On the Stability of the Periodic Waves for the Benney System[*]

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Abstract. We analyze the Benney model for interaction of short and long waves in resonant water wave interactions. Our particular interest is in the periodic traveling waves, which we construct and study in detail. The main results are that, for all natural values of the parameters, the periodic dnoidal waves are spectrally stable with respect to perturbations of the same period. For another natural set of parameters, we construct the snoidal waves, which exhibit instabilities, in the same setup. Our results are the first instability results in this context. On the other hand, the spectral stability established herein improves significantly on the work of Angulo, Corcho, and Hakkaev [*Adv. Difference Equ.*, 16 (2011), pp. 523–550], which established stability of the dnoidal waves, on a subset of parameter space, by relying on the Grillakis–Shatah theory. Our approach, which turns out to give definite answer for the entire domain of parameters, relies on the instability index theory, as developed by Kapitula, Kevrekidis, and Sandstede [*Phys. D*, 3–4 (2004), pp. 263–282]; Kapitula, Kevrekidis, and Sandstede [*Phys. D*, 195 (2004), no. 3–4, 263–282], *Phys. D*, 201 (2005), pp. 199–201]; Lin and Zeng [*Instability, Index Theorem, and Exponential Trichotomy for Linear Hamiltonian PDEs*, 2021]; and Pelinovsky [*Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461 (2005), pp. 783–812]. Interestingly, end even though the linearized operators are explicit, our spectral analysis requires subtle and detailed analysis of matrix Schrödinger operators in the periodic context, which support some interesting features.

Key words. spectral stability, periodic waves, Benney system

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1. Introduction.

We consider the following system of PDE:

$$(1.1) \qquad \begin{cases} iu_t + u_{xx} = uv + \beta|u|^2 u, & -T \le x \le T, t \in \mathbb{R}_+ \\ v_t = (|u|^2)_x, \end{cases}$$

where β is a real parameter, u is complex-valued function, and v is real-valued function. This system is introduced by Benney [9, 10], which models the interaction of short and long waves in resonant water waves interaction in a nonlinear medium.

The Cauchy problem on the whole line case for the system (1.1) was considered in [7, 14]. The existence and nonlinear stability of solitary waves was studied in [20, 27].

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We consider such model on a periodic background; that is, we impose a periodic boundary condition. The Cauchy problem for (1.1) has been previously considered in this context [3]. Let us pause for the moment and review the said paper, as it serves as a starting point for our investigation. More precisely, in [3], the authors have established, via the Fourier restriction method, that the problem is locally well-posed for data $(u_0, v_0) \in H^r[-T, T] \times H^s[-T, T]$ whenever $\max(0, r - 1) \leq s \leq \min(r, 2r - 1)$. In particular, Hadamard well-posedness holds in the spaces $H^{\frac{1}{2}}([-T, T]) \times L^2[-T, T]$ and also in the smaller space $H^1([-T, T]) \times L^2[-T, T]$. Interestingly, ill-posedness results (in the sense of nonuniformly continuous dependence on initial conditions) were also obtained in $H^r \times H^s$ whenever $r < 0$.

Here we consider the spectral stability of periodic traveling waves of dnoidal and snoidal type. We are interested in the stability of periodic traveling wave solutions of (1.1) with respect to perturbations that are periodic of the same period as the corresponding wave solutions.

We provide the relevant definitions of the various notion of stability below, but we would like to discuss the advances made in the last forty years in the area of stability of periodic traveling waves. Benjamin, in the seminal work [8], first considered the stability of the cnoidal solution as a periodic traveling wave of KdV. His results were later clarified and streamlined in [2], where the authors have made use of the Grillakis–Shatah–Strauss formalism. It is worth mentioning the work [1], where the author has addressed, in a similar manner, periodic waves for mKdV and NLS. In the important works [15, 16], the authors have considered the stability of more general families of solutions arising in the generalized KdV models.

More recently, in the works [4, 5], Angulo and Natali have developed a novel approach for studying periodic traveling waves for a general class of dispersive models which extracts the necessary spectral information, based on the so-called positivity theory for the multipliers. For other models, such as the Klein–Gordon–Schrodinger system, the Schrödinger–Boussinesq system, and the Schrodinger system, stability of periodic waves is obtained in [29, 30, 31, 6, 17]. In the context of standing waves, interesting contributions were made by Gallay and Haragus [18, 19]. While the results in [18] concern periodic waves in the context of NLS on the line, the results in [19] are more relevant to our discussion herein. Namely, rigorous stability analysis was developed to deal with quasi-periodic waves in the cubic NLS context both in the focusing and defocusing scenarios. All of these works rely, in one degree or another, on the Grillakis–Shatah–Strauss approach, which establishes orbital stability based on conservation laws. This almost universally requires a C^1 dependence on the wave speed parameters, which is not always easy to establish, so an ad hoc assumption in that regard is usually made.

As it turns out, one may study an almost equivalent stability property, namely, the spectral/linear stability; see Definition 1 below.[1] This is a fast-developing theory which has seen some spectacular advances in recent years [24, 25, 28, 33]. This approach has several advantages over the classical GSS approach. For example, one can study the spectral stability as a purely linear problem, without paying particular attention to the actual conservation laws; see (1.14) below. A second major advantage is that, when it comes to systems of coupled PDEs, it is just technically hard to deal with the conservation laws directly, as the linearized operators become nondiagonal matrix operators, which are harder to analyze.

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[1]In fact, under some generic conditions on the waves, one may convert such spectral stability statements into orbital stability results; see Theorem 5.2.11 in [26].

The stability of waves, especially in the context of systems of coupled PDE, especially in the spatially periodic context, is a challenging topic and an active area of research. We should point out that great progress was made in the last fifteen years regarding dispersive equations for scalar quantities—in that regard, we would like to mention the works [11], [12] for KdV-type models, while [13] established an index counting formula for abstract second-order-in-time models. Concerning systems of dispersive PDE, there are just a few results available in the literature about periodic waves. In fact, we are aware of just a few rigorous works on the subject: [22] deals with stability of periodic waves in systems by the index counting method, while [21] and [17] apply the more standard GSS formalism to the corresponding problem at hand. One explanation for the relative scarcity of rigorous analytical results in this context are the difficulties associated with the spectral analysis of the linearized operators in cases of systems.

Regarding the Benney system, which is the system of interest in this article, it was already considered in [3]. More specifically, the authors were able to construct a family of smooth periodic traveling waves of dnoidal type and show their orbital stability. This was done under certain conditions on β and by relying on the Grillakis–Shatah–Strauss approach. More specifically, they rely on the following conservation laws for the Benney system:

$$M(u) = \int_{-T}^{T} |u(t,x)|^2 dx,$$

$$E(u,v) = \int_{-T}^{T} \left[v(t,x)|u(t,x)|^2 + |u_x(t,x)|^2 + \frac{\beta}{2}|u(t,x)|^4 \right] dx,$$

$$P(u,v) = \int_{-T}^{T} \left[|v(t,x)|^2 + 2\Im(u(t,x)\overline{u}_x(t,x)) \right] dx.$$

In order to explain our spectral stability results in detail, we need to linearize the system (1.1) about the periodic traveling wave solutions. Then we need to obtain the required spectral information about the operator of linearization and investigate the index of stability k_{Ham}, as introduced in [28].

The paper is organized as follows. First, we construct the periodic traveling waves of dnoidal and snoidal type and set up the linearized problem for system (1.1). In section 2, we overview the index stability theory and investigate spectral properties of the operator of the linearization. In section 3, using the index counting theory, we analyze the stability of periodic traveling waves.

1.1. Periodic traveling waves.
In this section, we construct periodic waves of the form

$$u(t,x) = e^{i\omega t}e^{i\frac{c}{2}(x-ct)}\varphi(x-ct), \quad v(t,x) = \psi(x-ct),$$

for the Benney system (1.1). Plugging in (1.1), we get the following system:

$$(1.2) \qquad \begin{cases} \varphi'' - \left(\omega - \frac{c^2}{4}\right)\varphi = \varphi\psi + \beta\varphi^3, \\ -c\psi' = 2\varphi\varphi'. \end{cases}$$

The case $c = 0$ leads to semitrivial constant solutions φ, so we do not consider it herein. Henceforth, we assume $c \neq 0$. Integrating the second equation in (1.2), we get $\psi = -\frac{1}{c}\varphi^2 + \gamma$,

where γ is a constant of integration. Substituting ψ in the first equation of (1.2), we get the following equation for φ:

$$(1.3) \qquad \varphi'' - \sigma\varphi = \left(\beta - \frac{1}{c}\right)\varphi^3,$$

where we have introduced the important parameter $\sigma = \omega - \frac{c^2}{4} + \gamma$. Integrating, we get

$$(1.4) \qquad \varphi'^2 = \frac{1}{2}\left(\beta - \frac{1}{c}\right)\varphi^4 + \sigma\varphi^2 + a =: U(\varphi),$$

where a is a constant of integration. It is well known that φ is a periodic function provided that the energy level set $H(x; y) = a$ of the Hamiltonian system $dH = 0$,

$$H(x; y) = y^2 - \sigma x^2 + \frac{1}{2}\left(\frac{1}{c} - \beta\right)x^4,$$

contains an oval (a simple closed real curve free of critical points). Depending on the properties of the biquadratic polynomial $U(\varphi)$, we distinguish two cases which give rise to different explicit solutions, both in terms of the Jacobi elliptic functions.

1.1.1. Dnoidal solutions. Consider the case $\frac{1}{c} - \beta > 0$, $\sigma > 0$, and $a < 0$. Denote by $\varphi_0 > \varphi_1 > 0$ the positive roots of $-\varphi^4 + \frac{2c\sigma}{1-c\beta}\varphi^2 + \frac{2ca}{1-c\beta}$. Then the profile equation (1.4) takes the form

$$\varphi'^2 = \frac{1-c\beta}{2c}(\varphi_0^2 - \varphi^2)(\varphi^2 - \varphi_1^2).$$

Then $\varphi_1 < \varphi < \varphi_0$, and up to translation, the solution φ is given by

$$(1.5) \qquad \varphi(x) = \varphi_0 dn(\alpha x, \kappa),$$

where

$$(1.6) \qquad \varphi_0^2 + \varphi_1^2 = \frac{2c\sigma}{1-c\beta}, \quad \kappa^2 = \frac{\varphi_0^2 - \varphi_1^2}{\varphi_0^2}, \quad \alpha^2 = \frac{1-c\beta}{2c}\varphi_0^2 = \frac{\sigma}{2-\kappa^2}.$$

Since the period of dn is $2K(\kappa)$, the fundamental period of φ is $2T = \frac{2K(\kappa)}{\alpha}$.

The next case of consideration are the snoidal solutions.

1.1.2. Snoidal solutions. Let $\frac{1}{c} - \beta < 0$, $\sigma < 0$, and $a < 0$. Then

$$\varphi'^2 = \frac{c\beta - 1}{2c}(\varphi_0^2 - \varphi^2)\left(\frac{2c\sigma}{1-c\beta} - \varphi_0^2 - \varphi^2\right).$$

Up to translations, the solution is given by

$$(1.7) \qquad \varphi(x) = \varphi_0 sn(\alpha x, \kappa),$$

where

$$(1.8) \qquad \kappa^2 = \frac{(1-c\beta)\varphi_0^2}{2c\sigma - (1-c\beta)\varphi_0^2}, \quad \alpha^2 = -\frac{2c\sigma - (1-c\beta)\varphi_0^2}{2c} = -\frac{\sigma}{1+\kappa^2}.$$

Since the period of sn is $4K(\kappa)$, the fundamental period of φ is $2T = \frac{4K(\kappa)}{\alpha}$.

We formulate our findings in the following proposition.

Proposition 1. *Let (c, β, σ) be three real parameters and $\kappa \in (0, 1)$. Then we can identify the following families of solutions of* (1.4).

If $c \neq 0$ and $\beta < \frac{1}{c}, \sigma > 0$, then φ is a family of dnoidal solutions given by (1.5). *Its parameters are given by*

$$(1.9) \qquad \varphi_0^2 = \frac{2\sigma}{(2 - \kappa^2)(\frac{1}{c} - \beta)}, \quad \alpha^2 = \frac{\sigma}{2 - \kappa^2},$$

whereas its fundamental period is $2T = \frac{2K(\kappa)}{\alpha} = \frac{2K(\kappa)\sqrt{2-\kappa^2}}{\sqrt{\sigma}}$. Note that this is a three free parameter family, depending and uniquely determined by $(\frac{1}{c} - \beta, \sigma, \kappa) \in \mathbb{R}_+ \times \mathbb{R}_+ \times (0, 1)$.

If $c \neq 0$ and $\beta > \frac{1}{c}, \sigma < 0$, we obtain the snoidal family described in (1.7), *where*

$$(1.10) \qquad \varphi_0^2 = \frac{2\sigma\kappa^2}{(\frac{1}{c} - \beta)(1 + \kappa^2)}, \quad \alpha^2 = -\frac{\sigma}{1 + \kappa^2},$$

and the fundamental period is given by $2T = 4K(\kappa)\frac{\sqrt{1+\kappa^2}}{\sqrt{-\sigma}}$. This is also uniquely determined by three independent parameters as follows: $(\frac{1}{c} - \beta, \sigma, \kappa) \in \mathbb{R}_- \times \mathbb{R}_- \times (0, 1)$.

Now that we have identified the relevant nonlinear waves for the Benney model (1.1), we focus our attention to the corresponding linearized problem.

1.2. Linearized equations. We take the perturbation in the form

$$(1.11) \quad u(t, x) = e^{i\omega t} e^{i\frac{c}{2}(x-ct)}(\varphi(x - ct) + U(t, x - ct)), \quad v(t, x) = \psi(x - ct) + V(t, x - ct),$$

where $U(t, x)$ is complex-valued function and $V(t, x)$ is real-valued function. Plugging in the system (1.1), using (1.2), and ignoring all quadratic and higher-order terms yields a linear equation for (U, V). Furthermore, we split the real and imaginary parts of complex-valued function U as $U = P + iQ$, which allows us to rewrite the linearized problem as the following system:

$$(1.12) \qquad \begin{cases} -Q_t = -P_{xx} + \left(w - \frac{c^2}{4}\right)P + 3\beta\varphi^2 P + \varphi V + \psi P, \\ P_t = -Q_{xx} + \left(w - \frac{c^2}{4}\right)Q + \psi Q + \beta\varphi^2 Q, \\ V_t - cV_x = 2\partial_x(\varphi P). \end{cases}$$

Let us denote

$$\mathcal{J} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2\partial_x & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{H} := \begin{pmatrix} L_1 & \varphi & 0 \\ \varphi & \frac{c}{2} & 0 \\ 0 & 0 & L_2 \end{pmatrix},$$

where[2]

$$\begin{aligned} L_1 &= -\partial_x^2 + \sigma + \left(3\beta - \frac{1}{c}\right)\varphi^2, \\ L_2 &= -\partial_x^2 + \sigma + \left(\beta - \frac{1}{c}\right)\varphi^2. \end{aligned}$$

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[2] Note that the operator L_2 is the standard operator L_- if we were to consider the waves φ as solutions to the cubic NLS; see (1.3).

Then the system (1.12) can be written of the form

$$(1.13) \qquad \vec{Z}_t = \mathcal{J}\mathcal{H}\vec{Z}, \quad \vec{Z} = \begin{pmatrix} P \\ V \\ Q \end{pmatrix}.$$

The standard mapping into a time-independent problem $\vec{Z} \to e^{\lambda t}\vec{z}$ transforms the linear differential equation (1.13) into the eigenvalue problem

$$(1.14) \qquad \mathcal{J}\mathcal{H}\vec{z} = \lambda \vec{z}.$$

By general properties of Hamiltonian systems and the operators $\mathcal{J}, \mathcal{H}$ in particular, if λ is an eigenvalue of (1.14), then so are $\bar{\lambda}, -\lambda, -\bar{\lambda}$. We give now the following standard definition of spectral stability.

Definition 1. *We say that the wave φ is spectrally unstable if the eigenvalue problem (1.14) has a nontrivial solution $(\vec{u}, \lambda)$, so that $\vec{z} \neq 0, \vec{z} \in H^2[-T, T] \times H^1[-T, T] \times H^2[-T, T]$, and $\lambda : \Re\lambda > 0$.*

In the opposite case, that is, (1.14) has no nontrivial solutions, with $\Re\lambda > 0$, we say that the wave is spectrally stable.

Remark. The definition of linear stability is closely related to the one given in Definition 1 for spectral stability. More precisely, φ is a linearly stable wave if the flow of the differential equation (or equivalently the semigroup generated by $\mathcal{J}\mathcal{H}$) has Lyapunov exponent less than or equal to zero. Equivalently,

$$(1.15) \qquad \limsup_{t \to \infty} \frac{\ln\|\vec{U}(t)\|}{t} \leq 0$$

for each initial data $\vec{U}(0) \in H^2[-T, T] \times H^1[-T, T] \times H^2[-T, T]$. It is a standard fact that these two notions coincide in the case of periodic domains due to the fact that the spectrum of $\mathcal{J}\mathcal{H}$ consists of eigenvalues only. A general justification of (1.15) which applies to our case is provided in Theorem 2.2 of [28].

We are now ready to present our main results, which concern the spectral stability of the traveling periodic waves—of dnoidal and snoidal type.

1.3. Main results. The following is our main result, which concerns the stability of the dnoidal waves identified in Proposition 1.

Theorem 1 (stability of the dnoidal waves). *Let $\omega \in \mathbb{R}$ and $c \neq 0, \beta < \frac{1}{c}, \sigma > 0$. Then the Benney system (1.1) has a family of dnoidal solutions in the form*

$$\left(e^{i\omega t}e^{i\frac{c}{2}(x-ct)}\varphi(x-ct), \psi(x-ct) = \left(e^{i\omega t}e^{i\frac{c}{2}(x-ct)}\varphi(x-ct), -\frac{1}{c}\varphi^2(x-ct) + \sigma + \frac{c^2}{4} - \omega\right),\right.$$

where the dnoidal solutions φ are identified by (1.5), whose parameters are given by (1.9). These solutions are spatially periodic provided that

$$(1.16) \qquad c\frac{K(\kappa)\sqrt{2 - \kappa^2}}{\sqrt{\sigma}} \in 2\pi\mathbb{Z}.$$

Under these assumptions, the periodic waves are spectrally stable, in the sense of Definition 1, for all values of the parameters $\omega \in \mathbb{R}, \sigma > 0, \beta < \frac{1}{c}, \kappa \in (0,1)$, subject to (1.16).

Remark. In [3], the authors proved that dnoidal solutions are orbitally stable for $\beta \leq 0$ and for $\beta > 0$ and $8\beta\sigma - 3c(1 - \beta c)^2 \leq 0$. This is achieved by evaluating the number of negative eigenvalues of the operator of linearization around the periodic waves and number of positive eigenvalues of the Hessian of $d(\omega, c) = E(u, v) - \frac{c}{4}P(u, v) - \frac{\omega}{2}M(u, v)$. We extend this result herein to the whole domain of the parameters.

Our next result concerns the instability of the snoidal waves, also identified in Proposition 1.

Theorem 2 (instability of the snoidal solutions). *Let $\omega \in \mathbb{R}$ and $c \neq 0, \beta > \frac{1}{c}, \sigma < 0$. Then the Benney system has a family of snoidal solutions*

$$\left(e^{i\omega t}e^{i\frac{c}{2}(x-ct)}\varphi(x - ct), -\frac{1}{c}\varphi^2(x - ct) + \sigma + \frac{c^2}{4} - \omega \right),$$

where φ is described in (1.7), together with (1.10). These waves are periodic exactly when

$$(1.17) \qquad cK(\kappa)\frac{\sqrt{1 + \kappa^2}}{\sqrt{-\sigma}} \in \pi\mathbb{Z}.$$

The snoidal periodic waves are spectrally unstable (with at least one real and positive eigenvalue) for all values of the parameters $\omega \in \mathbb{R}, \sigma < 0, \beta > \frac{1}{c}, \kappa \in (0,1)$, subject to (1.17).

The plan for the paper as well as some major points are explained below. In section 2, we introduce the basics of the instability index theory. We also outline well-known results about the scalar linearized Schrödinger operators L_1, L_2 identified earlier as well as a related operator L, which plays significant role in our spectral analysis. This allows us to compute the Morse index of the operator $\mathcal{H}$ as well as the kernel and the generalized kernel of $\mathcal{J}\mathcal{H}$; see Proposition 3. In section 3, we deploy the instability index theory to reduce matters to the Morse index of a scalar two-by-two matrix D. For the dnoidal case, the computations here are involved since only one of the entries of D is (barely) explicitly computable, and it involves the construction of Green's function for the Schrödinger operator L^{-1}. This is, however, enough to conclude stability. In the snoidal case, one argues by computing selected (easier) quantities in the limit $0 < \beta - \frac{1}{c} << 1$, which allows one to conclude that real instability exists close to this limit. Then a continuation argument, coupled with an earlier rigidity argument about[3] $Ker(\mathcal{J}\mathcal{H})$, confirms that the real instability persists across the whole domain of parameters.

2. Preliminaries. We first review the basics of the instability index theory as developed in [24, 25, 28, 33].

2.1. Instability index count. We follow the notations and presentation in [24, 25], but the same results appears in [33], while the most general version can be found in [28]. Consider the Hamiltonian eigenvalue problem

$$(2.1) \qquad \mathcal{I}\mathcal{L}u = \lambda u,$$

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[3]Establishing that the generalized kernel of $\mathcal{J}\mathcal{H}$ remains five dimensional and, importantly, does not change across the parameter domain.

where $\mathcal{I}^* = -\mathcal{I}, \mathcal{L}^* = \mathcal{L}$ and $\mathcal{I}, \mathcal{H} : \overline{\mathcal{I}f} = \mathcal{I}\bar{f}, \quad \overline{\mathcal{H}f} = \mathcal{H}\bar{f}$; i.e. $\mathcal{I}, \mathcal{H}$ map real-valued elements into real-valued elements.

Introduce the Morse index of a self-adjoint, bounded from below operator S, by setting $n(S) = \#\{\lambda \in \sigma(S) : \lambda < 0\}$, counted with multiplicities. Let $k_r := \#\{\lambda \in \sigma_{pt.}(\mathcal{IL}) : \lambda > 0\}$ represent the number of positive real eigenvalues of $\mathcal{IL}$, counted with multiplicities $k_c := \#\{\lambda \in \sigma_{pt.}(\mathcal{IL}) : \Re\lambda > 0, \Im\lambda > 0\}$—the number of quadruplets of complex eigenvalues of $\mathcal{IL}$ with nonzero real and imaginary parts—whereas

$$k_i^- = \#\{i\lambda, \lambda > 0 : \mathcal{IL}f = i\lambda f, \langle \mathcal{L}f, f \rangle < 0\}$$

is the number of pairs of purely imaginary eigenvalues of negative Krein signature. Consider the generalized kernel of $\mathcal{JH}$:

$$gker(\mathcal{IL}) = span \cup_{l=1}^{\infty} \ker(\mathcal{IL})^l.$$

Under general conditions, described in [24], one has that $gker(\mathcal{IL})$ is finite dimensional, so one can take a basis,[4] say, $\eta_1, \ldots, \eta_N$. Then we introduce a symmetric matrix D by

$$D := \{\{D_{ij}\}_{i,j=1}^N \ : \ D_{ij} = \langle \mathcal{L}\eta_i, \eta_j \rangle\}.$$

We are now ready to state the main result of this section, namely, the following formula for the Hamiltonian index:

$$(2.2) \qquad\qquad k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{L}) - n(D).$$

Clearly, spectral stability for (2.1) follows from $k_{Ham} = 0$, but such a condition is not necessary for spectral stability. For example, one might encounter a situation where $k_{Ham} = 2$ but with $k_i^- = 1$, which is an example of spectrally stable configuration with a nonzero $K_{Ham.}$. On the other hand, it is clear that if k_{Ham} is an odd integer, then $k_r \geq 1$, guaranteeing instability.

2.2. Spectral information about $\mathcal{JH}$. Due to the results in section 2.1, it becomes clear that we need a determination of a basis of $gker(\mathcal{JH})$. It turns out that it is helpful to introduce another Schrödinger operator, namely,

$$L = -\partial_x^2 + \sigma + 3\left(\beta - \frac{1}{c}\right)\varphi^2.$$

For context, this is the well-known operator L_+ if we were to consider the waves as solutions to the standard cubic NLS; see (1.3).

2.2.1. The spectra of L, L_2. For self-adjoint operator H acting on $L_{per}^2[0; T]$ with domain $D(H) = H^2([0; T])$, we have that its spectrum is purely discrete:

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \ldots.$$

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[4]In the applications, one needs to have an explicit form of such a basis anyway before any determination of the stability can be made. In a way, we shall need to check the finite dimensionality of $gker(\mathcal{IL})$.

Eigenvalues λ_i, $i = 0, 1, 2...$, correspond to the periodic eigenvalues, while μ_i, $i = 0, 1, 2...$, correspond to the semiperiodic eigenvalues. Then we have that $Hf = \lambda f$ has a solution of period T if and only if $\lambda = \lambda_i$, $i = 0, 1, 2, ...$, and a solution of period $2T$ if and only if $\lambda = \lambda_i$, $\lambda = \mu_i$, $i = 0, 1, 2, ...$

We start with the observation that $L\varphi' = 0$, which is obtained by differentiating (1.3) respect to x. Also, $L_2\varphi = 0$, which is just a restatement of (1.3). It is actually helpful, for the rest of the argument to list the lowest few eigenvalues for both operators L, L_2, where φ is either the dnoidal solution (1.5) or the snoidal solution (1.7). In fact, matters reduce to the explicit Hill operators

$$\Lambda_1 = -\partial_y^2 + 6k^2 sn^2(y, k),$$
$$\Lambda_2 = -\partial_y^2 + 2k^2 sn^2(y, k).$$

It is well known that the first four eigenvalues of Λ_1 with periodic boundary conditions on $[0, 4K(k)]$ are simple. These eigenvalues and corresponding eigenfunctions are given by

$$\begin{cases} \nu_0 = 2 + 2\kappa^2 - 2\sqrt{1 - \kappa^2 + \kappa^4}, & \phi_0(y) = 1 - (1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4})sn^2(y, \kappa), \\ \nu_1 = 1 + \kappa^2, & \phi_1(y) = cn(y, \kappa)dn(y, \kappa) = sn'(y, \kappa), \\ \nu_2 = 1 + 4\kappa^2, & \phi_2(y) = sn(y, \kappa)dn(y, \kappa) = -cn'(y, \kappa), \\ \nu_3 = 4 + \kappa^2, & \phi_3(y) = sn(y, \kappa)cn(y, \kappa) = -\kappa^{-2}dn'(y, \kappa). \end{cases}$$

Regarding Λ_2, the first three eigenvalues and the corresponding eigenfunctions with periodic boundary conditions on $[0, 4K(k)]$ are simple, and

$$\begin{aligned} \epsilon_0 = k^2, & \quad \theta_0(y) = dn(y, k), \\ \epsilon_1 = 1, & \quad \theta_1(y) = cn(y, k), \\ \epsilon_2 = 1 + k^2, & \quad \theta_2(y) = sn(y, k). \end{aligned}$$

In the dnoidal case, using that $\kappa^2 sn^2 x + dn^2 x = 1$ and (1.5), (1.6), we get

$$(2.3) \qquad\qquad L = \alpha^2[\Lambda_1 - (4 + \kappa^2)].$$

Note that in this case, ν_0 and ν_3 correspond to the periodic eigenvalues, while ν_1 and ν_2 correspond to the semiperiodic eigenvalues. It follows that the first two eigenvalues of the operator L, equipped with periodic boundary condition on $[-T, T]$, are simple; zero is the second eigenvalue; and $n(L) = 1$. In the snoidal case, using (1.7) and (1.8), we have

$$(2.4) \qquad\qquad L = \alpha^2[\Lambda_1 - (1 + \kappa^2)].$$

It follows again that zero is the second eigenvalue, and $n(L) = 1$.

Regarding the operator L_2, in the dnoidal case, using again (1.5), (1.6), we have that

$$L_2 = \alpha^2[\Lambda_2 - k^2],$$

whence, using the spectral information available for Λ_2, we conclude $L_2 \geq 0$, $n(L_2) = 0$.

In the snoidal case, we have

$$L_2 = \alpha^2[\Lambda_2 - (1 + k^2)],$$

whence the spectral description of Λ_2 allows us to conclude that $n(L_2) = 2$, with a simple eigenvalue at zero. We collect our results about L, L_2 in the following proposition.

Proposition 2. *Let φ be either the dnoidal wave* (1.5) *or the snoidal wave* (1.7). *Then*
- *in both the dnoidal and snoidal cases, the Hill operator L, equipped with periodic boundary conditions on $[-T, T]$, has Morse index $n(L) = 1$ and $Ker[L] = span[\varphi']$;*
- *in the dnoidal case, the operator L_2 has Morse index $n(L_2) = 0$, $Ker[L_2] = span[\varphi]$;*
- *in the snoidal case, the operator L_2 has Morse index $n(L_2) = 2$, $Ker[L_2] = span[\varphi]$.*

We are now ready to describe the kernel and the generalized kernel of $\mathcal{J}\mathcal{H}$.

2.2.2. Generalized kernel of $\mathcal{J}\mathcal{H}$.

Proposition 3. *Let φ be either the dnoidal wave* (1.5) *or the snoidal wave* (1.7). *Then the kernel of $\mathcal{H}$ is two dimensional, namely,*

$$(2.5) \qquad Ker[\mathcal{H}] = span\left[\begin{pmatrix} \varphi' \\ -\frac{2}{c}\varphi\varphi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix}\right].$$

In addition, under the assumption

$$(2.6) \qquad \langle L^{-1}\varphi, \varphi \rangle \neq 0,$$

we can identify all the generalized eigenvectors as follows:

$$(2.7) \qquad gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{H}) = span\left[\begin{pmatrix} \frac{1}{2c(c\beta-1)}\varphi \\ -\frac{\beta}{c(c\beta-1)}\varphi^2 \\ L_2^{-1}\varphi'. \end{pmatrix}, \begin{pmatrix} -L^{-1}\varphi \\ \frac{2}{c}\varphi L^{-1}\varphi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right].$$

Proof. We start with $Ker[\mathcal{H}]$. We have that $\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \ker\mathcal{H}$ if

$$(2.8) \qquad \left| \begin{aligned} & L_1 f + \varphi g = 0, \\ & \varphi f + \frac{c}{2} g = 0, \\ & L_2 h = 0. \end{aligned} \right.$$

From the second equation of (2.8), we have $g = -\frac{2}{c}\varphi f$, and plugging in the first equation, we get

$$0 = L_1 f + \varphi g = -\partial_x^2 f + \sigma f + \left(3\beta - \frac{1}{c}\right)\varphi^2 - \frac{2}{c}\varphi^2 f = Lf.$$

From Proposition 2, we get that all solutions are multiples of $f = \varphi'$ and $g = -\frac{2}{c}\varphi\varphi'$. From Proposition 2, we know that $Ker(L_2) = span[\varphi]$, and so, from the third equation of (2.8),

we have that another vector in $Ker(\mathcal{H})$ is $h = \varphi$. This identifies $Ker(\mathcal{H})$ for us as the one presented in (2.5).

We now turn to a representation for $Ker(\mathcal{JH})$. Consider $Ker(\mathcal{JH}) \ominus Ker(\mathcal{H})$. We set the equations for $\left(\begin{smallmatrix} f \\ g \\ h \end{smallmatrix}\right) \in Ker(\mathcal{JH}) \ominus Ker(\mathcal{H})$. We need to solve $\mathcal{H}\left(\begin{smallmatrix} f \\ g \\ h \end{smallmatrix}\right) \in Ker(\mathcal{J}) = span\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$. This is equivalent to $h = 0$ and

$$(2.9) \qquad \left| \begin{array}{l} L_1 f + \varphi g = 0, \\ \varphi f + \frac{c}{2} g = 1. \end{array} \right.$$

Solving it implies in a similar manner

$$f = -\frac{2}{c} L^{-1}\varphi, \quad g = \frac{2}{c}\left(1 + \frac{2}{c}\varphi L^{-1}\varphi\right).$$

This yields an additional, third vector in the representation of $Ker(\mathcal{JH})$. More specifically, we obtain

$$(2.10) \qquad Ker(\mathcal{JH}) = span\left\{ \begin{pmatrix} \varphi' \\ -\frac{2}{c}\varphi\varphi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} -L^{-1}\varphi \\ 1 + \frac{2}{c}\varphi L^{-1}\varphi \\ 0 \end{pmatrix} \right\}.$$

We now work on identifying the adjoint/generalized eigenvectors. We start with the next level adjoints e-vectors, namely, $Ker((\mathcal{JH})^2)$. First, we consider the equation

$$\mathcal{JH}\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \varphi' \\ -\frac{2}{c}\varphi\varphi' \\ 0 \end{pmatrix}.$$

This has solutions which are all multiples of

$$\begin{aligned} f &= \frac{1}{c^2} L^{-1}[\varphi^3] = \frac{1}{2c(c\beta - 1)}\varphi, \\ g &= -\frac{2}{c^2}\left(\frac{\varphi^2}{2} + \frac{\varphi L^{-1}[\varphi^3]}{c}\right) = -\frac{\beta}{c(c\beta - 1)}\varphi^2, \\ h &= L_2^{-1}\varphi', \end{aligned}$$

where we have used the identity $L\varphi = 2(\beta - \frac{1}{c})\varphi^3$. This gives a new element $\vec{\xi} \in Ker((\mathcal{JH})^2) \ominus Ker(\mathcal{JH})$, namely,

$$\vec{\xi} := \begin{pmatrix} \frac{1}{2c(c\beta-1)}\varphi \\ -\frac{\beta}{c(c\beta-1)}\varphi^2 \\ L_2^{-1}\varphi' \end{pmatrix}.$$

Next, we solve

$$\mathcal{JH}\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix}.$$

We obtain that all solutions are multiples of the vector

$$
(2.11) \qquad f = -L^{-1}\varphi, \quad g = \frac{2}{c}\varphi L^{-1}\varphi, \quad h = 0.
$$

We compare this with a similar element, already present in $Ker(\mathcal{JH})$. We conclude that we can consider instead the following new element $\vec{\eta} \in Ker((\mathcal{JH})^2) \ominus Ker(\mathcal{JH})$:

$$
\vec{\eta} = \begin{pmatrix} -L^{-1}\varphi \\ 1 + \frac{2}{c}\varphi L^{-1}\varphi \\ 0 \end{pmatrix} - \begin{pmatrix} -L^{-1}\varphi \\ \frac{2}{c}\varphi L^{-1}\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
$$

Finally, we solve the equation for the third eigenvector, with unknown $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}$:

$$
(2.12) \qquad \mathcal{JH}\Psi = \begin{pmatrix} -L^{-1}\varphi \\ 1 + \frac{2}{c}\varphi L^{-1}\varphi \\ 0 \end{pmatrix}.
$$

Taking into account that $\mathcal{JH}\Psi = \begin{pmatrix} L_2\Psi_3 \\ * \\ * \end{pmatrix}$ necessitates the solvability condition $L^{-1}\varphi \perp Ker[L_2] = span[\varphi]$. This means that as long as $\langle L^{-1}\varphi, \varphi \rangle \neq 0$, there are no further elements of $Ker((\mathcal{JH})^2) \ominus Ker(\mathcal{JH})$. All in all, we have established that

$$
(2.13) \qquad Ker((\mathcal{JH})^2) \ominus Ker(\mathcal{JH}) = span[\vec{\xi}, \vec{\eta}].
$$

Next, we show that

$$
(2.14) \qquad Ker((\mathcal{JH})^3) \ominus Ker((\mathcal{JH})^2) = \{0\}.
$$

Note that combining (2.14) and (2.13) with (2.10) yields the formula (2.7). So, it remains to show (2.14). To this end, we need to show that the equation

$$
(2.15) \qquad \zeta_1\vec{\xi} + \zeta_2\vec{\eta} = \mathcal{JH}\Psi = \begin{pmatrix} L_2\Psi_3 \\ 2\partial_x(\varphi\Psi_1 + \frac{c}{2}\Psi_2) \\ * \end{pmatrix}
$$

has no solutions if $(\zeta_1, \zeta_2) \neq (0,0)$. Note that the first equation in (2.15) reads $L_2\Psi_3 = \frac{\zeta_1}{2c(c\beta-1)}\varphi$. As $Ker(L_2) = span[\varphi]$, this forces a solvability condition, $\langle \varphi, \frac{\zeta_1}{2c(c\beta-1)}\varphi \rangle = 0$, which is impossible, unless $\zeta_1 = 0$. Now that we know that $\zeta_1 = 0$, the second equation in (2.15) reads

$$
2\partial_x(\varphi\Psi_1 + \frac{c}{2}\Psi_2) = \zeta_2.
$$

This implies $\varphi\Psi_1 + \frac{c}{2}\Psi_2 = \frac{\zeta_2}{2}x + const$. The left-hand side of this identity is $2T$ periodic, while the right-hand side is never $2T$ periodic, unless $\zeta_2 = 0$. Thus, we conclude that $\zeta_2 = 0$ as well, which establishes (2.14).

This completes the proof of Proposition 3. ∎

Next, we compute the Morse index of $\mathcal{H}$.

2.3. Morse index of $\mathcal{H}$. In the next proposition, we compute the Morse index of $\mathcal{H}$.

Proposition 4. We have the following formula for the Morse index $n(\mathcal{H})$:
- *If φ is the dnoidal wave given by (1.5), then $n(\mathcal{H}) = 1$.*
- *For the snoidal case, i.e. φ is given by (1.7), we have $n(\mathcal{H}) = 3$.*

Proof. Denote $\mathcal{H}_0 := \begin{pmatrix} L_1 & \varphi \\ \varphi & \frac{c}{2} \end{pmatrix}$. Clearly, $n(\mathcal{H}) = n(\mathcal{H}_0) + n(L_2)$. Taking into account the computation of $n(L_2)$ in Proposition 2 (which yields $n(L_2) = 0$ in the dnoidal case and $n(L_2) = 2$ in the snoidal case), it remains to show that $n(\mathcal{H}_0) = 1$ in both cases under consideration.

To this end, observe that we have the following expression for the quadratic form associated to $\mathcal{H}_0$:

$$
\begin{aligned}
(2.16) \qquad \left\langle \mathcal{H}_0 \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle &= \langle L_1 f, f \rangle + 2\langle \varphi f, g \rangle + \tfrac{c}{2}\langle g, g \rangle \\
&= \langle Lf, f \rangle + \int_{-T}^{T} \left[\sqrt{\tfrac{2}{c}} f + \sqrt{\tfrac{c}{2}} g \right]^2 dx.
\end{aligned}
$$

First, we confirm that $\mathcal{H}_0$ has at least one negative eigenvalue. Recall from Proposition 2 that $n(L) = 1$. Let us denote by h the eigenfunction of L corresponding to the negative eigenvalue. For $f = h$ and $g := -\frac{2}{c}h$ in (2.16), we get

$$
\left\langle \mathcal{H}_0 \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle = \langle Lh, h \rangle < 0.
$$

Hence, $\mathcal{H}_0$ has a negative eigenvalue. Thus, selecting $f \perp h$ and using the max-min characterization of eigenvalues, we have that the second smallest eigenvalue λ_1 satisfies the estimate

$$
\lambda_1(\mathcal{H}_0) \geq \inf_{(f,g)\perp(h,0):\|f\|^2+\|g\|^2=1} \left\langle \mathcal{H}_0 \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle \geq \inf_{f\perp h, \|f\|\leq 1} \langle Lf, f \rangle \geq 0
$$

since L has $n(L) = 1$, and so $\inf_{f \perp h} \langle Lf, f \rangle \geq 0$. That is, $n(\mathcal{H}_0) = 1$. ∎

3. Stability analysis of the waves. We start by analyzing the stability of the dnoidal waves. Our starting point is the instability Krein index count (2.2). Thus, it remains to determine the Morse index of the matrix D associated with it. Recall that, under the assumption (2.6), we have identified

$$
\vec{\psi}_1 = \begin{pmatrix} \frac{1}{2c(c\beta-1)}\varphi \\ -\frac{\beta}{c(c\beta-1)}\varphi^2 \\ \\ L_2^{-1}\varphi'. \end{pmatrix} ; \quad \vec{\psi}_2 = \begin{pmatrix} -L^{-1}\varphi \\ \frac{2}{c}\varphi L^{-1}\varphi \\ 0 \end{pmatrix} ; \quad \vec{\psi}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ,
$$

so that $gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{H}) = span[\vec{\psi}_1, \vec{\psi}_2, \vec{\psi}_3]$. By direct computations, we have

$$
\mathcal{H}\vec{\psi}_1 = \begin{pmatrix} 0 \\ -\frac{1}{2c}\varphi^2 \\ \varphi' \end{pmatrix} ; \quad \mathcal{H}\vec{\psi}_2 = \begin{pmatrix} -\varphi \\ 0 \\ 0 \end{pmatrix} ; \quad \mathcal{H}\vec{\psi}_3 = \begin{pmatrix} \varphi \\ \frac{c}{2} \\ 0 \end{pmatrix}
$$

and

$$(3.1) \qquad D_{11} = \langle \mathcal{H}\psi_1, \psi_1 \rangle = \langle L_2^{-1}\varphi', \varphi' \rangle + \frac{\beta}{2c^2(c\beta - 1)} \langle \varphi^2, \varphi^2 \rangle,$$

$$(3.2) \qquad D_{12} = D_{21} = \langle \mathcal{H}\vec{\psi}_1, \vec{\psi}_2 \rangle = -\frac{1}{2c(c\beta - 1)} \langle \varphi, \varphi \rangle,$$

$$(3.3) \qquad D_{22} = \langle \mathcal{H}\vec{\psi}_2, \vec{\psi}_2 \rangle = \langle L^{-1}\varphi, \varphi \rangle, D_{33} = \langle \mathcal{H}\vec{\psi}_3, \vec{\psi}_3 \rangle = cT,$$

$$(3.4) \qquad D_{13} = D_{31} = \langle \mathcal{H}\vec{\psi}_1, \vec{\psi}_3 \rangle = -\frac{1}{2c} \langle \varphi, \varphi \rangle,$$

$$(3.5) \qquad D_{23} = D_{32} = \langle \mathcal{H}\vec{\psi}_2, \vec{\psi}_3 \rangle = 0.$$

3.1. Dnoidal waves. According to instability index count formula (2.2) and Proposition 4, which implies that $n(\mathcal{H}) = 1$, the stability analysis reduces to establishing that $n(D) = 1$. Indeed, in such a case, the right-hand side of (2.2) is zero and thus would rule out all potential instabilities on the left-hand side.

We proceed to evaluating the elements of the matrix D. In fact, we shall need to only compute $D_{22} = \langle L^{-1}\varphi, \varphi \rangle$, which we will now show is negative. To this end, start with the identity $L\varphi' = 0$. In order to construct Green's function for the operator L, we need a solution $\psi : L\psi = 0$. In principle, the following function provides such a solution:

$$(3.6) \qquad \psi(x) = \varphi'(x) \int^x \frac{1}{\varphi'^2(s)} ds, \quad \left| \begin{matrix} \varphi' & \psi \\ \varphi'' & \psi' \end{matrix} \right| = 1.$$

Unfortunately, as φ' has zeros in the interval of integration, this integral is not well defined. Instead, we use the standard roundabout way of making the definition of such integral well defined, which involves integration by parts. Specifically, we proceed by using the identities

$$\frac{1}{cn^2(y,\kappa)} = \frac{1}{dn(y,\kappa)} \frac{\partial}{\partial_y} \frac{sn(x,\kappa)}{cn(y,\kappa)}, \quad \frac{1}{sn^2(y,\kappa)} = -\frac{1}{dn(y,\kappa)} \frac{\partial}{\partial_y} \frac{cn(x,\kappa)}{sn(y,\kappa)}.$$

Integrating by parts yields the alternative, well-defined expression for ψ, which is formally equivalent to (3.6):

$$(3.7) \quad \psi(x) = \frac{1}{\alpha^2\kappa^2\varphi_0} \left[\frac{1 - 2sn^2(\alpha x, \kappa)}{dn(\alpha x, \kappa)} - \alpha\kappa^2 sn(\alpha x, \kappa) cn(\alpha x, \kappa) \int_0^x \frac{1 - 2sn^2(\alpha s, \kappa)}{dn^2(\alpha s, \kappa)} ds \right].$$

Thus, we may construct Green's function as follows:

$$L^{-1}f = \varphi' \int_0^x \psi(s)f(s)ds - \psi(s) \int_0^x \varphi'(s)f(s)s + C_f\psi(x),$$

where C_f is chosen, so that $L^{-1}f$ has the same period as φ. After integrating by parts, we get

$$(3.8) \qquad \langle L^{-1}\varphi, \varphi \rangle = -\langle \varphi^3, \psi \rangle + \frac{\varphi^2(T) + \varphi(0)^2}{2} \langle \varphi, \psi \rangle + C_\varphi \langle \varphi, \psi \rangle.$$
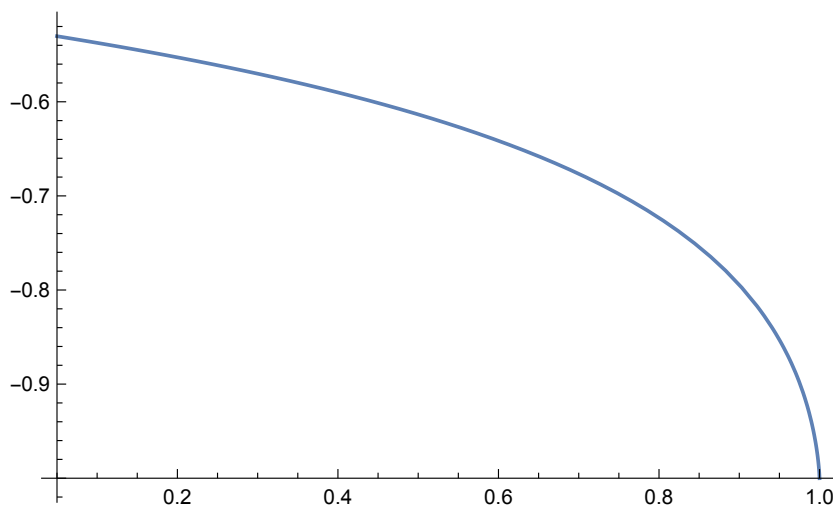
Figure 1. *Graph of* $\kappa \to \frac{E^2(\kappa) - (1-\kappa^2)K^2(\kappa)}{2(1-\kappa^2)K(\kappa) - (2-\kappa^2)E(\kappa)}$.

Integrating by parts yields

$$\langle \psi'', \varphi \rangle = 2\psi'(T)\varphi(T) + \langle \psi, \varphi'' \rangle.$$

Using that $L\varphi = 2(\beta - \frac{1}{c})\varphi^3$, we get

$$\langle \psi, \varphi^3 \rangle = \frac{c}{c\beta - 1}\psi'(T)\varphi(T).$$

Using that $\int_0^{K(\kappa)} \frac{1 - 2sn^2(x)}{dn^2(x)}dx = \frac{1}{\kappa^2(1-\kappa^2)}[2(1-\kappa^2)K(\kappa) - (2-\kappa^2)E(\kappa)]$, we get

$$\langle \varphi, \psi \rangle = \frac{1}{\alpha^3\kappa^2}[E(\kappa) - K(\kappa)],$$

(3.9) $$\langle \varphi^3, \psi \rangle = \frac{1}{\alpha}\frac{c}{c\beta - 1}[2(1-\kappa^2)K(\kappa) - (2-\kappa^2)E(\kappa)],$$

$$C_\varphi = -\frac{\varphi''(T)}{2\psi'(T)}\langle \varphi, \psi \rangle + \frac{\varphi^2(T) - \varphi^2(0)}{2}.$$

Taking into account $\frac{\varphi_0^2}{\alpha^2} = \frac{2c}{1-c\beta}$, we get

(3.10) $$D_{22} = \langle L^{-1}\varphi, \varphi \rangle = \frac{1}{\alpha}\frac{1}{\frac{1}{c} - \beta}\frac{E^2(\kappa) - (1-\kappa^2)K^2(\kappa)}{2(1-\kappa^2)K(\kappa) - (2-\kappa^2)E(\kappa)} < 0$$

(see Figure 1), so in particular, the condition (2.6) is satisfied. Also, since $D_{22} < 0$ for all values of the parameters, it is clear that $D_{22} = \langle De_2, e_2 \rangle \leq \inf_{\xi \in \mathbf{R}^3 : \|\xi\|=1}\langle D\xi, \xi \rangle$, whence[5] $n(D) \geq 1$. As discussed, this implies that the dnoidal waves are spectrally stable.

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[5]By the way, by (2.2), this actually implies that $n(D) = 1$.

3.2. Snoidal waves. According to the formulas (3.1), (3.2), (3.3), and (3.4), we shall need to compute $\langle L_2^{-1}\varphi', \varphi' \rangle$, $\langle L^{-1}\varphi, \varphi \rangle$, and $\int \varphi^2, \int \varphi^4$.

To this end, we start with the computation of $\langle L^{-1}\varphi, \varphi \rangle$. We have $L\varphi' = 0$ and $L\psi = 0$, where $\psi(x) = \varphi'(x) \int^x \frac{1}{\varphi'^2(s)} ds$. Using that

$$\frac{1}{cn^2(\alpha x)} = \frac{1}{\alpha dn(\alpha x)} \frac{\partial}{\partial x} \frac{sn(\alpha x)}{cn(\alpha x)},$$

we get the odd function ψ:

$$\psi(x) = \frac{1}{\varphi_0 \alpha^2 (1 - \kappa^2)} \left[sn(\alpha x) - \alpha \kappa^2 cn(\alpha x) dn(\alpha x) \int_0^x \frac{1 + sn^2(\alpha s)}{dn^2(\alpha s)} ds \right].$$

Integration by parts yields the formulas

$$\langle L^{-1}\varphi, \varphi \rangle = -\langle \varphi^3, \psi \rangle + C_\varphi \langle \varphi, \psi \rangle,$$
$$\langle \psi'', \varphi \rangle = -2\varphi'(T)\psi(T) + \langle \psi, \varphi'' \rangle.$$

A direct calculation shows that $L\varphi = 2\left(\beta - \frac{1}{c}\right)\varphi^3$, whence

$$\langle \varphi^3, \psi \rangle = -\frac{c}{c\beta - 1} \varphi'(T)\psi(T).$$

Now we have the relations

$$\begin{cases} \psi(T) = \frac{\kappa^2}{\varphi_0 \alpha^2 (1-\kappa^2)} \int_0^{2K(\kappa)} \frac{1+sn^2(x)}{dn^2(x)} dx, \\[2mm] \varphi'(T) = -\varphi_0 \alpha, \quad C_\varphi = -\frac{\varphi'(T)}{2\psi(T)} \langle \varphi, \psi \rangle, \\[2mm] \alpha^2 = -\frac{\sigma}{1+\kappa^2}, \quad \varphi_0^2 = \frac{2c\sigma\kappa^2}{(1-c\beta)(1+\kappa^2)}. \end{cases}$$

Integration by parts allows us to compute

$$\langle \varphi, \psi \rangle = \frac{1}{\alpha^3(1-\kappa^2)} \left[\int_0^{2K(\kappa)} sn^2(x) dx + \int_0^{2K(\kappa)} \frac{1+sn^2(x)}{dn^2(x)} dx - 2K(\kappa) \right].$$

Putting all this together, we have

$$\begin{cases} \langle \varphi^3, \psi \rangle = \frac{1}{\alpha} \frac{c}{c\beta-1} \frac{\kappa^2}{1-\kappa^2} \int_0^{2K(\kappa)} \frac{1+sn^2(x)}{dn^2(x)} dx, \\[3mm] C_\varphi \langle \varphi, \psi \rangle = \frac{1}{\alpha} \frac{c}{c\beta-1} \frac{1}{(1-\kappa^2) \int_0^{2K(\kappa)} \frac{1+sn^2(x)}{dn^2(x)} dx} \left[\int_0^{2K(\kappa)} sn^2(x) dx + \int_0^{2K(\kappa)} \frac{1+sn^2(x)}{dn^2(x)} dx - 2K(\kappa) \right]^2, \end{cases}$$

whence finally

(3.11)
$$\langle L^{-1}\varphi, \varphi \rangle = \frac{1}{\alpha} \frac{1}{\left(\beta - \frac{1}{c}\right)} F(\kappa),$$
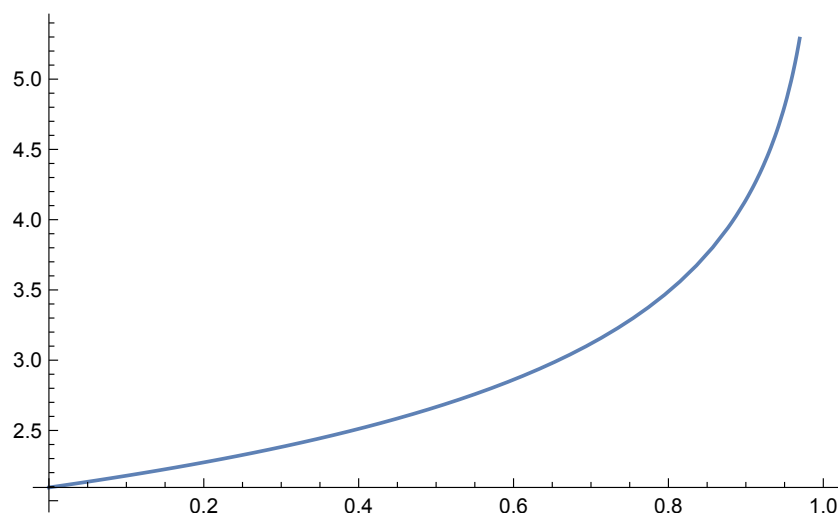
Figure 2. *Graph of $F(\kappa)$.*

where

$$F(\kappa) = \left[\frac{\left(\int_0^{2K(\kappa)} sn^2(x)dx + \int_0^{2K(\kappa)} \frac{1+sn^2(x)}{dn^2(x)}dx - 2K(\kappa)\right)^2}{(1-k^2)\int_0^{2K(\kappa)} \frac{1+sn^2(x)}{dn^2(x)}dx} - \frac{\kappa^2}{1-\kappa^2)} \int_0^{2K(\kappa)} \frac{1+sn^2(x)}{dn^2(x)}dx \right]$$

$$= 2K(\kappa) + 2E(\kappa)\left(-1 + \frac{\kappa^2 E(\kappa)}{(\kappa^2+1)E(\kappa) - (1-\kappa^2)K(\kappa)}\right).$$

We have plotted it using Mathematica; see Figure 2. From this, it becomes clear that $\langle L^{-1}\varphi, \varphi \rangle > 0$. In particular, the condition (2.6) holds, whence the conclusions of Proposition 3 hold.

We will now compute $\langle L_2^{-1}\varphi', \varphi' \rangle$. We have $L_2\varphi = 0$, and $\psi = \varphi \int^x \frac{1}{\varphi^2}ds$ is also solution of $L_2\psi = 0$. Using the identity

$$\frac{1}{sn^2(y,\kappa)} = -\frac{1}{\alpha dn(y,\kappa)}\frac{\partial}{\partial y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integration by parts, we can alternatively express ψ as follows:

$$\psi(x) = -\frac{1}{\alpha\varphi_0}\left[\frac{cn(\alpha x)}{dn(\alpha x)} - \alpha\kappa^2 sn(\alpha x,\kappa)\int_0^x \frac{cn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)}ds\right].$$

Using that φ is odd function and ψ is even function, we get

$$\langle L_2^{-1}\varphi', \varphi' \rangle = -\int_{-T}^T \varphi^2\varphi'\psi dx + C_{\varphi'}\int_{-T}^T \varphi'\psi dx,$$

$$C_{\varphi'} = -\frac{\varphi'(T)}{2\psi'(T)}\int_{-T}^T \varphi'\psi dx.$$

Hence,

$$\langle L_2^{-1}\varphi', \varphi' \rangle = -\int_{-T}^{T} \varphi^2 \varphi' \psi dx - \frac{\varphi'(T)}{2\psi'(T)} \left(\int_{-T}^{T} \varphi' \psi dx \right)^2.$$

In addition, we have

$$\begin{cases} \varphi'(T) = -\alpha\varphi_0, \quad \psi'(T) = -\frac{\kappa^2}{\varphi_0} \int_0^{2K(\kappa)} \frac{cn^2 x}{dn^2 x} dx, \\[2mm] \int_{-T}^{T} \varphi'(x)\psi(x)dx = -\frac{1}{\alpha} \left[\int_0^{2K(\kappa)} cn^2(x)dx + \int_0^{2K(\kappa)} \frac{cn^2(x)}{dn^2(x)} dx \right], \\[2mm] \int_{-T}^{T} \varphi^2\varphi'(x)\psi(x)dx = -\frac{\varphi_0^2}{\alpha} \left[2\int_0^{2K(\kappa)} sn^2(x)cn^2(x)dx + \frac{\kappa^2}{2} \int_0^{2K(\kappa)} \frac{sn^4(x)cn^2(x)}{dn^2(x)} dx \right], \\[2mm] \alpha^2 = -\frac{\sigma}{1+\kappa^2}, \quad \varphi_0^2 = \frac{2(-\sigma)\kappa^2}{(\beta-\frac{1}{c})(1+\kappa^2)}. \end{cases}$$

Putting all this together, we get

$$\langle L_2^{-1}\varphi', \varphi' \rangle = \frac{\varphi_0^2}{\alpha} \left[2\int_0^{2K(\kappa)} sn^2(x)cn^2(x)dx + \frac{\kappa^2}{2} \int_0^{2K(\kappa)} \frac{sn^4(x)cn^2(x)}{dn^2(x)} dx \right] -$$

$$- \frac{\varphi_0^2}{\alpha} \frac{\left(\int_0^{2K(\kappa)} cn^2(x)dx + \int_0^{2K(\kappa)} \frac{cn^2(x)}{dn^2(x)} dx \right)^2}{2\kappa^2 \int_0^{2K(\kappa)} \frac{cn^2 x}{dn^2 x} dx}.$$

Finally, we have

$$\langle \varphi, \varphi \rangle = \frac{2\varphi_0^2}{\alpha} \int_0^{2K(\kappa)} sn^2(x)dx,$$

$$\langle \varphi^2, \varphi^2 \rangle = \frac{2\varphi_0^4}{\alpha} \int_0^{2K(\kappa)} sn^4(x)dx.$$

We now compute $\det(D)$ in the regime $\beta = \frac{1}{c} + \epsilon, 0 < \epsilon << 1$. We will establish the following proposition, regarding the matrix D, introduced in (3.1), (3.2), (3.3), (3.4), (3.5).

Proposition 5. *Fix* $c \neq 0, \sigma < 0$. *Then there exists* $\epsilon_0 = \epsilon_0(c, \sigma) > 0$, *so that for all* $0 < \epsilon < \epsilon_0$ *and* $\beta = \frac{1}{c} + \epsilon$, *we have that* $\det(D) > 0$.

Before we proceed with the proof of Proposition 5, let us finish the proof of Theorem 2. That is, we show that the snoidal waves are spectrally unstable.

We argue as follows: For very small ϵ, we have from Proposition 5 that $\det(D) > 0$, whence the symmetric matrix D has either two negative eigenvalues and a positive one (in which case $n(D) = 2$) or three positive eigenvalues or $n(D) = 0$.

By (2.2), we conclude that either $k_{Ham} = 3 - 2 = 1$ or $k_{Ham.} = n(\mathcal{L}) - n(D) = 3 - 0 = 3$. This implies that there is at least one real instability. In fact, for systems with $k_{Ham} = 1$, this is obvious. If $k_{Ham.} = 3$, the possibilities are as follows: Three real instabilities, one real instability and two complex/oscillatory instabilities, and one real instability and a pair of purely imaginary eigenvalues of negative Krein signature. Unfortunately, the instability index

theory outlined in section 2.1 does not allow us to specify precisely which situation we find ourselves in, even for $\epsilon << 1$. We claim that we can nevertheless confirm that the waves are unstable in the sense that the eigenvalue problem (1.14) has at least one positive eigenvalue.

To this end, consider the parameters c, σ fixed and β as a bifurcation parameter. We start with the observation made above that for small $0 < \epsilon << 1$ (that is, β slightly bigger than $\frac{1}{c}$), we have at least one real unstable eigenvalue. Allowing the parameter $\beta > \frac{1}{c}$ to increase, the Krein index may of course change since our analysis showed that $n(D) = 0$ or $n(D) = 2$ only for $0 < \epsilon << 1$. But regardless of that, there will always be at least one real instability. This is due to Proposition 3, which asserts that the eigenvalue at zero is of algebraic multiplicity five for all values of the parameters, with three eigenvectors and two generalized eigenvectors described there. The only scenario for the real instability present at $\epsilon << 1$ to become stable is by passing through the zero generalized eigenspace for some intermediate value of β, which would have been detected by our analysis in Proposition 3. As we have shown, this does not happen. Thus, the real and positive eigenvalue is present for all $\beta > \frac{1}{c}$, and the snoidal waves φ are unstable. This completes the proof of Theorem 2, and it remains to establish Proposition 5.

3.3. Proof of Proposition 5. We first calculate $\det(D)$. By the specifics of it (see (3.1), (3.2), (3.3), (3.4), (3.5)), we have

$$\det(D) = D_{33} \det(\tilde{D}) - \left(\frac{1}{2c}\int \varphi^2\right)^2 D_{22},$$

where $\tilde{D} = \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix}$. Taking into account the form of $\varphi_0^2 = const.\epsilon^{-1}$, we have

$$\langle L_2^{-1}\varphi', \varphi'\rangle = const.\epsilon^{-1} + O(\epsilon^{-2}),$$

while $\langle \varphi^2, \varphi^2\rangle = const.\epsilon^{-2} + O(\epsilon^{-1})$. So, we can conclude that

$$D_{11} = \frac{\beta}{2c^2(c\beta - 1)}\langle \varphi^2, \varphi^2\rangle + \langle L_2^{-1}\varphi', \varphi'\rangle = \frac{\beta}{2c^2(c\beta - 1)}\langle \varphi^2, \varphi^2\rangle + O(\epsilon^{-1}),$$

whence

$$\det(\tilde{D}) = D_{11}D_{22} - D_{12}^2 =$$

$$= \frac{\beta\varphi_0^4}{\alpha c(c\beta - 1)}\left(\int_0^{2K(k\kappa)} sn^4(x)dx\right)\frac{F(\kappa)}{\alpha(c\beta - 1)} - \frac{\varphi_0^4}{\alpha^2 c^2(c\beta - 1)^2}\left(\int_0^{2K(k\kappa)} sn^2(x)dx\right)^2 + O(\epsilon^{-2})$$

$$= \frac{\varphi_0^4}{\alpha^2(c\beta - 1)^2}\left[\frac{\beta}{c}\left(\int_0^{2K(k\kappa)} sn^4(x)dx\right)F(\kappa) - \frac{1}{c^2}\left(\int_0^{2K(k\kappa)} sn^2(x)dx\right)^2\right] + O(\epsilon^{-2}).$$

Clearly, the first expression is of the form $const.\epsilon^{-4}$, and hence it is dominant in the regime $\beta = \frac{1}{c} + \epsilon, 0 < \epsilon << 1$. Furthermore, the assignment $\beta = \frac{1}{c} + \epsilon$ allows us to further extract a leading-order term as follows:

$$\det(\tilde{D}) = \frac{\varphi_0^4}{\alpha^2 c^2(c\beta - 1)^2}\left[\left(\int_0^{2K(k\kappa)} sn^4(x)dx\right)F(\kappa) - \left(\int_0^{2K(k\kappa)} sn^2(x)dx\right)^2\right] + O(\epsilon^{-3}).$$
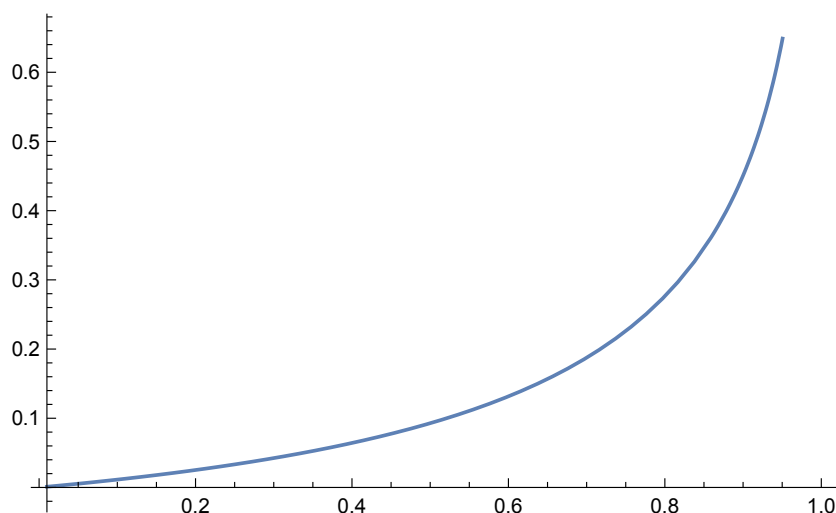
Figure 3. *Graph of $H(\kappa)$.*

We have computed this last function of κ in MATHEMATICA, and we have obtained the following explicit expression for it:

$$H(\kappa) = \left(\int_0^{2K(k\kappa)} sn^4(x)dx \right) F(\kappa) - \left(\int_0^{2K(k\kappa)} sn^2(x)dx \right)^2 =$$

$$= \frac{(2(\kappa^2+2)K(k) - 4(\kappa^2+1)E(k))\left(2K(k) + 2E(k)\left(\frac{\kappa^2 E(k)}{(\kappa^2-1)K(k)+(\kappa^2+1)E(k)} - 1\right)\right) - 12(E(k)-K(k))^2}{3\kappa^4}.$$

Plotting this leads to the conclusion $H[\kappa] > 0$ (see Figure 3) and thus to a leading-order

$$\det(\tilde{D}) = C(k, \sigma, c)\epsilon^{-4} + O(\epsilon^{-3})$$

as $\epsilon : 0 < \epsilon << 1$. In addition, observe that by (3.11), we have that

$$\left(\frac{1}{2c} \int \varphi^2 \right)^2 D_{22} = O(\epsilon^{-3}).$$

Accordingly, we have that

$$\det(D) = D_{33} \det(\tilde{D}) - \left(\frac{1}{2c} \int \varphi^2 \right)^2 D_{22} = C(k, \sigma, c)\epsilon^{-4} + O(\epsilon^{-3}),$$

whence $\det(D) > 0$ for all small enough $\epsilon > 0$. This completes the proof of Proposition 5.
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