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Periodic Staircase Matrices and Generalized Cluster Structures

Misha Gekhtman[1], Michael Shapiro[2,*] and Alek Vainshtein[3]

[1]Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA, [2]Department of Mathematics, Michigan State University, East Lansing, MI 48823, USA, and [3]Department of Mathematics and Department of Computer Science, University of Haifa, Haifa, Mount Carmel 31905, Israel

**Correspondence to be sent to: e-mail: mshapiro@msu.edu*

As is well known, cluster transformations in cluster structures of geometric type are often modeled on determinant identities, such as short Plücker relations, Desnanot–Jacobi identities, and their generalizations. We present a construction that plays a similar role in a description of generalized cluster transformations and discuss its applications to generalized cluster structures in GL_n compatible with a certain subclass of Belavin–Drinfeld Poisson–Lie brackets, in the Drinfeld double of GL_n, and in spaces of periodic difference operators.

1 Introduction

Since the discovery of cluster algebras in [4], many important algebraic varieties were shown to support a cluster structure in a sense that the coordinate rings of such variety are isomorphic to a cluster algebra or an upper cluster algebra. Lie theory and representation theory turned out to be a particularly rich source of varieties of this sort including but in no way limited to such examples as Grassmannians [5, 19], double Bruhat cells [1], and strata in flag varieties [16]. In all these examples,

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cluster transformations that connect distinguished coordinate charts within a ring of regular functions are modeled on three-term relations such as short Plücker relations, Desnanot–Jacobi identities, and their Lie-theoretic generalizations of the kind considered in [3]. This remains true even in the case of exotic cluster structures on GL_n considered in [8, 10] where cluster transformations can be obtained by applying Desnanot–Jacobi-type identities to certain structured matrices of a size far exceeding n.

On the other hand, as we have shown in [7, 9], there are situations when, in order to stay within a ring of regular functions, one has to employ generalized cluster transformations, that is, exchange relations in which the product of a cluster variable being removed and the variable that replaces it is equal to a multinomial expression in other cluster variables in the seed rather than a binomial expression appearing in the definition of the usual cluster transformation. Generalized cluster transformations of this kind were first considered in [2], and in [7, 9] we used them, in a more general form, to construct a generalized cluster structure in the standard Drinfeld double of GL_n and several related varieties. There, we had to rely on an $(n+1)$-term identity [9, Proposition 8.1] (see also Proposition 3.6 below) for certain polynomial functions on the space Mat_n of $n \times n$ matrices; this identity involved, as coefficients, conjugation invariant functions on Mat_n.

In this paper, we argue that in constructing generalized cluster structures, identities of the kind we employed in [7, 9] play a role similar to the one classical three-term determinantal identities do in a construction of usual cluster structures. To support this argument, we derive identity (3.7) that is associated with a class of infinite periodic block bidiagonal staircase matrices and that generalizes [9, Proposition 8.1]. We then present three examples in which our main identity is applied to construct an initial seed of a regular generalized cluster structure.

The paper is organized as follows. In Section 2, we review the definition of generalized cluster structures. Section 3 is devoted to the proof of the main identity (3.7) (Theorem 3.2). In the next three sections, we apply (3.7) to construct generalized cluster structures on the Drinfeld double of GL_n (Section 4), thus providing a construction alternative to the one presented in [7, 9], on the space of periodic band matrices (Section 5), and, in Section 6, on GL_6 equipped with a particular Poisson–Lie bracket arising in the Belavin–Drinfeld classification. In the latter case, the resulting generalized cluster structure is compatible with that Poisson bracket. The last section contains the proofs of several lemmas about the properties of certain minors of a periodic staircase matrices.

2 Generalized Cluster Structures

Following [9], we remind the definition of a generalized cluster structure represented by a quiver with multiplicities. Let $(Q, d_1, \dots, d_N)$ be a quiver on N mutable and M frozen vertices with positive integer multiplicities d_i at mutable vertices. A vertex is called *special* if its multiplicity is greater than 1. A frozen vertex is called *isolated* if it is not connected to any other vertices. Let $\mathbb{F}$ be the field of rational functions in $N+M$ independent variables with rational coefficients. There are M distinguished variables corresponding to frozen vertices; they are denoted $x_{N+1}, \dots, x_{N+M}$. The *coefficient group* is a free multiplicative abelian group of Laurent monomials in stable variables, and its integer group ring is $\bar{\mathbb{A}} = \mathbb{Z}[x_{N+1}^{\pm 1}, \dots, x_{N+M}^{\pm 1}]$ (we write $x^{\pm 1}$ instead of x, x^{-1}).

An *extended seed* (of *geometric type*) in $\mathbb{F}$ is a triple $\Sigma = (\mathbf{x}, Q, \mathcal{P})$, where $\mathbf{x} = (x_1, \dots, x_N, x_{N+1}, \dots, x_{N+M})$ is a transcendence basis of $\mathbb{F}$ over the field of fractions of $\bar{\mathbb{A}}$ and $\mathcal{P}$ is a set of N *strings*. The ith string is a collection of monomials $p_{ir} \in \mathbb{A} = \mathbb{Z}[x_{N+1}, \dots, x_{N+M}]$, $0 \le r \le d_i$, such that $p_{i0} = p_{id_i} = 1$; it is called *trivial* if $d_i = 1$, and hence both elements of the string are equal to one. The monomials p_{ir} are called *exchange coefficients*.

Given a seed as above, the *adjacent cluster* in direction k, $1 \le k \le N$, is defined by $\mathbf{x}' = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\}$, where the new cluster variable x'_k is given by the *generalized exchange relation*

$$x_k x'_k = \sum_{r=0}^{d_k} p_{kr} u_{k;>}^r v_{k;>}^{[r]} u_{k;<}^{d_k - r} v_{k;<}^{[d_k - r]}; \tag{2.1}$$

here $u_{k;>}$ and $u_{k;<}$, $1 \le k \le N$, are defined by

$$u_{k;>} = \prod_{k \to i \in Q} x_i, \qquad u_{k;<} = \prod_{i \to k \in Q} x_i,$$

where the products are taken over all edges between k and mutable vertices, and *stable τ-monomials* $v_{k;>}^{[r]}$ and $v_{k;<}^{[r]}$, $1 \le k \le N$, $0 \le r \le d_k$, defined by

$$v_{k;>}^{[r]} = \prod_{N+1 \le i \le N+M} x_i^{\lfloor r b_{ki} / d_k \rfloor}, \qquad v_{k;<}^{[r]} = \prod_{N+1 \le i \le N+M} x_i^{\lfloor r b_{ik} / d_k \rfloor}, \tag{2.2}$$

where b_{ki} is the number of edges from k to i and b_{ik} is the number of edges from i to k; here, as usual, the product over the empty set is assumed to be equal to 1. The right hand side of (2.1) is called a *generalized exchange polynomial.*

The standard definition of the *quiver mutation* in direction k is modified as follows: if both vertices i and j in a path $i \to k \to j$ are mutable, then this path contributes d_k edges $i \to j$ to the mutated quiver Q'; if one of the vertices i or j is frozen then the path contributes d_j or d_i edges $i \to j$ to Q'. The multiplicities at the vertices do not change. Note that isolated vertices remain isolated in Q'.

The *exchange coefficient mutation* in direction k is given by

$$p'_{ir} = \begin{cases} p_{i,d_i-r'} & \text{if } i = k; \\ p_{ir'} & \text{otherwise.} \end{cases} \tag{2.3}$$

Given an extended seed $\Sigma = (\mathbf{x}, Q, \mathcal{P})$, we say that a seed $\Sigma' = (\mathbf{x}', Q', \mathcal{P}')$ is *adjacent* to Σ (in direction k) if $\mathbf{x}'$, Q', and $\mathcal{P}'$ are as above. Two such seeds are *mutation equivalent* if they can be connected by a sequence of pairwise adjacent seeds. The set of all seeds mutation equivalent to Σ is called the *generalized cluster structure* (of geometric type) in $\mathbb{F}$ associated with Σ and denoted by $\mathcal{GC}(\Sigma)$.

Fix a ground ring $\widehat{\mathbb{A}}$ such that $\mathbb{A} \subseteq \widehat{\mathbb{A}} \subseteq \bar{\mathbb{A}}$. The *generalized upper cluster algebra* $\overline{\mathcal{A}}(\mathcal{GC}) = \overline{\mathcal{A}}(\mathcal{GC}(\Sigma))$ is the intersection of the rings of Laurent polynomials over $\widehat{\mathbb{A}}$ in cluster variables taken over all seeds in $\mathcal{GC}(\Sigma)$. Let V be a quasi-affine variety over $\mathbb{C}$, $\mathbb{C}(V)$ be the field of rational functions on V, and $\mathcal{O}(V)$ be the ring of regular functions on V. A generalized cluster structure $\mathcal{GC}(\Sigma)$ in $\mathbb{C}(V)$ is an embedding of $\mathbf{x}$ into $\mathbb{C}(V)$ that can be extended to a field isomorphism θ between $\mathbb{F} \otimes \mathbb{C}$ and $\mathbb{C}(V)$. It is called *regular on V* if any cluster variable in any cluster belongs to $\mathcal{O}(V)$, and *complete* if $\overline{\mathcal{A}}(\mathcal{GC})$ tensored with $\mathbb{C}$ is isomorphic to $\mathcal{O}(V)$. The choice of the ground ring is discussed in [9, Section 2.1].

The following proposition is borrowed from [9].

Proposition 2.1. Let V be a Zariski open subset in $\mathbb{C}^{N+M}$ and $\mathcal{GC} = \mathcal{GC}(\Sigma)$ be a generalized cluster structure in $\mathbb{C}(V)$ with N cluster and M stable variables such that

(i) there exists an extended cluster $\mathbf{x} = (x_1, \ldots, x_{N+M})$ in $\mathcal{GC}$ such that $\theta(x_i)$ is regular on V for $1 \le i \le N+M$, and $\theta(x_i)$ and $\theta(x_j)$ are coprime in $\mathcal{O}(V)$ for $1 \le i \ne j \le N+M$;

(ii) for any cluster variable x'_k, $1 \le k \le N$, obtained via the generalized exchange relation (2.1) applied to $\mathbf{x}$, $\theta(x'_k)$ is regular on V and coprime in $\mathcal{O}(V)$ with $\theta(x_k)$.

Then $\mathcal{GC}$ is a regular generalized cluster structure.

If additionally

(iii) each regular function on V belongs to $\theta(\overline{\mathcal{A}}_{\mathbb{C}}(\mathcal{GC}))$,

then $\overline{\mathcal{A}}_{\mathbb{C}}(\mathcal{GC})$ is naturally isomorphic to $\mathcal{O}(V)$.

Remark 2.2. (i) The definition above is a particular case of a more general definition of generalized cluster structures given in [9].

(ii) Quivers with multiplicities differ from weighted quivers introduced in [15].

3 Identity for Minors of a Periodic Staircase Matrix

Consider a periodic block bidiagonal matrix

$$L = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & & \\ & 0 & X & Y & 0 & & \\ & & 0 & X & Y & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{3.1}$$

where $X \in \mathrm{Mat}_n$ and $Y \in GL_n$ are matrices of the form

$$X = \begin{bmatrix} 0_{a\times b} & * \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} * & * \\ 0_{(n-a)\times b} & * \end{bmatrix}, \tag{3.2}$$

with $a > b+1 \ge 1$; the entries in the submatrices of X and Y denoted by $*$ can take arbitrary complex values. This choice ensures that L has a staircase shape. Below is an example of a dense submatrix of L for $n = 9$, $a = 5$, $b = 2$:

$$
\begin{array}{cccccccccccccccccccc}
x_{17} & \underline{x_{18}} & \underline{x_{19}} & y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & y_{16} & y_{17} & y_{18} & y_{19} \\
x_{27} & x_{28} & \underline{x_{29}} & \underline{y_{21}} & y_{22} & y_{23} & y_{24} & y_{25} & y_{26} & y_{27} & y_{28} & y_{29} \\
x_{37} & x_{38} & x_{39} & \underline{y_{31}} & \underline{y_{32}} & y_{33} & y_{34} & y_{35} & y_{36} & y_{37} & y_{38} & y_{39} \\
x_{47} & x_{48} & x_{49} & y_{41} & \underline{y_{42}} & \underline{y_{43}} & y_{44} & y_{45} & y_{46} & y_{47} & y_{48} & y_{49} \\
x_{57} & x_{58} & x_{59} & y_{51} & y_{52} & \underline{y_{53}} & \underline{y_{54}} & y_{55} & y_{56} & y_{57} & y_{58} & y_{59} \\
&&&&& y_{63} & \underline{y_{64}} & \underline{y_{65}} & y_{66} & y_{67} & y_{68} & y_{69} \\
&&&&& y_{73} & y_{74} & \underline{y_{75}} & \underline{y_{76}} & y_{77} & y_{78} & y_{79} \\
&&&&& y_{83} & y_{84} & y_{85} & \underline{y_{86}} & \underline{y_{87}} & y_{88} & y_{89} \\
&&&&& y_{93} & y_{94} & y_{95} & y_{96} & \underline{y_{97}} & \underline{y_{98}} & y_{99} \\
&&&&& x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & \underline{x_{18}} & \underline{x_{19}} & y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & y_{16} & y_{17} & y_{18} \\
&&&&& x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & \underline{x_{29}} & \underline{y_{21}} & y_{22} & y_{23} & y_{24} & y_{25} & y_{26} & y_{27} & y_{28} \\
&&&&& x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & x_{38} & x_{39} & \underline{y_{31}} & \underline{y_{32}} & y_{33} & y_{34} & y_{35} & y_{36} & y_{37} & y_{38} \\
&&&&& x_{43} & x_{44} & x_{45} & x_{46} & x_{47} & x_{48} & x_{49} & y_{41} & \underline{y_{42}} & \underline{y_{43}} & y_{44} & y_{45} & y_{46} & y_{47} & y_{48} \\
&&&&& x_{53} & x_{54} & x_{55} & x_{56} & x_{57} & x_{58} & x_{59} & y_{51} & y_{52} & \underline{y_{53}} & \underline{y_{54}} & y_{55} & y_{56} & y_{57} & y_{58} \\
&&&&&&&&&&&&&& y_{63} & \underline{y_{64}} & \underline{y_{65}} & y_{66} & y_{67} & y_{68} \\
&&&&&&&&&&&&&& y_{73} & y_{74} & \underline{y_{75}} & \underline{y_{76}} & y_{77} & y_{78} \\
&&&&&&&&&&&&&& y_{83} & y_{84} & y_{85} & \underline{y_{86}} & \underline{y_{87}} & y_{88} \\
&&&&&&&&&&&&&& y_{93} & y_{94} & y_{95} & y_{96} & \underline{y_{97}} & \underline{y_{98}} \\
&&&&&&&&&&&&&& x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & \underline{x_{18}} \\
&&&&&&&&&&&&&& x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} \\
&&&&&&&&&&&&&& x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & x_{38} \\
&&&&&&&&&&&&&& x_{43} & x_{44} & x_{45} & x_{46} & x_{47} & x_{48} \\
&&&&&&&&&&&&&& x_{53} & x_{54} & x_{55} & x_{56} & x_{57} & x_{58}
\end{array}
$$

Denote $k = a - b$. We say that a diagonal of L is *inner* if when it is viewed as the main diagonal of L then L is not block triangular. In the example above, there are two inner diagonals whose entries are underlined. In general, L has $a - b - 1 = k - 1$ inner diagonals. We define the *core* Φ of L as follows. Delete the 1st row in every block row of L, then in the resulting matrix pick the dense $((k-1)n + b) \times ((k-1)n + b)$ submatrix whose upper left entry is y_{21}, so that

$$
\Phi = \begin{bmatrix}
Y_{[2,n]} & & & & \\
X_{[2,n]} & Y_{[2,n]} & & & \\
& \ddots & \ddots & & \\
& & X_{[2,n]} & Y_{[2,n]} & \\
& & & X_{[2,a]} & Y_{[2,a]}^{[1,b]}
\end{bmatrix} \tag{3.3}
$$

(the Y-block in the lower right corner does not exist when $b = 0$). Here and in what follows, for two index sets I, J, we write A_I^J for the submatrix with rows indexed by I and columns indexed by J; if I (respectively, J) coincides with the set of all rows (respectively, columns) of A, it is omitted in the notation. Further, for $p \le q$, we denote by $[p, q]$ the set $p, p+1, \dots, q$; notation $[p, p]$ is shortened to $[p]$. For our example above, the core is a 20×20 matrix.

$$
\begin{array}{cccccccccccccccccccc}
\underline{y}_{21} & y_{22} & y_{23} & y_{24} & y_{25} & y_{26} & y_{27} & y_{28} & y_{29} & & & & & & & & & & & \\
\underline{y}_{31} & \underline{y}_{32} & y_{33} & y_{34} & y_{35} & y_{36} & y_{37} & y_{38} & y_{39} & & & & & & & & & & & \\
y_{41} & \underline{y}_{42} & \underline{y}_{43} & y_{44} & y_{45} & y_{46} & y_{47} & y_{48} & y_{49} & & & & & & & & & & & \\
y_{51} & y_{52} & \underline{y}_{53} & \underline{y}_{54} & y_{55} & y_{56} & y_{57} & y_{58} & y_{59} & & & & & & & & & & & \\
 & & y_{63} & \underline{y}_{64} & \underline{y}_{65} & y_{66} & y_{67} & y_{68} & y_{69} & & & & & & & & & & & \\
 & & y_{73} & y_{74} & \underline{y}_{75} & \underline{y}_{76} & y_{77} & y_{78} & y_{79} & & & & & & & & & & & \\
 & & y_{83} & y_{84} & y_{85} & \underline{y}_{86} & \underline{y}_{87} & y_{88} & y_{89} & & & & & & & & & & & \\
 & & y_{93} & y_{94} & y_{95} & y_{96} & \underline{y}_{97} & \underline{y}_{98} & y_{99} & & & & & & & & & & & \\
 & & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & \underline{x}_{29} & \underline{y}_{21} & y_{22} & y_{23} & y_{24} & y_{25} & y_{26} & y_{27} & y_{28} & y_{29} & & \\
 & & x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & x_{38} & x_{39} & \underline{y}_{31} & \underline{y}_{32} & y_{33} & y_{34} & y_{35} & y_{36} & y_{37} & y_{38} & y_{39} & & \\
 & & x_{43} & x_{44} & x_{45} & x_{46} & x_{47} & x_{48} & x_{49} & y_{41} & \underline{y}_{42} & \underline{y}_{43} & y_{44} & y_{45} & y_{46} & y_{47} & y_{48} & y_{49} & & \\
 & & x_{53} & x_{54} & x_{55} & x_{56} & x_{57} & x_{58} & x_{59} & y_{51} & y_{52} & \underline{y}_{53} & \underline{y}_{54} & y_{55} & y_{56} & y_{57} & y_{58} & y_{59} & & \\
 & & & & & & & & & & & y_{63} & \underline{y}_{64} & \underline{y}_{65} & y_{66} & y_{67} & y_{68} & y_{69} & & \\
 & & & & & & & & & & & y_{73} & y_{74} & \underline{y}_{75} & \underline{y}_{76} & y_{77} & y_{78} & y_{79} & & \\
 & & & & & & & & & & & y_{83} & y_{84} & y_{85} & \underline{y}_{86} & \underline{y}_{87} & y_{88} & y_{89} & & \\
 & & & & & & & & & & & y_{93} & y_{94} & y_{95} & y_{96} & \underline{y}_{97} & \underline{y}_{98} & y_{99} & & \\
 & & & & & & & & & & & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & \underline{x}_{29} & \underline{y}_{21} & y_{22} \\
 & & & & & & & & & & & x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & x_{38} & x_{39} & \underline{y}_{31} & \underline{y}_{32} \\
 & & & & & & & & & & & x_{43} & x_{44} & x_{45} & x_{46} & x_{47} & x_{48} & x_{49} & y_{41} & \underline{y}_{42} \\
 & & & & & & & & & & & x_{53} & x_{54} & x_{55} & x_{56} & x_{57} & x_{58} & x_{59} & y_{51} & y_{52}
\end{array}
$$

Consider n-element segments of inner diagonals in L obtained as intersections with a single block row. The main diagonal of Φ is made of the entries 2 to n of such segment belonging to the uppermost inner diagonal, followed by the entries 2 to n of the segment belonging to the next inner diagonal from the top, and so on, followed by entries 2 to n of the segment belonging to the lowest inner diagonal, followed by entries $x_{2,n-k+2}, \dots, x_{kn}, y_{k+1,1}, \dots, y_{ab}$. Consequently, each matrix entry that lies on an inner diagonal of L and does not belong to the 1st row of X or Y enters the main diagonal of Φ exactly once.

For $i = 1, \dots, (k-1)n + b$ let

$$\varphi_i = \det \Phi^{[i,(k-1)n+b]}_{[i,(k-1)n+b]} \tag{3.4}$$

be the trailing minors of Φ. In particular, $\varphi_1 = \det \Phi$ is called the *core determinant*. Additionally, we set $\varphi_{(k-1)n+b+1} = 1$.

We consider φ_i as polynomials in the entries of X and Y indicated by $*$ in (3.2). Our goal is to establish a generalized exchange relation for φ_1 that involves the coefficients of the characteristic polynomial $\det(\lambda X + \mu Y)$.

Denote

$$XY^{-1} = \begin{bmatrix} W \\ 0_{(n-a)\times n} \end{bmatrix}, \quad W^{[1,a]} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \quad Y^{[1,b]}_{[1,a]} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where W is $a \times n$, W_{11} is $k \times k$, W_{22} is $b \times b$, Y_1 is $k \times b$, and Y_2 is $b \times b$. Let

$$U = W_{11} - Y_1 Y_2^{-1} W_{21}.$$

If $b = 0$, we set $U = W_{11}$ and use a standard convention $\det Y_2 = 1$.

Lemma 3.1. For any λ, μ,

$$\det(\lambda Y + \mu X) = \lambda^{n-k} \det Y \det(\lambda \mathbf{1}_k + \mu U).$$

Proof. Let $t = \frac{\lambda}{\mu}$, then $\det(\lambda Y + \mu X) = \mu^n \det(tY + X)$. In turn,

$$\det(tY + X) = \det Y \det\left(t\mathbf{1}_n + \begin{bmatrix} W \\ 0 \end{bmatrix} \right) = t^{n-a} \det Y \det\left(t\mathbf{1}_a + W^{[1,a]}_{[1,a]} \right).$$

Note that $W^{[1,a]} Y^{[1,b]}_{[1,a]} = (WY)^{[1,b]} = X^{[1,b]}_{[1,a]} = 0$, and so

$$\begin{bmatrix} \mathbf{1}_k & -Y_1 Y_2^{-1} \\ 0 & \mathbf{1}_b \end{bmatrix} W^{[1,a]}_{[1,a]} \begin{bmatrix} \mathbf{1}_k & Y_1 Y_2^{-1} \\ 0 & \mathbf{1}_b \end{bmatrix} = \begin{bmatrix} U & 0 \\ \star & 0 \end{bmatrix}; \tag{3.5}$$

here and in what follows, exact expressions for submatrices denoted by $\star$ are not relevant for further discussion. Consequently,

$$\det(tY+X) = t^{n-a}\det Y \det\left(t\mathbf{1}_a + \begin{bmatrix} U & 0 \\ \star & 0 \end{bmatrix}\right) = t^{n-k}\det Y \det\left(t\mathbf{1}_k + U\right),$$

and the claim follows. ■

As an immediate corollary from Lemma 3.1, we can write

$$\det(\lambda Y + \mu X) = \lambda^{n-k}\sum_{i=0}^{k} c_i(X,Y)\mu^i\lambda^{k-i}, \tag{3.6}$$

where $c_i(X,Y)$ are polynomials in the entries of X and Y.

Theorem 3.2. The generalized exchange relation for the core determinant φ_1 is given by

$$\varphi_1\varphi_1^* = \sum_{i=0}^{k} c_i(X,Y)\left((-1)^{n-1}\det\bar{Y}\varphi_{n+1}\right)^i \varphi_2^{k-i}, \tag{3.7}$$

where φ_1^* is a polynomial in the entries of X and Y and $\bar{Y} = Y_{[2,n]}^{[2,n]}$.

Proof. We start from expressing functions φ_1, φ_2, and φ_{n+1} via U.

Lemma 3.3. The core determinant φ_1 can be written as

$$\varphi_1 = \varepsilon_1 (\det Y)^{k-1}\det Y_2 \det\left[U^{k-1}e_1 \dots U^2e_1\ Ue_1\ e_1\right],$$

where $\varepsilon_1 = (-1)^{n\frac{k(k-1)}{2}}$ and $e_1 = (1,0,\dots,0) \in \mathbb{C}^k$.

Lemma 3.4. The minor φ_2 can be written as

$$\varphi_2 = \varepsilon_2 (\det Y)^{k-2}\det\bar{Y}\det Y_2 \det\left[U^{k-2}v_\gamma\ U^{k-2}e_1 \dots U^2e_1\ Ue_1\ e_1\right],$$

where $\varepsilon_2 = -\varepsilon_1$ and $v_\gamma = U(e_2 + \gamma e_1)$ with

$$\gamma = \frac{\det Y_{1\cup[3,n]}^{[2,n]}}{\det\bar{Y}} \tag{3.8}$$

and $e_2 = (0,1,0,\dots,0) \in \mathbb{C}^k$.

Lemma 3.5. The minor φ_{n+1} can be written as

$$\varphi_{n+1} = \varepsilon_{n+1} (\det Y)^{k-2} \det Y_2 \det \left[U^{k-2}e_1 \; U^{k-3}v_\gamma \; U^{k-3}e_1 \dots U^2 e_1 \; Ue_1 \; e_1 \right],$$

where $\varepsilon_{n+1} = (-1)^{n(k-2)(k-3)/2}$ and v_γ is the same as in Lemma 3.4.

Proofs of Lemmas 3.3–3.5 are given in Section 7.1.

Now we can invoke a result proven in [9, Proposition 8.1].

Proposition 3.6. Let A be a complex $k \times k$ matrix. For $u, v \in \mathbb{C}^k$, define matrices

$$K(A;u) = \left[u \; Au \; A^2 u \dots A^{k-1}u \right],$$

$$K_1(A;u,v) = \left[v \; u \; Au \dots A^{k-2}u \right], \quad K_2(A;u,v) = \left[Av \; u \; Au \dots A^{k-2}u \right].$$

In addition, let w be the last row of the classical adjoint of $K_1(A;u,v)$, so that $wK_1(A;u,v) = \left(\det K_1(A;u,v)\right) e_k^T$. Define $K^*(A;u,v)$ to be the matrix with rows $w, wA, \dots, wA^{k-1}$. Then

$$\det\left(\det K_1(A;u,v)A - \det K_2(A;u,v)\mathbf{1}_k \right) = (-1)^{\frac{k(k-1)}{2}} \det K(A;u) \det K^*(A;u,v). \tag{3.9}$$

We will make use of the following properties of matrices K and K^*.

Lemma 3.7. (i) For any $\gamma \in \mathbb{C}$, there exists an invertible matrix A such that $\det K(A;e_1) = 0$, but $\det K^*(A;e_1, A^{-1}(e_2 + \gamma e_1)) \neq 0$.

(ii) Moreover, A can be chosen in such a way that all principal leading minors of A do not vanish.

The proof of the Lemma is given in Section 7.3.

Using notation introduced in Proposition 3.6, we can re-write the claims of Lemmas 3.3–3.5 as

$$\det K(U^{-1}; e_1) = \varepsilon_1 (\det Y)^{1-k} \left(\det Y_2\right)^{-1} (\det U)^{1-k} \varphi_1,$$

$$\det K_1(U^{-1}; e_1, v_\gamma) = \varepsilon_2 (\det Y)^{2-k} \left(\det \bar{Y}\right)^{-1} \left(\det Y_2\right)^{-1} (\det U)^{2-k} \varphi_2,$$

$$\det K_2(U^{-1}; e_1, v_\gamma) = -\varepsilon_{n+1} (\det Y)^{2-k} \left(\det Y_2\right)^{-1} (\det U)^{2-k} \varphi_{n+1}.$$

Consequently, the matrix in the left hand side of (3.9) equals

$$(\det Y)^{2-k}\left(\det \bar{Y}\right)^{-1}\left(\det Y_2\right)^{-1}(\det U)^{2-k}\left(\varepsilon_2\varphi_2\mathbf{1}_k+\varepsilon_{n+1}\det\bar{Y}\varphi_{n+1}U\right)U^{-1},$$

and (3.9) becomes

$$\begin{aligned}&\det\left(\varepsilon_2\varphi_2\mathbf{1}_k+\varepsilon_{n+1}\det\bar{Y}\varphi_{n+1}U\right)\\&\quad=(-1)^{\frac{k(k-1)}{2}}\varepsilon_1\varphi_1\frac{\det K^*(U^{-1};e_1,v)}{\det Y}c_k(X,Y)^{(k-1)(k-2)}\left(\det Y_2\right)^{k-1}\left(\det\bar{Y}\right)^k,\end{aligned}\tag{3.10}$$

since $c_k(X,Y)=\det Y\det U$.

Using Lemma 3.1 and equations (3.6) and (3.10), we get (3.7) with

$$\varphi_1^*=(-1)^{\frac{k(k-1)}{2}}\varepsilon_1\varepsilon_2^k\det K^*(U^{-1};e_1,v_\gamma)c_k(X,Y)^{(k-1)(k-2)}\left(\det Y_2\right)^{k-1}\left(\det\bar{Y}\right)^k.\tag{3.11}$$

Note that $\det K^*(U^{-1};e_1,v_\gamma)$ is a rational function of X,Y whose denominator can contain only powers of $\det Y$, $\det Y_2$, and $\det\bar{Y}$. It remains to establish that φ_1^* is a polynomial function of X and Y. By (3.7), this fact is an immediate corollary of the following statement.

Lemma 3.8. The core determinant $\varphi_1=\varphi_1(X,Y)$ is an irreducible polynomial in the entries of X and Y.

The proof of the Lemma is given in Section 7.2. ■

Remark 3.9. Infinite block Toeplitz matrices with finitely many diagonals, of which (3.1) is an example, are naturally associated with polynomial loops in GL_n. In fact, examples of applications of our construction considered in the next two sections can be viewed as two instances of generalized cluster structures on Poisson submanifolds in the space of polynomial loops with respect to the Poisson structure defined by the trigonometric R-matrix. This hints at a possibility to extend the construction to produce generalized cluster structures in a wider class of such Poisson submanifolds. We hope to pursue this line of inquiry in the future.

4 Example 1: A Generalized Cluster Structure on the Drinfeld Double of GL_n

In [7, 9], we presented a generalized cluster structure on the standard Drinfeld double $D(GL_n)=GL_n\times GL_n$ and studied its properties. In this section, we explain how the construction of Section 3 can be applied to obtain an alternative seed that gives rise to a generalized cluster structure on $D(GL_n)$. As discussed in Remark 4.4 below, this generalized cluster structure likely does not coincide with the one considered in [7, 9].

In this case, X and Y in (3.1) are arbitrary $n \times n$ matrices, and hence $b = 0$ and $a = k = n$. Consequently, the core Φ is an $N \times N$ matrix

$$\Phi = \Phi(X, Y) = \begin{pmatrix} Y_{[2,n]} & & & \\ X_{[2,n]} & Y_{[2,n]} & & \\ & \ddots & \ddots & \\ & & X_{[2,n]} & Y_{[2,n]} \\ & & & X_{[2,n]} \end{pmatrix}$$

with $N = (n-1)n$ and $\varphi_i = \det \Phi^{[i,N]}_{[i,N]}$. Further, we have $U = W = XY^{-1}$ and $\det(\lambda Y + \mu X) = \sum_{i=0}^{n} c_i(X,Y)\mu^i\lambda^{n-i}$.

Following [9], we define $g_{ij} = \det X^{[j,j+n-i]}_{[i,n]}$ for $1 \le j \le i \le n$, and, $h_{ij} = \det Y^{[j,n]}_{[i,i+n-j]}$ for $1 \le i \le j \le n$; note that $\varphi_i = g_{i-N+n-1,i-N+n-1}$ for $i > N - n + 1$, and that $h_{22} = \bar{Y}$. The family $\mathcal{F}_n$ of $2n^2$ functions in the ring of regular functions on $D(GL_n)$ is defined as

$$\mathcal{F}_n = \left\{ \{\varphi_i\}_{i=1}^{N-n+1};\ \{g_{ij}\}_{1\le j\le i\le n};\ \{h_{ij}\}_{1\le i\le j\le n};\ \{\tilde{c}_i\}_{i=1}^{n-1} \right\}$$

with $\tilde{c}_i(X,Y) = (-1)^{i(n-1)} c_i(X,Y)$ for $1 \le i \le n-1$.

The corresponding quiver Q_n is defined below and illustrated, for the $n = 4$ case, in Figure 1. It has $2n^2$ vertices corresponding to the functions in $\mathcal{F}_n$. The $n-1$ vertices corresponding to $\tilde{c}_i(X,Y)$, $1 \le i \le n-1$, are isolated; they are not shown. There are $2n$ frozen vertices corresponding to g_{i1}, $1 \le i \le n$, and h_{1j}, $1 \le j \le n$; they are shown as squares in the figure below. All vertices except for one are arranged into a $(2n-1) \times n$ grid; we will refer to vertices of the grid using their position in the grid numbered top to bottom and left to right. The edges of Q_n are $(i,j) \to (i+1, j+1)$ for $i = 1, \dots, 2n-2$, $j = 1, \dots, n-1$, $(i,j) \to (i, j-1)$, and $(i,j) \to (i-1, j)$ for $i = 2, \dots, 2n-1$, $j = 2, \dots, n$, and $(i,1) \to (i-1,1)$ for $i = 2, \dots, n$. Additionally, there is an oriented path

$$(n+1, n) \to (3,1) \to (n+2, n) \to (4,1) \to \cdots (n,1) \to (2n-1, n).$$

The edges in this path are depicted as dashed in Figure 1. The vertex $(2,1)$ is special; it is shown as a hexagon in the figure. The last remaining vertex of Q_n is placed to the left of the special vertex and there is an edge pointing from the former one to the latter.

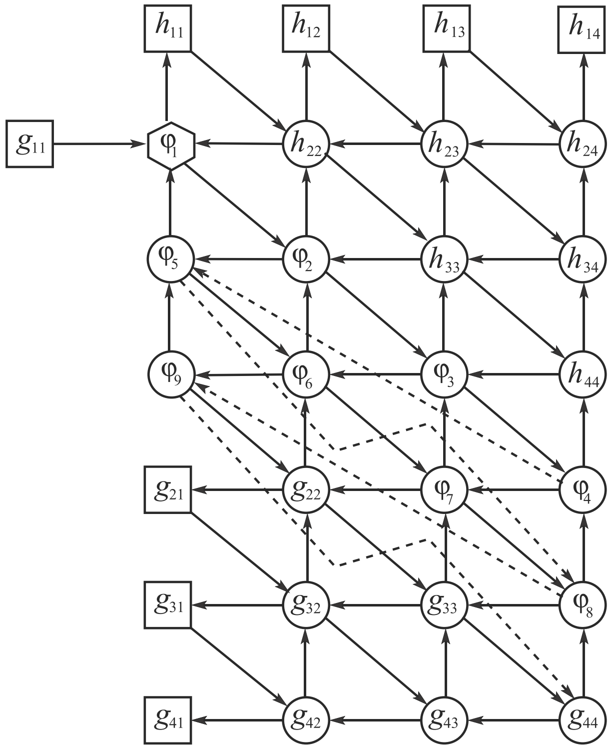

Fig. 1. Quiver Q_4.

Functions h_{ij} are attached to the vertices (i,j), $1 \le i \le j \le n$, and all vertices in the upper row of Q_n are frozen. Functions g_{ij} are attached to the vertices $(n+i-1,j)$, $1 \le j \le i \le n$, $(i,j) \ne (1,1)$, and all such vertices in the 1st column are frozen. The function g_{11} is attached to the vertex to the left of the special one, and this vertex is frozen. Functions φ_{kn+i} are attached to the vertices $(i+k+1,i)$ for $1 \le i \le n$, $0 \le k \le n-3$; the function φ_{N-n+1} is attached to the vertex $(n,1)$. All these vertices are mutable. The set of strings $\mathcal{P}_n$ contains a unique nontrivial string $(1, \tilde{c}_1(X,Y), \dots, \tilde{c}_{n-1}(X,Y), 1)$ corresponding to the unique special vertex.

Theorem 4.1. The extended seed $\Sigma_n = (\mathcal{F}_n, Q_n, \mathcal{P}_n)$ defines a regular generalized cluster structure on $D(GL_n)$.

Proof. We start with checking that relation (3.7) with $k = n$ indeed defines a generalized exchange relation as described in (2.1). The degree of the exchange relation is $d_n = n$, exchange coefficients are given by $p_{1r} = \tilde{c}_r(X,Y)$ for $r = 1, \dots, n-1$, the cluster

τ-monomials are $u_{1;>} = h_{22}\varphi_{n+1}$ and $u_{1;<} = \varphi_2$. The stable τ-monomials are defined as follows:

$$v_{1;>}^{[n]} = h_{11} = \det Y, \qquad v_{1;>}^{[r]} = 1 \quad \text{for } 0 \le r \le n-1,$$

and

$$v_{1;<}^{[n]} = g_{11} = \det X, \qquad v_{1;<}^{[r]} = 1 \quad \text{for } 0 \le r \le n-1.$$

Let us show that cluster transformations defined by the quiver Q_n produce regular functions. For the special vertex, this follows from Theorem 3.2. For the vertices corresponding to g_{ij} and h_{ij} with $i \ne j$, the claim is wellknown from the study of the standard cluster structure on GL_n. For other mutable vertices, we use determinantal identities often utilized for this purpose (see, e.g., [8], [9], [10]). The 1st is the Desnanot–Jacobi identity for minors of a square matrix A:

$$\det A \det A^{\hat\gamma\hat\delta}_{\hat\alpha\hat\beta} + \det A^{\hat\delta}_{\hat\alpha} \det A^{\hat\gamma}_{\hat\beta} = \det A^{\hat\gamma}_{\hat\alpha} \det A^{\hat\delta}_{\hat\beta}, \tag{4.1}$$

where "hatted" subscripts and superscripts indicate deleted rows and columns, respectively. The 2nd is a version of a short Plücker relation for an $m \times (m+1)$ matrix B:

$$\det B^{\hat\alpha\hat\beta}_{\hat\delta} \det B^{\hat\gamma} + \det B^{\hat\beta\hat\gamma}_{\hat\delta} \det B^{\hat\alpha} = \det B^{\hat\alpha\hat\gamma}_{\hat\delta} \det B^{\hat\beta}, \tag{4.2}$$

and the 3rd is the corollary of (4.2):

$$\begin{aligned} &\det B^{\hat1\hat m\widehat{m+1}}_{\hat1\hat2} \det B^{\hat1\hat2}_{\hat1} \det B^{\widehat{m+1}} + \det B^{\hat1\hat2\widehat{m+1}}_{\hat1\hat2} \det B^{\hat m\widehat{m+1}}_{\hat1} \det B^{\hat1} \\ &\quad = \det B^{\hat1\widehat{m+1}}_{\hat1} \left(\det B^{\hat1\hat m\widehat{m+1}}_{\hat1\hat2} \det B^{\hat2} - \det B^{\hat2\hat m\widehat{m+1}}_{\hat1\hat2} \det B^{\hat1} \right). \end{aligned} \tag{4.3}$$

In more detail, for functions φ_i with $2 \le i \le n-1$, we use (4.3) for the matrix $B = \begin{bmatrix} \Phi & e_N^T \end{bmatrix}^{[i-1,N+1]}_{[i-1,N]}$. For φ_n, we use (4.1) for the matrix $A = \Phi^{[n-1,N]}_{[n-1,N]}$ with parameters $\alpha = \gamma = 1$, $\beta = 2$, $\delta = N-n+2$. For functions φ_i with $n+1 \le i \le N-1$, we consider a perturbation $\Phi(\theta) = \Phi + \theta e_{(n-1)^2+1,(n-1)^2-1}$ of the core and use (4.3) for the matrix $B(\theta) = \Phi(\theta)^{[i-n-1,N]}_{[i-n,N]}$ (for $i = n+1$ the range of columns $[0,N]$ stands for Φ prepended with the previous column of the infinite periodic matrix (3.1); this column contains $X^{[n]}_{[2,n]}$ to the left of the uppermost copy of $Y_{[2,n]}$ in (3.3)). A direct check shows that the identity (4.3) for $B(\theta)$ yields a polynomial of degree 3 in θ that vanishes identically. The coefficient of this polynomial at θ is the exchange relation we are looking for. For φ_N, we use (4.2) for the matrix $B = \Phi^{[N-n-1,N]}_{[N-n,N]}$ with parameters $\alpha = \delta = 1$, $\beta = 2$, $\gamma = n+2$. For functions h_{ii} with $3 \le i \le n$, we consider a perturbation $\overline{\Phi}(\theta) = \begin{bmatrix} \Phi & e_N^T & e_N^T \end{bmatrix} + \theta e_{n-1,n+1} + \theta e_{N-1,N+1}$ and use (4.3) for the matrix $\overline{B}(\theta) = \overline{\Phi}(\theta)^{[i-1,N+2]}_{[i-2,N]}$. A direct check shows that the identity (4.3) for $\overline{B}(\theta)$ yields a polynomial of degree 4 in θ that vanishes identically. The coefficient of this polynomial at θ^2 is the exchange relation we are looking for. Finally, for h_{22}, we prepend a row $[Y_{[1]}\ 0]$ to the matrix $\overline{\Phi}(\theta)$ and proceed with the obtained matrix exactly as in the previous case.

By Proposition 2.1, it remains to check that any two functions in $\mathcal{F}_n$ are coprime and that for any nonfrozen $f \in \mathcal{F}_n$, the function f^* that replaces f after the mutation is coprime with f. The 1st claim above is an immediate corollary of the following statement.

Lemma 4.2. All functions in the family $\mathcal{F}_n$ are irreducible.

The proof of Lemma 4.2 is given in Section 7.2. The 2nd claim above is provided by the following statement.

Lemma 4.3. Every nonfrozen $f \in \mathcal{F}_n$ does not divide the corresponding f^*.

The proof of Lemma 4.3 is given in Section 7.3. ∎

Remark 4.4. (i) The regular generalized cluster structure described in Theorem 4.1 is complete in $\mathcal{O}(D(GL_n))$ and compatible with the standard Poisson–Lie bracket on $D(GL_n)$.

(ii) In [9], we used a different initial seed $\widetilde{\Sigma}_n$ to define a regular complete generalized cluster structure $\mathcal{GC}(\widetilde{\Sigma}_n)$ on $D(GL_n)$ compatible with the standard Poisson–Lie structure on $D(GL_n)$. Moreover, the sets of frozen variables for both structures coincide. However, for $n = 4$, the initial seed described above is not mutation equivalent to the one constructed in [9].

Details and proofs of assertions mentioned in the above remark are given in [11].

5 Example 2: Generalized Cluster Structure on Periodic Band Matrices.

In this section, we consider the case of L in (3.1) being a $(k+1)$ diagonal n-periodic band matrix with $k \le n$. In other words, L represents a periodic difference operator. Such operators play an important role in spectral theory; they also appear as Lax operators in the theory of integrable systems, such as periodic Toda lattices and their multicomponent analogues (see, e.g., [17]). More recently, periodic difference operators found applications that, in turn, proved to be related to the theory of cluster algebras, in particular, in the investigation of frieze patterns and pentagram maps and their generalizations [13, 18]. In this section, we will use Theorem 3.2 to construct a generalized cluster algebra structure on the space of periodic difference operators.

We choose Y in (3.2) to be a lower triangular band matrix with $k+1$ nonzero diagonals (including the main diagonal); consequently, X is an upper triangular with zeroes everywhere outside of $k \times k$ upper triangular block in the upper right corner. We assume that entries of the lowest and highest diagonals are all nonzero. X and Y are now $n \times n$ matrices of the form

$$X = \begin{bmatrix} 0 & \cdots & 0 & a_{11} & \cdots & a_{k1} \\ 0 & \cdots & 0 & 0 & a_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & \cdots & 0 & a_{1k} \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$
$$Y = \begin{bmatrix} a_{k+1,1} & 0 & \cdots & \cdots & \cdots & 0 \\ a_{k2} & a_{k+1,2} & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{1,k+1} & a_{2,k+1} & \cdots & a_{k+1,k+1} & 0 & \cdots \\ 0 & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{1n} & a_{2n} & \cdots & a_{k+1,n} \end{bmatrix}, \tag{5.1}$$

and we can choose $a = k$, $b = 0$. Consequently,

$$U = W_{11} = \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{1k} \end{bmatrix} \left(Y^{-1}\right)^{[1,k]}_{[n-k+1,n]}, \tag{5.2}$$

and hence

$$\det U = \frac{a_{11} \cdots a_{1n}}{a_{k+1,1} \cdots a_{k+1,n}}, \qquad c_k(X, Y) = a_{11} \cdots a_{1n}. \tag{5.3}$$

Furthermore, γ in (3.8) is equal to 0, and therefore $v_\gamma = Ue_2$.

The core Φ is a reducible $(k-1)n \times (k-1)n$ matrix, and for $i = 1, \ldots, (k-1)(n-1)$ we have $\varphi_i = \tilde{\varphi}_i a_{12} \cdots a_{1k}$ with

$$\tilde{\varphi}_i = \det \Phi^{[i,(k-1)(n-1)]}_{[i,(k-1)(n-1)]}. \tag{5.4}$$

Relation (3.7) can be rewritten as

$$\tilde{\varphi}_1\varphi_1^* = \left(a_{12}\cdots a_{1k}\right)^{k-1}\sum_{i=0}^{k} c_i(X,Y)\left((-1)^{n-1}\det\bar{Y}\tilde{\varphi}_{n+1}\right)^i\tilde{\varphi}_2^{k-i}, \tag{5.5}$$

for $k > 2$ and as

$$\tilde{\varphi}_1\varphi_1^* = c_0(X,Y)\tilde{\varphi}_2^2 a_{12} + (-1)^{n-1}c_1(X,Y)\det\bar{Y}\tilde{\varphi}_2 + c_2(X,Y)\left(\det\bar{Y}\right)^2 a_{12}^{-1}, \tag{5.6}$$

for $k = 2$, since in this case $\varphi_{n+1} = \varphi_{(k-1)n+b+1} = 1$ according to the convention introduced in Section 3. In both cases, φ_1^* is a polynomial function in matrix entries of X, Y, according to Theorem 3.2. Since $c_0(X,Y) = \det Y = a_{k+1,1}\det\bar{Y}$, the right hand side of (5.5) is divisible by $\det\bar{Y} = a_{k+1,2}\cdots a_{k+1,n}$. On the other hand, it is easy to see that $\tilde{\varphi}_1$ is not divisible by $a_{1i}, a_{k+1,i}$ for $i = 2,\dots,n$. This means that for $k > 2$,

$$\varphi_1^* = \left(a_{12}\cdots a_{1k}\right)^{k-1}\det\bar{Y}\tilde{\varphi}_1^*,$$

where $\tilde{\varphi}_1^*$ is a polynomial function in matrix entries of X and Y. Thus, (5.5) becomes

$$\tilde{\varphi}_1\tilde{\varphi}_1^* = a_{k+1,1}\tilde{\varphi}_2^k + \sum_{i=1}^{k}\tilde{c}_i(X,Y)(\det\bar{Y})^{i-1}\tilde{\varphi}_{n+1}^i\tilde{\varphi}_2^{k-i}, \tag{5.7}$$

where $\tilde{c}_i(X,Y) = (-1)^{i(n-1)}c_i(X,Y)$ for $1 \le i \le k$. In what follows, it will be convenient to introduce $\tilde{a}_{11} = (-1)^{k(n-1)}a_{11}$, so that $\tilde{c}_k(X,Y) = \tilde{a}_{11}a_{12}\cdots a_{1n}$.

Similarly, for $k = 2$, $\varphi_1^* = \tilde{\varphi}_1^*\det\bar{Y}$ where $\tilde{\varphi}_1^*$ is a polynomial function in matrix entries of X and Y, and (5.6) becomes

$$\tilde{\varphi}_1\tilde{\varphi}_1^* = a_{31}a_{12}\tilde{\varphi}_2^2 + \tilde{c}_1(X,Y)\tilde{\varphi}_2 + \tilde{c}_2(X,Y)\det\bar{Y} \tag{5.8}$$

with $\tilde{c}_1(X,Y) = (-1)^{n-1}c_1(X,Y)$ and $\tilde{c}_2(X,Y) = c_2(X,Y)/a_{12} = a_{11}a_{13}\cdots a_{1n}$.

For $k \le n$, denote by $\mathcal{L}_{kn}$ the space of periodic difference operators represented by n-periodic $(k+1)$-diagonal matrices with all entries of the lowest and the highest diagonals nonzero. A generalized cluster structure in the space of regular functions on $\mathcal{L}_{kn}$ is defined by the following data.

Consider the family $\mathcal{F}_{kn}$ of functions on $\mathcal{L}_{kn}$:

$$\mathcal{F}_{kn} = \left\{\{\tilde{\varphi}_i\}_{i=1}^{(k-1)(n-1)};\ \tilde{a}_{11};\ \{a_{1i}\}_{i=2}^{n};\ \{a_{k+1,i}\}_{i=1}^{n};\ \{\tilde{c}_i(X,Y)\}_{i=1}^{k-1}\right\}.$$

Let Q_{kn} be the quiver with $(k+1)n$ vertices, of which $k-1$ vertices are isolated and are not shown in the figure below, $(k+1)(n-1)$ are arranged in an $(n-1)\times(k+1)$ grid and denoted (i,j), $1 \le i \le n-1$, $1 \le j \le k+1$, and the remaining two are placed on top of the leftmost and the rightmost columns in the grid and denoted $(0,1)$ and $(0,k+1)$, respectively. All vertices in the leftmost and in the rightmost columns are frozen. The vertex $(1,k)$ is special, and its multiplicity equals k. All other vertices are regular mutable vertices.

The edge set of Q_{kn} consists of the edges $(i,j)\to(i+1,j)$ for $i=1,\dots,n-2$, $j=2,\dots,k$; $(i,j)\to(i,j-1)$ for $i=1,\dots,n-1$, $j=2,\dots,k$, $(i,j)\neq(1,k)$; $(i+1,j)\to(i,j+1)$ for $i=1,\dots,n-2$, $j=2,\dots,k$, shown by solid lines. In addition, there are edges $(n-1,3)\to(1,2)$, $(1,2)\to(n-1,4)$, $(n-1,4)\to(1,3),\dots,(1,k-1)\to(n-1,k+1)$ that form a directed path (shown by dotted lines). Save for this path, and the missing edge $(1,k)\to(1,k-1)$, mutable vertices of Q_{kn} form a mesh of consistently oriented triangles

Finally, there are edges between the special vertex $(1,k)$ and frozen vertices $(i,1)$, $(i,k+1)$ for $i=0,\dots n-1$. There are $k-1$ parallel edges between $(1,k)$ and $(i,k+1)$ for $i=1,\dots,n-1$, and one edge between $(1,k)$ and all other frozen vertices (including $(0,k+1)$). If $k>2$, all of these edges are directed towards $(1,k)$, and if $k=2$, the direction of the edge between $(1,1)$ and $(1,k)$ is reversed. Quiver Q_{47} is shown in Figure 2.

We attach functions $\tilde a_{11}, a_{12},\dots,a_{1n}$, in a top to bottom order, to the vertices of the leftmost column in Q_{kn}, and functions $a_{k+1,1},\dots,a_{k+1,n}$, in the same order, to the vertices of the rightmost column in Q_{kn}. Functions $\tilde\varphi_i$ are attached, in a top to bottom, right to left order, to the remaining vertices of Q_{kn}, starting with $\tilde\varphi_1$ attached to the special vertex $(1,k)$. The set of strings $\mathcal{P}_{kn}$ contains a unique nontrivial string $(1,\tilde c_1(X,Y),\dots,\tilde c_{k-1}(X,Y),1)$ corresponding to the unique special vertex.

Theorem 5.1. The extended seed $\Sigma_{kn}=(\mathcal{F}_{kn},Q_{kn},\mathcal{P}_{kn})$ defines a regular generalized cluster structure $\mathcal{GC}(\Sigma_{kn})$ on $\mathcal{L}_{kn}$.

Proof. Similarly to the proof of Theorem 4.1, let us check first that relation (5.7) indeed defines a generalized cluster transformation as described in (2.1). The degree of the exchange relation is $d_k=k$, exchange coefficients are given by $p_{1r}=\tilde c_r(X,Y)$ for $r=1,\dots,k-1$, and the cluster τ-monomials are $u_{1;>}=\tilde\varphi_2$ and $u_{1;<}=\tilde\varphi_{n+1}$ for $k>2$ (for $k=2$, $u_{1;<}=1$). The stable τ-monomials are defined as follows:

$$v_{1;>}^{[k]}=\begin{cases}a_{k+1,1} & \text{if } k>2,\\ a_{31}a_{12} & \text{if } k=2,\end{cases}\qquad v_{1;>}^{[r]}=1 \quad \text{for } 0\le r\le k-1,$$

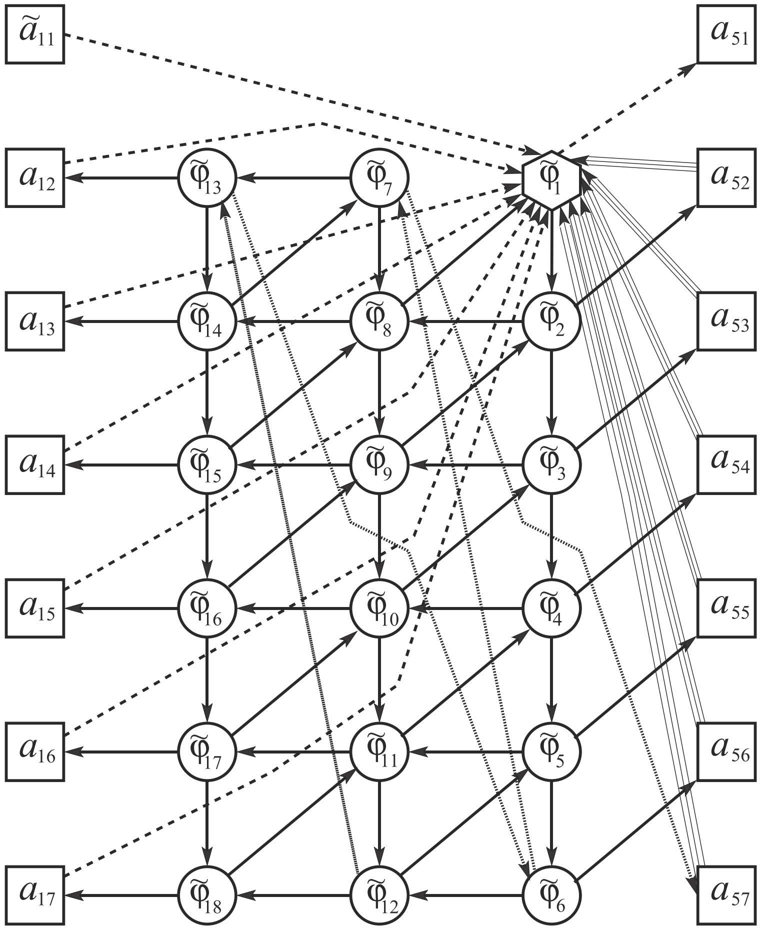

Fig. 2. Quiver Q_{47}.

and

$$v_{1;<}^{[k]} = \begin{cases} \tilde{a}_{11}a_{12}\dots a_{1n}a_{k+1,2}^{k-1}\dots a_{k+1,n}^{k-1} & \text{if } k>2, \\ a_{11}a_{13}\dots a_{1n}a_{k+1,2}\dots a_{k+1,n} & \text{if } k=2, \end{cases}$$

$$v_{1;<}^{[r]} = a_{k+1,2}^{r-1}\dots a_{k+1,n}^{r-1} \quad \text{for } 1 \le r \le k-1,$$

$$v_{1;<}^{[0]} = 1;$$

the expression for $v_{1;<}^{[r]}$ follows from (2.2) via $\lfloor (k-1)r/k \rfloor = r-1$.

Let us show that cluster transformations defined by the quiver Q_{kn} produce regular functions. For the special vertex, this follows from Theorem 3.2. For other mutable vertices, we use determinantal identities (4.1)–(4.3).

In more detail, consider a perturbation

$$\Phi(\theta) = \Phi + \theta \sum_{i=1}^{n-1} e_{(k-2)(n-1)+i,(k-2)(n-1)+i-2}$$

of the core. For every six-valent vertex (i,j) in Q_{kn} we apply (4.3) to the submatrix $B(\theta) = \Phi(\theta)^{[(k-j-1)(n-1)+i-2,(k-1)(n-1)]}_{[(k-j-1)(n-1)+i-1,(k-1)(n-1)]}$ of $\Phi(\theta)$ and get a polynomial identity of degree $3(n-1)$ in θ. The claim follows from considering the coefficient at θ^{n-1}. Indeed, the submatrix $\Phi^{[t,(k-1)(n-1)]}_{[t,(k-1)(n-1)]}$ that defines the function $\tilde{\varphi}_t$ coincides with the submatrix of Φ of the same size with the upper left corner at row $t-s(n-1)$ and column $t-sn$ for $s = 1, 2, \dots$. Note that for the function attached to (i,j) we have $t = (k-j)(n-1)+i$, and the result follows.

For vertices $(i, 2)$, $i = 1, \dots, n-1$, one needs to apply (4.2) to the submatrix $B = \Phi^{[(k-3)(n-1)+i-2,(k-1)(n-1)]}_{[(k-3)(n-1)+i-1,(k-1)(n-1)]}$ with $\alpha = \delta = 1$, $\beta = 2$, and γ being the last row. The same holds for the vertex $(n-1, 3)$ with $i = 0$. Finally, for vertices (i,k), $i = 2, \dots, n-1$, one needs to apply (4.1) to the submatrix $A = \Phi^{[i-1,(k-1)(n-1)]}_{[i-1,(k-1)(n-1)]}$ with $\alpha = \gamma = 1$, $\beta = 2$, and δ being the last column. The vertex $(1, k-1)$ is treated in the same way.

Similarly to the proof of Theorem 4.1, it remains to prove that all functions in $\mathcal{F}_{kn}$ are coprime and that each nonfrozen $f \in \mathcal{F}_{kn}$ is coprime with f^*. The 1st of the above claims is an immediate corollary of the following statement.

Lemma 5.2. All functions in the family $\mathcal{F}_{kn}$ are irreducible.

The proof of Lemma 5.2 is given in Section 7.2. The 2nd claim above is provided by the following statement.

Lemma 5.3. Every nonfrozen $f \in \mathcal{F}_{kn}$ does not divide the corresponding f^*.

The proof of Lemma 5.3 is given in Section 7.3. ∎

Remark 5.4. (i) The regular generalized cluster structure described in Theorem 5.1 is complete in $\mathcal{O}(\mathcal{L}_{kn})$.

(ii) Under certain mild nondegeneracy conditions, for any generalized cluster structure, there exists a compatible quadratic Poisson structure (see [9, Proposition 2.5] for details). This compatible Poisson structure coincides with a natural Poisson structure on the space of periodic finite difference operators considered in [13] and used in the proof of complete integrability of generalized pentagram maps.

Details and proofs of assertions mentioned in the above remark are given in [11].

Let us examine the case $k = 2$ in more detail. By [2, Theorem 2.7], the finite-type classification for generalized cluster structures coincides with that for usual cluster structures. Consequently, $\mathcal{GC}(\Sigma_{2n})$ is of type C_{n-1}. In [20], every cluster structure of finite type with principal coefficients was given a geometric realization in the ring of regular of functions on a reduced double Bruhat cell corresponding to a Coxeter element of the Weyl group and its inverse. In the A_n case, this double Bruhat cell consists of tridiagonal matrices A in SL_{n+1} with nonzero off-diagonal entries and with subdiagonal entries normalized to be equal to 1. Then, [20,Theorem 1.1] shows that the set of mutable cluster variables in such a realization coincides with the set of all dense principal minors of A.

We have the following analogue of [20, Theorem 1.1].

Proposition 5.5. The set of mutable cluster variables in $\mathcal{GC}\left(\Sigma_{2n}\right)$ coincides with the set of all distinct dense principal minors of $L \in \mathcal{L}_{2n}$ of size less than n.

Proof. Since $\mathcal{GC}\left(\Sigma_{2n}\right)$ is a generalized cluster structure of type C_{n-1}, the number of mutable cluster variables is $n(n-1)$, that is, the number of almost positive roots in C_{n-1}. Since this is also the number of distinct dense principal minors of $L \in \mathcal{L}_{2n}$ of size less than n, we only need to show that every such minor appears as a cluster variable in $\mathcal{GC}\left(\Sigma_{2n}\right)$. In the spirit of [20], we denote by $x_{[i,j]}$ the dense principal minor of L with diagonal entries $a_{2i}, a_{2,i+1}, \ldots, a_{2,j-1}, a_{2j}$, where either $1 \le i \le j \le n$, $(i,j) \ne (1,n)$, or $1 \le j < i-1 \le n-1$.

The initial cluster variables $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}$ are minors $x_{[i,n]}$, $i = 2, \ldots, n$, contained in an $(n-1) \times (n-1)$ tridiagonal matrix $\Phi^{[1,n-1]}_{[1,n-1]}$. If we treat, temporarily, $\tilde{\varphi}_1$ as a frozen variable, $\tilde{\varphi}_2, \ldots, \tilde{\varphi}_{n-1}$ form an initial cluster of a cluster structure of finite-type A_{n-2}, whose set of mutable cluster variables is the collection $x_{[i,j]}$, $2 \le i \le j \le n-1$, according to [20, Theorem 1.1]. (In [20], the corresponding tridiagonal matrix is normalized to have determinant 1, and also all the subdiagonal entries are equal to one; however, the calculation needed to obtain the desired result goes through without any modifications).

Next, we perform a generalized mutation from our initial cluster in direction 1 using (5.8). We claim that $\tilde{\varphi}_1^*$ is equal to $x_{[3,1]}$. Clearly, the degree of $\tilde{\varphi}_1^*$ in matrix entries of L is equal to $n-1$. By (3.11),

$$\varphi_1^* = (-1)^{n+1} \det K^*(U^{-1}; e_1, v_\gamma) \left(\det \bar{Y}\right)^2,$$

and so $\tilde{\varphi}_1^* = \varphi_1 / \det \bar{Y}$ is proportional to the numerator of $\det K^*(U^{-1}; e_1, v_\gamma)$ viewed as rational function in terms of entries of L with a coefficient that is a monomial in a_{3j},

$j = 1, \dots, n$. Since the degree of $x_{[3,1]}$ is $n-1$, we only need to show that $\det K^*(U^{-1}; e_1, v_\gamma)$ is proportional to $x_{[3,1]}$.

Recall that for band matrices $\gamma = 0$, and so v_γ defined in Lemma 3.4 is equal to Ue_2. Then w in Proposition 3.6 becomes $w = [-u_{22}, u_{12}]$, and we obtain $\det K^*(U^{-1}; e_1, v_\gamma) = -u_{12}$. By (5.2),

$$u_{12} = \frac{(-1)^{n+1} a_{31}}{\det Y} \left(a_{11} \det Y^{\widehat{1}\widehat{n-1}}_{\hat{1}\hat{2}} - a_{21} \det Y^{\hat{1}\hat{n}}_{\hat{1}\hat{2}} \right) = \frac{(-1)^n x_{[3,1]}}{\det Y},$$

and hence $\tilde{\varphi}_1^* = x_{[3,1]}$; here in the last equality we used the expansion of $x_{[3,1]}$ with respect to the last row.

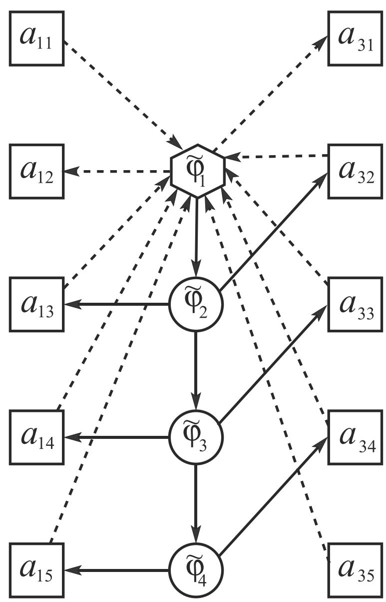

Fig. 3. Quiver Q_{25}.

After the generalized mutation, the quiver is transformed as follows: all edges incident to the special vertex change direction, edges pointing from the vertex corresponding to $\tilde{\varphi}_2$ to vertices corresponding to a_{13} and a_{32}, disappear, but new edges appear instead pointing to $\tilde{\varphi}_2$ from frozen vertices corresponding to $a_{11}, a_{14}, \dots, a_{1n}$ and $a_{33}, \dots, a_{3n}$ (cf. Figure 3). It is easy to check via (4.1) for the submatrix of L obtained from $\Phi^{[1,n-1]}_{[1,n-1]}$ by cyclically shifting 2nd indices of all entries a_{ij} up by 1 that mutation at the vertex $(2, 2)$ transforms $\tilde{\varphi}_2$ to $x_{[4,1]}$. Similarly, consequent mutations at the vertices $(3, 2), (4, 2), \dots, (n-1, 2)$ transform each $\tilde{\varphi}_i$ to $x_{[i+2,1]}$, $i = 3, \dots, n-2$, and $\tilde{\varphi}_{n-1}$ to $x_{[1,1]}$. Moreover, the resulting quiver coincides with the initial one. Clearly, we can perform a similar shift operation $n-2$ more times and recover the rest of functions $x_{[i,j]}$ as cluster variables. ■

Remark 5.6. Proposition 5.5 provides a geometric realization of generalized cluster structures of finite-type C_n. We should mention that generalized cluster algebras of this type but with *constant* exchange coefficients have been recently considered in [12] in the context of study of representations of the quantum loop algebra of sl_2 at roots of unity, and in [14, section 9], where they were realized as *Caldero–Chapoton algebras* associated with a special triangulation of a polygon with one orbifold point.

6 Example 3: Exotic Generalized Cluster Structure on GL_6

In [6], we initiated the study of cluster structures in the ring of regular functions on GL_n compatible with R-matrix Poisson–Lie brackets. Such brackets are classified by Belavin–Drinfeld triples $\Gamma = (\Gamma_1, \Gamma_2, \gamma : \Gamma_1 \to \Gamma_2)$, where Γ_1 and Γ_2 are subsets of the set of positive simple roots in the A_{n-1} root system and γ is a nilpotent isometry (see [6] for details). The cluster structures corresponding to nonempty Belavin–Drinfeld triples are called *exotic*. In [10], we treated the subclass of Belavin–Drinfeld triples that we called aperiodic. The 1st instance of a periodic Belavin–Drinfeld triple occurs for $n = 6$ with the triple Γ given by

$$\Gamma_1 = \{\alpha_1, \alpha_5\},\ \Gamma_2 = \{\alpha_2, \alpha_4\}, \quad \gamma(\alpha_1) = \alpha_2,\ \gamma(\alpha_5) = \alpha_4. \tag{6.1}$$

It will be convenient to denote elements of $D(GL_6)$ by (R, S). Following the construction described in [10], we consider a collection of matrices

$$\mathcal{L}_\Gamma(R,S) = \left\{R, R^{[1,2]}_{[5,6]}, R^{[1,4]}_{[3,6]}, R^{[1,3]}_{[4,6]}, S^6_1, S^{[2,6]}_{[1,5]}, S^{[4,6]}_{[1,3]}, L_1(R,S), L_2(R,S)\right\},$$

where $L_1 = L_1(R,S)$, $L_2 = L_2(R,S)$ both have a form (3.2), see Figure 4.

Here, 2×2 and 5×5 blocks featured in L_1 are submatrices $R^{[1,2]}_{[5,6]}$ and $S^{[2,6]}_{[1,5]}$, while 3×3 and 6×6 blocks featured in L_2 are $S^{[4,6]}_{[1,3]}$ and R. L_1 is 5-periodic, L_2 is 7-periodic, and each has one inner diagonal, which corresponds to $k = 2$ in (3.2). Overlaps between blocks in L_1, L_2 are prescribed by Γ (see [10] for details).

For $L_1(R,S)$, we choose

$$X_1 = \begin{bmatrix} S^{[2,6]}_{[4,5]} \\ 0_{3\times 5} \end{bmatrix}, \quad Y_1 = \begin{bmatrix} R^{[1,2]}_{[5,6]} & 0_{2\times 3} \\ S^{[2,3]}_{[1,3]} & S^{[4,6]}_{[1,3]} \end{bmatrix},$$

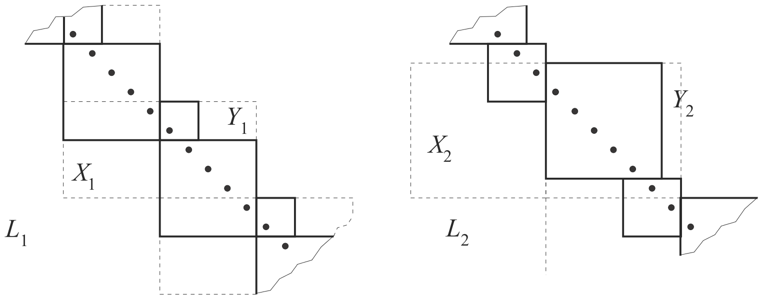

Fig. 4. Matrices L_1 and L_2.

which corresponds to $a = 2$, $b = 0$, while for $L_2(R, S)$ we choose

$$X_2 = \begin{bmatrix} 0_{2\times 4} & S^{[4,6]}_{[2,3]} \\ 0_{5\times 4} & 0_{5\times 3} \end{bmatrix}, \quad Y_2 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} & 0 \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} & 0 \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} & 0 \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & r_{46} & 0 \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} & r_{56} & 0 \\ r_{61} & r_{62} & r_{63} & r_{64} & r_{65} & r_{66} & 0 \\ 0 & 0 & 0 & 0 & s_{14} & s_{15} & s_{16} \end{bmatrix},$$

which corresponds to $a = 6$, $b = 4$. Thus, (3.3) results in

$$\Phi_1 = \begin{bmatrix} r_{61} & r_{62} & 0 & 0 & 0 \\ s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \end{bmatrix}$$

and

$$\Phi_2 = \begin{bmatrix} r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} & 0 \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} & 0 \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & r_{46} & 0 \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} & r_{56} & 0 \\ r_{61} & r_{62} & r_{63} & r_{64} & r_{65} & r_{66} & 0 \\ 0 & 0 & 0 & 0 & s_{14} & s_{15} & s_{16} \\ 0 & 0 & 0 & 0 & s_{34} & s_{35} & s_{36} \end{bmatrix}.$$

Consequently, (3.6) yields

$$\begin{aligned}\det\left(\lambda Y_1+\mu X_1\right)&=\lambda^3\left(\det S^{[2,6]}_{[1,5]}\mu^2+c_{11}(R,S)\lambda\mu+\det S^{[4,6]}_{[1,3]}\det R^{[1,2]}_{[5,6]}\lambda^2\right),\\ \det\left(\lambda Y_2+\mu X_2\right)&=\lambda^5\left(\det S^{[4,6]}_{[1,3]}\det R^{[1,4]}_{[3,6]}\mu^2+c_{21}(R,S)\lambda\mu+s_{16}\det R\lambda^2\right).\end{aligned}\tag{6.2}$$

Let us denote the functions associated with Φ_1, Φ_2 via (3.4) by φ_{1i}, $1\le i\le 5$, and φ_{2i}, $1\le i\le 7$, respectively. Taking into account that

$$\det\bar Y_1=r_{62}\det S^{[4,6]}_{[1,3]},\qquad \det\bar Y_2=s_{16}\det R^{[2,6]}_{[2,6]},$$

we obtain from (3.7) and (6.2)

$$\begin{aligned}\varphi_{11}\varphi^*_{11}&=\det S^{[2,6]}_{[1,5]}\det S^{[4,6]}_{[1,3]}r^2_{62}+c_{11}(R,S)r_{62}\varphi_{12}+\det R^{[1,2]}_{[5,6]}\varphi^2_{12},\\ \varphi_{21}\varphi^*_{21}&=s_{16}\det S^{[4,6]}_{[1,3]}\det R^{[1,4]}_{[3,6]}\left(\det R^{[2,6]}_{[2,6]}\right)^2+c_{21}(R,S)\det R^{[2,6]}_{[2,6]}\varphi_{22}+\det R\varphi^2_{22},\end{aligned}\tag{6.3}$$

where φ^*_{11} and φ^*_{21} are polynomial in the entries of R, S.

Recall that the family

$$\mathcal F^{\rm st}_n=\left\{\{g_{ij}(R)\}_{1\le j\le i\le n},\ \{h_{ij}(R)\}_{1\le i<j\le n}\right\}$$

with g_{ij} and h_{ij} defined in the previous section is a cluster for the standard cluster structure on GL_n that has a property that for every pair i,j of indices between 1 and n there is a unique function in $\mathcal F^{\rm st}_n$ represented by a minor whose upper left entry is r_{ij}. These functions are attached in a natural way to vertices of the corresponding quiver, $Q^{\rm st}_n$, that form an $n\times n$ grid with all the vertices in the 1st row and column frozen. The edges $(i,j)\to(i+1,j+1)$, $(i+1,j)\to(i,j)$, and $(i,j+1)\to(i,j)$ form a mesh of consistently oriented triangles (except that edges between frozen variables are ignored).

Let now $\mathcal F_\Gamma$ be the family of functions that consists of all distinct dense trailing minors of matrices that comprise $\mathcal L_\Gamma(R,R)$. Alternatively, we can describe $\mathcal F_\Gamma$ as

$$\begin{aligned}\mathcal F_\Gamma=\Big(\mathcal F^{\rm st}_6\setminus\big\{\{g_{i+1,i}(R)\}_{1\le i\le 5},\ g_{61}(R),\{h_{i,i+2}(R)\}_{1\le i\le 4},h_{15}(R),h_{26}(R)\big\}\Big)\\ \cup\left\{\{\varphi_{1i}(R,R)\}_{1\le i\le 4},\{\varphi_{2i}(R,R)\}_{1\le i\le 6}\right\}.\end{aligned}$$

Note that $\mathcal F_\Gamma$ contains only 34 functions in contrast with $\mathcal F^{\rm st}_6$, which contains 36. Specifically, none of the functions in $\mathcal F_\Gamma$ is represented as a minor whose upper left entry is r_{26} or r_{46}. All other r_{ij} do appear in this way, and so we attach them to the

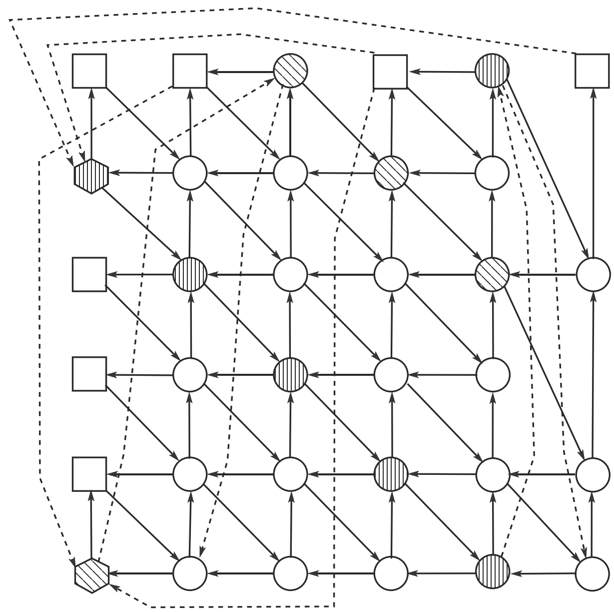

Fig. 5. Quiver Q_Γ.

corresponding nodes of a 6×6 grid that will serve as the vertex set of the quiver Q_Γ depicted in Figure 5. Here, the white vertices denote functions in the intersection $\mathcal{F}_\Gamma \cap \mathcal{F}_6^{\rm st}$, the ones with the vertical filling refer to φ_{1i}, and the ones with the diagonal filling, to φ_{2i}. The special vertices $(6, 1)$ and $(2, 1)$ correspond to φ_{11} and φ_{21}, respectively. Strings of exchange coefficients attached to these vertices are $(1, c_{11}(R, R), 1)$ and $(1, c_{21}(R, R), 1)$, respectively. These are the only nontrivial strings in the set of strings $\mathcal{P}_\Gamma$ that we associated with Q_Γ and $\mathtt{F}_\Gamma$. The corresponding generalized exchange relations are obtained from (6.3):

$$\varphi_{11}\varphi_{11}^* = h_{12}h_{14}g_{62}^2 + c_{11}(R, R)g_{62}\varphi_{12} + g_{51}\varphi_{12}^2,$$

$$\varphi_{21}\varphi_{21}^* = h_{16}h_{14}g_{31}g_{22}^2 + c_{21}(R, R)g_{22}\varphi_{22} + g_{11}\varphi_{22}^2.$$

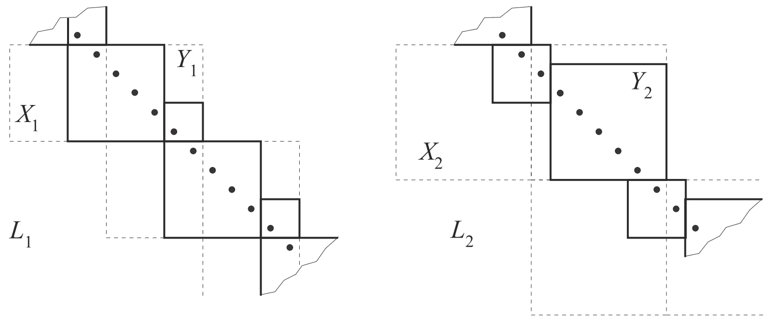

Fig. 6. Another partition of matrices L_1 and L_2.

Proposition 6.1. The seed $\Sigma_\Gamma = (\mathcal{F}_\Gamma, Q_\Gamma, \mathcal{P}_\Gamma)$ defines a regular complete generalized cluster structure in the ring of regular functions on GL_6. This structure is compatible with the Poisson–Lie bracket $\{\cdot,\cdot\}_\Gamma$ specified by Γ given by (6.1).

Proof. The proof is based on lengthy calculations, some of them straightforward, some *ad hoc*, and some relying on symbolic computations using Maple . In particular, the proof of regularity relies on Theorem 3.2 and identities (4.1),(4.2), and (4.3), just like in the proofs of Theorem 5.1 and Theorem 4.1. The proof of compatibility of Σ_Γ with $\{\cdot,\cdot\}_\Gamma$ is Maple assisted. To prove completeness, we constructed sequences of mutations that recover matrix entries x_{51}, x_{61} and x_{ij}, $i = 1,3,4,5,6$, $j = 2,3,4,6$, as cluster variables. For each of the remaining matrix entries, we found two Laurent polynomial expressions of the form $\frac{M}{f}$, where $M \in \overline{\mathcal{A}}(\Sigma_\Gamma)$ and f's entering two expressions for the same matrix element are coprime cluster variables. By [8, Lemma 8.3], this guarantees that matrix entries in question belong to $\overline{\mathcal{A}}(\Sigma_\Gamma)$. We omit the details of the proof since the general case of generalized cluster structures associated with Poisson brackets that arise in the Belavin–Drinfeld classification will be treated in a follow-up to [10]. ■

Note that the choice of the periodic staircase structure for the matrices L_1 and L_2 is not unique. Each one of them admits one more such structure, as shown in Figure 6.

Any pair of choices presented on Figures 4 and 6 gives rise to a regular complete generalized cluster structure compatible with the Poisson–Lie structure $\{\cdot,\cdot\}_\Gamma$. It is interesting to investigate whether the seeds thus obtained are mutation equivalent.

7 Properties of Core Minors

7.1 Expressing core minors via U

Proof of Lemma 3.3. Using block-column operations, we obtain

$$\varphi_1 = (\det Y)^{k-1} \det \begin{bmatrix} (\mathbf{1}_n)_{[2,n]} & & & & & \\ \begin{bmatrix} W_{[2,a]} \\ 0 \end{bmatrix} & (\mathbf{1}_n)_{[2,n]} & & & & \\ & \ddots & \ddots & & & \\ & & & \begin{bmatrix} W_{[2,a]} \\ 0 \end{bmatrix} & (\mathbf{1}_n)_{[2,n]} & \\ & & & & W_{[2,a]} & Y^{[1,b]}_{[2,a]} \end{bmatrix}.$$

In the 2nd determinant above, there are rows containing a single nonzero entry equal to 1. Removing these rows and corresponding columns, we can further rewrite it as

$$\begin{aligned} &\varepsilon \det \begin{bmatrix} W^{[1]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} & & & \\ & \ddots & \ddots & & \\ & & W^{[1,a]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} & \\ & & & W^{[1,a]}_{[2,a]} & Y^{[1,b]}_{[2,a]} \end{bmatrix} \\ &= \varepsilon \det Y_2 \det \begin{bmatrix} W^{[1]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} & & & \\ & \ddots & \ddots & & \\ & & W^{[1,a]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} & \\ & & & \begin{bmatrix} U_{[2,k]} & \star \\ W_{21} & W_{22} \end{bmatrix} & \begin{bmatrix} 0 \\ \mathbf{1}_b \end{bmatrix} \end{bmatrix} \\ &= \varepsilon \det Y_2 \det \begin{bmatrix} W^{[1]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} & & \\ & \ddots & \ddots & \\ & & W^{[1,a]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} \\ & & & \begin{bmatrix} U_{[2,k]} & \star \end{bmatrix} \end{bmatrix} \end{aligned} \tag{7.1}$$

with $\varepsilon = (-1)^{n-1+(n-a)([k/2]-1)}$. The 1st equality above is obtained by multiplying the last block row on the left by $\begin{bmatrix} \mathbf{1}_k & -Y_1Y_2^{-1} \\ 0 & \mathbf{1}_b \end{bmatrix}^{[2,a]}_{[2,a]}$.

Next, transform the matrix featured in (7.1) by multiplying the last block column on the right by $W^{[1,a]}$ and subtracting it from the previous one, then multiplying the $(k-2)$nd block column on the right by $W^{[1,a]}$ and subtracting it from the $(k-3)$rd one, etc., finally, multiplying the 2nd block column by $W^{[1]}$ and subtracting it from the 1st block column. The resulting matrix equals

$$\begin{bmatrix} 0 & (\mathbf{1}_a)_{[2,a]} & & \\ 0 & 0 & \ddots & \\ & & & (\mathbf{1}_a)_{[2,a]} \\ V_{k-2}^{[1]} & \cdots & V_1 & \begin{bmatrix} U_{[2,k]} & \star \end{bmatrix} \end{bmatrix} \tag{7.2}$$

with $V_i = (-1)^i \left[U_{[2,k]} \star\right] \left(W^{[1,a]}\right)^i$ for $i = 1, \ldots, k-2$. Note that for $j = 1, \ldots, k$ we have

$$\begin{aligned} V_i^{[j]} &= (-1)^i \left(\begin{bmatrix} \mathbf{1}_k & -Y_1 Y_2^{-1} \\ 0 & \mathbf{1}_b \end{bmatrix} \left(W^{[1,a]}\right)^{i+1} \right)_{[2,k]}^{[j]} \\ &= (-1)^i \left(\begin{bmatrix} \mathbf{1}_k & -Y_1 Y_2^{-1} \\ 0 & \mathbf{1}_b \end{bmatrix} \left(W^{[1,a]}\right)^{i+1} \begin{bmatrix} \mathbf{1}_k & -Y_1 Y_2^{-1} \\ 0 & \mathbf{1}_b \end{bmatrix} \right)_{[2,k]}^{[j]}, \end{aligned}$$

and hence (3.5) implies

$$V_i^{[j]} = (-1)^i \left(U^{i+1}\right)_{[2,k]}^{[j]}. \tag{7.3}$$

This means that the determinant of the matrix in (7.2) equals

$$\varepsilon' \det\left[V_{k-2}^{[1]} \ldots V_1^{[1]} U_{[2,k]}^{[1]}\right] = (-1)^{k-1} \varepsilon' \det\left[U^{k-1} e_1 \ldots U^2 e_1 \; U e_1 \; e_1\right]$$

with $\varepsilon' = (-1)^{(k-1)(k-2)/2+(a-1)([k/2]-1)}$, and the claim of the lemma follows since $(-1)^{k-1}\varepsilon\varepsilon' = (-1)^{k(k-1)/2+(n-1)[k/2]} = \varepsilon_1$. ■

Proof of Lemma 3.4. Define $\bar{W}$ via $X^{[2,n]}\bar{Y}^{-1} = \begin{bmatrix} \bar{W} \\ 0_{(n-a)\times(n-1)} \end{bmatrix}$. We proceed as in the proof of Lemma 3.3 and get

$$\varphi_2 = (\det Y)^{k-2} \det \bar{Y} \det \begin{bmatrix} (\mathbf{1}_{n-1})_{[3,n]} & & & & \\ \begin{bmatrix} \bar{W}_{[2,a]} \\ 0 \end{bmatrix} & (\mathbf{1}_n)_{[2,n]} & & & \\ & \ddots & \ddots & & \\ & & \begin{bmatrix} W_{[2,a]} \\ 0 \end{bmatrix} & (\mathbf{1}_n)_{[2,n]} & \\ & & & W^{[1,a]}_{[2,a]} & Y^{[1,b]}_{[2,a]} \end{bmatrix}$$

$$= -\varepsilon\, (\det Y)^{k-2} \det \bar{Y} \det Y_2 \det \begin{bmatrix} \bar{W}^{[1]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} & & \\ & \ddots & \ddots & \\ & & W^{[1,a]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} \\ & & & \begin{bmatrix} U_{[2,k]} & \star \end{bmatrix} \end{bmatrix},$$

where ε is the same as in (7.1). Similarly to the proof of Lemma 3.3, this yields

$$\varphi_2 = (-1)^{k-1}\varepsilon_1\, (\det Y)^{k-2} \det \bar{Y} \det Y_2 \det \left[w\ U^{k-2}e_1 \ldots U^2 e_1\ Ue_1\ e_1 \right],$$

where $w = (-1)^{k-2} \begin{bmatrix} U_{[2,k]} & \star \end{bmatrix} \left(W^{[1,a]}\right)^{k-3} \bar{W}^{[1]}$.

Next, factor Y as $Y = \begin{bmatrix} 1 & \star \\ 0 & \mathbf{1}_{n-1} \end{bmatrix} \begin{bmatrix} \star & 0 \\ \star & \bar{Y} \end{bmatrix}$. Then

$$XY^{-1} = \begin{bmatrix} 0\ X^{[2,n]} \end{bmatrix} \begin{bmatrix} \star & 0 \\ \star & \bar{Y}^{-1} \end{bmatrix} \begin{bmatrix} 1 & \star \\ 0 & \mathbf{1}_{n-1} \end{bmatrix}$$

implies $W = \begin{bmatrix} W^{[1]} & \bar{W} \end{bmatrix} \begin{bmatrix} 1 & \star \\ 0 & \mathbf{1}_{n-1} \end{bmatrix}$. Consequently, $\bar{W}^{[1]} = W^{[2]} + \gamma W^{[1]}$, where γ is given by (3.8). It remains to use (7.3) to get $w = (-1)^{k-2}U^{k-1}(e_2 + \gamma e_1) = (-1)^{k-2}U^{k-2}v$. ■

Proof of Lemma 3.5. Similar considerations show that

$$\varphi_{n+1} = (\det Y)^{k-2} \det \begin{bmatrix} (\mathbf{1}_n)_{[3,n]} & & & & \\ \begin{bmatrix} W_{[2,a]} \\ 0 \end{bmatrix} & (\mathbf{1}_n)_{[2,n]} & & & \\ & \ddots & \ddots & & \\ & & \begin{bmatrix} W_{[2,a]} \\ 0 \end{bmatrix} & (\mathbf{1}_n)_{[2,n]} & \\ & & & W_{[2,a]} & Y^{[1,b]}_{[2,a]} \end{bmatrix}$$

$$= \bar{\varepsilon}\, (\det Y)^{k-2} \det Y_2 \det \begin{bmatrix} W^{[1,2]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} & & \\ & \ddots & \ddots & \\ & & W^{[1,a]}_{[2,a]} & (\mathbf{1}_a)_{[2,a]} \\ & & & \begin{bmatrix} U_{[2,k]} & \star \end{bmatrix} \end{bmatrix}$$

with $\bar{\varepsilon} = (-1)^{(n-a)([k/2]-1)}$, similarly to (7.1). This leads to an analog of (7.2), which yields

$$\varphi_{n+1} = \bar{\varepsilon}\bar{\varepsilon}'\, (\det Y)^{k-2} \det Y_2 \det \left[U^{k-2}e_1 \; U^{k-2}e_2 \; U^{k-3}e_1 \ldots U^2 e_1 \; Ue_1 \; e_1 \right]$$

$$= \varepsilon_{n+1}\, (\det Y)^{k-2} \det Y_2 \det \left[U^{k-2}e_1 \; U^{k-3}v_\gamma \; U^{k-3}e_1 \ldots U^2 e_1 \; Ue_1 \; e_1 \right],$$

where $\bar{\varepsilon}' = (-1)^{(a-1)([k/2]-1)+k(k-1)/2-1}$. ■

7.2 Irreducibility of core minors

Proof of Lemma 3.8. For X, Y given by (3.2), we say that φ_1 is of *type* (n, k, b). The three parameters satisfy conditions

$$n \geq k + b, \qquad k \geq 2, \qquad b \geq 0.$$

The proof of irreducibility is based on induction on all the parameters.

For type $(n, 2, 0)$, the irreducibility of φ_1 is a well-known fact, since the corresponding core is an $n \times n$ matrix of independent variables. For type $(3, 3, 0)$, we have

$$\varphi_1 = \begin{vmatrix} y_{21} & y_{22} & y_{23} & 0 & 0 & 0 \\ y_{31} & y_{32} & y_{33} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & y_{21} & y_{22} & y_{23} \\ x_{31} & x_{32} & x_{33} & y_{31} & y_{32} & y_{33} \\ 0 & 0 & 0 & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 0 & x_{31} & x_{32} & x_{33} \end{vmatrix},$$

and its irreducibility can be verified by direct observation. In a similar way, one can treat the case $(4, 3, 0)$.

Let now φ_1 be of type $(n, 3, 0)$ with $n > 4$. Note that φ_1 is a homogeneous polynomial of degree 2 in each variable. Assume that $\varphi_1 = PQ$, then both P and Q are homogeneous. Let $y = y_{41}$. Note that the coefficient c_y at y^2 in φ_1 equals $\pm\varphi_1(X^{[2,n]}, Y^{[2,n]}_{[1,3]\cup[5,n]})$; the latter is of type $(n-1, 3, 0)$, and hence is irreducible by induction. Consequently, $P = c_y y^p + o(y^p)$ and $Q = y^q + o(y^q)$ with $p + q = 2$.

Further, c_y has degree 2 in $z = y_{52}$, and hence $\deg_z P = 2$, $\deg_z Q = 0$. Similarly to the above, the coefficient c_z at z^2 in φ_1 is an irreducible polynomial of degree 2 in y, and we conclude that $p = 2$, $q = 0$, and hence Q is a constant.

Let now φ_1 be of type $(n, k, 0)$ with $k > 3$. Note that φ_1 is a homogeneous polynomial of degree $k - 1$ in each variable. Assume that $\varphi_1 = PQ$, then both P and Q are homogeneous. Let $y = y_{21}$. Note that the coefficient at y^{k-1} in φ_1 equals $\pm\psi_1 \det Z_1$, where $\psi_1 = \varphi_1(\bar{X}, \bar{Y})$ with $\bar{X} = X^{[2,n]}_{[2,n]}$ and Z_1 is an $(n-1) \times (n-1)$ matrix $\begin{bmatrix} Y^{[2,n]}_{[k+1,n]} \\ X^{[2,n]}_{[2,k]} \end{bmatrix}$. Note that ψ_1 is a core determinant of type $(n-1, k-1, 0)$, and hence is irreducible by induction, whereas $\det Z_1$ is irreducible as the determinant of a matrix of independent variables. Consequently, either

(i) $P = \psi_1 y^p + o(y^p)$ and $Q = \pm \det Z_1 y^q + o(y^q)$ with $p + q = k - 1$, or

(ii) $P = \psi_1 \det Z_1 y^p + o(y^p)$ and $Q = \pm y^q + o(y^q)$ with $p + q = k - 1$.

In any case, the total degree of P is at least $(k-2)(n-1) + p$, and the total degree of Q is at most $n - 1 + q$.

Similarly to the treatment of the case $(n, 3, 0)$ above, we let $z = y_{32}$ and note that $\deg_z \psi_1 = k - 2$, and hence $\deg_z P \geq k - 2$ and $\deg_z Q \leq 1$. The same reasoning as above shows that the coefficient at z^{k-1} in φ_1 equals $\pm\psi' \det Z'$, where ψ' is a core determinant of type $(n-1, k-1, 0)$, and hence irreducible by induction, and $\det Z'$ is the determinant of an $(n-1)\times(n-1)$ matrix of independent variables. Consequently, $P = c'z^{p'} + o(p')$ with $p' \geq k-2$, and there are four possibilities for c':

(a) $c' = \alpha'\psi' \det Z'$,

(b) $c' = \alpha'\psi'$,

(c) $c' = \alpha' \det Z'$,

(d) $c' = \alpha'$,

where α' is a nonzero constant.

The last two possibilities are ruled out immediately, since they imply that the total degree of P is at most $n-1+p'$, which is strictly less than $(k-2)(n-1)+p$. In case (ia), the comparison of the two expressions for the total degree of P gives $(k-2)(n-1)+p = (k-1)(n-1)+p'$, which is equivalent to $p = n-1+p'$, and hence is impossible. Similarly, case (iib) yields $n-1+p = p'$, which can be satisfied only if $p=0$, $p' = k-1$, and $n=k$. However, $p=0$ in case (iib) means that $P = \psi_1 \det Z_1$, and hence $p' = \deg_z P = k-2$, a contradiction. In the remaining cases (ib) and (iia), we get $p = p' \geq k-2$ and $q = q' \leq 1$.

Assume first that $q=0$. In case (i), we get $Q = \pm \det Z_1$, and, simultaneously, $Q = \alpha' \det Z'$, a contradiction, since $Z' = \begin{bmatrix} Y^{1\cup[3,n]}_{[k+1,n]} \\ X^{1\cup[3,n]}_{[2,k]} \end{bmatrix}$. In case (ii), we get that Q is a constant. So, in what follows, we assume that $p = k-2$ and $q=1$.

Let now $t \neq y$ be an arbitrary entry in the 2nd row of Y or X. Applying the same reasoning as above, we get that $\deg_t P = k-2$, $\deg_t Q = 1$, and the coefficients at t^{k-2} in P and at t in Q have a similar structure, that is, all of them simultaneously look either as in case (i), or as in case (ii).

Assume that all coefficients are as in case (i). Note that ψ_1 and all its analogs do not depend on the entries of the 2nd row of Y. Consequently, one can write

$$P = \sum_{j=1}^{n} y_{2j}^{k-2}\alpha_j\psi_j + R + S,$$

where ψ_j are core determinants of type $(n-1, k-1, 0)$ depending on submatrices of X and Y, α_j are nonzero constants with $\alpha_1 = 1$, R contains all monomials in P that depend on entries in the 2nd row of Y that are not included in the 1st sum, and S contains only the monomials that do not depend on these entries. Besides, we can write

$$Q = \sum_{j=1}^{n} y_{2j}\beta_j \det Z_j + T,$$

where Z_j are $(n-1)\times(n-1)$ matrices built of the entries in the rows $[k+1,n]$ of Y and rows $[2,k]$ of X similarly to Z_1, $\beta_j=\pm\alpha_j^{-1}$, and T does not depend on the entries in the 2nd row of Y. Consequently, $TS=0$, since φ_1 does not contain monomials that do not depend on the entries of the 2nd row of Y. Note that S does not vanish since for every entry t in the 2nd row of X, $\deg_t P=k-2$ and the coefficient at t^{k-2} in P does not depend on the entries in the 2nd row of Y. Therefore, $T=0$ and

$$Q=\sum_{j=1}^{n} y_{2j}\beta_j \det Z_j. \tag{7.4}$$

Let us fix $t=x_{21}$. Recall that $\deg_t Q=1$. Similarly to the treatment of y above, $Q=\bar\beta_1 t\det\bar Y+o(t)$, where $\bar\beta_1$ is a nonzero constant. On the other hand, it follows from (7.4) that $Q=t\det\bar Z_1+o(t)$, where

$$\bar Z_1=\begin{bmatrix} \beta_2 y_{22} & \beta_3 y_{23} & \dots & \beta_n y_{2n}\\ y_{k+1,2} & y_{k+1,3} & \dots & y_{k+1,n}\\ \vdots & & & \vdots\\ y_{n2} & y_{n3} & \dots & y_{nn}\\ x_{32} & x_{33} & \dots & x_{3n}\\ \vdots & & & \vdots\\ x_{k2} & x_{k3} & \dots & x_{kn} \end{bmatrix},$$

a contradiction.

Assume now that all coefficients are as in case (ii). Then the same treatment as in case (i) leads to $Q=\sum_{j=1}^n y_{2j}\beta_j$ for some nonzero constants β_j, and hence the coefficient at x_{21} in Q vanishes, a contradiction.

To proceed further with the case $b>0$, we will need one more basic type, $(3,2,1)$, in which case

$$\varphi_1=\begin{vmatrix} y_{21} & y_{22} & y_{23} & 0\\ y_{31} & y_{32} & y_{33} & 0\\ 0 & x_{21} & x_{22} & y_{21}\\ 0 & x_{31} & x_{32} & y_{31} \end{vmatrix}$$

is irreducible via direct observation.

Let now φ_1 be of type (n,k,b) with $b>0$, and let $\varphi_1=PQ$. Put $y=y_{21}$ and note that $\deg_y\varphi_1=k$. It is easy to see that the coefficient $\bar\psi_2$ at y^k in φ_1 is itself a core determinant of type $(n-1,k,b-1)$, and hence is irreducible by induction. Consequently,

$P = y^p\bar{\psi}_2 + o(y^p)$ and $Q = \pm y^q + o(y^q)$ with $p + q = k$. In particular, the total degree of P is $(k-1)(n-1) + b - 1 + p$ and the total degree of Q is q.

Similarly, for $z = y_{31}$, we have $P = \alpha z^{p'}\bar{\psi}_3 + o(z^{p'})$ and $Q = \beta z^{q'} + o(z^{q'})$ with $p' + q' = k$, where $\bar{\psi}_3$ is a core determinant of type $(n-1, k, b-1)$ and $\alpha\beta = \pm 1$ (the opposite case would imply $(k-1)(n-1)b - 1 + p = q'$, which is impossible). Total degrees of $\bar{\psi}_2$ and $\bar{\psi}_3$ coincide, so $p = p'$ and $q = q'$. Consequently, $p > 0$, since otherwise $P = \bar{\psi}_2 = \alpha\bar{\psi}_3$, a contradiction.

Consider first the case $b = 1$ and $n = k + 1 > 3$. Let $t = y_{3n}$, then $\deg_t \varphi_1 = \deg_t \bar{\psi}_2 = k - 1$, and the coefficient at t^{k-1} in φ_1 equals $\bar{\psi} \det \bar{Z}$, where $\bar{\psi}$ is a core determinant of type $(k, k-1, 1)$ and hence is irreducible by induction, and $\bar{Z} = [X^{[1,k]}_{[2,k+1]}\ Y^{[1]}_{[2,k+1]}]$ and hence $\det \bar{Z}$ is irreducible as the determinant of a matrix of independent variables. Consequently, we have four possibilities similar to (a)–(d) above. The last two are ruled out via total degree comparison, since $k^2 - k + p > 2k$ for $k > 2$ and $p > 0$. The 2nd one yields $p = 1$, in which case $Q = \det \bar{Z}$ and hence $\deg_y Q = 1 < k - 1 = q$, a contradiction. The remaining case yields $p = k$ and $q = 0$, hence Q is a constant.

For $b = 1$ and $n > k + 1$ take $t = y_{n2}$ and note that $\deg_t \varphi_1 = k - 1$ and the coefficient $\bar{\psi}_n$ at t^{k-1} in φ_1 is a core determinant of type $(n-1, k, 1)$ and hence is irreducible by induction. Moreover, $\deg_y \bar{\psi}_n = k$, and hence $p = k$, $q = 0$ and Q is a constant.

Finally, for type (n, k, b) with $b > 1$, take $u = y_{32}$, then similarly to above, $P = \bar{\alpha}' u^{\bar{p}'} \bar{\psi}' + o(u^{\bar{p}'})$, where $\bar{\psi}'$ is a core determinant of type $(n-1, k, b-1)$ and hence is irreducible by induction. Moreover, $\deg_y \bar{\psi}' = k$, and hence $p = k$, $q = 0$ and Q is a constant. ■

Proof of Lemma 4.2. For φ_1, this fact is proved in Lemma 3.8 (cp. to the case of type $(n, n, 0)$). For other functions φ_i the proof is similar. It exploits the fact that one can find two variables y and z such that the coefficient at the highest degree of the variable in φ_i is either an irreducible polynomial or a product of two such polynomials and that the highest degree of z in φ_i equals to the highest degree of z in one of the above two polynomials for y.

In more detail, for $2 \le i \le n-1$, one takes $y = x_{n-1,i}$ and $z = x_{n,i+1}$. Then $\deg_y \varphi_i = \deg_z \varphi_i = n - 1$ and the coefficients at y^{n-1} and z^{n-1} in φ_i are equal to $\psi \det Z$, where ψ is φ_1 for the size $n-1$ and Z is an $(n-i)\times(n-i)$ matrix of independent variables.

For the case of φ_{pn+i}, $1 \le i \le n-1$, $1 \le p \le n-3$, one takes $y = x_{n-1,i}$ and $z = x_{n,i+1}$. Then, $\deg_y \varphi_{pn+i} = \deg_z \varphi_{pn+i} = n - p - 1$ and the coefficients at y^{n-p-1} and z^{n-p-1} in φ_{pn+i} are equal to $\varphi_{(p-1)(n-1)+i}$ for the size $n-1$.

Finally, for the case of φ_{pn}, $1 \le p \le n-2$, one takes $y = x_{n-1,n}$ and $z = y_{n1}$. Then $\deg_y \varphi_{pn} = n-p$ and $\deg_z \varphi_{pn} = n-p-1$, the coefficient at y^{n-p} in φ_{pn} equals $\varphi_{(p-1)(n-1)+1}$ for the size $n-1$, while the coefficient at z^{n-p-1} in φ_{pn} equals the product of $\varphi_{p(n-1)}$ for the size $n-1$ by the determinant of an $(n-1)\times(n-1)$ matrix of independent variables. Further details are left to the interested reader.

Irreducibility of the remaining functions in the family $\mathcal{F}_n$ is discussed in [9, Section 6.3]. ■

Proof of Lemma 5.2. Irreducibility of the functions in $\mathcal{F}_{2n}$ is trivial. For $k > 2$, we have to deal separately with functions $\tilde{c}_t$ and $\tilde{\varphi}_t$.

To prove irreducibility of $\tilde{c}_t$, $1 \le t \le k-1$, note that each such function is linear in all a_{ij}, $1 \le i \le k+1$, $1 \le j \le n$. Assume that $\tilde{c}_t = P_1 P_2$ and that P_1 is linear in $a_{k+1,1}$ (and hence P_2 does not depend on $a_{k+1,1}$). Moreover, P_1 depends linearly in all nonzero entries in the 1st row of X and in the 1st column of Y, whereas P_2 does not depend on any of these entries. Note that for any a_{ij} as above there exists a staircase sequence $a_{k+1,1} = a_{i_0 j_0}, a_{i_1 j_1}, \dots, a_{i_l j_l}, \dots, a_{i_r j_r} = a_{ij}$ such that $1 \le i_l \le k+1$, $1 \le j_l \le n$ for $1 < l < r$ and every consecutive pair $(a_{i_{l-1} j_{l-1}}, a_{i_l j_l})$ alternately lies in the same row or in the same column of the matrix $X+Y$. Moving along this sequence and applying the same reasoning as above, we consecutively get that P_1 is linear in $a_{i_1 j_1}, a_{i_2 j_2}, \dots, a_{ij}$, and hence P_2 does not depend on a_{ij}, which means that P_2 is a constant.

Irreducibility of $\tilde{\varphi}_t$ is proved similarly to the proof of Lemma 3.8. Below we sketch the proof for $\tilde{\varphi}_1$; cases $t > 1$ are treated in a similar way.

Let $\tilde{\varphi}_1$ be of type (n,k). Take $x = a_{k+1,n}$, then $\deg_x \tilde{\varphi}_1 = k-2$. The coefficient c^x at x^{k-2} in $\tilde{\varphi}_1$ equals $\psi_1^x a_{12} \cdots a_{1,k-1} D^x$ where $D^x = \det Y_{[k,n]}^{[1,n-k+1]}$. Consider ψ_1^x as a polynomial of degree $(k-2)(n-k-2)$ in variables $a_{1,k+1}, \dots, a_{1,n-1}$. The constant term of this polynomial is $\tilde{\varphi}_1$ of type $(n-1,k-1)$ for a shifted set of variables, and hence is irreducible by the induction hypothesis. Consequently, ψ_1^x is irreducible since it is homogeneous as a polynomial in all variables a_{ij}.

Assume that $\tilde{\varphi}_1 = P'P''$. It follows from above that $P' = x^d \psi_1^x R' + o(x^d)$ and $P'' = x^{k-2-d} R'' + o(x^{k-2-d})$ with $R'R'' = a_{12} \cdots a_{1,k-1} D^x$. Consequently, $\deg P' \ge (k-2)(n-2)+d$ and $\deg P'' \le n+k-d-3$.

Take $y = a_{k2}$, then $\deg_y \tilde{\varphi}_1 = k-1$ and $\deg_y P' \ge \deg_y \psi_1^x = k-2$, and hence $\deg_y P'' \le 1$. Further, the coefficient c^y at y^{k-1} in $\tilde{\varphi}_1$ equals $\psi_1^y a_{13} \cdots a_{1k} D^y$ where $D^y = \det Y_{[k+1,n]}^{[2,n-k+1]}$ and ψ_1^y has the same structure as ψ_1^x. If $\deg_y P'' = 0$, then P'' is a factor of

c^y, which is impossible. Consequently, $\deg_y P'' = 1$, and hence either

$$P' = y^{k-2}S' + o(y^{k-2}) \quad \text{and} \quad P'' = y\psi_1^y S'' + o(y) \quad \text{with} \quad S'S'' = a_{13}\cdots a_{1k}D^y$$

or

$$P' = y^{k-2}\psi_1^y S' + o(y^{k-2}) \quad \text{and} \quad P'' = yS'' + o(y) \quad \text{with} \quad S'S'' = a_{13}\cdots a_{1k}D^y.$$

In the 1st of the above two cases, we get $\deg P'' \ge (k-2)(n-2)+1$, which is strictly greater than $n+k-d-3$ for $k>3$. For $k=3$, either $\deg_x P' = 0$ or $\deg_x P'' = 0$, which is impossible for the same reason as $\deg_y P'' = 0$. Consequently, the 2nd case holds true, and hence $\deg_x P' = k-3$ and $\deg_x P' = 1$.Taking into account the reasoning above, we can write $P'' = xR'' + yS'' + T''$, where T'' does not depend on x and y.

Consider now the coefficient c at $x^{k-3}y^{k-1}$ in $\tilde{\varphi}_1$. On the one hand, c is equal to $c^{xy}R'S''$, where c^{xy} is the coefficient at y^{k-2} in ψ_1^x. The latter equals $\psi_1^{xy}a_{13}\cdots a_{1,k-1}D^{xy}$, where $D^{xy} = \det Y_{[k,n-1]}^{[2,n-k+1]}$. On the other hand, c is equal to $c^{yx}a_{13}\cdots a_{1k}D^y$, where c^{yx} is the coefficient at x^{k-3} in ψ_1^y. The latter equals $\psi_1^{yx}a_{13}\cdots a_{1,k-1}D^{yx}$, where $D^{yx} = \det Y_{[k,n]}^{[2,n-k+2]}$. It is easy to see that $\psi_1^{xy} = \psi_1^{yx}$, hence we arrive at $D^{xy}R'S'' = D^{yx}a_{13}\cdots a_{1k}D^y$, which is clearly impossible. ■

7.3 Coprimality results

Proof of Lemma 3.7 (i) Let A be a matrix with distinct nonzero eigenvalues: $A = C^{-1}\operatorname{diag}(\lambda_1,\dots,\lambda_k)C$ with $\det C \ne 0$. We follow the proof of [9, Proposition 8.1] and write

$$\det K(A;e_1) = \operatorname{Van}(\lambda_1,\dots,\lambda_k)\prod_{i=1}^k c_{i1}$$

and

$$\det K^*(A;e_1,A^{-1}(e_2+\gamma e_1)) = \operatorname{Van}(\lambda_1,\dots,\lambda_k)\prod_{i=1}^k w_i$$

where Van is the Vandermonde determinant and

$$w_i = \sum_{j\ne i} \pm(c_{j2}+\gamma c_{j1})\lambda_j^{-1}\operatorname{Van}(\lambda_1,\dots,\hat{\lambda}_i,\dots,\hat{\lambda}_j,\dots,\lambda_k)\prod_{m\ne i,j} c_{m1}.$$

Pick $c_{11} = 0$, then for any choice of $c_{i1} \neq 0$, $2 \leq i \leq k$, and $c_{12} \neq 0$ one has $\det K(A; e_1) = 0$ and

$$w_i = \pm c_{12}\lambda_1^{-1} \operatorname{Van}(\lambda_2, \ldots, \hat{\lambda}_i, \ldots, \lambda_k) \prod_{m \neq 1,i} c_{m1} \neq 0, \qquad 2 \leq i \leq k.$$

Next, pick $\lambda_i = t^i$. Then for t big enough the expression $\lambda_j^{-1} \operatorname{Van}(\lambda_2, \ldots, \hat{\lambda}_j, \ldots, \lambda_k)$ grows as $t^{\varkappa - j^2 + 2j}$ where $\varkappa$ depends only on k. Consequently, the leading term in the expression

$$w_1 = \sum_{j=2}^{k} \pm(c_{j2} + \gamma c_{j1})\lambda_j^{-1} \operatorname{Van}(\lambda_2, \ldots, \hat{\lambda}_j, \ldots, \lambda_k) \prod_{m \neq 1,j} c_{m1}$$

is obtained for $j = 2$, and it suffices to pick c_{22} such that $c_{22} + \gamma c_{21} \neq 0$ to guarantee $w_1 \neq 0$. The rest of c_{ij} can be picked arbitrarily to satisfy condition $\det C \neq 0$, which yields $\det K^*(A; e_1, A^{-1}(e_2 + \gamma e_1)) \neq 0$.

(ii) We have to refine the choice of c_{ij} made in the proof of part (i). Note that an arbitrary principal leading minor of A can be written as

$$\det A_I^I = \sum_K \pm \det(C^{-1})_I^K \prod_{i \in K} \lambda_i \det C_K^I = \frac{1}{\det C} \sum_K \pm \det C_{K^c}^{I^c} \prod_{i \in K} \lambda_i \det C_K^I,$$

where $I = \{1, 2, \ldots, |I|\}$ and $|K| = |I|$. Our choice of λ_i guarantees that for t big enough the leading term in the above expression is obtained when $K = \{k - |I| + 1, k - |I| + 2, \ldots, k\}$. Consequently, condition $\det A_I^I \neq 0$ is guaranteed by $\det C_K^I \neq 0$ and $\det C_{K^c}^{I^c} \neq 0$. Clearly, these conditions can be satisfied via a suitable choice of the entries c_{ij} distinct from c_{i1}, c_{12}, and c_{22}. ■

Proof of Lemma 4.3. The claim for $f = \varphi_1$ follows from Lemma 3.7(i). Indeed, fix an invertible Y such that $\det \bar{Y} \neq 0$ and define γ via (3.8). Next, pick A that satisfies the conditions in Lemma 3.7(i) and put $X = A^{-1}Y$. Consequently, $\varphi_1(X, Y) = \det K(A; e_1)(\det X)^{n-1} = 0$, while

$$\varphi_1^*(X, Y) = \pm \det K^*(A; e_1, A^{-1}\bar{v})(\det X)^{(n-1)(n-2)}(\det \bar{Y})^n \neq 0$$

via (3.11), and the claim follows.

For $f = \varphi_i$, $2 \leq i \leq n$, the claim is trivial, since in this range $\deg f^* < \deg f$. In the case $f = \varphi_i$, $n + 1 \leq i \leq N - n$, it follows from the explanations in the proof of Theorem 4.1 that $\varphi_i^* = \varphi_{i+n}\varphi_{i-n}^0 - \varphi_{i-n}\varphi_{i+n}^0$, where φ_t^0 is the minor of Φ obtained by replacing the

1st column of φ_t by the immediately preceding column. Consider the specialization that sets to zero the entry z (y_{st} or x_{st}) that occupies position (i,i) in Φ and all the entries of the matrices X and Y that lie in the same columns of Φ below z. It is easy to see that this specialization implies vanishing of φ_i and φ_{i+n}, since both minors acquire a zero column. However, the same specialization for φ_{i-n} and φ^0_{i+n} yields nontrivial polynomials since the coefficients at $\bar z^{n-p}$ in the 1st one and at $\bar z^{n-p-3}$ in the 2nd one are nontrivial, where $\bar z$ lies immediately above z in Φ and $p = \lfloor (i-1)/n \rfloor$. Consequently, $\tilde\varphi^*_i$ is not divisible by $\tilde\varphi_i$. The cases $f = g_{ii}$, $2 \le i \le n$, and $f = h_{ii}$, $3 \le i \le n$ are treated via the same specialization. For $f = h_{22}$, the specialization is given by $y_{nj} = 0$ for $2 \le j \le n$. ∎

Proof of Lemma 5.3. To prove the coprimality of $\tilde\varphi_t$ and $\tilde\varphi^*_t$, it suffices to check that the latter is not divisible by the former. For $t = 1$, we need the following statement.

Proposition 7.1. The image of the map $(X, Y) \mapsto U$ defined by (5.2) contains an arbitrary $k \times k$ matrix with nonzero trailing principal minors

Proof. In what follows B, B_1, $\bar B_1$ are $k \times k$ invertible upper triangular and N, $\bar N$, N_1 are $k \times k$ unipotent lower triangular. It suffices to show that for any B and N as above, there exist $n \times n$ matrices X, Y of the form (5.1) such that U defined by (5.2) is given by $U = BN$.

Let Y_0 of the form described in (5.1) be totally nonnegative with all combinatorially nontrivial minors nonzero and set $Y_1 = JY_0J$, where $J = \operatorname{diag}\left((-1)^i)_{i=0}^{n-1}\right)$. Then $M = \left(Y_1^{-1}\right)^{[1,k]}_{[n-k+1,n]}$ is totally nonnegative and invertible since $\det M = \det Y_1^{-1} \det Y_1{}^{[1,n-k]}_{[k+1,n]}$. Thus there exist N_1 and B_1 as above such that $M = N_1 B_1$.

It is not hard to see that there exists an invertible positive diagonal matrix D such that $NDB_1^{-1} = \bar B_1^{-1} \bar N$ for $\bar B_1$ and $\bar N$ as above. Let Y_2 be obtained via multiplying Y_1 on the left by an appropriate diagonal matrix so that that $\left(Y_2^{-1}\right)^{[1,k]}_{[n-k+1,n]} = MD^{-1}$. Now, let $Y = Y_2 \begin{bmatrix} \mathbf{1}_{n-k} & 0 \\ 0 & N_1 \bar N^{-1} \end{bmatrix}$ and $X = \begin{bmatrix} 0 & B\bar B_1^{-1} \\ 0 & 0 \end{bmatrix}$. Then X, Y are of the required form, $\left(Y^{-1}\right)^{[1,k]}_{[n-k+1,n]} = \bar N N_1^{-1} MD^{-1} = \bar N B_1 D^{-1}$ and (5.2) gives $U = B\bar B_1^{-1} \bar N B_1 D^{-1} = BNDB_1^{-1} B_1 D^{-1} = BN$. ∎

To complete the proof for $t = 1$, we invoke Lemma 3.7(ii), which guarantees that one can choose A in Lemma 3.7(i) in such a way that the trailing principal minors of A^{-1} are nonzero, and proceed as in the proof of Lemma 4.3.

For $2 \le t \le n$, the claim is trivial, since in this case $\deg \tilde\varphi^*_t < \deg \tilde\varphi_t$.

Let now $n < t < (k-2)(n-1)$. Then it follows from the explanations above that $\tilde{\varphi}^*_t = \tilde{\varphi}_{t+n}\tilde{\varphi}^0_{t-n} - \tilde{\varphi}_{t-n}\tilde{\varphi}^0_{t+n}$, same as in the previous section. Find the unique pair (i,j), $1 \le i \le n-1$, satisfying $t = (k-j)(n-1)+i$ and consider the specialization $a_{ij} = a_{i-1,j+1} = \cdots = a_{i-p,j+p} = \cdots = a_{1,j+i-1} = 0$, where the index $j+p$ is understood mod n and its value $j+p = 1$ is excluded. It is easy to see that this specialization implies vanishing of $\tilde{\varphi}_t$ and $\tilde{\varphi}_{t+n}$, since both minors acquire a zero column. However, the same specialization for $\tilde{\varphi}_{t-n}$ and $\tilde{\varphi}^0_{t+n}$ yields nontrivial polynomials since the coefficients at $a^j_{i+1,j-1}$ in the 1st one and at $a^{j-3}_{i+1,j-1}$ in the 2nd one are nontrivial. Consequently, $\tilde{\varphi}^*_t$ is not divisible by $\tilde{\varphi}_t$.

Finally, let $n \ge (k-2)(n-1)$. Then it follows from the explanations above that $\tilde{\varphi}^*_t = \tilde{\varphi}^0_{t-n}$, and the same specialization as above proves that $\tilde{\varphi}^*_t$ is not divisible by $\tilde{\varphi}_t$. ■

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