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Stability of periodic waves for the fractional KdV and NLS equations

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We consider the focussing fractional periodic Korteweg–deVries (fKdV) and fractional periodic non-linear Schrödinger equations (fNLS) equations, with L^2 sub-critical dispersion. In particular, this covers the case of the periodic KdV and Benjamin-Ono models. We construct two parameter family of bell-shaped travelling waves for KdV (standing waves for NLS), which are constrained minimizers of the Hamiltonian. We show in particular that for each $\lambda > 0$, there is a travelling wave solution to fKdV and fNLS $\phi : \|\phi\|^2_{L^2[-T,T]} = \lambda$, which is non-degenerate. We also show that the waves are spectrally stable and orbitally stable, provided the Cauchy problem is locally well-posed in $H^{\alpha/2}[-T,T]$ and a natural technical condition. This is done rigorously, without any *a priori* assumptions on the smoothness of the waves or the Lagrange multipliers.

Keywords: Fractional KdV; factional NLS; periodic waves; stability

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1. Introduction

In this work, we shall be concerned with the initial value problem for the fractional periodic KdV (fKdV) and the fractional periodic NLS (fNLS) model. More specifically, fKdV is

$$\begin{cases} u_t - \Lambda^\alpha u_x + (u^2)_x = 0, -T \leqslant x \leqslant T, \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

while the corresponding quadratic NLS problem,

$$\begin{cases} iu_t - \Lambda^\alpha u + |u|u = 0, -T \leqslant x \leqslant T. \\ u(0,x) = u_0(x) \end{cases} \tag{1.2}$$

Here, the fractional differentiation operator is defined, say for finite trigonometric polynomials via

$$\Lambda^\alpha\left[\sum_{k=-N}^{N} a_k e^{i\pi k(x/T)}\right] = \sum_{k=-N}^{N}\left(\frac{\pi|k|}{T}\right)^\alpha a_k e^{i\pi k(x/T)},$$

and then by extensions to all elements of $H^\alpha[-T,T]$, the standard Sobolev space, see § 2.

1.1. Well-posedness, conserved quantities and travelling waves

In both problems, we can in principle consider any $\alpha > 0$, although we will see that for meaningful results, one needs to restrict to $\alpha > \frac{1}{3}$. The cases $\alpha = 1$ and $\alpha = 2$ are of course classical and well-studied—these are the Benjamin-Ono and the KdV models respectively. The local and global well-posedness theory has been well developed for these standard cases, even for very low regularity data, see [**10**] for KdV and [**17**, **26**] for the Benjamin-Ono case. For non-integer α, we mention the relatively recent works [**13**, **20**] for the fKdV posed on the line, which provides global well-posedness in the energy space $H^{\alpha/2}(\mathbb{R})$. Note that the flow maps in the non-local cases, i.e. $\alpha < 2$, are generally not even uniformly continuous with respect to initial data. The state of affairs regarding the Cauchy problem for the fNLS (1.2) is as follows: the local well-posedness results is addressed in [**27**], for data in $H^s, s > \frac{3}{2} - \frac{5\alpha}{4}$, while global existence is in the energy space $H^{\alpha/2}(\mathbb{R}), \alpha > \frac{6}{7}$. We should also mention the recent article [**16**], which provides a norm inflation phenomena for certain Sobolev spaces with negative smoothness. An extensive review of the literature and numerical blow up scenarios for this and related models is in [**21**].

The existence of travelling waves (standing waves respectively) is another aspect of the theory, as it offers important information regarding the global dynamical properties of these models. In fact, such solutions (and the behaviour of the solutions starting with data close to them) provide the most important clues and indeed the skeleton of the full dynamic picture. This is why the problem for the existence and stability properties of travelling waves for these and related equations has played such a central role. In that regards, we mention [**7**], where the non-linear stability of the KdV travelling waves on the line was established. This was followed by [**8**], and the far reaching generalizations in [**12**, **23**]. Finally, we mention the work [**25**], where the authors have offered detailed description of the asymptotic stability of the solitons for the gKdV model.

A very satisfactory result, including uniqueness for the line soliton, holds for Benjamin-Ono model as well [**1**]. The asymptotic stability for the travelling waves for the BO model is established in [**19**]. Some other recent results for travelling waves on the line, for the non-local models $\alpha < 2$, as well as more general multipliers are in [**6**, **24**]. See also the book [**5**], where the approach for many of these results can be found.

In recent years, the periodic travelling waves for these and related models, together with their stability properties, were considered in numerous papers. Here is a list of some recent developments [2–4, 14, 22], which is certainly incomplete[1]. In most of these works, the waves are constructed either variationally or through some ODE-based methods[2]. The orbital stability considerations for these waves often involve some variation of the Benjamin's method [7]. Note that an essential ingredient in this approach is the so-called *non-degeneracy of the wave*, which roughly states that the kernel of the linearized operator $\mathcal{L}_+$, see (1.4) and (1.5) below, is spanned by ϕ'. This is also an important issue, which arises, when one analyses the uniqueness of the waves in (1.3) as well. We shall provide more details about the specifics of these works, as it pertains to our contribution below, see the discussion after theorem 1.4.

Let us record three important *formally* conserved quantities for the solutions of (1.1).

- the L^2 norm

$$\mathcal{P}(u) = \int_{-T}^{T} u^2(x)\,\mathrm{d}x$$

- the Hamiltonian

$$\mathcal{H}(u) = \frac{1}{2}\left\langle \Lambda^{\alpha/2}u, \Lambda^{\alpha/2}u \right\rangle - \frac{1}{3}\int_{-T}^{T} u^3(x)\,\mathrm{d}x$$

- the mass

$$\mathcal{M}(u) = \int_{-T}^{T} u(x)\,\mathrm{d}x$$

while clearly only $\mathcal{P}, \mathcal{H}$ are conserved on the solutions of (1.2). *Let us note however, that even for the cases where one has global well-posedness, it is generally not clear whether these quantities are actually conserved along the evolution, especially when one works with spaces with limited regularity, say $H^{\alpha/2}[-T,T]$.*

For travelling waves of (1.1), we take the ansatz $u(t,x) = \phi(x-\omega t)$, while for (1.2), we consider $u(t,x) = ei\omega t\phi(x)$. In addition, we will be interested in positive solutions ϕ only. In the case of (1.1), we obtain, after one integration, the profile equation

$$\Lambda^\alpha \phi + \omega\phi - \phi^2 + a = 0, \quad -T \leqslant x \leqslant T, \tag{1.3}$$

where $a \in \mathbb{R}$ is a constant of integration. In the case of (1.2), we obtain exact same equation, but with $a = 0$. Thus, we will generally consider (1.3) with any a, and sometimes we will refer specifically to the case $a = 0$ as it concerns the NLS problem (1.2). Another helpful reduction, that we would like to point out is the following scaling argument. More specifically, using the transformation $\phi(x) = T^{-\alpha}\Phi(x/T)$,

[1] In addition, there is quite a bit of recent works dealing with instabilities of such waves. We do not review these issues here, as our results pertain exclusively to stability.

[2] Although some of these waves are in fact explicit.

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where Φ now has period 2, leads us to the equation

$$\Lambda^\alpha \Phi + T^\alpha \omega \Phi - \Phi^2 + T^{2\alpha} a = 0.$$

Also, a moment thought reveals that the stability of Φ is equivalent to the stability of ϕ, in the context of (1.1) or (1.2). So, understanding the stability of Φ, as a function of the parameters ω, a on the basic interval $[-1, 1]$ can lead to the answer of the stability of waves defined on any interval [3] $[-T, T]$.

1.2. Linear and non-linear stability of the solitons

We now discuss the problem of the stability of the travelling waves $\phi_\omega(x - \omega t)$ for (1.1) and the standing waves $ei\omega t\phi_\omega$ for (1.2), provided they exist. For the fKdV case, taking the ansatz $u = \phi_\omega(x - \omega t) + v(t, x - \omega t)$, and for the fNLS case, we take $u = e^{i\omega t}[\phi_\omega + v_1(t, \cdot) + iv_2(t, \cdot)]$, where v_1, v_2 are taken to be real-valued. Plugging in (1.1) ((1.2) respectively) and ignoring $O(|v|^2)$ leads us to the linearized systems

$$v_t = \partial_x \mathcal{L}_+ v, \tag{1.4}$$

$$\vec{v}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix} \vec{v}, \tag{1.5}$$

where the linearized operators $\mathcal{L}_\pm$ take the form

$$\mathcal{L}_+ = \Lambda^\alpha + \omega - 2\phi, \quad \mathcal{L}_- = \Lambda^\alpha + \omega - \phi, \quad D(\mathcal{L}_\pm) = H^\alpha.$$

Passing to the eigenvalue ansatz $v \to e^{\lambda t} v$ yields the relevant eigenvalue problems

$$\partial_x \mathcal{L}_+ v = \lambda v, \tag{1.6}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix} \vec{v} = \lambda \vec{v}. \tag{1.7}$$

A straightforward comparison of $\mathcal{L}_\pm$ with the constant coefficient operator $\mathcal{L}^0 = \Lambda^\alpha + \omega$ and the fact that the spectra of the operators

$$\partial_x \mathcal{L}^0 \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}^0 & 0 \\ 0 & \mathcal{L}^0 \end{pmatrix}$$

consist entirely of eigenvalues of finite multiplicity implies, by Weyl's criteria (see e.g. corollary 44, [**29**]), that the spectral problems (1.6) and (1.7) have their entire spectrum filled with eigenvalues of finite multiplicity. With that in mind, we give the following definitions of stability.

DEFINITION 1.1. We say that the wave $\phi_\omega(x - \omega t)$ of the fKdV is spectrally stable, with respect to perturbations of the same period, if $\sigma(\partial_x \mathcal{L}_+) \subset \{\lambda : \Re\lambda \leqslant 0\}$. Alternatively, the eigenvalue problem (1.6) does not have solutions $(\lambda, v) : \Re\lambda >$

[3] So, henceforth, without loss of generality, we shall mostly restrict our attention to the case $T = 1$.

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$0, v \in D(\partial_x \mathcal{L}_+), v \neq 0$. For the fNLS problem, the spectral stability of $e^{i\omega t}\phi_\omega$ is understood as the absence of non-trivial solutions of (1.7), with $\Re\lambda > 0$.

The stronger notion of orbital stability for fKdV will be useful in the sequel. As we have mentioned above, the results in this direction are conditional upon well-posedness results, in addition to actual conservation of the momentum $\mathcal{P}(u)$ and the Hamiltonian $\mathcal{H}(u)$.

Assumption 1.2. Let ϕ be a solution of (1.3). Assume that there exists $\epsilon_0 > 0$ and a metric space (X, d_X), so that $X \subset \{g \in H^{\alpha/2}[-T, T] : d_X(g, \phi) < \epsilon_0\}$, with the following properties:

1. The solution map $g \to u_g$ is locally in time continuous in the metric d_X. That is, for each $\epsilon < \epsilon_0$ and $g : d_X(g, \phi) < \epsilon/2$, there exists $t_0 = t_0(g) > 0$, so that $\sup_{0<t<t_0} d_X(u_g(t), \phi) < \epsilon$.

2. All initial data $g \in X$ generates a global in time solution u_g of (1.1), so that $g \in C((0, \infty)), H^{\alpha/2}[-T, T])$.

3. For all $0 < t < \infty$, there is the conservation of momentum, Hamiltonian and mass

$$\mathcal{P}(u_g(t, \cdot)) = \mathcal{P}(g), \mathcal{H}(u_g(t, \cdot)) = \mathcal{H}(g), \mathcal{M}(u_g(t, \cdot)) = \mathcal{M}(g),$$

Loosely speaking, we require relatively strong well-posedness result to hold in a suitable subspace of $H^{\alpha/2}[-T, T]$, so that the relevant conserved quantities $\mathcal{P}, \mathcal{H}, \mathcal{M}$ are conserved along the evolution. For example, a global well-posedness in a space of sufficiently high regularity, say H^3, would be ideal, since then, the solutions to (1.1) are classical and the conservation laws calculations are justified. This holds for example, in the cases of KdV and Benjamin-Ono, but one then is restricted to taking only perturbations u_0, which are sufficiently smooth. One should also remember that generally speaking, in the cases $\alpha < 2$, the data-to-solution map is not uniform continuous in the scale of the Sobolev spaces (of any order!). This idiosyncrasy of the model (1.1) is not consequential for our results, as we only focus on having global unique solutions, which necessarily conserve $\mathcal{P}, \mathcal{H}, \mathcal{M}$.

Our next definition is about the orbital stability of the waves.

Definition 1.3. We say that ϕ is orbitally stable travelling wave for the fKdV problem, if for every $\epsilon > 0$, there exists $\delta > 0$, so that whenever $u_0 \in X$, $\|u_0 - \phi\|_{H^{\alpha/2}} < \delta$ and u_0 is real-valued, then the solution u is globally in $H^{\alpha/2}[-T, T]$ and

$$\sup_{t>0} \inf_{y \in [-T,T]} \|u(t, \cdot + y) - \phi(\cdot)\|_{H^{\alpha/2}[-T,T]} < \epsilon.$$

Similarly, orbital stability for fNLS is understood as follows: for every $\epsilon > 0$, there is $\delta > 0$, so that whenever $u_0 \in X$, $\|u_0 - \phi\|_{H^{\alpha/2}} < \delta$, there is a global solution $u(t, \cdot) \in H^{\alpha/2}[-T, T]$, so that

$$\sup_{t>0} \inf_{y \in [-T,T]} \inf_{\theta \in \mathbb{R}} \|u(t, \cdot + y) - e^{i\theta}\phi_\omega(\cdot)\|_{H^{\alpha/2}[-T,T]} < \epsilon.$$

1.3. Main result

We are now ready to state the main results of this paper.

THEOREM 1.4. *Let* $\alpha \in (\frac{1}{2}, 2]$, $T > 0$. *Then, for each* $\lambda > 0$ *and* $a \in \mathbb{R}$, *there is a bell-shaped and classical solution of* (1.3), $\phi_{\lambda,a} \in H^\infty[-T, T]$, *where* $\omega = \omega(\lambda, a, \phi_{\lambda,a})$ *and* $\int_{-T}^{T} \phi^2(x)\, dx = \lambda$.

In addition, the corresponding travelling wave solutions $\phi_{\omega_\lambda}(x - \omega_\lambda t)$ *of the fKdV equation* (1.1) *are non-degenerate, when* $a \neq \lambda/2T$ *and spectrally stable, in the sense of definition 1.1, assuming the technical condition*

$$\left\langle \mathcal{L}_+^{-1}\phi, \phi \right\rangle \neq 0. \tag{1.8}$$

Moreover, assuming global well-posedness as in assumption 1.2, the waves are also orbitally stable, when $a \neq \lambda/2T$.

Similarly, for $a = 0$, *the standing wave solutions* $e^{i\omega_\lambda t}\phi_{\omega_\lambda}$ *of the fNLS* (1.2) *are non-degenerate and spectrally stable, provided* (1.8) *holds. Under assumption 1.2, one can similarly upgrade the statements to orbital stability.*

Remarks:

- The properties of the maps $a \to \omega(a), a \to \phi_a$ are certainly of interest, for example continuity, differentiability and monotonicity properties, etc. We will henceforth suppress the dependence upon a from our notations for the sake of conciseness.
- It is somewhat implicit in the statement that the wave speed ω may depend on the particular solution ϕ. To clarify this important point, we cannot rule out a scenario where for a given $(\lambda, a) \in \mathbb{R}_+ \times \mathbb{R}$, there are two waves $\phi, \tilde{\phi} : \|\phi\|^2 = \lambda = \|\tilde{\phi}\|^2$ satisfying (1.3), with $\omega(\lambda, a, \phi,) \neq \omega(\lambda, a, \tilde{\phi})$.
- In relation to the previous point, $\lambda \to \omega(\lambda)$ may be a multi-valued mapping. On the other hand, in proposition 3.2 below, we clarify that on a full measure subset $\mathcal{A} \subset \mathbb{R}_+$, ω_λ is independent on the waves of our construction.
- The restrictions $a \neq (\lambda/2T)$ and (1.8) are likely artefacts of the argument, but we cannot remove it for the moment. In particular, (1.8) is certainly expected to hold, at least generically in the parameters, see [**28**] for further discussion.

In closing this introduction section, we should mention that ours is not the first work on the construction and stability analysis of travelling waves for the fKdV, although it appears to be first of its kind for fNLS. We discuss some recent results below.

1.4. Some recent results

We should mention that periodic waves, in the fKdV context, were previously constructed in [**14**]. In this work, the authors have used different variational construction, namely they construct the solutions subject to the constraint $\int_{-T}^{T} \phi^3(x)\, dx = const.$, which is why they can get to the larger range $\alpha > \frac{1}{3}$. In the stability arguments, the authors tacitly assume smoothness of the Lagrange multipliers[4] on the constraints, which simplifies matters quite a bit. Using the assumed smoothness, they show the orbital stability of the waves.

Our approach does not make use of any such assumptions. In fact, let us give an informal preview to our existence and stability results, together with the difficulties associated with various steps in the proof. We construct first, for each $\lambda > 0$, normalized waves, that is functions that minimize the modified energy $\mathcal{H}(u) + a\mathcal{M}(u)$ for fixed L^2 norm, $\|\phi\|_{L^2}^2 = \lambda$, see proposition 3.1 below. This procedure generates bell-shaped functions, with speeds $\omega_{a,\lambda,\phi}$ as Lagrange multipliers.

The smoothness (or even continuity) of the map $(a, \lambda) \to \omega_{a,\lambda}$ is a highly non-trivial issue. In fact, we show that $\lambda \to \omega_{a,\lambda}$ is non-decreasing, while the continuity and differentiability of this map remains an open question. Even more dramatically, the continuity, let alone the differentiability, of the Banach space valued mapping $\lambda \to \phi_\lambda$ remains an open and very challenging question. This is often an assumption, see [**5**] and also (5.2.47) on p. 139 in [**18**] where this is explicitly required. The issue was sidestepped as an obvious one in previous publications. While we accept that the continuity and even differentiability is very likely true, we would want to reiterate the fact that it is not obvious, except in the cases with scaling (i.e. when the problem is posed on $\mathbb{R}$, instead of $[-T, T]$), in which the relation $\omega \to \phi_\omega$ is explicit.

While we do not make any continuity/differentiability assumptions of the sort, we certainly would benefit from such smoothness properties. In fact, we prove some very modest results along these lines, see proposition 3.2 and lemma 3.3 below, which however turn out to suffice for our purposes. For example, a key step in the argument, is the *weak non-degeneracy* of ϕ, i.e. $\phi \perp \mathrm{Ker}[\mathcal{L}_+]$. Note that this is trivial[5], if one assumes the H^1 smoothness of the map $\omega \to \phi_\omega$. With the non-degeneracy at hand, one proceeds to establish that the waves are non-degenerate, in the sense that $\mathrm{Ker}[\mathcal{L}_+] = \mathrm{span}[\phi']$. This is then a crucial piece of information, which is needed in the proof of orbital stability for these waves.

The paper is organized as follows. In § 2, we first show that the distributional solutions of (1.3) are in fact H^∞. This is followed by a few useful lemmas, in particular the Sturm-Liouville theory in the fractional case, see lemma 2.6. In § 3, we present the variational construction, together with a selection of additional spectral properties for the operators $\mathcal{L}_\pm$, as well as properties of the Lagrange multipliers ω_λ. In § 4 we show the non-degeneracy of the waves - the proof proceeds in two steps, first we establish in lemma 4.1 the weak non-degeneracy, using properties of the functions m, ω. Next, we use the Sturm–Liouville theory available in this case to upgrade this to strong non-degeneracy—see lemma 4.3 and the final stages of the proof immediately after. We finish this section by establishing spectral stability for the waves—note that while the orbital (non-linear) stability results in the next section are stronger, they do require *a priori* well-posedness assumptions. Finally, in § 5, we show the orbital stability of the waves, both for fKdV and fNLS. Note that for that part, we employ a direct contradiction argument that does not require

[4]while on a more basic level, and as was discussed above, it is not at all clear why these multipliers are independent on the particular constrained minimizers.

[5]Indeed, taking *formally* derivatives in ω in (1.3) leads to $\mathcal{L}_+[\partial_\omega \phi] = -\phi$, whence $\phi \perp \mathrm{Ker}[\mathcal{L}_+]$.

continuity of the maps $\omega \to \phi_\omega$ or $\lambda \to \phi_\lambda$, as this is an open question as of this writing.

After the submission of this article, a new paper [28] has put forward an alternative approach for constructing the travelling waves of the fKdV, as well as analyzing their corresponding stability properties. It resorts, similar to [14], to a variational problem in the form

$$\inf_{u\in H^{\alpha/2}} \left\{ \int_{-\pi}^{\pi} |\Lambda^{\alpha/2} u|^2 + \omega u^2, \int_{-\pi}^{\pi} u^3 \, \mathrm{d}x = 1, \int_{-\pi}^{\pi} u(x) \, \mathrm{d}x = 0 \right\}. \tag{1.9}$$

The one parameter family local minimizers, parametrized by $\omega > -1$, turn out to be a single-lobe functions. In addition, they are produced for all $\alpha \in (\frac{1}{3}, 2]$, but with the mean-value zero condition imposed in the constraint. The corresponding Euler–Lagrange equation are of course exactly in the form (1.3), but what used to be a free constant a is now tied to the minimizer via $a = (1/2\pi) \int_{-\pi}^{\pi} \phi^2$. The authors offer in-depth stability analysis, and it turns out that some of their solutions are unstable[6], while some others are stable. In the stability considerations, the authors need to make the strong assumption of non-degeneracy for the waves[7], $\mathrm{Ker}[\mathcal{L}_+] = \mathrm{span}[\phi']$. Furthermore, the argument in [28] relies on the smoothness of the map $\omega \to \phi_\omega$, which is shown to hold only when the non-degeneracy hold in an open neighbourhood $(\omega_0 - \delta, \omega_0 + \delta)$.

On the other hand, even though the waves are different (the two parameter family constructed in theorem 1.4 consists of bell-shaped functions, while the minimizers of (1.9) are mean-value zero), some connections between the two can be drawn. In particular, it has been established that the global minimizers constructed in theorem 1.4 are in fact constants, for all sufficiently small values of the L^2 norm λ. The reader is invited to check the specific results in [28].

2. Preliminaries

We recall the definition of the Lebesgue spaces, introduced by the usual norms

$$\|f\|_{L^p[-T,T]} = \left(\int_{-T}^{T} |f(x)|^p \, \mathrm{d}x \right)^{1/p}, \ 1 \leqslant p < \infty.$$

For the Fourier coefficients, taken $\hat{f}(k) := (1/\sqrt{2T}) \int_{-T}^{T} f(x) e^{-i\pi k(x/T)} \, \mathrm{d}x$, one can define the $H^s, s \in \mathbb{R}$ norms via the standard

$$\|f\|_{H^s} = \left(\sum_{k=-\infty}^{\infty} (1 + |k|^2)^s |\hat{f}(k)|^2 \right)^{1/2}.$$

Also, introduce $H^\infty = \cap_{l=1}^{\infty} H^l$. Next, there is an interesting Sobolev embedding, which will be useful for us, see lemma 2.7 below,

$$\|f\|_{H^{-a}[-T,T]} \leqslant C_{a,T} \|f\|_{L^1[-T,T]}, \tag{2.1}$$

[6]Recall that here the range is $\alpha > \frac{1}{3}$, whereas in our construction it is $\alpha > \frac{1}{2}$
[7]even though they are able to check it rigorously in certain cases

whenever $a > \frac{1}{2}$. Indeed, we have

$$\left(\sum_k \frac{|\hat{f}(k)|^2}{< k >^{2a}} \right)^{1/2} \leqslant C_a \sup_k |\hat{f}(k)| \leqslant C_a \|f\|_{L^1[-T,T]}.$$

2.1. A posteriori smoothness of weak solutions of fractional elliptic equations

In this section, we show that predictably, weak solutions to elliptic equations must be smoother than initially required, as to be solutions in a stronger sense. We work with the underlying elliptic equation (1.3), but one can easily extend the results below, by simply following our scheme.

DEFINITION 2.1. We say that $\phi \in L^2[-T,T]$ is a distributional solution of (1.3), if for every test function $h \in H^\infty([-T,T])$, one has the identity

$$\langle \phi, \Lambda^\alpha h \rangle + \omega \langle \phi, h \rangle - \langle \phi^2, h \rangle + a \langle 1, \phi \rangle = 0.$$

Note that $\langle \phi^2, h \rangle$ makes sense, since $\phi^2 \in L^1[-T,T]$, while $h \in L^\infty[-T,T]$.

We have the following *a posteriori* smoothness result.

PROPOSITION 2.2. *Let $\alpha > \frac{1}{2}$. Then, the distributional solutions ϕ of (1.3) belong to $H^\infty([-T,T])$.*

Proof. We add $A\phi$ to both sides of (1.3), where A is a large positive constant, say $A = |\omega| + 1$. Thus, the equation becomes $A\phi + \phi^2 - a = (\Lambda^\alpha + \omega + A)\phi$. Note $\sigma(\Lambda^\alpha + \omega + A) = \{(\pi|k|/T)^\alpha + \omega + A, k = 0, \pm 1, \ldots\} \subset [1, \infty)$, whence $\Lambda^\alpha + \omega + A$ is invertible on $L^2[-T,T]$. Also, its inverse clearly improves the regularity of its input by α derivatives. In other words, $((\Lambda^\alpha + \omega + A)^{-1} : H^s \to H^{s+\alpha}$.

Introduce $\tilde{\phi} := (\Lambda^\alpha + \omega + A)^{-1}[A\phi + \phi^2 - a]$. This is of course nothing but the formal solution of (1.3), that is ϕ, but we are about to prove this rigorously. First, observe that since $(\Lambda^\alpha + \omega + A)^{-1} : L^2 \to H^\alpha$, we have that $\tilde{\phi} \in H^\alpha$. Then, for every test function h, we have

$$\left\langle \tilde{\phi}, (\Lambda^\alpha + \omega + A)h \right\rangle = \langle A\phi + \phi^2 - a, h \rangle = \langle \phi, (\Lambda^\alpha + \omega + A)h \rangle.$$

It follows that $\tilde{\phi} = \phi$, in sense of distributions, since $(\Lambda^\alpha + \omega + A)(H^\infty) = H^\infty$. Thus, $\phi \in H^\alpha$. One can now bootstrap this to H^∞, since once we know $\phi \in H^\alpha$, then $A\phi + \phi^2 - a \in H^\alpha$, because of Sobolev embedding. But then $(\Lambda^\alpha + \omega + A)^{-1} : H^\alpha \to H^{2\alpha}$, so $\phi = \tilde{\phi} \in H^{2\alpha}$ and so on. □

Note that the variational solutions that we produce will be distributional solutions of (1.3). Thus, such solutions will be in the class H^∞, as a consequence of the *a posteriori* smoothness results in proposition 2.2.

2.2. Some basic results of the instability index theory

In this section, we present some results about the solvability of eigenvalue problems of the form (1.6) and (1.7). In fact, there is a more general theory developed for more general eigenvalue problems of this type, we shall generally follow the presentation in [23] as it suits our purposes the best. Namely, considering an eigenvalue problem of the form

$$\mathcal{J}\mathcal{L}f = \lambda f, \tag{2.2}$$

under the assumptions, that there exists a real-valued Hilbert space $\mathcal{X}$ (with dot product $(\cdot,\cdot)$ and an dual pairing between $\mathcal{X}$ and $\mathcal{X}^*$ given by $\langle\cdot,\cdot\rangle$), so that $\mathcal{J}$, $\mathcal{L}$ are (generally) unbounded operators as follows

- $D(\mathcal{J}) \subset \mathcal{X}^*$ and $\mathcal{J} : D(\mathcal{J}) \to \mathcal{X}$, so that $\mathcal{J}^* = -\mathcal{J}$, in the sense that for every $u^*, v^* \in D(\mathcal{J}) \subset \mathcal{X}^*$, $\langle \mathcal{J}u^*, v^*\rangle = -\langle u^*, \mathcal{J}v^*\rangle$.
- $\mathcal{L} : \mathcal{X} \to \mathcal{X}^*$ is a bounded and symmetric operator, in the sense that for every $u, v \in \mathcal{X}$, $(u, v) \to \langle \mathcal{L}u, v\rangle$ is bounded and symmetric form on $\mathcal{X} \times \mathcal{X}$.
- $\mathrm{Ker}[\mathcal{L}]$ is finite dimensional and the following is an $\mathcal{L}$ invariant decomposition

  $$\mathcal{X} = \mathcal{X}_- \oplus \mathrm{Ker}[\mathcal{L}] \oplus \mathcal{X}_+, n(\mathcal{L}) := \dim(\mathcal{X}_-) < \infty,$$

  where $\mathcal{L}|_{\mathcal{X}_-} < 0$ and $\mathcal{L}|_{\mathcal{X}_+} \geqslant \delta > 0$. So, since $dim(\mathcal{X}_-) < \infty$, and at the cost of taking even smaller $\delta > 0$, $\langle \mathcal{L}u_-, u_-\rangle \leqslant -\delta\|u_-\|^2, \langle \mathcal{L}u_+, u_+\rangle \geqslant \delta\|u_+\|^2$, for every $u_\pm \in \mathcal{X}_\pm$.
- $$\{f \in \mathcal{X}^* : \langle f, u\rangle = 0, \forall u \in \mathcal{X}_- \oplus \mathcal{X}_+\} \subset D(\mathcal{J}).$$

Then, for the (finite dimensional) generalized kernel $g\mathrm{Ker}[\mathcal{L}] := \{u \in \mathcal{X} : (\mathcal{J}\mathcal{L})^k u = 0, k = 1, 2, \ldots\} \subset \mathrm{Ker}[\mathcal{L}]$, take the complement $\mathcal{Q}$, that is $g\mathrm{Ker}[\mathcal{L}] = \mathrm{Ker}[\mathcal{L}] \oplus \mathcal{Q}$. Introduce the non-negative integer

$$k_0^{\leqslant 0}(\mathcal{L}) := \max\{\dim(Z) : Z \text{ subspace of } \mathcal{Q} : \langle \mathcal{L}z, z\rangle < 0, \forall z \in Z\}.$$

Introduce for the following indices, counting different types of eigenvalues (all with their respective multiplicities),

$$\begin{aligned} k_{unstable} &= \#\{\lambda > 0 : \lambda \in \sigma_p(\mathcal{J}\mathcal{L})\}, \\ k_c &= \#\{\lambda, \Re\lambda > 0, \Im\lambda > 0 : \lambda \in \sigma_p(\mathcal{J}\mathcal{L})\}, \\ k_i^{\leqslant 0} &= \{\lambda \in i\mathbb{R}_+, \lambda \in \sigma_p(\mathcal{J}\mathcal{L}) \text{ with a negative Krein signature}\}. \end{aligned}$$

Now, theorem 2.3 in [23] asserts the relation,

$$k_{\text{unstable}} + 2k_c + 2k_i^{\leqslant 0} = n(\mathcal{L}) - k_0^{\leqslant 0}(\mathcal{L}), \tag{2.3}$$

see (2.9), [23] for precise definitions. In particular, if $n(\mathcal{L}) = 1$ and $k_0^{\leqslant 0}(\mathcal{L}) \geqslant 1$, we will be able to conclude from (2.3) that all the terms on the left are zero, implying spectral stability.

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Next, we discuss the particular setup in the cases (1.4) and (1.5), respectively. For the fKdV, that is for the spectral problem (1.4), we take $\mathcal{J} = \partial_x, D(\mathcal{J}) = H^1[-T,T]$, $\mathcal{L} = \mathcal{L}_+, D(\mathcal{L}) = H^\alpha[-T,T]$. The Hilbert space $\mathcal{X} := H^{\alpha/2}[-T,T]$, so that we have the required bounds $\langle \mathcal{L}u, v\rangle \leqslant C\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}$. Clearly, the other conditions will be satisfied, once we check that $\mathrm{Ker}[\mathcal{L}]$ and $\mathcal{X}_-$ are finite dimensional subspaces and $\mathrm{Ker}[\mathcal{L}] \subset H^1$.

For the fNLS spectral problem (1.5), we take

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

while

$$\mathcal{L} = (\begin{smallmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{smallmatrix}), D(\mathcal{L}) = H^\alpha \times H^\alpha,$$

while $\mathcal{X} = H^{\alpha/2} \times H^{\alpha/2}$. An easy corollary of this theory is

Corollary 2.3. *Assume that $n(\mathcal{L}_+) = 1$, while $\mathcal{L}_- \geqslant 0$. If in addition,*

- *Weak non-degeneracy holds, i.e. $\phi \perp \mathrm{Ker}[\mathcal{L}_+]$. In particular, $\mathcal{L}_+^{-1}\phi$ is well-defined.*
- *The Vakhitov–Kolokolov index is negative: $\left\langle \mathcal{L}_+^{-1}\phi, \phi\right\rangle < 0$*

then the eigenvalue problems (1.6) and (1.7) are spectrally stable.

Remark: The Vakhitov–Kolokolov criteria for spectral stability appeared first in [**31**] in the more recognizable form[8] $\partial_\omega\|\phi_\omega\|_{L^2}^2 > 0$. Note that this is equivalent to $\left\langle \mathcal{L}_+^{-1}\phi, \phi\right\rangle < 0$ above. This was later greatly generalized and popularized in [**12**].

2.3. A few useful lemmas

In this section, we present some lemmas, which will be used in the sequel. They are unrelated, so we put them in the order in which they are referred to in the text.

The generalized Polya–Szegö inequality is standard for the functions on $\mathbb{R}$, and it states that among all functions, the decreasingly rearranged ones have the smallest H^β norms, as long as $\beta \in (0,1]$. We need such result for periodic functions, one can find it for example in [**9**], lemma A.1.

Lemma 2.4. *[Generalized Polya–Szegö inequality] For any $\beta \in (0,1]$,*

$$\int_{-1}^{1} |\Lambda^\beta u(x)|^2 \, \mathrm{d}x \geqslant \int_{-1}^{1} |\Lambda^\beta u^*(x)|^2 \, \mathrm{d}x, \tag{2.4}$$

where u^* is the decreasing rearrangement of u. That is, for all $u \in H^\beta[-1,1]$, the corresponding rearrangement $u^* \in H^\beta[-1,1]$ and in addition (2.4) holds. Equality is achieved only when u is bell-shaped, i.e. $u = u^*$.

[8]It has to be mentioned that this was popular among physicists as an universal criteria for stability. This is slightly misleading, as this criteria is valid only under the additional assumption about the Morse index $n(\mathcal{L}_+) = 1$.

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The following lemma was proved in [30].

LEMMA 2.5. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function that satisfies*

$$\limsup_{\epsilon\to 0+} \sup_{\lambda\in(a,b)} \frac{f(\lambda+\epsilon)+f(\lambda-\epsilon)-2f(\lambda)}{\epsilon^2} \leqslant 0.$$

Then, f is concave down.

The next result is a variant of the well-known Sturm–Liouville oscillation theorem, but this time for fractional Schrödinger operator. It was first obtained for operators acting on the line $\mathbb{R}$ [11]. It was then extended for the periodic case, following similar ideas in [14] (for the lowest three eigenfunctions), and then in [15] for all eigenfunctions.

LEMMA 2.6. *Let $V : [-T,T] \to \mathbb{R}$ be a continuous function and $\alpha \in (0,2)$. Consider the self-adjoint fractional Schrödinger operator $\mathcal{H} = \Lambda^\alpha + V$ with domain $D(\mathcal{H}) = H^\alpha[-T,T]$. Let its spectrum*[9] *be ordered as follows*

$$\lambda_0(\mathcal{H}) < \lambda_1(\mathcal{H}) \leqslant \lambda_2(\mathcal{H}) \leqslant \dots$$

Then, the corresponding eigenfunctions $\psi_n : \mathcal{H}\psi_n = \lambda_n\psi_n$ have no more than $2n$ changes of sign in the interval $[-T,T]$.

The next lemma is about the mapping properties of Schrödinger operators $\mathcal{H}$ of the type described in lemma 2.6 and its inverses, whenever they exist. First, for every $\lambda \in \mathbb{R}$, $\lambda \notin \sigma(\mathcal{H})$, we have that $\mathcal{H} - \lambda : H^\alpha \to L^2$, whence $(\mathcal{H}-\lambda)^{-1} : L^2 \to D(\mathcal{H}) = H^\alpha$. By taking adjoints, we also have $(\mathcal{H}-\lambda)^{-1} : H^{-\alpha} \to L^2$, for $\lambda \in \mathbb{R} \cap \rho(\mathcal{H})$. Taking into account the embedding $L^1[-T,T] \hookrightarrow H^{-\alpha}[-T,T]$, i.e. (2.1), we have shown

LEMMA 2.7. *For $\alpha > \frac{1}{2}$, and $a \notin \sigma(\mathcal{H})$, we have $(\mathcal{H}-\lambda)^{-1} : L^1[-T,T] \to L^2[-T,T]$. In addition, supposing that for invariant subspace, $S \subset L^2[-T,T]$ of $\mathcal{H}$, we have that $\lambda \notin \sigma_S(\mathcal{H})$. That is, $(\mathcal{H}-\lambda)^{-1} : S \to S$. Then,*

$$\|(\mathcal{H}-\lambda)^{-1} f\|_{L^2\cap S} \leqslant C\|f\|_{L^1\cap S}$$

3. The variational construction

The classical way to produce solitary waves is to minimize energy, with respect to fixed L^2 norm. The result of this are the so-called normalized waves. In order to simplify the exposition, we shall work with $T = 1$. Later on, we easily reduce to this case by a simple rescaling argument.

[9]which consists entirely of eigenvalues with finite multiplicity.

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PROPOSITION 3.1. *Let $\alpha \in (\frac{1}{2}, 2]$ and $\lambda > 0, a \in \mathbb{R}$. Then, the minimization problem*

$$\begin{cases} \mathcal{E}_a[\varphi] := \frac{1}{2}\int_{-1}^{1} |\Lambda^{\alpha/2}\varphi(x)|^2\,\mathrm{d}x - \frac{1}{3}\int_{-1}^{1} |\varphi(x)|^3\,\mathrm{d}x + a\int_{-1}^{1} |\varphi(x)|\,\mathrm{d}x \\ \int_{-1}^{1} \varphi^2(x)\,\mathrm{d}x = \lambda \end{cases} \tag{3.1}$$

has a bell-shaped solution, $\varphi = \varphi_{a,\lambda}$. Moreover, $\varphi_{a,\lambda}$ satisfies, in a distributional sense, the Euler–Lagrange equation

$$\Lambda^\alpha \varphi + \omega\varphi - \varphi^2 + a = 0, -1 \leqslant x \leqslant 1, \tag{3.2}$$

for $\omega = \omega(\lambda, a; \varphi)$, given by the either of the two formulas

$$\omega_{\lambda,a} = \frac{\int_{-1}^{1} \varphi^3(x)\,\mathrm{d}x - \int_{-1}^{1} |\Lambda^{\alpha/2}\varphi(x)|^2\,\mathrm{d}x - a\int_{-1}^{1} \varphi(x)\,\mathrm{d}x}{\lambda}, \tag{3.3}$$

$$\omega_{\lambda,a} = \frac{\lambda - 2a}{\int_{-1}^{1} \varphi(x)\,\mathrm{d}x} \tag{3.4}$$

In addition, we have the following preliminary properties of the linearized operators

- *$\mathcal{L}_+$ has exactly one negative eigenvalue, denoted by $-\sigma_\lambda^2$, which is simple, with a corresponding eigenfunction χ_λ. In addition, $\mathcal{L}_+|_{\{\varphi_\lambda\}^\perp} \geqslant 0$.*
- *For $a = 0$, the operator $\mathcal{L}_- := \Lambda^\alpha + \omega - \varphi_\lambda \geqslant 0$ satisfies $Ker(\mathcal{L}_-) = \mathrm{span}[\varphi_\lambda]$ and for some $\delta > 0$, $\mathcal{L}_-|_{\{\varphi_\lambda\}^\perp} \geqslant \delta Id$.*
- *For $a < 0$, there exists $\delta > 0$, so that $\mathcal{L}_- \geqslant \delta Id$, while for $a > 0$, $\mathcal{L}_-$ has $n(\mathcal{L}_-) = 1$, with $\mathcal{L}_-|_{\{\varphi_\lambda\}^\perp} \geqslant \delta Id$. In particular, for $a \neq 0$, $0 \notin \sigma(\mathcal{L}_-)$, in other words $\mathcal{L}_-^{-1}$ exists.*

Remarks: In the statement above, it is implicit that the Lagrange multiplier ω may in fact depend on the particular minimizer φ as well. This is also related to the uniqueness issue for the solutions of the constrained minimization problem (3.1). More precisely, for given values of $\lambda > 0, a \in \mathbb{R}$, it is possible that there exist two solutions $\varphi_{\lambda,a}, \tilde{\varphi}_{\lambda,a}$ of (3.1). Each of them will certainly satisfy the Euler–Lagrange equation (3.2), but may be two different Lagrange multipliers $\omega, \tilde{\omega}$. We cannot rule out neither of these possibilities in this article.

Our next result prepares some background information, needed later on in the arguments, for the following function

$$m(\lambda) := \inf_{\int_{-1}^{1} \varphi^2(x)\,\mathrm{d}x = \lambda} \mathcal{E}_a[\varphi].$$

Note that it is not *a priori* clear why m is even finite for all $\lambda > 0$, but this is established below. Note that, m also depends on a, but we prefer not to emphasize this dependence in the notation.

PROPOSITION 3.2. *The function m has the following properties*

- *m is finite everywhere, that is $m(\lambda) > -\infty$ for every $\lambda > 0$,*
- *m is a locally Lipschitz, and its derivative, which exists at least a.e., is*

$$m'(\lambda) = -\frac{\omega_\lambda}{2}.$$

- *m is concave down.*

In particular, at all points in the full measure subset $\mathcal{A} := \{\lambda \in \mathbb{R}_+ : m'(\lambda) -$ exists$\}$, the function $\omega = \omega(\lambda)$ is independent on the concrete minimizer φ, as a derivative of $-2m(\lambda)$.

Regarding the function $\lambda \to \omega_\lambda$,

- *For $a \leqslant 0$, $\omega_\lambda > 0$ for all λ,*
- *For $a > 0$, $\omega_\lambda < 0, \lambda \in (0, 2a)$ and $\omega_\lambda > 0, \lambda \in (2a, \infty)$.*
- *$\lambda \to \omega_\lambda, \lambda \in \mathcal{A}$ is non-decreasing. In fact, its first derivative, which is guaranteed to exist at least a.e. in view of the monotonicity, satisfies*

$$\omega'(\lambda) > \frac{\sigma_\lambda^2}{2\langle \chi_\lambda, \varphi_\lambda\rangle^2} > 0,$$

where χ_λ is a normalized eigenfunction[10]*, corresponding to the negative eigenvalue for $\mathcal{L}_+$.*

- *Even outside of $\mathcal{A}$, the function $\lambda \to \omega(\lambda, \varphi_\lambda)$ is non-decreasing. More precisely, suppose $0 < \lambda_1 < \lambda_2$, with corresponding, possibly non-unique, minimizers $\varphi_{\lambda_1}, \varphi_{\lambda_2}$. Then,*

$$\omega(\lambda_1, \varphi_{\lambda_1}) \leqslant \omega(\lambda_2, \varphi_{\lambda_2}).$$

We prove these results over the course of the § 3.

3.1. Well-posedness and existence of minimizers for the variational problem (3.1)

Let $\epsilon > 0$ be an arbitrary real number. Then, by the Sobolev embedding, the Gagliardo–Nirenberg's and the Young's inequalities, we obtain that for any $\varphi \in H^{\alpha/2}$, satisfying the constraint $\int \varphi^2(x)\,\mathrm{d}x = \lambda$ and for $\alpha > \frac{1}{2}$, there exists $C = C_{\epsilon,\alpha}$,

$$\begin{aligned}\|\varphi\|_{L^3}^3 \leqslant& C\|\varphi\|_{H^{1/6}}^3 \leqslant C\|\varphi\|_{H^{\alpha/2}}^{1/\alpha}\|\varphi\|_{L^2}^{3-(1/\alpha)} \\ =& C\lambda^{3/2-(1/2\alpha)}\|\varphi\|_{H^{\alpha/2}}^{1/\alpha} \leqslant \epsilon\|\Lambda^{\alpha/2}\varphi\|_{L^2}^2 + C_{\epsilon,\alpha}\lambda^{(3\alpha-1)/(2\alpha-1)}. \end{aligned} \tag{3.5}$$

where in the last step, we have used the Young's inequality. Clearly, inf in the constrained minimization problem (3.3) is bounded from below, hence the problem is well-posed.

[10]Note that $\langle \chi_\lambda, \varphi_\lambda\rangle \neq 0$, since otherwise, due to $\mathcal{L}_+|_{\{\varphi_\lambda\}^\perp} \geqslant 0$, we get the contradiction $-\sigma_\lambda^2 = \langle \mathcal{L}_+\chi_\lambda, \chi_\lambda\rangle \geqslant 0$

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Pick a minimizing sequence, that is $\varphi_n \in H^{\alpha/2}$, so that $\mathcal{E}[\varphi_n] \to m(\lambda)$. We will show that the sequence is compact in L^2 and subsequently in all $L^p, p \in (2, \infty)$. We have that for all large enough n, $\mathcal{E}[\varphi_n] < m(\lambda) + 1$. It follows that

$$\|\Lambda^{\alpha/2}\varphi_n\|_{L^2}^2 \leqslant 2\mathcal{E}[\varphi_n] + \frac{2}{3}\int_{-1}^{1} \varphi_n^3(x)\,\mathrm{d}x - 2a\int_{-1}^{1} |\varphi_n(x)|\,\mathrm{d}x.$$

By (3.5) and Cauchy–Schwartz, the right-hand side can be estimated as follows

$$\|\Lambda^{\alpha/2}\varphi_n\|_{L^2}^2 \leqslant 2(m(\lambda) + 1) + \frac{2}{3}\left(\|\Lambda^{\alpha/2}\varphi_n\|_{L^2}^2 + C_{\epsilon,\lambda}\right) + 4|a|\sqrt{\lambda}. \tag{3.6}$$

Let us reiterate that this estimate holds whenever $\|\varphi\|^2 = \lambda$. Hiding $\|\Lambda^{\alpha/2}\varphi_n\|_{L^2}^2$ behind the left-hand side, leads to an *a priori* estimate

$$\sup_n \|\Lambda^{\alpha/2}\varphi_n\| \leqslant C_{\lambda,a}.$$

Note that since $H^{\alpha/2}[-1,1]$ compactly embeds into $L^2[-1,1]$, the sequence $\{\varphi_n\}$ is a compact in $L^2[-1,1]$. Take a convergent subsequence $\{\varphi_{n_k}\}_{k=1}^{\infty}$. We find that its limit $\varphi := \lim_k \varphi_{n_k}$ satisfies $\|\varphi\|_{L^2}^2 = \lambda$. By Gagliardo–Nirenberg's inequality, the same sequence is compact in any $L^p, p > 2$ space. By Hölder's and Gagliardo–Nirenberg, it is also compact in any $L^q, q \in [1, 4)$ (recall $\sup_n \|\varphi_n\|_{H^{\alpha}/2} < \infty$, $H^{\alpha/2} \hookrightarrow L^4$, as $\alpha > \frac{1}{2}$). In particular,

$$\int_{-1}^{1} \varphi_{n_k}(x)\,\mathrm{d}x \to \int_{-1}^{1} \varphi(x)\,\mathrm{d}x, \quad \int_{-1}^{1} |\varphi_{n_k}(x)|^3\,\mathrm{d}x \to \int_{-1}^{1} |\varphi(x)|^3\,\mathrm{d}x.$$

Finally, by the weak convergence in $H^{\alpha/2}$, $\varphi_{n_k} \to \varphi$, we have by the lower semi-continuity of the norms with respect to the weak convergence

$$\liminf_k \int_{-1}^{1} |\Lambda^{\alpha/2}\varphi_{n_k}(x)|^2\,\mathrm{d}x \geqslant \int_{-1}^{1} |\Lambda^{\alpha/2}\varphi(x)|^2\,\mathrm{d}x.$$

But then,

$$m(\lambda) = \lim_k \mathcal{E}[\varphi_{n_k}] \geqslant \liminf_k \mathcal{E}[\varphi_{n_k}] \geqslant \mathcal{E}[\varphi],$$

which is a contradiction (recall $\int \varphi^2(x)\,\mathrm{d}x = \lambda$), unless $\mathcal{E}[\varphi] = m(\lambda)$. In addition, since it must be that $\lim_k \int_{-1}^{1} |\Lambda^{\alpha/2}\varphi_{n_k}(x)|^2\,\mathrm{d}x = \int_{-1}^{1} |\Lambda^{\alpha/2}\varphi(x)|^2\,\mathrm{d}x$, it follows that $\lim_k \|\varphi_{n_k} - \varphi\|_{H^{\alpha/2}} = 0$. Thus, φ is a minimizer. We observe that the minimizer is necessarily bell-shaped, by the generalized Polya–Szegö's inequality (2.4).

Note that we have shown in particular that each minimizing sequence has an $H^{\alpha/2}$ convergent subsequence, which converges to a minimizer.

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3.2. Euler–Lagrange equation

We now derive the Euler–Lagrange equation (3.2). Let $\epsilon > 0$ and take any test function $h \in H^\infty[-1,1]$. Consider

$$g(\epsilon) := \mathcal{E}\left[\sqrt{\lambda}\frac{\varphi+\epsilon h}{\|\varphi+\epsilon h\|}\right] \geqslant g(0) = \mathcal{E}[\varphi].$$

Observe that the function $\epsilon \to \|\varphi+\epsilon h\|^q = (\lambda + 2\epsilon\langle\varphi, h\rangle + \epsilon^2\|h\|^2)^{q/2}$ is real-analytic in a neighbourhood of zero. The expansion of up to second term yields

$$\|\varphi+\epsilon h\|^q = \lambda^{q/2} + \epsilon q \lambda^{q/2-1}\langle\varphi, h\rangle + O(\epsilon^2),$$

whence

$$\frac{\lambda}{2\|\varphi+\epsilon h\|^2}\int_{-1}^{1}|\Lambda^{\alpha/2}(\varphi+\epsilon h)|^2\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1}|\Lambda^{\alpha/2}\varphi|^2\,\mathrm{d}x \\ + \epsilon\left[\left\langle\Lambda^\alpha\varphi - \frac{\|\Lambda^{\alpha/2}\varphi\|^2}{\lambda}\varphi, h\right\rangle\right] + O(\epsilon^2)]$$

$$\frac{\lambda^{3/2}}{3\|\varphi+\epsilon h\|^3}\int_{-1}^{1}(\varphi+\epsilon h)^3\,\mathrm{d}x = \frac{1}{3}\int_{-1}^{1}\varphi^3(x)\,\mathrm{d}x \\ + \epsilon\left[\langle\varphi^2, h\rangle - \frac{\langle\varphi, h\rangle}{\lambda}\int\varphi^3(x)\,\mathrm{d}x\right] + O(\epsilon^2),$$

$$\frac{a\sqrt{\lambda}}{\|\varphi+\epsilon h\|}\int_{-1}^{1}(\varphi(x)+\epsilon h(x))\,\mathrm{d}x = a\int_{-1}^{1}\varphi(x)\,\mathrm{d}x \\ + \epsilon\left[a\langle 1, h\rangle - \frac{a}{\lambda}\langle\varphi, h\rangle\right] + O(\epsilon^2).$$

Putting everything together, we obtain

$$g(\epsilon) = g(0) + \epsilon[\left\langle\Lambda^\alpha\varphi - \varphi^2 + a - \frac{\|\Lambda^{\alpha/2}\varphi\|^2 - \int\varphi^3 + a\int\varphi}{\lambda}\varphi, h\right\rangle + O(\epsilon^2).$$

It follows that (3.2) is satisfied, in a weak sense, with ω given by (3.3). The formula (3.4) is obtained by taking dot product of the Euler–Lagrange equation (3.2) with 1 and elementary algebraic manipulations.

We now turn our attention to the statements regarding the linearized operators $\mathcal{L}_\pm$.

3.3. Spectral properties of $\mathcal{L}_\pm$

We start with the spectral properties of $\mathcal{L}_+$. We use again the property that g attains its minimum at $\epsilon = 0$. In order to simplify the argument, take the test function h, so that $h \perp \varphi$, $\|h\|_{L^2} = 1$. Note that this implies $\|\varphi+\epsilon h\|^2_{L^2} = \lambda + \epsilon^2$, whence

$$\|\varphi + \epsilon h\|_{L^2}^q = \lambda^{q/2} + (q/2)\lambda^{\frac{q}{2}-1}\epsilon^2 + o(\epsilon^2).$$

The expansion of $g(\epsilon)$ around zero takes the form

$$\begin{aligned} g(\epsilon) = \mathcal{E}\left[\sqrt{\lambda}\frac{\varphi+\epsilon h}{\|\varphi+\epsilon h\|}\right] &= g(0) + \epsilon\left\langle \Lambda^\alpha \varphi - \varphi^2 + a, h\right\rangle \\ &+ \frac{1}{2}\left[\|\Lambda^{\alpha/2}\varphi\|^2 + \epsilon^2\|\Lambda^{\alpha/2}h\|^2)\right]\left[1 - \frac{1}{2\lambda}\epsilon^2\right] \\ &- \frac{1}{3}\left[\int \varphi^3(x) + 3\epsilon^2\left\langle \varphi h, h\right\rangle\right]\left[1 - \frac{3}{2\lambda}\epsilon^2\right] \mathrm{d}x \\ &+ a\left(\int \varphi(x)\,\mathrm{d}x\right)\left(1 - \frac{1}{2\lambda}\epsilon^2\right) + o(\epsilon^2). \end{aligned}$$

Clearly, $\left\langle \Lambda^\alpha \varphi - \varphi^2 + a, h\right\rangle = \left\langle \Lambda^\alpha \varphi - \varphi^2 + a + \omega\varphi, h\right\rangle = 0$, by the Euler–Lagrange equation. Thus, we can rewrite the last identity as

$$g(\epsilon) - g(0) = \frac{\epsilon^2}{2}\left(\left\langle \Lambda^\alpha h, h\right\rangle - 2\left\langle \varphi h, h\right\rangle + \omega\right) + o(\epsilon^2).$$

Recalling that $\|h\| = 1$, $(\langle \Lambda^\alpha h, h\rangle - 2\langle \varphi h, h\rangle + \omega) = \langle \mathcal{L}_+ h, h\rangle$. Since 0 is a local minimum for the function g, we conclude that $\langle \mathcal{L}_+ h, h\rangle \geqslant 0$. Thus,

$$\mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0,$$

whence we deduce that $\mathcal{L}_+$ has at most one negative eigenvalue, or $n(\mathcal{L}_+) \leqslant 1$. On the other hand, by differentiating the Euler–Lagrange equation in x, we obtain $\mathcal{L}_+[\varphi'] = 0$, hence zero is an eigenvalue. Note however that φ' changes sign in $[-1, 1]$, hence it is not the eigenfunction corresponding to the smallest eigenvalue. It follows that there is a negative eigenvalue or $n(\mathcal{L}_+) = 1$.

The claims about $\mathcal{L}_-$ follow easily in the case $a = 0$. By direct evaluation, $\mathcal{L}_-[\varphi] = 0$ (this is simply (3.2)), so 0 is an eigenvalue. Since $\mathcal{L}_-|_{\{\varphi\}^\perp} > \mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0$, we conclude that 0 is at the bottom of the spectrum.

In the case $a \neq 0$, we observe that $\mathcal{L}_-\varphi = -a$, this is again an instance of (3.2). Let now $a < 0$. Assuming that the smallest eigenvalue is $-\sigma^2, \sigma \geqslant 0$, take Ψ to be its (necessarily positive, according to Sturm–Liouville's theory) eigenfunction, $\mathcal{L}_-\Psi = -\sigma^2\Psi$. Take a dot product of this last identity with φ. We have

$$0 < -a\left\langle \Psi, 1\right\rangle = \left\langle \mathcal{L}_-\Psi, \varphi\right\rangle = -\sigma^2\left\langle \Psi, \varphi\right\rangle \leqslant 0,$$

all due to the $\Psi > 0, \varphi > 0$, $a < 0$. So, a contradiction is reached, which implies $\mathcal{L}_- > 0$.

In the case $a > 0$, we observe that $\langle \mathcal{L}_-\varphi, \varphi\rangle = -a\langle 1, \varphi\rangle < 0$, whence $\mathcal{L}_-$ has at least one negative eigenvalue. Since $\mathcal{L}_- > \mathcal{L}_+$ and $n(\mathcal{L}_+) = 1$, it follows that $\mathcal{L}_-$ has exactly one negative eigenvalue and moreover, $\mathcal{L}_-|\{\varphi\}^\perp > \mathcal{L}_+|\{\varphi\}^\perp \geqslant 0$, so $0 \notin \sigma(\mathcal{L}_-)$.

3.4. Properties of $m(\lambda), \omega(\lambda)$

Recall that we have shown that (3.1) is well-posed and solvable. We have also established a number of useful spectral properties of $\mathcal{L}_\pm$. We now turn to the proof of proposition 3.2.

We start with the observation that with the test function $u = \sqrt{\lambda/2}$, we arrive for the following (very rough) estimate for $m(\lambda)$, namely $m(\lambda) \leqslant -(\lambda^{3/2}/3\sqrt{2}) + a\sqrt{2\lambda}$. In addition, we have derived various *a priori* estimates on the minimizers in the form $\|\Lambda^{\alpha/2}\varphi_\lambda\|_{L^2} \leqslant C_\lambda$, see for example (3.6). If we insert the above estimate for $m(\lambda)$, we arrive at an explicitly computable and continuous in λ bound C_λ. In view of all this, we can set up the variational problem in the form

$$m(\lambda) := \inf_{\int_{-1}^1 u^2(x)\,\mathrm{d}x = \lambda : \|\Lambda^{\alpha/2}u\|_{L^2} \leqslant 2C_\lambda} \mathcal{E}[u].$$

Introducing the new variable $U : u = \sqrt{\lambda}U$, consider a new function

$$k(\lambda) := \frac{m(\lambda)}{\lambda} = \inf_{\int_{-1}^1 U^2(x)\,\mathrm{d}x = 1 : \|\Lambda^{\alpha/2}U\|_{L^2} \leqslant D_\lambda} \frac{1}{2}\int_{-1}^1 |\Lambda^{\alpha/2}U(x)|^2 - \frac{\sqrt{\lambda}}{3}\int_{-1}^1 |U(x)|^3 + \frac{a}{\sqrt{\lambda}}\int_{-1}^1 |U(x)|,$$

where $D_\lambda := 2\lambda^{-1/2}C_\lambda$ is also continuous.

We will now show that $\lambda \to k(\lambda)$ is locally Lipschitz, whence $m(\lambda)$ will be locally Lipschitz as well. Considering the functional over which we need to minimize for the construction of $k(\lambda+\delta)$, for small δ, we have for every U in the constrained set

$$\begin{aligned}
&\frac{1}{2}\int_{-1}^1 |\Lambda^{\alpha/2}U(x)|^2\,\mathrm{d}x - \frac{\sqrt{\lambda+\delta}}{3}\int_{-1}^1 |U(x)|^3\,\mathrm{d}x + a(\lambda+\delta)^{-1/2}\int_{-1}^1 |U(x)|\,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_{-1}^1 |\Lambda^{\alpha/2}U(x)|^2\,\mathrm{d}x - \frac{\sqrt{\lambda}}{3}\int_{-1}^1 |U(x)|^3\,\mathrm{d}x + a\lambda^{-1/2}\int_{-1}^1 |U(x)|\,\mathrm{d}x + E_{\delta,\lambda},
\end{aligned}$$

where

$$|E_{\delta,\lambda}| \leqslant C|\delta|(\lambda^{-1/2} + \lambda^{-3/2})(\|U\|_{L^3}^3 + \|U\|_{L^1}) \leqslant C|\delta|(1 + D_{\lambda+\delta}^3),$$

since we have assumed that U is in the constrained set for $k(\lambda+\delta)$ and hence by Hölder's and Sobolev embedding $\|U\|_{L^3} + \|U\|_{L^1} \leqslant C\|\Lambda^{\alpha/2}U\| \leqslant CD_{\lambda+\delta}$. Taking $\inf_{\int_{-1}^1 U^2(x)\,\mathrm{d}x = 1 : \|\Lambda^{\alpha/2}U\|_{L^2} \leqslant D_\lambda}$, we obtain

$$k(\lambda) - C|\delta|(1 + D_{\lambda+\delta}^3) \leqslant k(\lambda+\delta) \leqslant k(\lambda) + C|\delta|(1 + D_{\lambda+\delta}^3).$$

This implies Lipschitzness of the mapping $\lambda \to k(\lambda)$, once we take into account that $\lambda \to D_\lambda$ is continuous and hence locally bounded. Thus, $\lambda \to m(\lambda)$ is locally Lipschitz and it has a derivative almost everywhere. In fact, we can compute its derivative, whenever it exists, explicitly.

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Lemma 3.3. *The function m is differentiable a.e. in $\mathbb{R}_+$ with an a.e. defined derivative*

$$m'(\lambda) = -\frac{\omega_\lambda}{2}.$$

In particular, since m is absolutely continuous, it can be recovered from its a.e. derivative. That is, for every $0 < \lambda_1 < \lambda_2$, there is

$$m(\lambda_2) - m(\lambda_1) = -\frac{1}{2}\int_{\lambda_1}^{\lambda_2} \omega(\lambda)\,\mathrm{d}\lambda. \tag{3.7}$$

Finally, m is concave down. In particular, m is twice differentiable a.e. in λ and $m''(\lambda) \leqslant 0$. Moreover, for every $0 < \lambda_1 < \lambda_2 < \infty$, with corresponding possibly non-unique minimizers $\varphi_{\lambda_1}, \varphi_{\lambda_2}$

$$\omega(\lambda_1, \varphi_{\lambda_1}) \leqslant \omega(\lambda_2, \varphi_{\lambda_2}). \tag{3.8}$$

Remark: Note that the concavity of m implies that the function $\lambda \to \omega_\lambda = -2m'(\lambda)$, (which is defined a.e.) is non-decreasing. The property (3.8) is an extension of this, as it claims that even when $\lambda \to \omega$ may depend on the particular minimizer φ_λ, it is still a non-decreasing function of λ.

Proof. Starting with a minimizer φ_λ, we have by definition that for all $\epsilon \in \mathbb{R}$ and test functions h,

$$\mathcal{E}(\varphi_\lambda + \epsilon h) \geqslant m(\|\varphi_\lambda + \epsilon h\|^2).$$

But expanding in powers of ϵ, we see that

$$\begin{aligned}\mathcal{E}(\varphi_\lambda + \epsilon h) &= \mathcal{E}(\varphi_\lambda) - \epsilon\omega_\lambda \langle \varphi_\lambda, h\rangle + \frac{\epsilon^2}{2}\langle (\mathcal{L}_+ - \omega_\lambda)h, h\rangle + O(\epsilon^3),\\ m(\|\varphi_\lambda + \epsilon h\|^2) &= m(\lambda + 2\epsilon\langle \varphi_\lambda, h\rangle + \epsilon^2\|h\|^2).\end{aligned}$$

Taking into account $\mathcal{E}(\varphi_\lambda) = m(\lambda)$, we arrive at

$$m(\lambda + 2\epsilon\langle \varphi_\lambda, h\rangle + \epsilon^2\|h\|^2) \leqslant m(\lambda) - \epsilon\omega_\lambda\langle \varphi_\lambda, h\rangle + \frac{\epsilon^2}{2}\langle (\mathcal{L}_+ - \omega_\lambda)h, h\rangle + O(\epsilon^3). \tag{3.9}$$

Ignoring for a second all terms in the form $O(\epsilon^2)$, we can see that whenever $m'(\lambda)$ exists[11], we can compute it as follows fix $h = \varphi_\lambda$, for $\epsilon > 0$, divide (3.9) by $2\lambda\epsilon + \lambda\epsilon^2 > 0$ for $0 < \epsilon \ll 1$, so

$$\frac{m(\lambda + 2\lambda\epsilon + \lambda\epsilon^2) - m(\lambda)}{2\lambda\epsilon + \lambda\epsilon^2} \leqslant -\frac{\epsilon\lambda\omega_\lambda}{2\lambda\epsilon + \lambda\epsilon^2} + O(\epsilon), \tag{3.10}$$

It follows that $m'(\lambda) \leqslant -(\omega_\lambda/2)$. Similarly, for $\epsilon < 0$, we divide by $2\lambda\epsilon + \lambda\epsilon^2 < 0$ for $\epsilon < 0, |\epsilon| \ll 1$, so that after taking limit $\lim_{\epsilon\to 0-}$, we get the opposite inequality $m'(\lambda) \geqslant -(\omega_\lambda/2)$. Altogether, $m'(\lambda) = -(\omega_\lambda/2)$.

[11] which is at least a.e. at this point, since it was established that m is Lipschitz.

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Next, we show that m is concave down. To this end, apply (3.9) for $h = \chi_\lambda / 2\langle \chi_\lambda, \varphi_\lambda\rangle$, recalling $\chi_\lambda : \|\chi_\lambda\| = 1$ is the eigenfunction, corresponding to the negative eigenvalue of $\mathcal{L}_+$, that is $\mathcal{L}_+\chi_\lambda = -\sigma_\lambda^2\chi_\lambda$, to obtain

$$m(\lambda + \epsilon + \epsilon^2\|h\|^2) - m(\lambda) \leqslant -\frac{\epsilon}{2}\omega_\lambda - \frac{\epsilon^2}{2}\omega_\lambda\|h\|^2 - \frac{\epsilon^2}{2}\sigma_\lambda^2\|h\|^2 + O(\epsilon^3).$$

Introduce now $\delta := \epsilon + \epsilon^2\|h\|^2$, so that the previous inequality reads

$$m(\lambda + \delta) - m(\lambda) \leqslant -\frac{\epsilon_\delta}{2}\omega_\lambda - \frac{\epsilon_\delta^2}{2}\omega_\lambda\|h\|^2 - \frac{\epsilon_\delta^2}{2}\sigma_\lambda^2\|h\|^2 + O(\delta^3), \tag{3.11}$$

where ϵ_δ is given by the quadratic equation formula

$$\epsilon_\delta = \frac{-1 + \sqrt{1 + 4\delta\|h\|^2}}{2\|h\|^2} = \delta - \delta^2\|h\|^2 + O(\delta^3). \tag{3.12}$$

Applying (3.11) to $-\delta$ instead of δ and adding the result to (3.11) yields

$$\begin{aligned} m(\lambda + \delta) + m(\lambda - \delta) - 2m(\lambda) &\leqslant -\frac{\epsilon_\delta + \epsilon_{-\delta}}{2}\omega_\lambda - \frac{\epsilon_\delta^2 + \epsilon_{-\delta}^2}{2}\omega_\lambda\|h\|^2 \\ &- \frac{\epsilon_\delta^2 + \epsilon_{-\delta}^2}{2}\sigma_\lambda^2\|h\|^2 + O(\delta^3). \end{aligned} \tag{3.13}$$

Taking into account the asymptotics (3.12), we conclude

$$m(\lambda + \delta) + m(\lambda - \delta) - 2m(\lambda) \leqslant -\delta^2\sigma_\lambda^2\|h\|^2 + O(\delta^3). \tag{3.14}$$

Dividing by δ^2, taking $\sup_{\lambda\in(a,b)}$ on any interval $(a, b) \subset \mathbb{R}_+$ and taking a limit in $\delta \to 0+$ allows to conclude

$$\lim_{\delta\to 0+} \sup_{\lambda\in(a,b)} \frac{m(\lambda + \delta) + m(\lambda - \delta) - 2m(\lambda)}{\delta^2} \leqslant 0.$$

Invoking lemma 2.5, we derive that m is concave down on $\mathbb{R}_+$. This of course means that the $\omega(\lambda)$ is non-decreasing, differentiable a.e. in λ and from (3.14), we can in fact derive the estimate a.e. in λ

$$\omega'(\lambda) = -2m''(\lambda) > \frac{\sigma_\lambda^2}{2\langle \chi_\lambda, \varphi_\lambda\rangle^2} > 0.$$

Now that we know that m is concave down, it means that the function m has a left and right derivatives everywhere. Note that even when the function m does not have a derivative, we can still take limits in (3.10) (and its analogue for $\epsilon < 0$) to obtain

$$m'(\lambda+) \leqslant -\frac{\omega(\lambda, \varphi_\lambda)}{2} \leqslant m'(\lambda-). \tag{3.15}$$

In particular, for every $0 < \lambda_1 < \lambda_2 < \infty$, we have from (3.15)

$$\omega(\lambda_1, \varphi_{\lambda_1}) \leqslant -2m'(\lambda_1+) \leqslant -2m'(\lambda_2-) \leqslant \omega(\lambda_2, \varphi_{\lambda_2}).$$

Combining the last estimate with (3.15) provides a direct proof that m' is non-increasing function as well. □

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4. Non-degeneracy of the waves and spectral stability

Non-degeneracy of the waves not only plays an important role in the stability considerations, but it is of interest in its own. In particular, it always seems to be an important first step towards uniqueness of the waves, as solutions to the corresponding profile equations, e.g. (1.3). We start with a less ambitious task, which turns out to be the main step towards the non-degeneracy, we call it *weak non-degeneracy.*

4.1. Weak non-degeneracy of φ_λ

LEMMA 4.1. *Any constrained minimizers φ_λ produced in proposition 3.1 enjoys the weak non-degeneracy property, that is $\varphi_\lambda \perp \mathrm{Ker}[\mathcal{L}_+]$. In particular, $\mathcal{L}_+^{-1}\varphi$ is well-defined.*

Proof. We first establish that $\langle \mathcal{L}_+\varphi_\lambda, \varphi_\lambda\rangle < 0$. We have, using (3.3),

$$\langle \mathcal{L}_+\varphi_\lambda, \varphi_\lambda\rangle = \|\Lambda^{\alpha/2}\varphi_\lambda\|^2 + \omega\lambda - 2\int \varphi_\lambda^3 = -\int \varphi_\lambda^3 - a\int \varphi_\lambda.$$

This is clearly negative if $a \geqslant 0$. It is also clear that the sign of it is not easily determined, if $a < 0$.

We shall need to see that $m(0+) := \lim_{\lambda\to 0+} m(\lambda) = 0$. To this end, testing with the function $u = \sqrt{\lambda/2}$ yields a bound from above, $m(\lambda) \leqslant -(\lambda^{3/2}/3\sqrt{2}) + a\sqrt{2\lambda}$, which implies $m(0+) \leqslant 0$.

For the bound from below, we use the bound (3.5) for $\|u\|_{L^3}^3$ (recall that this bound requires $\alpha > \frac{1}{2}$). This implies that for appropriately small ϵ

$$m(\lambda) \geqslant \inf_{\|u\|^2=\lambda}\left[\frac{1}{4}\|\Lambda^{\alpha/2}u\|^2 - a\int_{-1}^{1}|u|\,\mathrm{d}x\right] - C\lambda^{\frac{3\alpha-1}{2\alpha-1}} \geqslant -C\big(\sqrt{\lambda} + \lambda^{\frac{3\alpha-1}{2\alpha-1}}\big).$$

Taking $\lim_{\lambda\to 0+}$ yields the bound $m(0+) \geqslant 0$, and subsequently $m(0+) = 0$.

Now, using (3.7), with $\lambda_1 = 0+$, and the fact that $\lambda \to \omega_\lambda$ is non-decreasing (i.e. the property (3.8)) and

$$-2m(\lambda) = \int_0^\lambda \omega(\mu)\,\mathrm{d}\mu \leqslant \lambda \lim_{\mu\to\lambda-}\omega(\mu) \leqslant \lambda\omega(\lambda, \varphi_\lambda).$$

Note that in the last inequality, we took into account that there are possibly multiple minimizers for the value of α, with possibly different $\omega(\lambda, \varphi_\lambda)$.

It follows that

$$\begin{aligned} 0 \leqslant 2m(\lambda) + \lambda\omega(\lambda, \varphi_\lambda) &= \left(\|\Lambda^{\alpha/2}\varphi_\lambda\|^2 - \frac{2}{3}\int \varphi_\lambda^3 + 2a\int \varphi_\lambda\right) \\ &\quad + \left(\int \varphi_\lambda^3 - \|\Lambda^{\alpha/2}\varphi_\lambda\|^2 - a\int \varphi_\lambda\right) \\ &= \frac{1}{3}\int \varphi_\lambda^3 + a\int \varphi_\lambda. \end{aligned}$$

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In other words,

$$-a\int \varphi_\lambda \leqslant \frac{1}{3}\int \varphi_\lambda^3. \tag{4.1}$$

We can now estimate, using (4.1),

$$\langle \mathcal{L}_+\varphi_\lambda, \varphi_\lambda\rangle = -\int \varphi_\lambda^3 - a\int \varphi_\lambda \leqslant -\frac{2}{3}\int \varphi_\lambda^3 < 0. \tag{4.2}$$

Thus, the inequality $\langle \mathcal{L}_+\varphi_\lambda, \varphi_\lambda\rangle < 0$ is established in the case $a < 0$ as well.

We now apply the property $\mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0$. More concretely, take $h \in \mathrm{Ker}[\mathcal{L}_+]$ and consider $\tilde{h} := h - \lambda^{-1}\langle h, \varphi_\lambda\rangle \varphi_\lambda \perp \varphi_\lambda$. It must be that

$$0 \leqslant \left\langle \mathcal{L}_+\tilde{h}, \tilde{h}\right\rangle = \lambda^{-2}\langle h, \varphi_\lambda\rangle^2 \langle \mathcal{L}_+\varphi, \varphi\rangle .$$

Assuming $\langle h, \varphi_\lambda\rangle \neq 0$, this leads to a contradiction, as the right-hand side is strictly negative. Thus, $\langle h, \varphi_\lambda\rangle = 0$ and the weak non-degeneracy is established. □

We now discuss a corollary of the weak non-degeneracy and the property (4.2).

Corollary 4.2.

$$\left\langle \mathcal{L}_+^{-1}\varphi, \varphi\right\rangle \leqslant 0. \tag{4.3}$$

Remark: The slightly stronger inequality $\left\langle \mathcal{L}_+^{-1}\varphi, \varphi\right\rangle < 0$ is exactly the Vakhitov–Kolokolov criteria for spectral stability, see corollary 2.3. In this sense, (4.3) establishes spectral stability, modulo an assumption of the type $\left\langle \mathcal{L}_+^{-1}\varphi, \varphi\right\rangle \neq 0$.

Proof. Lemma 4.1 established that $\mathcal{L}_+^{-1}\varphi$ is well defined. Since $\mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0$, we apply this to the vector $\eta := \mathcal{L}_+^{-1}\varphi - \|\varphi\|^{-2}\left\langle \mathcal{L}_+^{-1}\varphi, \varphi\right\rangle \varphi \perp \varphi$. In short, $0 \leqslant \langle \mathcal{L}_+\eta, \eta\rangle$, which simplifies, after some algebraic manipulations to

$$\left\langle \mathcal{L}_+^{-1}\varphi, \varphi\right\rangle \leqslant (\left\langle \mathcal{L}_+^{-1}\varphi, \varphi\right\rangle)^2 \frac{\langle \mathcal{L}_+\varphi, \varphi\rangle}{\|\varphi\|^6} \leqslant 0$$

by taking into account that $\langle \mathcal{L}_+\varphi, \varphi\rangle < 0$. □

4.2. Non-degeneracy of φ_λ: conclusion of the proof

We now continue with the goal of establishing that the wave φ_λ is non-degenerate, that is $\mathrm{Ker}[\mathcal{L}_+] = \mathrm{span}[\varphi'_\lambda]$. Note that we always have $\varphi'_\lambda \subset \mathrm{Ker}[\mathcal{L}_+]$. We claim that $\mathrm{Ker}[\mathcal{L}_+]$ is at most two dimensional. Indeed, we know already that $n(\mathcal{L}_+) = 1$, so $\lambda_0(\mathcal{L}_+) < 0$. Since 0 is an eigenvalue, it must be that $\lambda_1(\mathcal{H}) = 0$. By bell-shapedness, one of the corresponding eigenfunctions, φ'_λ is an odd function, which has exactly one zero, at $x = 0$. Since $\mathcal{L}_+$ is a fractional Schrödinger operator with even potential, the linearly independent eigenfunctions may be taken to be either even or odd.

By the Sturm–Liouville's theory for the fractional periodic Schrödinger operators, see lemma 2.6, we have that the eigenfunctions corresponding to the zero eigenvalue have at most two zeroes in $[-T, T]$. Clearly, there cannot be another odd eigenfunction (other than φ'_λ), since it would have to have exactly one zero, which happens at $x=0$, and as such, it cannot possibly be orthogonal to φ'_λ. Thus, there could be another eigenfunction, say $\Psi_\lambda : \|\Psi_\lambda\|_{L^2}=1$, which is even and which has exactly two zeros (since it cannot have one zero), at say $\pm b, b\in(0,T)$. Note that similar to proposition 2.2, it can be shown that $\Phi_\lambda \in H^\infty[-T,T]$. Thus, we have proved the following preliminary result

LEMMA 4.3. *For the fractional Schrödinger operator $\mathcal{L}_+$, we have that either $\mathrm{Ker}[\mathcal{L}_+]=\mathrm{span}[\varphi'_\lambda]$ or $\mathrm{Ker}[\mathcal{L}_+]=\mathrm{span}[\varphi'_\lambda,\Psi_\lambda]$, where $\Psi_\lambda:[-T,T]\to\mathbb{R}$ is a smooth even function, with exactly two zeroes, $\Psi_\lambda(-b)=\Psi_\lambda(b)=0$, with $\Psi_\lambda|_{(-b,b)}>0$, where $b\in(0,T)$.*

By direct calculations, $\mathcal{L}_+[1]=\omega-2\varphi_\lambda$. In particular $\omega-2\varphi_\lambda\perp\mathrm{Ker}[\mathcal{L}_+]$. On the other hand, $\varphi_\lambda\perp\mathrm{Ker}[\mathcal{L}_+]$ by lemma 4.1. It follows that $1\perp\mathrm{Ker}[\mathcal{L}_+]$, provided $\omega\neq 0$. Furthermore,

$$\mathcal{L}_+[\varphi_\lambda]=-\varphi_\lambda^2-a.$$

Thus, $-\varphi_\lambda^2-a\perp\mathrm{Ker}[\mathcal{L}_+]$, so in particular $\varphi_\lambda^2\perp\mathrm{Ker}[\mathcal{L}_+]$. But now, we consider the function $Q(x):=\varphi^2(x)-\varphi(b)\varphi(x)$. By construction $Q\perp\mathrm{Ker}[\mathcal{L}_+]$, so it must be that $\langle Q,\Psi_\lambda\rangle=0$. On the other hand, recall that φ_λ is bell-shaped, so $Q(x)=\varphi(x)(\varphi(x)-\varphi(b))$ is positive in $(-b,b)$ and it is negative in $b<|x|<1$. But this is exactly the behaviour of Ψ_λ, in fact $Q(x)\Psi_\lambda(x)\geqslant 0$ for $-T<x<T$. Thus, $\langle Q,\Psi_\lambda\rangle=0$ is impossible, a contradiction. Thus, φ_λ is non-degenerate, when $\omega\neq 0$. This is of course exactly the case when $a\neq\lambda/2$.

5. Orbital stability of the waves

We present the proof of the orbital stability, following a variation of the classical T.B. Benjamin's method. Here is a good point to discuss why the smoothness properties of the map $\lambda\to\varphi_\lambda$ matters a great deal.

Following Benjamin's original approach, the strategy is to consider first initial data $u_0\in H^{\alpha/2}:\|u_0-\varphi_\lambda\|_{H^{\alpha/2}}\ll 1$, but with the additional property $P(u_0)=P(\varphi_\lambda)=\lambda$. In the second step, one removes this assumption $P(u_0)=P(\varphi_\lambda)=\lambda$, that is, take $u_0:P(u_0)\neq P(\varphi_\lambda)$, while still close to φ_λ in $H^{\alpha/2}$ metric. It has to be noted that in the original work of Benjamin, as well as many subsequent works, this second step almost automatically reduces to the first one, *if the mapping $\lambda\to\varphi_\lambda$ is at least continuous as a Banach space valued mapping into L^2.*

In some instances, for example in the classical case of a single power non-linearity for problems posed on the line $\mathbb{R}$, the function $\lambda\to\varphi_\lambda$ is explicitly known by scaling arguments, and smooth by inspection, as stated. Virtually in all other cases, like for the waves constructed herein, scaling is not available and this becomes non-trivial. On the other hand, many authors feel that this is a natural assumption and they explicitly take this as an assumption (and even stronger assumptions like the differentiability in spaces stronger than L^2), while others tacitly assume it in their arguments. We emphasize once again that the proof presented herein does not make any explicit assumptions beyond what is already established rigorously in lemma 3.3.

Next, we present a general coercivity criteria. Versions of this result have previously appeared in the literature (see e.g. lemma 6.9, [**5**]).

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5.1. A general coercivity result

LEMMA 5.1. *Under the assumptions*

- $L_+|_{\{\varphi\}^\perp} \geqslant 0$,
- φ *is non-degenerate, i.e.* $\mathrm{Ker}[\mathcal{L}_+] = \mathrm{span}[\varphi']$,
- $\langle \mathcal{L}_+^{-1}\varphi, \varphi \rangle \neq 0$

the operator $\mathcal{L}_+$ *is coercive on* $H^{\alpha/2}$. *That is, there is* $\sigma > 0$, *so that*

$$\langle \mathcal{L}_+\eta, \eta \rangle \geqslant \sigma \|\eta\|^2_{H^{\alpha/2}}, \eta \perp \mathrm{span}[\varphi, \varphi']. \tag{5.1}$$

Remark: Recall that according to corollary 4.2, the conditions $\mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0$ and $\mathrm{Ker}[\mathcal{L}_+] = \mathrm{span}[\varphi']$ already imply $\langle \mathcal{L}_+^{-1}\varphi, \varphi \rangle \leqslant 0$.

Proof. We provide a proof along the lines of lemma 6.9, [5]. Namely, we show first that

$$\inf\{\langle \mathcal{L}_+ f, f\rangle : \|f\| = 1, \langle f, \varphi\rangle = 0, \langle f, \varphi'\rangle = 0\} > 0. \tag{5.2}$$

Clearly, the infimum above is greater or equal to zero, per the assumption $\mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0$. It is easy to see that there is minimizer f^* for it. Indeed, taking a minimizing sequence, say $\{f_n\}$, we obtain the formula

$$\langle \mathcal{L}_+ f_n, f_n\rangle = \|\Lambda^{\alpha/2} f_n\|^2 + \omega - 2\int \varphi f_n^2.$$

From it, since $|\int \varphi f_n^2| \leqslant \|\varphi\|_{L^\infty}$, we clearly conclude that $\sup_n \|\Lambda^{\alpha/2} f_n\|^2 < \infty$. By the compactness of the embedding $H^{\alpha/2} \hookrightarrow L^2$, we extract an L^2 convergent subsequence, converging to a minimizer $f^* \in H^{\alpha/2} : f^* \perp \varphi, \varphi'$.

Assume now for a contradiction that the minimization problem (5.2) has a minimum value of zero, i.e. $\langle \mathcal{L}_+ f^*, f^*\rangle = 0$. Writing the Euler–Lagrange equation for (5.2), we obtain, for some scalars μ, ν, κ,

$$\mathcal{L}_+ f^* = \mu f^* + \nu\varphi + \kappa\varphi'. \tag{5.3}$$

Taking dot product with φ' implies $\kappa = 0$. Taking dot product with f^* and since $\langle \mathcal{L}_+ f^*, f^*\rangle = 0, f^* \perp \varphi$, we conclude that $\mu = 0$ as well. It follows that $\mathcal{L}_+ f^* = \nu\varphi$, or

$$\mathcal{L}_+[f^* - \nu\mathcal{L}_+^{-1}\varphi] = 0.$$

So, since $\mathrm{Ker}[\mathcal{L}_+] = \mathrm{span}[\varphi']$, there exists γ, so that $f^* - \nu\mathcal{L}_+^{-1}\varphi = \gamma\varphi'$. Taking dot product with φ implies (since $f^* \perp \varphi$),

$$-\nu \langle \mathcal{L}_+^{-1}\varphi, \varphi \rangle = 0.$$

We now use the crucial assumption $\langle \mathcal{L}_+^{-1}\varphi, \varphi \rangle \neq 0$ to conclude that $\nu = 0$ as well. This leaves us with $f^* = \gamma\varphi'$, which implies $\gamma = 0$, since $f^* \perp \varphi'$. But then, $f^* = 0$, which is a contradiction, since $\|f^*\| = 1$.

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Thus, for some $\sigma_0 > 0$, $\inf_{\|f\|=1, f\perp\varphi,\varphi'} \langle \mathcal{L}_+ f, f\rangle \geqslant \sigma_0 > 0$ or equivalently

$$\langle \mathcal{L}_+ f, f\rangle \geqslant \sigma_0 \|f\|^2, f \perp \varphi, \varphi'. \tag{5.4}$$

Recall however, that we need a slightly stronger inequality, namely (5.1). This however follows easily from (5.4). Indeed, assume for a contradiction that (5.1) is false. Then, there is a $f_n : f_n \perp \varphi, \varphi'$, so that $\|f_n\|_{H^{\alpha/2}} = 1$, $\lim_n \langle \mathcal{L}_+ f_n, f_n\rangle = 0$. This is clearly inconsistent with (5.4), unless $\lim_n \|f_n\|_{L^2} = 0$. But then, since $1 = \|f_n\|^2_{H^{\alpha/2}} = \|\Lambda^{\alpha/2} f_n\|^2 + \|f_n\|^2$, it follows that

$$\lim_n \|\Lambda^{\alpha/2} f_n\|^2 = 1.$$

But then,

$$0 = \lim_n \langle \mathcal{L}_+ f_n, f_n\rangle = \lim_n [\|\Lambda^{\alpha/2} f_n\|^2 + \omega \|f_n\|^2 - 2\int \varphi f_n^2] = 1,$$

since $|\int \varphi f_n^2| \leqslant \|\varphi\|_{L^\infty} \|f_n\|^2 \to 0$. A contradiction is reached. This means that (5.1) holds. □

Next, we discuss the actual orbital stability statement. We start with the simpler fractional KdV case, as it presents itself with a single symmetry, namely space translation.

5.2. Orbital stability for the fKdV

Proposition 5.2. *Let φ be a wave, satisfying the profile equation,* (1.3). *Let the conditions* (1), (2), (3) *of assumption 1.2 are satisfied and in addition the following is satisfied*

- *The operator $\mathcal{L}_+ = \Lambda^\alpha + \omega - 2\varphi$ satisfies $\mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0$.*
- *φ is non-degenerate, i.e.* $\mathrm{Ker}[\mathcal{L}_+] = \mathrm{span}[\varphi']$.
- $\langle \mathcal{L}_+^{-1} \varphi, \varphi\rangle \neq 0$

Then, φ is orbitally stable. In particular, for every $\lambda > 0, a \neq \lambda/2$, the constrained minimizers φ_λ for the problem (3.1) *are orbitally stable.*

Proof. Our proof proceeds by a contradiction argument. More precisely, assuming that orbital stability does not hold, there is an $\epsilon_0 > 0$ and a sequence of initial data $u_n : \lim_n \|u_n - \varphi\|_{H^{\alpha/2}} = 0$, while for the corresponding solutions

$$\sup_{0\leqslant t<\infty} \inf_{r\in\mathbb{R}} \|u_n(t,\cdot) - \varphi(\cdot - r)\|_{H^{\alpha/2}} \geqslant \epsilon_0, \; n = 1, 2, \ldots \tag{5.5}$$

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Note the conservation of total energy

$$\begin{aligned} E[u] &= \mathcal{H}[u] + \frac{\omega}{2}\mathcal{P}[u] + a\mathcal{M}[u] \\ &= \frac{1}{2}\left[\int_{-T}^{T} |\Lambda^{\alpha/2} u(t,x)|^2 \,\mathrm{d}x + \omega \int_{-T}^{T} u^2(t,x)\,\mathrm{d}x \right] \\ &\quad - \frac{1}{3}\int_{-T}^{T} u^3(t,x) + a\int_{-T}^{T} u(t,x)\,\mathrm{d}x. \end{aligned}$$

Introduce

$$\epsilon_n := |\mathcal{E}(u_n(t)) - \mathcal{E}(\varphi)| + |\mathcal{P}(u_n(t)) - \mathcal{P}(\varphi)|,$$

which is conserved in time. Note that $\lim_n \epsilon_n = 0$, since $\lim_n \|u_n - \varphi\|_{H^{\alpha/2}} = 0$.

For $0 < \epsilon \ll 1$, consider a neighbourhood $\mathcal{U}_\varepsilon$ in the set of all real-valued functions, which are close to translations of φ. More precisely, introduce

$$\mathcal{U}_\varepsilon = \{u \in H^{\alpha/2}_{real}[-T,T] \ \ : \ \ \inf_{r\in\mathbb{R}} ||u - \varphi(\cdot - r)||_{H^{\alpha/2}} < \varepsilon\}.$$

By lemma 3.2, [**12**], see also lemma 7.7, p. 95 in [**5**], there exists $\epsilon_0(\varphi) > 0$, so that for all $0 < \epsilon < \epsilon_0(\varphi)$, there is a unique C^1 map $\beta : \mathcal{U}_\varepsilon \mapsto \mathbb{R}$ such that

$$\langle u(\cdot + \beta(u)), \varphi' \rangle = 0, \quad \beta(\varphi) = 0. \tag{5.6}$$

Since we need $\epsilon < \min(\epsilon_0(\varphi), \epsilon_0)$, take the new ϵ_0 to be $1/10 \min(\epsilon_0, \epsilon_0(\varphi), 1)$.

Fix for the moment $\epsilon < \epsilon_0 < 1$. By the continuity of the solution map (as required in assumption 1.2) and the map β, we have that there exists $t_n = t_n(\epsilon) > 0$, so that $\sup_{0\leqslant t<t_n} \|u_n(t,\cdot) - \varphi\|_{H^{\alpha/2}} < \epsilon/2$ and $\beta(u_n(t))$ is so close to $\beta(\varphi) = 0$, so that

$$\|\varphi - \varphi(\cdot - \beta(u_n(t)))\|_{H^{\alpha/2}} < \frac{\epsilon}{2}.$$

Consequently, for $t \in (0, t_n)$,

$$\begin{aligned} &\|u_n(t, \cdot + \beta(u_n(t))) - \varphi\|_{H^{\alpha/2}} = \|u_n(t,\cdot) - \varphi(\cdot - \beta(u_n(t)))\|_{H^{\alpha/2}} \\ &\quad \leqslant \|u_n(t,\cdot) - \varphi\|_{H^{\alpha/2}} + \|\varphi - \varphi(\cdot - \beta(u_n(t)))\|_{H^{\alpha/2}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Based on this, for large n and $\epsilon < \epsilon_0(\varphi)$, one may define $T_n^* = T_n^*(\epsilon) > 0$, so that

$$T_n^* = \sup\left\{\tau_0 : \sup_{0<t<\tau_0} \|u_n(t, \cdot + \beta(u_n(t))) - \varphi(\cdot)\|_{H^{\alpha/2}} < \epsilon \right\}.$$

The previous calculation implies $T_n^* \geqslant t_n$. Our goal is to show that for all sufficiently small ϵ, there exists N_ϵ, so that for all $n > N_\epsilon$, $T_n^* = \infty$, which will provide the required contradiction with (5.5).

We henceforth work with $t \in (0, T_n^*)$. Denote

$$\psi_n(t,\cdot) = u_n(t, \cdot + \beta(u_n(t))) - \varphi(\cdot) = \mu_n(t)\varphi + \eta_n(t,\cdot), \quad \eta_n \perp \varphi.$$

We have that

$$\mathcal{P}(u_n(t)) = \mathcal{P}(\varphi) + 2\langle \varphi, \mu_n(t)\varphi + \eta_n\rangle + \|\psi_n\|_{L^2}^2 = \mathcal{P}(\varphi) + 2\mu_n(t)\|\varphi\|^2 + \|\psi_n\|_{L^2}^2.$$

It follows that $2\mu_n\|\varphi\|^2 = \mathcal{P}(u_n) - \mathcal{P}(\varphi) - \|\psi_n\|_{L^2}^2$ whence

$$|\mu_n| \leqslant \frac{|\mathcal{P}(u_n) - \mathcal{P}(\varphi)| + \|\psi_n\|_{L^2}^2}{2\|\varphi\|^2} \leqslant C(\epsilon_n + \|\psi_n\|_{L^2}^2). \tag{5.7}$$

Since $E'(\varphi) = 0$, Taylor expanding and various Sobolev embedding estimates yield the formula

$$\begin{aligned} E(u_n(t)) - E(\varphi) &= E(u_n(t, \cdot + \beta(u_n(t)))) - E(\varphi) = E(\varphi + \psi_n(t)) - E(\varphi) \\ &= \frac{1}{2}\langle \mathcal{L}_+\psi_n(t), \psi_n(t)\rangle + O(\|\psi_n(t)\|_{H^{\alpha/2}}^3) \\ &= \frac{1}{2}\langle \mathcal{L}_+\eta_n(t), \eta_n(t)\rangle + \frac{1}{2}(\mu_n^2\langle \mathcal{L}_+\varphi, \varphi\rangle + 2\mu_n\langle \mathcal{L}_+\varphi, \eta_n\rangle) \\ &\quad + O(\|\psi_n(t)\|_{H^{\alpha/2}}^3). \end{aligned}$$

By construction, $\eta_n(t) \perp \varphi$. In addition, from (5.6), we have that

$$\langle \eta_n(t), \varphi'\rangle = \langle u_n(t, \cdot + \beta(u_n(t))) - \varphi - \mu_n\varphi, \varphi'\rangle = 0.$$

So, $\eta_n(t) \perp \operatorname{span}\{\varphi, \varphi'\}$. Then, by the requirements of proposition 5.2, $\mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0$. In addition, by the non-degeneracy, $\operatorname{Ker}[\mathcal{L}_+] = \operatorname{span}[\varphi']$. By lemma 5.1 and more specifically (5.1), we have

$$\langle \mathcal{L}_+\eta_n(t), \eta_n(t)\rangle \geqslant \kappa\|\eta_n(t)\|_{H^{\alpha/2}}^2. \tag{5.8}$$

Regarding the other terms, note $\|\psi_n\|^2 = |\mu_n|^2\|\varphi\|^2 + \|\eta_n\|^2 \geqslant \|\eta_n\|^2$, whence

$$\mu_n^2 + |\mu_n|\|\eta_n\|_{L^2} \leqslant C(\epsilon_n + \|\psi_n\|_{H^{\alpha/2}}^3).$$

Plugging this information into the expression for $E(u_n(0)) - E(\varphi) = E(u_n(t)) - E(\varphi)$, we arrive at

$$\frac{\kappa}{2}\|\eta_n(t)\|_{H^{\alpha/2}}^2 \leqslant C\epsilon_n + C\|\psi_n(t)\|_{H^{\alpha/2}}^3. \tag{5.9}$$

By the definition of η_n and (5.7), we have however for $t \in (0, T_n^*)$

$$\begin{aligned} \|\eta_n(t)\|_{H^{\alpha/2}} &\geqslant \|\psi_n - \mu_n\varphi\|_{H^{\alpha/2}} \geqslant \|\psi_n(t)\|_{H^{\alpha/2}} - |\mu_n|\|\varphi\|_{H^{\alpha/2}} \\ &\geqslant \|\psi_n(t)\|_{H^{\alpha/2}} - C(\epsilon_n + \|\psi_n(t)\|_{H^{\alpha/2}}^2), \end{aligned} \tag{5.10}$$

where the constant C appearing in the previous inequality depends on λ, φ, but not on t, n. At this point, select ϵ so small that $C\epsilon < \frac{1}{2}$. It follows that for these values of ϵ and $t \in (0, T_n^*)$, by (5.10),

$$\|\eta_n(t)\|_{H^{\alpha/2}} \geqslant \frac{1}{2}\|\psi_n(t)\|_{H^{\alpha/2}} - C\epsilon_n.$$

Plugging this back into (5.9), we obtain

$$\|\psi_n(t)\|^2_{H^{\alpha/2}} \leqslant C\epsilon_n + C\|\psi_n(t)\|^3_{H^{\alpha/2}}. \tag{5.11}$$

Again, for the new constant C that appears in (5.11), select ϵ still maybe smaller, so that $C\epsilon < \frac{1}{2}$, so that we can finally conclude from (5.11),

$$\|\psi_n(t)\|^2_{H^{\alpha/2}} \leqslant D\epsilon_n, \tag{5.12}$$

which is valid for such small ϵ, for all n and for all $t \in (0, T_n^*)$. But this means that $T_n^* = \infty$ for all large enough n. Indeed, for ϵ small as above, take n so large that $\sqrt{D\epsilon_n} \ll \epsilon$, which can be done since $\lim_n \epsilon_n = 0$. Assuming that $T_n^* < \infty$ means that

$$\sqrt{D\epsilon_n} \geqslant \limsup_{t\to T_n^*-} \|\psi_n(t)\|_{H^{\alpha/2}} \geqslant \epsilon,$$

a contradiction. So, $T_n^*(\epsilon) = \infty$ for all large enough n. This is now a contradiction with (5.5), once we pick ϵ small enough (in order to satisfy the previous two conditions and in addition $\epsilon \ll \epsilon_0$) and then n large enough so that $T_n^*(\epsilon) = \infty$. □

5.3. Stability for the fNLS standing waves

For this part of the argument, we take $a = 0$ in (3.2). We have similar to proposition 5.2.

PROPOSITION 5.3. *Let φ be a wave, satisfying assumption 1.2 and the following*

- *The operator $\mathcal{L}_+ = \Lambda^\alpha + \omega - 2\varphi$ satisfies $\mathcal{L}_+|_{\{\varphi\}^\perp} \geqslant 0$.*
- *φ is non-degenerate, i.e.* $\mathrm{Ker}[\mathcal{L}_+] = \mathrm{span}[\varphi']$.
- $\langle \mathcal{L}_+^{-1}\varphi, \varphi\rangle \neq 0$.
- *The operator $\mathcal{L}_- = \Lambda^\alpha + \omega - \varphi$ satisfies $\mathcal{L}_-|_{\{\varphi\}^\perp} \geqslant \kappa$, for some $\kappa > 0$.*

Then, φ is orbitally stable. In particular, for every $\lambda > 0$ and $a = 0$, the solutions $e^{i\omega t}\varphi_{\omega_\lambda}$ of (1.2), where φ_λ are constrained minimizers for the problem (3.1) are orbitally stable.

Proof. Note first that the first three assumptions, along with lemma 5.1, guarantee that there exists $\kappa > 0$, so that for

$$\langle \mathcal{L}_+\eta, \eta\rangle \geqslant \kappa\|\eta\|^2_{H^{\alpha/2}}, \ \eta \perp \varphi, \varphi', \tag{5.13}$$

$$\langle \mathcal{L}_-\zeta, \zeta\rangle \geqslant \kappa\|\zeta\|^2_{H^{\alpha/2}}, \ \zeta \perp \varphi. \tag{5.14}$$

The proof then proceeds again by a contradiction, as in proposition 5.2.

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Assuming that orbital stability fails, we conclude that there exists $\epsilon_0 > 0$ and a sequence of complex-valued initial data $u_n : \lim_n \|u_n - \varphi\|_{H^{\alpha/2}} = 0$, so that for the corresponding solutions stay away from (a translate and modulated versions of) φ. That is,

$$\sup_{0\leqslant t<\infty} \inf_{r,\theta\in\mathbb{R},} \|u_n(t,\cdot) - e^{i\theta}\varphi(\cdot - r)\|_{H^{\alpha/2}} \geqslant \epsilon_0. \tag{5.15}$$

Consider the set, for small enough ϵ

$$\mathcal{U}_\epsilon := \{u = v + iw : v, w \in H^{\alpha/2}_{real}[-T,T] : \inf_r [\|v - \varphi(\cdot - r)\|_{H^{\alpha/2}} + \|w\|_{H^{\alpha/2}}] < \epsilon\},$$

together with the well-defined map $\beta : \mathcal{U}_\epsilon \to \mathbb{R}$, so that

$$\langle v(\cdot + \beta(v)), \varphi'\rangle = 0. \tag{5.16}$$

Letting again $E(u) = \mathcal{H}(u) + \omega/2\mathcal{P}(u)$ and $\epsilon_n := |\mathcal{H}(u_n(t)) - \mathcal{H}(\varphi)| + |\mathcal{P}(u_n(t)) - \mathcal{P}(\varphi)|$, we observe again that ϵ_n is conserved and $\lim_n \epsilon_n = 0$, since $\lim_n \|u_n - \varphi\|_{H^{\alpha/2}} = 0$. Also, there is the variational identity

$$\frac{\delta E}{\delta u}[\varphi] = \frac{\delta \mathcal{H}}{\delta u}[\varphi] + \frac{\omega}{2}\frac{\delta \mathcal{P}}{\delta u}[\varphi] = 0.$$

We now define the appropriate translation and modulation parameters. The translation parameter is simply as before, $r_n(t) := \beta(v_n(t))$, while the modulation parameter $\theta_n(t)$ is determined from the relation

$$\langle w_n(t, \cdot + \beta(v_n(t))), \varphi\rangle = \sin(\theta_n(t))\|\varphi\|^2_{L^2}. \tag{5.17}$$

Note that while $(v, w) \in \mathcal{U}_\epsilon$, the expression on the left-hand side of (5.17) is $O(\epsilon)$, so $\theta_n(t)$ is taken to be the unique small solution of (5.17). More generally, under the *a priori* assumption that $u_n = v_n + iw_n$ belongs to the set $\mathcal{U}_\epsilon$, which we will eventually uphold for all times t under consideration, it follows that both $r_n(t) = O(\epsilon), \theta_n(t) = O(\epsilon)$ are uniquely determined.

Next, fix small enough $\epsilon > 0$, so that the map $\beta : \mathcal{U}_\epsilon \to \mathbb{R}$ is well defined and (5.16) holds. By the continuity of the solution map and the C^1 property of the map β, there exists $t_n = t_n(\epsilon) > 0$, so that $\sup_{t\in(0,t_n)} \|u_n(t,\cdot) - \varphi\| < \epsilon$. In particular,

$$\|v_n(t,\cdot) - \varphi\| \leqslant \|u_n(t,\cdot) - \varphi\| < \epsilon,$$

whence $\beta(v_n(t))$ is $O(\epsilon)$ close to $\beta(\varphi) = 0$ and $\theta_n(t) = O(\epsilon)$. Thus,

$$|e^{i\theta_n(t)} - 1|\|\varphi\|_{H^{\alpha/2}} < C_0\epsilon, \|\varphi - \varphi(\cdot - \beta(v_n(t)))\|_{H^{\alpha/2}} < C_0\epsilon,$$

for some constant $C_0 = C_0(\varphi)$. Thus, for $t \in (0, t_n)$,

$$\begin{aligned}\|u_n(t, \cdot + \beta(v_n(t)) - e^{i\theta_n(t)}\varphi\|_{H^{\alpha/2}} &\leqslant \|u_n(t,\cdot) - \varphi\|_{H^{\alpha/2}} + \|\varphi - \varphi(\cdot - \beta(v_n(t)))\|_{H^{\alpha/2}} \\ &\quad + |e^{i\theta_n(t)} - 1|\|\varphi\|_{H^{\alpha/2}} \leqslant (2C_0 + 1)\epsilon.\end{aligned}$$

Define

$$\begin{aligned}T^*_n = T^*_n(\epsilon) := \sup\{\tau_0 : \sup_{0<\tau<\tau_0} &\|u_n(\tau, \cdot + \beta(u_n(\tau))) \\ &- e^{i\theta_n(\tau)}\varphi(\cdot)\|_{H^{\alpha}/2} < 2(2C_0 + 1)\epsilon\},\end{aligned}$$

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so that the previous calculation implies $T_n^* \geqslant t_n > 0$. We will show that for all small enough ϵ, there exists $N = N_\epsilon$, so that for all $n > N_\epsilon$, $T_n^* = \infty$. This is in a contradiction with (5.15), by taking $\epsilon \ll \epsilon_0$ and correspondingly large N_ϵ.

Write for $t \in (0, T_n^*)$,

$$\psi_n(t) := u_n(t, \cdot + \beta(v_n(t)) - e^{i\theta_n(t)}\varphi = \\ = v_n(t, \cdot + \beta(v_n(t))) - \cos(\theta_n(t))\varphi + i(w_n(t, \cdot + \beta(v_n(t))) - \sin(\theta_n(t))\varphi).$$

Note that $t \in (0, T_n^*)$ implies $\|\psi_n(t)\|_{H^{\alpha/2}} < 2(2C_0 + 1)\epsilon$. Viewing $\psi_n(t)$ as a vector in the real and imaginary parts, we project over the vector $\begin{pmatrix} \varphi \\ 0 \end{pmatrix}$ and its orthogonal complement, whence we obtain the representation

$$\begin{pmatrix} v_n(t, \cdot + \beta(v_n(t))) - \cos(\theta_n(t))\varphi \\ w_n(t, \cdot + \beta(v_n(t))) - \sin(\theta_n(t))\varphi \end{pmatrix} = \mu_n(t) \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \\ + \begin{pmatrix} \eta_n(t) \\ \zeta_n(t) \end{pmatrix}, \begin{pmatrix} \eta_n(t) \\ \zeta_n(t) \end{pmatrix} \perp \begin{pmatrix} \varphi \\ 0 \end{pmatrix}. \tag{5.18}$$

By the construction of $\beta(v_n(t))$, we have by (5.16) that $\langle v_n(\cdot + \beta(v_n(t))), \varphi'(\cdot)\rangle = 0$, so taking dot product of the first equation in (5.18) with φ' yields $\langle \eta_n(t), \varphi'\rangle = 0$, or $\eta_n(t) \perp \varphi'$.

Furthermore, the condition $\begin{pmatrix} \eta_n(t) \\ \zeta_n(t) \end{pmatrix} \perp \begin{pmatrix} \varphi \\ 0 \end{pmatrix}$ is nothing, but $\eta_n(t) \perp \varphi$, so $\eta_n(t) \perp \varphi, \varphi'$. In addition, the choice of θ_n in (5.17) is equivalent to $w_n(t, \cdot + \beta(v_n(t))) - \sin(\theta_n(t))\varphi \perp \varphi$, which translates to exactly $\zeta_n(t) \perp \varphi$. It is then clear that

$$\mathcal{P}(u_n(t)) = \mathcal{P}(u_n(t, \cdot + \beta(v_n(t)))) = \mathcal{P}(\varphi) + 2\mu_n(t)\cos(\theta_n(t))\|\varphi\|_{L^2}^2 + \|\psi_n(t)\|_{L^2}^2. \tag{5.19}$$

Taking into account $\theta_n(t) = O(\epsilon)$ (so $\cos(\theta_n(t)) = 1 + O(\epsilon^2)$), we obtain from (5.19),

$$|\mu_n(t)| \leqslant \frac{|\mathcal{P}(u_n(t)) - \mathcal{P}(\varphi)| + \|\psi_n(t)\|_{L^2}^2}{2\cos(\theta_n(t))\|\varphi\|_{L^2}^2} \leqslant C(\epsilon_n + \|\psi_n(t)\|_{L^2}^2) \leqslant C(\epsilon_n + \epsilon^2), \tag{5.20}$$

where in the last inequality, we have used that $t \in (0, T_n^*)$. Note in addition, that taking L^2 norms in (5.18) and using the orthogonality relations yields

$$|\mu_n(t)|^2 + \|\zeta_n(t)\|^2 + \|\eta_n\|^2 = \|\psi_n(t)\|^2 \leqslant C\epsilon^2, \tag{5.21}$$

Now,

$$E[u_n(t)] - E[\varphi] = E[u_n(t, \cdot + \beta(v_n(t)))] - E[\varphi] = E[e^{i\theta_n(t)}\varphi + \psi_n] - E[\varphi] \\ = E[(\cos(\theta_n(t))\varphi + \mu_n\varphi + \eta_n) + i(\sin(\theta_n(t))\varphi + \zeta_n)] - E[\varphi].$$

Note

$$|(\cos(\theta_n)\varphi + \mu_n\varphi + \eta_n) + i(\sin(\theta_n)\varphi + \zeta_n)|^2 = \varphi^2 + 2\cos(\theta_n)\varphi(\mu_n\varphi + \eta_n) \\ + (\mu_n\varphi + \eta_n)^2 + 2\sin(\theta_n)\varphi\zeta_n + \zeta_n^2 = \varphi^2 + 2\varphi(\mu_n\varphi + \eta_n) + (\mu_n\varphi + \eta_n)^2 \\ + 2\sin(\theta_n)\varphi\zeta_n + \zeta_n^2 + O(\epsilon^2(|\zeta_n| + |\eta_n|)).$$

where we have used $\cos(\theta_n(t)) = 1 + O(\epsilon^2)$. By the relations (5.20), (5.21), $\mu_n^2 + |\mu_n||\eta_n| \leqslant C(\epsilon_n + \epsilon^3)$. It follows that

$$\begin{aligned} & E[(\cos(\theta_n(t))\varphi + \mu_n\varphi + \eta_n) + i(\sin(\theta_n(t))\varphi + \zeta_n)] \\ & \quad - E[\varphi] = \langle \Lambda^\alpha \varphi, \mu_n\varphi + \eta_n\rangle + \frac{1}{2}\langle \Lambda^\alpha \eta_n, \eta_n\rangle \\ & \quad + \sin(\theta_n)\langle \Lambda^\alpha\varphi, \zeta_n\rangle + \frac{1}{2}\langle \Lambda^\alpha \zeta_n, \zeta_n\rangle + \omega\langle \varphi, \mu_n\varphi + \eta_n + \sin(\theta_n)\zeta_n\rangle \\ & \quad + \frac{\omega}{2}(\langle \eta_n, \eta_n\rangle + \langle \zeta_n, \zeta_n\rangle) - \mu_n\langle \varphi^2, \varphi\rangle - \langle \varphi^2, \eta_n\rangle \\ & \quad - \langle \varphi, \eta_n^2\rangle - \sin(\theta_n)\langle \varphi^2, \zeta_n\rangle - \frac{1}{2}\langle \varphi, \zeta_n^2\rangle + O(\epsilon_n + \epsilon^3). \end{aligned}$$

By the profile equation, $\Lambda^\alpha\varphi + \omega\varphi - \varphi^2 = 0$, we can simplify the expression above

$$E[u_n(t)] - E[\varphi] \geqslant \frac{1}{2}[\langle \mathcal{L}_+\eta_n(t), \eta_n(t)\rangle + \langle \mathcal{L}_-\zeta_n(t), \zeta_n(t)\rangle] - C(\epsilon^3 + |\epsilon_n|). \quad (5.22)$$

As we have pointed out, $\eta_n(t) \perp span\{\varphi, \varphi'\}$, $\zeta_n(t) \perp \varphi$, so (5.13) and (5.14) above imply

$$\langle \mathcal{L}_+\eta_n, \eta_n\rangle + \langle \mathcal{L}_-\zeta_n, \zeta_n\rangle \geqslant \kappa[\|\eta_n\|^2_{H^{\alpha/2}} + \|\zeta_n\|^2_{H^{\alpha/2}}].$$

We conclude, by taking into account $|E[u_n(t)] - E[\varphi]| \leqslant \epsilon_n$, and $t \in (0, T_n^*)$

$$\kappa[\|\eta_n(t)\|^2_{H^{\alpha/2}} + \|\zeta_n(t)\|^2_{H^{\alpha/2}}] \leqslant C(\epsilon_n + \epsilon^3). \quad (5.23)$$

This implies however that for all $t \in (0, T_n^*)$, we have (again, using (5.20) for $\mu_n(t)$),

$$\|\psi_n(t)\|_{H^{\alpha/2}} \leqslant C\sqrt{\epsilon_n} + C\epsilon^{3/2}. \quad (5.24)$$

But then, for sufficiently small ϵ and for large enough n, we must have $T_n^* = \infty$. Indeed, otherwise

$$C\sqrt{\epsilon_n} + C\epsilon^{3/2} \geqslant \limsup_{t\to T_n^*-} \|\psi_n(t)\|_{H^{\alpha/2}} \geqslant C_1\epsilon.$$

Such an inequality clearly will not hold by selecting $\epsilon : C\sqrt{\epsilon} < C_1/2$ and then n so large that $\sqrt{\epsilon_n} \ll \epsilon$, which can be done since $\lim_n \epsilon_n = 0$. Thus a contradiction is reached and the waves are orbitally stable. □

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References

1 C. J. Amick and J. F. Toland. Uniqueness and related analytic properties for the Benjamin-Ono equation - a nonlinear Neumann problem in the plane. *Acta Math.* **167** (1991), 107–126.

2 J. Angulo Pava, J. L. Bona and M. Scialom. Stability of cnoidal waves. *Adv. Differential Equations.* **11** (2006), 1321–1374.

3 J. Angulo Pava. Nonlinear stability of periodic traveling waves olutions to the Schrödinger and the modified Korteweg-de Vries equations. *J. Differential Equ.* **235** (2007), 1–30.

4 J. Angulo Pava and F. Natali. Positivity properties of the Fourier transform and the stability of periodic travelling-wave solutions. *SIAM J. Math. Anal.* **40** (2008), 1123–1151.

5 J. Angulo Pava. Nonlinear dispersive equations. Existence and stability of solitary and periodic travelling wave solutions. In *Mathematical surveys and monographs*, vol. 156(Providence, RI: American Mathematical Society, 2009).

6 J. Angulo Pava, M. Scialom and C. Banquet. The regularized Benjamin-Ono and BBM equations: Well-posedness and nonlinear stability. *J. Differential Equ.* **250** (2011), 4011–4036.

7 T. B. Benjamin. The stability of solitary waves. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **328** (1972), 153–183.

8 J. L. Bona, P. E. Souganidis and W. A. Strauss. Stability and instability of solitary waves of Korteweg-de Vries type. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **411** (1987), 395–412.

9 K. Claasen and M. Johnson. Nondegeneracy and Stability of Antiperiodic Bound States for Fractional Nonlinear Schrödinger Equations. *J. Differential Equ.* **266** (2019), 5664–5712.

10 J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao. Sharp global well- posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$. *J. Amer. Math. Soc.* **16** (2003), 705–749.

11 R. Frank and E. Lenzmann. Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$. *Acta Math.* 210 (2013), 261–318.

12 M. Grillakis, J. Shatah and W. Strauss. Stability theory of solitary wavs in the presence of symmetry I. *J. Funct. Anal.* **74** (1987), 160–197.

13 S. Herr, A. Ionescu, C. Kenig and H. Koch. A para-differential renormalization technique for nonlinear dispersive equations. *Comm. Partial Differential Equ.* **35** (2010), 1827–1875.

14 V. Hur and M. Johnson. Stability of periodic traveling waves for nonlinear dispersive equations. *SIAM J. Math. Anal.* **47** (2015), 3528–3554.

15 V. Hur, M. Johnson and J. Martin. Oscillation estimates of eigenfunctions via the combinatorics of non-crossing partitions. Discrete Anal., 2017), Paper No. 13, 20 pp.

16 V. Hur. Norm inflation for equations of KdV type with fractional dispersion. *Differential Integral Equ.* **31** (2018), 833–850.

17 A. Ionescu and C. Kenig. Global well-posedness of the Benjamin-Ono equation in low-regularity spaces. *J. Amer. Math. Soc.* **20** (2007), 753–798.

18 T. Kapitula and K. Promislow. Spectral and dynamical stability of nonlinear waves. In *Applied mathematical sciences*, vol. 185 (New York: Springer, 2013).

19 C. Kenig and Y. Martel. Asymptotic stability of solitons for the Benjamin-Ono equation. *Rev. Mat. Iberoam.* **25** (2009), 909–970.

20 C. E. Kenig, Y. Martel and L. Robbiano. Local well-posedness and blow-up in the energy space for a class of L^2 critical dispersion generalized Benjamin-Ono equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **28** (2011), 853–887.

21 C. Klein, F. Linares, D. Pilod and J. C. Saut. On Whitham and related equations. *Stud. Appl. Math.* **140** (2018), 133–177.

22 U. Le and D. Pelinovski. Convergence of Petviashvili's method near periodic waves of the fractional Korteweg-De Vries equation. *SIAM J. Math. Anal.* **51** (2019), 2850–2883.

23 Z. Lin and C. Zeng. Instability, index theorem, and exponential trichotomy for Linear Hamiltonian PDEs, available at https://arxiv.org/abs/1703.04016, to appear as Mem. Amer. Math. Soc.

24 F. Linares, D. Pilod and J. C. Saut. Remarks on the orbital stability of ground state solutions of fKdV and related equations. *Adv. Differential Equ.* **20** (2015), 835–858.

https://doi.org/10.1017/prm.2020.54 Published online by Cambridge University Press

25 Y. Martel and F. Merle. Asymptotic stability of solitons for subcritical generalized KdV equations. *Arch. Ration. Mech. Anal.* **157** (2001), 219–254.

26 L. Molinet. Global well-posedness in L^2 for the periodic Benjamin-Ono equation. *Amer. J. Math.* **130** (2008), 635–683.

27 L. Molinet, D. Pilod and S. Vento. On well-posedness for some dispersive perturbations of Burgers equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **35** (2018), 1719–1756.

28 F. Natali, U. Le and D. Pelinovsky. New variational characterization of periodic waves in the fractional Korteweg-de Vries equation, available at arXiv:1907.01412.

29 S. Nonnenmacher. Lecture notes. Spectral Theory of self-adjoint operators, June 2017, University of Cardiff.

30 A. Stefanov. On the normalized ground states of second order PDE's with mixed power non-linearities. *Comm. Math. Phys.* **369** (2019), 929–971.

31 N. Vakhitov and A. A. Kolokolov. Stationary solutions of the wave equation in the medium with nonlinearity saturation. *Radiophys. Quantum Electron.* **16** (1973), 783–789.