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Order reduction of nonlinear quasi-periodic systems subjected to external excitations

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ABSTRACT

This paper presents order reduction techniques for nonlinear quasi-periodic systems subjected to external excitations. The order reduction techniques presented here are based on the Lyapunov–Perron (L–P) Transformation. For a class of non-resonant quasi-periodic systems, the L-P Transformation can convert a linear quasi-periodic system into a linear time-invariant one. This Linear Time-Invariant (LTI) system retains the dynamics of the original quasi-periodic system. Once this LTI system is obtained, the tools and techniques available for analysis of LTI systems can be used, and the results could be obtained for the original quasi-periodic system via the L–P Transformation. This approach is similar to the Lyapunov–Floquet (L–F) transformation to convert a linear time-periodic system into an LTI system and perform analysis and control.

Order reduction is a systematic way of constructing dynamical system models with relatively smaller states that accurately retain large-scale systems' essential dynamics. This work presents reduced-order modeling techniques for nonlinear quasi-periodic systems subjected to external excitations. The methods proposed here use the L–P Transformation that makes the linear part of transformed equations time-invariant. In this work, two order reduction techniques are suggested. The first method is simply an application of the well-known Guyan like reduction method to nonlinear systems. The second technique is based on the concept of an invariant manifold for quasi-periodic systems.

The 'quasi-periodic invariant manifold' based technique yields 'reducibility conditions'. These conditions (referred to in the perturbation literature as resonance conditions) help us understand the system's various types of resonant interactions. These resonances indicate energy interactions between the system states, nonlinearity, and external excitation. To retain the essential dynamical characteristics, one must preserve all these 'resonant' states in the reduced-order model. Thus, if the 'reducibility conditions' are satisfied then only, a nonlinear order reduction based on the quasi-periodic invariant manifold approach is possible. It is found that the invariant manifold approach yields good results. These methodologies are general and can be used for parametric study, sensitivity analysis, and controller design.

1. Introduction

Order reduction is constructing small order systems from the large-scale structure, which capture the dominant dynamics [1]. In the design and development process, engineers often analyze complex dynamical systems governed by a large set of integrodifferential (ordinary/partial) equations. These systems are complicated to solve analytically, and one must resort to numerical techniques [2]. While solving these dynamical systems numerically, one has to consider various issues like convergence, numerical truncation errors, and, most importantly, the limited computational resources and time. To simulate the dynamical system's response accurately within a reasonable amount of time, one can construct an equivalent small-scale system known as the 'reduced order model' that will approximate the large-scale system dynamics [3, 4]. This approach simplifies a sizeable dynamical system by replacing it with an equivalent small-scale system known as the Model Order Reduction (MOR).

Researchers have used order reduction of linear systems using multiple techniques. Some of the methods include error minimization [5], pole clustering [6], transfer function-based [7], and Pade approximation [8], to mention a few. For a comprehensive overview, we refer to Ref. [9]. In addition, nonlinear order reduction is studied by researchers from a structural point of view in the second-order and state–space form [10–12]. For order reduction of the time-periodic system, researchers utilized the L–F Transformation and performed order reduction. We refer to Refs. [13–17] for details on this approach.

The order Reduction procedure comprises the following steps.
1. A study of the large-scale system and identifying the dominant states pertaining to the dominant dynamics.

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2. Elimination of the non-dominant states either by simply neglecting their contribution or replacing them with the appropriate functions of dominant states.
3. Formulation of an equivalent reduced-order system consisting only of the dominant states.

This work concentrates on the 'automatic' order reduction of an important class of engineering systems known as the 'nonlinear quasi-periodic systems' [18–20]. These systems have quasi-periodic coefficients and in the state–space form expressed as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(\mathbf{x}, t) + \mathbf{F}(t) \tag{1}$$

where $\mathbf{A}(t)$ is an $n \times n$ matrix with quasi-periodic coefficients and $\mathbf{f}(\mathbf{x}, t)$ is a nonlinear function with monomials of $\mathbf{x}$, $\mathbf{F}(t)$ is the external deterministic excitation, and $\mathbf{x}$ is an n vector of appropriate dimensions. The objective of order reduction is to construct a reduced-order system

$$\dot{\mathbf{x}}_r = \mathbf{A}_r(t)\mathbf{x}_r + \mathbf{f}_r(\mathbf{x}_r, t) + \mathbf{F}_r(t) \tag{2}$$

that captures the essential dynamics of the large-scale system. It is noted that $\mathbf{A}(t)$ the matrix is quasi-periodic and contains incommensurate frequencies. To the best of the author's knowledge, no current techniques would allow direct order reduction from the Eq. (1) to the Eq. (2).

The reducibility of a linear quasi-periodic system has been the subject of research in the scientific community. Many excellent references discuss the reducibility of quasi-periodic systems [21–26]. Recently, Waswa and Redkar presented a technique based on L-F Transformation, state augmentation, and normal form to reduce linear quasi-periodic system into an LTI system [27]. Very recently, Subramanian and Redkar presented a method to compute L-P Transformation based on intuitive state augmentation and normal forms [28]. This technique can be utilized to calculate a closed-form expression for the L–P Transformation. In this paper, we use the L–P transformation-based approach presented in Refs. [28,29] to obtain the LTI representation of the quasi-periodic system and perform the order reduction. For clarity, it is noted that the reducibility of a quasi-periodic system means converting a linear quasi-periodic system into an LTI system of the same dimension. Order reduction means reducing the size (or the number of states) of the original quasi-periodic system or its LTI representation obtained via the L-P Transformation.

This paper briefly reviews the L–P transformation computation, conditions for reducibility, and applications. The linear and nonlinear order reduction techniques are outlined in section two. Section three presents an application- Mathieu Hill-type equation for which the L–P Transformation can be determined using the state augmentation and normal forms. The reduced-order models are constructed using linear and nonlinear techniques for this system. Section five discusses the resonant interaction. Section six presents examples of the effectiveness of the order reduction approach near resonances. In the end, in section five, discussions and conclusions are presented.

2. Mathematical preliminaries

2.1. Computation of the L-P transformation

A quasi-periodic dynamical system, without any external excitation, can be expressed as

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y}; \tag{3}$$

where $\mathbf{A}(t)$ is a $n \times n$ matrix containing a finite number (k) of incommensurable frequencies ($k \geq 2$)

$$\mathbf{A}(t) = \widetilde{A}(\omega_1 t, \dots\dots, \omega_k t) \quad \forall k \geq 2 \tag{4}$$

It is noted that $\mathbf{A}(t)$ is continuous and periodic in each argument. Still, the ratio of any two frequencies is irrational [25]. It can be observed that the dynamical system given by the Eq. (4) is linear, and the method of normal forms [30] is inapplicable. However, the parametric excitation terms can be considered fictitious states. Therefore, the linear nonautonomous system given by the equation can be expressed as a nonlinear autonomous system.

Consider the most general form of the Eq. (3) with quasi-periodicity provided by

$$\dot{\mathbf{x}} = (\mathbf{B}_0 + \mathbf{B}(t))\mathbf{x} \tag{5}$$

where $\mathbf{A}(t)$ expressed as a constant matrix $\mathbf{B}_0$ and a quasi-periodic matrix $\mathbf{B}(t)$ of appropriate dimensions. It is important to note that we assume the eigenvalues (λ) of $\mathbf{B}_0$ and the fundamental frequencies (ω) of $\mathbf{B}(t)$ satisfying the diophantine condition [31] to avoid small divisors

$$|\lambda_i - \lambda_j + \sqrt{-1}(k, \omega)| \geq \frac{c}{|k|^{\gamma}} \forall k \in \mathbb{Z}^r \backslash \{0\} \tag{6}$$

where $|k| = |k_1| + \cdots + |k_r|$ and c is a constant.

Typically, $\mathbf{B}(t)$ comprises of $\sum_{i=1}^{n}(a_i \mathrm{Cos}(\omega_i t) + b_i \mathrm{Sin}(\omega_i t))$ type terms where ω_i is the frequency of quasi-periodic excitation (c.f. Eq. (4)). Assuming $q_i = \mathrm{Cos}(\omega_i t)$ and $p_i = \mathrm{Sin}(\omega_i t)$ the Eq. (5) can be expressed as

$$\dot{\overline{\mathbf{x}}} = \overline{\mathbf{B}}_0\overline{\mathbf{x}} + \mathbf{f}(\overline{\mathbf{x}}) \tag{7}$$

where $\overline{\mathbf{x}} = [\mathbf{x}, \mathbf{p}, \mathbf{q}]^T$, $\mathbf{p} = [p_1, p_2, \dots, p_n]^T$, $\mathbf{q} = [q_1, q_2, \dots, q_n]^T$ and

Applying the modal Transformation $\overline{\mathbf{x}} = \mathbf{Mz}$, if $\mathbf{B}_0$ has semi-simple eigenvalues, the Eq. (7) is transformed into

$$\dot{\mathbf{z}} = \mathbf{Jz} + \mathbf{M}^{-1}\mathbf{f}(\mathbf{z}) \tag{8}$$

where $\mathbf{J}$ is the Jordan form of $\overline{\mathbf{B}}_0$ (assumed to have semi-simple eigenvalues). The diagonal elements of the $\mathbf{J}$ matrix contain the linear matrix's ($\overline{\mathbf{B}}_0$) eigenvalues and frequencies of parametric excitations $[\lambda_{n+1} \dots \lambda_{n+m}]$ that are incommensurate.

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_n & \\ & & & & \boldsymbol{\omega} \end{bmatrix}, \boldsymbol{\omega} = \begin{bmatrix} \lambda_{n+1} & & \\ & \ddots & \\ & & \lambda_{n+m} \end{bmatrix} \tag{9}$$

The system shown in the Eq. (8) is amenable to an application of Normal Forms [30]. A near identity transformation [30] (of the form given by Eq. (10)) is applied to the Eq. (8).

$$\mathbf{z} = \mathbf{v} + \mathbf{h}_r(\mathbf{v}) \tag{10}$$

where $\mathbf{h}_r(\mathbf{v})$ is a formal power series in $\mathbf{v}$ of degree r with T periodic coefficients that leads to

$$\dot{\mathbf{v}} = \mathbf{Jv} - \left[\frac{\partial \mathbf{h}_r(\mathbf{v})}{\partial(\mathbf{v})}\mathbf{Jv} - \mathbf{Jh}_r(\mathbf{v})\right] + \mathbf{f}_r(\mathbf{v}) \tag{11}$$

The higher-order nonlinear terms in the Eq. (11) are eliminated by considering the following condition

$$\frac{\partial \mathbf{h}_r(\mathbf{v})}{\partial(\mathbf{v})}\mathbf{Jv} - \mathbf{Jh}_r(\mathbf{v}) + \mathbf{f}_r(\mathbf{v}) = 0 \tag{12}$$

where

$$\mathbf{h}_r(\mathbf{v}) = \sum_{j=1}^{s}\sum_{\overline{m}_r}\sum_{v=-\alpha}^{v=\alpha} h_{j\overline{m}v}\,|\mathbf{v}|^m\,\mathbf{e}_j, \mathbf{f}_r(\mathbf{v}) = \sum_{j=1}^{s}\sum_{\overline{m}_r}\sum_{v=-\alpha}^{v=\alpha} f_{j\overline{m}v}\,|\mathbf{v}|^m\,\mathbf{e}_j,$$

$$m_r = (m_1, m_2..), \sum_{i=1}^{n} m_i = 2, |\mathbf{v}|^m = \upsilon_1^{m_1}\upsilon_2^{m2}..\upsilon_n^{m_n}$$

and $\mathbf{e}_j$ is the jth member of the natural basis

After solving the Eq. (12), the solvability expression for a given degree of nonlinearity can be expressed as

$$h_{jmv} = \frac{f_{jmv}}{\mathbf{m}_r.\lambda - \lambda_j} \tag{13}$$

where $\lambda = [\lambda_1, \lambda_2, \lambda_3 .... \lambda_n, \lambda_{n+1} \dots \lambda_{n+m}]^T$ are the eigenvalues of $\mathbf{J}$,

$$\mathbf{m}_r.\lambda - \lambda_j \neq 0 \tag{14}$$

Suppose the solvability condition [23] given by Eq. (14) is satisfied. In that case, one can obtain the linear equation given by the Eq. (15) and the near identify Transformation given by the Eq. (10).

$$\dot{\mathbf{v}} = \overline{\mathbf{J}}\mathbf{v} \tag{15}$$

The near identity transformation given by the Eq. (10) is wholly known in the non-resonant case and $\mathbf{h}_r(\mathbf{v})$ contains the terms that explicitly depend upon fictitious states $\mathbf{p}$ and $\mathbf{q}$. One can substitute these fictitious states in terms of their closed-form expression $q_i = \mathrm{Cos}(\omega_i t)$, and $p_i = \mathrm{Sin}(\omega_i t)$ that yields $\mathbf{h}_r(\mathbf{v}, t)$ leading to the following form of the near identity transformation

$$\mathbf{z} = [\mathbf{I} + \overline{\mathbf{Q}}(t)]\mathbf{v} \approx \widetilde{\mathbf{Q}}(t)\mathbf{v} \tag{16}$$

This Transformation is similar to the Lyapunov-Floquet (L-F) Transformation [14,32] but for quasi-periodic systems.

Limitations;

One important aspect is that this technique uses the normal form technique, which can be viewed as an extension of the higher-order averaging method [33,34] and has the same limitations as the averaging. This approach may not yield accurate results when the nonlinearity is very strong (i.e., very strong parametric excitation in the present case) or the linear term is absent (i.e., $\mathbf{B}_0 = \mathbf{0}$ c.f. Eq. (5)). On the other hand, to the best of the author's knowledge, this approach is the only approach that yields the L-P Transformation given by the Eq. (16) in a closed-form. The authors have successfully used this approach to analyze linear and nonlinear quasi-periodic systems.

2.2. Computation of the inverse of the L-P transformation

For parametrically excited quasi-periodic linear systems of the Eq. (3) form, the L-P Transformation is sufficient for analysis. However, the inverse of the L-P Transformation is needed for quasi-periodic nonlinear systems or quasi-periodic linear/nonlinear systems with deterministic or stochastic excitations. The L-P Transformation is a matrix where the matrix elements contain a truncated quasi-periodic Fourier series. Therefore, inverting a quasi-periodic matrix is not a trivial problem. This section presents two possible approaches to obtain the inverse L-P Transformation.

Symbolic Computation: In minimal cases, when the L-P Transformation matrix (given by Eq. (16)) is small (2×2) and contains only a few terms, Symbolic computation software like Mathematica or Maple may be able to find the inverse. However, the inverse computed with this direct approach should be checked for the following conditions.

$$\begin{aligned} &\widetilde{\mathbf{Q}}^{-1}(0) = \mathbf{I} \\ &\widetilde{\mathbf{Q}}^{-1}(t) \times \widetilde{\mathbf{Q}}(t) = \mathbf{I} \end{aligned} \tag{17}$$

The expression provided for $\mathbf{Q}^{-1}(t)$may need further simplification for ease in future use.

Neural Network: One can also use a dynamical method using a recurrent neural network proposed for inversion of the time-varying matrix. One could use the gradient method [35], Zhang dynamics [36–38], or Chen dynamics [37] to find an inverse. This section briefly presents the Zhang dynamics approach [36] that could be used for inverting the L-P Transformation.

Consider a time-varying matrix $\mathbf{Y}(t)$ with inverse $\mathbf{W}(t) = \mathbf{Y}^{-1}(t)$ so that the Eq. (18) is valid

$$\begin{aligned} &\mathbf{Y}(t)\mathbf{W}(t) = \mathbf{I} \\ &\mathbf{Y}(t)\mathbf{W}(t) - \mathbf{I} = \mathbf{0} \end{aligned} \tag{18}$$

We assume $\mathbf{Y}(t)$ is known and $\frac{d\mathbf{Y}(t)}{dt}$ exists. The objective is to find $\mathbf{W}(t)$ using the following equation

$$\mathbf{E}(\mathbf{W}(t), t) \equiv \mathbf{Y}(t)\mathbf{W}(t) - \mathbf{I} \tag{19}$$

where $\mathbf{E}(\mathbf{W}(t), t)$ is a matrix-valued error function. The derivative of the error function $\dot{\mathbf{E}}(\mathbf{W}(t), t)$ should be selected such that $\mathbf{E}(\mathbf{W}(t), t) \to \mathbf{0}$. Thus, $\dot{\mathbf{E}}(\mathbf{W}(t), t)$ can be chosen as

$$\frac{d\mathbf{E}(\mathbf{W}(t), t)}{dt} = -\Gamma \mathbf{F}(\mathbf{E}(\mathbf{W}(t), t)) \tag{20}$$

where Γ is a scaling factor for the convergence and $\mathbf{F}(\mathbf{E}(\mathbf{W}(t), t))$ is called an activation function or matrix mapping recurrent neural network.

Differentiating Eq. (19) w.r.t. time and substituting Eqs. (19) and (20) yields

$$\begin{aligned} &\mathbf{Y}(t)\dot{\mathbf{W}}(t) = -\dot{\mathbf{Y}}(t)\mathbf{W}(t) - \Gamma \mathbf{F}(\mathbf{E}(\mathbf{W}(t), t)) \\ &\mathbf{Y}(t)\dot{\mathbf{W}}(t) = -\dot{\mathbf{Y}}(t)\mathbf{W}(t) - \Gamma \mathbf{F}(\mathbf{Y}(t)\mathbf{W}(t) - \mathbf{I}) \end{aligned} \tag{21}$$

The Eq. (21) is a matrix differential equation that can be solved for $\mathbf{W}(t)$ using an appropriate initial condition. In the current paper, $\mathbf{Y}(t)$ is the L-P transformation matrix $\widetilde{\mathbf{Q}}(t)$ and $\mathbf{W}(t)$ is the inverse of L-P Transformation $\widetilde{\mathbf{Q}}^{-1}(t)$. Thus Eq. (21) can be written as

$$\widetilde{\mathbf{Q}}(t)\dot{\widetilde{\mathbf{Q}}}^{-1}(t) = -\dot{\widetilde{\mathbf{Q}}}(t)\widetilde{\mathbf{Q}}^{-1}(t) - \Gamma \mathbf{F}(\widetilde{\mathbf{Q}}(t)\widetilde{\mathbf{Q}}^{-1}(t) - \mathbf{I}) \tag{22}$$

One has to select an appropriate activation function and scaling constant $\Gamma\mathbf{F}()$ to achieve convergence. Then, the Eq. (22) can be numerically integrated with the initial condition $\widetilde{\mathbf{Q}}^{-1}(0) = \mathbf{I}$ to determine $\widetilde{\mathbf{Q}}^{-1}(t)$. For more details on the Zhang Neural Network, its application, and proof of convergence, we refer to Ref. [38].

3. Order reduction techniques

3.1. Order reduction via linear projection

Consider a nonlinear quasi-periodic system described by the Eq. (1). Applying the L-P Transformation $\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ produces

$$\dot{\mathbf{z}}(t) = \mathbf{J}\mathbf{z}(t) + \mathbf{Q}^{-1}(t)\mathbf{f}(\mathbf{z}, t) + \mathbf{Q}^{-1}(t)\mathbf{F}(t) \equiv \mathbf{J}\mathbf{z}(t) + \mathbf{w}(\mathbf{z}, t) + \overline{\mathbf{F}}(t) \tag{23}$$

where $\mathbf{J}$ is the constant matrix and $\mathbf{w}(\mathbf{z}, t)$ represents an appropriately defined nonlinear quasi-periodic vector consisting of monomials of z_j.

Again, the objective of order reduction is to replace the nonlinear quasi-periodic system given by Eq. (23) with an equivalent system provided by

$$\dot{\mathbf{z}}_r(t) = \mathbf{J}_r\mathbf{z}_r(t) + \mathbf{w}_r(\mathbf{z}_r, t) + \mathbf{F}_r(t) \tag{24}$$

We partition the Eq. (23) as

$$\begin{Bmatrix} \dot{\mathbf{z}}_r \\ \dot{\mathbf{z}}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{J}_r & 0 \\ 0 & \mathbf{J}_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_r \\ \mathbf{z}_s \end{Bmatrix} + \begin{Bmatrix} \mathbf{w}_r(\mathbf{z}_r, \mathbf{z}_s, t) \\ \mathbf{w}_s(\mathbf{z}_r, \mathbf{z}_s, t) \end{Bmatrix} + \begin{Bmatrix} \mathbf{F}_r(t) \\ \mathbf{F}_s(t) \end{Bmatrix} (a) \tag{25}$$

where $\mathbf{z}_s$ is an $(n-r)$ vector of non-dominant states, $\mathbf{J}_s$ is the matrix block of dimension $(n-r) \times (n-r)$ corresponding to the non-dominant states as defined earlier and $\mathbf{w}_r(\mathbf{z}_r, \mathbf{z}_s, t)$ and $\mathbf{w}_s(\mathbf{z}_r, \mathbf{z}_s, t)$ are the monomials of $\mathbf{z}$(of order i) with quasi-periodic coefficients.

In the linear technique, the contribution of the non-dominant states is considered insignificant and hence neglected. Thus, the reduced-order model is given by

$$\dot{\mathbf{z}}_r(t) = \mathbf{J}_r\mathbf{z}_r(t) + \mathbf{w}_r(\mathbf{z}_r, 0, t) + \mathbf{F}_r(t) \tag{26}$$

The Eq. (26) is the reduced-order model of the actual large-scale system described by the Eq. (25). The Eq. (26) can be integrated numerically and using the transformation $\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{T}\mathbf{z}_r(t)$, where $\mathbf{T} = \begin{bmatrix} \mathbf{I}_{r\times r} & \mathbf{0}_{r\times(n-r)} \end{bmatrix}^T$all the states in $\mathbf{x}$ can be recovered.

This linear projection technique is simple and easy to implement. It may or may not provide accurate results. The selection of dominant states depends upon the judgment of the analyst. It does not give a clear insight into system dynamics if the system behavior is complex and involves internal and parametric resonance.

3.2. Order reduction using invariant manifold

This methodology is based on the 'Invariant Manifold Theory'. According to this theory, there exists a relationship between the dominant "master" and the non-dominant "slave" states of the system. It is possible to replace (under certain conditions) the non-dominant states with dominant states. Thus, the order of the system can be reduced.

We assume that forcing frequency is incommensurate with the frequency of quasi-periodic parametric excitation. The constraint (or manifold governing) equations relating to 'master' and 'slave' states are complex. Still, they admit the solution in the form of asymptotic expansion. The relationship between the dominant and the non-dominant states of the system will involve contributions from the forcing and nonlinearity. If there are no resonances, replacing the non-dominant states with the dominant ones is possible.

Once again, consider a nonlinear quasi-periodic system given by the Eq. (23) in the L-P transformed domain that is further partitioned as the Eq. (25). After ordering and expanding the nonlinear terms, we obtain.

$$\begin{aligned}
\dot{\mathbf{z}}_r &= \mathbf{J}_r\mathbf{z}_r + \varepsilon\mathbf{w}_{r2}(\mathbf{z}_r,\mathbf{z}_s,t) + \varepsilon^2\mathbf{w}_{r3}(\mathbf{z}_r,\mathbf{z}_s,t) + \varepsilon^3\mathbf{w}_{r4}(\mathbf{z}_r,\mathbf{z}_s,t) + \cdots\\
&+\varepsilon^{i-1}\mathbf{w}_{ri}(\mathbf{z}_r,\mathbf{z}_s,t) + \varepsilon^{i-1}\mathbf{O}(|\mathbf{z}|^i) + \overline{\mathbf{F}}_r(t)(a)\\
\dot{\mathbf{z}}_s &= \mathbf{J}_s\mathbf{z}_s + \varepsilon\mathbf{w}_{s2}(\mathbf{z}_r,\mathbf{z}_s,t) + \varepsilon^2\mathbf{w}_{s3}(\mathbf{z}_r,\mathbf{z}_s,t) + \varepsilon^3\mathbf{w}_{s4}(\mathbf{z}_r,\mathbf{z}_s,t) + \cdots\\
&+\varepsilon^{i-1}\mathbf{w}_{si}(\mathbf{z}_r,\mathbf{z}_s,t) + \varepsilon^{i-1}\mathbf{O}(|\mathbf{z}|^i) + \overline{\mathbf{F}}_s(t)(b)
\end{aligned} \tag{27}$$

where $\varepsilon^{n-1}\mathbf{w}_{rn}(\mathbf{z},t)$ include the terms of monomials of order n in 'master' dynamics and $\varepsilon^{n-1}\mathbf{w}_{sn}(\mathbf{z},t)$ include terms of monomials of order n in 'slave' dynamics. In this approach, we assume a nonlinear relationship between the dominant $(\mathbf{z}_r)$ and the non-dominant $(\mathbf{z}_s)$ states as

$$\begin{aligned}
\mathbf{z}_s &= \overline{\mathbf{h}}_{01}(t) + \varepsilon(\overline{\mathbf{h}}_{02}(t) + \overline{\mathbf{h}}_{12}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{22}(\mathbf{z}_r,t))\\
&+\varepsilon^2(\overline{\mathbf{h}}_{03}(t) + \overline{\mathbf{h}}_{13}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{23}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{33}(\mathbf{z}_r,t))\\
&+\varepsilon^3(\overline{\mathbf{h}}_{04}(t) + \overline{\mathbf{h}}_{14}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{24}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{34}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{44}(\mathbf{z}_r,t)) + \cdots
\end{aligned} \tag{28}$$

Here $\overline{\mathbf{h}}_{ij}(\mathbf{z}_r,t)$ are the unknown quasi-periodic vector coefficients. Substitution of the Eq. (28) into (27) yields

$$\begin{aligned}
\dot{\mathbf{z}}_s &= \dot{\overline{\mathbf{h}}}_{01}(t) + \varepsilon\{\dot{\overline{\mathbf{h}}}_{02}(t) + \frac{\partial}{\partial t}(\overline{\mathbf{h}}_{12}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{22}(\mathbf{z}_r,t))\\
&+\frac{\partial}{\partial \mathbf{z}_r}(\overline{\mathbf{h}}_{12}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{22}(\mathbf{z}_r,t))\cdot\dot{\mathbf{z}}_r\}\\
&+\varepsilon^2\{\dot{\overline{\mathbf{h}}}_{03}(t) + \frac{\partial}{\partial t}(\overline{\mathbf{h}}_{13}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{23}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{33}(\mathbf{z}_r,t))\\
&+\frac{\partial}{\partial \mathbf{z}_r}(\overline{\mathbf{h}}_{13}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{23}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{33}(\mathbf{z}_r,t))\cdot\dot{\mathbf{z}}_r\}\\
&+\varepsilon^3\{\dot{\overline{\mathbf{h}}}_{04}(t) + \frac{\partial}{\partial t}(\overline{\mathbf{h}}_{14}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{24}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{34}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{44}(\mathbf{z}_r,t))\\
&+\frac{\partial}{\partial \mathbf{z}_r}(\overline{\mathbf{h}}_{14}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{24}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{34}(\mathbf{z}_r,t) + \overline{\mathbf{h}}_{44}(\mathbf{z}_r,t))\cdot\dot{\mathbf{z}}_r\} + \cdots
\end{aligned} \tag{29}$$

Dropping spatial and temporal arguments for brevity, Eq. ((27)-b) can be rewritten as

$$\begin{aligned}
\dot{\mathbf{z}}_s &= \mathbf{J}_s\cdot\{\overline{\mathbf{h}}_{01} + \varepsilon(\overline{\mathbf{h}}_{02} + \overline{\mathbf{h}}_{12} + \overline{\mathbf{h}}_{22}) + \varepsilon^2(\overline{\mathbf{h}}_{03} + \overline{\mathbf{h}}_{13} + \overline{\mathbf{h}}_{23} + \overline{\mathbf{h}}_{33})\\
&+\varepsilon^3(\overline{\mathbf{h}}_{04} + \overline{\mathbf{h}}_{14} + \overline{\mathbf{h}}_{24} + \overline{\mathbf{h}}_{34} + \overline{\mathbf{h}}_{444}) + \cdots\\
&\sum_{m=4}^{n}\varepsilon^m((\mathbf{h}_{0m+1} + \sum_{k=1}^{m+1}\mathbf{h}_{k\,m+1}),t)\\
&+\varepsilon\mathbf{w}_{s2}(\overline{\mathbf{h}}_{01} + \varepsilon(\overline{\mathbf{h}}_{02} + \overline{\mathbf{h}}_{12} + \overline{\mathbf{h}}_{22}) + \varepsilon^2(\overline{\mathbf{h}}_{03} + \overline{\mathbf{h}}_{13} + \overline{\mathbf{h}}_{23} + \overline{\mathbf{h}}_{33})\\
&+\varepsilon^3(\overline{\mathbf{h}}_{04} + \overline{\mathbf{h}}_{14} + \overline{\mathbf{h}}_{24} + \overline{\mathbf{h}}_{34} + \overline{\mathbf{h}}_{444}) + \cdots\\
&\sum_{m=4}^{n}\varepsilon^m(\mathbf{h}_{0m+1} + \sum_{k=1}^{m+2}\mathbf{h}_{k\,m+1}),t)\\
&+\varepsilon^2\mathbf{w}_{s3}(\overline{\mathbf{h}}_{01} + \varepsilon(\overline{\mathbf{h}}_{02} + \overline{\mathbf{h}}_{12} + \overline{\mathbf{h}}_{22}) + \varepsilon^2(\overline{\mathbf{h}}_{03} + \overline{\mathbf{h}}_{13} + \overline{\mathbf{h}}_{23} + \overline{\mathbf{h}}_{33})\\
&+\varepsilon^3(\overline{\mathbf{h}}_{04} + \overline{\mathbf{h}}_{14} + \overline{\mathbf{h}}_{24} + \overline{\mathbf{h}}_{34} + \overline{\mathbf{h}}_{44}) + \cdots)\}\\
&\sum_{m=4}^{n}\varepsilon^m(\mathbf{h}_{0m+1} + \sum_{k=1}^{m+1}\mathbf{h}_{k\,m+1}),t) + \varepsilon^3\mathbf{w}_{s4}(\overline{\mathbf{h}}_{01} + \varepsilon(\overline{\mathbf{h}}_{02} + \overline{\mathbf{h}}_{12} + \overline{\mathbf{h}}_{22})\\
&+\varepsilon^2(\overline{\mathbf{h}}_{03} + \overline{\mathbf{h}}_{13} + \overline{\mathbf{h}}_{23} + \overline{\mathbf{h}}_{33}) + \varepsilon^3(\overline{\mathbf{h}}_{04} + \overline{\mathbf{h}}_{14} + \overline{\mathbf{h}}_{24} + \overline{\mathbf{h}}_{34} + \overline{\mathbf{h}}_{444}) + \cdots\\
&\sum_{m=4}^{n}\varepsilon^m(\mathbf{h}_{0m+1} + \sum_{k=1}^{m+1}\mathbf{h}_{k\,m+1}),t)\\
&+\varepsilon^{i-1}\mathbf{w}_{si}(\overline{\mathbf{h}}_{01} + \varepsilon(\overline{\mathbf{h}}_{02} + \overline{\mathbf{h}}_{12} + \overline{\mathbf{h}}_{22}) + \varepsilon^2(\overline{\mathbf{h}}_{03} + \overline{\mathbf{h}}_{13} + \overline{\mathbf{h}}_{23} + \overline{\mathbf{h}}_{33})\\
&+\varepsilon^3(\overline{\mathbf{h}}_{04} + \overline{\mathbf{h}}_{14} + \overline{\mathbf{h}}_{24} + \overline{\mathbf{h}}_{34} + \overline{\mathbf{h}}_{444}) + \cdots\\
&\sum_{m=4}^{n}\varepsilon^m(\mathbf{h}_{0m+1} + \sum_{k=1}^{m+1}\mathbf{h}_{k\,m+1}),t)\}\\
&+\overline{\mathbf{F}}_s(t)
\end{aligned} \tag{30}$$

Substituting the Eqs. (27) and (28) into the Eq. (29) and equating it to the Eq. (30) yields a complex partial differential equation involving various orders of ε. By correlating the terms of the same order of ε, we obtain the equations, which need to be solved to determine $\mathbf{h}_{0m+1}(t)$ and $\mathbf{h}_{k\,m+1}(\mathbf{z}_m,t)$

Collecting the terms in order of ε^0 yields

$$\dot{\overline{\mathbf{h}}}_{01}(t) = \mathbf{J}_s\overline{\mathbf{h}}_{01}(t) + \overline{\mathbf{F}}_s(t) \tag{31}$$

The Eq. (31) is a linear equation involving pure temporal arguments. The solution of the Eq. (31) can be determined using the convolution integral [39] as

$$\overline{\mathbf{h}}_{01}(t) = \mathbf{e}^{\mathbf{J}_s t}\mathbf{h}_{01}(0) + \int_0^t \mathbf{e}^{\mathbf{J}_s(t-\tau)}\overline{\mathbf{F}}_s(\tau)d\tau \tag{32}$$

If the forcing $\mathbf{F}(t)$ is harmonic with frequency $k\omega_f$, then after the L-P Transformation, the frequency of harmonic excitation $\overline{\mathbf{F}}(t)$ becomes $\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{\infty}(\overline{\mathbf{p}}\cdot\boldsymbol{\omega}_p + k\omega_f)$, where $\boldsymbol{\omega}_p$ is the vector containing quasi-periodic frequencies in the L-P Transformation $\boldsymbol{\omega}_p = \{\omega_1 \quad \omega_2\}, \overline{\mathbf{p}} = \{p_1 \quad p_2\}^T$.

Expressing forcing in the most general form as

$$\overline{\mathbf{F}}_s(t) = \sum_{k=-\infty}^{+\infty}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{\infty}\mathbf{C}_{p_1p_2k}\mathbf{e}^{\bar{i}(\mathbf{p}\cdot\boldsymbol{\omega}_p + k\omega_f)t} \tag{33}$$

If the eigenvalues of $\mathbf{J}_s$ are purely imaginary and given by $\bar{\lambda}_p$; $p = 1,2,\ldots,s$ then the solution can be written as

$$\begin{aligned}
\overline{\mathbf{h}}_{01}(t) = \sum_{j=1}^{s}\sum_{p_1=-\infty}^{\infty}\sum_{p_2=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}&(C_{jp_1p_2k}\frac{\mathbf{e}^{\bar{i}(\bar{n}\omega_p + k\omega_f)t}}{\bar{i}(\mathbf{p}\cdot\boldsymbol{\omega}_p + k\omega_f - \lambda_j)}e_j\\
&- C_{jp_1p_2k}\frac{\mathbf{e}^{\bar{i}\lambda_j t}}{\bar{i}(\mathbf{p}\cdot\boldsymbol{\omega}_p + k\omega_f - \lambda_j)}e_j)
\end{aligned} \tag{34}$$

It can be seen that if $\mathbf{p}\cdot\boldsymbol{\omega}_p + k\omega_f - \lambda_j = 0$ for any combination, $\overline{\mathbf{h}}_{01}(t)$ cannot be found out, and the system is said to be in 'linear resonance'. This resonance is referred to as a 'primary' or a 'main resonance' in perturbation analysis.

Collecting the terms at the order of ε^1 yields

$$\dot{\overline{\mathbf{h}}}_{02}(t) = \mathbf{J}_s\overline{\mathbf{h}}_{02}(t) - \frac{\partial\overline{\mathbf{h}}_{12}}{\partial\mathbf{z}_r}\mathbf{F}_r + \mathbf{w}_{s2_0}(t) \tag{35}$$

$$\frac{\partial\overline{\mathbf{h}}_{12}}{\partial t} + \frac{\partial\overline{\mathbf{h}}_{12}}{\partial\mathbf{z}_r}\mathbf{J}_r\mathbf{z}_r + \frac{\partial\overline{\mathbf{h}}_{22}}{\partial\mathbf{z}_r}\mathbf{F}_r(t) - \mathbf{J}_s\overline{\mathbf{h}}_{12} = \mathbf{w}_{s2_1}(\mathbf{z}_r,t) \tag{36}$$

$$\frac{\partial\overline{\mathbf{h}}_{22}}{\partial t} + \frac{\partial\overline{\mathbf{h}}_{22}}{\partial\mathbf{z}_r}\mathbf{J}_r\mathbf{z}_r - \mathbf{J}_s\overline{\mathbf{h}}_{22} = \mathbf{w}_{s2_2}(\mathbf{z}_r,t) \tag{37}$$

It can be seen that Eqs. (35), (36) and (37) are coupled equations. However, the Eq. (37) can be solved independently. Assuming the most general form of nonlinearity and expanding the known and unknown terms in multiple Fourier series as

$$\overline{\mathbf{h}}_{22}(\mathbf{z}_r,t) = \sum_{j=1}^{s}\sum_{\overline{\mathbf{m}}}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{+\infty}h_{j\overline{m}p_1p_2v}\,e^{\bar{i}(\overline{\mathbf{p}}\cdot\boldsymbol{\omega})t}|\mathbf{z}_r|^{\mathbf{m}}e_j \tag{38}$$

$$\mathbf{w}_{s2_2}(\mathbf{z}_r,t)=\sum_{j=1}^{s}\sum_{\tilde{\mathbf{m}}}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{\infty}a_{j\tilde{m}p_1p_2v}e^{\bar{i}(\overline{\mathbf{p}}\cdot\boldsymbol{\omega})t}\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}e_j \tag{39}$$

$$\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}=z_1^{m_1}z_2^{m_2}\dots z_r^{m_r},\bar{i}=\sqrt{-1},m_1+\cdots\dots+m_r=2;$$
$$\boldsymbol{\omega}_p=\{\omega_1\quad\omega_2\},\overline{\mathbf{p}}=\{p_1\quad p_2\}^T$$

Collecting the terms to solve for the unknowns yields the '*reducibility condition*' given by the Eq. (40),

$$\bar{i}(\overline{\mathbf{p}}\cdot\boldsymbol{\omega})+\sum_{l=1}^{r}(m_l\lambda_l)-\bar{\lambda}_p\neq 0 \tag{40}$$

where all the terms appearing in the Eq. (40) are defined before.

In the absence of resonances $\overline{\mathbf{h}}_{22}(\mathbf{z}_r,t)$ can be obtained. At this stage, forcing frequency does not appear in the Eq. (40), implying no direct interaction between the nonlinearity and the external excitation. However, we construct the solution using Eqs. (35), and (36) forcing interacts with the nonlinearity, giving additional 'resonance conditions'.

To find out the solution to the Eq. (36), which contains the contribution from the nonlinearity $\mathbf{w}_{s2}(\mathbf{z}_r,t)$, we expand the known and the unknown terms in the multiple Fourier series of the form

$$\overline{\mathbf{h}}_{12}(\mathbf{z}_r,t)=\sum_{j=1}^{s}\sum_{\tilde{\mathbf{m}}}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{+\infty}\sum_{p_3=-\infty}^{+\infty}h_{j\tilde{m}p_1p_2v}\,e^{\bar{i}(\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})t}\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}e_j \tag{41}$$

$$\mathbf{w}_{rs_1}(\mathbf{z}_r,t)=\sum_{j=1}^{s}\sum_{\tilde{\mathbf{m}}}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{\infty}\sum_{p_3=-\infty}^{+\infty}a_{j\tilde{m}p_1p_2v}e^{\bar{i}(\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})t}\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}e_j \tag{42}$$

$$\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}=z_1^{m_1}z_2^{m_2}\dots z_r^{m_r},\bar{i}=\sqrt{-1},m_1+\cdots\dots+m_r=1;$$
$$\overline{\boldsymbol{\omega}}=\left\{\omega_1\quad\omega_2\quad\omega_f\right\};\tilde{\mathbf{p}}=\left\{p_1\quad p_2\quad p_3\right\}^T$$

A term-by-term comparison yields

$$h_{j\tilde{m}v}=\frac{a_{j\tilde{m}v}}{\bar{i}(\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})+\lambda_l-\bar{\lambda}_p} \tag{43}$$

The 'combined reducibility condition' can be expressed as

$$\bar{i}(\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})+\lambda_l-\bar{\lambda}_p\neq 0 \tag{44}$$

It can be observed that all the terms appearing in the Eq. (35) are free from spatial arguments, and they can be solved using convolution, as discussed before. The forcing terms $\mathbf{w}_{s20}(t)$(with a square type term in $\overline{\mathbf{h}}_{01}(t)$) appearing due to $\mathbf{w}_{s2}(\mathbf{z}_r,t)$ is known from the Eq. (32) , which can be expressed in the form

$$\mathbf{w}_{s2_0}(t)=\sum_{j=1}^{s}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{\infty}\sum_{p_3=-\infty}^{+\infty}a_{j\tilde{m}v}e^{\bar{i}(\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})t}e_j \tag{45}$$

where $\bar{i}=\sqrt{-1}$, $\overline{\boldsymbol{\omega}}=\left\{\omega_1\quad\omega_2\quad\omega_f\right\};\tilde{\mathbf{p}}=\left\{p_1\quad p_2\quad p_3\right\}^T$ and $\frac{\partial\overline{\mathbf{h}}_{12}}{\partial\mathbf{z}_r}\overline{\mathbf{F}}_r$ is known from the Eq. (34).

However, the Eq. (35) cannot be solved if $\lambda_s=\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}}$, which can be written as

$$\lambda_s=p_1\omega_p+p_2\omega_{nl}+p_3\omega_f \tag{46}$$

The exact combination will be determined by the kind of terms present in the forcing.

As we collect the terms in the order of ε^2, we obtain

$$\dot{\overline{\mathbf{h}}}_{03}(t)=\mathbf{J}_s\overline{\mathbf{h}}_{03}(t)+\frac{\partial\overline{\mathbf{h}}_{13}}{\partial\mathbf{z}_r}\overline{\mathbf{F}}_r(t)-\frac{\partial\overline{\mathbf{h}}_{12}}{\partial\mathbf{z}_r}\underbrace{\mathbf{w}_{m2_0}(t)}_{[\overline{\mathbf{h}}_{01}(t)]_2}+\underbrace{\mathbf{w}_{s2_{30}}(t)}_{\overline{\mathbf{h}}_{01}(t)\cdot\overline{\mathbf{h}}_{02}(t)}+\underbrace{\mathbf{w}_{s3_0}(t)}_{[\overline{\mathbf{h}}_{01}(t)]_3} \tag{47}$$

$$\frac{\partial\overline{\mathbf{h}}_{13}}{\partial t}+\frac{\partial\overline{\mathbf{h}}_{22}}{\partial\mathbf{z}_r}\underbrace{\mathbf{w}_{m2_0}(\mathbf{z}_r,t)}_{[\overline{\mathbf{h}}_{01}(t)]_2}+\frac{\partial\overline{\mathbf{h}}_{13}}{\partial\mathbf{z}_r}\mathbf{J}_r\mathbf{z}_r+\frac{\partial\overline{\mathbf{h}}_{23}}{\partial\mathbf{z}_r}\overline{\mathbf{F}}_r$$
$$=\mathbf{J}_s\overline{\mathbf{h}}_{13}+\underbrace{\mathbf{w}_{s2_{31}}(\mathbf{z}_r,t)}_{\overline{\mathbf{h}}_{02}(t)\cdot\mathbf{z}_r+\overline{\mathbf{h}}_{01}(t)\cdot\overline{\mathbf{h}}_{12}(\mathbf{z}_r,t)}+\underbrace{\mathbf{w}_{s3_1}(\mathbf{z}_r,t)}_{[\overline{\mathbf{h}}_{01}(t)]_2\cdot\mathbf{z}_r} \tag{48}$$

$$\frac{\partial\overline{\mathbf{h}}_{23}}{\partial t}+\frac{\partial\overline{\mathbf{h}}_{23}}{\partial\mathbf{z}_r}\mathbf{J}_r\mathbf{z}_r+\frac{\partial\overline{\mathbf{h}}_{33}}{\partial\mathbf{z}_r}\overline{\mathbf{F}}_r=\mathbf{J}_s\overline{\mathbf{h}}_{23}+\underbrace{\mathbf{w}_{s2_{32}}(\mathbf{z}_r,t)}_{\overline{\mathbf{h}}_{12}(\mathbf{z}_r,t)\cdot\mathbf{z}_r+\overline{\mathbf{h}}_{01}(t)\cdot\overline{\mathbf{h}}_{22}(\mathbf{z}_r,t)}+\underbrace{\mathbf{w}_{s3_2}(\mathbf{z}_r,t)}_{\overline{\mathbf{h}}_{01}(t)\cdot\mathbf{z}_r} \tag{49}$$

$$\frac{\partial\overline{\mathbf{h}}_{33}}{\partial t}+\frac{\partial\overline{\mathbf{h}}_{33}}{\partial\mathbf{z}_r}\mathbf{J}_r\mathbf{z}_r=\mathbf{J}_s\overline{\mathbf{h}}_{33}+\underbrace{\mathbf{w}_{s2_{33}}(\mathbf{z}_r,t)}_{\overline{\mathbf{h}}_{22}(\mathbf{z}_r,t)\cdot\mathbf{z}_r}+\mathbf{w}_{s3_3}(\mathbf{z}_r,t) \tag{50}$$

The Eq. (47) has only temporal arguments; the Eq. (48) is linear in spatial arguments; the Eq. (49) depends upon quadratic spatial arguments, and the Eq. (50) involves cubic spatial arguments. As before, one has to solve these equations sequentially.

It can be observed that the Eq. (50) can be solved independently and involves contribution from $\mathbf{w}_{s2}(\mathbf{z}_r,t)$ denoted by $\mathbf{w}_{s2_{33}}(\mathbf{z}_r,t)$. To solve this equation, we expand the known terms and unknown terms in the multiple Fourier series (*c.f.* Eq. (37))

$$\overline{\mathbf{h}}_{33}(\mathbf{z}_r,t)=\sum_{j=1}^{s}\sum_{\tilde{\mathbf{m}}}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{+\infty}h_{j\tilde{m}v}\,e^{\bar{i}(\overline{\mathbf{p}}\cdot\boldsymbol{\omega})t}\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}e_j \tag{51}$$

$$\mathbf{w}_{s3_3}(\mathbf{z}_r,t)=\sum_{j=1}^{s}\sum_{\tilde{\mathbf{m}}}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{\infty}a_{j\tilde{m}v}e^{\bar{i}(\overline{\mathbf{p}}\cdot\boldsymbol{\omega})t}\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}e_j \tag{52}$$

$$\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}=z_1^{m_1}z_2^{m_2}\dots z_r^{m_r},\bar{i}=\sqrt{-1},m_1+\cdots\dots+m_r=3;$$
$$\boldsymbol{\omega}=\{\omega_1\quad\omega_2\},\overline{\mathbf{p}}=\{p_1\quad p_2\}^T$$

It is possible to determine $\overline{\mathbf{h}}_{33}(\mathbf{z}_r,t)$ if the following' reducibility condition' is satisfied.

$$\bar{i}(\overline{\mathbf{p}}\cdot\boldsymbol{\omega})+\sum_{l=1}^{r}(m_l\lambda_l)-\bar{\lambda}_p\neq 0 \tag{53}$$

Once $\overline{\mathbf{h}}_{33}(\mathbf{z}_r,t)$ is known, we can solve the Eq. (49). This equation contains terms arising from the product of $\frac{\partial\overline{\mathbf{h}}_{33}(\mathbf{z}_r,t)}{\partial\mathbf{z}_r}\overline{\mathbf{F}}_r(t)$ (where $\overline{\mathbf{F}}_r(t)$ is the forcing on the master states), the contribution from quadratic $\mathbf{w}_{s2}(\mathbf{z}_r,t)$ nonlinearity (denoted by $\mathbf{w}_{s2_{32}}(\mathbf{z}_r,t)$), and contribution from cubic nonlinearity $\mathbf{w}_{s3}(\mathbf{z}_r,t)$(represented by $\mathbf{w}_{s3_2}(\mathbf{z}_r,t)$). As before, we expand the known terms (marked as $\mathbf{w}_{sk}(\mathbf{z}_r,t)$) and the unknown terms ($\mathbf{h}_{23}(\mathbf{z}_r,t)$ in multiple Fourier series of the form

$$\overline{\mathbf{h}}_{23}(\mathbf{z}_r,t)=\sum_{j=1}^{s}\sum_{\tilde{\mathbf{m}}}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{+\infty}\sum_{p_3=-\infty}^{+\infty}h_{j\tilde{m}p_1p_2p_3v}\,e^{\bar{i}(\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})t}\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}e_j \tag{54}$$

$$\mathbf{w}_{sk}(\mathbf{z}_r,t)=\sum_{j=1}^{s}\sum_{\tilde{\mathbf{m}}}\sum_{p_1=-\infty}^{+\infty}\sum_{p_2=-\infty}^{\infty}\sum_{p_3=-\infty}^{+\infty}a_{j\tilde{m}p_1p_2p_3v}e^{\bar{i}(\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})t}\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}e_j \tag{55}$$

$$\left|\mathbf{z}_r\right|^{\tilde{\mathbf{m}}}=z_1^{m_1}z_2^{m_2}\dots z_r^{m_r},\bar{i}=\sqrt{-1},m_1+\cdots\dots+m_r=2;$$
$$\overline{\boldsymbol{\omega}}=\left\{\omega_1\quad\omega_2\quad\omega_f\right\};\tilde{\mathbf{p}}=\left\{p_1\quad p_2\quad p_3\right\}$$

As before, we can obtain $\overline{\mathbf{h}}_{23}(\mathbf{z}_r,t)$ via term-by-term comparison if and only if the following 'combined reducibility condition' is satisfied.

$$\bar{i}(\tilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})+\sum_{l=1}^{r}(m_l\lambda_l)-\bar{\lambda}_p\neq 0 \tag{56}$$

It can be observed that this 'combined reducibility condition' involves a contribution from the forcing.

Once $\overline{\mathbf{h}}_{23}(\mathbf{z}_r,t)$ is known, we can solve the Eq. (48) which contains the contribution from $\mathbf{w}_{s3}(\mathbf{z}_r,t)$(denoted by $\mathbf{w}_{s3_1}(\mathbf{z}_r,t)$) and $\mathbf{w}_{s2}(\mathbf{z}_r,t)$ (represented by $\mathbf{w}_{s3_1}(\mathbf{z}_r,t)$) and $\frac{\partial\overline{\mathbf{h}}_{23}(\mathbf{z}_r,t)}{\partial\mathbf{z}_r}\overline{\mathbf{F}}_r(t)$ To determine $\overline{\mathbf{h}}_{13}(\mathbf{z}_r,t)$, we expand the known and the unknown terms in the multiple Fourier series of the form given by Eq. (41) and Eq. (42), respectively and obtain the '*combined reducibility condition*' given by Eq. (44).

Further, to obtain $\overline{\mathbf{h}}_{03}(t)$, we use the convolution theorem, as before. However, it can be seen from the Eq. (47) (which contains only temporal arguments), the forcing terms arise from $\frac{\partial\overline{\mathbf{h}}_{13}}{\partial\mathbf{z}_r}\overline{\mathbf{F}}_r(t)$ (forcing on the master states), and nonlinear terms $\mathbf{w}_{s2}(\mathbf{z}_r, t)$ $\mathbf{w}_{s3}(\mathbf{z}_r, t)$ and are denoted as $\mathbf{w}_{s_q}(t)$, which can be expressed as

$$\mathbf{w}_{s_q}(t) = \sum_{j=1}^{s} \sum_{p_1=-\infty}^{+\infty} \sum_{p_2=-\infty}^{\infty} \sum_{p_3=-\infty}^{+\infty} a_{j\overline{m}v} e^{\bar{i}(\widetilde{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}})t} e_j \tag{57}$$

where $\bar{i} = \sqrt{-1}$, $\overline{\boldsymbol{\omega}} = \{\omega_1 \quad \omega_2 \quad \omega_f\}$; $\widetilde{\mathbf{p}} = \{p_1 \quad p_2 \quad p_3\}^T$.

The Eq. (47)cannot be solved if

$$\lambda_s = \widetilde{\mathbf{p}} \cdot \overline{\boldsymbol{\omega}}. \tag{58}$$

It is possible to continue the procedure discussed above to construct the relationship between 'slave' and 'master' states to the desired order and recover various '*resonance conditions*' involving contributions from external excitation and nonlinearity at multiple orders.

4. Applications

Consider a coupled undamped Mathieu Hill-type nonlinear quasi-periodic system subjected to external excitation given by

$$\begin{aligned} &\ddot{x} + (a_1 + b_1 \cos\omega_1 t + c_1 \cos\omega_1 t)x + x^2 y = A_1 \cos(\omega t) \\ &\ddot{y} + (a_2 + b_2 \cos\omega_1 t + c_2 \cos\omega_2 t)x + y^2 x = A_2 \cos(\omega t) \end{aligned} \tag{59}$$

The Eq. (59) can be expressed as

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{A}}_1(t) & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{A}}_2(t) \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ x^2 y \\ 0 \\ y^2 x \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(\omega t) \\ 0 \\ \cos(\omega t) \end{bmatrix} \tag{60}$$

where $\widetilde{\mathbf{A}}_1(t) = \begin{bmatrix} 0 & 1 \\ -(a_1 + b_1\cos\omega_1 t + c_1\cos\omega_2 t) & 0 \end{bmatrix}$,

$\widetilde{\mathbf{A}}_2(t) = \begin{bmatrix} 0 & 1 \\ -(a_2 + b_2\cos\omega_1 t + c_2\cos\omega_2 t) & 0 \end{bmatrix}$

Applying the L-P Transformation $\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ and its inverse to the Eq. (60) yields an equation similar to the Eq. (23)

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} f_{11}(\mathbf{z}, t) \\ f_{12}(\mathbf{z}, t) \\ f_{21}(\mathbf{z}, t) \\ f_{22}(\mathbf{z}, t) \end{bmatrix} + \begin{bmatrix} F_{11}(t) \\ F_{12}(t) \\ F_{21}(t) \\ F_{22}(t) \end{bmatrix} \tag{61}$$

Where

$$\mathbf{J}_1 = \begin{bmatrix} \dfrac{\left(-\lambda_1 + \frac{b_1{}^2(C_2\lambda_3(-4\lambda_3+\lambda_3)+C_1\lambda_4(-4\lambda_1+\lambda_4))}{(4\lambda_1-\lambda_3)\lambda_3(4\lambda_1-\lambda_4)\lambda_4}\right)}{\sqrt{\lambda_1}} & 0 \\ 0 & \dfrac{\left(\lambda_1 + b_1{}^2\left(\frac{C_1}{4\lambda_1\lambda_3-\lambda_3{}^2} + \frac{C_2}{4\lambda_1\lambda_4-\lambda_4{}^2}\right)\right)}{\sqrt{\lambda_1}} \end{bmatrix}$$

$$\mathbf{J}_2 = \begin{bmatrix} \dfrac{\left(-\lambda_2 + \frac{b_2{}^2(C_3\lambda_3(-4\lambda_3+\lambda_3)+C_4\lambda_4(-4\lambda_1+\lambda_4))}{(4\lambda_2-\lambda_3)\lambda_3(4\lambda_2-\lambda_4)\lambda_4}\right)}{\sqrt{\lambda_2}} & 0 \\ 0 & \dfrac{\left(\lambda_2 + b_2{}^2\left(\frac{C_3}{4\lambda_2\lambda_3-\lambda_3{}^2} + \frac{C_4}{4\lambda_2\lambda_4-\lambda_4{}^2}\right)\right)}{\sqrt{\lambda_2}} \end{bmatrix}$$

$\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = \omega_1, \lambda_4 = \omega_2 C_1$ and C_2 are constant depending upon the initial conditions of the fictitious states. For more details on the computation of $\mathbf{J}$, we refer to Ref. [22]. At this point, we have to select master and slave states. Assuming eigenvalues of the $\mathbf{J}_1$ matrix are closer to ω (the frequency of external excitation), we choose $\mathbf{z}_r = \{z_1, z_2\}^T$ as the master states and $\mathbf{z}_s = \{z_3, z_4\}^T$ as the slave states and partition Eq. (61) similar to the Eq. (25).

$$\begin{Bmatrix} \dot{\mathbf{z}}_r \\ \dot{\mathbf{z}}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{J}_r & 0 \\ 0 & \mathbf{J}_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_r \\ \mathbf{z}_s \end{Bmatrix} + \begin{Bmatrix} \mathbf{w}_r(\mathbf{z}_r, \mathbf{z}_s, t) \\ \mathbf{w}_s(\mathbf{z}_r, \mathbf{z}_s, t) \end{Bmatrix} + \begin{Bmatrix} \mathbf{F}_r(t) \\ \mathbf{F}_s(t) \end{Bmatrix} (a) \tag{62}$$

$$\mathbf{J}_r = \mathbf{J}_1, \mathbf{J}_s = \mathbf{J}_2, \mathbf{w}_r(\mathbf{z}_r, \mathbf{z}_s, t) = \{f_{11}(\mathbf{z}, t), f_{12}(\mathbf{z}, t)\}^T,$$

Where $\mathbf{w}_s(\mathbf{z}_r, \mathbf{z}_s, t) = \{f_{21}(\mathbf{z}, t), f_{22}(\mathbf{z}, t)\}^T$

$$\mathbf{F}_r(t) = \{F_{11}(t), F_{12}(t)\}^T, \mathbf{F}_s(t) = \{F_{21}(t), F_{22}(t)\}^T$$

In this particular case,

$$\begin{aligned} &a_1 = 3, a_2 = 5, b_1 = b_2 = c_1 = c_2 = 2.5, \\ &\omega_1 = 2\pi \text{ rad/s}, \omega_2 = 7 \text{ rad/s}, \omega = 1 \text{ rad/s}, A_1 = 1, A_2 = 1 \end{aligned}$$

$\mathbf{J}_1$, $\mathbf{J}_2$ are given as

$$\mathbf{J}_1 = \begin{bmatrix} -1.78i & 0 \\ 0 & +1.78i \end{bmatrix}, \mathbf{J}_2 = \begin{bmatrix} -2.29i & 0 \\ 0 & +2.29i \end{bmatrix} \tag{63}$$

One can apply order reduction techniques discussed in Section 3.

(a) Order reduction using the linear method

Eq. (62) comprises of 4 states $\{z_1 \quad z_2 \quad z_3 \quad z_4\}^T$. Following the procedure outlined in Section 3.1, we neglect the contribution from the non-dominant states $\mathbf{z}_s = \{z_3 \quad z_4\}^T$, and the system dynamics is approximated by

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = [\mathbf{J}_1] \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} f_{11}(z_1, z_2, 0, 0, t) \\ f_{22}(z_1, z_2, 0, 0, t) \end{Bmatrix} + \begin{Bmatrix} F_{11}(t) \\ F_{12}(t) \end{Bmatrix} \tag{64}$$

The Eq. (64) is the reduced-order model of the system described by the Eq. (62). This reduced-order system is integrated numerically with typical initial conditions. All the states in $\mathbf{x}$ are obtained using the L-P$\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ Transformation. This solution is called linear reduced-order system response. This response can be compared with the original system's response calculated via numerical integration of the Eq. (59). The time trace comparison is shown in Figs. 1 and 2. Fig. 3 compares phase planes for the original and the reduced-order system via the linear technique. It can be noticed that the linear reduced-order model fails to capture the dynamics of the original system. One reason for this failure is that the slave states are also excited by forcing $\mathbf{F}_s(t)$ that is completely ignored in the reduced-order model. The linear order reduction technique may yield acceptable results when the eigenvalues corresponding to slave states have negative real parts or no forcing on slave states. However, in general, the linear order reduction approach for nonlinear quasi-periodic systems subjected to external excitation may not yield accurate results. For clarity, the Welch power spectrum for the original system response is compared with the Welch power spectrum for the linearly reduced system in Fig. 4. These power spectrums do not match, indicating that the original system's dynamics (frequency content) are not captured in the linearly reduced-order system.

(b) Order reduction using an invariant manifold

As discussed earlier, we try to relate the non-dominant states to the dominant states by a quasi-periodic nonlinear transformation. Suppose the system does not exhibit any resonances (like the case under consideration). In that case, the 'reducibility condition' is satisfied, and the system order can be reduced.

We start with the Eq. (62) and select the same states [$\mathbf{z}_r = \{z_1 \quad z_2\}^T$] as the dominant states and try to find a nonlinear quasi-periodic relationship of the form given by the Eq. (28) . For this particular example, the relationship between $\mathbf{z}_s$ and $\mathbf{z}_r$ are

$$\mathbf{z}_s = \sum_i \mathbf{h}_i(z_1, z_2, t) \equiv \mathbf{H}(z_1, z_2, t),\ s = 3, 4 \tag{65}$$

$$\text{where} \quad \mathbf{h}_i = \sum_{\overline{\mathbf{m}}} \overline{\mathbf{h}}_i(t) z_1^{m_1} \dots z_2^{m_2}, \overline{\mathbf{m}} = (m_1, m_2)^T, m_1 + m_2 = 3 \tag{66}$$

Here $\overline{\mathbf{h}}_i(t)$ are the unknown quasi-periodic vector coefficients. We substitute the Eq. (65) into the Eq. (62). After expanding $\overline{\mathbf{h}}_i(t)$ and $\mathbf{w}_s(z_r, t)(s = 3, 4)$ in the Fourier series and neglecting the terms of higher-order, we obtain the relationship between the dominant and the

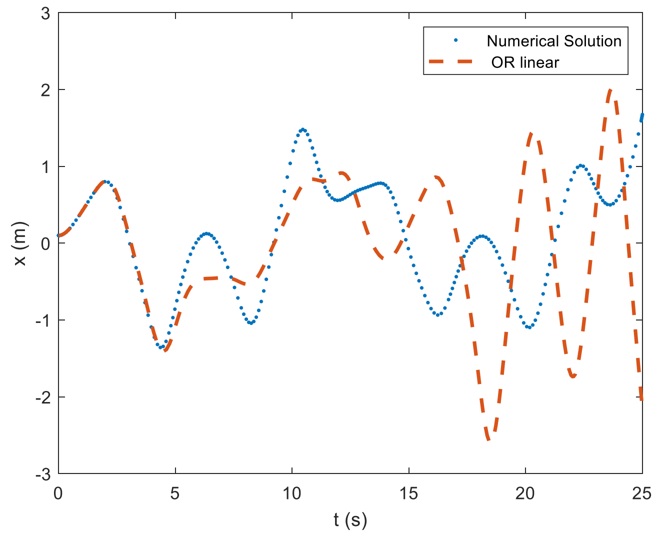

Fig. 1. Time trace comparisons of original and the reduced-order system via the linear approach $x(t)$ v/s time.

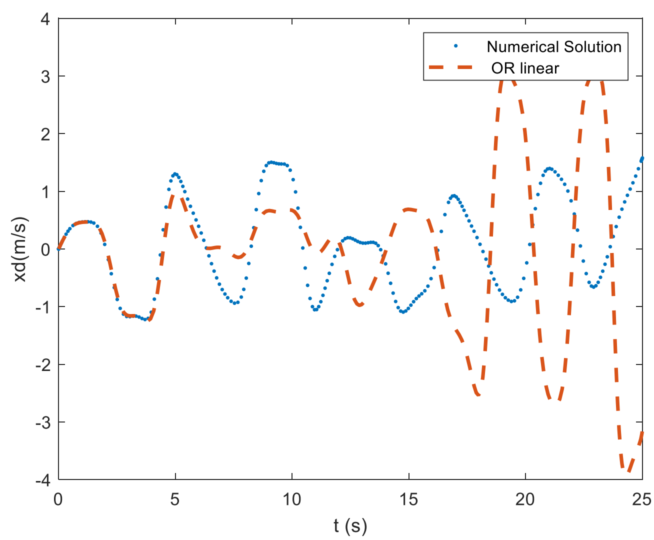

Fig. 2. Time trace comparisons of original and the reduced-order system via the linear approach $\dot{x}(t)$ v/s time.

non-dominant states as

$$z_3 = \mathbf{H}_1(z_1, z_2, t), \qquad z_4 = \mathbf{H}_2(z_1, z_2, t) \tag{67}$$

The Eq. (67) is substituted into Eq. (62)-a to get the reduced-order model as

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = [\mathbf{J}_1] \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} \underline{w}_1(z_1, z_2, t) \\ \underline{w}_2(z_1, z_2, t) \end{Bmatrix} + \begin{Bmatrix} F_{11}(t) \\ F_{12}(t) \end{Bmatrix} \tag{68}$$

The Eq. (68) is the reduced-order model of the system described by the Eq. (62). As before, this reduced-order system is integrated numerically with typical initial conditions. All the states in $\mathbf{x}$ are obtained using the L-P$\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ Transformation. This solution is called nonlinear reduced-order system response. Similar to the linear reduced-order system analysis, the nonlinear reduced-order system response can be compared with the original system's response calculated via numerical integration of the Eq. (58). The time trace comparison is shown in Figs. 5 and 6. Fig. 7 compares phase planes for the original and the reduced-order system via the nonlinear technique. It can be noticed that the nonlinear reduced-order model captures the

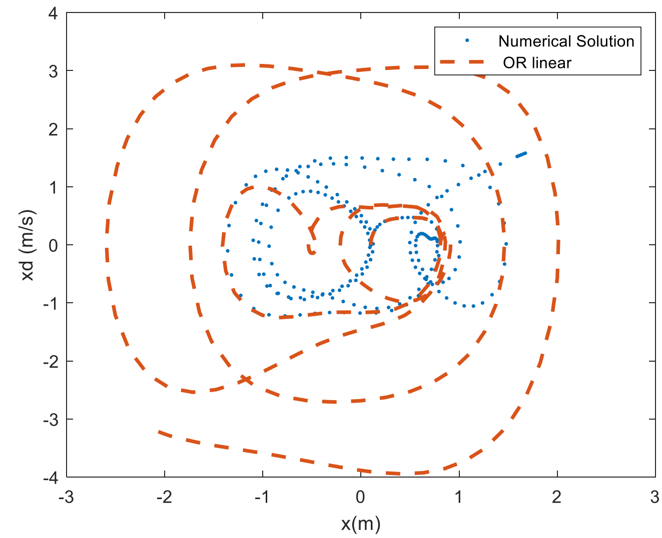

Fig. 3. Phase plane comparisons of original and the reduced-order system via the linear approach $x(t)$ v/s $\dot{x}(t)$.

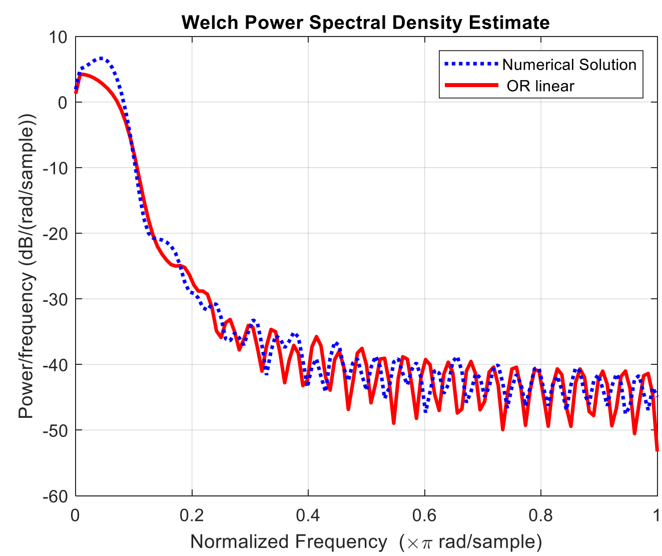

Fig. 4. Welch Power Spectral density comparison of original and the reduced-order system via the linear approach.

dynamics of the original system quite well. For additional insight, the Welch power spectrum for the original system response is compared with the Welch power spectrum for the nonlinearly reduced system in Fig. 8. These power spectrums match, indicating that the original system's dynamics (frequency content) are captured in the nonlinearly reduced-order system. These symbolic computations were performed using Mathematica™.

5. Resonant interactions

Nonlinear quasi-periodic systems can exhibit complex dynamics with resonant interactions. The system can exhibit primary resonance, secondary resonance, internal resonances, and combination resonances. The example presented in section four does not have any resonances. However, it is possible to recover the resonance conditions as the order reduction is carried out via the invariant manifold approach.

One can follow the procedure discussed in section three to obtain the relationship between 'slave' states (to be eliminated) and 'master' states (to be retained) to desired order and recover various

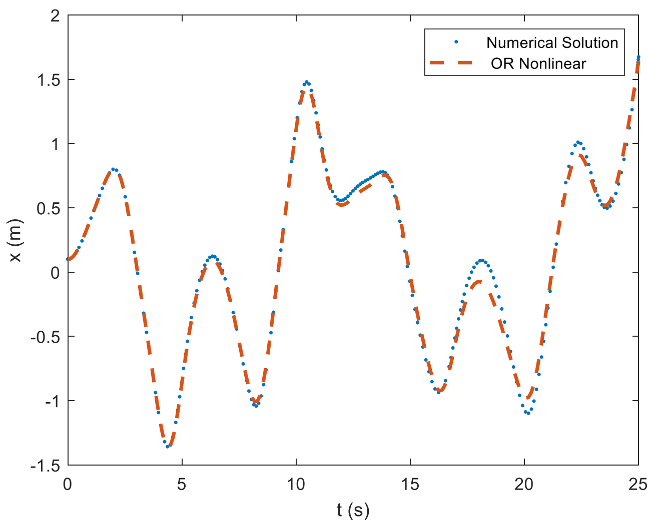

Fig. 5. Time trace comparisons of original and the reduced-order system via the nonlinear approach $x(t)$ v/s time.

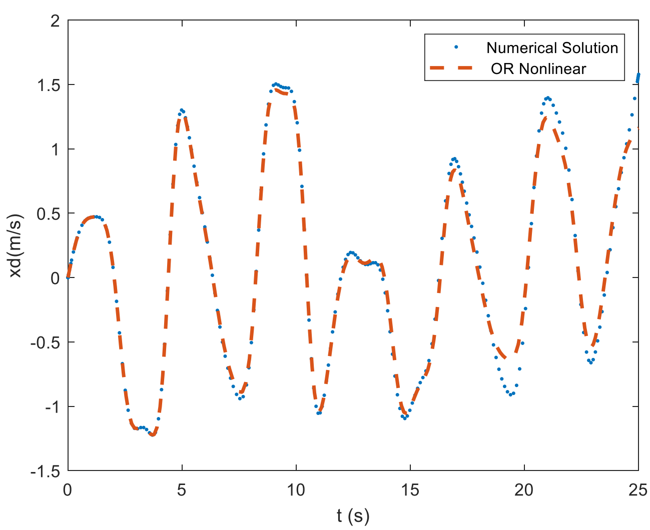

Fig. 6. Time trace comparisons of original and the reduced-order system via the nonlinear approach $\dot{x}(t)$ v/s time.

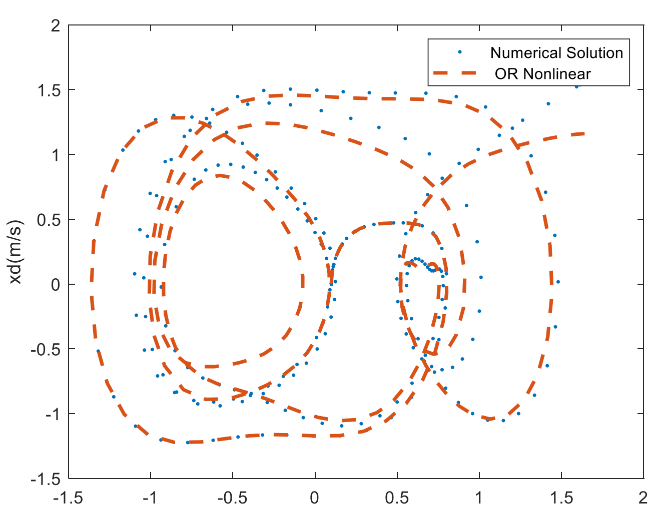

Fig. 7. Phase plane comparisons of original and the reduced-order system via the nonlinear approach $x(t)$ v/s $\dot{x}(t)$.

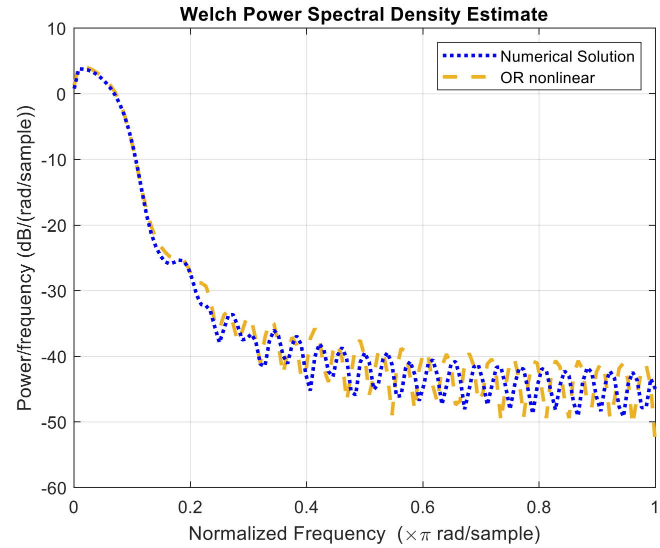

Fig. 8. Welch Power Spectral density comparison of original and the reduced-order system via the nonlinear approach.

resonance conditions involving contributions from external excitation and nonlinearity at multiple orders.

Typically, the states with eigenvalues (or exponents) that are close to the frequency external excitation indicate a stronger interaction and dominate the response and hence can be chosen as the master states. It is to be noted that in the absence of damping, the response of the resonant system is unbounded, and the system is said to be in primary resonance (c.f. Eq. (34)). For clarity, this resonance condition is given by

$$\mathbf{p}\cdot\boldsymbol{\omega}_p + k\omega_f - \lambda_j = 0 \tag{69}$$

where the terms appearing in the Eq. (69) are defined before. It can be noted that the resonance conditions imply a strong coupling leading to energy interaction in external forcing and various modes (or states) of the system. If the resonances are present, then the reducibility conditions discussed are not satisfied, and order reduction is not possible due to the presence of an irremovable coupling.

As the harmonic balance is carried out at ε (c.f. equations (35) to (39)), one yields the combined reducibility condition given by the Eq. (40) reproduced here for clarity.

$$\bar{i}(\overline{\mathbf{p}}\cdot\boldsymbol{\omega}) + \sum_{l=1}^{r}(m_l\lambda_l) - \overline{\lambda}_p \neq 0 \tag{70}$$

where all the terms appearing in the Eq. (70) are defined before. The Eq. (70) indicates the energy interaction between the master states and slave states. The absence of quasi-periodic parametric excitation $(\overline{\mathbf{p}}\cdot\boldsymbol{\omega}) = \mathbf{0}$ yields the classic secondary resonance condition discussed in perturbation literature. It is important to note that the L-P Transformation contributes to $(\overline{\mathbf{p}}\cdot\boldsymbol{\omega})$ terms in equations (69) and (70) leading to many resonance conditions due to quasi-periodic Fourier series representation of the Transformation.

Harmonic balance carried out in ε^2(c.f. equations (47) to (50)) leads to the resonance condition given by the Eq. (56). It is important to note that the Eq. (47) is purely temporal and cannot be solved if secondary resonances indicated by the Eq. (58) are present.

For a special case, when only cubic nonlinearity is present, the relationship between 'master' and 'slave' states can be expressed as

$$\mathbf{z}_s = \overline{\mathbf{h}}_{01}(t) + \varepsilon(\overline{\mathbf{h}}_{03}(t) + \overline{\mathbf{h}}_{13}(\mathbf{z}_r, t) + \overline{\mathbf{h}}_{23}(\mathbf{z}_r, t) + \overline{\mathbf{h}}_{33}(\mathbf{z}_r, t)) \tag{71}$$

Following the same procedure discussed above, when we collect the terms of ε^0 we obtain the Eq. (31) that can be solved by convolution.

However, as we collect the terms of the order ε^1, we obtain

$$\dot{\overline{\mathbf{h}}}_{03}(t) = \mathbf{J}_s \overline{\mathbf{h}}_{03}(t) + \frac{\partial \overline{\mathbf{h}}_{13}}{\partial \mathbf{z}_r} \overline{\mathbf{F}}_r(t) + \underbrace{\mathbf{w}_{s3_0}(t)}_{[\overline{\mathbf{h}}_{01}(t)]_3} \tag{72}$$

$$\frac{\partial \overline{\mathbf{h}}_{13}}{\partial t} + \frac{\partial \overline{\mathbf{h}}_{13}}{\partial \mathbf{z}_r} \mathbf{J}_r \mathbf{z}_r + \frac{\partial \overline{\mathbf{h}}_{23}}{\partial \mathbf{z}_r} \overline{\mathbf{F}}_r = \mathbf{J}_s \overline{\mathbf{h}}_{13} + \underbrace{\mathbf{w}_{s3_1}(\mathbf{z}_r, t)}_{[\overline{\mathbf{h}}_{01}(t)]_2 \cdot \mathbf{z}_r} \tag{73}$$

$$\frac{\partial \overline{\mathbf{h}}_{23}}{\partial t} + \frac{\partial \overline{\mathbf{h}}_{23}}{\partial \mathbf{z}_r} \mathbf{J}_r \mathbf{z}_r + \frac{\partial \overline{\mathbf{h}}_{33}}{\partial \mathbf{z}_r} \overline{\mathbf{F}}_r = \mathbf{J}_s \overline{\mathbf{h}}_{23} + \underbrace{\mathbf{w}_{s3_2}(\mathbf{z}_r, t)}_{\overline{\mathbf{h}}_{01}(t) \cdot [\mathbf{z}_r]_2} \tag{74}$$

$$\frac{\partial \overline{\mathbf{h}}_{33}}{\partial t} + \frac{\partial \overline{\mathbf{h}}_{33}}{\partial \mathbf{z}_r} \mathbf{J}_r \mathbf{z}_r = \mathbf{J}_s \overline{\mathbf{h}}_{33} + \mathbf{w}_{s3_3}(\mathbf{z}_r, t) \tag{75}$$

These equations can be solved by expanding the known and unknown terms in the multiple Fourier series of appropriate form and term by term comparison similar to equations ((50),(49),(47) and (48)) yield the '*reducibility condition*' and '*combined reducibility conditions*' similar to equations (53), (56),(44) and (58).

Let us consider a special case when forcing on the 'slave' states is zero and the Eq. (25) takes the form

$$\begin{Bmatrix} \dot{\mathbf{z}}_r \\ \dot{\mathbf{z}}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{J}_r & 0 \\ 0 & \mathbf{J}_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_r \\ \mathbf{z}_s \end{Bmatrix} + \begin{Bmatrix} \varepsilon \mathbf{w}_r(\mathbf{z}_r, \mathbf{z}_s, t) \\ \varepsilon \mathbf{w}_s(\mathbf{z}_r, \mathbf{z}_s, t) \end{Bmatrix} + \begin{Bmatrix} \overline{\mathbf{F}}_r(t) \\ 0 \end{Bmatrix} (a) \tag{76}$$

where all the terms appearing in the Eq. (76) are defined before. To obtain the invariant manifold expression, we express the relationship between 'master' and 'slave' states as (*c .f.* Eq. (28))

$$\begin{aligned} \mathbf{z}_s &= \varepsilon(\overline{\mathbf{h}}_{02}(t) + \overline{\mathbf{h}}_{12}(\mathbf{z}_r, t) + \overline{\mathbf{h}}_{22}(\mathbf{z}_r, t)) \\ &+ \varepsilon^2(\overline{\mathbf{h}}_{03}(t) + \overline{\mathbf{h}}_{13}(\mathbf{z}_r, t) + \overline{\mathbf{h}}_{23}(\mathbf{z}_r, t) + \overline{\mathbf{h}}_{33}(\mathbf{z}_r, t)) \\ &+ \varepsilon^3(\overline{\mathbf{h}}_{04}(t) + \overline{\mathbf{h}}_{14}(\mathbf{z}_r, t) + \overline{\mathbf{h}}_{24}(\mathbf{z}_r, t) + \overline{\mathbf{h}}_{34}(\mathbf{z}_r, t) + \overline{\mathbf{h}}_{44}(\mathbf{z}_r, t)) + \cdots \end{aligned} \tag{77}$$

As we follow the procedure to determine manifold constraint equation as described earlier, we obtain for ε^1(*c.f.* equations (35)–(37))

$$\dot{\overline{\mathbf{h}}}_{02}(t) = \mathbf{J}_s \overline{\mathbf{h}}_{02}(t) - \frac{\partial \overline{\mathbf{h}}_{12}}{\partial \mathbf{z}_r} \mathbf{F}_r \tag{78}$$

$$\frac{\partial \overline{\mathbf{h}}_{12}}{\partial t} + \frac{\partial \overline{\mathbf{h}}_{12}}{\partial \mathbf{z}_r} \mathbf{J}_r \mathbf{z}_r - \mathbf{J}_s \overline{\mathbf{h}}_{12} = -\frac{\partial \overline{\mathbf{h}}_{22}}{\partial \mathbf{z}_r} \mathbf{F}_r(t) \tag{79}$$

$$\frac{\partial \overline{\mathbf{h}}_{22}}{\partial t} + \frac{\partial \overline{\mathbf{h}}_{22}}{\partial \mathbf{z}_r} \mathbf{J}_r \mathbf{z}_r - \mathbf{J}_s \overline{\mathbf{h}}_{22} = \mathbf{w}_{s2_2}(\mathbf{z}_r, t) \tag{80}$$

where the Eq. (80) is the same as the Eq. (37). However, equations (78) and (79) do not include any contribution form $\mathbf{w}_{s2}(\mathbf{z}_r, t)$ as $\overline{\mathbf{h}}_{01}(t)$ (or $\mathbf{F}_s(t)$) is absent. However, the nonlinear interaction between 'master forcing' and 'slave' states give rise to term $\frac{\partial \overline{\mathbf{h}}_{22}}{\partial \mathbf{z}_r} \mathbf{F}_r(t)$ in the Eq. (79) and subsequently the term $\frac{\partial \overline{\mathbf{h}}_{12}}{\partial \mathbf{z}_r} \mathbf{F}_r$ in the Eq. (78), which are known. The Eq. (80) can be solved by expanding the known and unknown terms in the multiple Fourier series given by equations (38) and (39) yielding the same '*reducibility condition*' provided by the Eq. (40). Further, to solve the Eq. (79), we have to expand the known and unknown terms in the multiple Fourier series of the form

$$\overline{\mathbf{h}}_{12}(\mathbf{z}_r, t) = \sum_{j=1}^{s} \sum_{\overline{\mathbf{m}}} \sum_{p_1=-\infty}^{+\infty} \sum_{p_2=-\infty}^{+\infty} \sum_{p_3=-\infty}^{+\infty} h_{j\overline{m}v}\, e^{\bar{i}(\widetilde{\mathbf{p}} \cdot \overline{\boldsymbol{\omega}})t} \left|\mathbf{z}_r\right|^{\mathbf{m}}_{\sim} e_j \tag{81}$$

$$\frac{\partial \mathbf{h}_{22}}{\partial \mathbf{z}_r} \mathbf{F}_r(t) = \sum_{j=1}^{s} \sum_{\overline{\mathbf{m}}} \sum_{p_1=-\infty}^{+\infty} \sum_{p_2=-\infty}^{\infty} \sum_{p_3=-\infty}^{+\infty} a_{j\overline{m}v} e^{\bar{i}(\widetilde{\mathbf{p}} \cdot \overline{\boldsymbol{\omega}})t} \left|\mathbf{z}_r\right|^{\mathbf{m}}_{\sim} e_j \tag{82}$$

$$\left|\mathbf{z}_r\right|^{\mathbf{m}}_{\sim} = z_1^{m_1} z_2^{m_2} \ldots z_r^{m_r}, \bar{i} = \sqrt{-1}, m_1 + \cdots \ldots + m_r = 1;$$
$$\overline{\boldsymbol{\omega}} = \begin{Bmatrix} \omega_1 & \omega_2 & \omega_{f_m} \end{Bmatrix}; \widetilde{\mathbf{p}} = \begin{Bmatrix} p_1 & p_2 & p_3 \end{Bmatrix}$$

where ω_{f_m} is the frequency of 'master forcing' and all other terms are defined before. A term by term comparison yields the expression similar to Eq. (43) and '*combined reducibility condition*' given by (*c.f.* Eq. (70))

$$\bar{i}(p_1\omega_p + p_2\omega_{nl} + p_3\omega_{f_m}) + \lambda_l - \overline{\lambda}_p \neq 0 \tag{83}$$

To obtain $\mathbf{h}_{12}(\mathbf{z}_r, t)$ the '*combined reducibility condition*' given by the Eq. (83) should be satisfied. Let us consider a special case when $p_1 = p_2 = 0$. Thus, the Eq. (83) yields

$$\bar{i}(p_3\omega_{f_m}) + \lambda_l - \overline{\lambda}_p \neq 0 \tag{84}$$

which implies when the difference between the exponent of 'master' states and the 'slave' states equals the integral multiple of the 'master forcing' frequency order cannot be reduced. When the parametric excitation is small (and the exponents approach the natural frequency of the linearized autonomous system ω_n) Eq. (84) can be expressed as

$$\bar{i}(p_3\omega_{f_m}) + \omega_{n_r} - \omega_{n_s} \neq 0 \tag{85}$$

This result is well-known in the perturbation literature. Assuming all the '*combined reducibility conditions*' are satisfied, we obtain $\overline{\mathbf{h}}_{12}(\mathbf{z}_r, t)$ and solve the Eq. (78), which involves contribution from $\overline{\mathbf{h}}_{12}(\mathbf{z}_r, t)$ in the form $\frac{\partial \overline{\mathbf{h}}_{12}}{\partial \mathbf{z}_r} \overline{\mathbf{F}}_r(t)$(which depends upon time alone). The solution of the Eq. (78) can be obtained via the convolution provided $\lambda_s \neq \widetilde{\mathbf{p}} \cdot \overline{\boldsymbol{\omega}}$ ($\lambda_s \neq p_1\omega_p + p_2\omega_{nl} + p_3\omega_{f_m}$). For a particular case when $p_1 = p_2 = 0$, this condition can be written as

$$\lambda_s \neq p_3\omega_{f_m} \tag{86}$$

which states that the order cannot be reduced if the 'slave' states resonate with 'master forcing'. Again, under the small parametric excitation assumption, the condition given by the Eq. (86) can be expressed as

$$\omega_{n_s} \neq \bar{i}(p_3\omega_{f_m}) \tag{87}$$

which is obtained by various researchers via perturbation analysis. It is possible to continue to get the condition at a higher order of nonlinearity ($\varepsilon^2, \varepsilon^3, ..$). In conclusion, the results obtained by '*reducibility condition*' and '*combined reducibility condition*' contain the resonance conditions obtained via perturbation analysis.

6. Examples of resonant interaction

Consider the coupled undamped Mathieu Hill-type nonlinear quasi-periodic system subjected to external excitation given by Eq. (59). After the L-P Transformation, the system can be expressed as the Eq. (62), where all the terms appearing in the Eq. (62) are defined before. In this section, we consider the effectiveness of the order reduction technique as the system approaches the primary resonance (given by Eq. (69)), internal resonance (given by Eq. (70)), and secondary resonance given by Eq. (86)).

As mentioned earlier, a primary resonance condition indicates a tight coupling between the external forcing and master states, and the system response is dominated by the resonant modes. Consider the following system parameters

$$a_1 = 3, a_2 = 5, b_1 = b_2 = c_1 = c_2 = 2.5, \omega_1 = 2\pi \text{ rad/s},$$
$$\omega_2 = 7 \text{ rad/s}, \omega = 1.58 \text{ rad/s}, A_1 = 1, A_2 = 1$$

the $\mathbf{J}_1$, $\mathbf{J}_2$ are given as

$$\mathbf{J}_1 = \begin{bmatrix} -1.78i & 0 \\ 0 & +1.78i \end{bmatrix}, \mathbf{J}_2 = \begin{bmatrix} -2.29i & 0 \\ 0 & +2.29i \end{bmatrix}$$

It can be observed that for the given system parameters and external forcing, for $(\overline{\mathbf{p}} \cdot \boldsymbol{\omega}) = \mathbf{0}$ (the constant term in the L-P Transformation), the system is close to primary resonance.

$$k\omega_f - \lambda_j \approx 0$$

This leads to the "small divisor problem" as the denominator of terms in the Eq. (34) approaches zero. It can be observed that the total response

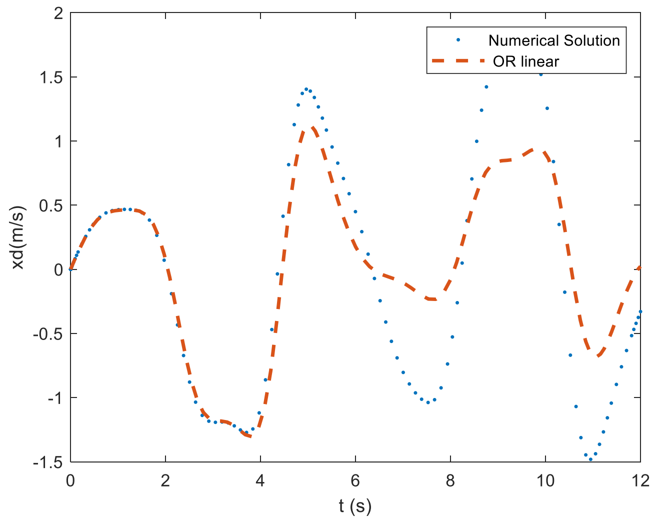

Fig. 9. Time trace comparisons of original and the reduced-order system via the linear approach $x(t)$ v/s time when the $\varepsilon = 0.2$ close to primary resonance.

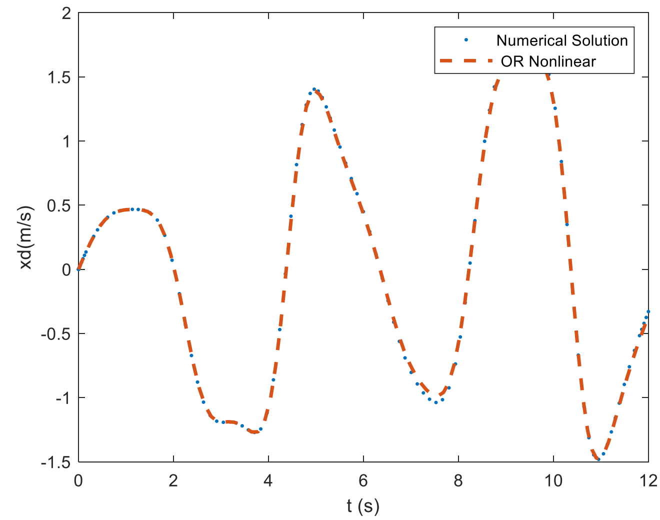

Fig. 10. Time trace comparisons of original and the reduced-order system via the nonlinear approach $x(t)$ v/s time when the $\varepsilon = 0.2$ close to primary resonance.

of the system is dominated by the resonant ("master") states. As long as the system response is bounded, the invariant manifold-based order reduction technique can capture the system's behavior to some degree. It can be noted that when the denominator of the terms in the Eq. (34) is zero, the response becomes infinite due to singularity. Close to the singularity, the order reduction results are dominated by the stability of the numerical integration algorithm and floating-point precision. Typically, based on our experience, when running on Intel-i7, 32 GB RAM, Mathematica 11, and MATLAB 2020b, the reduced-order model results were close to the original system when the master states were close to (but not in exact) resonance $k\omega_f - \lambda_j = \varepsilon, \varepsilon \geq 0.2$ when we used ODE solvers in MATLAB for numerical integration. As $\varepsilon \leq 0.2$, the numerical integration algorithms displayed warnings and could not meet integration tolerances.

The "small divisor problem" has received significant attention in the nonlinear dynamics community and studied by researchers using KAM theory [40], general direct methods (Siegel's method) [41], and their extensions. We are currently working on a paper that explicitly addresses order reduction of the quasi-periodic system near resonance where the "small divisor problem" is significant where one has to use the KAM theory and other analysis methods.

On the other hand, reduced-order model obtained via the linear approach did not capture the dynamics of the original system when the system was close to primary resonance. Figs. 9 and 10 show the time trace comparison of the original and reduced systems via linear and nonlinear methods, respectively. It can be observed that the time trace of the nonlinear reduced-order system is more in agreement with the original system. However, as the system is undamped, the response is amplified but bounded.

The system is said to be in internal resonance when the resonance condition given by the Eq. (88) is satisfied (c.f. Eq. (70))

$$\bar{i}(\overline{\mathbf{p}} \cdot \boldsymbol{\omega}) + \sum_{l=1}^{r} (m_l \lambda_l) - \overline{\lambda}_p = 0 \tag{88}$$

For $(\overline{\mathbf{p}} \cdot \boldsymbol{\omega}) = \mathbf{0}$(the constant term in the L-P Transformation) and the parameters

$$a_1 = 3, a_2 = 5, b_1 = c_1 = 2.5, b_2 = c_2 = 3.8, \omega_1 = 2\pi \text{ rad/s},$$
$$\omega_2 = 7 \text{ rad/s}, \omega = 1 \text{ rad/s}, A_1 = 1, A_2 = 1$$

the $\mathbf{J}_1$, $\mathbf{J}_2$, C_1 and C_2 are given as

$$\mathbf{J}_1 = \begin{bmatrix} -1.78i & 0 \\ 0 & +1.78i \end{bmatrix}, \mathbf{J}_2 = \begin{bmatrix} -5.1i & 0 \\ 0 & +5.1i \end{bmatrix}$$

c.f. Eq. (88), the system is close to 3:1 internal resonance given by

$$\sum_{l=1}^{r} (m_l \lambda_l) - \overline{\lambda}_p = 0; m = 3 \tag{89}$$

When the system is close to 3:1 internal resonance, similar to the primary resonance case, the denominator of the terms in the Eq. (40) approaches zero due to a "small divisor problem". When running on Intel-i7, 32 GB RAM, Mathematica 11, and MATLAB 2020b, the reduced-order model results were initially close to the original system when the master states were close to (but not in exact) internal resonance with the slave states $\sum_{l=1}^{r} (m_l \lambda_l) - \overline{\lambda}_p = \varepsilon, \varepsilon \geq 0.20$. However, as time increased, the time traces of the nonlinear reduced system and the original system diverged. The Poincare' plots showed a similar structure but were different in magnitude. The detailed discussion and sensitivity of the long-term response of the reduced-order system near resonance will be reported elsewhere. For $\varepsilon \leq 0.2$, the numerical integration algorithms displayed warnings and could not meet integration tolerances. The reduced-order model, via the linear approach, did not capture the dynamics of the original system when the system was close to internal resonance.

The validity of the order reduction approach was studied further when the system was close to secondary resonances, i.e., $\lambda_l - \overline{\lambda}_p < \varepsilon; \varepsilon = 0.2$ c.f. Eq. (84) and $\lambda_s - p_3\omega_{f_m} < \varepsilon; \varepsilon = 0.2$ c.f. Eq. (86), the numerical integration algorithms displayed warning and could not meet integration tolerances. Thus, for the typical system parameters used for demonstration in this paper $\varepsilon = 0.2$ appears to be a threshold where numerical solvers can maintain the integration tolerances. This threshold ε is dependent on the system parameters, type of numerical integration solver/software used, and machine precision. The validity of order reduction close to resonance and associated "small divisor problem" is an exciting topic and needs further attention to understand the convergence, resonance interactions, stability of numerical integration techniques, and limitations.

7. Discussion and conclusions

This paper presents a technique for obtaining a reduced-order model of a nonlinear quasi-periodic system subjected to external excitation. The central idea is to assume a quasi-periodic transformation with unknown coefficients between the master and the slave states. This Transformation can be determined by collecting the terms of the same order and solving them using harmonic balance. In the solution process, we obtain reducibility conditions that indicate resonances between system states, nonlinearity, and external excitation. The linear resonance condition is also obtained as we find the solution of quasi-periodic Transformation. One crucial point is deciding which states to retain and which ones to eliminate. Initially, one could start with the states corresponding to eigenvalues close to external excitation frequencies and start the order reduction process. The resonance interactions in the nonlinear quasi-periodic system are complex. In order reduction, one may see a "small divisor" problem. Such a case indicates resonant interaction, and these resonant states must be included in the master states. It can be noted that with the advent of symbolic software like Mathematica and Maple, the procedure for order reduction can be automated [42]. One can consider quasi-periodic and external excitation as fictitious states and carry out the order reduction. This approach is presented in Ref. [43], and further simplification via the method of normal form can be achieved as discussed in Ref. [44] for autonomous systems.

It is emphasized that the methods proposed in this work are applicable when the quasi-periodic system is almost reducible and meets the reducibility conditions given in Refs. [26,34]. It is essential to observe that the small divisor problem needs to be addressed during the computation of the L-P transformation and while performing the order reduction of the L-P transformed system. It is respectfully noted that a general L-P transformation for the quasi-periodic system, like the L-F transformation for the periodic system, is impossible as the Floquet type theory does not exist for the quasi-periodic system. However, almost reducibility can be proved under certain conditions for a class of quasi-periodic systems. These cases are discussed with rigorous mathematical proofs in Refs. [23–26]. It is noted that many engineering systems fall under this class of problems and are amenable to the approach presented in this paper.

For the reducible (or almost reducible) quasi-periodic systems, the reduced-order models based on the L-P transformation and invariant manifold approach will contain all the essential dynamics. The responses of the reduced-order system quantitatively and qualitatively are similar to the original system. This reduced-order system can be simplified using the method of normal forms. One can study this simplified system for bifurcation and control. The reduced-order system can be used to optimize essential parameters, study sensitivity, and design controllers.

NOMENCLATURE

$\mathbf{x}$ - n vector of states

$\mathbf{A}(t)$ - $n \times n$ time quasiperiodic matrix

$\mathbf{f}(\mathbf{x}, t)$ - nonlinear n vector such that $\mathbf{f}(0, t) = 0$

$\mathbf{Q}(t)$ - L-P transformation matrix of dimension $n \times n$

$\mathbf{M}$ - Modal matrix of dimension $n \times n$

$\mathbf{z}$ - n vector of the L-P transformed states

$\mathbf{z}_r$ - $r(r << n)$ vector of dominant states

$\mathbf{z}_s(t)$ - $s(s + r = n)$ vector of non-dominant (slave) states

$\mathbf{J}_r$ - $r \times r$ Jordan block corresponding to dominant states

$\mathbf{H}(\mathbf{z}_r, t)$ - Nonlinear quasi-periodic invariant manifold function relating the non-dominant states to dominant states

CRediT authorship contribution statement

Sandesh G. Bhat: Investigation, Software, Writing – original draft. **Susheelkumar Cherangara Subramanian:** Software, Investigation. **Sangram Redkar:** Methodology, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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