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On the instability of the Ruf–Sani solitons for the NLS with exponential nonlinearity

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ABSTRACT

We study the two dimensional non-linear Schrödinger equation with two types of exponential non-linearities. It is well-known by a work of Ruf–Sani (Ruf and Sani, 2013), that such models support solitary wave solutions, which are solutions of some constrained minimization problem. We show that the Ruf–Sani solitons are spectrally unstable and unstable by blow up.

1. Introduction

We consider the Schrödinger equation with focusing exponential nonlinearity

$$iu_t + \Delta u + f_\mu(u) = 0; \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2 \tag{1.1}$$

with

$$f_\mu(u) = (e^{4\pi|u|^2} - 1 - 4\pi\mu|u|^2)u, \quad \mu \in \{0,1\}.$$

Note that this model enjoys the conserved quantities

$$M(u) = \int_{\mathbb{R}^2} |u|^2 dx, E_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx - \int_{\mathbb{R}^2} F_\mu(u)dx,$$

where $F_\mu' = f_\mu, F_\mu(0) = 0$. Explicitly, $F_\mu(u) = \frac{1}{8\pi}\left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2\mu|u|^4\right), \quad \mu \in \{0,1\}$. Our work concentrates on the solitary waves of (1.1), namely the solutions in the form $u = e^{i\omega t}\phi, \omega > 0$, which clearly satisfy the profile problem

$$-\Delta\phi + \omega\phi = f_\mu(\phi). \tag{1.2}$$

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1.1. Ground states

There are various definitions of ground states, which may be adopted for such objects. The notion of ground state has to do with an underlying mode of variational construction. In our case, we shall exclusively consider the Ruf–Sani construction, [1], which solves a particular constrained variational problem. Here is the precise result, due to Ruf–Sani, [1].

Proposition 1 *([1], See also [2]). Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, which satisfies*

$$\lim_{t\to 0} \frac{f(t)}{t} = 0, \quad \limsup_{|t|\to+\infty} \frac{tf(t)}{e^{4\pi t^2}} > 0, \quad \lim_{|t|\to\infty} \frac{f(t)}{e^{\alpha t^2}} = \begin{cases} 0 & \alpha > 4\pi \\ +\infty & \alpha < 4\pi \end{cases}$$

For $F : F' = f, F(0) = 0$, and all $s \neq 0$, $0 < 2F(s) \leq sf(s)$. Then, the minimization problem

$$\begin{cases} \|\nabla u\| \to \min \\ \text{subject to } \ \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} F(u(x))dx = 0. \end{cases} \tag{1.3}$$

has a solution Q. Moreover, Q satisfies the following properties:

- *Q solves the Euler–Lagrange equation*

$$-\Delta Q + Q = f(Q) \tag{1.4}$$

- *Q is radially symmetric, $Q \in C^2 \cap L^\infty$, Q is exponentially decaying at $\pm\infty$.*
- *$0 < \|\nabla Q\| < 1$ and*

$$\frac{1}{2}\|Q\|^2 = \int_{\mathbb{R}^2} F(Q), \quad \|\nabla Q\|^2 + \|Q\|^2 = \int_{\mathbb{R}^2} f(Q)Q. \tag{1.5}$$

Remark. We would like to note that if one starts with a nice solution of the elliptic problem (1.4), then the relation (1.5) is nothing but the Pohozaev identity for such solutions and can be easily obtained by integration by parts, by taking dot product of (1.4) with $x \cdot \nabla Q$ and with Q respectively.

As a simple consequence of this result, we will obtain suitable solutions of (1.2). Indeed, for a fixed $\omega > 0$, set $f(z) := \frac{1}{\omega}(e^{4\pi z^2} - 1 - 4\pi\mu z^2)z, \mu \in \{0,1\}$, with the corresponding function $F(z) = \frac{1}{8\pi\omega}\left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2\mu|u|^4\right)$. We claim that the pair $f, F, \mu \in \{0,1\}$ satisfies the conditions in Proposition 1. For the case, $\mu = 0$, the only non trivial part of this statement is the inequality $2F(z) \leq zf(z)$, which can be seen by the expansion in McLaurin series

$$2F(z) = \frac{1}{4\pi\omega}\sum_{l=2}^{\infty} \frac{(4\pi z^2)^l}{l!} < \frac{1}{4\pi\omega}\sum_{l=2}^{\infty} \frac{(4\pi z^2)^l}{(l-1)!} = zf(z)$$

and similar for the case $\mu = 1$. We can thus infer the existence of a function Q_ω, as specified in Proposition 1. Moreover, the assignment $\phi_\omega(x) := Q_\omega(\sqrt{\omega}x)$ introduces a function, which is a solution of (1.2), since Q_ω solves the Euler–Lagrange equation (1.4), corresponding to the specific nonlinearity f_ω. We say that f is Schwarz symmetric (bell-shaped) if $f = f^*$, its decreasing rearrangement. The minimization problem (Ruf–Sani)

$$\inf\left\{\|\nabla u\|_2^2 : \ \frac{1}{2}\|u\|_2^2 - \int_{\mathbb{R}^2} F_\mu(u(x))dx = 0\right\}$$

always has a Schwarz-symmetric (bell-shaped) minimizer. For further results on decreasing rearrangements in this context, we refer to [3]. We are now ready to collect our findings about ϕ_ω in the following corollary.

Corollary 1. *For each $\omega > 0$ and $\mu \in \{0,1\}$, there exists a solution ϕ_ω of the elliptic problem (1.2). Moreover, $\phi_\omega \in C^2 \cap L^\infty$ is a bell-shaped function, $0 < \|\nabla\phi_\omega\|_{L^2} < 1$ and*

$$\frac{\omega}{2}\|\phi_\omega\|^2 = \int_{\mathbb{R}^2} F(\phi_\omega), \quad \|\nabla\phi_\omega\|^2 + \omega\|\phi_\omega\|^2 = \int_{\mathbb{R}^2} f(\phi_\omega)\phi_\omega. \tag{1.6}$$

Remark. We call the functions ϕ_ω the **Ruf–Sani solitons** associated with the nonlinear Schrödinger equation with exponential nonlinearity (1.1).

1.2. Main results

The main objective of this paper is to study further properties of the Ruf–Sani solitons. It is for example easy to compute the precise asymptotics at $\pm\infty$. Namely, it is a standard to obtain

$$\phi_\omega(x) = c\frac{e^{-\sqrt{\omega}|x|}}{\sqrt{|x|}} + o\left(\frac{e^{-\sqrt{\omega}|x|}}{|x|}\right), |x| \gg 1, x \in \mathbb{R}^2 \tag{1.7}$$

see for example Theorem 2, [4], which applies to any for general super-linear nonlinearity.

Next, we shall be interested in the properties of the linearized operators $\mathscr{L}_\pm$. For convenience, introduce functions g, G

$$g(z) = e^{4\pi z} - 1 - 4\pi\mu z \tag{1.8}$$
$$G(z) = \frac{e^{4\pi z} - 1 - 4\pi z - 8\pi^2\mu z^2}{4\pi}, \tag{1.9}$$

so that $f(z) = g(z^2)z$, $G(z^2) = 2F(z)$ and $G(0) = 0, G'(z) = g(z)$. In these variables,

$$\mathscr{L}_- = -\Delta + \omega - g(\phi_\omega^2) = -\Delta + \omega - (e^{4\pi\phi_\omega^2} - 1 - 4\pi\mu\phi_\omega^2);$$
$$\mathscr{L}_+ = -\Delta + \omega - (2\phi^2 g'(\phi^2) + g(\phi^2)) = -\Delta + \omega - (e^{4\pi\phi_\omega^2}(8\pi\phi_\omega^2 + 1) - 1 - 12\mu\pi\phi_\omega^2).$$

as these are paramount in the stability analysis of the waves ϕ_ω as solutions to (1.1), see (1.10) below.

In line with the expectations in the classical cases of power nonlinearities, we have the usual properties of $\mathscr{L}_\pm$. Recall that for a semi-bounded from below self-adjoint operator S with a finite dimensional negative subspace X_-, the Morse index is defined as follows $n(S) = dim(X_-) = \#\{\sigma_p(S) \cap (-\infty, 0)\}$.

Proposition 2. *Let $\omega > 0, \mu \in \{0,1\}$ and ϕ_ω are the Ruf–Sani solitons constructed in Corollary 1. Then, the Schrödinger operators $\mathscr{L}_\pm$ have the following properties*

- *$\mathscr{L}_- \geq 0$, with a simple eigenvalue at zero, $Ker[\mathscr{L}_-] = span[\phi_\omega]$.*
- *$\mathscr{L}_+$ has Morse index 1. That is, $n(\mathscr{L}_+) = 1$.*

Our next result concerns the instability of the Ruf–Sani waves. In order to put the results in the proper context, let us consider the linearization of the Schrödinger equation with exponential nonlinearity in a vicinity of the soliton $e^{i\omega t}\phi_\omega$. More precisely, take $u = e^{i\omega t}(\phi_\omega(x) + v) = e^{i\omega t}(\phi_\omega(x) + v_1 + iv_2)$ and plug this in (1.1). After ignoring all terms $O(|v|^2)$, we obtain the linearized system

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathscr{L}_+ & 0 \\ 0 & \mathscr{L}_- \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: \mathscr{J}\mathscr{L}\vec{v}. \tag{1.10}$$

Passing to a time independent problem, $\vec{v} \to e^{\lambda t}\vec{v}$, allows us to reduce matters to the eigenvalue problem

$$\mathscr{J}\mathscr{L}\vec{v} = \lambda\vec{v}. \tag{1.11}$$

Definition 1. We say that the wave ϕ_ω is spectrally unstable, if the eigenvalue problem (1.11) has a non-trivial solution $(\lambda, \vec{v}) : \Re\lambda > 0, \vec{v} \in D(\mathscr{L}), \vec{v} \neq 0$.

The next theorem is the main result of our work.

Theorem 1. *Let $\omega > 0, \mu \in \{0,1\}$. Then, the Ruf–Sani waves $e^{i\omega t}\phi_\omega$ are spectrally unstable, with a single real growing mode. These waves are also unstable by blow-up.*

Instability by blow up results in similar contexts have been obtained in [5,6].

2. Proof of Proposition 2

2.1. An alternative variational characterization of ϕ_ω

Specifically, taking into account that Q_ω is a constrained minimizer for (1.3), it is easy to see, by rescaling, that ϕ_ω is a constrained minimizer of

$$\begin{cases} \|\nabla u\|_{L^2} \to \min \\ \text{subject to } \ \omega\|u\|^2 - \frac{1}{4\pi}\int_{\mathbb{R}^2}\left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2\mu|u|^4\right)dx = 0. \end{cases} \tag{2.1}$$

We will show that reversing the roles of the constraints and the cost function produces the same outcome.

Lemma 1. *The constrained minimization problem*

$$\begin{cases} I[u] := \omega\|u\|^2 - \frac{1}{4\pi}\int_{\mathbb{R}^2}\left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2\mu|u|^4\right)dx \to \min \\ \text{subject to } \ \|\nabla u\|_{L^2} = \|\nabla\phi_\omega\|_{L^2} \end{cases} \tag{2.2}$$

has a solution and $I_{\min} := \inf_{\|\nabla u\| = \|\nabla\phi_\omega\|} I[u] = 0$. In particular, $u = \phi_\omega$ solves (2.2).

Proof. The argument is pretty straightforward and exploits the fact that ϕ_ω is a solution of (2.1). Indeed, as $u = \phi_\omega$ satisfies the constraint of (2.1) and (2.2), we have that

$$I_{\min} := \inf_{\|\nabla u\| = \|\nabla\phi_\omega\|} I[u] \le I[\phi_\omega] = 0$$

Note that, so far, we have not even ruled out the scenario $I_{\min} = -\infty$! We however claim that $I_{\min} = 0$, which means that $u = \phi_\omega$ is a solution to (2.2).

Assume, for a contradiction, that this is not the case. That is, assume $I_{\min} < 0$. Then, there exists $\tilde{\phi} \neq 0$, so that $\|\nabla\tilde{\phi}\| = \|\nabla\phi_\omega\|$, but $I[\tilde{\phi}] < 0$. Consider then the continuous function $h(\alpha) := I[\alpha\tilde{\phi}] : [0,1] \to \mathbb{R}$. Since[2]

$$h(\alpha) = \alpha^2\left[\omega\|\tilde{\phi}\|^2 - \frac{1}{4\pi}\sum_{l=2 \text{ or } l=3}^{\infty} \alpha^{2l-2}\int_{\mathbb{R}^2}\frac{(4\pi\tilde{\phi}^2)^l}{l!}\right],$$

it is clear that $h(\alpha) > 0$, for all $0 < \alpha \ll 1$. Since $h(1) = I[\tilde{\phi}] < 0$, it follows by continuity that for some $\tilde{\alpha} \in (0,1)$, we have that $I(\tilde{\alpha}\tilde{\phi}) = h(\tilde{\alpha}) = 0$. Thus, $\tilde{u} := \tilde{\alpha}\tilde{\phi}$ satisfies the constraints in (2.1). But then, we reach a contradiction, as $\|\nabla\phi_\omega\| \le \|\nabla\tilde{u}\| = \tilde{\alpha}\|\nabla\tilde{\phi}\| = \tilde{\alpha}\|\nabla\phi_\omega\|$. □

Our next task is to establish the spectral properties of $\mathscr{L}_\pm$, based on the fact that ϕ_ω is a constrained minimizer of (2.2).

[2] In the case $\mu = 0$, the sum runs from $l = 2$, while in the case $\mu = 1$, from $l = 3$.

2.2. The Morse index of $\mathscr{L}_+$ is exactly one

We consider a variation of the function ϕ_ω in (2.2), which has a built in property $\|\nabla u\| = \|\nabla \phi_\omega\|$. More specifically, for a test function h, consider

$$u_\epsilon = \|\nabla \phi_\omega\| \frac{\phi_\omega + \epsilon h}{\|\nabla(\phi_\omega + \epsilon h)\|},$$

which satisfies the constraints of (2.2). The function $m(\epsilon) := I[u_\epsilon]$ then has a minimum at $\epsilon = 0$, with $m(0) = 0$. The necessary condition $m'(0) = 0$ yields the Euler–Lagrange equation for this problem, which is, as expected, nothing but (1.2). Note that this follows, as we take into account the relation (1.6). Next, the necessary condition $m''(0) \geq 0$ can be written explicitly as well. However, this gets a bit technical, so we reduce our considerations to the case $h \perp \Delta \phi_\omega$, which is enough for our purposes, while simplifying the expressions. Indeed, we have

$$\|\nabla(\phi_\omega + \epsilon h)\|^2 = \|\nabla \phi_\omega\|^2 - 2\epsilon \langle \Delta \phi_\omega, h\rangle + \epsilon^2 \|\nabla h\|^2 = \|\nabla \phi_\omega\|^2 + \epsilon^2 \|\nabla h\|^2$$

Also, note that since already we have ensured the validity of $m(0) = m'(0) = 0$, it follows that $m(\epsilon) = const.\epsilon^2 + o(\epsilon^2)$. Thus, we can ignore all powers of ϵ in the expansion of $m(\epsilon)$. To this end,

$$\begin{aligned} m(\epsilon) = I[u_\epsilon] = \omega \|u_\epsilon\|_{L^2}^2 - \int_{\mathbb{R}^2} G(u_\epsilon^2) = \\ = \omega \frac{\|\phi_\omega\|^2 + 2\epsilon \langle \phi_\omega, h\rangle + \epsilon^2 \|h\|^2}{1 + \frac{\epsilon^2}{\|\nabla \phi_\omega\|^2}\|\nabla h\|^2} - \int_{\mathbb{R}^2} G\left(\phi_\omega^2 + 2\epsilon \phi_\omega h + \epsilon^2 \left(h^2 - \frac{\phi_\omega^2}{\|\nabla \phi_\omega\|^2}\|\nabla h\|^2\right)\right) = \\ = \epsilon^2 \langle \mathscr{L}_+ h, h\rangle + O(\epsilon^3), \end{aligned}$$

where we took into account (1.6). Clearly then, $\langle \mathscr{L}_+ h, h\rangle = \frac{m''(0)}{2} \geq 0$, for all $h \perp \Delta \phi_\omega$. In particular, $n(\mathscr{L}_+) \leq 1$. On the other hand,

$$\langle \mathscr{L}_+ \phi_\omega, \phi_\omega\rangle = \|\nabla \phi_\omega\|^2 + \omega \|\phi_\omega\|^2 - \int f(\phi_\omega^2) - 8\pi \int (e^{4\pi \phi^2} - \mu)\phi_\omega^4 dx = -8\pi \int (e^{4\pi \phi_\omega^2} - \mu)\phi_\omega^4 dx < 0,$$

which shows by the min–max characterization of the eigenvalues, that there is a negative eigenvalue, whence $n(\mathscr{L}_+) = 1$.

2.3. $\mathscr{L}_- \geq 0$ with a simple eigenvalue at zero

As $\mathscr{L}_-$ is a Schrödinger operator with a radial kernel, its action, can be decomposed, in a standard way, on the spaces of spherical harmonics. Indeed, Δ acts invariantly on each separate spherical harmonic space $\mathscr{X}_l, l = 0, \dots$ (composed of the eigenfunctions for a fixed eigenvalue $\lambda_l = -l^2$ for $\Delta_{\mathbf{S}^1}$). So, denoting

$$\mathscr{L}_{-,l} := \mathscr{L}_+|_{\mathscr{X}_l} = -\partial_{rr} - \frac{1}{r}\partial_r + \frac{l^2}{r^2} - g(\phi^2),$$

we obtain $\mathscr{L}_- = \oplus_{l=0}^\infty \mathscr{L}_{-,l}$, which can be thought of as acting on $L^2(rdr)$. Note that

$$\mathscr{L}_{-,0} < \mathscr{L}_{-,1} < \cdots .$$

Note that $\mathscr{L}_-[\phi_\omega] = 0$, which implies that $\mathscr{L}_{-,0}[\phi_\omega] = 0$. Since $\phi_\omega > 0$, this must be the ground state and the lowest eigenvalue for $\mathscr{L}_{-,0}$ is zero. It follows that $\mathscr{L}_- \geq \mathscr{L}_{-,0} \geq 0$, with a simple eigenvalue at zero, spanned by ϕ_ω.

3. Spectral instability of the Ruf–Sani solitons

One may provide a necessary and sufficient condition, even when ϕ_ω is not necessarily weakly non-degenerate, but under the conditions $\mathscr{L}_- \geq 0$, and $n(\mathscr{L}_+) = 1$, which we have established already. Indeed, this is done in Theorem 4.1, [7]. The precise result is the following

Proposition 3. *Suppose $\mathscr{L}_- \geq 0, Ker[\mathscr{L}_-] = span[\phi]$, and $n(\mathscr{L}_+) = 1$. Then, the spectral problem (1.11) is spectrally stable if and only if $\mathscr{L}_+|_{\{\phi\}^\perp} \geq 0$.*

Remark. Even though the result already exists as stated, see Theorem 4.1, [7], we provide the short proof for the readers convenience.

Proof. Assume spectral instability, then there exists $h_1, h_2 : \mathscr{L}_- h_2 = \lambda h_1, \mathscr{L}_+ h_1 = -\lambda h_2$. So, $\mathscr{L}_-\mathscr{L}_+ h_1 = -\lambda^2 h_1$. As $h_1 \in Range[\mathscr{L}_-], h_1 \neq 0$, it follows that $h_1 \perp \phi$. As $\mathscr{L}_-|_{\{\phi\}^\perp} \geq \delta > 0$, there is $\eta := \mathscr{L}_-^{-\frac{1}{2}} h_1 \in \{\phi\}^\perp$ is well-defined. Then

$$\sqrt{\mathscr{L}_-}\mathscr{L}_+\sqrt{\mathscr{L}_-}\eta = -\lambda^2 \eta.$$

It follows that $-\lambda^2$ is real, as an element of the spectra of the self-adjoint $\sqrt{\mathscr{L}_-}\mathscr{L}_+\sqrt{\mathscr{L}_-}$, whence $\lambda \in i\mathbb{R} \cup \mathbb{R}$. Thus, $\lambda > 0$ is real, as an unstable eigenvalue. But now

$$0 > -\lambda^2 \|\eta\|^2 = \langle \sqrt{\mathscr{L}_-}\mathscr{L}_+\sqrt{\mathscr{L}_-}\eta, \eta\rangle = \langle \mathscr{L}_+ h_1, h_1\rangle$$

whence the condition $\mathscr{L}_+|_{\{\phi\}^\perp} \geq 0$ fails.

Conversely, assume that $\mathscr{L}_+|_{\{\phi\}^\perp} \geq 0$ fails. Then, there is $\tilde{\eta} \in \{\phi\}^\perp$ so that $\langle \mathscr{L}_+\tilde{\eta}, \tilde{\eta}\rangle < 0$. Thus, there is $\eta : \sqrt{\mathscr{L}_-}\eta = \tilde{\eta}$. So,

$$\langle \sqrt{\mathscr{L}_-}\mathscr{L}_+\sqrt{\mathscr{L}_-}\eta, \eta\rangle < 0.$$

It follows that $\sqrt{\mathscr{L}_-}\mathscr{L}_+\sqrt{\mathscr{L}_-}$ has negative e-value, so $\sqrt{\mathscr{L}_-}\mathscr{L}_+\sqrt{\mathscr{L}_-}z = -\lambda^2 z, \lambda > 0$. Hence instability. □

This justifies the following sufficient condition for spectral instability.

Corollary 2. *If $n(\mathscr{L}_+) = 1$ and there exists $\Psi \perp \phi_\omega$, so that $\langle \mathscr{L}_+\Psi, \Psi\rangle < 0$, then the wave ϕ_ω is spectrally unstable, with exactly one unstable real mode.*

3.1. Proof of the spectral instability of the waves

Per the results of Proposition 3, it suffices to construct $\Psi \perp \phi_\omega$, so that $\langle \mathscr{L}_+\Psi, \Psi\rangle < 0$. To this end, set $\Psi := x\cdot\nabla\phi_\omega + \phi_\omega$. A direct calculation shows $\Psi \perp \phi_\omega$. Indeed,

$$\langle \Psi, \phi_\omega\rangle = \langle x\cdot\nabla\phi_\omega, \phi_\omega\rangle + \|\phi_\omega\|^2 = \sum_{j=1}^{2}\frac{1}{2}\int_{\mathbb{R}^2} x_j\partial_j\phi_\omega^2 + \|\phi_\omega\|^2 = 0.$$

It remains to calculate $\langle \mathscr{L}_+\Psi, \Psi\rangle$. Since $-\Delta(x\cdot\nabla f) = -x\cdot\nabla\Delta f - 2\Delta f$ and using the profile equation (1.2), we compute $\mathscr{L}_+(x\cdot\nabla\phi_\omega) = -2\Delta\phi_\omega$. So,

$$\begin{aligned}
&\langle \mathscr{L}_+\Psi, \Psi\rangle = \langle \mathscr{L}_+(x\cdot\nabla\phi_\omega), x\cdot\nabla\phi_\omega + \phi_\omega\rangle + \langle \phi_\omega, \mathscr{L}_+(x\cdot\nabla\phi_\omega)\rangle + \langle \mathscr{L}_+\phi, \phi\rangle = \\
&-2\langle \Delta\phi_\omega, x\cdot\nabla\phi_\omega\rangle + 4\|\nabla\phi_\omega\|^2 - 2\int_{\mathbb{R}^2} g'(\phi_\omega^2)\phi_\omega^4 dx = 4\|\nabla\phi_\omega\|^2 - 2\int_{\mathbb{R}^2} g'(\phi_\omega^2)\phi_\omega^4 dx = \\
&\int_{\mathbb{R}^2}\left(4f(\phi_\omega)\phi_\omega - 8F(\phi_\omega) - 2g'(\phi_\omega^2)\phi_\omega^4\right)dx = -\int_{\mathbb{R}^2}\left[e^{4\pi\phi^2}\left(8\pi\phi_\omega^4 + \frac{1}{\pi} - 4\phi_\omega^2\right) - \frac{1}{\pi}\right]dx.
\end{aligned}$$

where we have used that $\langle \Delta\phi_\omega, x\cdot\nabla\phi_\omega\rangle = 0$, $\mathscr{L}_+\phi_\omega = -2\phi_\omega^3 g'(\phi^2)$ and the Pohozaev identities (1.6) and (1.6). We will show momentarily that the integrand function,

$$e^{4\pi z^2}\left(8\pi z^4 + \frac{1}{\pi} - 4z^2\right) - \frac{1}{\pi} \geq 0 \tag{3.1}$$

whence $\langle \mathscr{L}_+ \Psi, \Psi\rangle < 0$.

According to Proposition 3, this implies the spectral instability of ϕ_ω. It remains to prove (3.1). Indeed, it suffices to show that $\chi(x) := 8\pi x^2 + \frac{1}{\pi} - 4x - \frac{e^{-4\pi x}}{\pi} \geq 0$ for all $x \geq 0$. However, note that $f(0) = f'(0) = 0$, while $f''(x) = 16\pi(1 - e^{-4\pi x}) \geq 0$, whence $f(x) > 0$ for all $x > 0$.

4. Instability by blow up

Before tackling the strong instability of the standing waves, we need to make sure that the following Cauchy problem:

$$\begin{cases} iu_t + \triangle u + f_\mu(u) = 0, \ (t,x) \in \mathbb{R}\times\mathbb{R}^2 \\ u(t,0) = u_0(x) \end{cases} \tag{4.1}$$

where u_0 is an initial data in $H^1(\mathbb{R}^2)$, has a unique solution for a time $T > 0$. (4.1) has been resolved in [8]. More precisely, the authors have proved the following result.

Lemma 4.1 (*Theorem 1.10, [8]*)**.** *Let $u_0 \in H^1(\mathbb{R}^2)$ such that $\|\nabla u_0\|_0 < 1$, then there exist a time $T > 0$ and a unique solution to the Cauchy problem (4.1) in the space $C_T(H^1(\mathbb{R}^2))$ with initial data u_0. Moreover $u \in L^4_T(C^{\frac{1}{2}}(\mathbb{R}^2))$, where C^α is the space of α-Hölder continuous functions endowed with the norm*

$$\|u\|_{C^\alpha} = \|u\|_\infty + \sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Additionally, if T^* denote the maximal time, i.e, $T^* = \sup\{T > 0,$ (4.1) has a solution on $[0,T]\}$, we say that a solution blows up at T^*, if $\lim_{t\nearrow T^*}\|\nabla u(t,.)\|_2^2 = 1$. We also have the mass and energy conservation for the solutions of (4.1), which take the form

$$M(u(t,.)) = M(u_0), E_\mu(u(t,.)) = E(u_0) \tag{4.2}$$

For $v \in H^1(\mathbb{R}^2)$, we define the action functional in the following way:

$$S_\mu(v) = E_\mu(v) + \frac{1}{2}M(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 - \int F_\mu(v)dx.$$

Note that $S_\mu(\Phi)$ can also be rewritten in the following way :

$$S_\mu(\Phi) = \inf\left\{S_\mu(v) : \ v \in H^1(\mathbb{R}^2)\setminus\{0\}, \ I_\mu(v) = 0\right\}.$$

Note that a ground state as defined by (1.3) minimizes the action functional in the following way:

$$S_\mu(\Phi) = \inf\left\{S_\mu(v) : \ v \in H^1(\mathbb{R}^2)\setminus\{0\}, \ P_\mu(v) = 0\right\}.$$

The following two quantities also play a crucial role in the study of strong instability:

$$\begin{aligned} P_\mu(v) &= \frac{1}{2}\|v\|_2^2 - \int F_\mu(v)dx, \\ I_\mu(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \int vf_\mu(v)dx = 2E_\mu(v) - \int vf_\mu(v) - 4F_\mu(v)dx. \end{aligned}$$

A virial property has also been shown in [2], More precisely, the authors showed that if u is a solution to (4.1), then

$$\frac{d^2}{dt^2}\|xu\|_2^2 = 8I_\mu(u), \quad \forall\, t \in [0, T^*).$$

Lastly, we introduce two classical sets that are important in the study of orbital stability as they have the desired invariance with respect to the flow of (4.1):

$$K_\mu^- = \left\{v \in H^1(\mathbb{R}^2)\backslash\{0\} : \; S_\mu(v) < S_\mu(\Phi),\; I_\mu(v) < 0\right\}$$
$$K_\mu^+ = \left\{v \in H^1(\mathbb{R}^2)\backslash\{0\} : \; S_\mu(v) < S_\mu(\Phi),\; I_\mu(v) > 0\right\}$$

The following lemma provides sufficient conditions for a finite time blow-up.

Lemma 4.2 (*Lemma 3.9, [2]*). *Let $\mu \in \{0,1\}$ and $u_0 \in H^1(\mathbb{R}^2)$ be such that $\|\nabla u_0\|_2^2 < 1$ and $E_\mu(u_0) \ge 0$. If*

$$u_0 \in K_\mu^-, \quad u_0 \in H^1(\mathbb{R}^2) \cap L^2(|x|^2 dx),$$

then the corresponding solution to (4.1) blows up in finite time.

The next result, also proved in [2], shows that appropriate and close rescales of the ground state satisfy the requirements of Lemma 4.2 and hence provide the instability by blow up claimed in Theorem 1.

Lemma 4.3 (*Lemma 3.13, [2]*). *Let $\mu \in \{0,1\}$, $\Phi_\lambda(x) := \lambda\Phi(\lambda x)$. There exists $\epsilon > 0$, so that for all $\lambda : 1 < \lambda < 1+\epsilon$, the following holds true:*

$$E_\mu(\Phi_\lambda) > 0,\; I_\mu(\Phi_\lambda) < 0, \quad S_\mu(\Phi_\lambda) < S_\mu(\Phi), \tag{4.3}$$
$$\|\nabla\Phi_\lambda\|_2^2 < 1, \quad \Phi_\lambda \in H^1(\mathbb{R}^2) \cap L^2(|x|^2 dx) \tag{4.4}$$

Remark. The precise statement of Lemma 3.13 in [2] requires a slight modification, but the result quoted here holds true due to the arguments presented there.

We are now ready to complete the proof of the strong instability of the waves Φ. Consider $\Phi_n(x) := \lambda_n\Phi(\lambda_n x)$, for a sequence $\lambda_n \to 1+$. Clearly, $\lim_n \|\Phi_n - \Phi\|_{H^1} = 0$. Also, the sequence Φ_n satisfies all the assumptions in Lemma 4.2, due to Lemma 4.3. Therefore, all solutions with initial data Φ_n blow up in finite time, whence we conclude the strong instability of the waves Φ.

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