Representing and computing the B-derivative of the piecewise-differentiable flow of a class of nonsmooth vector fields

George Council
Department of Mechanical Engineering
Carnegie Mellon University
Pittsburgh, PA, USA
(gcouncil@andrew.cmu.edu).

Shai Revzen
Department of Electrical Engineering
and Computer Science
University of Michigan
Ann Arbor, MI, USA
(shrevzen@umich.edu).

Samuel A. Burden*
Department of Electrical & Computer Engineering
University of Washington
Seattle, WA, USA
(sburden@uw.edu, http://faculty.uw.edu/sburden).

This paper concerns first-order approximation of the piecewise-differentiable flow generated by a class of nonsmooth vector fields. Specifically, we represent and compute the Bouligand (or B-)derivative of the piecewise-differentiable flow generated by a vector field with event-selected discontinuities. Our results are remarkably efficient: although there are factorially many "pieces" of the derivative, we provide an algorithm that evaluates its action on a tangent vector using polynomial time and space, and verify the algorithm's correctness by deriving a representation for the B-derivative that requires "only" exponential time and space to construct. We apply our methods in two classes of illustrative examples: piecewise-constant vector fields and mechanical systems subject to unilateral constraints.

1 Introduction

First-order approximations – i.e. derivatives – are a foundational tool for analysis and synthesis in smooth dynamical and control systems. For instance, derivatives play a crucial rôle in: stability via spectral [1, Ch. 8.3] or Lyapunov [2, Ch. 5] methods; controllability via linearization [1, Ch. 8.7] or Frobenius/Chow [2, Ch. 8/Ch. 11] techniques; optimality via stationarity [3, Ch. 1] or Pontryagin [4, Ch. 1] principles; identifiability via adaptation [5, Ch. 2] or Expectation-Maximization [6, Ch. 10] methods. These tools all depend on the existence of a computationally-amenable representation for the first-order approximation of smooth system dynamics – namely, the *Fréchet* (or F-)derivative of the system's smooth flow [7, Ch. 5.6], which derivative is a continuous linear function of tangent vectors.[1]

By definition, nonsmooth systems do not generally enjoy existence (let alone computational amenability) of first-order approximations. Restricting to the class of (so-called [8, Def. 1, 2]) *event-selected* C^r (EC^r) vector fields that (i) are smooth except along a finite number of surfaces of discontinuity and (ii) preclude *sliding* [9, 10] or *branching* [11, Def. 3.11] through a transversality condition, we obtain flows that are piecewise-differentiable [8, Thm. 4] (specifically, *piecewise-*C^r (PC^r) [12, Ch. 4.1]). By virtue of their piecewise-differentiability, these flows admit a first-order approximation, termed the *Bouligand* (or B-)derivative, which derivative is a continuous *piecewise*-linear function of tangent vectors [12, Ch. 3, 4]. This paper is concerned with the efficient representation and computation of this piecewise-linear first-order approximation.

Our contributions are twofold: (i) we construct a representation for the B-derivative of the PC^r flow generated by an EC^r vector field; (ii) we derive an algorithm that evaluates the B-derivative on a given tangent vector. Although there are factorially many "pieces" of the derivative, we (i) represent it using exponential time and space and (ii) compute it using polynomial time and space. In an effort to make our results as accessible and useful as possible, we provide a concise summary of the algorithm in Sec. 2 and apply our methods in Sec. 3 *before* rehearsing the technical background in Sec. 4 needed to derive the representation in Sec. 5 and

*This material is based upon work supported by the U. S. Army Research Laboratory and the U. S. Army Research Office under grant number W911NF-16-1-0158 and by the U. S. National Science Foundation Award #1836819 and #1924303.

[1] We emphasize both properties of the classical derivative since the generalized derivative we consider retains one while relaxing the other.

verify the algorithm's correctness in Sec. 6.

We emphasize that our methods are most useful when there are more than two surfaces of discontinuity, as representation and computation of first-order approximations in the 1- and 2-surface cases have been investigated extensively [13–18], and these cases do not benefit from the complexity savings touted above. Previously, we established existence of the piecewise-linear first-order approximation of the flow [8, Rem. 1] and provided an inefficient scheme to evaluate each of its "pieces" [8, Sec. 7] in the presence of an arbitrary number of surfaces of discontinuity. To the best of our knowledge, the present paper contains the first representation for the B-derivative of the PC^r flow of a general EC^r vector field and polynomial-time algorithm to compute it.

2 Algorithm

The goal of this paper is to obtain an algorithm that efficiently computes the derivative of a class of nonsmooth flows. This computational task and our solution are easy to describe, yet verifying the algorithm's correctness requires significant technical overhead. Thus, the remainder of this section will be devoted to specifying the algorithm and the problem it solves using minimal notation and terminology. Subsequent sections will provide technical details – which may be of interest in their own right – that prove the algorithm is correct.

2.1 Problem statement

Given a vector field $F : \mathbb{R}^d \to T\mathbb{R}^d$ and a trajectory $x : [0,\infty) \to \mathbb{R}^d$ satisfying[2]

$$\forall t \geq 0 : x_t = x_0 + \int_0^t F(x_\tau)\, d\tau, \tag{1}$$

our goal is to approximate how x_t varies with respect to x_0 to first order for a given $t > 0$. Formally, with $\phi : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ denoting the *flow* of F satisfying

$$\forall t \geq 0, x_0 \in \mathbb{R}^d : \phi_t(x_0) = x_0 + \int_0^t F(\phi_\tau(x_0))\, d\tau, \tag{2}$$

our goal is to evaluate the directional derivative $D\phi_t(x_0; \delta x_0)$ given $t > 0$, $\delta x_0 \in T_{x_0}\mathbb{R}^d$:

$$D\phi_t(x_0; \delta x_0) = \lim_{\alpha \to 0^+} \frac{1}{\alpha} \left(\phi_t(x_0 + \alpha\, \delta x_0) - \phi_t(x_0)\right) \tag{3}$$

Specifically, we seek to evaluate this derivative for vector fields that are smooth everywhere except a finite collection of surfaces where they are allowed to be discontinuous. We will first recall how to obtain the derivative in the presence of zero (Sec. 2.2) and one (Sec. 2.3) surfaces of discontinuity before presenting our algorithm, which is applicable in the presence of an arbitrary number of surfaces of discontinuity (Sec. 2.4).

[2] In this section (only), we will denote time dependence using subscripts rather than parentheses to minimize the notational overhead.

2.2 Continuously-differentiable vector field

If F is continuously differentiable on the trajectory x, the derivative $\delta x_t = D\phi_t(x_0; \delta x_0)$ satisfies the linear time-varying *variational equation* [19, Appendix B]

$$\forall t \geq 0 : \delta x_t = \delta x_0 + \int_0^t DF(x_\tau) \cdot \delta x_\tau\, d\tau, \tag{4}$$

whence $\delta x_t = D\phi_t(x_0; \delta x_0)$ can be approximated to any desired precision in polynomial time by applying numerical simulation algorithms [19, Ch. 4] to Eq. (1), Eq. (4).

2.3 Single surface of discontinuity

If F is continuously differentiable everywhere except a smooth codimension-1 submanifold $H \subset \mathbb{R}^d$ that intersects the trajectory x transversally at only one point x_s, $s \in (0,t)$, the continuous-time equation Eq. (4) is augmented by the discrete-time update [13, Eq. (58)],

$$\delta x_s^+ = \left(I_d + \frac{(F^+ - F^-) \cdot \eta^\top}{\eta^\top \cdot F^-}\right) \cdot \delta x_s^- = M \cdot \delta x_s^-, \tag{5}$$

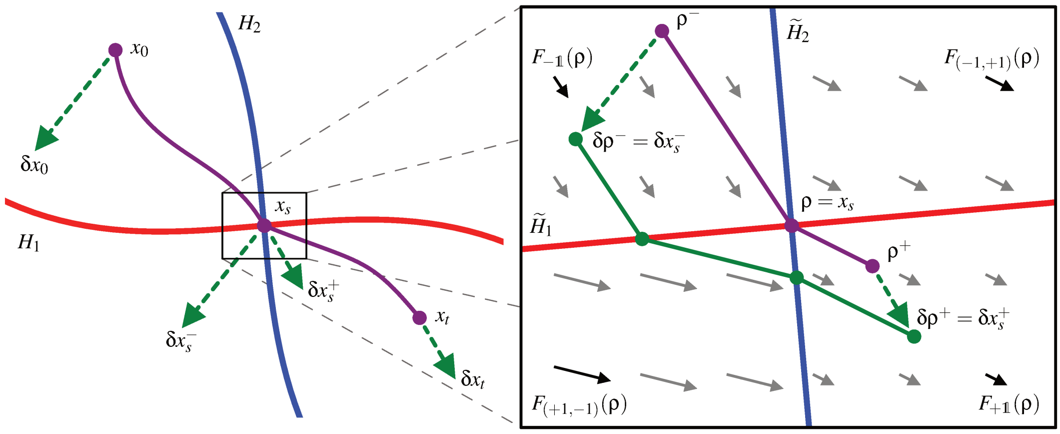
where $\delta x_s^\pm = \lim_{\tau \to s^\pm} \delta x_\tau$ and $F^\pm = \lim_{\tau \to s^\pm} F(x_\tau)$ denote the limiting values of δx_τ and $F(x_\tau)$ at s from the right $(+)$ and left $(-)$ and $\eta \in \mathbb{R}^d$ is any vector orthogonal to surface H at x_s; $M \in \mathbb{R}^{d \times d}$ is termed the *saltation matrix* [17, Eq. (2.76)], [20, Eq. (7.65)]. The desired derivative is

$$D\phi_t(x_0; \delta x_0) = D\phi_{t-s}(x_s) \cdot M \cdot D\phi_s(x_0) \cdot \delta x_0, \tag{6}$$

where $D\phi_{t-s}(x_s), D\phi_s(x_0) \in \mathbb{R}^{d \times d}$ can be approximated by simulating Eq. (2), Eq. (4) since the flow is smooth away from time s. Computing the saltation matrix M requires $O(d^2)$ time and space, but evaluating its action on δx_s^- in Eq. (5) requires only $O(d)$ time and space.

2.4 Multiple surfaces of discontinuity

If F is continuously differentiable everywhere except a finite set of smooth codimension-1 submanifolds $\{H_j\}_{j=1}^n$ that intersect the trajectory x transversally at only one point x_s (see Fig. 1(a) for an illustration when $n = 2$), $s \in (0,t)$, we showed in [8, Eq. (65)] that the discrete-time update Eq. (5) is applied once for each surface. However, the order in which the updates are applied, and the limiting values of the vector field used to determine each update's saltation matrix, depend on δx_0. If the surfaces intersect transversally, there are $n!$ different saltation matrices determined by 2^n vector field values, so considering all update orders requires factorial time and space. To make these observations precise and specify the notation employed in figs. 1 and 2, we formally



(a) continuous-time variational dynamics (4)
$\delta\dot{x}_\tau = DF(x_\tau)\cdot\delta x_\tau$

(b) discrete-time variational dynamics (8)
$\delta x_s^+ = \delta\rho^+ = B(\delta\rho^-) = B(\delta x_s^-)$

Fig. 1. Variational dynamics that determine B-derivative of planar EC^r vector field's PC^r flow (8). (a) Vector field $F : \mathbb{R}^2 \to T\mathbb{R}^2$ is smooth everywhere except smooth codimension-1 submanifolds $H_1, H_2 \subset \mathbb{R}^2$ that intersect transversally at $x_s \in \mathbb{R}^2$, generating a piecewise-differentiable flow $\phi : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\phi_\tau(x_0) = x_\tau$ for all $\tau \in [0,t]$, i.e. F is EC^r and ϕ is PC^r [8]. The B-derivative $D\phi_t(x_0;\delta x_0) = \delta x_t$ is determined as in (10) by the continuous-time variational dynamics $\delta\dot{x}_\tau = DF(x_\tau)\cdot\delta x_\tau$ and the discrete-time variational dynamics $\delta x_s^+ = B(\delta x_s^-)$. The algorithms in Fig. 2 evaluate the piecewise-linear function B using the auxiliary system in (b) determined by the tangent planes $\tilde{H}_1, \tilde{H}_2$ and vector field limits $F_b(\rho)$ in (9) for $b \in \{(-\mathbb{1},(+1,-1),(-1,+1),+\mathbb{1}\} = \{-1,+1\}^2$.

define the class of nonsmooth vector fields considered in this paper [8, Defs. 1, 2].

Definition 1. *(event-selected C^r (EC^r) vector field) A vector field $F : D \to TD$ defined on an open domain $D \subset \mathbb{R}^d$ is* event-selected C^r with respect to $h \in C^r(U,\mathbb{R}^n)$ at $\rho \in \mathbb{R}^d$ *if $U \subset D$ is an open neighborhood of ρ and:*

1. *(event functions) there exists $f > 0$ such that $Dh(x)\cdot F(x) \geq f$ for all $x \in U$;*
2. *(smooth extension) for all $b \in \{-1,+1\}^n = B_n$, with*

$$D_b = \left\{x \in U : b_j(h_j(x) - h_j(\rho)) \geq 0\right\}, \quad (7)$$

$F|_{\mathrm{Int}\,D_b}$ admits a C^r extension $F_b : U \to TU$.

Our algorithms in Fig. 2 compute

$$\delta x_s^+ = \delta\rho^+ = B(\delta\rho^-) = B(\delta x_s^-) \quad (8)$$

given $\delta\rho^- = \delta x_s^- \in \mathbb{R}^d$, normals $\{\eta_j = Dh_j(\rho)\}_{j=1}^n \subset \mathbb{R}^d$ at x_s to surfaces $\{H_j = h_j^{-1}(\rho)\}_{j=1}^n$, and a function $\Gamma : \{-1,+1\}^n \to \mathbb{R}^d$ that evaluates limits of F at $\rho = x_s$,

$$\forall b \in \{-1,+1\}^n : \Gamma(b) = F_b(\rho), \quad (9)$$

using the piecewise-constant dynamics illustrated in Fig. 1(b), which are the discrete-time analog of the continuous-time variational dynamics Eq. (4). Overall, the desired derivative is

$$D\phi_t(x_0;\delta x_0) = D\phi_{t-s}(x_s)\cdot B\left(D\phi_s(x_0)\cdot\delta x_0\right), \quad (10)$$

where $B : T_\rho\mathbb{R}^d \to T_\rho\mathbb{R}^d$ is the continuous piecewise-linear function defined by our algorithms in Fig. 2. Our algorithms require $O\left(n^2 d\right)$ time and $O(d)$ space to evaluate the directional derivative Eq. (3).

Assuming for the moment that these algorithms are correct, we emphasize that they achieve a dramatic reduction in the computational complexity of evaluating the B-derivative – from factorial to low-order polynomial – relative to naïve enumeration of all pieces of the B-derivative. However, despite the apparent simplicity of our algorithms (computationally and conceptually), verifying their correctness requires significant technical effort; the bulk of the present paper is devoted to this verification task.

3 Applications

To illustrate and validate our methods, we apply the algorithm from the preceding section to piecewise-constant vector fields in Sec. 3.1 and mechanical systems subject to unilateral constraints in Sec. 3.2.

3.1 Piecewise-constant vector field

Consider the vector field $F : \mathbb{R}^d \to T\mathbb{R}^d$ defined by

$$\dot{x} = F(x) = \mathbb{1} + \Delta(\mathrm{sign}(x)) \quad (11)$$

where $\Delta : B_d \to \mathbb{R}^d$; so long as all components of all vectors specified by Δ are larger than -1, i.e. $\min_{b\in B_d}[\Delta(b)]_j > -1$, F is event-selected C^∞ with respect to the identity function $h : \mathbb{R}^d \to \mathbb{R}^d$ defined by $h(q) = q$. We regard Eq. (11) as a canonical form for piecewise-constant event-selected C^∞ vector fields that are discontinuous across d subspaces, since any such vector field can be obtained by applying a linear

Algorithm 1 $\delta\rho^+ \leftarrow B(\delta\rho^-, \eta, \Gamma)$

$\delta t \leftarrow 0 \in \mathbb{R}$
$\delta\rho^+ \leftarrow \delta\rho^- \in \mathbb{R}^d$
$b \leftarrow -\mathbb{1} \in \{-1,+1\}^n$
while $b \neq +\mathbb{1}$ **do**
  for $j \in \{1,\ldots,n\}$ **do**
    $\tau_j \leftarrow -\left(\eta_j^\top \cdot \delta\rho^+\right) / \left(\eta_j^\top \cdot \Gamma(b)\right)$
  $j^* \leftarrow \arg\min_{j\in\{1,\ldots,n\}} \{\tau_j : b_j < 0\}$
  $\delta t \leftarrow \delta t + \tau_{j^*}$
  $\delta\rho^+ \leftarrow \delta\rho^+ + \tau_{j^*} \cdot \Gamma(b)$
  $b_{j^*} \leftarrow +1$
return $\delta\rho^+ - \delta t \cdot \Gamma(+\mathbb{1})$

Algorithm 2 `def B(dx,e,G):`

```
dt = 0
dx = np.array(dx)
b = -np.ones(len(e),dtype=np.int)
while np.any(b < 0) :
    tau = -np.dot(e,dx)/np.dot(e,G(b))
    tau[b > 0] = np.inf
    j = np.argmin(tau)
    dt += tau[j]
    dx += tau[j] * G(b)
     b[j] = +1
return dx - dt * G(b) # b == [+1,...,+1]
```

Fig. 2. Algorithms that evaluate the B-derivative of an EC^r vector field's PC^r flow written in pseudocode (*left*) and `Python` [21] sourcecode (*right*; requires `import numpy as np` [22]). These algorithms apply at a point $\rho \in \mathbb{R}^d$ where a vector field $F : \mathbb{R}^d \to T\mathbb{R}^d$ is event-selected C^r with respect to n surfaces (see Fig. 1 for an illustration when $d = n = 2$), and assume the following data is given:

tangent direction,	$\delta\rho^- \in T_\rho\mathbb{R}^d$,	`dx` – *array*, `dx.shape == (d,)`;
surface normals at ρ,	$\eta = \{\eta_j\}_{j=1}^n \subset \mathbb{R}^d$,	`e` – *array*, `e.shape == (n,d)`;
vector field limits (9),	$\Gamma : \{-1,+1\}^n \to \mathbb{R}^d$,	`G` – *function*, `G(b).shape == (d,)`.

change-of-coordinates to Eq. (11). In what follows, we focus on the trajectory that passes through the origin $\rho = 0$, which lies at the intersection of d surfaces of discontinuity for F. With $\rho^- = \rho - \frac{1}{2}F_{-\mathbb{1}}(\rho)$, $\rho^+ = \rho + \frac{1}{2}F_{+\mathbb{1}}(\rho)$, we note that ρ^- flows to ρ^+ through ρ in 1 (one) unit of time.

Our goal is to compute $D_x\phi(1,\rho^-;\delta\rho^-) \in T_{\rho^+}\mathbb{R}^d$ for a given $\delta\rho^- \in T_{\rho^-}\mathbb{R}^d$. In the general case, the desired derivative is piecewise-linear with (up to) $d!$ distinct pieces, providing a general test. In the special case where $\Delta(b) = -\delta \cdot b$ for all $b \in B_d$, $|\delta| < 1$, the desired derivative is linear [8, Eq. (86)],

$$D_x\phi(1,\rho^-;\delta\rho^-) = \frac{1-\delta}{1+\delta} \cdot \delta\rho^-, \tag{12}$$

providing a closed-form expression for comparison. Fig. 3 illustrates results from both cases with $d = 2$.

3.2 Mechanical systems subject to one-sided constraints

Consider a mechanical system whose configuration is subject to *unilateral* (i.e. one-sided) constraints. The dynamics of such systems have been studied extensively using the formalisms of complementarity [23, Sec. 3], measure differential inclusions [24, Sec. 3], hybrid systems [25, Sec. 2.4, 2.5], and geometric mechanics [26, Sec. 3]. Regardless of the chosen formalism, in a coordinate chart $Q \subset \mathbb{R}^d$ the dynamics governing q take the form

$$M(q)\ddot{q} = f(q,\dot{q}) \text{ subject to } a(q) \geq 0 \tag{13}$$

where: $M(q) \in \mathbb{R}^{d\times d}$ specifies the kinetic energy metric; $f(q,\dot{q}) \in \mathbb{R}^d$ specifies the internal, applied, and Coriolis forces; $a(q) \in \mathbb{R}^n$ specifies the *unilateral constraints* (we interpret $a(q) \geq 0$ componentwise); and we assume in what follows that M, f, and a are smooth functions. Different formalisms enforce the constraint $a(q) \geq 0$ in Eq. (13) differently, so we consider several cases in the following subsections.

3.2.1 Rigid constraints yield discontinuous flows

If constraints are enforced *rigidly* as in [23–25], meaning that they must be satisfied exactly, then the velocity must undergo impact (i.e. change discontinuously) whenever $\dot{q} \in T_qQ$ is such that $a_j(q) = 0$ and $Da_j(q) \cdot \dot{q} < 0$ for some $j \in \{1,\ldots,n\}$ [23, Sec. 2] [25, Eq. (23)] [24, Eq. (23)]. Unfortunately for our purposes, these discontinuities in the state vector $x = (q,\dot{q})$ cannot be modeled using an event-selected C^r vector field $\dot{x} = F(x)$, and the flow of such systems is generally discontinuous (although we note that the flow can be PC^r at non-impact times if the constraint surfaces intersect orthogonally [27], i.e. if the surface normals are orthogonal with respect to the inverse of the kinetic energy metric [24, Theorem 20]).

3.2.2 Soft conservative constraints yield Lipschitz-continuous vector fields, C^1 flows

We now consider the formalism in [26] that "softens" (i.e. approximately enforces) rigid constraints $a(q) \geq 0$ by augmenting the potential energy with *penalty functions* $\{v_j\}_{j=1}^n$ that scale quadratically with the degree of constraint

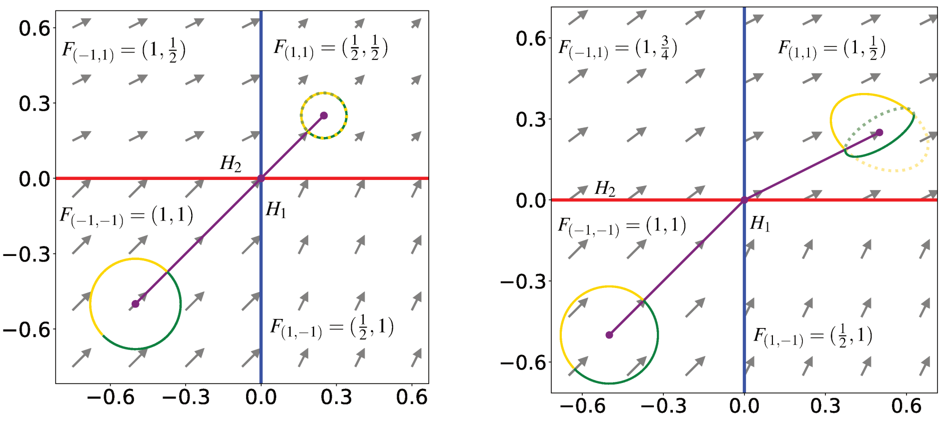


Fig. 3. B-derivative of planar instance of vector field from Sec. 3.1 in linear *(left)* and piecewise-linear *(right)* cases. The vector field F defined in Eq. (11) is piecewise-constant and discontinuous across the coordinate hyperplanes H_1, H_2, generating a piecewise-differentiable flow ϕ with B-derivative B. *(left)* The B-derivative is linear in the special case defined by Eq. (12). *(right)* The B-derivative is continuous and piecewise-linear in general, so a ball of initial conditions flows to a piecewise-ellipsoid (solid lines).

violation [26, Eq. (12)],

$$\forall j \in \{1, \ldots, n\} : v_j(q) = \begin{cases} 0, & a_j(q) \geq 0 \\ \frac{1}{2}\kappa_j a_j^2(q), & a_j(q) < 0 \end{cases} \qquad (14)$$

In essence, each rigid constraint $a_j(q) \geq 0$ is replaced by a spring with stiffness κ_j, leading to the unconstrained dynamics [26, Eq. (14)].

$$\begin{aligned} M(q)\ddot{q} &= f(q,\dot{q},u) - \sum_{j=1}^{n} Dv_j(q)^\top \\ &= f(q,\dot{q},u) - \sum_{j=1}^{n} \left\{ (\kappa_j a_j(q)) \cdot Da_j(q)^\top : a_j(q) < 0 \right\} \end{aligned} \qquad (15)$$

As shown by [28, Thm. 3], trajectories of Eq. (15) converge to those of Eq. (13) in the rigid limit (i.e. as stiffnesses go to infinity). Importantly for our purposes, the dynamics in Eq. (15) can be modeled using an event-selected vector field along trajectories that pass transversally through the constraint surfaces, whence our algorithms can compute the B-derivative of the flow. However, the vector field Eq. (15) in this case is (locally Lipschitz) continuous, hence the B-derivative is trivial (all non-identity terms in Eq. (57) are zero), whence the flow is continuously-differentiable (C^1).

3.2.3 Soft dissipative constraints yield EC^r vector fields, C^1 flows

We now augment the unconstrained dynamics Eq. (15) with dissipation as in [26]:

$$\begin{aligned} &M(q)\ddot{q} = \\ &\quad f(q,\dot{q},u) - \sum_{j=1}^{n} \left\{ \alpha_j(q,\dot{q}) \cdot Da_j(q)^\top : a_j(q) < 0 \right\} \end{aligned} \qquad (16)$$

where $\alpha_j(q,\dot{q}) = \kappa_j a_j(q) + \beta_j Da_j(q) \cdot \dot{q}$; in essence, each constraint penalty is augmented by a spring-damper that is only active when the constraint is violated as in studies involving contact with complex geometry [29] or terrain [30]. The dynamics in Eq. (16) can be modeled using an event-selected vector field along trajectories that pass transversally through the constraint surfaces, and the vector field is discontinuous along the constraint surfaces. However, we can show that the flow of Eq. (16) is continuously-differentiable (C^1) along *any* trajectory that passes transversally through constraint surfaces. Indeed, letting $x = (q,\dot{q})$ denote the state of the system so that $\dot{x} = (\dot{q},\ddot{q}) = F(x)$ is determined by Eq. (16), the saltation matrix Eq. (5) associated with each constraint a_j has the form

$$I + \frac{1}{Da_j(q) \cdot \dot{q}} \begin{bmatrix} 0 \\ \pm\alpha_j(q,\dot{q}) \end{bmatrix} \begin{bmatrix} Da_j(q) & 0 \end{bmatrix} \qquad (17)$$

where the sign in the column vector is determined by whether the constraint is activating ($-$) or deactivating ($+$). Since matrices of the form in Eq. (17) commute, the saltation matrices associated with simultaneous activation and/or deactivation of multiple constraints are all equal, whence the flow of Eq. (16) is continuously-differentiable (C^1) along any trajectory that passes transversally through constraint surfaces.

3.2.4 Example (vertical-plane biped)

To ground the preceding observations, we consider the vertical-plane biped illustrated in Fig. 4(*left*) that falls under the influence of gravity toward a substrate. The biped body has mass m and moment-of-inertia J; we let $(x,y) \in \mathbb{R}^2$ denote the position of its center-of-mass in the plane and $\theta \in S^1$ denote its rotation. Two rigid massless limbs of length ℓ protrude at an angle of $\pm\psi$ with respect to vertical from the body's center-of-mass above a smooth substrate whose height is a quadratic function of horizontal position, yielding

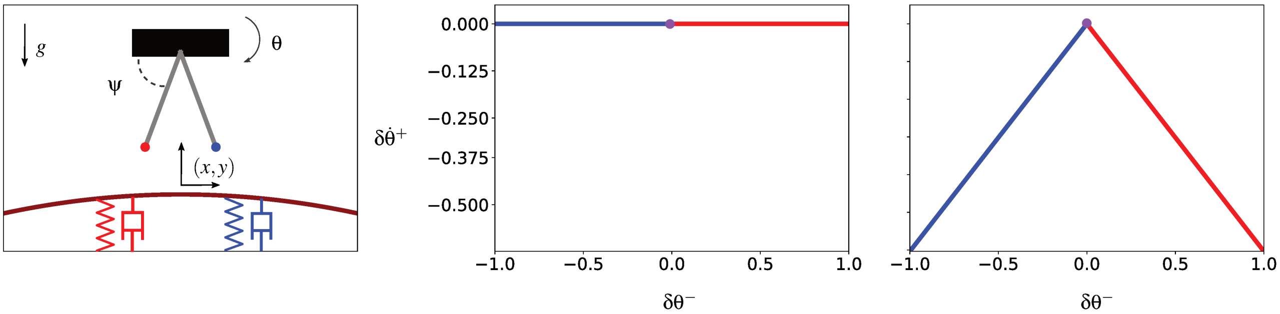


Fig. 4. Vertical-plane biped, a 3-degree-of-freedom mechanical system subject to unilateral constraints (Sec. 3.2.4), consists of a planar body with two rigid massless legs falling under the influence of gravity toward a substrate (*left*). The system's flow can be C^1 (*center*) or PC^r (*right*) depending on how forces vary as limbs contact substrate.

unilateral constraints

$$\begin{aligned} a_1(x,y,\theta) &= -y - (x + \ell\cos(\theta-\psi))^2 - \ell\sin(\theta-\psi), \\ a_2(x,y,\theta) &= -y - (x + \ell\cos(\theta+\psi))^2 - \ell\sin(\theta+\psi). \end{aligned} \tag{18}$$

We consider the smoothness of the system's flow along a trajectory that activates both constraints simultaneously. The formalism in Eq. (16) yields continuously-differentiable flow for this system as illustrated in Fig. 4(*center*).

To obtain a flow that is piecewise-differentiable but *not* continuously-differentiable, we modify the damping coefficients in Eq. (16) using the following logic[3]: $\beta_1 = \beta_2 = \frac{1}{2}$ if $a_1(q) < 0$ or $a_2(q) \geq 0$ (*exclusive* or); $\beta_1 = \beta_2 = 1$ if $a_1(q) < 0$ and $a_2(q) < 0$. The saltation matrices obtained from different sequences of constraint activations (left foot reaches substrate before right foot or vice-versa) are distinct:

$$M_{(\text{left},\text{right})} - M_{(\text{right},\text{left})} = \begin{bmatrix} 0 & 0 & 0 & 0\;0\;0 \\ 0 & 0 & 0 & 0\;0\;0 \\ 0 & 0 & 0 & 0\;0\;0 \\ 0 & 0 & 0 & 0\;0\;0 \\ -4\beta\cos(\psi) & 0 & -2\beta(\sin(2\psi)+\cos(\psi)) & 0\;0\;0 \\ 0 & 0 & 0 & 0\;0\;0 \end{bmatrix} \tag{19}$$

The piecewise-linear B-derivative of the system's flow is illustrated in Fig. 4(*right*). Sourcecode for this example is provided in SM.

4 Background

To verify correctness of the algorithms specified in Sec. 2, we utilize the representation of piecewise-affine functions from [33], elements of the theory of piecewise-differentiable functions from [12], and results about the class of nonsmooth flows under consideration from [8]. In an effort to make this paper self-contained (i.e. to save the reader from needing to cross-reference multiple citations to follow our derivations), we include a substantial amount of background details in this section. The expert reader may wish to skim or skip this section, returning only if questions arise in subsequent sections.

4.1 Polyhedral theory

We let $0_d \in \mathbb{R}^d$ denote the vector of zeros, $\mathbb{1}_n \in \mathbb{R}^n$ the vector of ones, and $I_d \in \mathbb{R}^{d\times d}$ the identity matrix; when dimensions are clear from context, we suppress subscripts. The vectorized signum function $\operatorname{sign} : \mathbb{R}^d \to \{-1,+1\}^d$ is defined for all $x \in \mathbb{R}^d$, $j \in \{1,\ldots,d\}$ by

$$[\operatorname{sign}(x)]_j = \operatorname{sign}(x_j) = \begin{cases} -1, & x_j < 0; \\ +1, & x_j \geq 0. \end{cases} \tag{20}$$

If $A \in \mathbb{R}^{\ell\times m}$ and $B \in \mathbb{R}^{m\times n}$ then $A \cdot B \in \mathbb{R}^{\ell\times n}$ denotes matrix multiplication. Given a subset $S \subset \mathbb{R}^d$, we let $\operatorname{aff} S$, $\operatorname{cone} S$, $\operatorname{conv} S$ denote the *affine span*, *cone span*, and *convex hull* of S, respectively [12, Sec. 2.1.1]. The *dimension* of a convex set S is defined to be the dimension of its affine span, $\dim S = \dim \operatorname{aff} S$. A nonempty set $S \subset \mathbb{R}^d$ is called a *polyhedron* [12, Sec. 2.1.2] if there exists $A \in \mathbb{R}^{m\times d}$, $b \in \mathbb{R}^m$ such that $S = \{x \in \mathbb{R}^d : A\cdot x \leq b\}$; note that S is closed and convex. The linear subspace $\mathcal{L} = \{x \in \mathbb{R}^d : A \cdot x = 0\}$ is called the *lineality space* of S.

4.2 Piecewise-affine functions

We will represent a piecewise-affine function using a *triangulation* (Z^-, Z^+, Δ) [33, Sec. 3.1] that consists of a combinatorial simplicial complex Δ whose vertex set is in 1-to-1 correspondence with each of the finite sets of vectors $Z^- \subset \mathbb{R}^d$, $Z^+ \subset \mathbb{R}^c$. For our purposes,[4] a *combinatorial simplicial complex* Δ is a collection of finite sets $\Delta = \{\Delta_\omega\}_{\omega\in\Omega}$ such that $S \subset \Delta_\omega \implies S \in \Delta$ for all $\omega \in \Omega$; we call $\bigcup_{\omega\in\Omega}\Delta_\omega$ the *vertex set* of Δ. We assume that, for

[3] Although we introduce this logic purely for illustrative purposes, we note that non-trivial dependence of forcing on the set of active constraints could be implemented physically using clutches [31] or actuators [32].

[4] There are more general definitions of ([complete] semi-)simplicial complexes and the closely-related concept of Δ-complexes in the literature [34, Ch. 2.1], [33, App. A.3.1]. Since we employ these concepts primarily in service of parameterizing piecewise-affine functions as in [33, Sec. 3.1], we adopt the (relatively restrictive) definitions of *combinatorial* and *geometric* simplicial complexes from [33, Sec. 2.2.1] in what follows.

every $\omega \in \Omega$, the collections of vectors $Z_\omega^\pm \subset Z^\pm$ determined by Δ_ω are *affinely independent* [33, Sec. 2.1.1] so that $\Delta_\omega^\pm = \operatorname{conv} Z_\omega^\pm$ are $(\#(\Delta_\omega) - 1)$-dimensional geometric simplices [33, Claim 2.9] where $\Delta_\omega^- \subset \mathbb{R}^d$, $\Delta_\omega^+ \subset \mathbb{R}^c$. We assume further that, for every $\omega, \omega' \in \Omega$, the collections of vectors $Z_{\omega,\omega'}^\pm \subset Z^\pm$ determined by $\Delta_\omega \cap \Delta_{\omega'}$ coincide with $Z_\omega^\pm \cap Z_{\omega'}^\pm \subset Z^\pm$ so that $\Delta^\pm = \{\Delta_\omega^\pm\}_{\omega\in\Omega}$ are *geometric simplicial complexes* [33, Sec. 2.2.1]. With these assumptions in place, the correspondence between Z^- and Z^+ determined by the triangulation (Z^-, Z^+, Δ) uniquely defines a piecewise-affine function $P : |\Delta^-| \to |\Delta^+|$ using the construction from [33, Sec. 3.1] where $|\Delta^-| = \bigcup_{\omega\in\Omega} \Delta_\omega^- \subset \mathbb{R}^d$, $|\Delta^+| = \bigcup_{\omega\in\Omega} \Delta_\omega^+ \subset \mathbb{R}^c$ are termed the *carriers* [12, Sec. 2.2.1] of the geometric simplicial complexes $\Delta^\pm$.

4.3 Piecewise-linear functions

If a piecewise-affine function $P : \mathbb{R}^d \to \mathbb{R}^c$ is *positively homogeneous*, that is,

$$\forall \alpha \geq 0, v \in \mathbb{R}^d : P(\alpha \cdot v) = \alpha \cdot P(v), \tag{21}$$

then P is *piecewise-linear* [12, Prop. 2.2.1]. In this case, P admits a *conical subdivision* [12, Prop. 2.2.3], that is, there exists a finite collection $\Sigma = \{\Sigma_\omega\}_{\omega\in\Omega}$ such that: (i) $\Sigma_\omega \subset \mathbb{R}^d$ is a d-dimensional *polyhedral cone* for each $\omega \in \Omega$;[5] (ii) the Σ_ω's cover $\mathbb{R}^d$;[6] and (iii) the intersection $\Sigma_\omega \cap \Sigma_{\omega'}$ is either empty or a *proper face* of both polyhedral cones for each $\omega, \omega' \in \Omega$.[7]

4.4 Piecewise-differentiable (PC^r) functions

(This section is largely repeated from [8, Sec. 3.2].) The notion of piecewise–differentiability we employ was originally introduced in [35]; since the monograph [12] provides a more recent and comprehensive exposition, we adopt the notational conventions therein. Let $r \in \mathbb{N} \cup \{\infty\}$ and $D \subset \mathbb{R}^d$ be open. A continuous function $f : D \to \mathbb{R}^c$ is called *piecewise-C^r* if for every $x_0 \in D$ there exists an open set $U \subset D$ containing x_0 and a finite collection $\{f_j : U \to \mathbb{R}^c\}_{j\in J}$ of C^r functions such that for all $x \in U$ we have $f(x) \in \{f_j(x)\}_{j\in J}$. The functions $\{f_j\}_{j\in J}$ are called *selection functions* for $f|_U$, and f is said to be a *continuous selection* of $\{f_j\}_{j\in J}$ on U. A selection function f_j is said to be *active* at $x \in U$ if $f(x) = f_j(x)$. We let $PC^r(D, \mathbb{R}^c)$ denote the set of piecewise-C^r functions from D to $\mathbb{R}^c$. Note that PC^r is closed under composition. The definition of piecewise-C^r may at first appear unrelated to the intuition that a function ought to be piecewise-differentiable precisely if its "domain can be partitioned locally into a finite number of regions relative to which smoothness holds" [36, Sec. 1]. However, as shown in [36, Thm. 2], piecewise-C^r functions are always piecewise-differentiable in this intuitive sense.

Piecewise-differentiable functions possess a first–order approximation $Df : TD \to T\mathbb{R}^c$ called the *Bouligand derivative* (or B–derivative) [12, Ch. 3]; this is the content of [12, Lemma 4.1.3]. Significantly, this B–derivative obeys generalizations of many techniques familiar from calculus, including the Chain Rule [12, Thm 3.1.1], Fundamental Theorem of Calculus [12, Prop. 3.1.1], and Implicit Function Theorem [37, Cor. 20]. We let $Df(x; \delta x)$ denote the B–derivative of f evaluated on the tangent vector $\delta x \in T_x D$. The B-derivative is positively homogeneous, i.e. $\forall \delta x \in T_x D, \lambda \geq 0 : Df(x; \lambda \delta x) = \lambda Df(x; \delta x)$, and coincides with the directional derivative of f in the $\delta x \in T_x D$ direction. In addition, the B-derivative $Df(x) : T_x D \to T_{f(x)}\mathbb{R}^c$ of f at $x \in D$ is a continuous selection of the derivatives of the selection functions active at x [12, Prop. 4.1.3],

$$\forall \delta x \in T_x D : Df(x; \delta x) \in \{Df_j(x) \cdot \delta x\}_{j\in J}. \tag{22}$$

However, the function Df is generally *not* continuous at $(x, \delta x) \in TD$; if it is, then f is C^1 at x [12, Prop. 3.1.2].

4.5 Event-selected C^r (EC^r) vector fields and PC^r flows

Vector fields with discontinuous right-hand-sides and their associated flows have been studied extensively [38]. In definition 1 [8, Defs. 1, 2], a special class of so-called *event-selected C^r* (EC^r) vector fields were defined which are allowed to be discontinuous along a finite number of codimension-1 submanifolds but do not exhibit *sliding* [10] along these submanifolds, and are C^r elsewhere. Importantly, as shown in [8, Thm. 5], an event-selected C^r vector field $F : \mathbb{R}^d \to T\mathbb{R}^d$ generates a piecewise-differentiable flow, that is, there exists a function $\phi : \mathcal{F} \to \mathbb{R}^d$ that is *piecewise-C^r* ($\phi \in PC^r$) in the sense defined in [12, Sec. 4.1] (summarized in Sec. 4.4) where $\mathcal{F} \subset \mathbb{R} \times \mathbb{R}^d$ and

$$\forall (t, x) \in \mathcal{F} : \phi(t, x) = x + \int_0^t F(\phi(s, x))ds. \tag{23}$$

Since ϕ is PC^r, it admits a first-order approximation $D\phi : T\mathcal{F} \to T\mathbb{R}^d$ termed the *Bouligand* (or *B-*)derivative [12, Sec. 3.1], which is a continuous piecewise-linear function of tangent vectors at every $(t, x) \in \mathcal{F}$, that is, the directional derivative $D\phi(t, x) : T_{(t,x)}\mathcal{F} \to T_{\phi(t,x)}\mathbb{R}^d$ is continuous and piecewise-linear for all $(t, x) \in \mathcal{F}$.

4.6 B-derivative of an EC^r vector field's PC^r flow

Suppose $F : \mathbb{R}^d \to T\mathbb{R}^d$ is an EC^r vector field with PC^r flow $\phi : \mathcal{F} \to \mathbb{R}^d$. Given a tangent vector $(\delta t, \delta x) \in T_{(t,x)}\mathcal{F}$, it was shown in [8, Sec. 7.1.4] that the value of the B-derivative $D\phi(t, x; \delta t, \delta x) \in T_{\phi(t,x)}\mathbb{R}^d$ can be obtained by solving a jump-linear-time-varying differential equation [8, Eq. (70)], where the "jump" arises from a matrix Ξ_ω determined by the sequence ω in which the perturbed initial state $x + \alpha\,\delta x$ crosses

[5] i.e. $\Sigma_\omega = \left\{\sum_{j=1}^{\ell_\omega} \alpha_j v_j^\omega : \{\alpha_j\}_{j=1}^{\ell_\omega} \subset [0, \infty)\right\}$, some $\{v_j\}_{j=1}^{\ell_\omega} \subset \mathbb{R}^d$ [12, Thm. 2.1.1], and $\dim \Sigma_\omega = d$

[6] i.e. $\bigcup_{\omega\in\Omega} \Sigma_\omega = \mathbb{R}^d$

[7] i.e. $\Sigma_\omega \cap \Sigma_{\omega'} = \left\{\sum_{j=1}^{\ell_{\omega,\omega'}} \alpha_j v_j^{\omega,\omega'} : \{\alpha_j\}_{j=1}^{\ell_{\omega,\omega'}} \subset [0, \infty)\right\}$, some $\left\{v_j^{\omega,\omega'}\right\}_{j=1}^{\ell_{\omega,\omega'}} \subset \left\{v_j^\omega\right\}_{j=1}^{\ell_\omega} \cup \left\{v_j^{\omega'}\right\}_{j=1}^{\ell_{\omega'}}$

the surfaces of discontinuity of the vector field F for small $\alpha > 0$ [8, Eq. (67)]. However, [8] did not provide a representation of the piecewise-linear operator $D\phi(t,x)$ (and, to the best of our knowledge, neither has subsequent work). The key theoretical contribution of this paper, obtained in Sec. 5, is a representation of the B-derivative with respect to state, $D_x\phi(t,x)$, using a *triangulation* of its domain and codomain as defined in [33, Sec. 3.1] (and recalled in Sec. 4.2).

To inform the triangulation of the B-derivative $D_x\phi(t,x)$, we recall the values it takes on. Since the flow $\phi : \mathcal{F} \to \mathbb{R}^d$ is piecewise-C^r (PC^r), it is a *continuous selection* of a finite collection of C^r functions $\left\{\phi_\omega : \mathcal{F}_\omega \to \mathbb{R}^d\right\}_{\omega\in\Omega}$ near $(t,x) \in \mathcal{F}$, where $\mathcal{F}_\omega \subset \mathcal{F}$ is an open set containing (t,x) for each $\omega \in \Omega$ [12, Sec. 4.1], and the B-derivative $D_x\phi(t,x)$ is a continuous selection of the classical (*Fréchet* or *F*-)derivatives $\{D_x\phi_\omega(t,x)\}_{\omega\in\Omega}$ [12, Prop. 4.1.3], that is,

$$\forall \delta x \in W_\omega \subset T_x\mathbb{R}^d : D_x\phi(t,x;\delta x) = D_x\phi_\omega(t,x)\cdot\delta x, \tag{24}$$

where $W_\omega \subset T_x\mathbb{R}^d$ is the subset of tangent vectors where the selection function $D_x\phi_\omega$ is *essentially active* [12, Prop. 4.1.1]. If $s,t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ are such that $0 < s < t$ and the vector field F is C^r on $\phi([0,t]\setminus\{s\},x)$, i.e. the trajectory initialized at $x \in \mathbb{R}^d$ encounters exactly one discontinuity of F at $\rho = \phi(s,x)$ on the time interval $[0,t]$, then $D_x\phi_\omega(t,x)$ has the form

$$\begin{aligned} &D_x\phi_\omega(t,x) = \\ &\quad D_x\phi(t-s,\rho)\cdot\left[F_{+\mathbb{1}}(\rho)\ I_d\right]\cdot\Xi_\omega\cdot\begin{bmatrix}0_d^\top\\ I_d\end{bmatrix}\cdot D_x\phi(s,x)\end{aligned} \tag{25}$$

where $F_{+\mathbb{1}}$ is the C^r extension of $F|_{\mathrm{Int}D_{+\mathbb{1}}}$ that exists by virtue of condition 2 in Def. 1 and $\Xi_\omega \in \mathbb{R}^{(d+1)\times(d+1)}$ is the matrix from [8, Eq. (67)] corresponding to the selection function index $\omega \in \Omega$. In what follows, we will work in circumstances where the selection functions are indexed by the symmetric permutation group over n elements, i.e. $\Omega = S_n$, and combine Eq. (24) and Eq. (25) as

$$\begin{aligned} &\forall \delta x \in W_\sigma \subset T_x\mathbb{R}^d : \\ &\quad D_x\phi(t,x;\delta x) = D_x\phi(t-s,\rho)\cdot M_\sigma\cdot D_x\phi(s,x)\cdot\delta x\end{aligned} \tag{26}$$

where the *saltation matrix*[8] $M_\sigma \in \mathbb{R}^{d\times d}$ corresponding to index σ is defined by

$$M_\sigma = \left[F_{+\mathbb{1}}(\rho)\ I_d\right]\cdot\Xi_\sigma\cdot\begin{bmatrix}0_d^\top\\ I_d\end{bmatrix}. \tag{27}$$

4.7 Local approximation of an EC^r vector field

Suppose vector field $F : \mathbb{R}^d \to T\mathbb{R}^d$ is event-selected C^r with respect to $h \in C^r(U,\mathbb{R}^n)$ at $\rho \in U \subset \mathbb{R}^d$. For $b \in B_n = \{-1,+1\}^n$ let

$$\widetilde{D}_b = \left\{x \in \mathbb{R}^d : b_j Dh_j(\rho)(x-\rho) \geq 0\right\} \tag{28}$$

and consider piecewise-constant vector field $\widetilde{F} : \mathbb{R}^d \to T\mathbb{R}^d$ defined by

$$\forall b \in B_n,\ x \in \widetilde{D}_b : \widetilde{F}(x) = F_b(\rho) \tag{29}$$

where F_b is the C^r extension of $F|_{\mathrm{Int}D_b}$ that exists by virtue of condition 2 in Def. 1. Note that $\widetilde{F}$ is event-selected C^r with respect to the affine function $\widetilde{h}$ defined by

$$\forall x \in \mathbb{R}^d : \widetilde{h}(x) = Dh(\rho)(x-\rho), \tag{30}$$

whence it generates a piecewise-differentiable flow $\widetilde{\phi} : \widetilde{\mathcal{F}} \to \mathbb{R}^d$ where $\widetilde{\mathcal{F}} = \mathbb{R}\times\mathbb{R}^d$. In [8, Sec. 7.1.3], $\widetilde{F}$ was referred to as the *sampled* vector field since it is obtained by "sampling" the selection functions F_b that define F near ρ, and it was noted that the function $\widetilde{\phi}$ is piecewise-affine and it approximates the original vector field's flow ϕ near ρ. We will leverage the algebraic properties of $\widetilde{\phi}$ and its relationship to ϕ in what follows to obtain our results.

4.8 Time-to-impact for an EC^r vector field and its local approximation

Suppose vector field $F : \mathbb{R}^d \to T\mathbb{R}^d$ is event-selected C^r with respect to $h \in C^r(U,\mathbb{R}^n)$ at $\rho \in U \subset \mathbb{R}^d$, and let $\phi \in PC^r(\mathcal{F},\mathbb{R}^d)$ be its piecewise-differentiable flow. Then [8, Thm. 7] ensures there exists a piecewise-differentiable *time-to-impact* function $\tau \in PC^r(U,\mathbb{R}^n)$ such that, $\forall x \in U, j \in \{1,\ldots,n\}$,

$$\phi(\tau_j(x),x) \in H_j = h_j^{-1}(h_j(\rho)), \tag{31}$$

i.e. x flows to the surface H_j in time $\tau_j(x)$. Similarly, applying [8, Thm. 7] to the sampled vector field $\widetilde{F} : \mathbb{R}^d \to T\mathbb{R}^d$ and piecewise-affine flow $\widetilde{\phi} : \widetilde{\mathcal{F}} \to \mathbb{R}^d$ associated with F at ρ constructed in Sec. 4.7 ensures there exists a piecewise-affine time-to-impact function $\widetilde{\tau} : \mathbb{R}^d \to \mathbb{R}^n$ such that, $\forall x \in \mathbb{R}^d, j \in \{1,\ldots,n\}$,

$$\widetilde{\phi}(\widetilde{\tau}_j(x),x) \in \widetilde{H}_j = \rho + \ker Dh_j(\rho), \tag{32}$$

i.e. the point x flows to the affine subspace $\widetilde{H}_j$ in time $\widetilde{\tau}_j(x)$.

5 Representation

Our main theoretical result is an explicit representation for the Bouligand (or B-)derivative of the piecewise-differentiable flow generated by an event-selected C^r vector

[8] $\Xi_\sigma \in \mathbb{R}^{(d+1)\times(d+1)}$ is referred to as a saltation matrix in [8, Sec. 7.1.4], but this usage is inconsistent with the original definition in [13].

field. To that end, let $F : \mathbb{R}^d \to T\mathbb{R}^d$ be an event-selected C^r vector field and $\phi : \mathcal{F} \to \mathbb{R}^d$ its piecewise-differentiable flow. In what follows, we will assume that $s,t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ are such that $0 < s < t$ and the vector field F is C^r on $\phi([0,t] \setminus \{s\}, x)$. Although a general trajectory can encounter more than one point of discontinuity for F, such points are isolated [8, Lem. 6], so the Chain Rule for B-differentiable functions [12, Thm. 3.1.1] can be applied to triangulate the desired flow derivative by composing the triangulated flow derivatives associated with each point. Thus, without loss of generality, we restrict our attention to portions of trajectories that encounter one point of discontinuity for F, which point lies at the intersection of n surfaces of discontinuity for F. We assume $n > 1$ because at least two surfaces are needed for our results to be useful: when $n = 1$ the desired B-derivative is linear [13], so it may be represented and employed in computations as a matrix.

The B-derivative $D_x\phi(t,x) : T_x\mathbb{R}^d \to T_{\phi(t,x)}\mathbb{R}^d$ we seek is a continuous piecewise-linear function, so it can be parsimoniously represented using a *triangulation* [33, Sec. 3.1], that is, a combinatorial simplicial complex (as defined in Sec. 4.2) each of whose vertices are associated with a pair of (tangent) vectors – one each in the domain and codomain of $D_x\phi(t,x)$. We will obtain this triangulation via an indirect route: in Sec. 5.1, we triangulate the piecewise-affine flow $\widetilde{\phi}$ introduced in Sec. 4.7; in Sec. 5.2, we differentiate our representation of $\widetilde{\phi}$ to obtain a triangulation of the B-derivative $D_x\widetilde{\phi}$; in Sec. 5.3, we show how the B-derivative $D_x\phi$ can be obtained from $D_x\widetilde{\phi}$, providing a triangulation of the desired derivative.

5.1 Triangulation

The goal of this subsection is to triangulate the piecewise-affine flow $\widetilde{\phi}$ introduced in Sec. 4.7. To that end, let $\rho = \phi(s,x)$ and suppose[9] $\operatorname{rank} Dh(\rho) = n$ so $\{\delta\rho \in T_\rho\mathbb{R}^d : b = \operatorname{sign} Dh(\rho)\cdot\delta\rho\}$ has nonempty interior for each $b \in \{-1,+1\}^n = B_n$. Letting $\mathcal{K} = \ker Dh(\rho) \subset T_\rho\mathbb{R}^d$ denote the kernel of $Dh(\rho)$ and $\mathcal{K}^\perp$ its orthogonal complement, for each $b \in B_n$ there exists a unique[10,11] $\zeta_b \in \mathcal{K}^\perp + \{\rho\}$ such that

$$Dh_{b>0}(\rho)(\zeta_b - \rho) = 0,\ Dh_{b<0}(\rho)(\zeta_b + F_b(\rho) - \rho) = 0 \tag{33}$$

where $h_{b>0}$ (respectively, $h_{b<0}$) denotes the function obtained by selecting components h_j of h for which $b_j = +1$ (respectively, $b_j = -1$). The vectors defined by Eq. (33) have special significance for the piecewise-affine flow $\widetilde{\phi}$ introduced in Sec. 4.7 (see Fig. 5(a)):

$$\forall b \in B_n : \zeta_b \in \widetilde{D}_{-\mathbb{1}},\ \widetilde{\phi}(1,\zeta_b) = \zeta_b + F_b(\rho) \in \widetilde{D}_{+\mathbb{1}}, \tag{34}$$

that is, the point ζ_b lies "before" all event surface tangent planes and flows in 1 (one) unit of time to $\zeta_b + F_b(\rho)$ which lies "after" all event surface tangent planes (neither "before" nor "after" should be interpreted strictly). We denote the collections of these vectors as follows:

$$Z^- = \{\zeta_b\}_{b\in B_n},\ Z^+ = \{\zeta_b + F_b(\rho)\}_{b\in B_n}. \tag{35}$$

In what follows, it will be convenient to use an element $\sigma \in S_n$ of the symmetric permutation group over n elements to specify $n+1$ elements of $b \in B_n$ as follows: for each $k \in \{0,\dots,n\}$, let $\sigma(\{0,\dots,k\}) \subset \{1,\dots,n\}$ specify the unique $b \in B_n$ whose j-th component is $+1$ if and only if $j \in \sigma(\{0,\dots,k\})$. Note that this identification yields, with some abuse of notation, $\sigma(\{0\}) = -\mathbb{1}$, $\sigma(\{0,\dots,n\}) = +\mathbb{1}$. Finally, note that the following are linearly independent collections of vectors:

$$\left\{\zeta_{\sigma(\{0,\dots,k\})} - \rho\right\}_{k=0}^{n-1} \tag{36a}$$

$$\left\{\zeta_{\sigma(\{0,\dots,k\})} + F_{\sigma(\{0,\dots,k\})}(\rho) - \rho\right\}_{k=1}^{n} \tag{36b}$$

This fact is easily verified for Eq. (36a) in coordinates where $Dh(\rho) = \left[I_n\ 0_{n\times(d-n)}\right]$, whence the fact follows for Eq. (36b) by Eq. (36a) and Eq. (34) via [8, Cor. 5(c)] (time-t flow of an EC^r vector field is a homeomorphism of the state space).

Let Δ be the combinatorial simplicial complex over vertices B_n with maximal n-simplices indexed by $\sigma \in S_n$ via

$$\Delta_\sigma = \{\sigma(\{0,\dots,k\})\}_{k=0}^{n} \in \Delta \tag{37}$$

where we regard $\sigma(\{0,\dots,k\})$ as an element of B_n using the same abuse of notation employed in Eq. (36). By associating each vertex $b \in B_n$ with the vector $\zeta_b \in Z^- \subset \mathbb{R}^d$, every n-simplex Δ_σ determines an n-dimensional geometric simplex $\Delta_\sigma^- \subset \mathbb{R}^d$, the dimensionality of which is ensured by Eq. (36a); similarly, Eq. (36b) ensures that associating each $b \in B_n$ with $(\zeta_b + F_b(\rho)) \in Z^+ \subset \mathbb{R}^d$ determines an n-dimensional geometric simplex $\Delta_\sigma^+ \subset \mathbb{R}^d$ from each n-simplex Δ_σ. Refer to Fig. 5(b) for an illustration when $n = 2$. The triple (Z^-, Z^+, Δ) parameterizes a continuous piecewise-affine homeomorphism $P : |\Delta^-| \to |\Delta^+|$ using the construction from [33, Sec. 3.1] (summarized in Sec. 4.2), where $|\Delta^\pm| = \bigcup_{\sigma\in S_n} \Delta_\sigma^\pm \subset \mathbb{R}^d$ denote the *carriers* of the geometric simplicial complexes $\Delta^\pm$.

We now show that the piecewise-affine function P constructed above is the non-linear part of the time-1 flow of the sampled system $\widetilde{\phi}_1$ restricted to $|\Delta^-|$. For each $\sigma \in S_n$ we extend the n-dimensional geometric simplex Δ_σ^- determined by the n-simplex Δ_σ via direct sum with the $(d-n)$-dimensional subspace $\mathcal{K}$ to obtain a d-dimensional polyhedron Σ_σ (see Fig. 5(c)), and let $|\Sigma| = \bigcup_{\sigma\in S_n} \Sigma_\sigma$. Note that $\mathcal{K}$ is a subset of the lineality space of Σ_σ for each $\sigma \in S_n$.

[9]As observed in [8, Sec. 7.1.5], first-order approximations of an EC^r vector field's PC^r flow are not affected by flow between surfaces that are tangent at ρ, so we assume such redundancy has been removed.

[10]Here and in what follows we mildly abuse notation via the natural vector space isomorphism $\mathbb{R}^d \simeq T_\rho\mathbb{R}^d$.

[11]$\operatorname{rank} Dh(\rho) = n$ ensures uniqueness since (i) $\mathcal{K}^\perp$ is n-dimensional, (ii) the rows of $Dh(\rho)$ are linearly independent, and hence (iii) there are n independent equations in the n unknowns needed to specify ζ_b in Eq. (33).

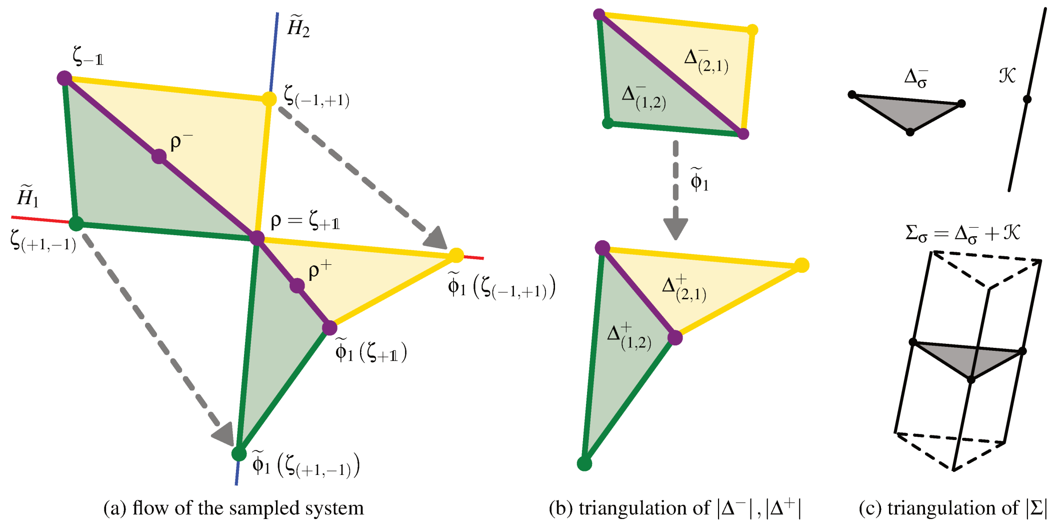


Fig. 5. Triangulation of time-1 flow $\widetilde{\phi}_1$ of *sampled system* associated with planar EC^r vector field. (a) For each $b \in \{-1,+1\}^2$, the point ζ_b defined by (33) flows from $\widetilde{D}_{-\mathbb{1}}$ to $\widetilde{D}_{+\mathbb{1}}$ in 1 (one) unit of time via the *sampled system* illustrated in Fig. 1(b) and defined in Sec. 4.7. (b) The sets $\{\zeta_{-\mathbb{1}}, \zeta_{+\mathbb{1}}, \zeta_{(+1,-1)}\}$, $\{\zeta_{-\mathbb{1}}, \zeta_{+\mathbb{1}}, \zeta_{(-1,+1)}\}$ indexed by (37) define geometric simplices $\Delta^-_{(1,2)}$, $\Delta^-_{(2,1)}$ that pass through subspaces $\widetilde{H}_1, \widetilde{H}_2$ in the same order. (c) For each $\sigma \in \{(1,2),(2,1)\}$, the set Σ_σ is a direct sum of Δ^-_σ with subspace $\mathcal{K}$.

Lemma 1. *$\widetilde{\phi}_1|_{|\Sigma|}$ is piecewise-affine and*

$$\forall z \in |\Delta^-|, \xi \in \mathcal{K} : \widetilde{\phi}_1(z+\xi) = P(z) + \xi. \tag{38}$$

Proof. This proof will proceed in two steps: (i) show that $\widetilde{\phi}_1(z) = P(z)$ for all $z \in |\Delta^-|$; (ii) show that $\widetilde{\phi}_1(z+\xi) = \widetilde{\phi}_1(z) + \xi$ for all $z \in |\Delta^-|, \xi \in \mathcal{K}$.

(i) Recall from Eq. (34) that $\widetilde{\phi}_1|_{Z^-} = P|_{Z^-}$ where Z^- is the vertex set for the geometric simplicial complex Δ^-. For each $\sigma \in S_n$ let $Z_\sigma = \{\zeta_b\}_{b \in \Delta_\sigma}$ denote the vertex set of the n-dimensional geometric simplex Δ^-_σ. Then we claim that each $z \in \Delta^-_\sigma$ passes through the same sequence of transition surfaces as each $\zeta_b \in Z_\sigma$. To verify this claim, we use the piecewise-affine *time-to-impact* function $\widetilde{\tau} : \mathbb{R}^d \to \mathbb{R}^n$ from Sec. 4.8. Note that ζ_b impacts affine subspace $\widetilde{H}_j$ at time 1 if $b_j = -1$ and at time 0 if $b_j = +1$, i.e.

$$\widetilde{\tau}_j(\zeta_b) = \begin{cases} 1, & b_j = -1; \\ 0, & b_j = +1. \end{cases} \tag{39}$$

Convex combination $\alpha\zeta_b + (1-\alpha)\zeta_{b'}$, $\alpha \in (0,1)$, $b, b' \in \Delta_\sigma$, impacts $\widetilde{H}_j$ at time $\widetilde{\tau}_j(\alpha\zeta_b + (1-\alpha)\zeta_{b'})$ that is: 0 if $b_j = b'_j = +1$, 1 if $b_j = b'_j = -1$, and between 0 and 1 otherwise. More generally, any point $z \in \Delta^-_\sigma$ is a convex combination of the vertices Z_σ, whence it impacts surfaces in the order prescribed by σ, so $\forall z \in \Delta^-_\sigma$:

$$0 \le \widetilde{\tau}_{\sigma(1)}(z) \le \widetilde{\tau}_{\sigma(2)}(z) \le \cdots \le \widetilde{\tau}_{\sigma(n)}(z) < 1. \tag{40}$$

Thus, $\widetilde{\phi}_1|_{\Delta^-_\sigma}$ is affine and agrees with $P|_{\Delta^-_\sigma}$. Since $|\Delta^-| = \bigcup_{\sigma \in S_n} \Delta^-_\sigma$, we have $\widetilde{\phi}_1|_{|\Delta^-|} = P$.

(ii) We now show that the piecewise-affine map $\widetilde{\phi}_1$ is indifferent to every $\xi \in \mathcal{K} = \ker Dh(\rho)$, so for every $z \in |\Delta^-|$:

$$\begin{aligned} \widetilde{\phi}_1(z+\xi) &= \widetilde{\phi}_1(\rho + (z+\xi-\rho)) && (41a)\\ &= \widetilde{\phi}_1(\rho) + D\widetilde{\phi}_1(\rho; z+\xi-\rho) && (41b)\\ &= \widetilde{\phi}_1(\rho) + D\widetilde{\phi}_1(\rho; z-\rho) + \xi && (41c)\\ &= \widetilde{\phi}_1(z) + \xi. && (41d) \end{aligned}$$

Indeed: Eq. (41a) since $z+\xi = \rho + (z+\xi-\rho)$; Eq. (41b) since $\widetilde{\phi}_1$ is affine on the segment $\{\rho + \alpha(z+\xi-\rho) : \alpha \in [0,1]\}$; Eq. (41c) since each piece of the continuous piecewise-linear B-derivative $D\widetilde{\phi}_1(\rho)$ is specified by a saltation matrix (as recalled in Sec. 4.4) that is the product of matrices of the form $(I_d + g \cdot Dh_j(\rho))$ [8, Eq. (60)], thus $\xi \in \mathcal{K} = \ker Dh(\rho)$ is transformed by I_d; Eq. (41d) for the same reason as Eq. (41b). □

5.2 B-derivative of $\widetilde{\phi}$

The goal of this subsection is to differentiate the representation of $\widetilde{\phi}$ from Sec. 5.1 to obtain a triangulation of the B-derivative $D\widetilde{\phi}_1 : T_{\rho^-}\mathbb{R}^d \to T_{\rho^+}\mathbb{R}^d$ between the following two points:

$$\rho^- = \rho - \frac{1}{2}F_{-\mathbb{1}}(\rho),\ \rho^+ = \widetilde{\phi}(1,\rho^-) = \rho + \frac{1}{2}F_{+\mathbb{1}}(\rho). \tag{42}$$

Lemma 2. *$B = D\widetilde{\phi}_1(\rho^-) : T_{\rho^-}\mathbb{R}^d \to T_{\rho^+}\mathbb{R}^d$ satisfies:*

1. *B specifies how $\widetilde{\phi}_1$ varies relative to $\widetilde{\phi}_1(\rho^-)$,*

$$\forall x \in |\Sigma| : \widetilde{\phi}_1(x) = \widetilde{\phi}_1(\rho^-) + B(x - \rho^-); \tag{43}$$

2. *B is piecewise-linear with conical subdivision*

$$\Sigma' = \left\{\Sigma'_\sigma = \operatorname{cone}\left(\Sigma_\sigma - \rho^-\right) : \sigma \in S_n\right\}; \quad (44)$$

3. $B|_{\Sigma'_\sigma}$ *is linear for all* $\sigma \in S_n$ *and* $\forall \delta\rho \in \Sigma'_\sigma$:

$$B(\delta\rho) = M_\sigma \cdot \delta\rho; \quad (45)$$

4. $\mathcal{L} = \mathcal{K} + \operatorname{span} F_{-\mathbb{1}}(\rho)$ *is a* $(d-n+1)$*-dimensional lineality space for* Σ' *and* $\forall \sigma \in S_n$:

$$\Sigma'_\sigma = \mathcal{L} + \operatorname{cone}\left\{\Pi^\perp_{\mathcal{L}} \cdot (\zeta_{\sigma(\{0,\ldots,k\})} - \rho)\right\}_{k=1}^{n-1}, \quad (46)$$

where $\Pi^\perp_{\mathcal{L}}$ *is the orthogonal projection onto* $\mathcal{L}^\perp$*;*

5. $B|_{\mathcal{L}}$ *is linear and* $\forall \delta\rho \in T_{\rho^-}\mathbb{R}^d$:

$$B(\delta\rho) = B(\Pi_{\mathcal{L}} \cdot \delta\rho) + B\left(\Pi^\perp_{\mathcal{L}} \cdot \delta\rho\right), \quad (47)$$

where $\Pi_{\mathcal{L}}$ *is the orthogonal projection onto* $\mathcal{L}$.

Proof. Each point follows from straightforward application of results in [12]: (1.), (2.), and (3.) are conclusions (4.), (3.), and (2.), respectively, of [12, Prop. 2.2.6]; (4.) follows from the definitions of lineality space [12, Sec. 2.1.2] and the ζ_b's Eq. (33); (5.) is a restatement of [12, Lem. 2.3.2]. □

5.3 B-derivative of ϕ

The goal of this subsection is to show that the piecewise-linear function B triangulated in Sec. 5.2 gives the non-linear part of the desired B-derivative $D_x\phi(t,x)$ and[12]

$$W_\sigma = D_x\phi(s,x)^{-1}\left(\Sigma'_\sigma\right) \subset T_x\mathbb{R}^d \quad (48)$$

is the cone of tangent vectors where the saltation matrix M_σ is active in Eq. (26).

Theorem 1. *Suppose vector field* $F : \mathbb{R}^d \to T\mathbb{R}^d$ *is event-selected* C^r *with respect to* $h : \mathbb{R}^d \to \mathbb{R}^n$ *at* ρ*. Let* $\phi : \mathcal{F} \to \mathbb{R}^d$ *be the* PC^r *flow of* F *and* $s,t \in \mathbb{R}$, $x \in \mathbb{R}^d$ *be such that* $0 < s < t$ *and* F *is* C^r *on* $\phi([0,t]\setminus\{s\},x) \subset \mathbb{R}^d$*. Then with* $\rho = \phi(s,x)$*, the B-derivative of the flow* ϕ *with respect to state,* $D_x\phi(t,x) : T_x\mathbb{R}^d \to T_{\phi(t,x)}\mathbb{R}^d$*, is given* $\forall \delta x \in W_\sigma \subset T_x\mathbb{R}^d$ *by*

$$D_x\phi(t,x;\delta x) = D_x\phi(t-s,\rho)\cdot B(D_x\phi(s,x)\cdot\delta x), \quad (49a)$$
$$D_x\phi(t,x;\delta x) = D_x\phi(t-s,\rho)\cdot M_\sigma \cdot D_x\phi(s,x)\cdot\delta x, \quad (49b)$$

where B *is the continuous piecewise-linear function from Lemma 2,* W_σ *is the cone from Eq. (48), and* M_σ *is the saltation matrix from Eq. (27).*

[12]Here and in what follows we mildly abuse notation via the natural vector space isomorphisms $\mathbb{R}^d \simeq T_{\rho^-}\mathbb{R}^d \simeq T_{\rho^+}\mathbb{R}^d \simeq T_\rho\mathbb{R}^d$.

Proof. Note that Eq. (49a) follows from Eq. (49b) by Eq. (45), and the fact that "pieces" of the B-derivative $D_x\phi(t,x)$ are determined by the collection of saltation matrices $\{M_\sigma\}_{\sigma\in S_n}$ was recalled in Sec. 4.4. Thus, to establish Eq. (49b) what remains to be shown is that M_σ is the active "piece" for all $\delta x \in W_\sigma$, i.e. that $\{W_\sigma\}_{\sigma\in S_n}$ is a conical subdivision for the piecewise-linear operator $D_x\phi(t,x)$, with W_σ as defined in Eq. (48).

Given $\delta x \in \operatorname{Int} W_\sigma$ let $\delta\rho = D_x\phi(s,x)\cdot\delta x \in \operatorname{Int}\Sigma'_\sigma$ so that

$$\widetilde{\tau}_{\sigma(1)}(\rho+\delta\rho) < \widetilde{\tau}_{\sigma(2)}(\rho+\delta\rho) < \cdots < \widetilde{\tau}_{\sigma(n)}(\rho+\delta\rho) \quad (50)$$

where $\widetilde{\tau}$ is the time-to-impact function for the sampled system as defined in Eq. (32). Note that $D_x\phi(t,x)$ is linear on $\operatorname{span} F(x)$, i.e. $\forall \alpha \in \mathbb{R}$

$$D_x\phi(t,x;\delta x + \alpha F(x)) = D_x\phi(t,x;\delta x) + \alpha F(\phi(t,x)), \quad (51)$$

so without loss of generality we assume $\delta\rho \in \operatorname{Int}\widetilde{D}_{-\mathbb{1}}$ by translating δx in the $-F(x)$ direction. We claim for all $\alpha > 0$ sufficiently small that $\phi(t, x+\alpha\,\delta x)$ passes through the event surfaces with the same sequence as $\widetilde{\phi}(1,\rho+\alpha\,\delta\rho)$, i.e. that

$$\tau_{\sigma(1)}(x+\alpha\,\delta x) < \tau_{\sigma(2)}(x+\alpha\,\delta x) < \cdots < \tau_{\sigma(n)}(x+\alpha\,\delta x), \quad (52)$$

where τ is the time-to-impact function defined in Eq. (31). To see this, note that $\forall k \in \{1,\ldots,n\}$:

$$\tau_{\sigma(k)}(x+\alpha\,\delta x) - \tau_{\sigma(k)}(x) \quad (53a)$$
$$= D\tau_{\sigma(k)}(x;\alpha\,\delta x) + O\left(\alpha^2\right) \quad (53b)$$
$$= D\widetilde{\tau}_{\sigma(k)}(\rho;\alpha\,\delta\rho) + O\left(\alpha^2\right) \quad (53c)$$
$$= \widetilde{\tau}_{\sigma(k)}(\rho+\alpha\,\delta\rho) - \widetilde{\tau}_{\sigma(k)}(\rho) + O\left(\alpha^2\right) \quad (53d)$$

where: Eq. (53b) since τ is PC^r; Eq. (53c) since $\delta\rho = D_x\phi(s,x)\cdot\delta x$ and $D\tau(x;\delta x)$, $D\widetilde{\tau}(\rho;\delta\rho)$ are are determined by the same data, namely, $Dh_{\sigma(k)}(\rho)$ and $F_{-\mathbb{1}}(\rho)$; Eq. (53d) since $\delta\rho \in \Sigma'_\sigma$. Combining the approximation Eq. (53) with Eq. (50) yields Eq. (52) as desired.

We conclude that $\{W_\sigma\}_{\sigma\in S_n}$ is a conical subdivision for the piecewise-linear operator $D_x\phi(t,x)$, which verifies Eq. (49) and completes the proof. □

Remark 1. *The only non-classical part of the B-derivative of the flow in Eq. (49a) is the piecewise-linear function B. Although there are* $n!$ *pieces of B in general, we explicitly represent all pieces using a triangulation of* 2^n *sample points defined in Eq. (35), achieving a substantial reduction – from factorial to "merely" exponential – of the information needed to represent the first-order approximation of the flow. Note that B implicitly determines the transition sequence* σ *associated with the perturbation direction* δx *in Eq. (49a), whereas this sequence must be explicitly specified to select the appropriate saltation matrix* M_σ *in Eq. (49b).*

6 Computation

We now attend to the complexity of the computational tasks required to construct or evaluate the B-derivative representation from the preceding section. To that end, let $F : \mathbb{R}^d \to T\mathbb{R}^d$ be an event-selected C^r vector field with respect to $h \in C^r(\mathbb{R}^d, \mathbb{R}^n)$ and $\phi : \mathcal{F} \to \mathbb{R}^d$ its piecewise-C^r flow, and assume $s,t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ are such that $0 < s < t$, $\rho = \phi(s,x)$, and the vector field F is C^r on $\phi([0,t] \setminus \{s\}, x)$.

We seek to compute $D_x\phi(t,x;\delta x)$ given $\delta x \in T_x\mathbb{R}^d$. Since Eq. (49a) from Theorem 1 yields

$$D_x\phi(t,x;\delta x) = D_x\phi(t-s,x) \cdot B(D_x\phi(s,x)\cdot\delta x) \tag{54}$$

where $B : T_\rho\mathbb{R}^d \to T_\rho\mathbb{R}^d$, the crux of the computation is

$$\delta\rho^+ = B(\delta\rho^-) \tag{55}$$

where $\delta\rho^- = D_x\phi(s,x)\cdot\delta x$. In fact, Lemma 2 offers further simplification via Eq. (47): since $B = B\circ\Pi_{\mathcal{L}} + B\circ\Pi_{\mathcal{L}}^\perp$ where $B\circ\Pi_{\mathcal{L}}$ is the linear function

$$B\circ\Pi_{\mathcal{L}}\cdot\delta\rho^- = \left(I_d + (F_{+\mathbb{1}}(\rho) - F_{-\mathbb{1}}(\rho))\cdot\frac{F_{-\mathbb{1}}(\rho)^\top}{\|F_{-\mathbb{1}}(\rho)\|^2}\right)\cdot\Pi_{\mathcal{L}}\cdot\delta\rho^-, \tag{56}$$

only the piecewise-linear function $B\circ\Pi_{\mathcal{L}}^\perp$ (equivalently, the restriction $B|_{\mathcal{L}^\perp}$) requires special consideration. In what follows, we will assume the following data, needed to construct the *sampled system* illustrated in Fig. 1(b), is given: linearly-independent normal vectors for the surfaces of discontinuity, i.e. $Dh(\rho) \in \mathbb{R}^{n\times d}$ with $\operatorname{rank} Dh(\rho) = n$; limiting values of the vector field at the point of intersection, i.e. $F_b(\rho) \in T_\rho\mathbb{R}^d$ for each $b \in B_n$; and F-derivatives of the continuously-differentiable parts of the flow, i.e. $D_x\phi(s,x), D_x\phi(t-s,x) \in \mathbb{R}^{d\times d}$.

6.1 Constructing the B-derivative

Lemma 2 demonstrates that there are $n!$ pieces of the piecewise-linear function B, namely, the collection of saltation matrices $\{M_\sigma\}_{\sigma\in S_n}$ in Eq. (45) that are active on the corresponding polyhedral cones in the conical subdivision $\Sigma' = \{\Sigma'_\sigma\}_{\sigma\in S_n}$ in Eq. (44). These polyhedral cones are generated by the 2^{n-1} points $\{\zeta_b : b \in B_n \setminus \{-\mathbb{1}, +\mathbb{1}\}\}$ in Eq. (46). For each $b \in B_n$, the point $\zeta_b \in \mathcal{K}^\perp + \{\rho\}$ where $\mathcal{K} = \ker Dh(\rho)$ can be determined by solving the n affine equations with n unknowns in Eq. (33). Given $\sigma \in S_n$, the linear piece $B|_{\mathcal{L}^\perp\cap\Sigma'_\sigma}$ can be constructed using the *saltation matrix* [8, Sec. 7.1.6] since $B(\delta\rho^-) = M_\sigma\cdot\delta\rho^-$ for all $\delta\rho^- \in \mathcal{L}^\perp\cap\Sigma'_\sigma$ where[13]

$$M_\sigma = \prod_{k=0}^{n-1}\left(I_d + \frac{\left(F_{\sigma(0:k+1)}(\rho) - F_{\sigma(0:k)}(\rho)\right)}{Dh_{\sigma(0:k)}(\rho)\cdot F_{\sigma(0:k)}(\rho)}\cdot Dh_{\sigma(0:k)}(\rho)\right), \tag{57}$$

or using *barycentric coordinates* [33, Eq. (3.1)] since $B(\delta\rho^-) = Z_\sigma^+\cdot(Z_\sigma^-)^\dagger\cdot\delta\rho^-$ for all $\delta\rho^- \in \mathcal{L}^\perp\cap\Sigma'_\sigma$ where

$$Z_\sigma^\pm = \left[z^\pm_{\sigma(0:1)}\ z^\pm_{\sigma(0:2)}\ \cdots\ z^\pm_{\sigma(0:n-1)}\right] \in \mathbb{R}^{d\times(n-1)}, \tag{58}$$

$$\forall b \in \Delta'_\sigma : z_b^- = \Pi_{\mathcal{L}}^\perp\cdot(\zeta_b - \rho),\ z_b^+ = B|_{\mathcal{L}^\perp}(z_b^-), \tag{59}$$

$$\Delta'_\sigma = \{\sigma(\{0,1,\ldots,k\})\}_{k=1}^{n-1}; \tag{60}$$

note that the pseudo-inverse $(Z_\sigma^-)^\dagger$ is injective on $\mathcal{L}^\perp\cap\Sigma'_\sigma$ by Eq. (36a) and Eq. (46). Although the matrices $M_\sigma, Z_\sigma^+\cdot(Z_\sigma^-)^\dagger \in \mathbb{R}^{d\times d}$ define the same linear transformation on the $(n-1)$-dimensional cone $\mathcal{L}^\perp\cap\Sigma'_\sigma$, they are generally not the same matrix. We conclude by noting that constructing the saltation matrix in Eq. (57) requires $O(nd^2)$ time and $O(d^2)$ space, whereas constructing the Barycentric coordinates in Eq. (58) requires $O(n^2d^2)$ time and $O(d^2)$ space (although evaluating the expression $Z_\sigma^+\cdot(Z_\sigma^-)^\dagger\cdot\delta\rho^-$ requires only $O(nd^2)$ time given $Z_\sigma^\pm$).

6.2 Evaluating the B-derivative

One obvious strategy to evaluate B on $\delta\rho^- \in T_\rho\mathbb{R}^d$ is to (i) determine $\sigma \in S_n$ such that $\delta\rho^- \in \Sigma'_\sigma$ then (ii) apply the corresponding saltation matrix or barycentric coordinates calculation from the preceding section. The general formulation of (i), termed the *point location* problem in the computational geometry literature, is "essentially open" [39, Sec. 6.5]. For an arrangement of m hyperplanes in $\mathbb{R}^d$, queries can be answered in $O(d\log m)$ time at the expense of $O(m^d)$ space [40]. In our context, the conical subdivision Σ' in Eq. (46) is determined by an arrangement of $m = O(n!^2)$ hyperplanes, so this general-purpose algorithm has time complexity $O(d\log n!) = O(dn\log n)$ and space complexity $O(n!^d)$.

The relationship established by Eq. (43) between the desired B-derivative and the flow of the *sampled system* illustrated in Fig. 1(b) suggests a different strategy, summarized in Fig. 2, with slightly worse $O(n^2d)$ time complexity but dramatically superior $O(d)$ space complexity. To understand the strategy, interpret the tangent vector $\delta\rho^- \in T_{\rho^-}\mathbb{R}^d$ as a perturbation away from the point $\rho^- = \rho - \frac{1}{2}F_{-\mathbb{1}}(\rho)$ that

[13] We mildly abuse notation as in Sec. 5.1 by using $\sigma \in S_n$ to specify $n+1$ elements of $b \in B_n$: for each $k \in \{0,\ldots,n\}$, we let $\sigma(0:k) = \sigma(\{0,\ldots,k\}) \subset \{1,\ldots,n\}$ specify the unique $b \in B_n$ whose j-th component is $+1$ if and only if $j \in \sigma(\{0,\ldots,k\})$.

flows through ρ to $\rho^+ = \rho + \frac{1}{2}F_{+\mathbb{1}}(\rho)$ in one unit of time and observe that[14] $\delta\rho^+ = \widetilde{\phi}_1(\rho^- + \delta\rho^-) - \rho^+ = B(\delta\rho^-)$ as in Eq. (43). The flow of the sampled system $\widetilde{\phi}_1$ is piecewise-affine, and can be evaluated on a given perturbation vector $\delta\rho^-$ by performing a sequence of n affine projections (one for each of the affine subspaces $\left\{\widetilde{H}_j\right\}_{j=1}^n$ where $\widetilde{F}$ is discontinuous) specified by the permutation $\sigma \in S_n$ for which $\delta\rho^- \in \Sigma'_\sigma$. Fortuitously, the sequence σ can be determined inductively as follows. First, define

$$\begin{aligned}
\delta t_1 &= 0, \\
\delta\rho_1 &= \delta\rho^-, \\
\sigma(1) &= \arg\min\left\{-\frac{Dh_j(\rho)\cdot\delta\rho_1}{Dh_j(\rho)\cdot F_{-\mathbb{1}}(\rho)} : j \in \{1,\ldots,n\}\right\}, \\
\tau_1 &= -\frac{Dh_{\sigma(1)}(\rho)\cdot\delta\rho_1}{Dh_{\sigma(1)}(\rho)\cdot F_{-\mathbb{1}}(\rho)}.
\end{aligned} \tag{61}$$

Then for $k \in \{1,\ldots,n-1\}$ inductively define

$$\begin{aligned}
\delta t_{k+1} &= \delta t_k + \tau_k, \\
\delta\rho_{k+1} &= \delta\rho_k + \tau_k \cdot F_{\sigma(\{0,\ldots,k-1\})}(\rho), \\
\sigma(k+1) &= \arg\min\left\{-\frac{Dh_j(\rho)\cdot\delta\rho_{k+1}}{Dh_j(\rho)\cdot F_{\sigma(\{0,\ldots,k\})}(\rho)}\right. \\
&\qquad\left. : j \in \{1,\ldots,n\}\setminus\sigma(\{1,\ldots,k\})\right\}, \\
\tau_{k+1} &= -\frac{Dh_{\sigma(k+1)}(\rho)\cdot\delta\rho_{k+1}}{Dh_{\sigma(k+1)}(\rho)\cdot F_{\sigma(\{0,\ldots,k\})}(\rho)}.
\end{aligned} \tag{62}$$

Finally, set $\delta\rho^+ = \delta\rho_n - (\delta t_n + \tau_n)\cdot F_{+\mathbb{1}}(\rho)$. By construction, $\delta\rho^- \in \Sigma'_\sigma$ and $\delta\rho^+ = B(\delta\rho^-)$. This strategy is succinctly summarized in pseudocode and sourcecode in Fig. 2; its time complexity is $O\left(n^2 d\right)$ since there are n steps in the induction and each step requires $O(n)$ dot products between d-vectors. The space complexity is $O(d)$ since each step in the induction requires $O(d)$ storage and data from preceding steps can be forgotten or overwritten.

We conclude by noting that, if a general-purpose algorithm is employed to solve the point location problem in $O(d\,n\log n)$ time to obtain the sequence $\sigma \in S_n$, then the induction described in the preceding paragraph can be simplified by skipping the steps that determine $\sigma(1)$ and $\sigma(k+1)$ from Eq. (61) and Eq. (62). This simplification reduces the time complexity of the induction to $O(nd)$, so the overall algorithm retains the $O(d\,n\log n)$ time complexity of the general-purpose point-location algorithm (at the expense of the superexponential $O\left(n!^d\right)$ space complexity of the point location algorithm). We are pessimistic these asymptotic complexities can be improved in general.

[14] This equation only holds when $\|\delta\rho^-\|$ is small enough to ensure $\rho^- + \delta\rho^- \in \widetilde{D}_{-\mathbb{1}}$ and $\rho^+ + \delta\rho^+ \in \widetilde{D}_{+\mathbb{1}}$; since the B-derivative is positively-homogeneous, we impose this restriction without loss of generality.

7 Conclusion

We constructed a representation for the *Bouligand* (or *B-*)derivative of the *piecewise-C^r (PC^r) flow* generated by an *event-selected C^r (EC^r) vector field* and applied the representation to derive a polynomial-time algorithm to evaluate the B-derivative on a given tangent vector. Our results provide a foundation that may support future work generalizing classical analysis and synthesis techniques for smooth control systems to the class of nonsmooth systems considered here. In particular, we envision applying our results to design and control the class of mechanical systems subject to unilateral constraints that arise in models of robot locomotion and manipulation.

References

[1] Hespanha, J. P., 2009. *Linear systems theory*. Princeton University Press.

[2] Sastry, S. S., 1999. *Nonlinear Systems: Analysis, Stability, and Control*. Springer.

[3] Bertsekas, D. P., 1999. *Nonlinear Programming*, 2nd ed. Athena Scientific.

[4] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F., 1962. *The mathematical theory of optimal processes (translated by KN Trirogoff)*. John Wiley & Sons.

[5] Sastry, S., and Bodson, M., 1989. *Adaptive control: stability, convergence, and robustness*. Prentice Hall.

[6] Ljung, L., 1999. *System identification: theory for the user*. Prentice-Hall.

[7] Polak, E., 1997. *Optimization: algorithms and consistent approximations*. Springer-Verlag.

[8] Burden, S. A., Sastry, S. S., Koditschek, D. E., and Revzen, S., 2016. "Event-selected vector field discontinuities yield piecewise-differentiable flows". *SIAM Journal on Applied Dynamical Systems,* **15**(2), pp. 1227–1267. doi:http://dx.doi.org/10.1137/15M1016588.

[9] Utkin, V., 1977. "Variable structure systems with sliding modes". *IEEE Transactions on Automatic Control,* **22**(2), pp. 212–222. doi:http://dx.doi.org/10.1109/TAC.1977.1101446.

[10] Jeffrey, M. R., 2014. "Dynamics at a switching intersection: hierarchy, isonomy, and multiple sliding". *SIAM Journal on Applied Dynamical Systems,* **13**(3), pp. 1082–1105. doi:http://dx.doi.org/10.1137/13093368X.

[11] Simic, S. N., Johansson, K. H., Lygeros, J., and Sastry, S., 2005. "Towards a geometric theory of hybrid systems". *Dynamics of Continuous, Discrete & Impulsive Systems. Series B. Applications & Algorithms,* **12**(5-6), pp. 649–687.

[12] Scholtes, S., 2012. *Introduction to piecewise differentiable equations*. Springer-Verlag. doi:http://dx.doi.org/10.1007/978-1-4614-4340-7.

[13] Aizerman, M. A., and Gantmacher, F. R., 1958. "Determination of stability by linear approximation of a periodic solution of a system of differential equa-

tions with discontinuous right-hand sides". *The Quarterly Journal of Mechanics and Applied Mathematics,* **11**(4), pp. 385–398. doi:http://dx.doi.org/10.1093/qjmam/11.4.385.

[14] Ivanov, A. P., 1998. "The stability of periodic solutions of discontinuous systems that intersect several surfaces of discontinuity". *Journal of Applied Mathematics and Mechanics,* **62**(5), pp. 677–685. doi:http://dx.doi.org/10.1016/S0021-8928(98)00087-2.

[15] Hiskens, I. A., and Pai, M. A., 2000. "Trajectory sensitivity analysis of hybrid systems". *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications,* **47**(2), pp. 204–220. doi:http://dx.doi.org/10.1109/81.828574.

[16] Dieci, L., and Lopez, L., 2011. "Fundamental matrix solutions of piecewise smooth differential systems". *Mathematics and Computers in Simulation,* **81**(5), pp. 932–953. doi:http://dx.doi.org/10.1016/j.matcom.2010.10.012.

[17] di Bernardo, M., Budd, C., Champneys, A. R., and Kowalczyk, P., 2008. *Piecewise-smooth dynamical systems: theory and applications*, Vol. 163. Springer Science + Business Media.

[18] Bizzarri, F., Brambilla, A., and Storti Gajani, G., 2013. "Lyapunov exponents computation for hybrid neurons". *Journal of Computational Neuroscience,* **35**(2), pp. 201–212. doi:http://dx.doi.org/10.1007/s10827-013-0448-6.

[19] Parker, T. S., and Chua, L. O., 1989. *Practical numerical algorithms for chaotic systems*. Springer.

[20] Leine, R. I., and Nijmeijer, H., 2013. *Dynamics and bifurcations of non-smooth mechanical systems*, Vol. 18. Springer Science + Business Media.

[21] Python Software Foundation. Python language reference, version 3.7.

[22] Oliphant, T. E., 2006. *A guide to NumPy*, Vol. 1. Trelgol Publishing USA.

[23] Lötstedt, P., 1982. "Mechanical systems of rigid bodies subject to unilateral constraints". *SIAM Journal on Applied Mathematics,* **42**(2), pp. 281–296. doi:http://dx.doi.org/10.1137/0142022.

[24] Ballard, P., 2000. "The dynamics of discrete mechanical systems with perfect unilateral constraints". *Archive for Rational Mechanics and Analysis,* **154**(3), pp. 199–274. doi:http://dx.doi.org/10.1007/s002050000105.

[25] Johnson, A. M., Burden, S. A., and Koditschek, D. E., 2016. "A hybrid systems model for simple manipulation and self-manipulation systems". *The International Journal of Robotics Research,* **35**(11), pp. 1354–1392. doi:http://dx.doi.org/10.1177/0278364916639380.

[26] Eldering, J., and Jacobs, H., 2016. "The role of symmetry and dissipation in biolocomotion". *SIAM Journal on Applied Dynamical Systems,* **15**(1), Jan., pp. 24–59. doi:http://dx.doi.org/10.1137/140970914.

[27] Pace, A. M., and Burden, S. A., 2017. "Piecewise-differentiable trajectory outcomes in mechanical systems subject to unilateral constraints". In Hybrid Systems: Computation and Control (HSCC), pp. 243–252. doi:http://dx.doi.org/10.1145/3049797.3049807.

[28] Tornambe, A., 1999. "Modeling and control of impact in mechanical systems: theory and experimental results". *IEEE Transactions on Automatic Control,* **44**(2), Feb., pp. 294–309. doi:http://dx.doi.org/10.1109/9.746255.

[29] Elandt, R., Drumwright, E., Sherman, M., and Ruina, A., 2019. "A pressure field model for fast, robust approximation of net contact force and moment between nominally rigid objects". In IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), pp. 8238–8245. doi:http://dx.doi.org/10.1109/IROS40897.2019.8968548.

[30] Aguilar, J., and Goldman, D. I., 2015. "Robophysical study of jumping dynamics on granular media". *Nature physics,* **12**(3), Nov., p. nphys3568. doi:http://dx.doi.org/10.1038/nphys3568.

[31] Collins, S. H., Wiggin, M. B., and Sawicki, G. S., 2015. "Reducing the energy cost of human walking using an unpowered exoskeleton". *Nature,* **522**(7555), June, pp. 212–215. doi:http://dx.doi.org/10.1038/nature14288.

[32] Sreenath, K., Park, H.-W., Poulakakis, I., and Grizzle, J. W., 2011. "A Compliant Hybrid Zero Dynamics Controller for Stable, Efficient and Fast Bipedal Walking on MABEL". *The International Journal of Robotics Research,* **30**(9), Aug., pp. 1170–1193. doi:http://dx.doi.org/10.1177/0278364910379882.

[33] Groff, R. E., 2003. "Piecewise linear homeomorphisms for approximation of invertible maps". PhD thesis, University of Michigan.

[34] Hatcher, A., 2002. *Algebraic topology*. Cambridge University Press.

[35] Robinson, S. M., 1987. "Local structure of feasible sets in nonlinear programming, part III: stability and sensitivity". *Nonlinear Analysis and Optimization,* **30**, pp. 45–66. doi:http://dx.doi.org/10.1007/BFb0121154.

[36] Rockafellar, R. T., 2003. "A property of piecewise smooth functions". *Computational Optimization and Applications,* **25**(1/3), pp. 247–250. doi:http://dx.doi.org/10.1023/A:1022921624832.

[37] Ralph, D., and Scholtes, S., 1997. "Sensitivity analysis of composite piecewise smooth equations". *Mathematical Programming,* **76**(3), pp. 593–612. doi:http://dx.doi.org/10.1007/BF02614400.

[38] Filippov, A. F., 1988. *Differential equations with discontinuous righthand sides*. Springer.

[39] de Berg, M., van Kreveld, M., Overmars, M., and Schwarzkopf, O., 2000. *Computational Geometry: Algorithms and Applications*. Springer.

[40] Chazelle, B., and Friedman, J., 1994. "Point location among hyperplanes and unidirectional ray-shooting". *Computational Geometry,* **4**(2), pp. 53–62. doi:http://dx.doi.org/10.1016/0925-7721(94)90009-4.

