

# OPTIMAL CONTROL OF PARAMETERIZED STATIONARY MAXWELL'S SYSTEM: REDUCED BASIS, CONVERGENCE ANALYSIS, AND A POSTERIORI ERROR ESTIMATES

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**ABSTRACT.** We consider an optimal control problem governed by parameterized stationary Maxwell's system with the Gauss's law. The parameters enter through dielectric, magnetic permeability, and charge density. Moreover, the parameter set is assumed to be compact. We discretize the electric field by a finite element method and use variational discretization concept for the control. We present a reduced basis method for the optimal control problem and establish the uniform convergence of the reduced order solutions to that of the original full-dimensional problem provided that the snapshot parameter sample is dense in the parameter set, with an appropriate parameter separability rule. Finally, we establish the absolute a posteriori error estimator for the reduced order solutions and the corresponding cost functions in terms of the state and adjoint residuals.

**1. Introduction.** Maxwell's equations with the Gauss's law play a central role in many day-to-day applications. However, the underlying coefficients in these equations, such as dielectric, magnetic permeability, and charge density contain parameters which must be inferred from experiments or treated as random variables. In many cases, these parameterized equations must be queried for different parameters, many times over and thus the problem quickly becomes intractable. This issue is only exasperated when dealing with optimization problems with such parameterized equations as constraints. The goal of this paper is to present numerical

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analysis, using reduced basis method, for the optimal control problem governed by stationary Maxwell's system with the Gauss's law as constraints.

We discretize these equations using a finite element method and carry out a variational discretization for the control. The finite element system for the PDE is a parameterized constrained saddle point system. It can be very expensive to solve, especially on fine meshes and for many parameter queries (cf. [23, 37, 18, 19]). From a reduced basis point of view, one needs a surrogate model for the system. Furthermore, since the reduced basis approach considers a suboptimal problem, convergence analysis and error estimates for the reduced order solution to that of the original full-dimensional problem are crucial, which are investigated in the present paper.

For completeness, we mention that the optimal control problems governed by the *non-parameterized* Maxwell systems have attracted a great deal of attention from many scientists in the last decades. For surveys on the subject, we refer the reader to, e.g., [7, 10, 31, 35, 36, 40, 41, 42, 43, 44] and the references therein. The construction of the reduced basis methods for parameterized Maxwell systems can be found in [6, 11, 12, 20, 21, 22]. Moreover, an incomplete list of references that considers the analysis of parameterized optimal control problems (not Maxwell) can be found in [4, 15, 27, 29, 30, 34, 38].

To the best of our knowledge, the optimal control of such a system is not known. The main results of our paper are contained in Theorem 5.5, where we prove the uniform convergence of reduced order solution, to the optimal control problem, to that of the original full-dimensional problem, and in Theorem 6.3 where we establish the absolute a posteriori error estimator for the reduced order solutions. Numerical implementation will be part of a future work.

The remainder of the paper is organized as follows. In Section 2, we state the problem under consideration. Section 3 is devoted to some functional spaces and the finite element method for the system 1. Primal reduced basis approach for the optimal control problem and first order optimality conditions are presented in Section 4. Convergence analysis and a posteriori error estimates for the reduced basis approximations are respectively discussed in Section 5 and Section 6.

**2. Problem formulation.** Let  $\Omega \subset \mathbb{R}^3$  be an open, bounded, nonempty domain with a connected Lipschitz-continuous boundary. Moreover, let  $\mathcal{P} \subset \mathbb{R}^p$  denotes the set of parameters, we assume that  $\mathcal{P}$  is compact. In this paper we deal with the following  $\mu$ -parameterized stationary Maxwell's system fulfilled by the electric field  $\mathbf{E}$ :

$$\begin{cases} \nabla \times (\sigma^{-1}(\mathbf{x}; \mu) \nabla \times \mathbf{E}(\mathbf{x}; \mu)) &= \epsilon(\mathbf{x}; \mu) \mathbf{u}(\mathbf{x}), & (\mathbf{x}; \mu) \in \Omega \times \mathcal{P}, \\ \nabla \cdot (\epsilon(\mathbf{x}; \mu) \mathbf{E}(\mathbf{x}; \mu)) &= \rho(\mathbf{x}; \mu), & (\mathbf{x}; \mu) \in \Omega \times \mathcal{P}, \\ \mathbf{E}(\mathbf{x}; \mu) \times \vec{\mathbf{n}}(\mathbf{x}) &= \mathbf{0}, & (\mathbf{x}; \mu) \in \partial\Omega \times \mathcal{P}, \end{cases} \quad (1)$$

where  $\vec{\mathbf{n}} := \vec{\mathbf{n}}(\mathbf{x})$  is the unit outward normal on  $\partial\Omega$ . In 1 the dielectric  $\epsilon := \epsilon(\mathbf{x}; \mu)$ , the magnetic permeability  $\sigma := \sigma(\mathbf{x}; \mu)$  and the charge density  $\rho := \rho(\mathbf{x}; \mu)$  are assumed to be known with

$$\underline{\rho} \leq \rho(\mathbf{x}; \mu) \leq \bar{\rho}, \quad \underline{\epsilon} \leq \epsilon(\mathbf{x}; \mu) \leq \bar{\epsilon} \quad \text{and} \quad \underline{\sigma} \leq \sigma(\mathbf{x}; \mu) \leq \bar{\sigma} \quad (2)$$

a.e. in  $\mathbf{x} \in \Omega$ , all  $\mu \in \mathcal{P}$  for some given constants  $\underline{\rho}, \bar{\rho}, \underline{\epsilon}, \bar{\epsilon}, \underline{\sigma}$ , and  $\bar{\sigma}$  independent of both  $\mathbf{x}$  and  $\mu$ , where  $\underline{\epsilon} > 0$  and  $\underline{\sigma} > 0$ . Furthermore, we assume that Gauss's law is applied to the current source, i.e.

$$\nabla \cdot (\epsilon(\mathbf{x}; \mu) \mathbf{u}(\mathbf{x})) = 0, \quad (\mathbf{x}; \mu) \in \Omega \times \mathcal{P}. \quad (3)$$

The function  $\mathbf{u}$  denotes the control variable. For

$$\mathbf{u} \in \mathbf{U}_{ad}(\mu) := \left\{ \mathbf{u} := (u_i)_{i=1}^3 \in \mathbf{L}^2(\Omega) := (L^2(\Omega))^3 \mid \nabla \cdot (\epsilon \mathbf{u}) = 0 \text{ and } \underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}} \right\}$$

given, we solve [1](#) for the electric field  $\mathbf{E} := \mathbf{E}(\mathbf{x}, \mathbf{u}; \mu) := \mathbf{E}(\mathbf{u}; \mu) \in \mathbf{H}_0(\text{curl}; \Omega)$  depending on  $\mathbf{u}$  and the parameter  $\mu$  as well (see [Section 3](#) for the definition of functional spaces). Here  $\underline{\mathbf{u}} := (\underline{u}_i)_{i=1}^3$  and  $\bar{\mathbf{u}} := (\bar{u}_i)_{i=1}^3 \in \mathbb{R}^3$  are the given lower and upper bounds of the control with  $\underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}}$  referring to the inequality  $u_i \leq u_i(\mathbf{x}) \leq \bar{u}_i$  for  $i = 1, 2, 3$  and a.e in  $\Omega$ . Therefore, for any given  $\mu \in \mathcal{P}$  we define the *control-to-state* operator  $\mathbf{E} : \mathbf{U}_{ad}(\mu) \rightarrow \mathbf{H}_0(\text{curl}; \Omega)$  that maps each  $\mathbf{u}$  to the unique weak solution  $\mathbf{E}(\mathbf{u}; \mu)$  of [1](#). We note that the admissible set  $\mathbf{U}_{ad}(\mu)$  depends on  $\epsilon(\mu)$ , which guarantees that the control-to-state operator  $\mathbf{E}$  is well defined (cf., e.g., [\[44\]](#)).

Let  $D$  be a non-empty measurable subset of  $\Omega$  and  $\mathbf{E}_d(\mu) \in \mathbf{L}^2(D)$ ,  $\mathbf{u}_d(\mu) \in \mathbf{L}^2(\Omega)$  respectively be the desired state and control, both of which can be dependent on parameters. In this paper we consider the parameterized control problem

$$\min_{(\mathbf{u}, \mathbf{E}) \in \mathbf{U}_{ad}(\mu) \times \mathbf{H}_0(\text{curl}; \Omega)} J(\mathbf{u}, \mathbf{E}; \mu), \quad (\mathbb{P}_e)$$

where<sup>[1](#)</sup> the cost functional is defined as

$$J(\mathbf{u}, \mathbf{E}; \mu) := \frac{1}{2} \|\sqrt{\epsilon(\mu)}(\mathbf{E}(\mu) - \mathbf{E}_d(\mu))\|_{\mathbf{L}^2(D)}^2 + \frac{\alpha}{2} \|\sqrt{\epsilon(\mu)}(\mathbf{u} - \mathbf{u}_d(\mu))\|_{\mathbf{L}^2(\Omega)}^2$$

and  $\alpha > 0$  is the regularization parameter. We assume that the desired state and control are uniformly  $\mathbf{L}^2$ -bounded with respect to the parameter, i.e.

$$\|\mathbf{E}_d(\mu)\|_{\mathbf{L}^2(D)} \leq e_d \quad \text{and} \quad \|\mathbf{u}_d(\mu)\|_{\mathbf{L}^2(\Omega)} \leq u_d \quad (4)$$

for all  $\mu \in \mathcal{P}$  with  $e_d$  and  $u_d$  some positive constants. Furthermore,  $\mathbf{E}_d$  fulfills the Gauss's law in  $D$ , i.e.,

$$\nabla \cdot (\epsilon(\mu) \mathbf{E}_d(\mu)) = \rho(\mu) \text{ in } D.$$

Let  $(\mathcal{E}_h, V_h)$  be the finite element space associated with the system [1](#) and  $\mathbf{E}_h(\mu)$  be the finite element approximation of  $\mathbf{E}(\mu)$  (cf. [Subsection 3.3](#)). Adopting the variational discretization concept introduced in [\[24\]](#) (where control is not directly discretized), we approximate the “exact” problem  $(\mathbb{P}_e)$  by the discrete one

$$\min_{(\mathbf{u}, \mathbf{E}_h) \in \mathbf{U}_{ad}(\mu) \times \mathcal{E}_h} J(\mathbf{u}, \mathbf{E}_h; \mu), \quad (\mathbb{P}_h)$$

subject to

$$\begin{cases} (\sigma^{-1}(\mu) \nabla \times \mathbf{E}_h(\mu), \nabla \times \boldsymbol{\Phi}_h)_{\mathbf{L}^2(\Omega)} &= (\epsilon(\mu) \mathbf{u}, \boldsymbol{\Phi}_h)_{\mathbf{L}^2(\Omega)} \\ (\epsilon(\mu) \mathbf{E}_h(\mu), \nabla \phi_h)_{\mathbf{L}^2(\Omega)} &= -(\rho(\mu), \phi_h)_{L^2(\Omega)} \end{cases} \quad (5)$$

for all  $(\boldsymbol{\Phi}_h, \phi_h) \in \mathcal{E}_h \times V_h$ .

As mentioned in the introduction, the constrained saddle point system [5](#) may be expensive to solve. Our goal is to create a reduced basis method for  $(\mathbb{P}_h)$ , prove its convergence, and derive a posteriori error estimates.

**3. Preliminaries.** We start this section by presenting the definition of functional spaces which are utilized in the paper, for more details one can consult [\[2, 32\]](#). Well-posedness and finite element discretization, including a priori error estimates of [1](#) are given in [Subsections 3.2 and 3.3](#), respectively.

<sup>1</sup>The subscript  $e$  in the problem  $(\mathbb{P}_e)$  refers to “exact”.

**3.1. Functional spaces.** In this paper bold typeface is used to indicate a point in  $\mathbb{R}^3$ , a (three-dimensional) vector-valued function or a Hilbert space of vector-valued functions.

In what follows we denote by  $C$  a generic positive constant which is independent of the mesh size  $h$  and the parameter  $\mu$ . Notice that the value of  $C$  can differ from one occurrence to another. Moreover  $C(\mu)$  is a generic positive constant which is independent of  $h$  but it can depend on  $\mu$ , its value also can differ from one occurrence to another.

The Hilbert spaces

$$\begin{aligned}\mathbf{H}(\text{div}; \Omega) &:= \{ \Phi \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \Phi \in L^2(\Omega) \} \quad \text{and} \\ \mathbf{H}(\text{curl}; \Omega) &:= \{ \Phi \in \mathbf{L}^2(\Omega) \mid \nabla \times \Phi \in \mathbf{L}^2(\Omega) \}\end{aligned}$$

are respectively equipped the inner product

$$\begin{aligned}(\Phi, \Psi)_{\mathbf{H}(\text{div}; \Omega)} &:= (\Phi, \Psi)_{\mathbf{L}^2(\Omega)} + (\nabla \cdot \Phi, \nabla \cdot \Psi)_{L^2(\Omega)} \quad \text{and} \\ (\Phi, \Psi)_{\mathbf{H}(\text{curl}; \Omega)} &:= (\Phi, \Psi)_{\mathbf{L}^2(\Omega)} + (\nabla \times \Phi, \nabla \times \Psi)_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

The *normal trace operator*  $\gamma_n(\Phi) := \vec{n} \cdot \Phi|_{\partial\Omega}$  for all  $\Phi \in \mathbf{C}^\infty(\bar{\Omega})$  can be extended to a surjective, continuous linear map from  $\mathbf{H}(\text{div}; \Omega) \rightarrow H^{-1/2}(\partial\Omega) := (H^{1/2}(\partial\Omega))^*$  such that Green's formula (cf. [32, Chapter 3])

$$(\nabla \cdot \Phi, \phi)_{L^2(\Omega)} = -(\Phi, \nabla \phi)_{\mathbf{L}^2(\Omega)} + \langle \gamma_n(\Phi), \phi \rangle_{(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))} \quad (6)$$

holds true for all  $\Phi \in \mathbf{H}(\text{div}; \Omega)$  and  $\phi \in H^1(\Omega)$ . The *tangential trace operator*  $\gamma_t(\Phi) := \vec{n} \times \Phi|_{\partial\Omega}$  for all  $\Phi \in \mathbf{C}^\infty(\bar{\Omega})$  can be also extended to a continuous linear map from  $\mathbf{H}(\text{curl}; \Omega) \rightarrow \mathbf{H}^{-1/2}(\partial\Omega)$ . Further, Green's formula [32, Theorem 3.29]

$$(\nabla \times \Phi, \Psi)_{\mathbf{L}^2(\Omega)} = (\Phi, \nabla \times \Psi)_{\mathbf{L}^2(\Omega)} + \langle \gamma_t(\Phi), \Psi \rangle_{(\mathbf{H}^{-1/2}(\partial\Omega), \mathbf{H}^{1/2}(\partial\Omega))} \quad (7)$$

holds true for all  $\Phi \in \mathbf{H}(\text{curl}; \Omega)$  and  $\Psi \in \mathbf{H}^1(\Omega)$ .

We conclude this subsection by the following definition

$$\mathbf{H}_0(\text{div}; \Omega) := \overline{\mathbf{C}_c^\infty(\Omega)}^{\mathbf{H}(\text{div}; \Omega)} \quad \text{and} \quad \mathbf{H}_0(\text{curl}; \Omega) := \overline{\mathbf{C}_c^\infty(\Omega)}^{\mathbf{H}(\text{curl}; \Omega)},$$

where the closures are respectively taken with respect to the norm of the space  $\mathbf{H}(\text{div}; \Omega)$  and  $\mathbf{H}(\text{curl}; \Omega)$  and  $\mathbf{C}_c^\infty(\Omega)$  is the space of all infinitely continuously differentiable functions with compact support in  $\Omega$ . Notice that

$$\begin{aligned}\mathbf{H}_0(\text{div}; \Omega) &:= \{ \Phi \in \mathbf{H}(\text{div}; \Omega) \mid \gamma_n(\Phi) = 0 \}, \\ \mathbf{H}_0(\text{curl}; \Omega) &:= \{ \Phi \in \mathbf{H}(\text{curl}; \Omega) \mid \gamma_t(\Phi) = 0 \}.\end{aligned}$$

**3.2. Variational formulation of the system.** For any given  $\mu \in \mathcal{P}$  and  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  an element  $\mathbf{E} := \mathbf{E}(\mu) := \mathbf{E}(\mathbf{u}; \mu) \in \mathbf{H}_0(\text{curl}; \Omega)$  is said to be a weak solution of **1** if

$$\begin{cases} (\sigma^{-1}(\mu) \nabla \times \mathbf{E}(\mu), \nabla \times \Phi)_{\mathbf{L}^2(\Omega)} &= (\epsilon(\mu) \mathbf{u}, \Phi)_{\mathbf{L}^2(\Omega)}, \quad \forall \Phi \in \mathbf{H}_0(\text{curl}; \Omega) \\ (\epsilon(\mu) \mathbf{E}(\mu), \nabla \phi)_{\mathbf{L}^2(\Omega)} &= -(\rho(\mu), \phi)_{L^2(\Omega)}, \quad \forall \phi \in H_0^1(\Omega). \end{cases} \quad (8)$$

The first equation in **8** is obtained by multiplying the first equation of **1** with  $\Phi \in \mathbf{H}_0(\text{curl}; \Omega)$ , and then using the identity **7**. The second equation of **8** is obtained by using the second equation in **1** and the Green's formula **6**.

We define by

$$\mathbf{V}(\mu) := \{ \tau \in \mathbf{H}_0(\text{curl}; \Omega) \mid \nabla \cdot (\epsilon \tau) = 0 \}.$$

For any fixed  $\mu \in \mathcal{P}$ , due to the connectedness of the boundary  $\partial\Omega$ , we have

$$\begin{aligned} & \{\boldsymbol{\tau} \in \mathbf{V}(\mu) \mid \nabla \times \boldsymbol{\tau} = \mathbf{0}\} \\ &= \{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) \mid \nabla \times \boldsymbol{\tau} = \mathbf{0}, \nabla \cdot (\epsilon \boldsymbol{\tau}) = 0, \vec{\mathbf{n}} \times \boldsymbol{\tau}|_{\partial\Omega} = \mathbf{0}\} = \{\mathbf{0}\} \end{aligned}$$

(cf., e.g., [14, 44]). An application of Peetre's lemma (see, [28, Lemma 2]) then yields that there is a positive constant  $C(\mu)$  such that

$$\|\boldsymbol{\tau}\|_{\mathbf{L}^2(\Omega)} \leq C(\mu) \|\nabla \times \boldsymbol{\tau}\|_{\mathbf{L}^2(\Omega)} \quad (9)$$

holds true for  $\boldsymbol{\tau} \in \mathbf{V}(\mu)$ . With the aid of the condition 2, we obtain the coercivity condition

$$\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \leq C(\mu) (\sigma^{-1} \nabla \times \mathbf{v}, \nabla \times \mathbf{v})_{\mathbf{L}^2(\Omega)} \quad (10)$$

for all  $\mathbf{v} \in \mathbf{V}(\mu)$ .

Therefore, due to the standard theory of the mixed variational problems (see, e.g., [9, 17]), we conclude that the system 8 attains a unique solution  $\mathbf{E} = \mathbf{E}(\mathbf{u}; \mu) \in \mathbf{H}_0(\mathbf{curl}; \Omega)$  which satisfies

$$\begin{aligned} \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl};\Omega)} &\leq C(\mu) (\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\rho\|_{L^2(\Omega)}) \\ &\leq C(\mu) |\Omega|^{1/2} (\bar{\rho} + \|\bar{\mathbf{u}}\|_{\mathbb{R}^3}) := C_{\mathbf{E}}(\mu). \end{aligned} \quad (11)$$

We therefore can define for any fixed  $\mu \in \mathcal{P}$  the map

$$S_\mu : \mathbf{U}_{ad}(\mu) \rightarrow \mathbf{H}_0(\mathbf{curl}; \Omega) \quad \text{with} \quad \mathbf{u} \mapsto S_\mu(\mathbf{u}) := \mathbf{E}(\mathbf{u})$$

and for any fixed  $\mathbf{u} \in \mathbf{U}_{ad}(\mu)$  the one

$$S_{\mathbf{u}} : \mathcal{P} \rightarrow \mathbf{H}_0(\mathbf{curl}; \Omega) \quad \text{with} \quad \mu \mapsto S_{\mathbf{u}}(\mu) := \mathbf{E}(\mu).$$

**3.3. Finite element discretization.** Hereafter, we assume  $(\mathcal{T}_h)_{h>0}$  is a quasi-uniform family of regular triangulations of the domain  $\Omega$  with the mesh size  $h$  (cf. [8]). For discretization of the state variable solving the system 8 let us denote the Nédélec finite element spaces (cf. [33])

$$\begin{aligned} \mathcal{E}_h &:= \{\mathbf{E}_h \in \mathbf{H}_0(\mathbf{curl}; \Omega) \mid \mathbf{E}_h|_T = \mathbf{a}_T + \mathbf{b}_T \times \mathbf{x}, \forall T \in \mathcal{T}_h \text{ with } \mathbf{a}_T, \mathbf{b}_T \in \mathbb{R}^3\}, \\ V_h &:= \{\phi_h \in H_0^1(\Omega) \mid \phi_h|_T = a_T + \mathbf{b}_T \cdot \mathbf{x}, \forall T \in \mathcal{T}_h \text{ with } a_T \in \mathbb{R}, \mathbf{b}_T \in \mathbb{R}^3\}, \end{aligned}$$

where  $\nabla V_h \subset \mathcal{E}_h$ .

The discrete variational formulation corresponding to the system 8 then reads: find  $\mathbf{E}_h \in \mathcal{E}_h$  such that 5 is satisfied for all  $(\Phi_h, \phi_h) \in \mathcal{E}_h \times V_h$ . Similar to 10, since the discrete Poincaré-Friedrichs-type inequality (cf. [26, Theorem 4.7], [32, Chapter 7])

$$\|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \leq C(\mu) \|\nabla \times \mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \quad (12)$$

is satisfied for all discrete  $\epsilon$ -divergence-free functions, i.e.

$$\mathbf{v}_h \in \mathcal{D}_h^\epsilon(\mu) := \left\{ \mathbf{E}_h \in \mathcal{E}_h \mid (\epsilon \mathbf{E}_h, \nabla \phi_h)_{\mathbf{L}^2(\Omega)} = 0 \text{ for all } \phi_h \in V_h \right\}, \quad (13)$$

we have that

$$\|\mathbf{v}_h\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \leq C(\mu) (\sigma^{-1} \nabla \times \mathbf{v}_h, \nabla \times \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} \quad (14)$$

for all  $\mathbf{v}_h \in \mathcal{D}_h^\epsilon(\mu)$ . Therefore, we conclude that the system 5 has a unique solution  $\mathbf{E}_h \in \mathcal{E}_h$  satisfying the estimate  $\|\mathbf{E}_h\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C_{\mathbf{E}}(\mu)$ .

To analyze the  $\mu$ -dependent optimal control problem, we assume that the Poincaré constant  $C(\mu)$  appearing in [10](#) and [14](#) is uniformly bounded with respect to  $\mu$ , i.e. there is a positive constant  $C$  such that

$$C(\mu) \leq C$$

for all  $\mu \in \mathcal{P}$ . Therefore, we arrive at a priori estimate for the states

$$\max \left( \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl};\Omega)}, \|\mathbf{E}_h\|_{\mathbf{H}(\mathbf{curl};\Omega)} \right) \leq C_{\mathbf{E}},$$

where the constant  $C_{\mathbf{E}}$  is independent of  $\mu$ .

For all  $s \geq 0$  we denote by (cf. [\[13\]](#))

$$\mathbf{H}^s(\mathbf{curl};\Omega) := \{ \Phi \in \mathbf{H}^s(\Omega) \mid \nabla \times \Phi \in \mathbf{H}^s(\Omega) \}.$$

Equipped with the norm

$$\|\Phi\|_{\mathbf{H}^s(\mathbf{curl};\Omega)} := \left( \|\Phi\|_{\mathbf{H}^s(\Omega)}^2 + \|\nabla \times \Phi\|_{\mathbf{H}^s(\Omega)}^2 \right)^{1/2},$$

it is a Banach space. Before going further, we state the following result.

**Theorem 3.1.** *For any given  $\mu \in \mathcal{P}$  let  $\mathbf{E}(\mu)$  and  $\mathbf{E}_h(\mu)$  be the unique solution to [8](#) and [5](#), respectively. Then:*

(i) *There holds the limit*

$$\lim_{h \rightarrow 0} \|\nabla \times (\mathbf{E}(\mu) - \mathbf{E}_h(\mu))\|_{\mathbf{L}^2(\Omega)} = 0.$$

(ii) *In addition  $\epsilon(\mu), \sigma^{-1}(\mu) \in W^{1,\infty}(\Omega)$  we get the regularity  $\mathbf{E}(\mu) \in \mathbf{H}^s(\mathbf{curl};\Omega)$  for some  $s \in (1/2, 1]$ . Furthermore, there exist constants  $\nu, \nu' \in (1/2, 1]$  such that the estimates*

$$\begin{aligned} \|\mathbf{E}(\mu) - \mathbf{E}_h(\mu)\|_{\mathbf{H}(\mathbf{curl};\Omega)} &\leq Ch^s \|\mathbf{E}(\mu)\|_{\mathbf{H}^s(\mathbf{curl};\Omega)} \\ \|\nabla \cdot \epsilon(\mathbf{E}(\mu) - \mathbf{E}_h(\mu))\|_{H^{-\nu}(\Omega)} &\leq Ch^{\nu+\nu'-1} \left( \|\nabla \times \mathbf{E}(\mu)\|_{\mathbf{L}^2(\Omega)} + \|\rho(\mu)\|_{H^{\nu'-1}(\Omega)} \right) \end{aligned}$$

hold true.

*Proof.* The regularity  $\mathbf{E} \in \mathbf{H}^s(\mathbf{curl};\Omega)$  follows from [\[13, Lemma 3.6\]](#). Further, the assertion is based on standard arguments, it is therefore omitted here.  $\square$

**4. Primal reduced basis approach.** By standard arguments (see, e.g., [\[25, 39\]](#)), one can verify that the problem  $(\mathbb{P}_e)$  attains a unique solution for each the parameter  $\mu \in \mathcal{P}$ . Furthermore, we can derive the following, for instance using Lagrangian approach, first order optimality system satisfied by the optimal control, state and adjoint.

**Theorem 4.1.** *The pair  $(\mathbf{u}_e^*, \mathbf{E}_e^*) \in \mathbf{U}_{ad}(\mu) \times \mathbf{H}_0(\mathbf{curl};\Omega)$  is the unique solution<sup>2</sup> of the problem  $(\mathbb{P}_e)$  if and only if there exists an adjoint state  $\mathbf{F}_e^* \in \mathbf{H}_0(\mathbf{curl};\Omega)$*

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<sup>2</sup>The superscript \* refers to “optimality”.

such that the triple  $(\mathbf{u}_e^*, \mathbf{E}_e^*, \mathbf{F}_e^*)$  satisfies the system

$$(\sigma^{-1}(\mu) \nabla \times \mathbf{E}_e^*(\mu), \nabla \times \Phi)_{\mathbf{L}^2(\Omega)} = (\epsilon(\mu) \mathbf{u}_e^*, \Phi)_{\mathbf{L}^2(\Omega)}, \quad (15a)$$

$$(\epsilon(\mu) \mathbf{E}_e^*(\mu), \nabla \phi)_{\mathbf{L}^2(\Omega)} = -(\rho(\mu), \phi)_{L^2(\Omega)}, \quad (15b)$$

$$(\sigma^{-1}(\mu) \nabla \times \mathbf{F}_e^*(\mu), \nabla \times \Phi)_{\mathbf{L}^2(\Omega)} = (\epsilon(\mu) (\mathbf{E}_e^*(\mu) - \mathbf{E}_d(\mu)), \Phi)_{\mathbf{L}^2(D)}, \quad (15c)$$

$$(\epsilon(\mu) \mathbf{F}_e^*(\mu), \nabla \phi)_{\mathbf{L}^2(\Omega)} = 0, \quad (15d)$$

$$\left( \epsilon(\mu) (\mathbf{u} - \mathbf{u}_e^*), \mathbf{u}_d(\mu) - \frac{1}{\alpha} \mathbf{F}_e^*(\mu) - \mathbf{u}_e^* \right)_{\mathbf{L}^2(\Omega)} \leq 0 \quad (15e)$$

for all  $(\Phi, \phi, \mathbf{u}) \in \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega) \times \mathbf{U}_{ad}(\mu)$ .

Notice that the following inequality holds true for all  $\mu \in \mathcal{P}$

$$\|\mathbf{F}_e^*(\mu)\|_{\mathbf{L}^2(\Omega)} \leq C(e_d + C_{\mathbf{E}}) := C_{\mathbf{F}} \quad (16)$$

Based on the finite element approach in Subsection 3.3, next we approximate the “exact” problem  $(\mathbb{P}_e)$  by the discrete one  $(\mathbb{P}_h)$ . Then, the associated first order optimality system for the problem  $(\mathbb{P}_h)$  reads:

**Theorem 4.2.** *The pair  $(\mathbf{u}_h^*, \mathbf{E}_h^*) \in \mathbf{U}_{ad}(\mu) \times \mathcal{E}_h$  is the unique solution of the problem  $(\mathbb{P}_h)$  if and only if there exists an adjoint state  $\mathbf{F}_h^* \in \mathcal{E}_h$  such that the triple  $(\mathbf{u}_h^*, \mathbf{E}_h^*, \mathbf{F}_h^*)$  satisfies the system*

$$(\sigma^{-1}(\mu) \nabla \times \mathbf{E}_h^*(\mu), \nabla \times \Phi_h)_{\mathbf{L}^2(\Omega)} = (\epsilon(\mu) \mathbf{u}_h^*, \Phi_h)_{\mathbf{L}^2(\Omega)}, \quad (17a)$$

$$(\epsilon(\mu) \mathbf{E}_h^*(\mu), \nabla \phi_h)_{\mathbf{L}^2(\Omega)} = -(\rho(\mu), \phi_h)_{L^2(\Omega)}, \quad (17b)$$

$$(\sigma^{-1}(\mu) \nabla \times \mathbf{F}_h^*(\mu), \nabla \times \Phi_h)_{\mathbf{L}^2(\Omega)} = (\epsilon(\mu) (\mathbf{E}_h^*(\mu) - \mathbf{E}_d(\mu)), \Phi_h)_{\mathbf{L}^2(D)}, \quad (17c)$$

$$(\epsilon(\mu) \mathbf{F}_h^*(\mu), \nabla \phi_h)_{\mathbf{L}^2(\Omega)} = 0, \quad (17d)$$

$$\left( \epsilon(\mu) (\mathbf{u} - \mathbf{u}_h^*), \mathbf{u}_d(\mu) - \frac{1}{\alpha} \mathbf{F}_h^*(\mu) - \mathbf{u}_h^* \right)_{\mathbf{L}^2(\Omega)} \leq 0 \quad (17e)$$

for all  $(\Phi_h, \phi_h, \mathbf{u}) \in \mathcal{E}_h \times V_h \times \mathbf{U}_{ad}(\mu)$ .

The above optimality system 17a–17e consists of several sets of variational equations and inequalities which may be computationally expensive. Thus the surrogate model approach will be considered next, where the original full-dimensional problem is replaced by a reduced order approximation.

Assume that we are given the reduced basis spaces

$$(\mathcal{E}_N, V_N) \subset (\mathcal{E}_h, V_h).$$

Furthermore, to guarantee the existence of a solution to the reduced order constraint system, we assume that the coercivity condition 14 is fulfilled for all

$$\mathbf{v}_N \in \mathcal{D}_N^\epsilon(\mu) := \left\{ \mathbf{E}_N \in \mathcal{E}_h \mid (\epsilon \mathbf{E}_N, \nabla \phi_N)_{\mathbf{L}^2(\Omega)} = 0 \text{ for all } \phi_N \in V_N \right\}. \quad (18)$$

We can then consider the reduced basis problem

$$\min_{(\mathbf{u}, \mathbf{E}_N) \in \mathbf{U}_{ad}(\mu) \times \mathcal{E}_N} J(\mathbf{u}, \mathbf{E}_N; \mu), \quad (\mathbb{P}_N)$$

subject to

$$\begin{cases} (\sigma^{-1}(\mu) \nabla \times \mathbf{E}_N(\mu), \nabla \times \Phi_N)_{\mathbf{L}^2(\Omega)} &= (\epsilon(\mu) \mathbf{u}, \Phi_N)_{\mathbf{L}^2(\Omega)} \\ (\epsilon(\mu) \mathbf{E}_N(\mu), \nabla \phi_N)_{\mathbf{L}^2(\Omega)} &= -(\rho(\mu), \phi_N)_{L^2(\Omega)} \end{cases} \quad (19)$$

for all  $(\Phi_N, \phi_N) \in \mathcal{E}_N \times V_N$ . The associated first order optimality system reads:

**Theorem 4.3.** *The pair  $(\mathbf{u}_N^*, \mathbf{E}_N^*) \in \mathbf{U}_{ad}(\mu) \times \mathcal{E}_N$  is the unique solution of the problem  $(\mathbb{P}_N)$  if and only if there exists an adjoint state  $\mathbf{F}_N^* \in \mathcal{E}_N$  such that the triple  $(\mathbf{u}_N^*, \mathbf{E}_N^*, \mathbf{F}_N^*)$  satisfies the system*

$$(\sigma^{-1}(\mu) \nabla \times \mathbf{E}_N^*(\mu), \nabla \times \Phi_N)_{\mathbf{L}^2(\Omega)} = (\epsilon(\mu) \mathbf{u}_N^*, \Phi_N)_{\mathbf{L}^2(\Omega)}, \quad (20a)$$

$$(\epsilon(\mu) \mathbf{E}_N^*(\mu), \nabla \phi_N)_{\mathbf{L}^2(\Omega)} = -(\rho(\mu), \phi_N)_{L^2(\Omega)}, \quad (20b)$$

$$(\sigma^{-1}(\mu) \nabla \times \mathbf{F}_N^*(\mu), \nabla \times \Phi_N)_{\mathbf{L}^2(\Omega)} = (\epsilon(\mu) (\mathbf{E}_N^*(\mu) - \mathbf{E}_d(\mu)), \Phi_N)_{\mathbf{L}^2(D)}, \quad (20c)$$

$$(\epsilon(\mu) \mathbf{F}_N^*(\mu), \nabla \phi_N)_{\mathbf{L}^2(\Omega)} = 0, \quad (20d)$$

$$\left( \epsilon(\mu) (\mathbf{u} - \mathbf{u}_N^*), \mathbf{u}_d(\mu) - \frac{1}{\alpha} \mathbf{F}_N^*(\mu) - \mathbf{u}_N^* \right)_{\mathbf{L}^2(\Omega)} \leq 0 \quad (20e)$$

for all  $(\Phi_N, \phi_N, \mathbf{u}) \in \mathcal{E}_N \times V_N \times \mathbf{U}_{ad}(\mu)$ .

We conclude this section by performing the greedy sampling procedure [19, 23, 37] applied to the problem under consideration. Note that, by the discrete Helmholtz decomposition (see, [32, Section 7.2.1]), for all  $z_h^\epsilon \in \mathcal{E}_h$ , there exists a unique pair  $(z_h^1, H(z_h^\epsilon)) \in \mathcal{D}_h^1 \times V_h$  such that  $z_h^\epsilon = z_h^1 + \nabla H(z_h^\epsilon)$ .

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**Algorithm 1** Greedy procedure

---

Choose  $\mathcal{S}_{train} \subset \mathcal{P}$ , an arbitrary  $\mu^1 \in \mathcal{S}_{train}$ ,  $\epsilon_{tol} > 0$  and  $N_{max} \in \mathbb{N}$

Set  $N := 1$ , and

$$\mathcal{E}_N := \text{span}\{\mathbf{E}_h^*(\mu^N), \mathbf{F}_h^*(\mu^N), \nabla H(\mathbf{E}_h^*(\mu^N)), \nabla H(\mathbf{F}_h^*(\mu^N))\}$$

$$V_N := \text{span}\{H(\mathbf{E}_h^*(\mu^N)), H(\mathbf{F}_h^*(\mu^N))\}$$

**while**  $\max_{\mu \in \mathcal{S}_{train}} \Delta_N(\mathcal{E}_N, V_N; \mu) > \epsilon_{tol}$  and  $N \leq N_{max}$  **do**

$$\mu^{N+1} := \arg \max_{\mu \in \mathcal{S}_{train}} \Delta_N(\mathcal{E}_N, V_N; \mu)$$

$$\mathcal{E}_{N+1} := \text{span}\{\{\mathbf{E}_h^*(\mu^{N+1}), \mathbf{F}_h^*(\mu^{N+1}), \nabla H(\mathbf{E}_h^*(\mu^{N+1})), \nabla H(\mathbf{F}_h^*(\mu^{N+1}))\} \cup \mathcal{E}_N\}$$

$$V_{N+1} := \text{span}\{\{H(\mathbf{E}_h^*(\mu^{N+1})), H(\mathbf{F}_h^*(\mu^{N+1}))\} \cup V_N\}$$

$$N := N + 1$$

**end while**

---

In Algorithm 1, the sampling parameter set  $\mathcal{S}_{train} \subset \mathcal{P}$  is finite, but rich enough so that it is a good approximation of the full parameter set  $\mathcal{P}$ . The initial parameter  $\mu^1$  is chosen arbitrarily in  $\mathcal{S}_{train}$ ,  $\epsilon_{tol}$  is a desired error tolerance and  $N_{max}$  is the maximum number of iterations. The pair  $\{\mathbf{E}_h^*(\mu^N), \mathbf{F}_h^*(\mu^N)\}$  is the optimal state and adjoint state defined by the optimality system 17a-17e at the parameter  $\mu = \mu^N$ . The quantity  $\Delta_N(\mathcal{E}_N, V_N; \mu)$  is an error estimator between solutions of the problem  $(\mathbb{P}_h)$  and the reduced one  $(\mathbb{P}_N)$  at the given parameter  $\mu$ , that will be described in detail in Section 6.

**5. Convergence of the reduced basis method.** Our aim in this section is to investigate the uniform convergence

$$\lim_{N \rightarrow \infty} \sup_{\mu \in \mathcal{P}} \|\mathbf{u}_h^*(\mu) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} = 0$$

of reduced basis optimal solutions to the original one. To do so, we assume that the snapshot parameter sample  $\mathcal{P}_N := \{\mu^1, \dots, \mu^N\}$  is dense in the compact set  $\mathcal{P}$ ,



i.e. the Hausdorff distance

$$\kappa_N := \sup_{\mu \in \mathcal{P}} \text{dist}(\mu, \mathcal{P}_N)$$

tends to zero as  $N$  to infinity, where  $\text{dist}(\mu, \mathcal{P}_N) := \inf_{\mu' \in \mathcal{P}_N} \|\mu - \mu'\|_{\mathbb{R}^p}$ .

A crucial property for an efficiently computational procedure of reduced basis approaches is the parameter separability that can be defined as follows (cf. [16]). Such separability conditions, for instance, can be obtained by using the Empirical Interpolation Method (EIM) [5], see also [3].

**Definition 5.1.** Assume that the functions  $\sigma$ ,  $\epsilon$ , the desired control and state  $\mathbf{u}_b$  and  $\mathbf{E}_d$  admit the expansions

$$\begin{aligned} \sigma^{-1}(\cdot; \mu) &:= \sum_{q=1}^{Q^\sigma} \Theta_q^\sigma(\mu) \sigma_q^{-1}(\cdot), & \epsilon(\cdot; \mu) &:= \sum_{q=1}^{Q^\epsilon} \Theta_q^\epsilon(\mu) \epsilon_q(\cdot), \\ \mathbf{u}_d(\cdot; \mu) &:= \sum_{q=1}^{Q^{\mathbf{u}_d}} \Theta_q^{\mathbf{u}_d}(\mu) \mathbf{u}_{dq}(\cdot), & \mathbf{E}_d(\cdot; \mu) &:= \sum_{q=1}^{Q^{\mathbf{E}_d}} \Theta_q^{\mathbf{E}_d}(\mu) \mathbf{E}_{dq}(\cdot), \end{aligned}$$

where  $Q^\sigma$ ,  $Q^\epsilon$ ,  $Q^{\mathbf{u}_d}$  and  $Q^{\mathbf{E}_d}$  are finite positive integers, the functions  $\Theta_q^\sigma, \Theta_q^\epsilon, \Theta_q^{\mathbf{u}_d}$  and  $\Theta_q^{\mathbf{E}_d} : \mathcal{P} \rightarrow \mathbb{R}$ , while the functions  $\sigma_q^{-1}, \epsilon_q, \mathbf{u}_{dq} : \Omega \rightarrow \mathbb{R}$  as well as  $\mathbf{E}_d : D \rightarrow \mathbb{R}$  are independent of the parameter  $\mu$ .

Due to 2 and 4, we assume that

$$\sigma_q^{-1} \in [\bar{\sigma}^{-1}, \underline{\sigma}^{-1}], \quad \epsilon_q \in [\underline{\epsilon}, \bar{\epsilon}], \quad \|\mathbf{u}_{dq}\|_{\mathbf{L}^2(\Omega)} \leq u_d \quad \text{and} \quad \|\mathbf{E}_{dq}\|_{\mathbf{L}^2(D)} \leq e_d$$

for all the index  $q$ . We start with some auxiliary results.

**Lemma 5.2.** (i) For any given  $\mu \in \mathcal{P}$  the inequality

$$\|S_{\mathbf{u}^1}(\mu) - S_{\mathbf{u}^2}(\mu)\|_{\mathbf{H}(\text{curl}; \Omega)} \leq C \|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathbf{L}^2(\Omega)}$$

is satisfied for all  $\mathbf{u}^1, \mathbf{u}^2 \in \mathbf{U}_{ad}(\mu)$ .

(ii) For any fixed  $\mathbf{u} \in \mathbf{U}_{ad}(\mu)$  the estimate

$$\begin{aligned} &\|S_{\mathbf{u}}(\mu^1) - S_{\mathbf{u}}(\mu^2)\|_{\mathbf{H}(\text{curl}; \Omega)} \\ &\leq C \left( \sum_{q=1}^{Q^\sigma} |\Theta_q^\sigma(\mu^1) - \Theta_q^\sigma(\mu^2)|^2 \right)^{1/2} + C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^1) - \Theta_q^\epsilon(\mu^2)|^2 \right)^{1/2} \end{aligned}$$

holds true.

*Proof.* (i) For any  $\Phi \in \mathbf{H}_0(\text{curl}; \Omega)$  from the system 8 have that

$$(\sigma^{-1}(\mu) \nabla \times (S_{\mathbf{u}^1}(\mu) - S_{\mathbf{u}^2}(\mu)), \nabla \times \Phi)_{\mathbf{L}^2(\Omega)} = (\epsilon(\mu)(\mathbf{u}^1 - \mathbf{u}^2), \Phi)_{\mathbf{L}^2(\Omega)}.$$

Taking  $\Phi = S_{\mathbf{u}^1}(\mu) - S_{\mathbf{u}^2}(\mu)$ , with the aid of 2 and 10, we obtain

$$\|S_{\mathbf{u}^1}(\mu) - S_{\mathbf{u}^2}(\mu)\|_{\mathbf{H}(\text{curl}; \Omega)}^2 \leq C \|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathbf{L}^2(\Omega)} \|S_{\mathbf{u}^1}(\mu) - S_{\mathbf{u}^2}(\mu)\|_{\mathbf{L}^2(\Omega)},$$

which yields the desired inequality.

(ii) Likewise, we get

$$\begin{aligned} &(\sigma^{-1}(\mu^1) \nabla \times (S_{\mathbf{u}}(\mu^1) - S_{\mathbf{u}}(\mu^2)), \nabla \times \Phi)_{\mathbf{L}^2(\Omega)} \\ &= (\epsilon(\mu^1) \mathbf{u}, \Phi)_{\mathbf{L}^2(\Omega)} - (\sigma^{-1}(\mu^1) \nabla \times S_{\mathbf{u}}(\mu^2), \nabla \times \Phi)_{\mathbf{L}^2(\Omega)} \\ &= ((\sigma^{-1}(\mu^2) - \sigma^{-1}(\mu^1)) \nabla \times S_{\mathbf{u}}(\mu^2), \nabla \times \Phi)_{\mathbf{L}^2(\Omega)} + ((\epsilon(\mu^1) - \epsilon(\mu^2)) \mathbf{u}, \Phi)_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Taking  $\Phi = S_{\mathbf{u}}(\mu^1) - S_{\mathbf{u}}(\mu^2)$ , we thus obtain

$$\begin{aligned} \|S_{\mathbf{u}}(\mu^1) - S_{\mathbf{u}}(\mu^2)\|_{\mathbf{H}(\mathbf{curl}; \Omega)} &\leq C \left( \sum_{q=1}^{Q^\sigma} |\Theta_q^\sigma(\mu^1) - \Theta_q^\sigma(\mu^2)|^2 \right)^{1/2} \|S_{\mathbf{u}}(\mu^2)\|_{\mathbf{H}(\mathbf{curl}; \Omega)} \\ &\quad + C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^1) - \Theta_q^\epsilon(\mu^2)|^2 \right)^{1/2} \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \end{aligned}$$

which together with 11 yield the desired inequality. The proof completes.  $\square$

**Lemma 5.3.** *Let  $\mathbf{u}_e^*(\mu)$  be the solution of the problem  $(\mathbb{P}_e)$  associated with the parameter  $\mu \in \mathcal{P}$ . Then, the estimate*

$$\|\mathbf{u}_e^*(\mu^1) - \mathbf{u}_e^*(\mu^2)\|_{\mathbf{L}^2(\Omega)} \leq \mathcal{B}(\mu^1, \mu^2)$$

is satisfied for all  $\mu^1, \mu^2 \in \mathcal{P}$ , where

$$\begin{aligned} \mathcal{B}(\mu^1, \mu^2) &= C\alpha^{1/2} \left( \sum_{q=1}^{Q^{\mathbf{u}_d}} |\Theta_q^{\mathbf{u}_d}(\mu^1) - \Theta_q^{\mathbf{u}_d}(\mu^2)|^2 \right)^{1/2} \\ &\quad + C\alpha^{-1/2} \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^1) - \Theta_q^\epsilon(\mu^2)|^2 \right)^{1/2} + C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^2) - \Theta_q^\epsilon(\mu^1)|^2 \right)^{1/4} \\ &\quad + C \left( \sum_{q=1}^{Q^\sigma} |\Theta_q^\sigma(\mu^1) - \Theta_q^\sigma(\mu^2)|^2 \right)^{1/4} + C \left( \sum_{q=1}^{Q^{\mathbf{E}_d}} |\Theta_q^{\mathbf{E}_d}(\mu^1) - \Theta_q^{\mathbf{E}_d}(\mu^2)|^2 \right)^{1/4}. \end{aligned}$$

*Proof.* By the variational inequality 15e, we have

$$\begin{aligned} \left( \epsilon(\mu^1)(\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)), \mathbf{u}_d(\mu^1) - \frac{1}{\alpha} \mathbf{F}_e^*(\mu^1) - \mathbf{u}_e^*(\mu^1) \right)_{\mathbf{L}^2(\Omega)} &\leq 0 \\ \left( \epsilon(\mu^2)(\mathbf{u}_e^*(\mu^1) - \mathbf{u}_e^*(\mu^2)), \mathbf{u}_d(\mu^2) - \frac{1}{\alpha} \mathbf{F}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^2) \right)_{\mathbf{L}^2(\Omega)} &\leq 0 \end{aligned}$$

which yield

$$\begin{aligned} \alpha \epsilon \|\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)\|_{\mathbf{L}^2(\Omega)}^2 &\leq (\epsilon(\mu^1)(\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)), \alpha(\mathbf{u}_d(\mu^2) - \mathbf{u}_d(\mu^1)))_{\mathbf{L}^2(\Omega)} \\ &\quad + ((\epsilon(\mu^2) - \epsilon(\mu^1))(\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)), \alpha \mathbf{u}_d(\mu^2) - \mathbf{F}_e^*(\mu^2) - \alpha \mathbf{u}_e^*(\mu^2))_{\mathbf{L}^2(\Omega)} \\ &\quad + (\epsilon(\mu^1)(\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)), \mathbf{F}_e^*(\mu^1) - \mathbf{F}_e^*(\mu^2))_{\mathbf{L}^2(\Omega)} := I_1 + I_2 + I_3. \end{aligned}$$

We bound for the terms  $I_1$ ,  $I_2$  and  $I_3$ . First, we get

$$\begin{aligned} I_1 &\leq C\alpha \|\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)\|_{\mathbf{L}^2(\Omega)} \left( \sum_{q=1}^{Q^{\mathbf{u}_d}} |\Theta_q^{\mathbf{u}_d}(\mu^2) - \Theta_q^{\mathbf{u}_d}(\mu^1)|^2 \right)^{1/2} \\ &\leq 4^{-1} \alpha \epsilon \|\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)\|_{\mathbf{L}^2(\Omega)}^2 + C\alpha \sum_{q=1}^{Q^{\mathbf{u}_d}} |\Theta_q^{\mathbf{u}_d}(\mu^2) - \Theta_q^{\mathbf{u}_d}(\mu^1)|^2 \end{aligned}$$

and, by 16,

$$\begin{aligned}
I_2 &\leq \|(\epsilon(\mu^2) - \epsilon(\mu^1))(\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1))\|_{\mathbf{L}^2(\Omega)} \cdot \\
&\quad \cdot (\alpha \|\mathbf{u}_d(\mu^2)\|_{\mathbf{L}^2(\Omega)} + C_{\mathbf{F}} + \alpha \|\mathbf{u}_e^*(\mu^2)\|_{\mathbf{L}^2(\Omega)}) \\
&\leq C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^2) - \Theta_q^\epsilon(\mu^1)|^2 \right)^{1/2} \|\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)\|_{\mathbf{L}^2(\Omega)} \\
&\leq 4^{-1} \alpha \epsilon \|\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)\|_{\mathbf{L}^2(\Omega)}^2 \\
&\quad + C \alpha^{-1} \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^2) - \Theta_q^\epsilon(\mu^1)|^2.
\end{aligned}$$

For  $I_3$  we write

$$\begin{aligned}
I_3 &= ((\epsilon(\mu^1) - \epsilon(\mu^2))\mathbf{u}_e^*(\mu^1), \mathbf{F}_e^*(\mu^2))_{\mathbf{L}^2(\Omega)} - ((\epsilon(\mu^1) - \epsilon(\mu^2))\mathbf{u}_e^*(\mu^2), \mathbf{F}_e^*(\mu^2))_{\mathbf{L}^2(\Omega)} \\
&\quad + (\epsilon(\mu^2)\mathbf{u}_e^*(\mu^1), \mathbf{F}_e^*(\mu^2))_{\mathbf{L}^2(\Omega)} - (\epsilon(\mu^2)\mathbf{u}_e^*(\mu^2), \mathbf{F}_e^*(\mu^2))_{\mathbf{L}^2(\Omega)} \\
&\quad + (\epsilon(\mu^1)(\mathbf{E}_e^*(\mu^1) - \mathbf{E}_d(\mu^1)), S_{\mathbf{u}_e^*(\mu^2)}(\mu^1) - \mathbf{E}_e^*(\mu^1))_{\mathbf{L}^2(D)} \\
&\leq C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^2) - \Theta_q^\epsilon(\mu^1)|^2 \right)^{1/2} \\
&\quad + (\epsilon(\mu^2)(\mathbf{E}_e^*(\mu^2) - \mathbf{E}_d(\mu^2)), S_{\mathbf{u}_e^*(\mu^1)}(\mu^2) - \mathbf{E}_e^*(\mu^2))_{\mathbf{L}^2(D)} \\
&\quad + (\epsilon(\mu^1)(\mathbf{E}_e^*(\mu^1) - \mathbf{E}_d(\mu^1)), S_{\mathbf{u}_e^*(\mu^2)}(\mu^1) - \mathbf{E}_e^*(\mu^1))_{\mathbf{L}^2(D)} \\
&:= I_3^1 + I_3^2 + I_3^3
\end{aligned}$$

with

$$\begin{aligned}
I_3^2 + I_3^3 &= ((\epsilon(\mu^1) - \epsilon(\mu^2))(\mathbf{E}_e^*(\mu^1) - \mathbf{E}_d(\mu^1)), S_{\mathbf{u}_e^*(\mu^2)}(\mu^1) - \mathbf{E}_e^*(\mu^1))_{\mathbf{L}^2(D)} \\
&\quad + (\epsilon(\mu^2)(\mathbf{E}_e^*(\mu^1) - \mathbf{E}_d(\mu^1)), S_{\mathbf{u}_e^*(\mu^2)}(\mu^1) - S_{\mathbf{u}_e^*(\mu^2)}(\mu^2))_{\mathbf{L}^2(D)} \\
&\quad + (\epsilon(\mu^2)(\mathbf{E}_e^*(\mu^2) - \mathbf{E}_d(\mu^2)), S_{\mathbf{u}_e^*(\mu^1)}(\mu^2) - S_{\mathbf{u}_e^*(\mu^1)}(\mu^1))_{\mathbf{L}^2(D)} \\
&\quad + (\epsilon(\mu^2)(\mathbf{E}_e^*(\mu^1) - \mathbf{E}_d(\mu^1)), S_{\mathbf{u}_e^*(\mu^2)}(\mu^2) - \mathbf{E}_e^*(\mu^1))_{\mathbf{L}^2(D)} \\
&\quad + (\epsilon(\mu^2)(\mathbf{E}_e^*(\mu^2) - \mathbf{E}_d(\mu^2)), S_{\mathbf{u}_e^*(\mu^1)}(\mu^1) - \mathbf{E}_e^*(\mu^2))_{\mathbf{L}^2(D)} \\
&:= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Since  $\mathbf{E}_e^*(\mu^1) = S_{\mathbf{u}_e^*(\mu^1)}(\mu^1)$ , we with the aid of 4, 11 and 5.2 get

$$\begin{aligned}
J_1 &\leq C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^1) - \Theta_q^\epsilon(\mu^2)|^2 \right)^{1/2} (\|S_{\mathbf{u}_e^*(\mu^2)}(\mu^1) - S_{\mathbf{u}_e^*(\mu^1)}(\mu^1)\|_{\mathbf{L}^2(D)}) \\
&\leq C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^1) - \Theta_q^\epsilon(\mu^2)|^2 \right)^{1/2} \|\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)\|_{\mathbf{L}^2(\Omega)} \\
&\leq 4^{-1} \alpha \epsilon \|\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)\|_{\mathbf{L}^2(\Omega)}^2 \\
&\quad + C \alpha^{-1} \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^1) - \Theta_q^\epsilon(\mu^2)|^2
\end{aligned}$$

and

$$J_2 + J_3 \leq C \left( \sum_{q=1}^{Q^\sigma} |\Theta_q^\sigma(\mu^1) - \Theta_q^\sigma(\mu^2)|^2 \right)^{1/2} + C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^1) - \Theta_q^\epsilon(\mu^2)|^2 \right)^{1/2}$$

and

$$\begin{aligned} J_4 + J_5 &= (\epsilon(\mu^2)(\mathbf{E}_e^*(\mu^1) - \mathbf{E}_d(\mu^1)), \mathbf{E}_e^*(\mu^2) - \mathbf{E}_e^*(\mu^1))_{\mathbf{L}^2(D)} \\ &\quad + (\epsilon(\mu^2)(\mathbf{E}_e^*(\mu^2) - \mathbf{E}_d(\mu^1)), \mathbf{E}_e^*(\mu^1) - \mathbf{E}_e^*(\mu^2))_{\mathbf{L}^2(D)} \\ &\quad + (\epsilon(\mu^2)(\mathbf{E}_d(\mu^1) - \mathbf{E}_d(\mu^2)), \mathbf{E}_e^*(\mu^1) - \mathbf{E}_e^*(\mu^2))_{\mathbf{L}^2(D)} \\ &\leq -\epsilon \|\mathbf{E}_e^*(\mu^2) - \mathbf{E}_e^*(\mu^1)\|_{\mathbf{L}^2(D)}^2 \\ &\quad + (\epsilon(\mu^2)(\mathbf{E}_d(\mu^1) - \mathbf{E}_d(\mu^2)), \mathbf{E}_e^*(\mu^1) - \mathbf{E}_e^*(\mu^2))_{\mathbf{L}^2(D)} \\ &\leq C \|\mathbf{E}_d(\mu^1) - \mathbf{E}_d(\mu^2)\|_{\mathbf{L}^2(D)} \\ &\leq C \left( \sum_{q=1}^{Q^{\mathbf{E}_d}} |\Theta_q^{\mathbf{E}_d}(\mu^1) - \Theta_q^{\mathbf{E}_d}(\mu^2)|^2 \right)^{1/2}. \end{aligned}$$

Therefore, we arrive at

$$\begin{aligned} I_3 &\leq 4^{-1} \alpha \epsilon \|\mathbf{u}_e^*(\mu^2) - \mathbf{u}_e^*(\mu^1)\|_{\mathbf{L}^2(\Omega)}^2 + C \alpha^{-1} \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^2) - \Theta_q^\epsilon(\mu^1)|^2 \\ &\quad + C \left( \sum_{q=1}^{Q^\epsilon} |\Theta_q^\epsilon(\mu^2) - \Theta_q^\epsilon(\mu^1)|^2 \right)^{1/2} + C \left( \sum_{q=1}^{Q^\sigma} |\Theta_q^\sigma(\mu^1) - \Theta_q^\sigma(\mu^2)|^2 \right)^{1/2} \\ &\quad + C \left( \sum_{q=1}^{Q^{\mathbf{E}_d}} |\Theta_q^{\mathbf{E}_d}(\mu^1) - \Theta_q^{\mathbf{E}_d}(\mu^2)|^2 \right)^{1/2}. \end{aligned}$$

The desired estimate follows from the bounds for  $I_1$ ,  $I_2$ ,  $I_3$ , which finishes the proof.  $\square$

Now we state the similar results for the finite dimensional approximation problem  $(\mathbb{P}_h)$  and the reduced basis approach  $(\mathbb{P}_N)$ , their proofs follow exactly as in the continuous case  $(\mathbb{P}_e)$ , therefore omitted here.

**Lemma 5.4.** *Let  $\mathbf{u}_h^*(\mu)$  and  $\mathbf{u}_N^*(\mu)$  respectively be the solution of the problems  $(\mathbb{P}_h)$  and  $(\mathbb{P}_N)$  at the given parameter  $\mu \in \mathcal{P}$ . Then, the estimates*

$$\|\mathbf{u}_h^*(\mu^1) - \mathbf{u}_h^*(\mu^2)\|_{\mathbf{L}^2(\Omega)} \leq \mathcal{B}(\mu^1, \mu^2) \quad \text{and} \quad \|\mathbf{u}_N^*(\mu^1) - \mathbf{u}_N^*(\mu^2)\|_{\mathbf{L}^2(\Omega)} \leq \mathcal{B}(\mu^1, \mu^2)$$

hold true for all  $\mu^1, \mu^2 \in \mathcal{P}$ .

We are in the position to state the main result of this section on the uniform convergence of reduced order solutions. To do so, we assume  $\mathbf{u}_h^*(\mu) = \mathbf{u}_N^*(\mu)$  for  $\mu$  belonging to the parameter sample  $\mathcal{P}_N$ . For the state equation, this assumption is the basic consistency property of an Reduced Basis scheme, which simply put is the reproduction of solutions (cf. [18, Proposition 2.20]). For a justification of this assumption for optimal control problems, see [1, pp. A282].

**Theorem 5.5.** Assume that the functions  $\Theta_q^\sigma, \Theta_q^\epsilon, \Theta_q^{\mathbf{u}^d}, \Theta_q^{\mathbf{E}^d} : \mathcal{P} \rightarrow \mathbb{R}$  are Hölder continuous, i.e.

$$\begin{aligned} |\Theta_q^\sigma(\mu^1) - \Theta_q^\sigma(\mu^2)| &\leq L^\sigma \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^\sigma} \\ |\Theta_q^\epsilon(\mu^1) - \Theta_q^\epsilon(\mu^2)| &\leq L^\epsilon \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^\epsilon} \\ |\Theta_q^{\mathbf{u}^d}(\mu^1) - \Theta_q^{\mathbf{u}^d}(\mu^2)| &\leq L^{\mathbf{u}^d} \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^{\mathbf{u}^d}} \\ |\Theta_q^{\mathbf{E}^d}(\mu^1) - \Theta_q^{\mathbf{E}^d}(\mu^2)| &\leq L^{\mathbf{E}^d} \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^{\mathbf{E}^d}} \end{aligned}$$

for all  $\mu^1, \mu^2 \in \mathcal{P}$  and all the index  $q$  with some positive constants  $L^\sigma, L^\epsilon, L^{\mathbf{u}^d}, L^{\mathbf{E}^d}$  and  $\gamma^\sigma, \gamma^\epsilon, \gamma^{\mathbf{u}^d}, \gamma^{\mathbf{E}^d}$ . For any given  $\mu \in \mathcal{P}$  let  $\mathbf{u}_h^*(\mu)$  and  $\mathbf{u}_N^*(\mu)$  be the solutions of the problems  $(\mathbb{P}_h)$  and  $(\mathbb{P}_N)$ , respectively. Then the estimate

$$\|\mathbf{u}_h^*(\mu) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} \leq C \kappa_N^\gamma$$

is established, where  $\gamma := \frac{1}{2} \min(\gamma^\sigma, \gamma^\epsilon, 2\gamma^{\mathbf{u}^d}, \gamma^{\mathbf{E}^d}) > 0$ .

*Proof.* For all  $\mu^1, \mu^2 \in \mathcal{P}$  we deduce from Lemma 5.4 that

$$\begin{aligned} &\max \left( \|\mathbf{u}_h^*(\mu^1) - \mathbf{u}_h^*(\mu^2)\|_{\mathbf{L}^2(\Omega)}, \|\mathbf{u}_N^*(\mu^1) - \mathbf{u}_N^*(\mu^2)\|_{\mathbf{L}(\Omega)}^2 \right) \\ &\leq C \left( \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^\epsilon} + \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^\epsilon/2} \right) \\ &\quad + C \left( \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^{\mathbf{u}^d}} + \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^{\mathbf{E}^d}/2} + \|\mu^1 - \mu^2\|_{\mathbb{R}^p}^{\gamma^{\mathbf{E}^d}/2} \right), \end{aligned}$$

where the positive constant  $C$  is independent of the parameters. For any fixed  $\mu \in \mathcal{P}$ , since the set  $\mathcal{P}_N$  is finite, there exists  $\mu^* \in \arg \min_{\mu' \in \mathcal{P}_N} \|\mu - \mu'\|_{\mathbb{R}^p}$ . By  $\mu^* \in \mathcal{P}_N$ , we get  $\mathbf{u}_h^*(\mu^*) = \mathbf{u}_N^*(\mu^*)$  and therefore obtain that

$$\begin{aligned} &\|\mathbf{u}_h^*(\mu) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} \\ &= \|\mathbf{u}_h^*(\mu) - \mathbf{u}_h^*(\mu^*) + \mathbf{u}_h^*(\mu^*) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} \\ &\leq \|\mathbf{u}_h^*(\mu) - \mathbf{u}_h^*(\mu^*)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h^*(\mu^*) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} \\ &\leq C \kappa_N^\gamma, \end{aligned}$$

which finishes the proof.  $\square$

**6. A posteriori error estimation.** In the greedy sampling procedure, a possible choice for the error estimator is that

$$\Delta_N(\mathcal{E}_N, V_N; \mu) = \|\mathbf{u}_h^*(\mu) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)}.$$

However, this estimator depends on  $\mathbf{u}_h^*$ , i.e. the full-dimensional problem  $(\mathbb{P}_h)$ . In view of a posteriori error estimates we wish to construct an error estimator which is independent of the solution to  $(\mathbb{P}_h)$ .

For a given  $\mu \in \mathcal{P}$  let  $(\mathbf{u}_N^*(\mu), \mathbf{E}_N^*(\mu), \mathbf{F}_N^*(\mu))$  satisfy the system 20a–20e. We consider  $\widehat{\mathbf{E}}_h(\mu) \in \mathcal{E}_h$  and  $\widehat{\mathbf{F}}_h(\mu) \in \mathcal{E}_h$  defined by

$$\begin{cases} \left( \sigma^{-1}(\mu) \nabla \times \widehat{\mathbf{E}}_h(\mu), \nabla \times \Phi_h \right)_{\mathbf{L}^2(\Omega)} &= (\epsilon(\mu) \mathbf{u}_N^*(\mu), \Phi_h)_{\mathbf{L}^2(\Omega)} \\ \left( \epsilon(\mu) \widehat{\mathbf{E}}_h(\mu), \nabla \phi_h \right)_{\mathbf{L}^2(\Omega)} &= -(\rho(\mu), \phi_h)_{L^2(\Omega)} \end{cases} \quad (21)$$

and

$$\begin{cases} \left( \sigma^{-1}(\mu) \nabla \times \widehat{\mathbf{F}}_h(\mu), \nabla \times \Phi_h \right)_{\mathbf{L}^2(\Omega)} &= (\epsilon(\mu) (\mathbf{E}_N^*(\mu) - \mathbf{E}_d(\mu)), \Phi_h)_{\mathbf{L}^2(D)} \\ \left( \epsilon(\mu) \widehat{\mathbf{F}}_h(\mu), \nabla \phi_h \right)_{\mathbf{L}^2(\Omega)} &= 0 \end{cases} \quad (22)$$

for all  $(\Phi_h, \phi_h) \in \mathcal{E}_h \times V_h$ , respectively. We further define the residuals  $R_{\mathbf{E}} := R_{\mathbf{E}}(\cdot; \mu) \in \mathcal{E}_h^*$  and  $R_{\mathbf{F}} := R_{\mathbf{F}}(\cdot; \mu) \in \mathcal{E}_h^*$  via the following identities

$$\begin{aligned} R_{\mathbf{E}}(\Phi_h; \mu) &:= (\epsilon(\mu) \mathbf{u}_N^*(\mu), \Phi_h)_{\mathbf{L}^2(\Omega)} - (\sigma^{-1}(\mu) \nabla \times \mathbf{E}_N^*(\mu), \nabla \times \Phi_h)_{\mathbf{L}^2(\Omega)} \\ R_{\mathbf{F}}(\Phi_h; \mu) &:= (\epsilon(\mu) (\mathbf{E}_N^*(\mu) - \mathbf{E}_d(\mu)), \Phi_h)_{\mathbf{L}^2(D)} - (\sigma^{-1}(\mu) \nabla \times \mathbf{F}_N^*(\mu), \nabla \times \Phi_h)_{\mathbf{L}^2(\Omega)} \end{aligned}$$

for all  $\Phi_h \in \mathcal{E}_h$ .

To begin, we present some auxiliary results.

**Lemma 6.1.** *Let  $(\mathbf{u}_h^*(\mu), \mathbf{E}_h^*(\mu), \mathbf{F}_h^*(\mu))$  and  $(\mathbf{u}_N^*(\mu), \mathbf{E}_N^*(\mu), \mathbf{F}_N^*(\mu))$  respectively satisfy the systems 17a–17e and 20a–20e at a given  $\mu \in \mathcal{P}$ . Then the following inequalities*

$$\begin{aligned} \|\mathbf{E}_h^*(\mu) - \widehat{\mathbf{E}}_h(\mu)\|_{\mathbf{H}(\text{curl}; \Omega)} &\leq C \|\mathbf{u}_h^*(\mu) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} \\ \|\mathbf{F}_h^*(\mu) - \widehat{\mathbf{F}}_h(\mu)\|_{\mathbf{H}(\text{curl}; \Omega)} &\leq C \|\mathbf{E}_h^*(\mu) - \mathbf{E}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} \end{aligned}$$

are satisfied.

*Proof.* By 14, we have

$$\begin{aligned} \|\mathbf{E}_h^* - \widehat{\mathbf{E}}_h\|_{\mathbf{H}(\text{curl}; \Omega)}^2 &\leq C (\sigma^{-1} \nabla \times (\mathbf{E}_h^* - \widehat{\mathbf{E}}_h), \nabla \times (\mathbf{E}_h^* - \widehat{\mathbf{E}}_h))_{\mathbf{L}^2(\Omega)} \\ &= C \left( \epsilon (\mathbf{u}_h^* - \mathbf{u}_N^*), \mathbf{E}_h^* - \widehat{\mathbf{E}}_h \right)_{\mathbf{L}^2(\Omega)} \\ &\leq C \|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)} \|\mathbf{E}_h^* - \widehat{\mathbf{E}}_h\|_{\mathbf{H}(\text{curl}; \Omega)} \end{aligned}$$

which implies the first inequality. Likewise, we get

$$\begin{aligned} \|\mathbf{F}_h^* - \widehat{\mathbf{F}}_h\|_{\mathbf{H}(\text{curl}; \Omega)}^2 &\leq C (\sigma^{-1} \nabla \times (\mathbf{F}_h^* - \widehat{\mathbf{F}}_h), \nabla \times (\mathbf{F}_h^* - \widehat{\mathbf{F}}_h))_{\mathbf{L}^2(\Omega)} \\ &= C \left( \epsilon (\mathbf{E}_h^* - \mathbf{E}_N^*), \mathbf{F}_h^* - \widehat{\mathbf{F}}_h \right)_{\mathbf{L}^2(D)} \\ &\leq C \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{L}^2(\Omega)} \|\mathbf{F}_h^* - \widehat{\mathbf{F}}_h\|_{\mathbf{H}(\text{curl}; \Omega)}. \end{aligned}$$

The proof completes.  $\square$

**Lemma 6.2.** *The inequalities*

$$\begin{aligned} C_1 \|R_{\mathbf{E}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*} &\leq \|\mathbf{E}_N^*(\mu) - \widehat{\mathbf{E}}_h(\mu)\|_{\mathbf{H}(\text{curl}; \Omega)} \leq C_2 \|R_{\mathbf{E}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*} \\ C_1 \|R_{\mathbf{F}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*} &\leq \|\mathbf{F}_N^*(\mu) - \widehat{\mathbf{F}}_h(\mu)\|_{\mathbf{H}(\text{curl}; \Omega)} \leq C_2 \|R_{\mathbf{F}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*} \end{aligned}$$

hold true, where the positive constants  $C_1$  and  $C_2$  are independent of  $h$  and  $\mu$ .

*Proof.* From the assumed coercivity condition (cf. 14 and 18) for reduced basis, we have that

$$\begin{aligned} \|\mathbf{E}_N^* - \widehat{\mathbf{E}}_h\|_{\mathbf{H}(\text{curl}; \Omega)}^2 &\leq C \left( \sigma^{-1} \nabla \times (\mathbf{E}_N^* - \widehat{\mathbf{E}}_h), \nabla \times (\mathbf{E}_N^* - \widehat{\mathbf{E}}_h) \right)_{\mathbf{L}^2(\Omega)} \\ &= C \left( \epsilon \mathbf{u}_N^*, \widehat{\mathbf{E}}_h - \mathbf{E}_N^* \right)_{\mathbf{L}^2(\Omega)} \\ &\quad - C \left( \sigma^{-1} \nabla \times \mathbf{E}_N^*, \nabla \times (\widehat{\mathbf{E}}_h - \mathbf{E}_N^*) \right)_{\mathbf{L}^2(\Omega)} \\ &= C R_{\mathbf{E}}(\widehat{\mathbf{E}}_h - \mathbf{E}_N^*) \\ &\leq C \|R_{\mathbf{E}}\|_{\mathbf{H}(\text{curl}; \Omega)^*} \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)}. \end{aligned}$$

To show the lower bound, we first take  $r_{\mathbf{E}} \in \mathcal{E}_h$  such that

$$R_{\mathbf{E}}(\Phi_h) = (r_{\mathbf{E}}; \Phi_h)_{\mathbf{H}(\text{curl}; \Omega)}, \quad \forall \Phi_h \in \mathcal{E}_h \text{ and } \|r_{\mathbf{E}}\|_{\mathbf{H}(\text{curl}; \Omega)} = \|R_{\mathbf{E}}\|_{\mathbf{H}(\text{curl}; \Omega)^*},$$

by Riesz Representation Theorem. In view of the above argument, we arrive at

$$\begin{aligned}
\|R_{\mathbf{E}}\|_{\mathbf{H}(\mathbf{curl};\Omega)^*}^2 &= (r_{\mathbf{E}}, r_{\mathbf{E}})_{\mathbf{H}(\mathbf{curl};\Omega)} \\
&= R_{\mathbf{E}}(r_{\mathbf{E}}) \\
&= (\epsilon \mathbf{u}_N^*, r_{\mathbf{E}})_{\mathbf{L}^2(\Omega)} - (\sigma^{-1} \nabla \times \mathbf{E}_N^*, \nabla \times r_{\mathbf{E}})_{\mathbf{L}^2(\Omega)} \\
&= \left( \sigma^{-1} \nabla \times (\widehat{\mathbf{E}}_h - \mathbf{E}_N^*), \nabla \times r_{\mathbf{E}} \right)_{\mathbf{L}^2(\Omega)} \\
&\leq \underline{\sigma}^{-1} \|r_{\mathbf{E}}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\mathbf{curl};\Omega)} \\
&= \underline{\sigma}^{-1} \|R_{\mathbf{E}}\|_{\mathbf{H}(\mathbf{curl};\Omega)^*} \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\mathbf{curl};\Omega)}.
\end{aligned}$$

The remaining inequalities follows by the same arguments, therefore omitted here.  $\square$

We now state the main results of the section.

**Theorem 6.3.** *Let  $(\mathbf{u}_h^*(\mu), \mathbf{E}_h^*(\mu), \mathbf{F}_h^*(\mu))$  and  $(\mathbf{u}_N^*(\mu), \mathbf{E}_N^*(\mu), \mathbf{F}_N^*(\mu))$  satisfy the systems 17a–17e and 20a–20e at a given  $\mu \in \mathcal{P}$ , respectively. Then the estimates*

$$\begin{aligned}
\underline{C} &\leq \|\mathbf{u}_h^*(\mu) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}_h^*(\mu) - \mathbf{E}_N^*(\mu)\|_{\mathbf{H}(\mathbf{curl};\Omega)} \\
&\quad + \|\mathbf{F}_h^*(\mu) - \mathbf{F}_N^*(\mu)\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq \overline{C}
\end{aligned}$$

are satisfied, where

$$\overline{C} = \overline{C}_{\mathbf{E}} \|R_{\mathbf{E}}(\cdot; \mu)\|_{\mathbf{H}(\mathbf{curl};\Omega)^*} + \overline{C}_{\mathbf{F}} \|R_{\mathbf{F}}(\cdot; \mu)\|_{\mathbf{H}(\mathbf{curl};\Omega)^*}$$

and

$$\underline{C} = \underline{C}_{\mathbf{E}} \|R_{\mathbf{E}}(\cdot; \mu)\|_{\mathbf{H}(\mathbf{curl};\Omega)^*} + \underline{C}_{\mathbf{F}} \|R_{\mathbf{F}}(\cdot; \mu)\|_{\mathbf{H}(\mathbf{curl};\Omega)^*}$$

with the positive constants  $\overline{C}_{\mathbf{E}}, \overline{C}_{\mathbf{F}}, \underline{C}_{\mathbf{E}}, \underline{C}_{\mathbf{F}}$  independent of  $h$  and  $\mu$ .

*Proof.* First we establish the upper bound. By variational inequalities 17e and 20e, we have

$$\left( \epsilon(\mathbf{u}_N^* - \mathbf{u}_h^*), \mathbf{u}_d - \frac{1}{\alpha} \mathbf{F}_h^* - \mathbf{u}_h^* \right)_{\mathbf{L}^2(\Omega)} \leq 0, \quad \left( \epsilon(\mathbf{u}_h^* - \mathbf{u}_N^*), \mathbf{u}_d - \frac{1}{\alpha} \mathbf{F}_N^* - \mathbf{u}_N^* \right)_{\mathbf{L}^2(\Omega)} \leq 0$$

which implies that

$$\begin{aligned}
\alpha \underline{C} \|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)}^2 &\leq (\epsilon(\mathbf{u}_N^* - \mathbf{u}_h^*), \mathbf{F}_h^* - \mathbf{F}_N^*)_{\mathbf{L}^2(\Omega)} \\
&= \left( \epsilon(\mathbf{u}_N^* - \mathbf{u}_h^*), \mathbf{F}_h^* - \widehat{\mathbf{F}}_h \right)_{\mathbf{L}^2(\Omega)} + \left( \epsilon(\mathbf{u}_N^* - \mathbf{u}_h^*), \widehat{\mathbf{F}}_h - \mathbf{F}_N^* \right)_{\mathbf{L}^2(\Omega)}.
\end{aligned} \tag{23}$$

Now we from 21, 17a, 17c and 22 get

$$\begin{aligned}
& \left( \epsilon(\mathbf{u}_N^* - \mathbf{u}_h^*), \mathbf{F}_h^* - \widehat{\mathbf{F}}_h \right)_{\mathbf{L}^2(\Omega)} \\
&= \left( \sigma^{-1} \nabla \times (\widehat{\mathbf{E}}_h - \mathbf{E}_h^*), \nabla \times (\mathbf{F}_h^* - \widehat{\mathbf{F}}_h) \right)_{\mathbf{L}^2(\Omega)} \\
&= \left( \epsilon(\mathbf{E}_h^* - \mathbf{E}_N^*), \widehat{\mathbf{E}}_h - \mathbf{E}_h^* \right)_{\mathbf{L}^2(D)} \\
&= \left( \epsilon(\mathbf{E}_h^* - \mathbf{E}_N^*), \widehat{\mathbf{E}}_h - \mathbf{E}_N^* + \mathbf{E}_N^* - \mathbf{E}_h^* \right)_{\mathbf{L}^2(D)} \\
&\leq \bar{\epsilon} \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{L}^2(D)} \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{L}^2(D)} - \underline{\epsilon} \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{L}^2(D)}^2 \\
&\leq 4^{-1} \bar{\epsilon}^2 \underline{\epsilon}^{-1} \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{L}^2(D)}^2 + \underline{\epsilon} \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{L}^2(D)}^2 - \underline{\epsilon} \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{L}^2(D)}^2 \\
&\leq C \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)}^2
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
& \left( \epsilon(\mathbf{u}_N^* - \mathbf{u}_h^*), \widehat{\mathbf{F}}_h - \mathbf{F}_N^* \right)_{\mathbf{L}^2(\Omega)} \\
&\leq 2^{-1} \alpha \underline{\epsilon} \|\mathbf{u}_N^* - \mathbf{u}_h^*\|_{\mathbf{L}^2(\Omega)}^2 + 2^{-1} \alpha^{-1} \underline{\epsilon}^{-1} \bar{\epsilon}^2 \|\widehat{\mathbf{F}}_h - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)}^2.
\end{aligned} \tag{25}$$

We thus have from 23–25 that

$$\begin{aligned}
& \|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)} \\
&\leq C \left( \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\widehat{\mathbf{F}}_h - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \right).
\end{aligned} \tag{26}$$

An application of Lemma 6.1 and 26 yield

$$\begin{aligned}
\|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} &\leq \|\mathbf{E}_h^* - \widehat{\mathbf{E}}_h\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \\
&\leq C \|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)} + \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \\
&\leq C \left( \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\widehat{\mathbf{F}}_h - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \right).
\end{aligned} \tag{27}$$

Furthermore, using Lemma 6.1 again, the last inequality gives

$$\begin{aligned}
\|\mathbf{F}_h^* - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} &\leq \|\mathbf{F}_h^* - \widehat{\mathbf{F}}_h\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\widehat{\mathbf{F}}_h - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \\
&\leq C \left( \|\widehat{\mathbf{E}}_h - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\widehat{\mathbf{F}}_h - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \right).
\end{aligned} \tag{28}$$

Combining the inequalities 26–28 with Lemma 6.2, we therefore arrive at the upper bound. Next, we will derive for the lower bound. We from Lemma 6.1 and Lemma 6.2 get that

$$\begin{aligned}
C_1 \|R_{\mathbf{E}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*} &\leq \|\widehat{\mathbf{E}}_h - \mathbf{E}_h^*\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \\
&\leq C \|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \\
&\leq \max(2, C) \left( \|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)} + 2^{-1} \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \right)
\end{aligned}$$

and

$$\begin{aligned}
C_1 \|R_{\mathbf{F}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*} &\leq \|\widehat{\mathbf{F}}_h - \mathbf{F}_h^*\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\mathbf{F}_h^* - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \\
&\leq C \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\mathbf{F}_h^* - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \\
&\leq \max(1, 2C) \left( 2^{-1} \|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} + \|\mathbf{F}_h^* - \mathbf{F}_N^*\|_{\mathbf{H}(\text{curl}; \Omega)} \right)
\end{aligned}$$

which completes the proof.  $\square$

We aim towards an error bound for the cost functional.



**Theorem 6.4.** *Let  $(\mathbf{u}_h^*(\mu), \mathbf{E}_h^*(\mu), \mathbf{F}_h^*(\mu))$  and  $(\mathbf{u}_N^*(\mu), \mathbf{E}_N^*(\mu), \mathbf{F}_N^*(\mu))$  satisfy the systems 17a–17e and 20a–20e at a given  $\mu \in \mathcal{P}$ , respectively. Then,*

$$|J(\mathbf{u}_h^*, \mathbf{E}_h^*; \mu) - J(\mathbf{u}_N^*, \mathbf{E}_N^*; \mu)| \leq C^J,$$

with

$$C^J = C_{\mathbf{E}}^J \|R_{\mathbf{E}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*} + C_{\mathbf{F}}^J \|R_{\mathbf{F}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*}$$

and the positive constants  $C^J, C_{\mathbf{E}}^J, C_{\mathbf{F}}^J$  independent of  $h$  and  $\mu$ .

*Proof.* We get that

$$\begin{aligned} & 2 |J(\mathbf{u}_h^*, \mathbf{E}_h^*; \mu) - J(\mathbf{u}_N^*, \mathbf{E}_N^*; \mu)| \\ &= (\epsilon(\mathbf{E}_h^* - \mathbf{E}_N^*), \mathbf{E}_h^* + \mathbf{E}_N^* - 2\mathbf{E}_d)_{\mathbf{L}^2(D)} \\ & \quad + \alpha (\epsilon(\mathbf{u}_h^* - \mathbf{u}_N^*), \mathbf{u}_h^* + \mathbf{u}_N^* - 2\mathbf{u}_d)_{\mathbf{L}^2(\Omega)} \\ & \leq C (\|\mathbf{E}_h^* - \mathbf{E}_N^*\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)}). \end{aligned} \quad (29)$$

The desired inequality follows directly from the last inequality together with 27, 26 and Lemma 6.2, which finishes the proof.  $\square$

We derive a posteriori error estimators.

**Theorem 6.5.** *The absolute a posteriori error estimator*

$$\|\mathbf{u}_h^*(\mu) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)} \leq \Delta_N^{ab}(\mu) \quad (30)$$

is established, where

$$\Delta_N^{ab}(\mu) := C (\|R_{\mathbf{E}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*} + \|R_{\mathbf{F}}(\cdot; \mu)\|_{\mathbf{H}(\text{curl}; \Omega)^*}).$$

Furthermore, in case  $\frac{2\Delta_N^{ab}}{\|\mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)}} \leq 1$  we have the relative a posteriori error estimator

$$\frac{\|\mathbf{u}_h^*(\mu) - \mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)}}{\|\mathbf{u}_h^*(\mu)\|_{\mathbf{L}^2(\Omega)}} \leq \Delta_N^{re}(\mu) := \frac{2\Delta_N^{ab}(\mu)}{\|\mathbf{u}_N^*(\mu)\|_{\mathbf{L}^2(\Omega)}}. \quad (31)$$

*Proof.* By 26 and Lemma 6.2 we obtain 30. It remains to show 31. We get

$$\|\mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)} - \|\mathbf{u}_h^*\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)} \leq \Delta_N^{ab} \leq 2^{-1} \|\mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)}.$$

and arrive at  $2^{-1} \|\mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}_h^*\|_{\mathbf{L}^2(\Omega)}$ . Therefore, we conclude

$$\frac{\|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)}}{\|\mathbf{u}_h^*\|_{\mathbf{L}^2(\Omega)}} \leq \frac{2\|\mathbf{u}_h^* - \mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)}}{\|\mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)}} \frac{2\Delta_N^{ab}}{\|\mathbf{u}_N^*\|_{\mathbf{L}^2(\Omega)}},$$

which finishes the proof.  $\square$

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