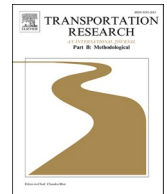




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# A multiple discrete continuous extreme value choice (MDCEV) model with a linear utility profile for the outside good recognizing positive consumption constraints

Chandra R. Bhat<sup>a,b,\*</sup>, Aupal Mondal<sup>a</sup>, Abdul Rawoof Pinjari<sup>c</sup>, Shobhit Saxena<sup>d</sup>,  
Ram M. Pendyala<sup>e</sup>

<sup>a</sup> Department of Civil, Architectural and Environmental Engineering, The University of Texas at Austin, 301 E. Dean Keeton St. Stop C1761, Austin, TX 78712, USA

<sup>b</sup> The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

<sup>c</sup> Department of Civil Engineering, Center for Infrastructure, Sustainable Transportation, and Urban Planning (CiSTUP), Indian Institute of Science (IISc), Bangalore 560012, India

<sup>d</sup> Department of Civil Engineering, Indian Institute of Science (IISc), Bangalore 560012, India

<sup>e</sup> Arizona State University, School of Sustainable Engineering and the Built Environment, 660 S. College Avenue, Tempe, AZ 85287, USA

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## ABSTRACT

A variant of the traditional multiple discrete-continuous extreme value (MDCEV) model that obviates the need to have budget information, labeled as the  $L\gamma$ -profile MDCEV model, has been proposed recently. This new model structure breaks the strong linkage between the discrete and continuous choice dimensions of decision-making. But recent studies show that this  $L\gamma$ -profile model may not work well in situations when, even if the budget is unobserved, the budget is known to be finite and small in magnitude. The reason is that the formulation, while ensuring the positivity of consumptions of the inside goods (that may or may not be consumed), does not guarantee, within the model formulation and estimation itself, the positivity of the consumption of the essential outside good. In this paper, we develop a formulation based on a reverse Gumbel structure for the stochastic terms in the utility functions of alternatives that develops a closed-form probability expression, while also accommodating the positivity requirement for the outside good. The ability of our proposed Budget-based Reverse Generalized  $L\gamma$ -profile model (labeled the BR- $GL\gamma$ -profile model) to recover true underlying model parameters is assessed. Our results clearly point to the benefit of employing the proposed model (relative to extant linear outside utility profile models in the literature) in empirical contexts when there is reason to believe that a finite ceiling applies to the budget (even if the budget is unobserved) or if the budget is actually available. In the latter case when the budget is available, our proposed model is a serious contender to the traditional  $\gamma$ -profile-MDCEV model and will generally outperform the traditional  $\gamma$ -profile-MDCEV when the consumption share of the outside good is high.

\* Corresponding author at: Department of Civil, Architectural and Environmental Engineering, The University of Texas at Austin, 301 E. Dean Keeton St. Stop C1761, Austin, TX 78712, USA.

E-mail address: [bhat@mail.utexas.edu](mailto:bhat@mail.utexas.edu) (C.R. Bhat).

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## 1. Introduction

Many consumer choice situations involving a portfolio or package choice of elemental alternatives, along with the amount of a continuous quantity to allocate to the chosen elemental alternatives, lend themselves nicely to analysis using a direct utility maximization approach (see [Wales and Woodland, 1983](#)). [Bhat \(2005, 2008\)](#) coined the term multiple discrete-continuous (MDC) choices for such situations, because these situations allow for the possibility of the choice of multiple elemental alternatives, including both a discrete element as well as a continuous element. A particularly appealing closed-form model structure following the MDC paradigm is the MDC extreme value (MDCEV) model ([Bhat, 2005, 2008](#)), which has now been applied in a wide variety of choice contexts (see, for example, [Ma et al., 2019](#); [Shin et al., 2019](#); [Varghese and Jana, 2019](#); [Mouter et al., 2021](#)).

In recent years, a variant of the MDCEV, based on employing a linear utility structure for one or more outside goods (that are essential and always consumed), has received increasing attention (see [Bhat, 2018](#); [Bhat et al., 2020](#); [Saxena et al., 2021](#)). Such a structure has the advantage of not needing the budget quantity, which indeed may be unobserved in many situations, as well as facilitates the modeling of multiple discrete-grouped (MDG) data where the amounts of consumptions are observed in grouped categories rather than in continuous form. It also loosens the strong tie between the discrete and continuous choice dimensions embedded in the traditional MDCEV model. This linear utility form (for the outside good) MDCEV model is a neat structure, but, as indicated in [Saxena et al. \(2021\)](#), may not work well in terms of data fit and prediction ability when the overall budget amount, even if unobserved, is known to be small. On the other hand, the formulation does very well when the budgets are large. The reason is that the formulation, while ensuring the positivity of consumptions of the inside goods (that may or may not be consumed), does not guarantee, within the model formulation and estimation itself, the positivity of the consumption of the essential outside good.<sup>1</sup> In addition, the estimators used thus far for the formulation with a linear utility form for the outside good do not explicitly recognize the budget constraint during estimation. But the probability that the consumption of the outside good will be positive (and that the budget constraint is met) increases as the total value of the budget increases, and tends to the value of 1 for the situation when the budget tends to infinity.

In a related effort, [Mondal and Bhat \(2021\)](#) recently proposed a reverse Gumbel distribution for the stochastic elements in the utility of alternatives (that is, a Type-1 extreme value Gumbel form based on the limiting distribution of the minimum of random variables rather than the traditional Type-1 extreme value Gumbel form based on the limiting distribution of the maximum of random variables), while maintaining a linear utility structure for the outside good. The main motivation for the use of the reverse Gumbel with the linear outside good utility structure is that it leads to a closed-form probability expression for the MDC consumption pattern, regardless of the number of linear budget constraints that dictate the MDC choice. Again, this formulation is useful for the case when budgets are not observed for each (or any) of the constraints. However, the same issues of bias in parameter estimates, poor fit to data, and poor prediction ability permeate through this alternative formulation if the budgets along any of the constraints determining choice are small in magnitude.

In this paper, we develop a formulation, also based on the reverse Gumbel structure for the stochastic terms in the utility functions of alternatives, that develops a closed-form probability expression while also accommodating the positivity requirement for the outside good. This is done through a truncation scheme that still yields a compact and closed-form likelihood expression. Importantly, the procedure works with both observed and unobserved budgets. In the case of observed budgets, a linear utility profile for the outside good can provide better results than the traditional MDCEV in cases when a very high proportion of the budget is allocated to the outside good (see [Bhat, 2018](#) for a detailed explanation). And, in the case of unobserved budgets, it can place a finite limit value (a ceiling) on what a reasonable budget may be.

The rest of this paper is structured as follows: The next section lays out the microeconomic framework and statistical specification of the proposed model. [Section 3](#) presents the forecasting approach for the model, while [Section 4](#) examines the performance of the proposed model using simulation experiments. [Section 5](#) presents an empirical application of the proposed model. [Section 6](#) summarizes the paper and identifies future research directions.

## 2. The linear outside good utility profile MDCEV model structure

Assume without any loss of generality that the essential Hicksian composite outside good is the first good. Consider the generalized version of the  $L_\gamma$ -profile utility functional form (which we will refer to as the  $GL_\gamma$  profile) as presented in [Eq. \(19\) of Bhat et al. \(2020\)](#) (this variant helps provide additional flexibility than would be possible otherwise, as discussed later in [Section 2.5](#)). Assuming that the budget information and the continuous consumption values for a sample are available, the MDC formulation is written as:

$$U(x) = \psi_1^{1-\alpha} x_1 + \sum_{k=2}^K \frac{\gamma_k}{\alpha} \psi_k^{1-\alpha} \left\{ \left( \frac{x_k}{\gamma_k} + 1 \right)^\alpha - 1 \right\} \quad (1)$$

$$s.t. \quad x_1 + \sum_{k=2}^K p_k x_k = E,$$

<sup>1</sup> It is not necessary that the outside good consumption must be above zero in all consumption situations. However, most MDC choice modeling applications in the literature involve situations where the outside good is essential (in that some part of the budget is always allocated to it). Therefore, it is important to formulate models that ensure a positive consumption for the outside good.

where the utility function  $U(\mathbf{x})$  is quasi-concave, increasing and continuously differentiable,  $\mathbf{x} \geq 0$  is the consumption quantity ( $\mathbf{x}$  is a vector of dimension  $(K \times 1)$  with elements  $x_k$ ), and  $\psi_k$  and  $\gamma_k$  are parameters associated with good  $k$ . The parameter  $\alpha$  is a fixed satiation parameter across all the inside goods (however, note that the effective satiation is different across the inside goods because of the presence of the good-specific  $\gamma_k$  parameter). The constraint in Eq. (1) is the linear budget constraint, where  $E$  is the total expenditure across all goods  $k$  ( $k = 1, 2, \dots, K$ ) and  $p_k > 0$  is the unit price of good  $k$  (with  $p_1 = 1$  to represent the numeraire nature of the first essential good). The function  $U(\mathbf{x})$  in Eq. (1) is a valid utility function if  $\psi_k > 0$  and  $\gamma_k > 0$  for all  $k$ , and  $\alpha \leq 1$  (As discussed in detail in Bhat, 2008, Section 6.1,  $\alpha$  can take negative values, but this creates instability in estimation; thus, it is common practice to require  $\alpha$  to be positive, which implies that  $0 \leq \alpha \leq 1$ ).  $\psi_k$  represents the baseline marginal utility, and  $\gamma_k$  is the vehicle to introduce corner solutions (that is, zero consumption) for the inside goods ( $k = 2, 3, \dots, K$ ), but also serves the role of a satiation parameter through translation (higher values of  $\gamma_k$  imply less satiation). The  $\alpha$  term represents an exponential satiation effect that is held fixed across all the inside goods (as explained in Bhat, 2008, it is difficult empirically to identify a separate  $\alpha$  satiation effect for each inside good, while also having separate  $\gamma_k$  satiation effects; further, the  $\alpha$  satiation effect in the utility profile form of Eq. (1) would not be identifiable unless there is price variation across the goods, as discussed later). Even so, for stability, in specific empirical contexts, it may be necessary to normalize  $\alpha$  to a specific value (such as, say  $\alpha \rightarrow 0$ , in which case the expression in Eq. (1) takes the usual  $L_\gamma$  profile for the inside goods).<sup>2</sup> There is no  $\gamma_1$  term for the first outside good because it is, by definition, always consumed. Further, as in the traditional MDCEV, we maintain the assumption that there are no cost economies of scale in the purchase of goods; that is, we will continue to retain the assumption that the unit price of a good remains constant regardless of the quantity of good consumed.

### 2.1. Optimal allocation and identification issues

To ensure the non-negativity of the baseline marginal utility, while also allowing it to vary across individuals based on observed and unobserved characteristics,  $\psi_k$  is usually parameterized as follows:

$$\psi_k = \exp(\beta' \mathbf{z}_k + \varepsilon_k), k = 1, 2, \dots, K, \quad (2)$$

where  $\mathbf{z}_k$  is a set of attributes that characterize alternative  $k$  and the decision maker (including a constant), and  $\varepsilon_k$  captures the idiosyncratic (unobserved) characteristics that impact the baseline utility of good  $k$ . A constant cannot be identified in the  $\beta$  term for one of the  $K$  alternatives. Similarly, individual-specific variables are introduced in the vector  $\mathbf{z}_k$  for  $(K-1)$  alternatives, with the remaining alternative serving as the base.

To find the optimal allocation of goods, the Lagrangian is constructed and the first order equations are derived based on the Karush-Kuhn-Tucker (KKT) conditions. The Lagrangian function for the model, when combined with the budget constraint, is:

$$L = U(\mathbf{x}) + \lambda \left( E - \sum_{k=1}^K p_k x_k \right), \quad (3)$$

where  $\lambda$  is a Lagrangian multiplier for the constraint. The KKT first order conditions for optimal consumption allocations ( $x_k^*$ ) are as follows:

$$\begin{aligned} \psi_1^{1-\alpha} - \lambda &= 0; \\ \left[ \psi_k \left( \frac{x_k^*}{\gamma_k} + 1 \right)^{-1} \right]^{1-\alpha} - \lambda p_k &= 0 \text{ if consumption} = x_k^* (x_k^* > 0), k = 2, 3, \dots, K, \\ [\psi_k]^{1-\alpha} - \lambda p_k &< 0 \text{ if } x_k^* = 0, k = 2, 3, \dots, K \end{aligned} \quad (4)$$

Substituting  $\psi_1^{1-\alpha} = \lambda$  into the latter two equations, using the statistical specification for the baseline preference functions from Eq. (2), defining  $\sigma = (1 - \alpha)$ , and taking logarithms, we get:

$$\begin{aligned} \ln \psi_k - \ln \left( \frac{x_k^*}{\gamma_k} + 1 \right) &= \ln \psi_1 + \frac{1}{\sigma} \ln p_k \text{ if consumption} = x_k^* (x_k^* > 0), k = 2, 3, \dots, K, \\ \ln \psi_k &< \ln \psi_1 + \frac{1}{\sigma} \ln p_k \text{ if } x_k^* = 0, k = 2, 3, \dots, K. \end{aligned} \quad (5)$$

After some additional algebraic operations, Eq. (5) may be written in terms of error differences between each inside good and the outside good as:

<sup>2</sup> The  $L_\gamma$  profile function form for inside good  $k$  takes the form  $\gamma_k \psi_k \ln \left( \frac{x_k}{\gamma_k} + 1 \right)$ .

$$\begin{aligned}
\eta_k &= \overleftarrow{V}_k - \overleftarrow{V}_1, \quad \eta_k = \varepsilon_k - \varepsilon_1, \quad \text{if consumption is equal to } x_k^* (k = 2, 3, \dots, K), \text{ where } x_k^* > 0 \\
\eta_k &< \overleftarrow{V}_{k0} - \overleftarrow{V}_1, \quad \eta_k = \varepsilon_k - \varepsilon_1, \quad \text{if } x_k^* = 0 (k = 2, 3, \dots, K), \text{ where} \\
V_k &= \ln \left( \frac{x_k^*}{\gamma_k} + 1 \right) - \beta' z_k, \quad \overleftarrow{V}_k = V_k + \frac{1}{\sigma} \ln p_k, \quad V_{k0} = -\beta' z_k, \quad \overleftarrow{V}_{k0} = V_{k0} + \frac{1}{\sigma} \ln p_k \quad (k = 2, 3, \dots, K), \text{ and} \\
\overleftarrow{V}_1 &= \overleftarrow{V}_{10} = V_1 = -\beta' z_1.
\end{aligned} \tag{6}$$

An important note based on the above equation system is that, if there is no price variation across the inside goods (that is,  $p_k = 1$  for all inside goods ( $k = 2, 3, \dots, K$ ), in addition to the numeraire price  $p_1 = 1$  of the outside good), the  $(1/\sigma) \ln p_k$  drops out entirely from the KKT conditions. This implies that, in our linear outside good utility structure,  $\sigma$  will be estimable only if there is price variation. An obvious normalization, in the absence of price variation, is to set  $\sigma = 1$ , which is equivalent to setting the parameter  $\alpha$  in the  $GL\gamma$  profile of Eq. (1) to the value of 0. But, in the presence of price variation, the reciprocal of  $\sigma$  is the coefficient on the  $\ln p_k$  parameter, allowing estimation of  $\sigma = 1 - \alpha$  in our utility profile (even so, it may be necessary in some contexts to pre-set  $\sigma = 1$  (that is,  $\alpha = 0$ ) for stability).

## 2.2. Condition for positive allocations for consumed goods

The linear outside good utility basis of the model above has the advantage of model estimation when there is no budget information. This is because the outside good consumption,  $x_1^*$ , does not appear in the KKT conditions of Eq. (4). The  $GL\gamma$ -profile MDCEV formulation above guarantees the positivity of the consumed inside goods. To see this, from Eq. (4), we can write:

$x_k^* = \left( \frac{\psi_k}{\psi_1 (p_k)^{(1/\sigma)}} - 1 \right) \gamma_k$  if  $x_k^* > 0$  ( $k = 2, 3, \dots, K$ ). But, from the inequality condition of Eq. (4), it should also be true that  $\left( \frac{\psi_k}{\psi_1 (p_k)^{(1/\sigma)}} > 1 \right)$ , because otherwise  $x_k^* = 0$  ( $k = 2, 3, \dots, K$ ). Thus, if the model predicts that  $x_k^* > 0$  ( $k = 2, 3, \dots, K$ ), the predicted consumption value will be positive. However, there is no guarantee in the formulation above that  $x_1^*$  will be positive for finite budgets. The implicit assumption in the linear profile outside good MDCEV models, made explicit in Mondal and Bhat (2021) and Saxena et al. (2021), is that the budget, while being unobserved, is very large and tends toward infinity. This is in contrast to Bhat's (2008) original non-linear utility MDCEV Model, where the primal feasibility condition of positive consumption of all goods (including the outside good), given a budget, is immediately satisfied based on the complementary slackness KKT first-order stochastic conditions (see footnote eight of Pinjari and Bhat, 2021). Further, in the case of finite observed budgets, the linear profile outside good MDCEV model structures used thus far in the literature do not accommodate the positivity requirement on the outside good (notwithstanding the fact that the  $L\gamma$ MDCEV model structure was conceived for the case of unobserved budgets, with an implicit assumption of very large budgets). To show this, if the budget were  $E$  and only the first  $M$  inside goods are consumed, the consumption of the inside good in the  $GL\gamma$ -profile model would be given by:

$$x_1^* = E - \sum_{k=2}^{M+1} p_k \left( \frac{\psi_k}{\psi_1 (p_k)^{(1/\sigma)}} - 1 \right) \gamma_k. \tag{7}$$

If the above outside good consumption is to be positive, it must be true that

$$\psi_1 > \frac{\sum_{k=2}^{M+1} \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^{M+1} p_k \gamma_k}, \quad \text{where } \delta = \frac{\sigma - 1}{\sigma} = \frac{-\alpha}{1 - \alpha}. \tag{8}$$

However, there is nothing in the KKT conditions of Eq. (4) that maintains this restriction. Essentially, when budgets are observed (or even when budgets are unobserved, but there is some reasonable ceiling for the budgets), the likelihood expression in all the linear utility outside good profile MDCEV model will be based on stochastic KKT conditions that provide a possible optimal solution (this solution being the set of estimated model parameters) that then has to be checked for primal feasibility to be declared as the true optimal point (primal feasibility here refers to the requirement that the outside good consumption be strictly positive; that is, that Eq. (8) holds). In effect, the model estimation is one step toward optimal consumption determination, which then needs to be vetted through a back-end forecasting stochastic truncation process to satisfy primal feasibility and obtain true optimal consumptions. Of course, when the budgets are large (moving toward infinity), the denominator of the expression in Eq. (8) also moves toward infinity, and the condition of Eq. (8) will be immediately met in the estimation process because of the already existing requirement that  $\psi_1 > 0$  (as maintained through the exponential specification for  $\psi_1$ ). Thus, as budgets become large, there is less need to consider any error truncation operations (for the error term  $\varepsilon_1$  in the inside good utility, given the error terms  $\varepsilon_k$  for the consumed inside good utilities) during forecasting, because positivity of the outside good will be near-guaranteed during the estimation step itself. On the other hand, when budgets are tight, there would be more need for truncation operations during forecasting. Thus, while positivity of the outside good can also be guaranteed during forecasting, this is done post-estimation. This can, and generally will, lead to biased parameter estimates, relatively poor model fit and poor predictions, because the likelihood of observed consumptions is maximized (in the maximum likelihood estimation process) while allowing a non-zero probability of infeasible consumptions (see Saxena et al., 2021).

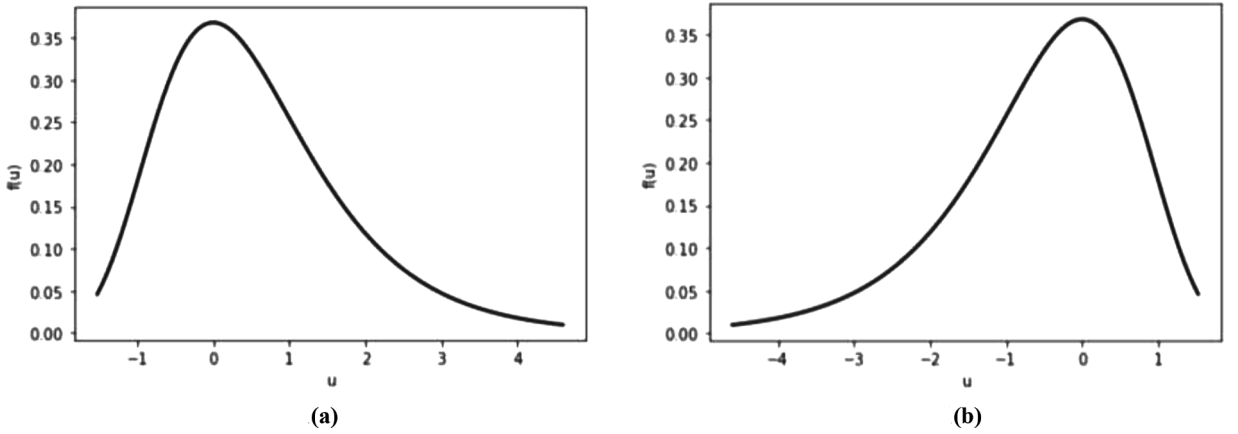


Fig. 1. (a) Type-1 extreme value (Maximum) or Gumbel distribution; (b) Type-1 extreme value (Minimum) or reverse-Gumbel distribution.

For instance, the maximization process in estimation may provide parameters that are such that it assigns non-zero probability density values to consumption patterns that drive the outside good consumption to zero or negative values (that is, non-zero and potentially high likelihood of infeasible consumption patterns). So, while this issue can be corrected in forecasting when budgets are observed, the model parameters themselves would not be as appropriate as when only the feasible consumption patterns are explicitly considered in the estimation phase itself (thus providing parameter estimates that correctly assign non-zero probability density values to only the feasible consumption patterns).

To summarize, we then need to develop a likelihood function based on the stochastic KKT conditions of Eq. (4), but while also maintaining the restriction of Eq. (8). Note that maintaining this restriction automatically ensures that the budget constraint is met (during estimation), for the restriction is obtained from a combination of two primal feasibility constraints – the budget constraint in Eq. (7) and the positivity of the consumption value of the outside good.

### 2.3. Statistical specification

The specification of the model is completed once assumptions are made regarding the joint distribution of the  $\varepsilon_k$  terms. The  $L\gamma$ MDCEV specification of Bhat (2018) and Bhat et al. (2020) uses the Type-1 extreme value (or Gumbel) distribution with non-standardized scale (based on the limiting distribution of the maximum of random variables). Unfortunately, doing so makes it difficult to maintain the restriction of Eq. (8) and certainly does not result in a closed-form expression. However, it is possible to develop a closed-form model accommodating the restriction of Eq. (8) if we assume a Gumbel distribution based on the limiting distribution of the minimum of random variables for the  $\varepsilon_k$  terms, and assume a standardized scale. That is, assume that the error terms  $\varepsilon_k$  are independent and identically distributed (IID) with a standard reverse-Gumbel distribution. The density functions of the standard Gumbel and standard reverse-Gumbel are plotted in Fig. 1a and b; as can be observed from these two figures, the reverse-Gumbel is obtained by reflection of the Gumbel about the y-axis. The probability density function and the cumulative density function of the standard reverse-Gumbel distribution are provided below.

$$f_{\varepsilon_k}(u) = e^{-e^u} \cdot e^u \text{ and } F_{\varepsilon_k}(u) = \text{Prob}(\varepsilon_k < u) = 1 - e^{-e^u} \text{ for } k = 1, 2, 3, \dots, K. \quad (9)$$

Based on the above reverse Gumbel distribution form for each error term, it is easy to see that one can write the joint multivariate survival distribution function (SDF) for the error terms  $\eta_k = \varepsilon_k - \varepsilon_1$  as follows (see Appendix A for the derivation through straightforward integration)<sup>3</sup>:

$$S_{\eta}(w_2, w_3, \dots, w_K) = \text{Prob}(\eta_2 > w_2, \eta_3 > w_3, \dots, \eta_K > w_K) = \frac{1}{(1 + \sum_{k=2}^K e^{w_k})}. \quad (10)$$

The multivariate cumulative distribution function (CDF) of the  $\eta$  vector can be written as a function of the SDFs corresponding to the random variates as follows:

$$F_{\eta}(w_2, w_3, \dots, w_K) = \text{Prob}(\eta_2 < w_2, \eta_3 < w_3, \dots, \eta_K < w_K) = 1 + \sum_{D \subset \{2, \dots, K\}; |D| \geq 1} (-1)^{|D|} S_D(w_D), \quad (11)$$

where  $S_D(\cdot)$  is the SDF of dimension  $D$ ,  $D$  represents a specific combination of the  $\eta$  terms (representing a specific sub-vector of the  $\eta$  vector; there are a total of  $(K-2) + C(K-2, 2) + C(K-2, 3) + \dots + C(K-2, K-2) = 2^{K-2} - 1$  possible combinations,  $|D|$  is the

<sup>3</sup> The  $\eta_k$  error terms are essentially multivariate logistically distributed with a correlation of 0.5, with the SDF expression as given below.

cardinality of the specific combination  $D$ , and  $\mathbf{w}_D$  is a sub-vector of the vector  $\mathbf{w} = (w_3, w_4, \dots, w_K)$  with the appropriate elements corresponding to the combination  $D$  extracted.

#### 2.4. Probability expressions ignoring positivity of outside good (the reverse-GL $\gamma$ MDCEV model or the R-GL $\gamma$ MDCEV model)

For presentation compactness, define  $\tilde{V}_{k0} = \frac{\leftrightarrow}{V_{k0}} - \frac{\leftrightarrow}{V_{10}}$  and  $\tilde{V}_k = \frac{\leftrightarrow}{V_k} - \frac{\leftrightarrow}{V_1}$  ( $k = 2, 3, \dots, K$ ). If the restriction in Eq. (8) is ignored, based on the KKT conditions, we get the expression below in the Reverse Gumbel-GL $\gamma$ (or the R-GL $\gamma$ ) MDCEV model for the consumption pattern where the first  $M$  inside goods are consumed at levels  $x_k^*$  ( $k = 2, 3, \dots, M+1$ ):

$$\begin{aligned}
 & P(x_2^*, \dots, x_{M+1}^*, 0, 0, \dots, 0) \\
 &= |J| \int_{\eta_{M+2}=-\infty}^{\tilde{\eta}_{M+2}=\tilde{V}_{M+2,0}} \int_{\eta_{M+3}=-\infty}^{\tilde{\eta}_{M+3}=\tilde{V}_{M+3,0}} \dots \int_{\eta_K=-\infty}^{\tilde{\eta}_K=\tilde{V}_{K,0}} f_\eta(\tilde{V}_2, \tilde{V}_3, \dots, \tilde{V}_{M+1}, \eta_{M+2}, \eta_{M+3}, \dots, \eta_K) d\eta_{M+2} d\eta_{M+3}, \dots, d\eta_K \\
 &= |J| \left. \frac{\partial^M F_\eta(\eta_2, \eta_3, \dots, \eta_{M+1}, \tilde{V}_{M+2,0}, \tilde{V}_{M+3,0}, \dots, \tilde{V}_{K,0})}{\partial \eta_2 \partial \eta_3 \dots \partial \eta_{M+1}} \right|_{\eta_2=\tilde{V}_2, \eta_3=\tilde{V}_3, \dots, \eta_{M+1}=\tilde{V}_{M+1}} \\
 &= |J| M! \left[ \frac{\exp\left(\sum_{i=1}^{M+1} \frac{\leftrightarrow}{V_i}\right)}{\left(\sum_{k=1}^{M+1} \exp\left(\frac{\leftrightarrow}{V_k}\right)\right)^{M+1}} + \sum_{D \subset \{M+2, M+3, \dots, K\}, |D| \geq 1} (-1)^{|D|} \frac{\exp\left(\sum_{i=1}^{M+1} \frac{\leftrightarrow}{V_i}\right)}{\left(\sum_{k=1}^{M+1} \exp\left(\frac{\leftrightarrow}{V_k}\right) + \sum_{k \in D} \exp\left(\frac{\leftrightarrow}{V_{k0}}\right)\right)^{M+1}} \right],
 \end{aligned} \tag{12}$$

where  $|J| = \left[ \prod_{i=2}^{M+1} f_i \right]$ ,  $f_i = \left( \frac{1}{x_i^* + \gamma_i} \right)$ . The probability that all the inside goods are consumed at levels  $x_2^*, x_3^*, \dots, x_K^*$  is:

$$\begin{aligned}
 & P(x_2^*, x_3^*, \dots, x_K^*) \\
 &= |J| f_\eta(V_2, V_3, V_4, \dots, V_K) = |J| M! \frac{\exp\left(\sum_{i=1}^K \frac{\leftrightarrow}{V_i}\right)}{\left(\sum_{k=1}^K \exp\left(\frac{\leftrightarrow}{V_k}\right)\right)^K}.
 \end{aligned} \tag{13}$$

As one would expect, the expression in Eq. (13) is the same probability as what would have been obtained had the traditional Gumbel distribution been used for the  $\varepsilon_k$  rather than our reverse Gumbel, because the density function of the differenced Gumbel error terms remains the same (which is the multivariate logistic distribution with 0.5 correlation). However, the probability expression is the same only for the probability of all goods being consumed (that is, only for the case represented by Eq. (13)). For the cases where some inside goods are consumed and some are not (as in Eq. (12)), or all inside goods are not consumed (see below), the probability expressions will differ between using the traditional Gumbel and the reverse Gumbel, because of the integration spaces being different. The probability that none of the inside goods are consumed is:

$$P(0, \dots, 0) = 1 + \sum_{D \subset \{2, \dots, K\}, |D| \geq 1} (-1)^{|D|} \frac{1}{\left(1 + \sum_{k \in D} e^{\tilde{V}_{k0} - \tilde{V}_{10}}\right)} = 1 + \sum_{D \subset \{2, \dots, K\}, |D| \geq 1} (-1)^{|D|} \frac{e^{\tilde{V}_{10}}}{\left(e^{\tilde{V}_{10}} + \sum_{k \in D} e^{\tilde{V}_{k0}}\right)} \tag{14}$$

#### 2.5. Probability expressions considering positivity requirement of outside good (the budget-based R-GL $\gamma$ or BR-GL $\gamma$ MDCEV model)

Taking the logarithm of Eq. (8), we get the condition for the positivity of the outside good as follows, given that the first  $M$  inside goods are consumed:

$$-V_1 + \varepsilon_1 > \ln \left( \sum_{k=2}^{M+1} [\exp(-V_{k0} + \varepsilon_k)] \gamma_k p_k^\delta \right) - \ln \left( E + \sum_{k=2}^{M+1} p_k \gamma_k \right). \tag{15}$$

Substituting  $h_k = [\exp(-V_{k0})](\gamma_k p_k^\delta)$  and re-arranging, the condition may be re-written as:

$$\left[ \varepsilon_1 - \ln \left( \sum_{k=2}^{M+1} [h_k e^{\varepsilon_k}] \right) \right] > G, \text{ where } G = V_1 - \ln \left( E + \sum_{k=2}^{M+1} p_k \gamma_k \right). \tag{16}$$

The probability of the condition above is:

$$P(x_1^* > 0) = P\left\{\left[\varepsilon_1 - \ln\left(\sum_{k=2}^{M+1} [h_k e^{\varepsilon_k}]\right)\right] > G\right\}. \quad (17)$$

Interestingly, the above probability has a closed-form solution. This is because of a surprisingly elegant property that the survival distribution function (SDF) of the difference between a reverse Gumbel distribution and the logarithm of the weighted sum of independent exponentially distributed random variables (note that  $\exp(-\varepsilon_k)$  is exponentially distributed, as long as  $\varepsilon_k$  is standard reverse-Gumbel) has a closed form (see [Appendix B](#) for the derivation).<sup>4</sup> That is,

$$P(x_1^* > 0) = P\left\{\left[\varepsilon_1 - \ln\left(\sum_{k=2}^{M+1} [h_k e^{\varepsilon_k}]\right)\right] > G\right\} = \frac{1}{\prod_{k=2}^{M+1} [1 + h_k e^G]}. \quad (18)$$

Finally, we can write the probability expressions for the consumption pattern with positive outside good consumption by a simple truncation mechanism as follows (for a derivation of the expressions below, please see [Appendix C](#)):

$$\begin{aligned} P(x_2^*, x_3^*, \dots, x_{M+1}^*, 0, \dots, 0, 0) | x_1^* > 0 &= \frac{P(x_2^*, x_3^*, \dots, x_{M+1}^*, 0, \dots, 0, 0)}{P(x_1^* > 0)} \\ P(x_2^*, x_3^*, \dots, x_{M+1}^*, x_{M+2}^*, \dots, x_K^*) | x_1^* > 0 &= \frac{P(x_2^*, x_3^*, \dots, x_{M+1}^*, x_{M+2}^*, \dots, x_K^*)}{P(x_1^* > 0)} \\ P(0, 0, 0, \dots, 0, \dots, 0, \dots, 0, \dots, 0) | x_1^* > 0 &= P(0, 0, 0, \dots, 0, \dots, 0, \dots, 0, \dots, 0). \end{aligned} \quad (19)$$

Substituting the expression from earlier for the untruncated probabilities in the numerator of the expressions above provides the necessary closed-form expressions.

## 2.6. Revisiting the truncation condition

The linear outside good utility function in [Bhat \(2018\)](#) and [Bhat et al. \(2020\)](#) is valuable as a model in the case when budgets are not observable. However, as discussed in [Saxena et al. \(2021\)](#), this linear outside good utility model will perform well only when there is reasonable support for budgets being very large, or equivalently, for the investment in the outside good(s) being much larger than the investment in the inside goods. The reason for this should be clear from the truncation probability in [Eq. \(18\)](#). As the budget along the constraint becomes larger (that is, as  $E$  becomes larger), so does  $\ln(E + \sum_{k=2}^{M+1} p_k \gamma_k)$ . And, correspondingly,  $G$  becomes negative and larger and larger in magnitude. Thus, when the budget  $E$  tends toward infinity (becomes very large),  $G \rightarrow -\infty$ . From [Eq. \(18\)](#),  $P(x_1^* > 0) \rightarrow 1$ . Thus, estimating without any truncation correction will not affect the accuracy of the model results with very high (tending toward infinity) budgets. Effectively, when the budget is high and allocation to the inside goods is relatively small, there is little need for truncation. On the other hand, with small overall budgets, the truncation correction probability  $P(x_1^* > 0)$  will be sizeable, and thus estimation without truncation can create problems.

As just discussed, in cases when the budget information is not available, but it also is not reasonable to assume that the budget is very large, the linear outside good utility function will not perform well. In such a case, one way to proceed would be to set a finite upper limit value as an approximation to the budget within the context of the proposed BR-GL $\gamma$ MDCEV model.

## 2.7. Discrete consumption probability expressions

The discrete consumption probability expressions are useful when comparing, after the models are estimated, the discrete consumption performance of our proposed BR-GL $\gamma$ MDCEV profile models with the traditional  $\gamma$ -profile MDCEV model of [Bhat \(2008\)](#). We first present the discrete consumption probability expression for the R-GL $\gamma$ MDCEV Model for each possible consumption bundle. For the R-GL $\gamma$ MDCEV Model, we may write:

$$\begin{aligned} P(d_2 = 1, d_3 = 1, \dots, d_{M+1} = 1, d_{M+2} = 0, \dots, d_{K-1} = 0, d_K = 0) \\ = \int_{\eta_2 = \tilde{V}_{2,0}}^{\infty} \int_{\eta_3 = \tilde{V}_{3,0}}^{\infty} \dots \int_{\eta_{M+1} = \tilde{V}_{M+1,0}}^{\infty} \int_{\eta_{M+2} = \tilde{V}_{M+2,0}}^{\infty} \dots \int_{\eta_{K-1} = \tilde{V}_{K-1,0}}^{\infty} \int_{\eta_K = \tilde{V}_{K,0}}^{\infty} f(\eta_2, \eta_3, \dots, \eta_K) d\eta_K d\eta_{K-1}, \dots, d\eta_2, \end{aligned} \quad (20)$$

where  $f(\eta_2, \eta_3, \dots, \eta_K)$  represents the multivariate density function (pdf) of the random variates  $\eta_2, \eta_3, \dots, \eta_K$ . The above expression may be expressed as:

<sup>4</sup> The property here applies only if each  $\varepsilon_k$  ( $k = 1, 2, \dots, K$ ) is standard reverse-Gumbel. But, by including an extra parameter  $\alpha$  in the utility function of [Eq. \(1\)](#) (rather than pre-imposing the traditional  $L\gamma$ -MDCEV profile that constrains  $\alpha = 0$ ), we recover a level of flexibility in the model through the  $\alpha$  parameter, even as we constrain the scale to be standardized. Of course, as discussed in [Section 2.1](#),  $\alpha$  is estimable in our formulation only if there is price variation across the inside goods.



$$P(d_2 = 1, d_3 = 1, \dots, d_{M+1} = 1, d_{M+2} = 0, \dots, d_{K-1} = 0, d_K = 0) \\ = S_M(\tilde{V}_{2,0}, \tilde{V}_{3,0}, \dots, \tilde{V}_{M+1,0}) + \sum_{D \subset \{M+2, \dots, K-1, K\}, |D| \geq 1} (-1)^{|D|} S_{M+|D|}(\tilde{V}_{2,0}, \tilde{V}_{3,0}, \dots, \tilde{V}_{M+1,0}, \tilde{V}_{D,0}), \quad (21)$$

where  $S_N(\cdot)$  for any dimension  $N$  is the multivariate survival distribution function given by Eq. (10),  $D$  represents a specific combination of the non-consumed goods (there are a total of  $2^{K-M-1} - 1$  possible combinations of the non-consumed goods),  $|D|$  is the cardinality of the specific combination  $D$ , and  $\tilde{V}_{D,0}$  is a vector with elements  $\tilde{V}_{j_d,0}$  of the non-consumed goods  $j_d$  appearing in combination  $D$ . The discrete consumption probability for the case of none of the inside goods being consumed is already provided in Eq. (14), while the discrete consumption probability for the case of all the inside goods being consumed is given by:

$$P(d_2 = 1, d_3 = 1, \dots, d_{M+1} = 1, d_{M+2} = 1, \dots, d_{K-1} = 1, d_K = 1) = S_{K-1}(\tilde{V}_{2,0}, \tilde{V}_{3,0}, \dots, \tilde{V}_{K,0}). \quad (22)$$

The corresponding expressions for the BR-GL $\gamma$ MDCEV model may then be obtained by dividing the expressions above by  $P(x_1^* > 0)$ .

### 3. Forecasting procedure

The forecasting procedure described in this section is based on the BR-GL $\gamma$ MDCEV model.

Given: The input data  $\mathbf{z}_k$  and  $p_k$ , and estimates of the model parameters  $(\beta', \gamma_2, \gamma_3, \dots, \gamma_K, \sigma')$ , where  $\sigma = 1 - \alpha$ . As earlier,  $\delta = \frac{\sigma-1}{\sigma} = \frac{\alpha}{1-\alpha}$ .

- Step 1: Draw  $K$  independent realizations of  $\varepsilon_k$  (say  $\mu_k$ ), one for each good  $k$  ( $k = 1, 2, \dots, K$ ) from the reverse extreme value distribution with location parameter of 0 and the scale parameter equal to one; label this distribution as REV(0,1).
- Step 2: Compute  $H_{k,0} = \mu_k - \tilde{V}_{k,0}$  for each inside good  $k = 2, 3, \dots, K$  using the inputs, and set  $H_{1,0}$  for the outside good to be an arbitrary value higher than the maximum of the  $H_{k,0}$  values across the inside goods.
- Step 3: Re-order the goods in descending order of  $H_{k,0}$ ; let  $\mathbf{G}$  be the vector of the re-ordered indices of the outside and inside goods (with the outside good appearing as the first entry and the ordering of the inside goods starting from position 2); set a new index  $m$  ( $m = 1, 2, \dots, K$ ) for this new ordering of the outside and inside goods. Let  $\tilde{\mathbf{H}}_0$  be the re-ordered vector of values of  $H_{k,0}$  so that  $\tilde{\mathbf{H}}_0 = (H_{1,0}, \tilde{H}_{2,0}, \dots, \tilde{H}_{m,0}, \dots, \tilde{H}_{K-1,0}, \tilde{H}_{K,0})$ , where  $\tilde{H}_{m,0} = \max_{k \in G[1:m-1]} (H_{k,0})$  for  $m = 2, 3, \dots, K$ . The notation  $k \neq G[1:m-1]$  denotes all goods  $k$  that are not in locations from the first spot (for the outside good) to the spot  $m-1$  in the vector  $\mathbf{G}$ .
- Step 4: Set  $M = 2$ .
- Step 5: If  $\mu_1 > \tilde{H}_{M,0}$ , set the consumptions of all the re-ordered inside goods  $m = M$  to  $m = K$  to zero. STOP.
- Step 6: If  $\mu_1 < \tilde{H}_{M,0}$ , compute  $\psi_M = \exp(\beta' \mathbf{z}_M + \mu_M)$ .
- Step 7: If  $\mu_1 > \ln\left(\frac{\sum_{k=2}^M \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^M p_k \gamma_k}\right) - \beta' \mathbf{z}_1$ , declare the inside good  $M$  as being selected for consumption and forecast the continuous value of consumption as follows:  $x_M^* = [\exp(\mu_M - \mu_1 - \tilde{V}_{M0}) - 1] \gamma_M$ . Set  $M = M + 1$ . Go to Step 5.
- Step 8: If  $\mu_1 < \ln\left(\frac{\sum_{k=2}^M \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^M p_k \gamma_k}\right) - \beta' \mathbf{z}_1$ , declare the inside good  $M$  as not being selected for consumption. STOP.

In cases when the budget information is not available, but is known to be finite, the analyst may set a finite upper limit value as an approximation to the budget not only in estimation (see Section 2.6), but also set that same finite value for  $E$  in the forecasting algorithm above.

An interesting insight from the forecasting procedure is that, unlike the case of the  $L\gamma$ -profile utility with infinite budgets where the consumption intensity of any inside good is independent of the price or attributes of other inside goods (see Saxena et al., 2021), there is cross-alternative demand dependency in our proposed BR-GL $\gamma$ MDCEV model. That is, a change in price (or any other attribute) of one inside good will impact the demand of the other inside goods in both the discrete and continuous dimensions of consumption. This is straightforward to note for the discrete dimension from Step 7 of the forecasting procedure above. In the following discussion, we will focus on a price increase, though the discussion is equally relevant to changes in other alternative attributes. First, for  $0 \leq \sigma \leq 1$ ,

$\delta < 0$ . Now consider the case of  $M = 3$  at step 7. The discrete consumption condition for  $M = 3$  is  $\mu_1 > \ln\left(\frac{\psi_2 \gamma_2 (p_2)^\delta + \psi_3 \gamma_3 (p_3)^\delta}{E + p_2 \gamma_2 + p_3 \gamma_3}\right) - \beta' \mathbf{z}_1$ . An increase in the price  $p_2$  of the first consumed inside good decreases the numerical value of the right side (note that, because  $\delta < 0$ , when  $p_2$  increases, the numerator in the first part of the above expression decreases, while the denominator of the first part of the same expression increases). Thus, the likelihood that the condition above will hold increases, implying that an increase in the price of one inside good will increase the discrete consumption probability of other inside goods. To show the cross price-demand effects for the

continuous consumptions, from the optimality conditions, it should be true that  $x_k^* = \left(\frac{\psi_k}{\psi_1 (p_k)^{(1/\sigma)}} - 1\right) \gamma_k$  if  $x_k^* > 0$  ( $k = 2, 3, \dots, K$ ).

Following the notation in the forecasting algorithm, let the discrete consumption condition in step 7 hold for  $M = 2$ . Then, using the index  $m$  for the ordered listing of goods as in step 3, and  $x_m^*$  for the continuous consumption of the  $m^{\text{th}}$  ordered good if consumed,



$$x_2^* = \left( \frac{\psi_2}{\psi_1(p_2)^{(1/\sigma)}} - 1 \right) \gamma_2.$$

Thus, as  $p_2$  increases,  $(p_2)^{(1/\sigma)}$  increases (because  $0 < \sigma < 1$ , given  $\sigma = (1 - \alpha)$  and  $0 \leq \alpha \leq 1$ ), and therefore, as expected,  $x_2^*$  decreases. Also,

$$p_2 x_2^* = p_2 \left( \frac{\psi_2}{\psi_1(p_2)^{(1/\sigma)}} - 1 \right) \gamma_2.$$

The first derivative of  $p_2 x_2^*$  with respect to price  $p_2$  is:

$$\frac{\partial(p_2 x_2^*)}{\partial p_2} = \left( \frac{\psi_2}{\psi_1(p_2)^{(1/\sigma)}} \left( 1 - \frac{1}{\sigma} \right) - 1 \right) \gamma_2.$$

The right side of the expression above is negative because  $0 < \sigma < 1$ . In other words, as  $p_2$  increases,  $p_2 x_2^*$  decreases. Now let  $M = 3$  be selected for consumption based on the discrete consumption condition of the first part of Step 7. Then the budget constraint will be:

$$p_2 x_2^* + p_3 x_3^* < E.$$

As  $p_2 x_2^*$  decreases when  $p_2$  increases, a larger quantity of  $x_3^*$  can now be consumed at a given price  $p_3$ . Another way to see this directly in the forecasting procedure is that, with an increase in  $p_2$ , a higher draw of  $\mu_3$  is possible in step 6 (thus increasing  $\psi_3$ ), while still adhering to the discrete consumption condition in step 7 for  $M = 3$ . Then, the continuous consumption value of  $x_3^* = [\exp(\mu_3 - \mu_1 - \tilde{V}_{30}) - 1] \gamma_3$  can be higher than earlier. Therefore, an increase in price of one of the inside goods (with no changes in prices in any of the other inside goods) will lead to an increase in both the discrete and continuous consumptions of other inside goods.

#### 4. Simulation evaluation

The simulation exercises undertaken in this section examine the effect of varying budgets (implicitly changing the proportion of consumption of the outside good relative to the combined consumption on the inside goods) on the performance of different models, as discussed below.

##### 4.1. Experimental design

In the design, we generate a sample of 3000 observations with four alternatives and two independent variables in the  $\mathbf{z}_{qk}$  vector in the baseline utility for each alternative (we introduce the subscript  $q$  for individuals here;  $q = 1, 2, \dots, 3000$ ).<sup>5</sup> For this simulation experiment, we consider a constant, but only in the baseline preference for the outside good. We set the coefficient on this constant to 0.75 (that is,  $\beta_0 = 0.75$ ). Of the two independent variables, the first is a dummy variable, while the other is a continuous variable (the use of alternative specific variables in the inside goods are suppressed to allow for a parsimonious specification for the ease of presentation of the simulation results). That is, consider the following for the  $\mathbf{z}_{qk}$  vectors ( $k = 1$  is the outside good):

$$\mathbf{z}_{q1} = [1, 0, 0], \mathbf{z}_{q2} = [0, y_q, \tilde{z}_{q2}], \mathbf{z}_{q3} = [0, y_q, \tilde{z}_{q3}], \text{ and } \mathbf{z}_{q4} = [0, y_q, \tilde{z}_{q4}]. \quad (23)$$

For the dummy variable ( $y_q$ ) in  $\mathbf{z}_{qk}$  ( $k = 2, 3, 4$ ), we treat this as an individual-specific variable (that does not vary across alternatives). To construct this dummy variable, 3000 independent values are drawn from the standard uniform distribution. If the value drawn is less than 0.5, the value of ‘0’ is assigned to the dummy variable. Otherwise, the value of ‘1’ is assigned. The coefficients on this dummy variable are specified to be 0 for the first two inside alternatives ( $k = 2, 3$ ) and 1.0 for the third inside good ( $k = 4$ ). Thus, a single parameter  $\beta_1 (= 1.0)$  is to be estimated for the dummy variable. The values for the continuous variable  $\tilde{z}_{q2}$  are drawn from a standard univariate normal distribution, while the corresponding values  $\tilde{z}_{q3}$  and  $\tilde{z}_{q4}$  are drawn from a univariate normal distribution with mean 0.5 and standard deviation of 1. The parameter  $\beta_2$  on this continuous variable is specified to be 1.25 ( $\beta_2 = 1.25$ ). We will consider the case of no price variation in this paper, and so the value of  $\alpha$  is normalized to zero (and not estimated; doing so implies that our budget-based reverse-GL $\gamma$  model and the reverse- $L\gamma$  profile model but with a given finite budget  $E$  are identical, because the scale of the error terms is set to 1 in the reverse- $L\gamma$  profile). Furthermore, the satiation parameter for the first inside good is set to  $e^{0.75}$  (that is,  $\gamma_2 = 2.117$ ). The corresponding satiation parameter values for the second and third inside goods are set at  $e^1$  (that is,  $\gamma_3 = 2.718$  and  $\gamma_4 = 2.718$ ). Once generated, the independent variable values are held fixed in the entire rest of the simulation exercise.

<sup>5</sup> To keep the discussion tight in terms of the data generation process and also to avoid clutter in the presentation of the detailed simulation results, we have limited the number of alternatives to four and the number of independent variables to two. However, we have also undertaken a similar simulation exercise with eight alternatives and 19 independent variables. The substantive results from this more extended simulation exercise are the same as those from the four-alternative case discussed here. These results are available upon request from the authors.

#### 4.2. Comparing the reverse $GL\gamma$ -profile (or the $R-GL\gamma$ -profile) with the budget-based $R-GL\gamma$ -profile (or the $BR-GL\gamma$ -profile)

As a recap, the reverse  $GL\gamma$ -profile (or the  $R-GL\gamma$ -profile) MDCEV model employs a linear baseline utility for the outside good and uses a reverse Gumbel stochastic term in the baseline utilities of the goods. We compare this  $R-GL\gamma$ -profile model with the budget-based  $R-GL\gamma$ -profile (or the  $BR-GL\gamma$ -profile) of this paper, which accommodates the case of observed budgets or the case of unobserved but “known-to-be finite” budgets, while also expressly recognizing the positivity constraint for the consumption of the outside good at the estimation stage. For the comparison of these two models, the budget values are varied from a low of 50 units to a high of 1000 units, with intermediate values of 250, 500, and 750 units (for a total of five budget values). Since the  $R-GL\gamma$ -profile model does not explicitly consider the positive consumption of the outside good, we should expect a deterioration in the performance of the  $R-GL\gamma$ -profile MDCEV model at the low budget values while the  $BR-GL\gamma$ -profile MDCEV model, which recognizes the budget constraint, should do reasonably well at all the budget values.

Using the design presented in the previous section, we generate the consumption quantity vector  $\mathbf{x}_q^*$  for each individual using the forecasting algorithm for the  $BR-GL\gamma$ -profile MDCEV model (as discussed in Section 3). The parameters to be estimated from the data generating process correspond to  $\theta = [\beta_0 = 0.75, \beta_1 = 1.0, \beta_2 = 1.25, \gamma_2 = 2.117, \gamma_3 = 2.718, \gamma_4 = 2.718]'$ . For each of the five values of total budget considered (ranging from 50 units to 1000 units), the data generation process is undertaken 500 times with different realizations of the  $\varepsilon_k$  vector (for each individual) to generate 500 different data sets (for a total of 2500 data generations of 3000 observations each). For each of the 2500 datasets, we estimate the  $R-L\gamma$ -profile and the  $BR-L\gamma$ -profile models. The performances of the models in recovering the “true” parameters, their standard errors, as well as predicting the consumption values are evaluated as discussed in Section 4.4.<sup>6</sup>

#### 4.3. Performance metrics

The performances of the models in recovering the “true” parameters, their standard errors, as well as predicting the consumption values are evaluated as follows:

- (1) For each of the two simulation experiments, estimate the parameters using each of the two models for each of the 2500 data sets. Estimate the standard errors. For each model in each simulation experiment, and for each budget level (in the first experiment) and each outside good constant value (in the second experiment), do the following:
- (2) Compute the mean estimate for each model parameter across the 500 data sets to obtain a *mean estimate*. Compute the *absolute percentage (finite sample) bias (APB)* of the estimator as:

$$APB(\%) = \left| \frac{\text{mean estimate} - \text{true value}}{\text{true value}} \right| \times 100.$$

- (3) Compute the standard deviation of each parameter estimate across the 500 datasets, and label this as the *finite sample standard deviation or FSSD* (essentially, this is the empirical standard error). Compute the FSSD as a percentage of the true value of each parameter.
- (4) Compute the mean standard error for each model parameter across the 500 datasets, and label this as *the asymptotic standard error or ASE* (essentially this is the standard error of the distribution of the estimator as the sample size gets large, and is a theoretical approximation to the FSSD).
- (5) Next, to evaluate the accuracy of the asymptotic standard error formula, compute the APB associated with the ASE of the estimator as:

$$APBASE(\%) = \left| \frac{ASE - FSSD}{FSSD} \right| \times 100$$

- (6) Examine the data fit at a disaggregate level by comparing the log-likelihood values at convergence of the models. A rigorous statistical test of data fit cannot be undertaken using traditional nested likelihood ratio tests, because the models are not nested forms of each other. But the model with the higher log-likelihood value is to be preferred, because all the models have the same number of estimated parameters. Based on the log-likelihood values for each of the 500 runs (corresponding to the 500 datasets), compute a mean log-likelihood value. A comparison of the mean log-likelihood values at convergence provides an evaluation of the overall multiple discrete-continuous (MDC) component of fit.
- (7) In addition to the MDC component of fit, for each of the 500 datasets, compute the effective log-likelihood value for the pure multiple discrete consumption component using Eqs. (14), (21), and (22) for the  $R-GL\gamma$ -profile model, and by dividing the  $R$

<sup>6</sup> The budget allocation to the inside goods (as a percentage of the total budget) varied approximately between a mean value (across the 500 data sets) of 31% for the budget of 50 to 7% for the budget of 1000 (the specific mean values for the other budget values were as follows: 21% for the budget of 250, 16% for the budget of 500, and 11% for the budget of 750).

$GL\gamma$ -profile expressions by  $P(x_1^* > 0)$  for the BR- $GL\gamma$ -profile model. Compute the predicted probability of the observed discrete choice for each observation (which can be one of eight discrete choice combinations based on whether or not each of the inside goods ( $k = 2, 3, 4$ ) is consumed or not) at the converged values, and compute the corresponding predictive log-likelihood function value for the pure discrete component. Again, based on the log-likelihood values for each of the 500 runs (corresponding to the 500 datasets), compute a mean predictive discrete consumption-based log-likelihood value. A comparison of the mean log-likelihood values at convergence provides an evaluation of the discrete component of fit. At the disaggregate level, compute the average probability of correct prediction for the discrete consumption for each individual, and then compute an average across all individuals. This average probability of correct prediction at a dataset-level is then averaged across the 500 datasets to obtain a single average probability of correct prediction.

- (8) Finally, at the aggregate level, examine model fit at both the discrete consumption level as well as the continuous consumption level. For the discrete level, for each dataset, predict the aggregate share of individuals participating in each of the eight possible discrete outcomes, and compare these predicted shares with the actual percentages of individuals in each combination (using the weighted *mean absolute percentage error* or MAPE statistic, which is the MAPE for each combination weighted by the actual percentage shares of individuals participating in each combination). Next, compute the average of the weighted MAPE statistic across the 500 datasets. For the continuous consumption level, for each dataset, compute an aggregate mean (across observations) of the observation-level continuous consumptions for each of the goods using step (7) of the forecasting algorithm with 1000 error vector replications per individual observation), and compute an MAPE by comparing the mean of the predicted aggregate consumption of each of the goods with the corresponding actual mean value of consumption of the good (ignoring zero consumptions based on the discrete choice, so this MAPE corresponds to consumption conditional on a positive discrete consumption decision). Then, average the dataset-level MAPE (across the 500 datasets) to obtain an overall MAPE for the continuous consumption quantity.

#### 4.4. Simulation results

Table 1 provides the parameter recovery results for the comparative study between the R- $GL\gamma$ -profile with the BR- $GL\gamma$ -profile. For each of the six parameters to be estimated, the first row provides the true value, followed by the estimate obtained and the following metrics for each estimate: APB, FSSD, ASE, and APBASE. The first set of numeric columns refers to the BR- $GL\gamma$ -profile, while the last set of columns corresponds to the R- $GL\gamma$ -profile (each sub-column corresponds to one of the five different budgets). The results from the table indicate that the BR- $GL\gamma$ -profile model (that recognizes the positivity condition on the outside good during estimation) consistently outperforms the R- $GL\gamma$ -profile model (that ignores the positivity of the outside good during estimation). As one would expect, this difference is particularly discernible at lower budget levels; in particular, the mean APB for the BR- $GL\gamma$ -profile model (of 15.752%; see penultimate row of Table 1) in the case of the budget with 50 units is about half the mean APB for the corresponding R- $GL\gamma$ -profile (of 28.941%). This is, because, as discussed earlier, the truncation correction probability term becomes more sizeable (and discernibly less than the value of one) at low budget values.<sup>7</sup> The APBASE term across both the models are comparable, although the BR- $GL\gamma$ -profile model again performs slightly better overall, and in particular in the lower budget cases.

In addition to evaluating the model's ability to accurately recover parameters, we also provide data fit measures at an aggregate as well as disaggregate level. Table 2 presents the results of the likelihood-based data fit measures (first row panel) and the non-likelihood based data fit measures (second row panel). Across all such metrics, the proposed BR- $GL\gamma$ -profile model outperforms the R- $GL\gamma$ -profile model. This is particularly observable, as expected, for the metrics corresponding to the budget level of 50 units. The likelihood metric at convergence is far superior to the BR- $GL\gamma$ -profile at 50 units. The other entries in Table 2 indicate that the difference in the two models is particularly so for the continuous consumption values. For example, at the aggregate level of continuous consumptions, the MAPE is 19.74% for the case of a budget level of 50 for the BR- $GL\gamma$ -profile model, while the corresponding MAPE is 38.60% for the R- $GL\gamma$ -profile model. On the other hand, the discrete consumption predictions, based on the likelihood or non-likelihood data fit measures, are not very different between the two models for any budget value, reflecting the fact that all the linear profile-based models loosen the tie between the discrete and continuous consumptions.

Overall, in terms of parameter recovery ability as well as likelihood and non-likelihood fit measures, the proposed BR- $GL\gamma$ -profile model performs definitively better than the non-budget based R- $GL\gamma$ -profile model, particularly for low budget scenarios.

<sup>7</sup> The difference in APB between the budget of 50 and the budget of 250 within each of the BR- $GL\gamma$ -profile and R- $GL\gamma$ -profile models may appear quite large. The difference in performance between the budget of 50 and 250 for the R- $GL\gamma$ -profile model is to be expected, because, with the budget of 50 and no correction for negative outside good consumptions (that is, no correction for infeasible consumption patterns), the parameters are likely to be way off relative to the case of the higher budget of 250. But it might seem surprising that this happens even for the BR- $GL\gamma$ -profile, which only allows for a feasible solution space. In fact, there is a pattern of consistent reduction in the APB going from a budget of 50 to a budget of 1000 even for the BR- $GL\gamma$ -profile model. The reason is that, as discussed in Section 2.6, with small overall budgets, the truncation correction probability  $P(x_1^* > 0)$  will be sizeable. And as the truncation correction probability increases, there is more non-linearity introduced in the likelihood function, making it more difficult to accurately recover the parameters, leading to the higher APB at lower budgets even for the BR- $GL\gamma$ -profile model. Nonetheless, it is clear that not only does the BR- $GL\gamma$ -profile model outperform the R- $GL\gamma$ -profile models at all budgets, but also that the BR- $GL\gamma$ -profile model performs much more respectably than the R- $GL\gamma$ -profile even at low budget levels.

**Table 1**  
Parameter recovery results for experiment 1.

Parameters	Metrics	BR-GL $\gamma$ -profile					R-GL $\gamma$ -profile				
		Budget	50	250	500	750	1000	50	250	500	750
$\beta_0$	True value	0.750	0.750	0.750	0.750	0.750	0.750	0.750	0.750	0.750	0.750
	Estimate	0.920	0.833	0.801	0.780	0.756	1.152	0.918	0.844	0.816	0.799
	APB (%)	22.667	11.067	6.800	4.000	0.800	53.663	22.403	12.516	8.857	6.471
	FSSD	0.032	0.030	0.031	0.031	0.029	0.032	0.029	0.027	0.031	0.029
	ASE	0.035	0.033	0.032	0.032	0.032	0.038	0.034	0.033	0.032	0.032
	APBASE (%)	9.250	9.315	4.109	3.173	8.107	17.974	14.012	18.578	5.382	10.515
$\beta_1$	True value	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Estimate	0.889	0.942	0.965	0.978	1.001	0.731	0.887	0.937	0.960	0.965
	APB (%)	11.100	5.800	3.500	2.200	0.100	26.896	11.254	6.317	4.027	3.525
	FSSD	0.052	0.047	0.049	0.045	0.047	0.050	0.046	0.048	0.044	0.048
	ASE	0.045	0.045	0.046	0.046	0.046	0.045	0.045	0.045	0.046	0.046
	APBASE (%)	13.205	3.464	5.820	1.725	2.586	11.264	1.812	4.486	2.947	4.044
$\beta_2$	True value	1.250	1.250	1.250	1.250	1.250	1.250	1.250	1.250	1.250	1.250
	Estimate	1.132	1.180	1.205	1.239	1.246	0.971	1.138	1.188	1.207	1.217
	APB (%)	9.440	5.600	3.617	0.880	0.320	22.351	8.972	4.934	3.407	2.650
	FSSD	0.022	0.023	0.021	0.020	0.021	0.018	0.022	0.022	0.019	0.020
	ASE	0.021	0.020	0.020	0.020	0.019	0.021	0.020	0.020	0.019	0.019
	APBASE (%)	5.689	14.836	8.289	2.711	6.866	16.854	9.505	12.395	0.721	4.268
$\gamma_2$	True value	2.117	2.117	2.117	2.117	2.117	2.117	2.117	2.117	2.117	2.117
	Estimate	2.462	2.335	2.264	2.205	2.155	2.547	2.276	2.230	2.182	2.164
	APB (%)	16.300	10.300	6.928	4.153	1.816	20.295	7.528	5.344	3.060	2.220
	FSSD	0.050	0.046	0.046	0.043	0.046	0.058	0.045	0.046	0.043	0.046
	ASE	0.047	0.045	0.044	0.044	0.044	0.048	0.045	0.044	0.044	0.044
	APBASE (%)	5.420	1.624	4.335	3.127	2.985	18.118	1.077	3.591	3.750	5.031
$\gamma_3$	True value	2.718	2.718	2.718	2.718	2.718	2.718	2.718	2.718	2.718	2.718
	Estimate	3.133	3.000	2.929	2.860	2.843	3.299	2.989	2.876	2.824	2.814
	APB (%)	15.270	10.385	7.760	5.223	4.582	21.375	9.972	5.814	3.905	3.547
	FSSD	0.046	0.044	0.039	0.044	0.043	0.050	0.043	0.038	0.043	0.041
	ASE	0.044	0.042	0.041	0.041	0.041	0.044	0.041	0.041	0.041	0.041
	APBASE (%)	5.495	6.587	6.628	5.471	3.749	12.674	5.159	7.797	4.623	0.354
$\gamma_4$	True value	2.718	2.718	2.718	2.718	2.718	2.718	2.718	2.718	2.718	2.718
	Estimate	3.254	3.122	3.020	2.914	2.881	3.508	3.083	2.948	2.866	2.849
	APB (%)	19.734	14.878	11.100	7.205	6.011	29.068	13.417	8.476	5.458	4.829
	FSSD	0.049	0.045	0.043	0.038	0.044	0.046	0.051	0.042	0.037	0.046
	ASE	0.044	0.042	0.042	0.042	0.043	0.043	0.042	0.042	0.042	0.042
	APBASE (%)	11.844	6.794	1.517	12.158	2.256	5.046	17.500	0.323	13.259	8.708
	Mean APB	15.752	9.672	6.618	3.943	2.272	28.941	12.258	7.233	4.786	3.874
	Mean APBASE	8.484	7.104	5.116	4.727	4.425	13.655	8.177	7.862	5.113	5.487

## 5. Empirical application

In this section, we demonstrate an application of our proposed model to the case of employed individuals' weekly activity participation.

### 5.1. Sample description

The data source for the empirical application is drawn from the Dutch Longitudinal Internet Studies for the Social Sciences (LISS) panel, which is a probability sample of Dutch households based on the country's population register. The panel, administered via the internet by CentERdata ([www.lissdata.nl](http://www.lissdata.nl)), is a standard social monthly survey undertaken in 2009, 2010, and 2012. In this study we focus on the data from the last wave (October 2012). The survey included questions about individuals' week-long activity participation and respondents' reported time allocation to various activities (including work) during the immediate seven days prior to the survey (Cherchye et al., 2012). The weekly time use data is complemented with socio-demographic information drawn from the LISS panel.

The sample used in our analysis includes individuals who are the sole workers within their respective households. The time-use decisions of such individuals are likely to be distinctly different from other unemployed individuals or employed individuals in a multi-worker household. Several consistency checks were performed to obtain the estimation sample of 1193 workers, the details of which can be found in Astroza et al. (2017). The focus of our analysis is the weekly time-use decisions of these individuals, subject to the weekly time budget constraint of 168 hours. We consider the following five non-work, non-education, and non-sleep activities as the inside goods (the percentage of individuals participating in each of these five activities, and the average weekly hours for those who participate in each of the activities, is also provided next to the activities):

- (1) *Household chores*, such as cleaning, laundry, shopping, cooking, gardening, odd jobs, car washing, and care for children or parents, but not personal care. (98.1% participation rate, with an average of 16.85 weekly hours among participants)

**Table 2**

Data fit measures for simulation experiment.

Data Fit Measure	BR-GL $\gamma$ -profile					R-GL $\gamma$ -profile				
	50	250	500	750	1000	50	250	500	750	1000
<b>Likelihood based data fit measures</b>										
Log-likelihood value at convergence	-13,446.90	-18,282.60	-19,364.70	-19,801.50	-20,103.30	-13,616.50	-18,386.70	-19,429.10	-19,847.80	-20,139.90
Predictive log-likelihood for discrete consumption	-4711.45	-4737.96	-4676.05	-4639.91	-4639.79	-4719.17	-4736.01	-4676.35	-4641.83	-4636.86
<b>Non-likelihood based disaggregate data fit measure</b>										
Average probability of correct prediction	0.280	0.269	0.274	0.277	0.276	0.277	0.269	0.274	0.276	0.277
<b>Non-likelihood based aggregate data fit measures</b>										
Weighted MAPE for aggregate shares	11.11	5.94	4.24	4.26	3.75	11.91	6.11	4.34	4.32	3.67
Overall MAPE for continuous consumption quantity	19.74	7.60	6.71	5.87	7.08	38.60	12.49	8.48	6.82	7.43

- (2) *Personal care*, such as time on washing, dressing, meeting biological needs (excluding sleep), visiting the hairdresser, and seeing the doctor (52.1% participation rate, with an average of 6.23 weekly hours among participants). Note that all individuals spent some time on personal care, but the personal care time here refers to time *beyond* what may be considered mandatory personal care time (based on an assumption that individuals spend about one hour per day on mandatory personal care activities, which then gets included as a component of the outside good).<sup>8</sup>
- (3) *Administrative chores and assistance*, such as managing own family finances and helping family/non-family members (93.2% participation rate, with an average of 7.54 weekly hours among participants).
- (4) *Leisure*, including in-home and out-of-home recreational activities, such as watching TV, reading, practicing sports, hobbies, visiting family or friends, going out, walking the dog, cycling, and being intimate (94.3% participation rate, with an average of 26.18 weekly hours among participants).
- (5) *Social*, including religious activities, civic and volunteer activities, and attending social gatherings. (42.5% participation rate, with an average of 11.67 weekly hours among participants).

The multiple discrete-continuous dependent variable corresponds to weekly participation and weekly time investment in each of the above five inside activity purposes. The outside good constitutes all remaining time, including work, education, travel, and sleep. Also, the unit price for time use in each of the inside activities is unity since the decision variables themselves represent time investments.

## 5.2. Model results

In this section, we demonstrate an application of the proposed BR-GLy-profile MDCEV model rather than provide an extensive commentary on substantive interpretations and policy implications. But, within the context of the data available, we explored alternative variable specifications to arrive at the best possible specification (including considering alternative functional forms for continuous independent variables such as income and age, including a linear form, piecewise linear forms in the form of spline functions, and dummy variable specifications for different groupings). The final variable specification was based on statistical significance testing as well as intuitive reasoning based on the results of earlier studies.

The results of our empirical application are provided in Table 3, and are discussed below by variable category. The coefficients represent the impact of variables on the logarithm of the baseline preference (that is, they correspond to the  $\beta$  vector elements in Eq. (2)) except for the satiation effects discussed later).

### 5.2.1. Individual characteristics

Our results suggest that women are more likely than men to partake in household chores relative to social, leisure, and administrative chores. Earlier family time-use studies have clearly established a gender asymmetry in household responsibilities, even when women work full-time outside the home (see Bernardo et al., 2015; Bernstein, 2015; Cerrato and Cifre, 2018). While this asymmetry has been attributed to the continued societal expectation that household chores rest squarely on the shoulders of women, there is also literature that suggests that women see the responsibility of household chores as a source of identity and power, and are reluctant to relinquish such responsibilities (see Martinez and Paterna, 2009; Vieira et al., 2019). Table 3 also indicates that women have a higher propensity for participation in personal care, a finding that is consistent with women placing more emphasis on their appearance than men. In the psychology and gender development literature (see, for example, Mafra et al., 2020; Quittkat et al., 2019; Borland and Akram, 2007), this need to look good has been associated with socially-learned behavior (through exposure to marketing campaigns of a feminine image) as well as traced to an evolutionary explanation of women using their own looks to provide themselves a competitive physical edge to attract the most desirable males, thereby attaining some amount of social power themselves (through the social power of the “desirable” male) in what has been a male-dominated society for much of human existence.

Age is also found to be a key determinant in individuals’ time use behavior, with those in the age group of 45 years or younger generally partaking less in all the “inside” activity purposes. Younger individuals and those who are middle-aged are in the formative and rising years of their careers, and are likely to spend more time at work (part of the outside activity) than other non-career and financial/retirement planning activities (see Olmo-Sánchez and Maeso-González, 2014; Regitz-Zagrosek, 2012; Henager and Cude, 2016). Of course, those in the middle age group may be in relatively settled relationships (see, for example, Williams et al., 2016), leading to a rise in participation in household chores and leisure activities to the same intensity level as those older than 45 years of age, as reflected in the absence of a coefficient for “household chores” and “leisure” purposes corresponding to those in the “30–45 years” age group.

### 5.2.2. Household demographics

Household size has a positive effect on the baseline preference for administrative chores and assistance, reflecting added finance planning obligations and household responsibilities (including assistance to friends/family members) in large-sized households. Moreover, larger families provide more opportunity to interact and partake in social activities. But household composition also matters, in addition to household size. Specifically, the presence of children (less than 15 years of age) increases participation in

<sup>8</sup> The one hour per day assumption for mandatory personal care is based on Lee (2008), who indicates that grooming, which typically is done in the morning, takes up, on average, a little more than 30 min for men and about 45 min for women every morning.

**Table 3**Empirical application results (using the BR-GL $\gamma$ profile).

Variables	Coefficient estimates (t-stats)				
	Household chores	Personal care	Admin. chores and assistance	Leisure	Social
<b>Individual characteristics</b>					
Female	0.489 (4.66)	0.693 (7.50)	–	–	–
Age (Base: More than 45 years)					
Below 30 years	-0.624 (-1.81)	-0.403 (-1.26)	-0.595 (-2.01)	-0.715 (-1.49)	-0.314 (-0.98)
30–45 years	–	-0.291 (-2.74)	-0.545 (-5.83)	–	-0.270 (-2.77)
<b>Household sociodemographic</b>					
Household size	–	–	0.145 (3.89)	–	0.138 (4.01)
Presence of child(ren)	0.607 (3.37)	–	–	-0.361 (-2.90)	–
Weekly household income (Base: Greater than equal 750 Euros)					
Less than 500 Euros	–	–	0.223 (2.06)	–	0.400 (4.10)
500–749 Euros	–	–	0.170 (1.76)	–	0.331 (3.96)
Baseline preference constant	2.890 (16.29)	4.135 (17.27)	1.590 (5.68)	3.811 (20.22)	-0.024 (-0.16)
<b>Satiation effects</b>					
Female	–	-0.284 (-1.82)	–	–	–
Age (Base: More than 45 years)					
Below 30 years	–	0.341 (1.33)	–	0.706 (1.67)	0.810 (3.55)
30–45 years	–	0.315 (1.87)	–	–	0.470 (3.31)
Household size	0.138 (1.83)	–	–	–	–
Satiation constant	0.841 (7.04)	-1.676 (-5.87)	1.196 (9.16)	1.884 (21.40)	1.269 (1.53)

household chores and lowers the propensity to participate in leisure. These results are not surprising, as child-care related activities take priority for parents at this life-cycle stage, and also has been shown to lead to time poverty/social exclusion among working parents (see for example, [Bernardo et al., 2015](#); [Craig and Brown, 2016](#)).

Our results also indicate that a lower household income (for weekly income levels of 750 or less euros relative to higher income levels) leads to increased participation in administrative chores (family finances related activities and helping family members) and social activities. The latter result is not surprising, since social activities may be perceived as a low-cost recreational outlet for low-income families. Beside, this effect may also be proxying for the effect of the closer-knit extended family and community unit of socialization among immigrants in the Netherlands, who generally earn less than domestic-born citizens.

### 5.2.3. Baseline preference constants

The baseline preference constants do not have any substantive interpretations, and simply serve as instruments to better fit the discrete participation rate and continuous consumption values of the inside goods.

### 5.2.4. Satiation effects through $\gamma_k$ parameters

To allow heterogeneity in the parameters across individuals, while also guaranteeing the positivity of the parameters, they are parameterized as  $\gamma_k = \exp(\delta_k \omega_k)$ . The estimates in [Table 3](#) for the satiation effects correspond to the elements of the  $\delta_k$  vector. A positive value for a  $\delta_k$  element implies that an increase in the corresponding element of the  $\omega_k$  vector increases  $\gamma_k$ , which has the result of reducing satiation effects and increasing the continuous consumption quantity of alternative  $k$  (conditional on consumption of alternative  $k$ ). On the other hand, a negative value for a  $\delta_k$  element implies that an increase in the corresponding element of the  $\omega_k$  vector decreases  $\gamma_k$ , which has the result of increasing satiation effects and decreasing the continuous consumption quantity of alternative  $k$  (conditional on consumption of alternative  $k$ ). The specification related to the satiation parameters are available in the bottom panel of [Table 3](#).

Interestingly, the satiation parameter results suggest that while women have a higher propensity to participate in personal care activities, this does not necessarily translate to longer participation durations subject to participation (in fact, there is a marginally significant negative effect of the “female” variable on satiation for personal care in [Table 3](#)). That is, there is little difference between men and women in time investment, among individuals who partake in personal care during the week. The age effects on satiation reveal that, while younger individuals are less likely to partake in personal care, administrative chores, and social activities during the week, they participate for longer periods in these activity purposes if they participate. This may be reflecting a justification effect or a “fixed cost” effect, wherein once the relatively time-poor young individuals decide to participate in these activities, they decide to invest a good amount of time in it. Finally, as the number of individuals in a household increases, not only does participation in household chores increase, but so does the time invested in household chores.

The constants in the satiation effects (last row of [Table 3](#)) generally reflect the high duration of time investment in leisure activity



and the low duration of time investment in personal care activity. These constants also adjust for the sample range of explanatory variables and the magnitudes of the estimated baseline preferences to provide the best fit for the continuous consumption values.

### 5.3. Data fit measures

In this section, we examine the data fit measures of three models for the empirical time-use data. The three models are the proposed BR-GL $\gamma$ -profile model, the R-GL $\gamma$ -profile model, and the traditional  $\gamma$ -profile-based MDCEV model. The last of these; the traditional  $\gamma$ -profile-based MDCEV model; employs a non-linear baseline utility for the outside good, requires an observed budget, and guarantees positivity of all goods that are predicted to be consumed. This model uses the usual extreme value error based on the limiting distribution of the maximum of random variables in the baseline utilities of the goods. As discussed in detail by Bhat (2018), the model is known to tie the continuous predictions (how much of an inside good to consume) and the discrete predictions (whether an inside good will be consumed) very tightly, leading to possibly poor predictions of the discrete choice, especially when the consumption of the outside good is very large. On the other hand, the BR-GL $\gamma$ -profile, which relaxes the strong tie between the discrete and continuous predictions, may do better on the discrete predictions than the traditional  $\gamma$ -profile-MDCEV model, especially at high consumptions of the outside good. However, it may also produce worse continuous predictions than the traditional  $\gamma$ -profile-MDCEV model for the consumed goods. Thus, a comparison of the BR-GL $\gamma$ -profile is undertaken with the  $\gamma$ -profile-based MDCEV model, in addition to a comparison between the BR-GL $\gamma$ -profile and R-GL $\gamma$ -profile models.

#### 5.3.1. Likelihood-based data fit measures

The likelihood-based data fit measures in terms of log-likelihood at convergence, predictive log-likelihood value at the discrete consumption level as well as the average probability of correct prediction for all the three models are provided in Table 4. Our proposed model outperforms the R-GL $\gamma$ -profile model in all the above metrics, highlighting the value of considering the positivity of the outside good consumption in estimation. However, the traditional  $\gamma$ -profile-based model performs better in terms of the overall MDC fit as observed from the marginally better log-likelihood convergence value, although our proposed model does substantially better in terms of the predictive discrete log-likelihood measure and average probability of correct prediction at the discrete consumption level. To evaluate and compare the performance of these models further, we also examine the non-likelihood based aggregate fit measures discussed next.

#### 5.3.2. Non-likelihood based data fit measures

The aggregate-level fit measures for the three models are shown in Table 5. For ease of presentation, we provide the pairwise predictions of activity participation at the disaggregate level for the five activities in our application (based on whether or not an individual participates in each of these five activities, there are a total of  $2^5 = 32$  activity-combinations; however, to make our presentation simple and to avoid clutter, we only provide pairwise predictions of activity participation, which corresponds to 10 possible combinations). For each of the three models (the proposed BR-GL $\gamma$ -profile model, the R-GL $\gamma$ -profile model and the traditional  $\gamma$ -profile-based MDCEV model), the predicted number of individuals participating in each pairwise combination at the discrete level is computed and provided in the top panel of Table 5. Our proposed model with a weighted MAPE (weighted with respect to the actual observed shares) value of just over 11% outperforms both the R-GL $\gamma$ -profile model (MAPE of 16.5%) and the traditional  $\gamma$ -profile-based MDCEV model (MAPE of 21%), reinforcing the superior performance of our proposed model in the discrete dimension based on the likelihood based fit measures.

The aggregate fit measures in the bottom panel of Table 5 correspond to the conditional continuous consumption dimension (that is, the average predicted continuous values; in our context, these values are the number of hours in a week for which an individual engages in the respective “inside” activity, given that an individual decides to participate in that activity). The proposed BR-GL $\gamma$ -profile model with the weighted MAPE of 27.73% performs much better than the R-GL $\gamma$ -profile model which has a MAPE of over 41%. However, the traditional  $\gamma$ -profile-based MDCEV model with a weighted MAPE of just over 20% provides the best prediction along the continuous consumption quantity (conditional on discrete participation).

#### 5.3.3. A summary discussion

The motivation for this paper was to propose a model that accommodates the positivity constraint on the outside good during estimation of the linear outside good utility model form. In the case when the budget is unobserved and may be expected to be large, the R-GL $\gamma$ -profile model is the model to use. But if there is reason to believe that a finite ceiling applies to the budget, even if the budget is unobserved, the BR-GL $\gamma$ -profile should be the model to use. And, if budget is available, finite, and the investment in the inside goods is not small relative to the investment in the outside good, our proposed BR-GL $\gamma$ -profile must be the preferred model relative to the R-GL $\gamma$ -profile model. These results are clearly evident in the superior performance of our proposed model relative to the R-GL $\gamma$ -profile in

**Table 4**  
Likelihood-based data fit measures for empirical study.

Metrics	BR-GL $\gamma$ -profile	R-GL $\gamma$ -profile	Traditional $\gamma$ -profile
Log-likelihood at convergence	-18,646.6	-19,450.6	-18,575.0
Predictive log-likelihood at the discrete consumption level	-2320.3	-2422.3	-2782.9
Average probability of correct prediction	0.211	0.203	0.178

**Table 5**

Data fit measures for empirical study.

Discrete choice consumption: Number of individuals with consumption in outside good and joint participation in...	Actual number	BR-GL $\gamma$ -profile	R-GL $\gamma$ -profile	Traditional $\gamma$ -profile
Household chores, Personal care	615	656	697	425
Household chores, Administrative chores	1091	1094	1097	1056
Household chores, Leisure	1120	1120	1121	1094
Household chores, Social	499	552	610	270
Personal care, Administrative chores	581	636	677	377
Personal care, Leisure	601	644	686	407
Personal care, Social	243	401	451	111
Administrative chores, Leisure	991	1067	1069	874
Administrative chores, Social	1060	741	655	753
Leisure, Social	474	545	601	381
Weighted Mean Absolute Percentage Error	–	11.3%	16.5%	21.0%
<b>Continuous consumption (conditional on positive discrete choice consumption)</b>	<b>Observed</b>	<b>BR-GL<math>\gamma</math>-profile</b>	<b>R-GL<math>\gamma</math>-profile</b>	<b>Traditional <math>\gamma</math>-profile</b>
Household chores	16.85	14.12	9.88	21.22
Personal care	6.23	3.58	3.27	7.54
Administrative chores	7.54	5.22	4.87	10.52
Leisure	26.18	19.52	15.80	28.32
Social	11.67	7.04	6.14	14.63
Weighted Mean Absolute Percentage Error	–	27.73%	41.64%	20.10%

both the simulation experiment as well as in our empirical demonstration.

In the case when the budget information is available, the best approach would be to estimate both our proposed BR-GL $\gamma$ -profile as well as a traditional  $\gamma$ -profile-MDCEV model. The advantage of our BR-GL $\gamma$ -profile is that it disentangles the discrete and continuous consumption decisions, which, in general, will provide better discrete choice predictions, especially when the budget is large and the outside good takes up a substantial share of the continuous consumption. This is discussed at length in Bhat (2018). However, the continuous consumption predictions from our BR-GL $\gamma$ -profile may be better or may be worse than the traditional  $\gamma$ -profile-MDCEV model, depending upon the empirical context. If it turns up that the BR-GL $\gamma$ -profile provides a better fit at the discrete level as well as the continuous level relative to the traditional  $\gamma$ -profile-MDCEV model, the choice would be clear. But, if the BR-GL $\gamma$ -profile provides a better fit at the discrete level, but not as good a fit as the traditional  $\gamma$ -profile-MDCEV model at the continuous level, the decision may be rather subjective. In such a situation, the analyst will have to examine the relative performances at both the discrete level and the continuous level, and make a final determination based on the context of the study and the relative priorities for the accuracy of the discrete and continuous predictions. While much more extensive investigations in different simulation/empirical contexts is needed to make additional definitive remarks on the performance of our proposed model and the traditional  $\gamma$ -profile-MDCEV, our preliminary explorations suggest that even in cases when the latter model performs better than our proposed model at the continuous level, the performance difference may not be by much. However, the traditional  $\gamma$ -profile-MDCEV model can perform much worse than our proposed model at the discrete level.

## 6. Conclusions

The traditional MDCEV model has now been widely used in a number of empirical contexts to analyze consumer discrete-continuous decisions. However, it is applicable only for cases when the budget is observed, and the model formulation also very closely ties the discrete and continuous decisions. More recently, a variant of the traditional MDCEV, based on adopting a linear utility form for the outside good, has received some attention. Labeled as the  $L\gamma$ -profile model, this new model structure not only does away with the need to observe budgets, but also breaks the strong linkage between the discrete and continuous choice dimensions of decision-making. But recent studies show that this  $L\gamma$ -profile model may not work well in situations when, even if the budget is unobserved, the budget is known to be finite and small in magnitude. The reason is that the formulation, while ensuring the positivity of consumptions of the inside goods (that may or may not be consumed), does not guarantee, within the model formulation and estimation itself, the positivity of the consumption of the essential outside good.

In this paper, we have developed a formulation, based on a reverse Gumbel structure for the stochastic terms in the utility functions of alternatives, that develops a closed-form probability expression, while also accommodating the positivity requirement for the outside good. This is done through a truncation scheme that still yields an elegant closed-form expression. Importantly, the procedure works with both observed and unobserved budgets. The ability of our proposed Budget-based Reverse Generalized  $L\gamma$ -profile model (labeled the BR-GL $\gamma$ -profile model) to recover true underlying model parameters is subsequently compared with that of the linear outside good utility model without the outside good positivity consideration (labeled the R-GL $\gamma$ -profile model). This evaluation is undertaken using an experimental set-up with varying budget levels. In addition, we demonstrate an application of our proposed BR-GL $\gamma$ -profile model to the weekly time-use decisions of individuals using the 2012 wave of the LISS (Longitudinal Internet Studies for the Social Sciences) Dutch panel data, compare the data fit of the proposed model with the R-GL $\gamma$ -profile model and the traditional  $\gamma$ -profile-MDCEV models.

Our results clearly point to the distinct benefit of employing our proposed BR-GL $\gamma$ -profile model (over the linear outside utility

profile models proposed thus far and employed in the literature) in empirical contexts where there is reason to believe that a finite ceiling applies to the budget (even if the budget is unobserved) or if the budget is actually available. In the latter case, our proposed model is a serious contender to the traditional  $\gamma$ -profile-MDCEV model. In such a case, it would be best to estimate both our proposed model and the traditional model, before making a final determination of which model to use.

Future research should focus on approaches to include the proposed truncation scheme into Bhat's (2018) flexible MDCEV model form as well as develop methods that ensure that the resulting truncation-based flexible MDCEV model also conforms to global utility-maximizing behavior across the multiple discrete and continuous consumption choices), (), (.

## Declaration of Competing Interest

None.

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## APPENDIX A. Integration to arrive at the multivariate survival distribution function

To show that the multivariate survival function collapses to a closed-form expression as shown in Eq. (10), we start off with the probability expression as below:

$$\begin{aligned} S_{\eta}(w_2, w_3, \dots, w_K) &= \text{Prob}(\eta_2 > w_2, \eta_3 > w_3, \dots, \eta_K > w_K) \\ &= \text{Prob}(\varepsilon_2 > w_2 + \varepsilon_1, \varepsilon_3 > w_3 + \varepsilon_1, \varepsilon_4 > w_4 + \varepsilon_1, \dots, \varepsilon_K > w_K + \varepsilon_1) \text{ since } \eta_k = \varepsilon_k - \varepsilon_1. \end{aligned}$$

Based on the property of the standard reverse-Gumbel distribution, we can write the above probability as

$$= \int_{\varepsilon_1 = -\infty}^{+\infty} \prod_{k=2}^K e^{-e^{[w_k + \varepsilon_1]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} d\varepsilon_1. \quad (\text{A.1})$$

The integrand above can be simplified as follows:

$$\begin{aligned} &\prod_{k=2}^K e^{-e^{[w_k + \varepsilon_1]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} \\ &= e^{-\sum_{k=2}^K e^{[w_k + \varepsilon_1]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} \\ &= e^{-\left(e^{\varepsilon_1} + \sum_{k=2}^K e^{\varepsilon_1} e^{w_k}\right)} e^{\varepsilon_1} \\ &= e^{-\left[e^{\varepsilon_1} \left(1 + \sum_{k=2}^K e^{w_k}\right)\right]} e^{\varepsilon_1} \end{aligned}$$

Therefore, the integration in Eq. (A.1) can be re-written as,

$$\int_{\varepsilon_1 = -\infty}^{+\infty} e^{-\left[e^{\varepsilon_1} \left(1 + \sum_{k=2}^K e^{w_k}\right)\right]} e^{\varepsilon_1} d\varepsilon_1.$$

To evaluate this integration, let  $s = e^{\varepsilon_1} (1 + \sum_{k=2}^K e^{w_k})$ .

Therefore,  $ds = e^{\varepsilon_1} (1 + \sum_{k=2}^K e^{w_k}) d\varepsilon_1$ .

Then the integration takes the following form (ignoring the limits for the moment),

$$\begin{aligned} &\int \frac{e^{-s} ds}{\left(1 + \sum_{k=2}^K e^{w_k}\right)} \\ &= -\frac{e^{-s}}{\left(1 + \sum_{k=2}^K e^{w_k}\right)}. \end{aligned}$$

Now, evaluating the limits, we have,

$$\left. \frac{e^{-e^{\varepsilon_1} \left(1 + \sum_{k=2}^K e^{w_k}\right)}}{\left(1 + \sum_{k=2}^K e^{w_k}\right)} \right|_{\varepsilon_1=-\infty}^{\varepsilon_1=+\infty} = -\frac{1}{\left(1 + \sum_{k=2}^K e^{w_k}\right)} [0 - 1].$$

Therefore, Eq. (A.1) takes the following result:

$$\int_{\varepsilon_1=-\infty}^{+\infty} \prod_{k=2}^K e^{-e^{[w_k + \varepsilon_1]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} d\varepsilon_1 = \frac{1}{\left(1 + \sum_{k=2}^K e^{w_k}\right)}.$$

Which is exactly the expression in Eq. (10).

## APPENDIX B. Derivation of the closed-form expression for the probability condition related to the positivity of the outside good consumption

We start off with the probability expression given in Eq. (17) of the text.

$$\begin{aligned} P(x_1^* > 0) &= P\left\{\left[\varepsilon_1 - \ln\left(\sum_{k=2}^{M+1} [h_k e^{\varepsilon_k}]\right)\right] > G\right\} \\ &= P\left\{\varepsilon_1 > G + \ln\left(\sum_{k=2}^{M+1} [h_k e^{\varepsilon_k}]\right)\right\} \end{aligned} \quad (\text{B.1})$$

Based on the property of the standard reverse-Gumbel distribution, we can write the above probability expression as

$$\begin{aligned} &= \int_{\varepsilon_{M+1}=-\infty}^{+\infty} \dots \int_{\varepsilon_3=-\infty}^{+\infty} \int_{\varepsilon_2=-\infty}^{+\infty} e^{-e^{\left[G + \ln\left(\sum_{k=2}^{M+1} [h_k e^{\varepsilon_k}]\right)\right]}} e^{-e^{\varepsilon_2}} e^{\varepsilon_2} d\varepsilon_2 e^{-e^{\varepsilon_3}} e^{\varepsilon_3} d\varepsilon_3 \dots e^{-e^{\varepsilon_{M+1}}} e^{\varepsilon_{M+1}} d\varepsilon_{M+1} \\ &= \int_{\varepsilon_{M+1}=-\infty}^{+\infty} \dots \int_{\varepsilon_3=-\infty}^{+\infty} \int_{\varepsilon_2=-\infty}^{+\infty} e^{-e^{G \sum_{k=2}^{M+1} [h_k e^{\varepsilon_k}]}} e^{-e^{\varepsilon_2}} e^{\varepsilon_2} d\varepsilon_2 e^{-e^{\varepsilon_3}} e^{\varepsilon_3} d\varepsilon_3 \dots e^{-e^{\varepsilon_{M+1}}} e^{\varepsilon_{M+1}} d\varepsilon_{M+1} \\ &= \int_{\varepsilon_{M+1}=-\infty}^{+\infty} \dots \int_{\varepsilon_3=-\infty}^{+\infty} \int_{\varepsilon_2=-\infty}^{+\infty} e^{-e^{G(h_2 e^{\varepsilon_2} + h_3 e^{\varepsilon_3} + \dots + h_{M+1} e^{\varepsilon_{M+1}})}} e^{-e^{\varepsilon_2}} e^{\varepsilon_2} d\varepsilon_2 e^{-e^{\varepsilon_3}} e^{\varepsilon_3} d\varepsilon_3 \dots e^{-e^{\varepsilon_{M+1}}} e^{\varepsilon_{M+1}} d\varepsilon_{M+1} \end{aligned} \quad (\text{B.2})$$

Given that the random variables  $\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{M+1}$  are independent, the integration in Eq. (B.2) can be re-written as,

$$\begin{aligned} &= \int_{\varepsilon_2=-\infty}^{+\infty} e^{-e^{G(h_2 e^{\varepsilon_2})}} e^{-e^{\varepsilon_2}} e^{\varepsilon_2} d\varepsilon_2 \int_{\varepsilon_3=-\infty}^{+\infty} e^{-e^{G(h_3 e^{\varepsilon_3})}} e^{-e^{\varepsilon_3}} e^{\varepsilon_3} d\varepsilon_3 \dots \int_{\varepsilon_{M+1}=-\infty}^{+\infty} e^{-e^{G(h_{M+1} e^{\varepsilon_{M+1}})}} e^{-e^{\varepsilon_{M+1}}} e^{\varepsilon_{M+1}} d\varepsilon_{M+1} \\ &= I_{\varepsilon_2}, I_{\varepsilon_3}, \dots, I_{\varepsilon_{M+1}} \text{ (say)} \end{aligned}$$

We solve the first integration  $I_{\varepsilon_2}$  as below:

$$\begin{aligned} I_{\varepsilon_2} &= \int_{\varepsilon_2=-\infty}^{+\infty} e^{-e^{G(h_2 e^{\varepsilon_2})}} e^{-e^{\varepsilon_2}} e^{\varepsilon_2} d\varepsilon_2 \\ &= \int_{\varepsilon_2=-\infty}^{+\infty} e^{-e^{\varepsilon_2} (1 + e^{G h_2})} e^{\varepsilon_2} d\varepsilon_2 \end{aligned}$$

To evaluate this integration, let  $s = e^{\varepsilon_2} (1 + e^{G h_2})$ .

Therefore,  $ds = e^{\varepsilon_2} (1 + e^{G h_2}) d\varepsilon_2$ .

The first integration then takes the following form (ignoring the limits for the moment),

$$I_{\varepsilon_2} = \int \frac{e^{-s} ds}{(1 + e^{G h_2})}.$$

This is a straightforward integration to solve, which results in

$$I_{\varepsilon_2} = -\frac{e^{-s}}{(1 + e^{G h_2})}.$$

Now, evaluating the limits [note that  $(1 + e^{G h_k})$  for all  $k (= 2, 3, \dots, M+1)$  is always positive by definition], we have:

$$I_{\varepsilon_2} = -\frac{e^{-e^{\varepsilon_2}(1+e^G h_2)}}{(1+e^G h_2)} \Big|_{\varepsilon_2=-\infty}^{\varepsilon_2=+\infty} = -\frac{1}{(1+e^G h_2)} [0-1].$$

Therefore,

$$I_{\varepsilon_2} = \frac{1}{(1+e^G h_2)}.$$

Similarly, following the exact same approach, we have,

$$I_{\varepsilon_3} = \frac{1}{(1+e^G h_3)}, I_{\varepsilon_{M+1}} = \frac{1}{(1+e^G h_{M+1})} \text{ and for general, } I_{\varepsilon_k} = \frac{1}{(1+e^G h_k)} \text{ for } k(=2, 3, \dots, M+1),$$

Therefore, the probability expression in Eq. (B.1) results into the following closed-form expression.

$$\begin{aligned} &= P\left\{\varepsilon_1 > G + \ln\left(\sum_{k=2}^{M+1} [h_k e^{\varepsilon_k}]\right)\right\} \\ &= I_{\varepsilon_2} \cdot I_{\varepsilon_3} \cdots I_{\varepsilon_{M+1}} \\ &= \frac{1}{(1+e^G h_2)} \cdot \frac{1}{(1+e^G h_3)} \cdots \frac{1}{(1+e^G h_{M+1})} \\ &= \frac{1}{\prod_{k=2}^{M+1} [1 + h_k e^G]} \end{aligned}$$

This is exactly Eq. (18) in the text.

#### APPENDIX C. Derivation of the conditional likelihood expression that ensures positive outside good consumption

Consider the following probability expression that ensures positive consumption of the outside good:

$$P((x_2^*, x_3^*, \dots, x_{M+1}^*, 0, \dots, 0, 0) | x_1^* > 0) = P\left((x_2^*, x_3^*, \dots, x_{M+1}^*, 0, \dots, 0, 0) | \psi_1 > \frac{\sum_{k=2}^{M+1} \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^{M+1} p_k \gamma_k}\right) \quad (C1)$$

The set of optimal consumptions in the above expression can be equivalently represented using the corresponding set of KKT conditions from Eq. (4) in the text. Hence, the above expression can be written as:

$$P((x_2^*, x_3^*, \dots, x_{M+1}^*, 0, \dots, 0, 0) | x_1^* > 0) = P\left(\left(\left[\psi_k \left(\frac{x_k^*}{\gamma_k} + 1\right)^{-1}\right]^{1-\alpha} = \lambda p_k; \quad [\psi_k]^{1-\alpha} < \lambda p_k \quad \forall k=\{2, 3, \dots, M+1\}\right) \mid \psi_1 > \frac{\sum_{k=2}^{M+1} \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^{M+1} p_k \gamma_k}\right) \quad (C2)$$

Expanding the above conditional probability expression, we get:

$$P((x_2^*, x_3^*, \dots, x_{M+1}^*, 0, \dots, 0, 0) | x_1^* > 0) = \frac{P\left(\left[\psi_k \left(\frac{x_k^*}{\gamma_k} + 1\right)^{-1}\right]^{1-\alpha} = \lambda p_k \text{ AND } [\psi_k]^{1-\alpha} < \lambda p_k \text{ AND } \psi_1 > \frac{\sum_{k=2}^{M+1} \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^{M+1} p_k \gamma_k} \mid \psi_1 > \frac{\sum_{k=2}^{M+1} \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^{M+1} p_k \gamma_k}\right)}{P\left(\psi_1 > \frac{\sum_{k=2}^{M+1} \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^{M+1} p_k \gamma_k}\right)} \quad (C3)$$

The numerator in the above expression is the joint likelihood of the KKT conditions in Eq. (4) and the truncation condition in Eq. (8) necessary for ensuring positive consumption of the outside good. Note that Eq. (8) implies truncation on the distributions of baseline preference parameters of only the chosen alternatives. Therefore, the numerator of the above expression includes redundant conditions specific to the stochastic parameters of the chosen alternatives. To remove such redundancies from the numerator, consider the sets of

conditions specific to only the chosen alternatives. That is, consider the following set of conditions from the numerator (after substituting  $\psi_1^{1-\alpha}$  for  $\lambda$ ):

$$\left[ \psi_k \left( \frac{x_k^*}{\gamma_k} + 1 \right)^{-1} \right]^{1-\alpha} = \psi_1^{1-\alpha} p_k, \quad k = 2, 3, \dots, M+1, \text{ and,} \quad (C4)$$

$$\psi_1 > \frac{\sum_{k=2}^{M+1} \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^{M+1} p_k \gamma_k}, \text{ where } \delta = \frac{-\alpha}{1-\alpha} \quad (C5)$$

From Eq. (C4), the expression for  $\psi_k$  is  $\psi_1 \left( \frac{x_k^*}{\gamma_k} + 1 \right) p_k^{\frac{1}{1-\alpha}}$ , which can be fed into Eq. (C5) to rewrite the latter equation as:

$$\psi_1 > \frac{\sum_{k=2}^{M+1} \psi_1 \gamma_k \left( \frac{x_k^*}{\gamma_k} + 1 \right) p_k}{E + \sum_{k=2}^{M+1} p_k \gamma_k} \quad (C6)$$

Simplifying the expression in Eq. (C6), we get

$$E + \sum_{k=2}^{M+1} p_k \gamma_k > \sum_{k=2}^{M+1} \gamma_k \left( \frac{x_k^*}{\gamma_k} + 1 \right) p_k \Rightarrow E > \sum_{k=2}^{M+1} p_k x_k^* \quad (C7)$$

The above condition is always true in observed data, and therefore the condition in Eq. (C5) becomes redundant in the numerator of Eq. (C3). Hence, the expression in Eq. (C3) can be written as:

$$P((x_2^*, x_3^*, \dots, x_{M+1}^*, 0, \dots, 0, 0) | x_1^* > 0) = \frac{P\left(\left[\psi_k \left(\frac{x_k^*}{\gamma_k} + 1\right)^{-1}\right]^{1-\alpha} = \lambda p_k \text{ AND } [\psi_k]^{1-\alpha} < \lambda p_k \mid \forall k=\{2,3,\dots,M+1\}\right)}{P\left(\psi_1 > \frac{\sum_{k=2}^{M+1} \psi_k \gamma_k (p_k)^\delta}{E + \sum_{k=2}^{M+1} p_k \gamma_k}\right)}, \text{ or } \frac{P(x_2^*, x_3^*, \dots, x_{M+1}^*, 0, \dots, 0, 0)}{P(x_1^* > 0)}$$

## Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:[10.1016/j.trb.2021.12.013](https://doi.org/10.1016/j.trb.2021.12.013).

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