R}^n)$ and hence functions in the class $H^s(\mathbf{R}^n)$ are automatically continuous.

2.2 The self-adjoint operators $(-\Delta)^s + \lambda - c\delta_0$

In this section, we introduce the necessary self-adjoint extensions of the operators formally introduced as $(-\Delta)^s + \lambda - c\delta_0$. There has been quite a bit of recent work on the subject, see [1,2,4,7,10], among others. In these papers, various (and sometimes all) self-adjoint extensions of such objects have been studied, under many different assumptions on the parameters. As dictated by the results of Proposition 1, we work under the assumption $s > \frac{n}{2}$. Incidentally, this simplifies matters quite a bit, in the sense that the self-adjoint extension, which generates the standard quadratic form, is canonical.

More specifically, for given constants $\lambda > 0, c > 0$, we introduce the following skew-symmetric quadratic form

$$\mathcal{Q}_c(f, g) = \langle \sqrt{((-\Delta)^s + \lambda) f}, \sqrt{((-\Delta)^s + \lambda) g} \rangle - c f(0) \bar{g}(0), \; f, g \in D(\mathcal{Q})$$

with domain $D(\mathcal{Q}) = H^s(\mathbf{R}^n)$. Note that since $D(\mathcal{Q}) \subset C(\mathbf{R}^n)$, the values $f(0), g(0)$ make sense. In addition, the form $\mathcal{Q}$ is bounded from below. This is a consequence of the Sobolev embedding $H^\alpha(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n), \alpha > \frac{n}{2}$. Indeed, choose $\alpha : \frac{n}{2} < \alpha < s$ and estimate via the Sobolev and the Gagliardo-Nirenberg's inequalities

$$\begin{aligned} \mathcal{Q}_c(f, f) &\geq c_\lambda \|f\|_{H^s}^2 - k_\alpha \|f\|_{H^\alpha}^2 \geq c_\lambda \|f\|_{H^s}^2 - k_\alpha \left(\frac{c_\lambda}{2k_\alpha} \|f\|_{H^s}^2 + d_{\alpha,\lambda} \|f\|_{L^2}^2\right) \\ &\geq D_{\alpha,\lambda} \|f\|_{\dot{H}^s}^2 - M_{\alpha,\lambda} \|f\|_{L^2}^2. \end{aligned}$$

In addition, $\mathcal{Q}$ is closed form, as $\|f\|_{H^s}^2 \sim \mathcal{Q}(f, f) + M\|f\|^2$, for large enough M. According to the standard theory for quadratic forms, see Theorem VIII.15 in [22], there is an unique self-adjoint operator $\mathcal{L}_c$, so that

$$D(\mathcal{L}_c) \subset D(\mathcal{Q}), \;\; \mathcal{D}_c(f, g) = \langle \mathcal{L}_c f, g \rangle, \;\; \forall f \in D(\mathcal{L}_c), g \in D(\mathcal{Q}).$$

Identifying the exact form of $\mathcal{L}_c$ may not be an easy task, in general. In our case, this is not so hard, as the operator has been essentially constructed in previous works, see [10] for the one dimensional case. We follow their notations and approach. To this end, introduce the Green's function of the operator $(-\Delta)^s + \lambda$, namely the function $\mathcal{G}_s^\lambda$, so that

$$((-\Delta)^s + \lambda)\mathcal{G}_s^\lambda = \delta_0.$$

By taking the Fourier transform, we can write the following formula for $\mathcal{G}_s^\lambda$

$$\widehat{\mathcal{G}_s^\lambda}(\xi) = \frac{1}{(2\pi|\xi|)^{2s} + \lambda}.$$

Clearly, since $s > \frac{n}{2}$, $\mathcal{G}_s^\lambda \in H^s(\mathbf{R}^n) \subset C(\mathbf{R}^n)$. Introduce the domain of the operator $\mathcal{L}_c$ as

$$D(\mathcal{L}_c) = \{\psi \in H^s(\mathbf{R}^n) : \psi = g + c\psi(0)\mathcal{G}_s^\lambda,\ g \in H^{2s}(\mathbf{R}^n)\} \subset H^s(\mathbf{R}^n). \tag{2.8}$$

With this domain, its action is defined as

$$\mathcal{L}_c\psi := ((-\Delta)^s + \lambda)g. \tag{2.9}$$

Note that for $\psi \in D(\mathcal{L}_c)$ and $h \in H^s(\mathbf{R}^n) = D(\mathcal{Q})$, we have

$$\begin{aligned}\langle \mathcal{L}_c\psi, h\rangle &= \langle ((-\Delta)^s + \lambda)g, h\rangle = \langle \sqrt{(-\Delta)^s + \lambda}\psi, \sqrt{(-\Delta)^s + \lambda}h\rangle - c\psi(0)\langle ((-\Delta)^s + \lambda)\mathcal{G}_s^\lambda, h\rangle \\ &= \langle \sqrt{(-\Delta)^s + \lambda}\psi, \sqrt{(-\Delta)^s + \lambda}h\rangle - c\psi(0)\bar{h}(0) = \mathcal{Q}_c(\psi, h).\end{aligned}$$

Thus, $\mathcal{L}_c$ is a closed symmetric operator, with a quadratic form precisely $\mathcal{Q}$. Note that the role of the constant λ in the definition is to ensure that the function $\widehat{\mathcal{G}_s^\lambda}$ has no singularity at $\xi = 0$. We now need to show that $\mathcal{L}_c$ is precisely the unique self-adjoint operator with this property.

Lemma 1 *The closed symmetric operator $\mathcal{L}_c$, with domain given in* (2.8) *and whose action is defined in* (2.9)*, is self-adjoint.*

Proof For technical reasons, let us first assume the condition

$$c\mathcal{G}_s^\lambda(0) \neq 1. \tag{2.10}$$

With that, we work on a different representation on $D(\mathcal{L}_c)$. More precisely, we would like to write ψ purely in terms of g. To this end, we evaluate the identity relating ψ and g at $x = 0$. We obtain the equation for $\psi(0)$

$$\psi(0) = g(0) + c\psi(0)\mathcal{G}_s^\lambda(0).$$

This equation has a solution, under the condition (2.10),

$$\psi(0) = \frac{g(0)}{1 - c\mathcal{G}_s^\lambda(0)}. \tag{2.11}$$

One can now write, for $c \neq \frac{1}{\mathcal{G}_s^\lambda(0)}$,

$$D(\mathcal{L}_c) = \{\psi \in L^2(\mathbf{R}^n) : \psi = g + c\mathcal{G}_s^\lambda \frac{g(0)}{1 - c\mathcal{G}_s^\lambda(0)}, g \in H^{2s}(\mathbf{R}^n)\},$$

which describes $D(\mathcal{L}_c)$ purely in terms of an arbitrary function $g \in H^{2s}(\mathbf{R}^n)$.

In order to show that $\mathcal{L}_c = \mathcal{L}_c^*$, it suffices to show that it has a real number in its resolvent set, see Corollary on p. 137, [21]. To this end, let $M >> 1$, and we will show that $-M - \lambda \in \rho(\mathcal{L}_c)$. Let $f \in L^2(\mathbf{R}^n)$ is arbitrary and consider

$$(\mathcal{L}_c + M - \lambda)\psi = f. \tag{2.12}$$

This is of course equivalent to the equation $((-\Delta)^s + M)g = f$, where

$$\psi = g + c\mathcal{G}_s^\lambda \frac{g(0)}{1 - c\mathcal{G}_s^\lambda(0)}.$$

which has the unique solution $g = ((-\Delta)^s + M)^{-1} f \in H^{2s}(\mathbf{R}^n)$. Thus, we can uniquely solve (2.12) as follows

$$\psi = g + c\mathcal{G}_s^\lambda \frac{g(0)}{1 - c\mathcal{G}_s^\lambda(0)}, \quad g = ((-\Delta)^s + M)^{-1} f \in H^{2s}(\mathbf{R}^n).$$

In terms of estimates $\|g\|_{H^{2s}} \leq C_M \|f\|_{L^2}$ and consequently

$$\|\psi\|_{L^2} \leq \|g\|_{L^2} + C|g(0)| \leq \|g\|_{H^s} \leq C_M \|f\|_{L^2}.$$

This shows that all $\mathcal{L}_c$, with c satisfying (2.10) are self-adjoint. What about c, which fails (2.10)? In this case

$$1 = c\mathcal{G}_s^\lambda(0) = c \int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \lambda} d\xi$$

It follows that for every $\tilde{\lambda} \neq \lambda$, say $\tilde{\lambda} > \lambda$, we have that $c\mathcal{G}_s^{\tilde{\lambda}}(0) \neq 1$. Thus, following the scheme described in the previous arguments, the operator $\mathcal{L}_c^{\tilde{\lambda}}$, formally defined through $(-\Delta)^s + \tilde{\lambda} - c\delta_0$ is self-adjoint. This means that $\mathcal{L}_c = \mathcal{L}_c^\lambda = \mathcal{L}_c^{\tilde{\lambda}} + (\lambda - \tilde{\lambda})Id$, is self-adjoint as well. □

Note that as a result of the definition of $D(\mathcal{L}_c)$, we give the following important formula, for the action on functions $\psi \in H^s(\mathbf{R}^n)$, with $\psi(0) = 0$. Namely,

$$\mathcal{Q}_c(\psi, \psi) = \|(-\Delta)^{\frac{s}{2}} \psi\|_{L^2}^2 + \lambda \|\psi\|_{L^2}^2. \tag{2.13}$$

3 Variational construction of the waves ϕ_ω and spectral consequences

We first construct, in a variational manner, some approximate solutions to the elliptic profile problem (1.5). This will turn out to be important in our subsequent considerations.

3.1 Variational constructions

Let $\omega, \sigma > 0$. For a radial function $V : \mathbf{R}^n \to \mathbf{R}$ as before[7] and $N >> 1$, consider the functional

$$I_{\omega,N}[u] = \frac{\int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2 dx + \omega \int_{\mathbb{R}^n} u^2 dx}{\left(\int_{\mathbb{R}^n} N^n V(Nx)|u|^{2\sigma+2}dx\right)^{\frac{1}{\sigma+1}}},$$

and the corresponding unconstrained variational problem $I_{\omega,N}[u] \to \min$. Clearly, $I_{\omega,N}[u] > 0$, so its optimal value is well-defined

$$m_N(\omega) := \inf_{u \in \mathcal{S}, u \neq 0} I_{\omega,N}[u].$$

Proposition 4 *Let $s > \frac{n}{2}$. Then the unconstrained minimization problem*

$$I_{\omega,N}[u] \to \min \tag{3.1}$$

has a real-valued solution $\phi_N \in H^s(\mathbf{R}^n) \cap L^\infty$, in particular $m_N(\omega) > 0$. Moreover, ϕ_N may be chosen to satisfy

$$N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2} dx = 1.$$

Finally, ϕ_N satisfies the Euler-Lagrange equation

$$(-\Delta)^s \phi_N + \omega\phi_N - m_N(\omega) N^n V(Nx)|\phi_N|^{2\sigma}\phi_N = 0 \tag{3.2}$$

in distributional sense.

Proof Since $\|V\|_{L^1} = 1$, we have for $u \in H^s(\mathbf{R}^n) \subset L^\infty$,

$$\left(N^n \int_{\mathbf{R}^n} V(Nx)|u(x)|^{2\sigma+2}dx\right)^{\frac{1}{\sigma+1}} \leq \|u\|^2_{L^\infty(\mathbf{R}^n)} \leq C\|u\|^2_{H^s(\mathbf{R}^n)}, \tag{3.3}$$

[7] That is, V is non-negative, radial, smooth and supported on the unit ball $\mathbf{B} \subset \mathbf{R}^n$, with $\int_{\mathbf{B}} V(x)dx = 1$.

whence (3.1) is a well-posed variational problem and $m_N(\omega) > 0$. Next, due to dilation properties of the functional $I_{\omega,N}$, we can assume that the infimum is taken only over functions with the normalization property

$$N^n \int_{\mathbb{R}^n} V(Nx)|u(x)|^{2\sigma+2} dx = 1.$$

Let u_k be a minimizing sequence such that $\int_{\mathbb{R}^n} N^n V(Nx)|u_k|^{2\sigma+2} dx = 1$ and hence

$$\lim_k (\|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2}^2 + \omega \|u_k\|_{L^2}^2) = m_N(\omega).$$

By weak compactness, we can select a weakly convergent subsequence (which we assume is just $\{u_k\}$), $u_k \rightharpoonup u$. By the lower semi-continuity of the norms, with respect to weak convergence

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \omega \|u\|_{L^2}^2 \leq \liminf_k (\|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2}^2 + \omega \|u_k\|_{L^2}^2) = m_N(\omega). \quad (3.4)$$

We now show that $\{u_k\}$ is pre-compact in $C(\mathbf{B})$. Indeed, since $s > \frac{n}{2}$, we have by the Sobolev embedding that

$$\|u_k\|_{C^\gamma(\mathbf{R}^n)} \leq C \|u_k\|_{H^s}, \quad (3.5)$$

for $0 < \gamma < \{s - \frac{n}{2}\}$. Consequently, u_k are uniformly Hölder-continuous, hence equicontinuous as elements of $C(\mathbf{B})$. Also, $\{u_k\}$ is a totally bounded by (3.5). By Arzelà-Ascoli, we have that $\{u_k\}_{k=1}^\infty$ is pre-compact in $C(\mathbf{B})$. That is, for a subsequence, which we again assume it is just u_k, we have that $u_k \rightrightarrows_{\mathbf{B}} u$. It is now clear that

$$1 = \lim_k N^n \int_{\mathbf{R}^n} V(Nx)|u_k(x)|^{2\sigma+2} dx = N^n \int_{\mathbf{R}^n} V(Nx)|u(x)|^{2\sigma+2} dx. \quad (3.6)$$

Thus, by (3.4)and (3.6), we conclude that $I_{\omega,N}[u] \leq m_N(\omega)$. This, by the definition of $m_N(\omega)$ means that $I_{\omega,N}[u] = m_N(\omega)$. In particular,

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \omega \|u\|_{L^2}^2 = m_N(\omega),$$

so u actually solves the minimization problem (3.1). This is the solution ϕ_N, that we were interested in.

Next, we show that the minimizer satisfies the Euler Lagrange equation. To this end, we take an arbitrary test function h and let $\epsilon > 0$ consider $u = \phi_N + \epsilon h$, and recall that

$$\int N^n V(Nx)|\phi_N|^{2\sigma+2} dx = 1.$$

Since ϕ_N is a minimizer we have that $I_{\omega,N}[u] \geq m_N(\omega)$. Expanding in powers of ϵ, we obtain

$$\int |(-\Delta)^{s/2}(\phi_N + \epsilon h)|^2 dx + \omega \int (\phi_N + \epsilon h) dx = m_N(\omega) + 2\epsilon \langle ((-\Delta)^s + \omega)\phi_N, h\rangle + O(\epsilon^2).$$

Similarly,

$$\int V(Nx)|\phi_N+\epsilon h|^{2\sigma+2}dx = \int V(Nx)|\phi_N|^{2\sigma+2}dx + (2\sigma+2)\epsilon\int V(Nx)|\phi_N|^{2\sigma}\phi_N h + O(\epsilon^2)$$
$$= 1 + (2\sigma+2)\epsilon\int V(Nx)|\phi_N|^{2\sigma}\phi_N h + O(\epsilon^2).$$

Thus, after simplifying, we arrive at

$$I_{\omega,N} = \frac{m_N(\omega) + 2\epsilon\langle((-\Delta)^s+\omega)\phi_N, h\rangle + O(\epsilon^2)}{1+2\epsilon\int N^n V(Nx)|\phi_N|^{2\sigma}\phi_N h dx + O(\epsilon^2)}$$
$$= m_N(\omega) + 2\epsilon\langle((-\Delta)^s+\omega)\phi_N - m_N(\omega)N^n V(Nx)|\phi_N|^{2\sigma}\phi_N, h\rangle + O(\epsilon^2).$$

Since this hold for any arbitrary test function h and any $\epsilon > 0$ we have that ϕ_N solves (3.2). □

Next, we have the following technical result.

Lemma 2 *There exists constants $C_1(\omega)$, $C_2(\omega)$, but independent on N, so that*

$$C_1(\omega) \le m_N(\omega) \le C_2(\omega).$$

Furthermore, the sequence $\{\phi_N\}_{N=1}^{\infty}$, is a pre-compact in every set of the form $C(K)$, where K is a compact subset of $\mathbf{R}^n$.

Proof The lower bound, with a constant independent on N follows from (3.3). The upper bound follows by testing against a concrete function like $u_0(x) = e^{-|x|^2}$. Since $\frac{1}{3} < u_0(x) \le 1$, on the support of $V(Nx)$, $N \ge 1$, we have that

$$m_N(\omega) \le I_{\omega,N}[u_0] \le 9\left(\|(-\Delta)^{\frac{s}{2}}u_0\|_{L^2}^2 + \omega\|u_0\|_{L^2}^2\right) =: C_2(\omega).$$

Next, since ϕ_N satisfy $N^n\int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2}dx = 1$, we have that $I_{\omega,N}[\phi_N] = \|(-\Delta)^{\frac{s}{2}}\phi_N\|_{L^2}^2 + \|\phi_N\|_{L^2}^2 = m_N(\omega)$. Thus, by Sobolev embedding

$$\|\phi_N\|_{C^\gamma(\mathbf{R}^n)} \le C\|\phi_N\|_{H^s} \le C(\omega)m_N(\omega) \le C_3(\omega).$$

for $0 < \gamma < \min\{1, s-\frac{n}{2}\}$. It follows by Arzela-Ascolli's theorem that for each compact $K \subset \mathbf{R}^n$, $\{\phi_N\}$ is pre-compact in $C(K)$. □

Clearly, Lemma 2 allows us to take a convergent (sub) sequence as $N \to \infty$. We wish to learn what the limit is expected to be. It turns out that it is nothing but the minimizer for the Sobolev inequality $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$. We justify that in the next section.

3.2 Relation to the minimizers for the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$

For $s > \frac{n}{2}$, $\omega > 0$, we study the functional

$$J_\omega[u] = \frac{\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \omega \|u\|_{L^2}^2}{\|u\|_{L^\infty}^2}$$

and the corresponding minimization problem $J_\omega[u] \to \min$. Finally, denote

$$c^2(\omega) := \inf_{u \in \mathcal{S}: u \neq 0} J_\omega[u].$$

The described optimization problem has a clear analytical interpretation, namely that c is the exact constant in the Sobolev embedding estimate

$$c(\omega)\|u\|_{L^\infty} \leq |||u|||_{H^s} := \sqrt{\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \omega \|u\|_{L^2}^2}.$$

We know from the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$ that c is well-defined and we can alternatively introduce it as follows $c(\omega) = \sup\{C > 0 : C\|u\|_{L^\infty} \leq |||u|||_{H^s}, \forall u \in \mathcal{S}\}$.

Another useful observation is that one can assume, without loss of generality, that in the infimum procedure described above, $\|u\|_{L^\infty}$ is replaced by $|u(0)|$. That is,

$$c^2(\omega) = \inf_{u \in H^s: u(0) \neq 0} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \omega \|u\|_{L^2}^2}{|u(0)|^2}.$$

Lemma 3 *Let $s > \frac{n}{2}, \omega > 0$ and $\gamma < \min(1, s - \frac{n}{2})$. Then, there exists $C = C(s, \omega, \gamma)$, so that*

$$c^2(\omega) \leq m_N(\omega) \leq c^2(\omega) + C N^{-\gamma} \tag{3.7}$$

Proof By (3.3), we see that for every $N \geq 1$, $I_{\omega,N} \geq J_\omega$, whence $m_N(\omega) \geq c^2(\omega)$.

For the opposite inequality, observe first that since $m_N(\omega) \leq C_2(\omega)$, we can take

$$m_N(\omega) = \inf_{u \in \mathcal{S}: u \neq 0} I_{\omega,N}[u] = \inf_{N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2} dx = 1;\ \ |||u|||_{H^s} \leq 10 C_2} I_{\omega,N}[u].$$

So, let $u \in H^s : N^n \int_{\mathbf{R}^n} V(Nx)|u(x)|^{2\sigma+2} dx = 1;\ \ |||u|||_{H^s} \leq 10C_2$. Recall that for every $q > 1$, there is C_q, so that for $a > 0, b > 0$ $|a^q - b^q| \leq C_q |a - b|(a^{q-1} + b^{q-1})$. As a consequence, by Sobolev embedding together with the definition

$$\|u\|_{C^\gamma} := \sup_{x, y: x - y \neq 0} \frac{|u(x) - u(y)|}{|x - y|^\gamma},$$

we have that,

$$\left| |u(x)|^{2\sigma+2} - |u(0)|^{2\sigma+2} \right| \leq C_\sigma |u(x) - u(0)| \|u\|_{L^\infty}^{2\sigma+1} \leq C_{\gamma,\sigma} |x|^\gamma \|u\|_{C^\gamma(\mathbf{R}^n)}^{2\sigma+1} \leq C_{\gamma,\sigma} |x|^\gamma \|u\|_{H^s}^{2\sigma+1}.$$

Since $|||u|||_{H^s} \leq 10C_2$, we conclude

$$\left| |u(x)|^{2\sigma+2} - |u(0)|^{2\sigma+2} \right| \leq C_{\gamma,\sigma,\omega} |x|^\gamma. \tag{3.8}$$

It follows that

$$\begin{aligned} \left| |u(0)|^{2\sigma+2} - 1 \right| &= \left| |u(0)^{2\sigma+2} - N^n \int_{\mathbf{R}^n} V(Nx)|u(x)|^{2\sigma+2} dx \right| \\ &= N^n \left| \int_{\mathbf{R}^n} V(Nx)[|u(x)|^{2\sigma+2} - |u(0)|^{2\sigma+2} dx \right| \leq C_{\gamma,\sigma,\omega} N^n \int_{\mathbf{R}^n} V(Nx)|x|^\gamma dx \\ &\leq C_{\gamma,\sigma,\omega} N^{-\gamma} \int_{\mathbf{R}^n} V(y)|y|^\gamma dy \leq C_{\gamma,\sigma,\omega} N^{-\gamma}, \end{aligned}$$

so $|u(0)| \leq 1 + C_{\gamma,\sigma,\omega} N^{-\gamma}$. Finally,

$$\begin{aligned} m_N(\omega) &= \inf_{N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2} dx = 1;\ |||u|||_{H^s} \leq 10C_2} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \omega \|u\|_{L^2}^2 \leq \\ &\leq (1 + C_{\gamma,\sigma,\omega} N^{-\gamma}) \inf_{|||u|||_{H^s} \leq 10C_2, u(0) \neq 0} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \omega \|u\|_{L^2}^2}{|u(0)|^2} \leq c^2 + C_{\gamma,\sigma,\omega} N^{-\gamma}. \end{aligned}$$

□

We now take limit as $N \to \infty$. In view of our discussion so far, it is not surprising that this procedure yields the minimizers for the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$. In turn, this allows us to present an explicit formula for the solutions of (1.5) and to interpret them as minimizers of the Sobolev embedding problem.

3.3 Description of the solutions for the profile equation (1.5)

Lemma 4 *Let* $s > \frac{n}{2}, \omega > 0$. *Then, for every constant* $C \neq 0$, *the function*

$$\hat{\phi}(\xi) = \frac{C}{(2\pi|\xi|)^{2s} + \omega}, \tag{3.9}$$

is a minimizer of the problem $\min_{u \in H^s} J_\omega[u]$. *In particular, the optimal Sobolev constant is given by the formula*

$$c^2(\omega) = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega} d\xi \right)^{-1}.$$

Proof From Lemma 3, it follows that $\lim_N m_N(\omega) = c^2(\omega)$. In addition, as we have pointed out, maximizers can be taken, with the property $\|\phi_N\|_{H^s} \leq C(\omega)$. As $H^s(\mathbf{R}^n)$ embeds in $C^\gamma(\mathbf{R}^n)$ for $0 < \gamma < s - \frac{n}{2}$ and this is compact embedding on bounded domains, we can select

$$\phi_N : N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2} dx = 1,$$

so that ϕ_N is uniformly convergent, on the compact subsets of $\mathbf{R}^n$ to $\phi \in H^s(\mathbf{R}^n)$.

We will show that $\phi(0) = 1$ and ϕ is in the form (3.9). We have, for each $N \geq 1$,

$$\begin{aligned}\left|1 - |\phi(0)|^{2\sigma+2}\right| &\leq N^n \int_{\mathbf{R}^n} V(Nx) \left||\phi_N(x)|^{2\sigma+2} - |\phi(0)|^{2\sigma+2}\right| dx \\ &\leq C_\sigma(\|\phi_N\|_{L^\infty}^{2\sigma+1} + |\phi(0)|^{2\sigma+1}) N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x) - \phi(0)| dx.\end{aligned}$$

But $\|\phi_N\|_{L^\infty} \leq \|\phi_N\|_{H^s} < C(\omega)$, while

$$|\phi_N(x) - \phi(0)| \leq |\phi_N(x) - \phi_N(0)| + |\phi_N(0) - \phi(0)| \leq C_\gamma |x|^\gamma + |\phi_N(0) - \phi(0)|.$$

Plugging this back in our estimate for $|1 - |\phi(0)|^{2\sigma+2}|$, we obtain, for each $0 < \gamma < s - \frac{n}{2}$,

$$|1 - |\phi(0)|^{2\sigma+2}| \leq C|\phi_N(0) - \phi(0)| + CN^n \int_{\mathbf{R}^n} V(Nx)|x|^\gamma dx \leq C|\phi_N(0) - \phi(0)| + CN^{-\gamma}.$$

Clearly, the expression on the right goes to zero as $N \to \infty$, as $\phi_N \rightrightarrows_{\mathbf{B}} \phi$. By adjusting the sign of ϕ_N, if necessary, this implies that we can take $\phi(0) = \lim_N \phi_N(0) = 1$.

Next, ϕ_N satisfies the Euler-Lagrange Eq. (3.2). Test this equation with ψ. We obtain

$$\langle \phi_N, ((-\Delta)^s + \omega)\psi \rangle = m_N(\omega) N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N|^{2\sigma} \phi_N(x)\psi(x) dx. \tag{3.10}$$

Taking limits in N then yields, after taking into account $\phi(0) = 1$,

$$\langle \phi, ((-\Delta)^s + \omega)\psi \rangle = c^2(\omega)\psi(0). \tag{3.11}$$

In other words, ϕ satisfies the equation

$$((-\Delta)^s + \omega)\phi - c^2\delta_0 = 0. \tag{3.12}$$

in a distributional sense.

By taking ψ in (3.10) to be an appropriate approximation of the function $\mathcal{G}_s^\omega(\cdot + x)$, we conclude that

$$\phi(x) = const.\mathcal{G}_s^\omega(x)$$

which is of course the same as (3.9). Additionally, by testing (3.12) by ϕ itself, we obtain

$$\|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 = c^2\phi(0)^2 = c^2.$$

This shows that ϕ is a minimizer for $\min_{u\in H^s} J_\omega[u]$ and so any function in the form (3.9) is one as well. Also,

$$c^2(\omega) = \frac{\|(-\Delta)^{\frac{s}{2}}\mathcal{G}_s^\omega\|_{L^2}^2 + \omega\|\mathcal{G}_s^\omega\|_{L^2}^2}{(\mathcal{G}_s^\omega(0))^2} = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi\right)^{-1}. \tag{3.13}$$

□

We now state a result that describes the solutions of (1.5).

Lemma 5 *The non-trivial solutions to (1.5), with $\phi(0) > 0$ are given by*

$$\hat{\phi}(\xi) = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi\right)^{-(1+\frac{1}{2\sigma})} \frac{1}{(2\pi|\xi|)^{2s}+\omega}. \tag{3.14}$$

Proof We can proceed as in the proof of Lemma 4 to see that

$$\hat{\phi}(\xi) = |\phi(0)|^{2\sigma}\phi(0)\frac{1}{(2\pi|\xi|)^{2s}+\omega}.$$

In order to determine $\phi(0)$, we apply the inverse Fourier transform to obtain an equation for it as follows

$$\phi(0) = \int_{\mathbf{R}^n} \hat{\phi}(\xi)d\xi = |\phi(0)|^{2\sigma}\phi(0)\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi.$$

It follows that

$$|\phi(0)|^{2\sigma} = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi\right)^{-1},$$

which proves the claim. □

3.4 The spectrum of $(-\Delta)^s + \omega - \mu\delta_0$

In this section, we develop some tools to study the bottom of the spectrum of the operators $(-\Delta)^s + \omega - \mu\delta_0$, depending on the value of μ. More specifically, we have the following result.

Proposition 5 *Let $s > \frac{n}{2}, \omega > 0$ and $L_\mu = (-\Delta)^s + \omega - \mu\delta_0$ be the self-adjoint operator introduced in Lemma 1. Then,*

- *If $\mu > c^2(\omega)$, the operator L_μ has one simple negative eigenvalue, $-\lambda_{\omega,\mu} < 0$, with eigenfunction $\Psi_0 : \widehat{\Psi}_0(\xi) = \frac{1}{(2\pi|\xi|)^{2s}+\omega+\lambda_{\omega,\mu}}$. For the rest of the spectrum*

$$\sigma(L_\mu)\setminus\{-\lambda_{\omega,\mu}\} \subset [\omega,\infty).$$

In particular, $L_\mu|_{\{\Psi_0\}^\perp} \geq \omega$.

- *If* $\mu = c^2(\omega)$, $L_\mu \geq 0$, 0 *is a simple eigenvalue, with an eigenfunction* Ψ_0 *defined as above. For the rest of the spectrum, there is* $\sigma(L_\mu)\setminus\{0\} \subset [\omega, \infty)$. *In particular,* $L_\mu|_{\{\Psi_0\}^\perp} \geq \omega$.
- *If* $\mu < c^2(\omega)$, *there is a simple eigenvalue* $\lambda_{\omega,\mu} \in (0, \omega)$, *with eigenfunction* $\Psi_0 : \widehat{\Psi}_0(\xi) = \frac{1}{(2\pi|\xi|)^{2s}+\omega-\lambda_{\omega,\mu}}$ *and* $\sigma(L_\mu) \setminus \{\lambda_{\omega,\mu}\} \subset [\omega, \infty)$. *In particular,* $L_\mu|_{\{\Psi_0\}^\perp} \geq \lambda_{\omega,\mu} > 0$.

Proof Assume first $\mu > c^2$. We would like to formally analyze the eigenvalue problem associated with the lowest eigenvalue of L_μ. So, we are looking for $f \neq 0$, $f \in D(L_\mu)$, so that $L_\mu f = -\lambda f$ for some $\lambda > 0$. This is the equation

$$((-\Delta)^s + \omega + \lambda)f = \mu f(0)\delta_0. \tag{3.15}$$

Arguing as in the proof of Lemma 4, by taking Fourier transform etc., we find that all possible solutions are in the form

$$\hat{f}(\xi) = \frac{\mu f(0)}{(2\pi|\xi|)^{2s} + \omega + \lambda}.$$

Clearly, $f \in D(L_\mu)$ and we need to see that there exists $\lambda > 0$, so that it solves (3.15). To this end, we have

$$f(0) = \int_{\mathbf{R}^n} \hat{f}(\xi)d\xi = \mu f(0)\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda}d\xi.$$

As we seek non-trivial solutions f (and hence $f(0) \neq 0$), this amounts to finding λ, so that for the given ω, we have

$$\mu\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda}d\xi = 1. \tag{3.16}$$

We claim that under the condition $\mu > c^2$, there is exactly one solution $\lambda = \lambda_{\omega,\mu} \in (0, \infty)$. Indeed, consider the continuous and decreasing function

$$h(\lambda) := \mu\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda}d\xi - 1.$$

Computing its limits at the ends of the interval

$$\lim_{\lambda\to 0+} h(\lambda) = \mu\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega}d\xi - 1 = \frac{\mu}{c^2} - 1 > 0, \lim_{\lambda\to+\infty} h(\lambda) = -1,$$

implies that there is a unique eigenvalue $\lambda_{\omega,\mu} > 0$. Moreover, the corresponding eigenfunction is, up to a multiplicative constant

$$\widehat{\Psi}_0(\xi) = \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda_{\omega,\mu}}.$$

We now prove the statement about the rest of the spectrum. Consider the spectral decomposition of the self-adjoint operator L_μ. Assume for a contradiction that for any $\delta > 0$, we have that $\sigma(L_\mu)\cap(-\lambda_{\omega,\mu}+\delta, \omega-\delta) \neq \emptyset$. Let $\Psi \in Image(\mathbb{P}_{(-\lambda_{\omega,\mu}+\delta,\omega-\delta)})$ (i.e. $\Psi = \mathbb{P}_{(-\lambda_{\omega,\mu}+\delta,\omega-\delta)}\Psi$) and then normalize it, that is $\|\Psi\|_{L^2} = 1$. As $\Psi_0(0) = \int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega+\lambda_{\omega,\mu}}d\xi > 0$, consider the well-defined element of $D(L_\mu)$,

$$\tilde{\Psi} := \Psi - \frac{\Psi(0)}{\Psi_0(0)}\Psi_0.$$

Note that $\tilde{\Psi}(0) = 0$, so according to (2.13), we have,

$$\langle L_\mu\tilde{\Psi}, \tilde{\Psi}\rangle = \|(-\Delta)^{\frac{s}{2}}\tilde{\Psi}\|^2_{L^2} + \omega\|\tilde{\Psi}\|^2_{L^2} \geq \omega\|\tilde{\Psi}\|^2_{L^2} \geq \omega\|\Psi\|^2_{L^2} = \omega.$$

where we have used that $\Psi \perp \Psi_0$, and hence $\|\tilde{\Psi}\|^2_{L^2} = \|\Psi\|^2_{L^2} + \frac{\Psi^2(0)}{\Psi_0^2(0)}\|\Psi_0\|^2_{L^2} \geq \|\Psi\|^2_{L^2} = 1$.

On the other hand, again by $\Psi \perp \Psi_0$, $L_\mu\Psi \perp \Psi_0$, and the properties of the spectral projections,

$$\langle L_\mu\tilde{\Psi}, \tilde{\Psi}\rangle = \langle L_\mu\Psi, \Psi\rangle + \frac{\Psi^2(0)}{\Psi_0^2(0)}\langle L_\mu\Psi_0, \Psi_0\rangle \leq (\omega-\delta) - \lambda_{\omega,\mu}\frac{\Psi^2(0)}{\Psi_0^2(0)} \leq \omega - \delta.$$

Clearly, the two estimates that we have obtained for $\langle L_\mu\tilde{\Psi}, \tilde{\Psi}\rangle$ are contradictory, which is due to the assumption $\sigma(L_\mu)\cap(-\lambda_{\omega,\mu}, \omega-\delta) \neq \emptyset$. Thus, $\sigma(L_\mu)\cap(-\lambda_{\omega,\mu}, \omega) = \emptyset$. In other words, $\sigma(L_\mu) \setminus \{-\lambda_{\omega,\mu}\} \subset [\omega, \infty)$, which was the claim.

The proof for $\mu = c^2$ goes along similar lines. Indeed, for any test function $\Psi \in H^s$, we have

$$\langle L_\mu\Psi, \Psi\rangle = \|(-\Delta)^{\frac{s}{2}}\Psi\|^2_{L^2} + \omega\|\Psi\|^2_{L^2} - c_s^2|\Psi(0)|^2 \geq 0,$$

by the definition of $c^2 = \inf J_\omega[\Psi]$. Hence, $L_\mu \geq 0$. Furthermore, by direct inspection $L_\mu[\mathcal{G}^s_\omega] = 0$, whence 0 is an eigenvalue (and it would have to be at the bottom of the spectrum). Finally, $\sigma(L_\mu) \setminus \{0\} \subset [\omega, \infty)$ is shown in the exact same way as in the case $\mu > c^2$.

For the case $\mu < c^2$, we can similarly identify an unique $\lambda_{\omega,\mu} \in (0, \omega)$, so that

$$\mu\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega - \lambda}d\xi = 1.$$

This $\lambda_{\omega,\mu} > 0$ is an eigenvalue for L_μ, with eigenfunction, $\Psi_0 : \widehat{\Psi}_0(\xi) = \frac{1}{(2\pi|\xi|)^{2s}+\omega-\lambda}$. Moreover, $\sigma(L_\mu) \setminus \{\lambda_{\omega,\mu}\} \subset [\omega, \infty)$ is proved in the same fashion as above. □

Note that the operators $\mathcal{L}_\pm$ have the form

$$\mathcal{L}_- = (-\Delta)^s + \omega - |\phi(0)|^{2\sigma}\delta_0 = (-\Delta)^s + \omega - c^2(\omega)\delta_0$$

$$\mathcal{L}_+ = (-\Delta)^s + \omega - (2\sigma + 1)c^2(\omega)\delta_0.$$

As a direct consequence of the results of Proposition 5, we have the following corollary.

Corollary 1 *Let* $s > \frac{n}{2}$, $\omega > 0$, $\sigma > 0$. *Then,*

- $\mathcal{L}_- \geq 0$, 0 *is a simple eigenvalue, with eigenfunction* $\mathcal{G}_s^\omega$ *and*

$$\sigma(\mathcal{L}_-) \setminus \{0\} \subset [\omega, \infty).$$

  Also, $\mathcal{L}_-|_{\{\mathcal{G}_s^\omega\}^\perp} \geq \omega$.
- $\mathcal{L}_+$ *has a simple negative eigenvalue, with an eigenfunction* Ψ_0. *Also,*

$$\mathcal{L}_+|_{\{\Psi_0\}^\perp} \geq \omega > 0.$$

4 Stability of the waves

In this section, we identify the regions of stability for the waves. We start with a short introduction in the theory of the Hamiltonian instability index, as developed in [15–17].

4.1 The Hamiltonian instability index theory

We are concerned with a Hamiltonian eigenvalue problem of the form

$$\mathcal{I}\mathcal{K}f = \lambda f, \tag{4.1}$$

where $\mathcal{I}^* = -\mathcal{I}, \mathcal{K}^* = \mathcal{K}$, $\mathcal{I}$ is bounded and invertible, so that $\mathcal{I}^{-1} : Ker(\mathcal{K}) \to Ker(\mathcal{K})^\perp$.

We would like to analyze the number of unstable eigenvalues of the eigenvalue problem (4.1). To this end, we assume that the Morse index of $\mathcal{K}$ is finite, that is,

$$n(\mathcal{K}) = \#\{\mu \in \sigma_{p.p.}(\mathcal{K}), \mu < 0\} < \infty$$

and $dim(Ker(\mathcal{K})) < \infty$, say $Ker(\mathcal{K}) = span\{\psi_j, j = 1, \ldots, N\}$. Introduce a scalar matrix $\mathcal{D}$, with entries[8]

$$\mathcal{D}_{ij} = \langle \mathcal{K}^{-1}\mathcal{I}^{-1}\psi_i, \mathcal{I}^{-1}\psi_j\rangle$$

Introduce the following three integers: k_r is the number of real and positive solutions λ in (4.1), accounting for the real unstable modes, then k_c is the number of solutions λ in (4.1) with positive real part. Finally, $k_0^{\leq 0}$ denotes the dimension of the marginally stable directions, corresponding to purely imaginary eigenvalue with negative Krein

[8] Note that since $\mathcal{I}^{-1} : Ker(\mathcal{K}) \to Ker(\mathcal{K})^\perp$, the operator $\mathcal{K}^{-1}$ is well-defined on $\mathcal{I}^{-1}\psi$.

index—that is eigenvalues $i\mu_j : \mathcal{J}\mathcal{K}\Psi_j = i\mu_j\Psi_j$, with $\langle \mathcal{K}\Psi_j, \Psi_j\rangle < 0$. Then, by [15–17], we have the formula

$$k_r + k_c + k_0^{\leq 0} = n(\mathcal{L}) - n(\mathcal{D}). \tag{4.2}$$

Note that by Hamiltonian symmetry considerations, both $k_c, k_0^{\leq 0}$ are even and non-negative integers. A very immediate corollary of the considerations above is the following statement, which is often referred to as the Vakhitov-Kolokolov stability criteria.

Corollary 2 *Let $\mathcal{K}$ be self-adjoint, with $n(\mathcal{K}) = 1$, $dim(Ker(\mathcal{K})) = 1$, say $Ker(\mathcal{K}) = span\{\Psi\}$. Assume that $\mathcal{I}$ also satisfy the assumptions listed above. Then, the Hamiltonian eigenvalue problem (4.1) is stable if and only if*

$$\langle \mathcal{K}^{-1}\mathcal{I}^{-1}\Psi, \mathcal{I}^{-1}\Psi\rangle < 0. \tag{4.3}$$

Indeed, in such a setup, the matrix $\mathcal{D}$ is a one dimensional matrix. Also, the right-hand side of (4.2) is either 0 or 1, whence $k_r = n(\mathcal{L}) - n(\mathcal{D}) = 1 - n(\mathcal{D})$ and stability is equivalent to $n(\mathcal{D}) = 1$, which is exactly the condition (4.3).

4.2 Instability index count for (1.6)

In our specific case, we need to apply the instability index counting theory to the eigenvalue problem (1.6). Recall that $\mathcal{J}^* = -\mathcal{J} = \mathcal{J}^{-1}$, while $\mathcal{L} = \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix}$, whence

$$n(\mathcal{L}) = n(\mathcal{L}_+) + n(\mathcal{L}_-) = 1 + 0 = 1,$$

due to the results of Corollary 1. Also, again by the description in Corollary 1,

$$Ker(\mathcal{L}) = \begin{pmatrix} Ker(\mathcal{L}_-) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Ker(\mathcal{L}_+) \end{pmatrix} = span\begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix}.$$

It follows that Corollary 2 is applicable to the eigenvalue problem (1.6), and in fact the spectral stability of it is equivalent to the condition

$$\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle < 0. \tag{4.4}$$

Since, $\phi_\omega = c\mathcal{G}_s^\omega$, it suffices to compute $\langle \mathcal{L}_+^{-1}\mathcal{G}_s^\omega, \mathcal{G}_s^\omega\rangle$. We accomplish this in the following proposition.

Proposition 6 *Let $n \geq 1$, $\omega > 0$, $\sigma > 0$ and $s > \frac{n}{2}$. Then,*

$$sgn\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle = sgn\langle \mathcal{L}_+^{-1}\mathcal{G}_s^\omega, \mathcal{G}_s^\omega\rangle = sgn\left(\sigma - \frac{2s-n}{n}\right).$$

In particular, the waves ϕ_ω are spectrally stable if and only if

$$0 < \sigma < \frac{2s}{n} - 1.$$

Proof We first need to find $\mathcal{L}_+^{-1}\mathcal{G}_s^\omega$. That is, we need to solve $\mathcal{L}_+\psi = \mathcal{G}_s^\omega$. Based on the formula (2.9) however, we need to solve

$$\mathcal{G}_s^\omega = \mathcal{L}_+\psi = ((-\Delta)^s + \omega)g$$

whence, we can actually find g pretty easily by taking Fourier transform. Namely,

$$((2\pi|\xi|)^{2s} + \omega)\hat{g}(\xi) = \widehat{\mathcal{G}_s^\omega}(\xi) = \frac{1}{(2\pi|\xi|)^{2s} + \omega}.$$

It follows that

$$\hat{g}(\xi) = \frac{1}{((2\pi|\xi|)^{2s} + \omega)^2},$$

or equivalently $g = \mathcal{G}_s^\omega * \mathcal{G}_s^\omega$. We can now proceed to find ψ from (2.11). Namely, taking into account that $\mathcal{L}_+ = (-\Delta)^s + \omega - (2\sigma+1)c^2$, we compute

$$\psi = g + (2\sigma+1)c^2 \frac{g(0)}{1 - (2\sigma+1)c^2\mathcal{G}_s^\omega(0)} \mathcal{G}_s^\omega.$$

Note however that $g(0) = \mathcal{G}_s^\omega * \mathcal{G}_s^\omega(0) = \|\mathcal{G}_s^\omega\|_{L^2}^2$. Also, according to (3.13), $c_s^2\mathcal{G}_s^\omega(0) = 1$, so

$$\psi = \mathcal{G}_s^\omega * \mathcal{G}_s^\omega - \frac{2\sigma+1}{2\sigma} \frac{\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^2} d\xi}{\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi} \mathcal{G}_\omega^s.$$

So,

$$\langle \mathcal{L}_+^{-1}\mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle = \langle \psi, \mathcal{G}_s^\omega \rangle = \langle \mathcal{G}_s^\omega * \mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle - \frac{2\sigma+1}{2\sigma} \frac{\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^2} d\xi}{\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi} \langle \mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle =$$

$$= \int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s} + \omega)^3} d\xi - \frac{2\sigma+1}{2\sigma} \frac{\left(\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^2} d\xi\right)^2}{\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi}.$$

So, it remains to compute

$$\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s} + \omega)^j} d\xi, \; j = 1, 2, 3.$$

We have done in the Appendix, see Proposition 9. More specifically, substituting the formulas (A.1), (A.2), (A.3) in the expression for $\langle \mathcal{L}_+^{-1} \mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle$, we obtain

$$\begin{aligned}\langle \mathcal{L}_+^{-1} \mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle &= \frac{\pi |\mathbf{S}^{n-1}| \omega^{\frac{n}{2s}-3}}{4s(2\pi)^n \sin(\frac{n\pi}{2s})} \left(\left(1-\frac{n}{2s}\right)\left(2-\frac{n}{2s}\right) - \frac{2\sigma+1}{\sigma}\left(1-\frac{n}{2s}\right)^2 \right) = \\ &= \frac{n\pi |\mathbf{S}^{n-1}| \omega^{\frac{n}{2s}-3}}{8s^2\sigma(2\pi)^n \sin(\frac{n\pi}{2s})} \left(1-\frac{n}{2s}\right)\left(\sigma - \frac{2s-n}{n}\right).\end{aligned}$$

Note that, as $s > \frac{n}{2}$, only the last term in the expression changes sign over the parameter space. We have this established Proposition 6 in full. ◻

Having the above spectral properties of the operator $\mathcal{L}_\pm$, we have one last step before arriving at the orbital stability of the wave. More specifically, we need to argue the coerciveness of $\mathcal{L}_\pm$ on the space $H^s(\mathbf{R}^n)$. To that end we have the following proposition.

Proposition 7 *Let $s > \frac{n}{2}$, $\omega > 0$, $\langle \mathcal{L}_+^{-1} \phi_\omega, \phi_\omega \rangle < 0$. Then, the operator $\mathcal{L}_+$ is coercive on $\{\phi_\omega\}^\perp$. That is, there exists $\delta > 0$, so that for all*

$$\langle \mathcal{L}_+ \Psi, \Psi \rangle \geq \delta \|\Psi\|_{H^s}^2, \quad \forall \Psi \perp \phi_\omega. \tag{4.5}$$

Proof This is a version of a well-known lemma in the theory, see for example Lemma 6.7 and Lemma 6.9 in [8]. Recall that we have already showed $Ker[\mathcal{L}_+] = \{0\}$ and $n(\mathcal{L}_+) = 1$. According[9] to Lemma 6.4, [8], under these conditions for $\mathcal{L}_+$, we have that for any $g \perp \phi_\omega$,

$$\langle \mathcal{L}_+ g, g \rangle \geq 0. \tag{4.6}$$

Consider the associated constrained minimization problem

$$\inf_{\|f\|=1, f \perp \phi_\omega} \langle \mathcal{L}_+ f, f \rangle \tag{4.7}$$

and set

$$\alpha := \inf\{\langle \mathcal{L}_+ f, f \rangle : f \perp \phi_\omega, \|f\|_{L^2} = 1\} \geq 0.$$

We will show that $\alpha > 0$. Assume for a contradiction that $\alpha = 0$.

Take a minimizing sequence $f_k : \|f_k\| = 1$, $f_k \perp \phi_\omega$, so that

$$\alpha = \lim_k \langle \mathcal{L}_+ f_k, f_k \rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} f_k\|^2 + \omega - (2\sigma+1)c^2 |f_k(0)|^2].$$

However, by Sobolev embedding and the Gagliardo-Nirenberg's inequalities, recall $\|f_k\|_{L^2} = 1$, we have that for all $\beta : \frac{n}{2} < \beta < s$ and for all $\epsilon > 0$,

$$|f(0)| \leq \|f\|_{L^\infty} \leq C_\beta(\|f\|_{\dot{H}^\beta} + C\|f\|_{L^2} \leq C_\beta \|f\|_{\dot{H}^s}^{\frac{\beta}{s}} \|f\|_{L^2}^{1-\frac{\beta}{s}} + C\|f\|_{L^2} \leq \epsilon \|f\|_{\dot{H}^s} + C_\epsilon \|f\|_{L^2}.$$

[9] And this is already explicit in a much earlier work by Weinstein.

Applying this estimate, we obtain a lower bound for $\langle \mathcal{L}_+ f_k, f_k \rangle$ (recall $\|f_k\|_{L^2} = 1$), as follows

$$\langle \mathcal{L}_+ f_k, f_k \rangle \geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} f_k\|^2 - C.$$

Since, $\alpha = \lim_k \langle \mathcal{L}_+ f_k, f_k \rangle$, this implies that $\sup_k \|(-\Delta)^{\frac{s}{2}} f_k\|^2 < \infty$. This means that we can select a subsequence of $\{f_k\}$ (denoted by the same), so that f_k converges weakly to $f \in H^s(\mathbf{R}^n)$. In addition, by the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow C^\gamma(\mathbf{R}^n)$, $\gamma < s - \frac{n}{2}$, we can, as we have done previously, without loss of generality assume that $f_n \rightrightarrows f$ on the compact subsets of $\mathbf{R}^n$. In particular, $\lim_k f_k(0) = f(0)$. Note that by the weak convergence, $\langle f, \phi_\omega \rangle = \lim_k \langle f_k, \phi_\omega \rangle = 0$, so $f \perp \phi_\omega$ and

$$\liminf_k \|(-\Delta)^{\frac{s}{2}} f_k\|^2 \geq \|(-\Delta)^{\frac{s}{2}} f\|^2, \quad \|f\|_{L^2} \leq \liminf \|f_k\|_{L^2} = 1. \tag{4.8}$$

It follows that

$$\langle \mathcal{L}_+ f, f \rangle \leq \liminf_k \langle \mathcal{L}_+ f_k, f_k \rangle = 0. \tag{4.9}$$

But by (4.6), and since $f \perp \phi_\omega$, we have that $\langle \mathcal{L}_+ f, f \rangle \geq 0$. It follows that $0 = \langle \mathcal{L}_+ f, f \rangle = \lim_k \langle \mathcal{L}_+ f_k, f_k \rangle$. This means that all inequalities in (4.8) and (4.9) are equalities and in particular

$$\lim_k \|(-\Delta)^{\frac{s}{2}} f_k\|_{L^2} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2},$$
$$\lim_k \|f_k\|_{L^2} = \|f\|_{L^2}.$$

These last identities, in addition to the H^s weak convergence f_k to f, implies strong convergence, that is $\lim_k \|f_k - f\|_{H^s} = 0$. In particular, $\|f\|_{L^2} = \lim_k \|f_k\|_{L^2} = 1$. In other words, f is a minimizer for the constrained minimization problem (4.7). Write the Euler-Lagrange equation for f

$$\mathcal{L}_+ f = df + c\phi_\omega. \tag{4.10}$$

Taking dot product with f and taking into account $\langle \mathcal{L}_+ f, f \rangle = 0$, $f \neq 0$ and $f \perp \phi_\omega$ implies that $d = 0$. This means that $f = c\mathcal{L}_+^{-1}\phi_\omega$. But then, $0 = \langle \mathcal{L}_+ f, f \rangle = c^2 \langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega \rangle$. Since $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega \rangle \neq 0$, it follows $c = 0$. But then, since $Ker[\mathcal{L}_+] = \{0\}$, (4.10) implies that $f = 0$, which is a contradiction. Thus, we have shown that $\alpha > 0$. As a consequence,

$$\langle \mathcal{L}_+ \Psi, \Psi \rangle \geq \alpha \|\Psi\|_{L^2}^2, \quad \forall \Psi \perp \phi_\omega. \tag{4.11}$$

Note that (4.5) is however stronger than (4.11), as it involves $\|\cdot\|_{H^s}$ norms on the right-hand side. Nevertheless, we show that it is relatively straightforward to deduce it from (4.11). Indeed, assume for a contradiction in (4.5), that $g_k : \|g_k\|_{H^s} = 1$, $g_k \perp \phi_\omega$, so that $\lim_k \langle \mathcal{L}_+ g_k, g_k \rangle = 0$.

Taking into account (4.11), this is only possible if $\lim_k \|g_k\|_{L^2} = 0$. So,

$$1 = \lim_k [\|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2 + \|g_k\|_{L^2}^2] = \lim_k \|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2.$$

Note that by (2.1), we have that for all $0 < \delta < s - \frac{n}{2}$, we have that

$$|g_k(0)| \le \|g_k\|_{L^\infty} \le C(\|g_k\|_{\dot{H}^{\frac{n}{2}+\delta}} + \|g_k\|_{\dot{H}^{\frac{n}{2}-\delta}}) \le C(\|g_k\|_{\dot{H}^s}^{\frac{\frac{n}{2}+\delta}{s}} \|g_k\|_{L^2}^{1-\frac{\frac{n}{2}+\delta}{s}} + \|g_k\|_{\dot{H}^s}^{\frac{\frac{n}{2}-\delta}{s}} \|g_k\|_{L^2}^{1-\frac{\frac{n}{2}-\delta}{s}},$$

whence $\lim_k \|g_k(0)| = 0$. But then, we achieve a contradiction, since

$$0 = \lim_k \langle \mathcal{L}_+ g_k, g_k \rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2 + \omega \|g_k\|_{L^2}^2 - (2\sigma + 1)c_s^2 |g_k(0)|^2] = 1.$$

□

4.3 Orbital stability

In this section, we prove that the spectrally stable solutions are in fact orbitally stable. There is, in general, a straightforward way to obtain orbital stability, based on spectral stability, see for example Theorem 5.2.11, [15]. While this is the case in general, we are dealing with non-standard linearized operators and their domains. In particular, the Assumption 5.2.5 a) on p. 136, [15] does not apply. Thus, we need to consider a direct proof, based on the Benjamin's approach.

As was established already, the case $0 < \sigma < \frac{2s}{n} - 1$ represents the spectrally stable waves, which we now analyze for orbital stability.

Proposition 8 *Let $\omega > 0$, $n \ge 1$, $s > \frac{n}{2}$, $0 < \sigma < \frac{2s}{n} - 1$ and the key assumptions (1), (2) are satisfied. Then $e^{i\omega t}\phi_\omega$ is orbitally stable solution of* (1.2).

Proof Let us outline first what the consequences of our assumptions are. By Proposition 6, we have that $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega \rangle < 0$, which by Proposition 7 means that the coercivity estimate (4.5) holds. By Corollary 1, $Ker(\mathcal{L}_+) = \{0\}$, that is the wave ϕ_ω is non-degenerate.

We now concentrate on the orbital stability. Our proof is by a contradiction argument. That is, assume there is $\epsilon_0 > 0$ and a sequence of initial data u_k : $\lim_k \|u_k - \phi\|_{H^s(\mathbb{R}^n)} = 0$, so that

$$\sup_{0 \le t < \infty} \inf_{\theta \in \mathbf{R}} \|u_k(t, \cdot) - e^{-i\theta}\phi\|_{H^s} \ge \epsilon_0. \tag{4.12}$$

Using the conserved quantities (1.3) and (1.4), we define a new conserved quantity

$$\mathcal{E}[u] := E[u] + \frac{\omega}{2} M[u],$$
$$\epsilon_k := |\mathcal{E}[u_k(t)] - \mathcal{E}[\phi_\omega]]| + |M[u_k(t)] - M[\phi_\omega]]|,$$

and for all $\epsilon > 0$,

$$t_k := \sup\{\tau : \sup_{0<t<\tau} \|u_k(t) - \phi\|_{H^s(\mathbb{R}^n)} < \epsilon\}.$$

Note that ϵ_k is conserved and $\lim_k \epsilon_k = 0$. By the assumption that local well-posedness holds, it must be that $t_k > 0$.

Consider $t \in (0, t_k)$ and let $u_k = v_k + i w_k$ and $\|w_k(t)\|_{H^s(\mathbb{R}^n)} \leq 2\|u_k - \phi\|_{H^s(\mathbb{R}^n)} < \epsilon$. This leads to the definition of the modulation parameter $\theta_k(t)$ such that $w_k + \sin\theta_k(t)\phi \perp \phi$, that is,

$$-\sin(\theta_k(t))\|\phi\|_{L^2} = \langle w_k(t), \phi\rangle. \tag{4.13}$$

By Cauchy-Schwartz we have $|\langle w_k(t), \phi\rangle| \leq \epsilon\|\phi\|_{L^2}$ and this means there is an unique small solution $\theta_k(t)$ of (4.13), with $|\theta_k(t)| \leq \epsilon$. Also

$$\|u_k(t,\cdot) - e^{-i\theta_k(t)}\phi\|_{H^s} \leq \|u_k(t,\cdot) - \phi\|_{H^s} + |e^{-i\theta_k(t)} - 1|\|\phi\|_{H^s} \leq C(\|\phi\|_{H^s})\epsilon.$$

Now define

$$T_k := \sup\{\tau : \sup_{0<t<\tau} \|u_k(t,\cdot) - e^{-i\theta_k(t)}\varphi(\cdot)\|_{H^s(\mathbb{R}^n)} < 2C\epsilon\}.$$

Clearly $0 < t_k < T_k$. From this we see that to get contradiction of (4.12) it is enough to show that for all $\epsilon > 0$ and large k, $T_k = \infty$. To that end let $t \in (0, T_k)$ write

$$\psi_k = u_k - e^{-i\theta_k(t)}\phi = v_k + i w_k - e^{-i\theta_k(t)}\phi,$$

and decompose into real and imaginary part of ψ_k and projecting on $\begin{pmatrix}\phi\\0\end{pmatrix}$ yield

$$\begin{pmatrix} v_k(t,\cdot) - \cos(\theta_k(t))\phi \\ w_k(t,\cdot) + \sin(\theta_k(t))\phi \end{pmatrix} = \mu_k(t)\begin{pmatrix}\phi\\0\end{pmatrix} + \begin{pmatrix}\eta_k(t,\cdot)\\ \zeta_k(t,\cdot)\end{pmatrix}, \quad \begin{pmatrix}\eta_k(t,\cdot)\\ \zeta_k(t,\cdot)\end{pmatrix} \perp \begin{pmatrix}\phi\\0\end{pmatrix}. \tag{4.14}$$

By the choice of θ_k we have $\zeta_k \perp \phi$, and from the above decomposition we also have $\eta_k \perp \phi$. So taking the L^2 norm of (4.14) we have

$$|\mu_k(t)|^2\|\phi\|_{L^2}^2 + \|\eta_k(t)\|_{L^2}^2 + \|\zeta_k(t)\|_{L^2}^2 = \|\psi_k(t)\|_{L^2}^2 \leq 4C^2\epsilon^2. \tag{4.15}$$

Next we take advantage of the two conserved quantities, to that end we consider the mass

$$\begin{aligned} M[u_k(t)] &= \int_{\mathbf{R}^n} |e^{-i\theta_k(t)}\phi + \psi_k(t)|^2 dx = M[\phi] + \|\psi_k(t,\cdot)\|_{L^2}^2 + 2\int_{\mathbf{R}^n} \phi(x)\Re[e^{-i\theta_k(t)}\psi_k(t,x)]dx \\ &= M[\phi] + \|\psi_k(t,\cdot)\|_{L^2}^2 + 2\mu_k(t)\cos(\theta_k(t))\|\phi\|_{L^2}^2. \end{aligned}$$

Here we use the fact that $w_k + \sin\theta_k(t)\phi \perp \phi$ and $\eta_k \perp \phi$. Solving for $\mu_k(t)$ and since $|\theta_k|$ is very small and $\|\psi_k(t,\cdot)\|_{L^2} \le 2C\epsilon$, in $t : 0 < t < T_k$ we have

$$|\mu_k(t)| \le \frac{|M[u_k(t)] - M[\phi]| + \|\psi_k(t,\cdot)\|_{L^2}^2}{2\cos(\theta_k(t)\|\phi\|_{L^2}^2} \le C(\epsilon_k + \|\psi_k(t,\cdot)\|_{L^2}^2) \le C(\epsilon_k + \epsilon^2). \tag{4.16}$$

Now we will expand $\mathcal{E}[u_k(t)] - \mathcal{E}[\phi]$ but first for any small perturbations of the wave $\alpha_1 + i\alpha_2 \in H^s(\mathbf{R}^n)$ and using (1.5) we have

$$E[\phi + (\alpha_1 + i\alpha_2)] - E[\phi] = \frac{1}{2}[\langle \mathcal{L}_+\alpha_1, \alpha_1\rangle + \langle \mathcal{L}_-\alpha_2, \alpha_2\rangle] + Err[\alpha_1, \alpha_2], \tag{4.17}$$

where

$$\begin{aligned}
&|Err[\alpha_1, \alpha_2]| \le C|((\phi(0) + \alpha_1(0))^2 + \alpha_2^2(0))^{\sigma+1} - \phi(0)^{2\sigma+2} \\
&- (2\sigma+2)\phi(0)^{2\sigma+1}\alpha_1(0) - \frac{(2\sigma+2)(2\sigma+1)}{2}\phi^{2\sigma}(0)\alpha_1^2(0) - (2\sigma+2)\phi^{2\sigma}(0)\alpha_2^2(0)| \\
&\le C(\|\phi\|_{L^\infty})(|\alpha_1(0)| + |\alpha_2(0)|)^{\min(2\sigma+2,3)}.
\end{aligned}$$

Note that

$$e^{i\theta_k(t)}\psi_k = [\cos(\theta_k)(\mu_k\phi + \eta_k) - \sin(\theta_k)\zeta_k] + i\,[\cos(\theta_k)\zeta_k + \sin(\theta_k)(\mu_k\phi + \eta_k)].$$

Now apply the expansion (4.17) with

$$\alpha_1 = \cos(\theta_k)(\mu_k\phi + \eta_k) - \sin(\theta_k)\zeta_k, \alpha_2 = \cos(\theta_k)\zeta_k + \sin(\theta_k)(\mu_k\phi + \eta_k)$$

together with (4.15), we see that $\|\alpha_1\|_{H^s} + \|\alpha_2\|_{H^s} \le C\epsilon$. So, we can bound the contribution of $|Err[\alpha_1, \alpha_2]|$ as follows

$$|Err[\alpha_1, \alpha_2]| \le C\epsilon^{\min(2\sigma,1)}(\|\alpha_1\|_{H^s}^2 + \|\alpha_2\|_{H^s}^2). \tag{4.18}$$

By the Sobolev embedding, $\mathcal{L}_-\phi = 0$ and $\mathcal{L}_+ = \mathcal{L}_- - 2\sigma|\phi(0)|^{2\sigma}\delta$ together with (4.15) and (4.16) we have

$$\begin{aligned}
\langle \mathcal{L}_+\alpha_1, \alpha_1\rangle &\ge \langle \mathcal{L}_+\eta_k, \eta_k\rangle - C(\epsilon^3 + \epsilon_k + \epsilon^2(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s}) + \epsilon(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s})^2) \\
\langle \mathcal{L}_-\alpha_2, \alpha_2\rangle &\ge \langle \mathcal{L}_-\zeta_k, \zeta_k\rangle - C(\epsilon^3 + \epsilon_k + \epsilon^2(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s}) + \epsilon(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s})^2).
\end{aligned}$$

Taking advantage of the coercivity of $\mathcal{L}_-$ and $\mathcal{L}_+$, which was established in Proposition 5, we have that for some $\kappa > 0$ and since $\eta_k, \zeta_k \perp \phi$ together with some algebraic manipulations yield

$$\|\eta_k(t)\|_{H^s}^2 + \|\zeta_k(t)\|_{H^s}^2 \le C(\epsilon^3 + \epsilon_k). \tag{4.19}$$

Here C is independent of ϵ and k. This implies that $T_k^* = \infty$, since if we assume that $T_k^* < \infty$, then

$$2C_0\epsilon = \limsup_{t\to T_k^*-} \|\psi_k(t)\|_{H^s} \leq C(|\mu_k(t)| + \|\eta_k(t)\|_{H^s} + \|\zeta_k(t)\|_{H^s}) \leq C(\epsilon^{\frac{3}{2}} + \sqrt{\epsilon_k}). \tag{4.20}$$

which is a contradiction, if ϵ is so that $C_0\epsilon > C\epsilon^{\frac{3}{2}}$ and then k is so large, and hence ϵ_k is so small, that $C_0\epsilon > C\sqrt{\epsilon_k}$, which certainly contradicts (4.20). Hence the wave is orbitally stable. □

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

APPENDIX A. The integrals $\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^j} d\xi$

Herein, we compute the integrals that arise in the calculation of the Vakhitov-Kolokolov index in Proposition 6.

Proposition 9 *For* $\omega > 0$, *we have*

$$\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi = \frac{\pi|\mathbf{S}^{n-1}|}{2s(2\pi)^n} \frac{\omega^{\frac{n}{2s}-1}}{\sin(\frac{n\pi}{2s})} \tag{A.1}$$

$$\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^2} d\xi = \frac{\pi|\mathbf{S}^{n-1}|}{2s(2\pi)^n} \left(1-\frac{n}{2s}\right) \frac{\omega^{\frac{n}{2s}-2}}{\sin(\frac{n\pi}{2s})} \tag{A.2}$$

$$\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^3} d\xi = \frac{\pi|\mathbf{S}^{n-1}|}{4s(2\pi)^n} \left(1-\frac{n}{2s}\right)\left(2-\frac{n}{2s}\right) \frac{\omega^{\frac{n}{2s}-3}}{\sin(\frac{n\pi}{2s})}. \tag{A.3}$$

Proof We easily pass to integrals in the radial variable as follows

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^j} d\xi &= |\mathbf{S}^{n-1}| \int_0^\infty \frac{\rho^{n-1}}{((2\pi\rho)^{2s}+\omega)^j} d\rho \\ &= |\mathbf{S}^{n-1}| \frac{\omega^{\frac{n}{2s}-j}}{2s(2\pi)^n} \int_0^\infty \frac{\rho^{\frac{n}{2s}-1}}{(\rho+1)^j} d\rho = \\ &= |\mathbf{S}^{n-1}| \frac{\omega^{\frac{n}{2s}-j}}{2s(2\pi)^n} \int_{-\infty}^\infty \frac{e^{t\frac{n}{2s}}}{(e^t+1)^j} dt.\end{aligned}$$

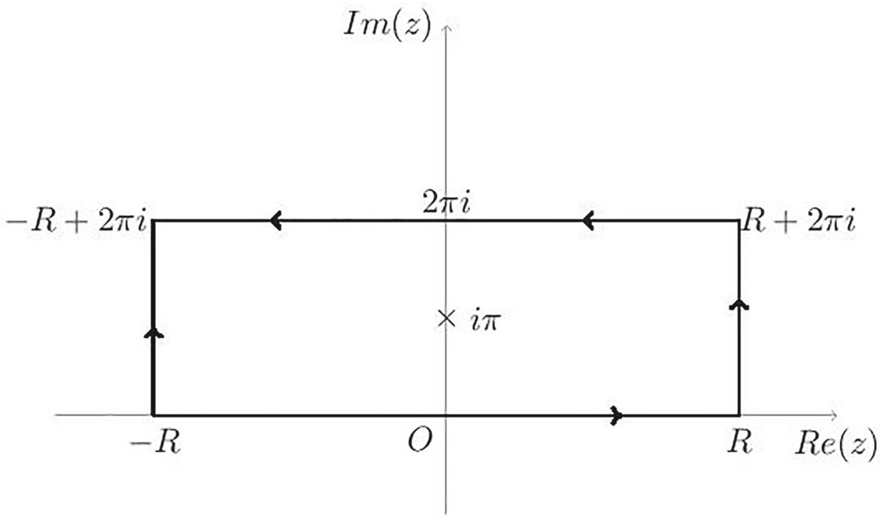

Fig. 1 Contour of integration

So, with $a := \frac{n}{2s} \in (0, 1)$, matters are clearly reduced to computing the integrals

$$\int_{-\infty}^{\infty} \frac{e^{ta}}{(e^t + 1)^j} dt,$$

for $a \in (0, 1)$, $j = 1, 2, 3$. In order to compute this integral, we use the residue theorem formula

$$\int_{\gamma_R} \frac{e^{az}}{(e^z + 1)^j} dz = 2\pi i Res \left(\frac{e^{az}}{(e^z + 1)^j}, \pi i \right).$$

where $R >> 1$, and $\gamma_R = \gamma_R^1 \cup \gamma_R^2 \cup \gamma_R^3 \cup \gamma_R^4$, and the curves γ_r^m, $m = 1, 2, 3, 4$ are given, together with their orientation as follows. Here γ_R is given in Fig. 1.

$$\gamma_R^1 = \{x \in (-R, R)\}, \gamma_R^2 = \{R + ih, h \in [0, 2\pi]\},$$
$$\gamma_R^3 = \{x + 2\pi i, x \in (R, -R)\}, \gamma_R^4 = \{-R + ih, h \in [2\pi, 0]\}.$$

Clearly,

$$\int_{\gamma_R^1} \frac{e^{az}}{(e^z + 1)^j} dz + \int_{\gamma_R^3} \frac{e^{az}}{(e^z + 1)^j} dz = (1 - e^{2\pi a i}) \int_{-R}^{R} \frac{e^{ta}}{(e^t + 1)^j} dt,$$

while for $R >> 1$,

$$\left| \int_{\gamma_R^2} \frac{e^{az}}{(e^z + 1)^j} dz \right| \leq C \frac{e^{Ra}}{(e^R - 1)^j}, \quad \left| \int_{\gamma_R^4} \frac{e^{az}}{(e^z + 1)^j} dz \right| \leq C e^{-aR}.$$

It follows that

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{az}}{(e^z+1)^j}dz=(1-e^{2\pi ai})\int_{-\infty}^{\infty}\frac{e^{ta}}{(e^t+1)^j}dt.$$

It remains to compute the residues associated with this complex integration. This is a straightforward calculation, the results of which are below

$$Res\left(\frac{e^{az}}{e^z+1},\pi i\right)=-e^{ia\pi} \tag{A4}$$

$$Res\left(\frac{e^{az}}{(e^z+1)^2},\pi i\right)=-(1-a)e^{ia\pi} \tag{A5}$$

$$Res\left(\frac{e^{az}}{(e^z+1)^3},\pi i\right)=-\frac{1}{2}(2-a)(1-a)e^{ia\pi}. \tag{A6}$$

The formulas (A.1), (A.2), (A.3) follow by substituting these expressions in the residue formulas and taking $R\to\infty$. □

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