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A finite element method for degenerate two-phase flow in porous media. Part II: Convergence

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Abstract: Convergence of a finite element method with mass-lumping and flux upwinding is formulated for solving the immiscible two-phase flow problem in porous media. The method approximates directly the wetting phase pressure and saturation, which are the primary unknowns. Well-posedness is obtained in [J. Numer. Math., 29(2), 2021]. Theoretical convergence is proved via a compactness argument. The numerical phase saturation converges strongly to a weak solution in L^2 in space and in time whereas the numerical phase pressures converge strongly to weak solutions in L^2 in space almost everywhere in time. The proof is not straightforward because of the degeneracy of the phase mobilities and the unboundedness of the derivative of the capillary pressure.

Keywords: convergence, degenerate coefficients, unbounded capillary pressure, weak solution

Classification: 65M12, 65M60

1 Introduction

This work establishes convergence via a compactness argument of a simplicial, first order finite element scheme for an immiscible two-phase flow problem in porous media. This scheme, set in a polyhedral domain, directly approximates the wetting phase pressure and saturation. The numerical method uses mass lumping to compute the integrals, in a fashion similar to that of the formulation proposed in [5]. The relationship between the mass lumped finite element method and finite volume methods has been highlighted in [4]. The finite element method with mass lumping allows for structured or unstructured finite element triangulations with the restriction that each angle be not larger than $\pi/2$ in view of preserving the maximum principle. In contrast to finite volume methods, no orthogonality constraint is required on the mesh. Our scheme utilizes a special upwinding to compute the fluxes, based on the nodal values of solutions.

Because of the generality of the problem (degeneracy of the phase mobilities and unboundedness of the derivative of the capillary pressure), the numerical analysis of the scheme is convoluted and technical and it is presented in two parts. The well-posedness of the scheme and the maximum principle for the saturation were shown in [7] thanks to upwinding. This present work is the second part of the analysis and its objective is to show strong convergence of the discrete saturation in L^2 in space and time, and strong convergence of the discrete pressures in L^2 in space, almost everywhere in time, to the weak solutions of the problem (see Theorem 1.1). A priori bounds on the phase pressures are difficult to obtain because of the degeneracy of the coefficients (phase mobilities) and the unboundedness of the capillary pressure. The argument is to first bound the sum of the phase pressure and an intermediate function g, which is a weighted primitive of the capillary pressure. Next a priori bounds for g are derived: the proof is technical and given in Section 2.

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Section 3 presents weak convergence, compactness in time and convergence of the numerical solutions. The last step of the convergence proof is to pass to the limit in the scheme, which is done in Section 4.

1.1 Statement of the problem

We consider the two-phase flow problem, with unknowns s and p_w as wetting phase saturation and pressure, in a domain Ω (polygon in 2D or polyhedron in 3D) and over the time interval $(0, T)$:

$$\partial_t(\varphi s) - \nabla\cdot(\eta_w(s)\nabla p_w) = f_w(s_{\rm in})\overline{q} - f_w(s)\underline{q} \tag{1.1}$$

$$-\partial_t(\varphi s) - \nabla\cdot(\eta_o(s)\nabla p_o) = f_o(s_{\rm in})\overline{q} - f_o(s)\underline{q} \tag{1.2}$$

complemented by a natural boundary condition, $\eta_w(s)\nabla p_w\cdot\mathbf{n} = 0$, $\eta_o(s)\nabla p_o\cdot\mathbf{n} = 0$, almost everywhere on $\partial\Omega\times]0,T[$, and an initial condition, $s(\cdot,0) = s^0$ almost everywhere in Ω, with $0 \leqslant s^0 \leqslant 1$. The wetting (resp., non-wetting) phase pressure, mobility and fractional flow are p_w, η_w, and f_w (resp., p_o, η_o, and f_o). The given flow rates are $\overline{q}$, $\underline{q}$; φ is the porosity and $s_{\rm in}$ is a given input saturation satisfying $0 \leqslant s_{\rm in} \leqslant 1$. The capillary pressure and fractional flows are defined by

$$\forall s\in[0,1],\quad p_c(s) = p_o - p_w,\quad f_w(s) = \frac{\eta_w(s)}{\eta_w(s)+\eta_o(s)},\quad f_o(s) = 1 - f_w(s). \tag{1.3}$$

Because the phase mobilities are degenerate when they are evaluated at some values of the saturations and the derivative of the capillary pressure is unbounded, this system of two coupled nonlinear partial differential equations has coefficients that vanish in parts of the domain, resulting in a loss of ellipticity; this degeneracy makes the numerical analysis challenging.

We present the assumptions made on the data. The porosity φ is piecewise constant in space, independent of time, positive, bounded and uniformly bounded away from zero. The mobility of the wetting phase $\eta_w \geqslant 0$ (resp., non-wetting phase $\eta_o \geqslant 0$) is continuous and increasing (resp., decreasing) over the interval [0,1]. This implies that the function f_w is increasing and the function f_o is decreasing. The capillary pressure p_c is a continuous, strictly decreasing function in $W^{1,1}(0,1)$. The flow rates at the injection and production wells, $\overline{q}, \underline{q} \in L^2(\Omega\times]0,T[)$ satisfy

$$\overline{q}\geqslant 0,\quad \underline{q}\geqslant 0,\quad \int_\Omega \overline{q} = \int_\Omega \underline{q}. \tag{1.4}$$

In addition, the mobilities $\eta_\alpha \in W^{1,\infty}(0,1)$, $\alpha = w, o$, satisfy $\eta_w(0) = \eta_o(1) = 0$,

$$\eta_w(s) + \eta_o(s) \geqslant \eta_* \quad \forall s\in[0,1] \tag{1.5}$$

for a positive constant η_*. Furthermore, we assume that there are constants, α_o, α_w, α_3 in the interval $]0,1]$, positive constants β_3, β_4, and constants $\vartheta_o > 1$ and $\vartheta_w > 1$ such that for all $x\in]0,1[$,

$$\alpha_w x^{\vartheta_w-1} \leqslant \eta_w'(x) \leqslant \frac{1}{\alpha_w}x^{\vartheta_w-1} \tag{1.6}$$

$$\alpha_o(1-x)^{\vartheta_o-1} \leqslant -\eta_o'(x) \leqslant \frac{1}{\alpha_o}(1-x)^{\vartheta_o-1} \tag{1.7}$$

$$\frac{1}{\alpha_3}x^{\beta_3-1}(1-x)^{\beta_4-1} \geqslant -p_c'(x) \geqslant \alpha_3 x^{\beta_3-1}(1-x)^{\beta_4-1}. \tag{1.8}$$

From (1.6) and (1.7), we deduce, respectively, for all $x\in]0,1[$,

$$\frac{\alpha_w}{\vartheta_w}x^{\vartheta_w} \leqslant \eta_w(x) \leqslant \frac{1}{\alpha_w\vartheta_w}x^{\vartheta_w},\qquad \frac{\alpha_o}{\vartheta_o}(1-x)^{\vartheta_o} \leqslant \eta_o(x) \leqslant \frac{1}{\alpha_o\vartheta_o}(1-x)^{\vartheta_o}. \tag{1.9}$$

We sum these two inequalities and denote by ℓ the resulting lower bound. It is easy to check that ℓ is a non-negative continuous function of x on $[0,1]$, hence uniformly continuous, therefore bounded away from zero. Thus there exists a positive constant $C_{\min}$ such that

$$\forall x\in[0,1],\quad C_{\min} \leqslant \ell(x) \leqslant C_{\max},\quad C_{\max} = \max_{x\in[0,1]}\left(\frac{1}{\alpha_w\vartheta_w}x^{\vartheta_w} + \frac{1}{\alpha_o\vartheta_o}(1-x)^{\vartheta_o}\right). \tag{1.10}$$

Although the numerical scheme studied below does not discretize the global pressure, following [3], its convergence proof uses a number of auxiliary functions related to the global pressure. We introduce the primitive g_c of p_c, $g_c \in \mathcal{C}^1([0,1])$, g and the auxiliary pressures p_{wg}, p_{wo}, for all $x \in [0,1]$:

$$g_c(x) = \int_x^1 p_c(s)\,\mathrm{d}s, \qquad g(x) = -\int_0^x \frac{\eta_w(s)\eta_o(s)}{\eta_w(s)+\eta_o(s)} p_c'(s)\,\mathrm{d}s \tag{1.11}$$

$$p_{wg}(x) = \int_0^x f_o(s)p_c'(s)\,\mathrm{d}s, \qquad p_{og}(x) = \int_0^x f_w(s)p_c'(s)\,\mathrm{d}s. \tag{1.12}$$

Let $Q = \Omega \times]0,T[$. In [2], the weak problem is formally: Find s in $L^\infty(Q)$ with $g(s)$ in $L^2(0,T;H^1(\Omega))$, p_α, $\alpha = w, o$, in $L^2(Q)$ with both $p_w + p_{wg}(s)$ and $p_o - p_{og}(s)$ in $L^2(0,T;H^1(\Omega))$, satisfying for all $v \in \mathcal{C}^2(Q)$ that vanish at $t = T$,

$$\begin{aligned}
-\int_Q \varphi\, s\,\partial_t v + \int_Q \big(\eta_w(s)\nabla(p_w + p_{wg}(s)) + \nabla g(s)\big)\cdot\nabla v &= \int_\Omega \varphi\, s^0 v(0) + \int_Q \big(f_w(s_{\text{in}})\overline{q} - f_w(s)\underline{q}\big)v\\
\int_Q \varphi\, s\,\partial_t v + \int_Q \big(\eta_o(s)\nabla(p_o - p_{og}(s)) - \nabla g(s)\big)\cdot\nabla v &= -\int_\Omega \varphi\, s^0 v(0) + \int_Q \big(f_o(s_{\text{in}})\overline{q} - f_o(s)\underline{q}\big)v.
\end{aligned} \tag{1.13}$$

1.2 Scheme

Let Ω be a Lipschitz polyhedron and $\mathcal{T}_h$ a regular simplicial mesh, with h denoting the mesh size. Let X_h denote the finite element space of order one,

$$X_h = \{v_h \in \mathcal{C}^0(\overline{\Omega})\,;\ \forall K \in \mathcal{T}_h,\ v_h|_K \in \mathbb{P}_1\}. \tag{1.14}$$

Let M be the dimension of X_h and let φ_i be the Lagrange basis function that takes the value 1 at node i and the value 0 at the other nodes. For any function $V_h \in X_h$, we write $V_h = \sum_i V^i\varphi_i$, where V^i is the nodal value at node i. Let $I_h \in \mathcal{L}(\mathcal{C}^0(\overline{\Omega}); X_h)$ denote the Lagrange interpolation operator. Let $\tau = T/N$ be the time step, and $t_n = n\tau$, the discrete times, $0 \leqslant n \leqslant N$. The set of primary unknowns is the discrete wetting phase saturation and the discrete wetting phase pressure, S_h^n and $P_{w,h}^n$, defined pointwise at time t_n by:

$$S_h^n = \sum_{i=1}^M S^{n,i}\varphi_i, \quad P_{w,h}^n = \sum_{i=1}^M P_w^{n,i}\varphi_i, \quad 1 \leqslant n \leqslant N$$

where $S^{n,i}$ and $P_w^{n,i}$ are the nodal values of the discrete saturation and wetting phase pressure. Then the discrete non-wetting phase pressure $P_{o,h}^n$ defined by

$$P_{o,h}^n = \sum_{i=1}^M P_o^{n,i}\varphi_i, \quad 1 \leqslant n \leqslant N, \qquad P_o^{n,i} = p_c(S^{n,i}) + P_w^{n,i} \quad \forall i$$

is a secondary unknown. As usual, it is convenient to associate time functions $S_{h,\tau}$, $P_{\alpha,h,\tau}$ with the sequences indexed by n. These are piecewise constant in time in $]0,T[$, for instance

$$P_{\alpha,h,\tau}(t,x) = P_{\alpha,h}^n(x), \quad \alpha = w, o \quad \forall (x,t) \in \Omega \times]t_{n-1}, t_n]. \tag{1.15}$$

To enforce the maximum principle, the mesh is assumed to satisfy the property [1]:

$$\forall K, \quad \int_K \nabla\varphi_i \cdot \nabla\varphi_j \leqslant 0 \quad \forall i \neq j. \tag{1.16}$$

For a given node i, we denote by Δ_i the union of elements sharing the node i and by $\mathcal{N}(i)$ the set of indices of all the nodes in Δ_i. In the spirit of [8], we define

$$c_{ij} = \int_{\Delta_i\cap\Delta_j} |\nabla\varphi_i \cdot \nabla\varphi_j| \quad \forall i, j. \tag{1.17}$$

We introduce the following form:

$$\forall W_h, U_h, V_h, Z_h \in X_h, \quad [Z_h, W_h; V_h, U_h]_h = \sum_{i,j=1}^{M} U^i c_{ij} \widetilde{W}^{ij}(V^j - V^i) \tag{1.18}$$

where $\widetilde{W}^{ij}$ is either equal to W^i or to W^j, depending on Z_h as follows:

$$\widetilde{W}^{ij} = \begin{cases} W^i, & Z^i > Z^j \\ W^j, & Z^i < Z^j \\ \max(W^i, W^j), & Z^i = Z^j. \end{cases} \tag{1.19}$$

With the above notation, the finite element scheme is: Find S_h^n, P_{wh}^n, and $P_{o,h}^n$ in X_h, for $1 \leqslant n \leqslant N$, solution of, for all ϑ_h in X_h,

$$\frac{1}{\tau}(S_h^n - S_h^{n-1}, \vartheta_h)_h^{\varphi} - [P_{w,h}^n, I_h(\eta_w(S_h^n)); P_{w,h}^n, \vartheta_h]_h = (I_h(f_w(s_{\text{in},h}^n))\overline{q}_h^n - I_h(f_w(S_h^n))\underline{q}_h^n, \vartheta_h)_h \tag{1.20}$$

$$-\frac{1}{\tau}(S_h^n - S_h^{n-1}, \vartheta_h)_h^{\varphi} - [P_{o,h}^n, I_h(\eta_o(S_h^n)); P_{o,h}^n, \vartheta_h]_h = (I_h(f_o(s_{\text{in},h}^n))\overline{q}_h^n - I_h(f_o(S_h^n))\underline{q}_h^n, \vartheta_h)_h \tag{1.21}$$

$$P_{o,h}^n - P_{w,h}^n = I_h(p_c(S_h^n)) \tag{1.22}$$

$$(P_{w,h}^n, 1)_h = 0. \tag{1.23}$$

The inner-products are defined by:

$$\forall U_h, V_h \in X_h, \quad (U_h, V_h)_h = \sum_{i=1}^{M} m_i U^i V^i, \quad (U_h, V_h)_h^{\varphi} = \sum_{i=1}^{M} \widetilde{m}_i(\varphi) U^i V^i \tag{1.24}$$

with (using the notation $|\mathcal{O}|$ for the measure of any region $\mathcal{O}$)

$$m_i = \frac{1}{d+1}|\Delta_i|, \qquad \widetilde{m}_i(\varphi) = \frac{1}{d+1} \sum_{K \in \Delta_i} \varphi|_K |K|.$$

The initial saturation is $S_h^0 = r_h(s^0)$, where r_h is a nodal approximation operator defined at each node i for any function $g \in L^1(\Omega)$ by

$$(r_h(g))^i = \frac{1}{|\Delta_i|} \int_{\Delta_i} g, \quad 1 \leqslant i \leqslant M. \tag{1.25}$$

We define the time and space average operators ρ_τ and ρ_h by:

$$\rho_\tau(f)^n = \rho_\tau(f)|_{]t_{n-1},t_n]} = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f, \qquad \rho_h(f)|_K = \rho_K(f) = \frac{1}{|K|} \int_K f. \tag{1.26}$$

The input saturation s_{in} is approximated in space and time by $s_{\text{in,h},\tau} = \rho_\tau(r_h(s_{\text{in}}))$. In order to preserve (1.4), the functions $\overline{q}$ and $\underline{q}$ are approximated as follows

$$\overline{q}_{h,\tau} = \rho_\tau\left(r_h(\overline{q}) - \frac{1}{|\Omega|} \int_\Omega (r_h(\overline{q}) - \overline{q})\right), \qquad \underline{q}_{h,\tau} = \rho_\tau\left(r_h(\underline{q}) - \frac{1}{|\Omega|} \int_\Omega (r_h(\underline{q}) - \underline{q})\right). \tag{1.27}$$

The main result of this paper is the following convergence theorem. For this, we define

$$U_{w,h,\tau} = P_{w,h,\tau} + I_h(p_{wg}(S_{h,\tau})), \quad U_{o,h,\tau} = P_{o,h,\tau} - I_h(p_{og}(S_{h,\tau})). \tag{1.28}$$

Theorem 1.1 (main result). *The discrete solutions converge up to subsequences as follows:*

$$\lim_{(h,\tau)\to(0,0)} S_{h,\tau} = s \quad \textit{strongly in } L^r(Q), \quad 2 \leqslant r < \infty$$

$$\lim_{(h,\tau)\to(0,0)} P_{\alpha,h,\tau} = p_\alpha \quad \textit{strongly in } L^r(\Omega),\ a.e.\ in\ ]0,T[, \ \alpha = w, o, \quad 2 \leqslant r < 6$$

$$\lim_{(h,\tau)\to(0,0)} P_{\alpha,h,\tau} = p_\alpha \quad \textit{weakly in } L^2(Q), \ \alpha = w, o$$

where $p_w + p_{wg}(s)$, $p_o - p_{wg}(s)$, and s solve the weak formulation (1.13). *In addition,*

$$\lim_{(h,\tau)\to(0,0)} U_{w,h,\tau} = p_w + p_{wg}(s), \qquad \lim_{(h,\tau)\to(0,0)} U_{o,h,\tau} = p_o - p_{og}(s), \qquad \textit{weakly in } L^2(0,T;H^1(\Omega))$$

and $U_{w,h,\tau}$ (resp., $U_{o,h,\tau}$) converges strongly in $L^r(\Omega)$ for almost every t in $]0,T[$, $2 \leqslant r < 6$.

The proof of the theorem requires several steps that are covered in the remaining of this work. All constants below are independent of h and τ.

2 A priori pressure bounds

First a priori bounds were shown in [7]:

$$\eta_* \|\nabla U_{\alpha,h,\tau}\|^2_{L^2(Q)} \leqslant C, \quad \alpha = w, o. \tag{2.1}$$

In view of (1.13), it remains to derive a bound for the gradient of $g(S_{h,\tau})$. More precisely, we will prove the following theorem.

Theorem 2.1. *There exists a constant C, independent of h and τ, such that*

$$\|\nabla(I_h(g(S_{h,\tau})))\|_{L^2(Q)} \leqslant C. \tag{2.2}$$

Estimating the gradient of $g(S_{h,\tau})$ is a long and intricate process; it is based on the fact that

$$|g(S^{n,j}) - g(S^{n,i})|^2 \leqslant C(f_w(S^{n,i}) - f_w(S^{n,j}))(g(S^{n,i}) - g(S^{n,j}))$$

see (2.54). Therefore, we must derive a bound for the product of the gradients of g and f_w. This is split into several steps.

2.1 A preliminary inequality

Our starting step is the following inequality.

Proposition 2.1. There exists a constant C_1 independent of h and τ such that

$$R_1 := -\sum_{n=1}^{N} \tau \sum_{\alpha=o,w} [P^n_{\alpha,h}, \eta_\alpha(S^n_{\alpha,h}); f_\alpha(S^n_h), P^n_{\alpha,h}]_h \tag{2.3}$$

satisfies $|R_1| \leqslant C_1$.

Proof. By testing (1.20) with $I_h(f_w(S^n_h))$ and (1.21) with $I_h(f_o(S^n_h))$, adding the resulting equalities, and multiplying by τ, we obtain

$$\begin{aligned}\sum_{n=1}^{N} (S^n_h - S^{n-1}_h, f_w(S^n_h) - f_o(S^n_h))^\varphi_h - \sum_{n=1}^{N} \tau \sum_{\alpha=o,w} [P^n_{\alpha,h}, \eta_\alpha(S^n_{\alpha,h}); f_\alpha(S^n_h), P^n_{\alpha,h}]_h \\ = \int_0^T \Big((\bar{q}_{h,\tau}, \sum_{\alpha=o,w} f_\alpha(s_{\text{in},h,\tau}) f_\alpha(S_{h,\tau}))_h - (\underline{q}_{h,\tau}, \sum_{\alpha=o,w} f^2_\alpha(S_{h,\tau}))_h \Big) \leqslant 4\|\bar{q}\|_{L^1(Q)}\end{aligned} \tag{2.4}$$

in view of (1.3) and (1.4). To control the time difference of $S_{h,\tau}$, we introduce the global flux defined by

$$\forall x \in [0,1], \quad G(x) = \int_0^x (f_w(s) - f_o(s))\,\mathrm{d}s \tag{2.5}$$

and we write

$$(S^n_h - S^{n-1}_h)(f_w(S^n_h) - f_o(S^n_h)) = (S^n_h - S^{n-1}_h) G'(S^n_h).$$

But by (1.3), $G'(x) = 2f_w(x) - 1$ is increasing. Hence, we easily check that

$$G(S^n_h) - G(S^{n-1}_h) \leqslant (S^n_h - S^{n-1}_h) G'(S^n_h).$$

Thus, the properties of φ imply

$$\sum_{n=1}^{N} (S^n_h - S^{n-1}_h, f_w(S^n_h) - f_o(S^n_h))^\varphi_h \geqslant (G(S^N_h), 1)^\varphi_h - (G(S^0_h), 1)^\varphi_h.$$

But the boundedness of $S_{h,\tau}$, the continuity of f_α, and the properties of φ imply

$$\left|(G(S_h^N), 1)_h^\varphi - (G(S_h^0), 1)_h^\varphi\right| \leqslant C'$$

with a constant C' independent of h and τ. By substituting these inequalities into (2.4) we derive (2.3) with $C_1 = 4\,\|\overline{q}\|_{L^1(Q)} + C'$. □

2.2 Some discrete total flux inequalities

In this section, it is convenient to rewrite (1.20) and (1.21) as an equivalent formulation that involves the nodal values $S^{n,i}$ and $P_w^{n,i}$ (see [7]):

$$\frac{\widetilde{m}_i(\varphi)}{\tau}(S^{n,i} - S^{n-1,i}) - \sum_{j=1}^{M} c_{ij}\eta_w(S_w^{n,ij})(P_w^{n,j} - P_w^{n,i}) = m_i\left(f_w(s_{\text{in}}^{n,i})\overline{q}^{n,i} - f_w(S^{n,i})\underline{q}^{n,i}\right) \tag{2.6}$$

$$-\frac{\widetilde{m}_i(\varphi)}{\tau}(S^{n,i} - S^{n-1,i}) - \sum_{j=1}^{M} c_{ij}\eta_o(S_o^{n,ij})(P_o^{n,j} - P_o^{n,i}) = m_i\left(f_o(s_{\text{in}}^{n,i})\overline{q}^{n,i} - f_o(S^{n,i})\underline{q}^{n,i}\right). \tag{2.7}$$

Here i runs from 1 to $M - 1$ in (2.6) and from 1 to M in (2.7). Following [3], the sum of the equations (2.6) and (2.7) suggests defining a discrete anti-symmetric upwinded total flux,

$$F^{n,ij} = -\eta_w(S_w^{n,ij})(P_w^{n,j} - P_w^{n,i}) - \eta_o(S_o^{n,ij})(P_o^{n,j} - P_o^{n,i}) \tag{2.8}$$

that satisfies

$$\sum_{j\neq i, j\in\mathcal{N}(i)} c_{ij}F^{n,ij} = m_i(\overline{q}^{n,i} - \underline{q}^{n,i}). \tag{2.9}$$

By multiplying (2.9) with $\tau f_\alpha^2(S^{n,i})$, and summing, we obtain for $\alpha = w, o$:

$$\left|\sum_{n=1}^{N} \tau \sum_{i,j=1}^{M} f_\alpha^2(S^{n,i})c_{ij}F^{n,ij}\right| \leqslant 2\,\|\overline{q}\|_{L^1(Q)}. \tag{2.10}$$

To simplify some of the calculations below, it is convenient to drop the time superscript n, when there is no ambiguity, and restore it when needed. By using the relation (1.22), $F^{i,j}$ can also be written as

$$\begin{aligned} F^{ij} &= -\left(\eta_w(S_w^{ij}) + \eta_o(S_o^{ij})\right)(P_w^j - P_w^i) - \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) \\ &= -\left(\eta_w(S_w^{ij}) + \eta_o(S_o^{ij})\right)(P_o^j - P_o^i) + \eta_w(S_w^{ij})(p_c(S^j) - p_c(S^i)). \end{aligned} \tag{2.11}$$

In order to insert it into (2.3), we bring forward F^{ij} in the expressions for $\eta_\alpha(S_\alpha^{ij})(P_\alpha^j - P_\alpha^i)$, $\alpha = w, o$. Starting from the identity

$$\begin{aligned} \eta_w(S_w^{ij})(P_w^j - P_w^i) = f_w(S_w^{ij})\Big[&\left(\eta_w(S_w^{ij}) + \eta_o(S_o^{ij})\right)(P_w^j - P_w^i) + \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) \\ &- \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) + \left(\eta_o(S_w^{ij}) - \eta_o(S_o^{ij})\right)(P_w^j - P_w^i)\Big] \end{aligned}$$

the expression (2.11) leads to

$$\eta_w(S_w^{ij})(P_w^j - P_w^i) = f_w(S_w^{ij})\Big[-F^{ij} - \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) + \left(\eta_o(S_w^{ij}) - \eta_o(S_o^{ij})\right)(P_w^j - P_w^i)\Big]. \tag{2.12}$$

Similarly,

$$\eta_o(S_o^{ij})(P_o^j - P_o^i) = f_o(S_o^{ij})\Big[-F^{ij} + \eta_w(S_w^{ij})(p_c(S^j) - p_c(S^i)) + \left(\eta_w(S_o^{ij}) - \eta_w(S_w^{ij})\right)(P_o^j - P_o^i)\Big]. \tag{2.13}$$

We also introduce the anti-symmetric quantities that collect the terms other than F^{ij} in (2.12) and (2.13),

$$C_w^{ij} = \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) - \left(\eta_o(S_w^{ij}) - \eta_o(S_o^{ij})\right)(P_w^j - P_w^i) \tag{2.14}$$

$$C_o^{ij} = -\eta_w(S_w^{ij})(p_c(S^j) - p_c(S^i)) - (\eta_w(S_o^{ij}) - \eta_w(S_w^{ij}))(P_o^j - P_o^i). \tag{2.15}$$

With this notation, we have

$$\eta_\alpha(S_\alpha^{ij})(P_\alpha^j - P_\alpha^i) = f_\alpha(S_\alpha^{ij})[-F^{ij} - C_\alpha^{ij}], \quad \alpha = w, o.$$

Thus, the term that is summed over i in (2.3) has the expression

$$-\sum_{\alpha=w,o} f_\alpha(S^i) \sum_{j\neq i, j\in\mathcal{N}(i)} c_{ij}\eta_\alpha(S_\alpha^{ij})(P_\alpha^j - P_\alpha^i) = \sum_{\alpha=w,o} f_\alpha(S^i) \sum_{j\neq i, j\in\mathcal{N}(i)} c_{ij} f_\alpha(S_\alpha^{ij})(F^{ij} + C_\alpha^{ij}). \tag{2.16}$$

Now, we reintroduce the superscript n and to simplify, we set

$$A_{1,i,n} = \sum_{\alpha=w,o} f_\alpha(S^{n,i}) \sum_{j=1}^{M} c_{ij} f_\alpha(S_\alpha^{n,ij}) F^{n,ij} \tag{2.17}$$

$$A_{\alpha,i,n} = f_\alpha(S^{n,i}) \sum_{j=1}^{M} c_{ij} f_\alpha(S_\alpha^{n,ij}) C_\alpha^{n,ij}. \tag{2.18}$$

With this notation, our next proposition is derived by substituting (2.16)–(2.18) into (2.3).

Proposition 2.2. We have, with the quantity R_1 of (2.3),

$$\sum_{n=1}^{N} \tau \sum_{i=1}^{M} A_{1,i,n} + \sum_{n=1}^{N} \tau \sum_{i=1}^{M} \sum_{\alpha=w,o} A_{\alpha,i,n} = R_1. \tag{2.19}$$

We must transform suitably each term in this sum to bring forward g. Let us start with the first term of (2.19), i.e., the combination of the discrete total flux.

2.3 Combination of the discrete total flux

To simplify, let A_1 denote the first term,

$$A_1 = \sum_{n=1}^{N} \tau \sum_{i,j=1}^{M} \sum_{\alpha=w,o} \Big[f_\alpha(S^{n,i}) c_{ij} f_\alpha(S_\alpha^{n,ij}) F^{n,ij} \Big].$$

Inspired by (2.10), we introduce the difference

$$A_2 = A_1 - \sum_{n=1}^{N} \tau \sum_{i,j=1}^{M} (f_w^2(S^{n,i}) + f_o^2(S^{n,i})) c_{ij} F^{n,ij}.$$

Clearly, A_2 collects the discrepancies arising from the upwinding,

$$A_2 = \sum_{n=1}^{N} \tau \sum_{i,j=1}^{M} \sum_{\alpha=w,o} \Big[f_\alpha(S^{n,i}) c_{ij} \big(f_\alpha(S_\alpha^{n,ij}) - f_\alpha(S^{n,i}) \big) F^{n,ij} \Big]. \tag{2.20}$$

As (2.10) yields

$$A_1 = A_2 + R_2, \qquad |R_2| \leqslant 4\, \|\overline{q}\|_{L^1(Q)} \tag{2.21}$$

a bound for A_1 stems from a bound for A_2. To this end, in view of (2.20), it is useful to consider the five subsets of indices $j \in \mathcal{N}(i)$, $j \neq i$, union and intersection:

$$\begin{aligned}
\mathcal{N}_w(i) &= \{j \in \mathcal{N}(i)\,;\ P_w^{n,j} > P_w^{n,i}\}, \quad \mathcal{N}_o(i) = \{j \in \mathcal{N}(i)\,;\ P_o^{n,j} > P_o^{n,i}\}\\
\mathcal{N}_{w,S}(i) &= \{j \in \mathcal{N}(i),\ j \neq i\,;\ P_w^{n,j} = P_w^{n,i},\ S^{n,j} \geqslant S^{n,i}\}\\
\mathcal{N}_{o,S}(i) &= \{j \in \mathcal{N}(i),\ j \neq i\,;\ P_o^{n,j} = P_o^{n,i},\ S^{n,j} \leqslant S^{n,i}\}\\
\mathcal{UN}(i) &= \mathcal{N}_w(i) \cup \mathcal{N}_o(i) \cup \mathcal{N}_{w,S}(i) \cup \mathcal{N}_{o,S}(i)\\
\mathcal{N}_{\mathcal{F}}(i) &= \{j \in \mathcal{N}(i)\,;\ P_w^{n,i} > P_w^{n,j},\ P_o^{n,i} > P_o^{n,j}\}.
\end{aligned} \tag{2.22}$$

Strictly speaking, these subsets should we written with the superscript n, but we omit it for the sake of simplicity. Then we have the following estimate for A_2.

Proposition 2.3. We have,

$$A_2 = -\frac{1}{2}\sum_{n=1}^{N}\tau\sum_{i=1}^{M}\sum_{j\in\mathcal{UN}(i)} c_{ij}\big(f_w(S^{n,j}) - f_w(S^{n,i})\big)^2 F^{n,ij} + R_3 \tag{2.23}$$

where the remainder R_3 satisfies $|R_3| \leqslant 2\,\|\overline{q}\|_{L^1(Q)}$.

Proof. Let us drop the superscript n. By definition, $f_w(S_w^{ij}) - f_w(S^i) = 0$ when $P_w^i > P_w^j$ and when $P_w^i = P_w^j$ and $S^i > S^j$. Similarly, $f_o(S_o^{ij}) - f_o(S^i) = 0$ when $P_o^i > P_o^j$ and when $P_o^i = P_o^j$ and $S^i < S^j$. Therefore, the nth term in A_2, say a_2, reduces to

$$a_2 = \sum_{i=1}^{M}\sum_{\alpha=w,o} f_\alpha(S^i)\sum_{j\in\mathcal{N}_\alpha(i)\cup\mathcal{N}_{\alpha,S}(i)} c_{ij}\big(f_\alpha(S^j) - f_\alpha(S^i)\big)F^{ij}.$$

By expanding the products, this can be written

$$a_2 = -\frac{1}{2}\sum_{i=1}^{M}\sum_{\alpha=w,o}\sum_{j\in\mathcal{N}_\alpha(i)\cup\mathcal{N}_{\alpha,S}(i)} c_{ij}\big(f_\alpha^2(S^i) - f_\alpha^2(S^j) + (f_\alpha(S^i) - f_\alpha(S^j))^2\big)F^{ij}. \tag{2.24}$$

Since c_{ij} vanishes when j is not a neighbor of i, we have, by interchanging i and j and using the anti-symmetry of F^{ij} and the symmetry of c_{ij},

$$-\sum_{i=1}^{M}\sum_{j\in\mathcal{N}_w(i)} c_{ij}f_w^2(S^j)F^{ij} = \sum_{i=1,j=1,P_w^j<P_w^i}^{M} c_{ij}f_w^2(S^i)F^{ij}. \tag{2.25}$$

Similarly,

$$-\sum_{i=1}^{M}\sum_{j\in\mathcal{N}_{w,S}(i)} c_{ij}f_w^2(S^j)F^{ij} = \sum_{i=1,j=1,P_w^i=P_w^j,S^i\geqslant S^j}^{M} c_{ij}f_w^2(S^i)F^{ij}. \tag{2.26}$$

Hence

$$-\frac{1}{2}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}_w(i)} c_{ij}\big(f_w^2(S^i) - f_w^2(S^j)\big)F^{ij} = -\frac{1}{2}\sum_{i=1,j=1,P_w^i\neq P_w^j}^{M} c_{ij}f_w^2(S^i)F^{ij}$$

and

$$-\frac{1}{2}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}_{w,S}(i)} c_{ij}\big(f_w^2(S^i) - f_w^2(S^j)\big)F^{ij} = -\frac{1}{2}\sum_{i=1,j=1,P_w^i=P_w^j}^{M} c_{ij}f_w^2(S^i)F^{ij}$$

because there is no contribution from the indices i, j such that $P_w^i = P_w^j$, $S^i = S^j$ since in this case the factor $F^{ij} = 0$. The same is true for the non-wetting phase. Thus

$$-\frac{1}{2}\sum_{\alpha=w,o}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}_\alpha(i)\cup\mathcal{N}_{\alpha,S}(i)} c_{ij}\big(f_\alpha^2(S^i) - f_\alpha^2(S^j)\big)F^{ij} = -\frac{1}{2}\sum_{\alpha=w,o}\sum_{i=1,j=1}^{M} c_{ij}f_\alpha^2(S^i)F^{ij}.$$

By comparing with (2.10), we see that

$$\left|\frac{1}{2}\sum_{n=1}^{N}\tau\sum_{i=1}^{M}\sum_{\alpha=w,o}\sum_{j\in\mathcal{N}_\alpha(i)\cup\mathcal{N}_{\alpha,S}(i)} c_{ij}\big(f_\alpha^2(S^{n,i}) - f_\alpha^2(S^{n,j})\big)F^{n,ij}\right| \leqslant 2\|\overline{q}\|_{L^1(Q)}. \tag{2.27}$$

This and the equality

$$\big(f_o(S^{n,j}) - f_o(S^{n,i})\big)^2 = \big(f_w(S^{n,j}) - f_w(S^{n,i})\big)^2$$

readily imply (2.23). □

Now, we set

$$A^{ij} = c_{ij}\big(f_w(S^j) - f_w(S^i)\big)^2 F^{ij}, \qquad a_3 = -\frac{1}{2}\sum_{i=1}^{M}\sum_{j\in\mathcal{UN}(i)} A^{ij}. \tag{2.28}$$

The next proposition simplifies the expression for a_3.

Proposition 2.4. We have

$$a_3 = \sum_{i=1}^{M} \sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} c_{ij} \big(f_w(S^j) - f_w(S^i) \big)^2 F^{ij}. \tag{2.29}$$

Proof. By expanding the indices in the set $\mathcal{UN}(i)$, interchanging the indices i and j, and using the anti-symmetry of A^{ij}, we derive

$$a_3 = \frac{1}{2} \Bigg(\bigg(\sum_{\alpha=w,o} \sum_{P_\alpha^i > P_\alpha^j} A^{ij} \bigg) + \sum_{P_w^i = P_w^j, S^j \leqslant S^i} A^{ij} + \sum_{P_o^i = P_o^j, S^i \leqslant S^j} A^{ij} \Bigg).$$

Now, we split the first two sums above as follows:

$$\sum_{\alpha=w,o} \sum_{P_\alpha^i > P_\alpha^j} A^{ij} = 2 \sum_{P_w^i > P_w^j, P_o^i > P_o^j} A^{ij} + \sum_{P_w^i > P_w^j, P_o^i \leqslant P_o^j} A^{ij} + \sum_{P_o^i > P_o^j, P_w^i \leqslant P_w^j} A^{ij}.$$

This leads to

$$a_3 = \sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} A^{ij} + \frac{1}{2} \Bigg(\sum_{P_w^i > P_w^j, P_o^i \leqslant P_o^j} A^{ij} + \sum_{P_o^i > P_o^j, P_w^i \leqslant P_w^j} A^{ij} + \sum_{P_w^i = P_w^j, S^j \leqslant S^i} A^{ij} + \sum_{P_o^i = P_o^j, S^i \leqslant S^j} A^{ij} \Bigg).$$

The anti-symmetry of A^{ij} gives

$$\sum_{P_w^i > P_w^j, P_o^i \leqslant P_o^j} A^{ij} = - \sum_{P_w^j > P_w^i, P_o^j < P_o^i} A^{ij} - \sum_{P_w^j > P_w^i, P_o^j = P_o^i} A^{ij}.$$

By substituting and applying twice again the anti-symmetry of A^{ij}, we derive

$$a_3 = \sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} A^{ij} + \frac{1}{2} \Bigg(\sum_{P_o^i > P_o^j, P_w^i = P_w^j} A^{ij} + \sum_{P_w^i > P_w^j, P_o^i = P_o^j} A^{ij} - \sum_{P_w^i = P_w^j, S^i \leqslant S^j} A^{ij} - \sum_{P_o^i = P_o^j, S^j \leqslant S^i} A^{ij} \Bigg). \tag{2.30}$$

Note that

$$\sum_{P_o^i > P_o^j, P_w^i = P_w^j} A^{ij} = \sum_{P_o^i \geqslant P_o^j, P_w^i = P_w^j} A^{ij}$$

since the additional term is zero. Therefore, the fact that p_c is strictly decreasing yields

$$\sum_{P_o^i > P_o^j, P_w^i = P_w^j} A^{ij} = \sum_{P_w^i = P_w^j, S^i \leqslant S^j} A^{ij}, \qquad \sum_{P_w^i > P_w^j, P_o^i = P_o^j} A^{ij} = \sum_{P_o^i = P_o^j, S^i \geqslant S^j} A^{ij}.$$

Thus all terms multiplying $1/2$ in (2.30) are cancelled and we recover (2.29). □

By applying (2.21) and Propositions 2.3 and 2.4, A_1 has the following expression.

Proposition 2.5. We have

$$\sum_{n=1}^{N} \tau \sum_{i,j=1}^{M} \sum_{\alpha=w,o} \Big[f_\alpha(S^{n,i}) c_{ij} f_\alpha(S_\alpha^{n,ij}) F^{n,ij} \Big] = \sum_{n=1}^{N} \tau \sum_{i=1}^{M} \sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} c_{ij} \big(f_w(S^{n,j}) - f_w(S^{n,i}) \big)^2 F^{n,ij} + R_4 \tag{2.31}$$

where

$$|R_4| \leqslant 6 \, \|\overline{q}\|_{L^1(Q)}. \tag{2.32}$$

This settles the contribution of the first term of (2.19); the second terms are handled in the next subsection.

2.4 Terms involving the capillary pressure and mobility

These are the terms $A_{\alpha,i,n}$ defined in (2.18). By virtue of the anti-symmetry of C_α^{ij}, we can write for $\alpha = w, o$:

$$\sum_{i,j=1}^{M} f_\alpha(S^i) c_{ij} f_\alpha(S_\alpha^{ij}) \, C_\alpha^{ij} = -\frac{1}{2} \sum_{i,j=1}^{M} \big(f_\alpha(S^j) - f_\alpha(S^i) \big) c_{ij} f_\alpha(S_\alpha^{ij}) \, C_\alpha^{ij}. \tag{2.33}$$

Owing to (1.3), the term with $\alpha = o$ in the right-hand side is $\frac{1}{2}\sum_{i,j=1}^{M}(f_w(S^j) - f_w(S^i))c_{ij}f_o(S_o^{ij})\,C_o^{ij}$. Therefore,

$$\sum_{\alpha=w,o}\sum_{i=1}^{M} A_{\alpha,i,n} = \frac{1}{2}\sum_{i,j=1}^{M} c_{ij}\big(f_w(S^{n,j}) - f_w(S^{n,i})\big)\big(-f_w(S_w^{n,ij})\,C_w^{n,ij} + f_o(S_o^{n,ij})\,C_o^{n,ij}\big). \tag{2.34}$$

Let K^{ij} denote the symmetric term

$$K^{ij} := c_{ij}\big(f_w(S^{n,j}) - f_w(S^{n,i})\big)\big(-f_w(S_w^{n,ij})\,C_w^{n,ij} + f_o(S_o^{n,ij})\,C_o^{n,ij}\big);$$

by virtue of this symmetry, we have

$$\sum_{\alpha=w,o}\sum_{i=1}^{M} A_{\alpha,i,n} = \sum_{P_w^{n,i} > P_w^{n,j}} K^{n,ij} + \frac{1}{2}\sum_{P_w^{n,i} = P_w^{n,j}} K^{n,ij}. \tag{2.35}$$

2.5 Combining all terms

The next lemma follows by substituting (2.31) and (2.35) into (2.19).

Lemma 2.1. *We have*

$$\begin{aligned}
-\sum_{n=1}^{N}\tau\sum_{\alpha=o,w}[P_{\alpha,h}^n, \eta_\alpha(S_{\alpha,h}^n); f_\alpha(S_h^n), P_{\alpha,h}^n]_h &= \sum_{n=1}^{N}\tau\Bigg[\sum_{i=1}^{M}\Bigg(\sum_{j\in\mathcal{N}_{\mathcal{F}}(i)} c_{ij}\big(f_w(S^{n,j}) - f_w(S^{n,i})\big)^2 F^{n,ij}\\
&\quad - \sum_{P_w^{n,i} > P_w^{n,j}} c_{ij}\big(f_w(S^{n,j}) - f_w(S^{n,i})\big)\big(f_w(S_w^{n,ij})\,C_w^{n,ij} - f_o(S_o^{n,ij})\,C_o^{n,ij}\big)\\
&\quad - \frac{1}{2}\sum_{P_w^{n,i} = P_w^{n,j}} c_{ij}\big(f_w(S^{n,j}) - f_w(S^{n,i})\big)\big(f_w(S_w^{n,ij})\,C_w^{n,ij} - f_o(S_o^{n,ij})\,C_o^{n,ij}\big)\Bigg)\Bigg] + R_4
\end{aligned} \tag{2.36}$$

with R_4 bounded by (2.32).

Thus, to bring forward g, we must suitably combine the terms of the above sum over i, and this is done by examining all pairs of indices (i, j) involved in (2.36), i.e., the pairs of indices in the following sets: (i) $P_w^i > P_w^j$ and $P_o^i > P_o^j$, (ii) $P_w^i > P_w^j$ and $P_o^i < P_o^j$, (iii) $P_w^i > P_w^j$ and $P_o^i = P_o^j$, (iv) $P_w^i = P_w^j$ and $P_o^i > P_o^j$, (v) $P_w^i = P_w^j$ and $P_o^i < P_o^j$. Note that the sixth case that would be $P_w^i = P_w^j$ and $P_o^i = P_o^j$ brings no information because it implies that $S^i = S^j$.

For the argument below, we shall use the following proposition, due to the continuity of $\eta_o f_w$ and $\eta_w f_o$ and the fact they do not change sign between S^i and S^j.

Proposition 2.6. For each indices i and j, there exist (non unique) points α and α' between S^i and S^j such that

$$g(S^j) - g(S^i) = -\eta_o(\alpha)f_w(\alpha)(p_c(S^j) - p_c(S^i)) = -\eta_w(\alpha')f_o(\alpha')(p_c(S^j) - p_c(S^i)). \tag{2.37}$$

To simplify, the superscript n is dropped. We now state several propositions. For brevity, their proofs are skipped and can be found in report [6].

Proposition 2.7. Let $P_w^i > P_w^j$ and $P_o^i > P_o^j$; then the factor of τ in (2.36) satisfies

$$\begin{aligned}
c_{ij}\big(f_w(S^j) - f_w(S^i)\big)\Big(\big(f_w(S^j) - f_w(S^i)\big)F^{ij} - \big(f_w(S^i)C_w^{ij} - f_o(S^i)C_o^{ij}\big)\Big)&\\
\geqslant c_{ij}\big(f_w(S^j) - f_w(S^i)\big)\big(g(S^j) - g(S^i)\big).&
\end{aligned} \tag{2.38}$$

Proposition 2.8. Let $P_w^i > P_w^j$ and $P_o^i < P_o^j$; then the factor of τ in (2.36) satisfies

$$-c_{ij}\big(f_w(S^j) - f_w(S^i)\big)\big(f_w(S^i)C_w^{ij} - f_o(S^j)C_o^{ij}\big) \geqslant c_{ij}\big(f_w(S^j) - f_w(S^i)\big)\big(g(S^j) - g(S^i)\big). \tag{2.39}$$

Proposition 2.9. Let $P_w^i > P_w^j$ and $P_o^i = P_o^j$; then the factor of τ in (2.36) satisfies

$$-c_{ij}(f_w(S^j) - f_w(S^i))(f_w(S^i)C_w^{ij} - f_o(S^j)C_o^{ij}) \geqslant c_{ij}(f_w(S^j) - f_w(S^i))(g(S^j) - g(S^i)). \tag{2.40}$$

Proposition 2.10. Let $P_w^i = P_w^j$ and $P_o^i > P_o^j$; then the factor of τ in (2.36) satisfies

$$-\frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(f_w(S^j)C_w^{ij} - f_o(S^i)C_o^{ij}) \geqslant \frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(g(S^j) - g(S^i)). \tag{2.41}$$

Proposition 2.11. Let $P_w^i = P_w^j$ and $P_o^i < P_o^j$; then the factor of τ in (2.36) satisfies

$$-\frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(f_w(S^i)C_w^{ij} - f_o(S^j)C_o^{ij}) \geqslant \frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(g(S^j) - g(S^i)). \tag{2.42}$$

2.6 Auxiliary bound for the gradient of *g*

The following theorem is the first outcome of this section.

Theorem 2.2. *There exists a constant C, independent of h and τ, such that*

$$\left| \int_Q \nabla(I_h(f_\alpha(S_{h,\tau}))) \cdot \nabla(I_h(g(S_{h,\tau}))) \right| \leqslant C, \quad \alpha = w, o. \tag{2.43}$$

Proof. Owing to (1.3), it suffices to prove (2.43) when $\alpha = w$. By applying Propositions 2.7–2.11 to Lemma 2.1 and combining with Proposition 2.1, we readily derive that

$$\sum_{n=1}^{N} \tau \sum_{i=1}^{M} \sum_{j \in \mathcal{N}(i), P_w^{n,i} \geqslant P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})) \leqslant C \tag{2.44}$$

with a constant C independent of h and τ. Therefore, (2.43) will follow if we bound the summand for all j such that $P_w^{n,i} < P_w^{n,j}$. But the symmetry of the summand implies that

$$\begin{aligned}\sum_{i=1}^{M} \sum_{j \in \mathcal{N}(i), P_w^{n,i} < P_w^{n,j}} & c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})) \\ &= \sum_{i=1}^{M} \sum_{j \in \mathcal{N}(i), P_w^{n,i} > P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})).\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega \nabla(I_h(f_w(S_h^n))) \cdot \nabla(I_h(g(S_h^n))) = 2 &\sum_{i=1}^{M} \sum_{j \in \mathcal{N}(i), P_w^{n,i} > P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})) \\ &+ \sum_{i=1}^{M} \sum_{j \in \mathcal{N}(i), P_w^{n,i} = P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i}))\end{aligned}$$

and (2.43), with another constant C, follows by substituting this equality into (2.44). □

2.7 Bound for the gradient of *g*

Lemma 2.2. *There is a positive constant C such that*

$$\forall x \in [0, 1], \quad g'(x) \leqslant C f_w'(x). \tag{2.45}$$

Proof. Considering (1.5) and (1.9), we infer

$$0 < g'(x) \leqslant \frac{1}{\eta_* \alpha_3} \frac{1}{\alpha_w \vartheta_w} \frac{1}{\alpha_o \vartheta_o} x^{\vartheta_w - 1 + \beta_3} (1-x)^{\vartheta_o - 1 + \beta_4} \tag{2.46}$$

thus implying that g' is a bounded function, i.e., g is Lipschitz continuous. Note that the Lipschitz constant L of g is bounded by

$$L \leqslant \frac{1}{\eta_* \alpha_3} \frac{1}{\alpha_w \vartheta_w} \frac{1}{\alpha_o \vartheta_o} \max_{x \in [0,1]} (x^{\vartheta_w - 1 + \beta_3} (1-x)^{\vartheta_o - 1 + \beta_4}). \tag{2.47}$$

On the other hand, (1.8)–(1.10) yield for all $x \in]0,1[$,

$$g'(x) \geqslant \frac{\alpha_3}{C_{\max}} \frac{\alpha_w}{\vartheta_w} \frac{\alpha_o}{\vartheta_o} x^{\vartheta_w - 1 + \beta_3} (1-x)^{\vartheta_o - 1 + \beta_4} > 0. \tag{2.48}$$

Thus $g \in W^{1,\infty}(0,1)$ is a strictly monotonic increasing function on $[0,1]$ with range $[0,\beta]$ for some $\beta > 0$, hence invertible with inverse $g^{-1} \in W^{1,\infty}(0,\beta)$.

Now, we turn to f_w. By definition, we have

$$f_w'(x) = \frac{1}{(\eta_w(x) + \eta_o(x))^2} (\eta_o(x)\eta_w'(x) - \eta_w(x)\eta_o'(x)). \tag{2.49}$$

The inequalities (1.6)–(1.10) imply that

$$f_w'(x) \geqslant \frac{1}{C_{\max}^2} \alpha_o \alpha_w \left[\frac{1}{\vartheta_o} x^{\vartheta_w - 1} (1-x)^{\vartheta_o} + \frac{1}{\vartheta_w} x^{\vartheta_w} (1-x)^{\vartheta_o - 1} \right].$$

Thus,

$$\forall x \in \left[0, \frac{3}{4}\right], \quad f_w'(x) \geqslant \frac{\alpha_o \alpha_w}{C_{\max}^2 \vartheta_o} \left(\frac{1}{4}\right)^{\vartheta_o} x^{\vartheta_w - 1} \tag{2.50}$$

and

$$\forall x \in \left[\frac{1}{4}, 1\right], \quad f_w'(x) \geqslant \frac{\alpha_o \alpha_w}{C_{\max}^2 \vartheta_w} \left(\frac{1}{4}\right)^{\vartheta_w} (1-x)^{\vartheta_o - 1}. \tag{2.51}$$

Let us use these results to compare g' and f_w'. It follows from (2.46) that

$$\forall x \in \left[0, \frac{3}{4}\right], \quad g'(x) \leqslant \left(\frac{1}{\eta_* \alpha_3} \frac{1}{\alpha_w \vartheta_w} \frac{1}{\alpha_o \vartheta_o} \frac{\vartheta_o C_{\max}^2}{\alpha_o \alpha_w} \right) \frac{\alpha_o \alpha_w}{C_{\max}^2 \vartheta_o} x^{\vartheta_w - 1}$$

and by setting

$$C_1 = \left(\frac{1}{\eta_* \alpha_3} \frac{1}{\alpha_w \vartheta_w} \frac{1}{\alpha_o \vartheta_o} 4^{\vartheta_o} \frac{\vartheta_o C_{\max}^2}{\alpha_o \alpha_w} \right)$$

and comparing with (2.50), we obtain

$$\forall x \in \left[0, \frac{3}{4}\right], \quad g'(x) \leqslant C_1 f_w'(x). \tag{2.52}$$

Similarly,

$$\forall x \in \left[\frac{1}{4}, 1\right], \quad g'(x) \leqslant \left(\frac{1}{\eta_* \alpha_3} \frac{1}{\alpha_w \vartheta_w} \frac{1}{\alpha_o \vartheta_o} \frac{\vartheta_w C_{\max}^2}{\alpha_o \alpha_w} \right) \frac{\alpha_o \alpha_w}{C_{\max}^2 \vartheta_w} (1-x)^{\vartheta_o - 1}$$

so that, by setting

$$C_2 = \left(\frac{1}{\eta_* \alpha_3} \frac{1}{\alpha_w \vartheta_w} \frac{1}{\alpha_o \vartheta_o} 4^{\vartheta_w} \frac{\vartheta_w C_{\max}^2}{\alpha_o \alpha_w} \right)$$

and comparing with (2.51), we deduce

$$\forall x \in \left[\frac{1}{4}, 1\right], \quad g'(x) \leqslant C_2 f_w'(x). \tag{2.53}$$

This leads to the desired relation (2.45) with $C = \max(C_1, C_2)$. □

We are now ready to show Theorem 2.1, which follows by combining (2.45) with (2.43). Let (i,j) be any pair of indices. If $S^{n,i} \leqslant S^{n,j}$, then by (2.45),

$$f_w(S^{n,j}) - f_w(S^{n,i}) = \int_{S^{n,i}}^{S^{n,j}} f_w'(x)\,\mathrm{d}x \geqslant C \int_{S^{n,i}}^{S^{n,j}} g'(x)\,\mathrm{d}x = C(g(S^{n,j}) - g(S^{n,i})).$$

As g is increasing, we have $g(S^{n,j}) - g(S^{n,i}) \geqslant 0$. Therefore

$$(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})) \geqslant C|g(S^{n,j}) - g(S^{n,i})|^2. \tag{2.54}$$

By changing both signs, the same result holds when $S^{n,j} < S^{n,i}$. Then (2.2) follows from (2.43).

3 Convergence

The interpolants of $p_{\alpha g}(S_{h,\tau})$, $g(S_{h,\tau})$, and $p_c(S_{h,\tau})$ play an important part in this work, see Theorem 2.1, and (1.22). Therefore, we begin by studying convergence properties first of $I_h(g(S_{h,\tau}))$ and $I_h(p_{\alpha g}(S_{h,\tau}))$, $\alpha = w, o$, and next $I_h(p_c(S_{h,\tau}))$. Some results will stem from an interesting relation between differences in values of $S_{h,\tau}$ and $g(S_{h,\tau})$.

3.1 Properties of $I_h(g(S_{h,\tau}))$ and $I_h(p_{\alpha g}(S_{h,\tau}))$, $\alpha = w, o$

3.1.1 Convergence properties of $I_h(g(S_{h,\tau}))$

Using the fact that the finite element basis functions and the discrete saturation are bounded below and above by 0 and 1, respectively, there exist constants C, D, E, independent of n, h, and τ, such that

$$\|g(S_{h,\tau})(t_n)\|_{L^2(\Omega)} \leqslant C\|I_h(g(S_{h,\tau})(t_n))\|_h \leqslant D\|I_h(g(S_{h,\tau})(t_n))\|_h^{\varphi} \leqslant E\|I_h(g(S_{h,\tau})(t_n))\|_{L^2(\Omega)}. \tag{3.1}$$

These inequalities carry over to the norm in $L^2(Q)$. Now, let us prove the following convergence property of $I_h(g(S_{h,\tau}))$.

Lemma 3.1. *Under the assumptions of Theorem* 2.1, *we have*

$$\lim_{(h,\tau)\to(0,0)} \|g(S_{h,\tau}) - I_h(g(S_{h,\tau}))\|_{L^2(Q)} = 0. \tag{3.2}$$

Proof. For any $\boldsymbol{x}$ in any element K of $\mathcal{T}_h$, we have

$$I_h(g(S_{h,\tau}))(\boldsymbol{x}, t_n) - g(S_{h,\tau})(\boldsymbol{x}, t_n) = \sum_{i=1}^{d+1} g(S^{n,i})\varphi_i(\boldsymbol{x}) - g\left(\sum_{i=1}^{d+1} S^{n,i}\varphi_i(\boldsymbol{x})\right)$$

where $1 \leqslant i \leqslant d+1$ are the local numbers of the nodes. As $S_{h,\tau}$ is a polynomial of degree one in K, it attains its maximum and its minimum in space at vertices of K, say $g(S^{n,\ell})$ and $g(S^{n,r})$ are its maximum and minimum, respectively. Thus, recalling that g is a nonnegative monotonically increasing function,

$$\sum_{i=1}^{d+1} g(S^{n,i})\varphi_i(\boldsymbol{x}) \leqslant g(S^{n,\ell}), \qquad g\left(\sum_{i=1}^{d+1} S^{n,i}\varphi_i(\boldsymbol{x})\right) \geqslant g(S^{n,r}).$$

Hence

$$\|I_h(g(S_{h,\tau})) - g(S_{h,\tau})\|_{L^2(Q)}^2 \leqslant \sum_{n=1}^{N} \tau \sum_{K\in\mathcal{T}_h} |K|\,|g(S^{n,\ell}) - g(S^{n,r})|^2. \tag{3.3}$$

For any node i, let $\varkappa_i$ denotes the maximum of $|K|$ over all elements K in Δ_i. Then we can readily check that

$$\sum_{n=1}^{N} \tau \sum_{K\in\mathcal{T}_h} |K|\,|g(S^{n,\ell}) - g(S^{n,r})|^2 \leqslant C \sum_{n=1}^{N} \tau \sum_{i=1}^{M} \varkappa_i \sum_{j\in\mathcal{N}(i)} |g(S^{n,j}) - g(S^{n,i})|^2$$

where C is a bound for the maximum number of elements that share a common edge, bound independent of h and τ by virtue of the regularity of the mesh. Now, recall the classical formula in each d-simplex K,

$$\int_K |\nabla\varphi_i \cdot \nabla\varphi_j| = \frac{1}{d^2}\frac{|F_i|\,|F_j|}{|K|}|\boldsymbol{n}_i \cdot \boldsymbol{n}_j| \tag{3.4}$$

where F_i is the face opposite to the vertex $\boldsymbol{a}_i$ and $\boldsymbol{n}_i$ is the exterior (to K) unit normal to the face F_i. The regularity of the mesh implies that there exists a constant C_0, independent of h and τ, such that $|\boldsymbol{n}_i \cdot \boldsymbol{n}_j| \geqslant C_0$. Hence, using again the regularity of the mesh, we obtain

$$\int_K |\nabla\varphi_i \cdot \nabla\varphi_j| \geqslant C\, h_K^{d-2}$$

and denoting by ρ_{ij} the minimum of h_K for all K in $\Delta_i \cap \Delta_j$, we deduce

$$c_{ij} \geqslant C\rho_{ij}^{d-2} \tag{3.5}$$

with another constant C independent of h and τ. By collecting these results, we derive

$$\|I_h(g(S_{h,\tau})) - g(S_{h,\tau})\|^2_{L^2(Q)} \leqslant C\sum_{n=1}^{N}\tau\sum_{i=1}^{M}\varkappa_i\sum_{j\in\mathcal{N}(i)}\left(\frac{1}{\rho_{ij}^{d-2}}\right)c_{ij}|g(S^{n,j}) - g(S^{n,i})|^2. \tag{3.6}$$

With another application of the regularity of the mesh, this becomes

$$\|I_h(g(S_{h,\tau})) - g(S_{h,\tau})\|^2_{L^2(Q)} \leqslant Ch^2\|\nabla(I_h(g(S_{h,\tau})))\|^2_{L^2(Q)} \tag{3.7}$$

(note that the power of h is independent of the dimension) and the limit (3.2) follows from Theorem 2.1. □

3.1.2 Relation between $g(S^{n,j}) - g(S^{n,i})$ and $S^{n,j} - S^{n,i}$

Here, we derive an upper bound for $S^{n,j} - S^{n,i}$ in terms of $g(S^{n,j}) - g(S^{n,i})$.

Lemma 3.2. *There exists a constant C, independent of h and τ, such that for all i, j, n,*

$$|S^{n,j} - S^{n,i}| \leqslant C\,|g(S^{n,j}) - g(S^{n,i})|^{1/r} \tag{3.8}$$

where $r = \max(\vartheta_o + \beta_4, \vartheta_w + \beta_3) > 1$.

Proof. To simplify, we set $c = S^{n,i}$, $d = S^{n,j}$ and assume $c < d$. From (2.48), it follows that

$$g(d) - g(c) \geqslant \frac{\alpha_3}{C_{\max}}\frac{\alpha_w}{\vartheta_w}\frac{\alpha_o}{\vartheta_o}\int_c^d x^{\vartheta_w+\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1}. \tag{3.9}$$

For the sake of brevity, we do not specify the constant factor in (3.9) and write

$$g(d) - g(c) \geqslant C_1\int_c^d x^{\vartheta_w+\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1}.$$

Now, we argue according to the positions of c and d. There are four cases.

1. If $1/8 \leqslant c \leqslant 7/8$, then (3.9) gives

$$\begin{aligned} g(d) - g(c) &\geqslant C_1\left(\frac{1}{8}\right)^{\vartheta_w+\beta_3-1}\int_c^d (1-x)^{\vartheta_o+\beta_4-1} \\ &= C_1\left(\frac{1}{8}\right)^{\vartheta_w+\beta_3-1}\frac{1}{\vartheta_o+\beta_4}\Big((1-c)^{\vartheta_o+\beta_4} - (1-d)^{\vartheta_o+\beta_4}\Big). \end{aligned}$$

But

$$\begin{aligned} (1-c)^{\vartheta_o+\beta_4} - (1-d)^{\vartheta_o+\beta_4} &= (1-c)^{\vartheta_o+\beta_4-1}(d-c) + (1-d)\big((1-c)^{\vartheta_o+\beta_4-1} - (1-d)^{\vartheta_o+\beta_4-1}\big) \\ &\geqslant (d-c)\left(\frac{1}{8}\right)^{\vartheta_o+\beta_4-1}. \end{aligned}$$

Hence

$$g(d) - g(c) \geqslant \frac{C_1}{\vartheta_o+\beta_4}\left(\frac{1}{8}\right)^{\vartheta_w+\beta_3+\vartheta_o+\beta_4-2}(d-c). \tag{3.10}$$

2. If $c > 7/8$, then $d > 7/8$ and (3.9) gives

$$g(d) - g(c) \geqslant C_1 \left(\frac{7}{8}\right)^{\vartheta_w+\beta_3-1} \frac{1}{\vartheta_o + \beta_4}\left((1-c)^{\vartheta_o+\beta_4} - (1-d)^{\vartheta_o+\beta_4}\right).$$

Let us set $a = 1 - d$, $b = d - c$, $\gamma = \vartheta_o + \beta_4 - 1 > 0$. We can also write

$$(1-c)^{\vartheta_o+\beta_4} - (1-d)^{\vartheta_o+\beta_4} = a^{\gamma+1}\left(\left(1 + \frac{b}{a}\right)^{\gamma+1} - 1\right).$$

It is easy to check that the function

$$x \mapsto (1+x)^{\gamma+1} - 1 - x^{\gamma+1}$$

vanishes at $x = 0$ and is strictly monotonic increasing, hence is strictly positive for $x > 0$. Hence

$$a^{\gamma+1}\left(\left(1 + \frac{b}{a}\right)^{\gamma+1} - 1\right) > a^{\gamma+1}\left(\frac{b}{a}\right)^{\gamma+1} = b^{\gamma+1}.$$

Thus

$$(1-c)^{\vartheta_o+\beta_4} - (1-d)^{\vartheta_o+\beta_4} \geqslant (d-c)^{\vartheta_o+\beta_4}$$

and

$$g(d) - g(c) \geqslant C_1 \left(\frac{7}{8}\right)^{\vartheta_w+\beta_3-1} \frac{1}{\vartheta_o + \beta_4}(d-c)^{\vartheta_o+\beta_4}. \tag{3.11}$$

3. If $c < 1/8$ and $d \leqslant 7/8$, then the integrand $1 - x \geqslant 1 - d \geqslant 1/8$ and by the above argument,

$$\begin{aligned} g(d) - g(c) &\geqslant C_1 \left(\frac{1}{8}\right)^{\vartheta_o+\beta_4-1} \frac{1}{\vartheta_w + \beta_3}\left(d^{\vartheta_w+\beta_3} - c^{\vartheta_w+\beta_3}\right) \\ &\geqslant C_1 \left(\frac{1}{8}\right)^{\vartheta_o+\beta_4-1} \frac{1}{\vartheta_w + \beta_3}(d-c)^{\vartheta_w+\beta_3}. \end{aligned} \tag{3.12}$$

4. If $c < 1/8$ and $d > 7/8$, then $c < (d-c)/6 < (d-c)/2 < d$. Therefore, we can write

$$\begin{aligned} g(d) - g(c) &\geqslant C_1 \int_{(d-c)/6}^{(d-c)/2} x^{\vartheta_w+\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1} \\ &\geqslant C_1 \left(\frac{1}{2}\right)^{\vartheta_o+\beta_4-1} \frac{1}{\vartheta_w + \beta_3}\left(\left(\frac{1}{2}(d-c)\right)^{\vartheta_w+\beta_3} - \left(\frac{1}{6}(d-c)\right)^{\vartheta_w+\beta_3}\right) \\ &\geqslant C_1 \left(\frac{1}{2}\right)^{\vartheta_o+\vartheta_w+\beta_3+\beta_4-1} \frac{1}{\vartheta_w + \beta_3}\left(1 - \left(\frac{1}{3}\right)^{\vartheta_w+\beta_3}\right)(d-c)^{\vartheta_w+\beta_3}. \end{aligned} \tag{3.13}$$

Since $d - c \leqslant 1$, $\vartheta_o + \beta_4 > 1$, and $\vartheta_w + \beta_3 > 1$, we have in all cases

$$g(d) - g(c) \geqslant C_2 (d-c)^{\max(\vartheta_o+\beta_4, \vartheta_w+\beta_3)}$$

where C_2 is the minimum of the constant factors in (3.10)–(3.13). □

The convergence to zero of the differences $I_h(p_{\alpha g}(S_{h,\tau})) - p_{\alpha g}(S_{h,\tau})$, $\alpha = w, o$, follows from this lemma and Theorem 2.1.

Lemma 3.3. *There exists a constant C, independent of h and τ, such that*

$$\|I_h(p_{\alpha g}(S_{h,\tau})) - p_{\alpha g}(S_{h,\tau})\|_{L^2(Q)} \leqslant C\, h^{\gamma_\alpha}, \quad \alpha = w, o \tag{3.14}$$

where $\gamma_w = \beta_3/r$, $\gamma_o = \beta_4/r$ and in both cases, r is the exponent of Lemma 3.2.

Proof. Let us start with $\alpha = w$. Arguing as in the proof of Lemma 3.1, with $-p_{wg}$ (monotonic increasing) instead of g, the analogue of (3.6) holds for $-p_{wg}(S_{h,\tau})$, with the same notation

$$\|I_h(p_{wg}(S_{h,\tau})) - p_{wg}(S_{h,\tau})\|^2_{L^2(Q)} \leqslant C \sum_{n=1}^{N} \tau \sum_{i=1}^{M} \sum_{j\in\mathcal{N}(i)} \left(\frac{\varkappa_i}{c_{ij}}\right) c_{ij} |p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})|^2 \tag{3.15}$$

and the result will stem from an adequate upper bound for $p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})$, for all neighbors j of i. To this end, we proceed as in Lemma 3.2. Let $c = S^{n,i}$, $d = S^{n,j}$, and suppose again that $c < d$; then by (1.12), (1.8), (1.9), and (1.10),

$$|p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})| \leqslant \frac{1}{C_{\min}\alpha_3\alpha_o\vartheta_o} \int_c^d x^{\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1} \tag{3.16}$$

that we write as

$$|p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})| \leqslant C_1' \int_c^d x^{\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1}.$$

Here, the discussion reduces to three cases.

1. If $1/8 \leqslant c \leqslant 7/8$, since $\vartheta_o + \beta_4 - 1 > 0$,

$$\int_c^d x^{\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1} \leqslant 8^{1-\beta_3} \int_c^d (1-x)^{\vartheta_o+\beta_4-1} \leqslant 8^{1-\beta_3}(d-c). \tag{3.17}$$

2. Likewise, if $c > 7/8$,

$$\int_c^d x^{\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1} \leqslant \left(\frac{8}{7}\right)^{1-\beta_3} (d-c). \tag{3.18}$$

3. If $c < 1/8$,

$$\int_c^d x^{\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1} \leqslant \int_c^d x^{\beta_3-1} = \frac{1}{\beta_3}(d^{\beta_3} - c^{\beta_3}) \leqslant \frac{1}{\beta_3}(d-c)^{\beta_3}. \tag{3.19}$$

Indeed, by Jensen's inequality, valid for $0 < \beta_3 \leqslant 1$,

$$d = c + (d-c) \leqslant (c^{\beta_3} + (d-c)^{\beta_3})^{1/\beta_3}, \quad \text{i.e., } d^{\beta_3} \leqslant c^{\beta_3} + (d-c)^{\beta_3}.$$

Consequently, in all cases,

$$|p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})| \leqslant C_2' |S^{n,j} - S^{n,i}|^{\beta_3}. \tag{3.20}$$

Thus, by substituting into (3.15), applying Lemma 3.2, and setting $\gamma_w = \beta_3/r$, we infer

$$\|I_h(p_{wg}(S_{h,\tau})) - p_{wg}(S_{h,\tau})\|^2_{L^2(Q)} \leqslant C \sum_{n=1}^{N}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}(i)} \frac{\varkappa_i}{c_{ij}} \tau c_{ij} |S^{n,j} - S^{n,i}|^{2\beta_3} \leqslant C \sum_{n=1}^{N}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}(i)} \frac{\varkappa_i}{c_{ij}} \tau c_{ij} A_{ij}^{2\gamma_w}$$

where $A_{ij} = |g(S^{n,j}) - g(S^{n,i})|$. Note that $r > \beta_3$, hence $\gamma_w < 1$. Then

$$\begin{aligned}\|I_h(p_{wg}(S_{h,\tau})) - p_{wg}(S_{h,\tau})\|^2_{L^2(Q)} &\leqslant C \sum_{n=1}^{N}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}(i)} \frac{\varkappa_i}{c_{ij}} (\tau c_{ij})^{1-\gamma_w} (\tau c_{ij})^{\gamma_w} A_{ij}^{2\gamma_w} \\ &\leqslant C \left(\sum_{n=1}^{N}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}(i)} \tau c_{ij} A_{ij}^2\right)^{\gamma_w} \left(\sum_{n=1}^{N}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}(i)} \left(\frac{\varkappa_i}{c_{ij}}\right)^{1/(1-\gamma_w)} \tau c_{ij}\right)^{1-\gamma_w}.\end{aligned}$$

But

$$\left(\sum_{n=1}^{N}\sum_{i=1}^{M}\sum_{j\in\mathcal{N}(i)} \left(\frac{\varkappa_i}{c_{ij}}\right)^{1/(1-\gamma_w)} \tau c_{ij}\right)^{1-\gamma_w} \leqslant C(T|\Omega|)^{1-\gamma_w} \sup_{i,j}\left(\frac{\varkappa_i}{c_{ij}}\right)^{\gamma_w}$$

and (3.14) with $\alpha = w$ follows from (3.5), the regularity of the mesh, and Theorem 2.1. When $\alpha = o$, the proof is based on fact that $-p_{og}$ is nonnegative, monotonically increasing, and satisfies the inequality

$$-p_{og}(x) \leqslant \frac{1}{C_{\min}} \frac{1}{\alpha_3\alpha_w\vartheta_w} \int_0^x x^{\vartheta_w+\beta_3-1}(1-x)^{\beta_4-1}.$$

By comparing with (3.16), we see that the above argument carries over to p_{og} with β_3 replaced by β_4. □

We remark that owing to (1.3),

$$\forall x \in [0,1], \quad p_{wg}(x) + p_{og}(x) = \int_0^x p_c'(s)\,ds = p_c(x) - p_c(0). \tag{3.21}$$

Finally, with the notation of Lemma 3.3, the following bound regarding $p_c(S_{h,\tau})$ follows from (3.14) and (3.21), and the fact that $p_c(0)$ is a constant:

$$\|I_h(p_c(S_{h,\tau})) - p_c(S_{h,\tau})\|_{L^2(Q)} \leqslant C\,h^{\gamma} \tag{3.22}$$

where $\gamma = \frac{1}{r}\min(\beta_3, \beta_4)$.

3.2 Weak convergence

The saturation satisfies the maximum principle [7]:

$$0 \leqslant S_{h,\tau} \leqslant 1. \tag{3.23}$$

The bound (3.23) implies that there exists a function $\overline{s} \in L^\infty(Q)$ and a subsequence of (h,τ) not indicated, such that

$$\lim_{(h,\tau)\to(0,0)} S_{h,\tau} = \overline{s} \quad \text{weakly* in } L^\infty(Q). \tag{3.24}$$

Proposition 3.1. The limit function $\overline{s}$ satisfies

$$\forall (x,t) \text{ a.e. in } Q, \quad 0 \leqslant \overline{s}(x,t) \leqslant 1. \tag{3.25}$$

Proof. The convergence (3.24) means that for all $\psi \in L^1(Q)$,

$$\int_Q S_{h,\tau}\psi \to \int_Q \overline{s}\psi, \qquad \int_Q (1 - S_{h,\tau})\psi \to \int_Q (1-\overline{s})\psi.$$

We argue by contradiction. Suppose that $\overline{s} > 1$ on a set of positive measure, say D, and take $\psi = (\overline{s}-1)_+$, the positive part of $\overline{s} - 1$. Then

$$0 \leqslant \int_Q (1 - S_{h,\tau})\psi \to \int_Q (1-\overline{s})(\overline{s}-1)_+ = \int_D (1-\overline{s})(\overline{s}-1)_+$$

thus contradicting the fact that $(1-\overline{s}) < 0$ on D. This proves that $\overline{s} \leqslant 1$. The proof that $\overline{s} \geqslant 0$ is similar. □

The bound (2.1) yields weak convergence, up to a subsequence, of the gradient of $U_{\alpha,h,\tau}$. We can deduce weak convergence of the sequences themselves by applying the generalized Poincaré inequality

$$\forall v \in H^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leqslant C\left(\left|\int_\Omega v\right| + \|\nabla v\|_{L^2(\Omega)}\right) \tag{3.26}$$

Indeed,

$$\int_\Omega U_{w,h,\tau} = (U_{w,h,\tau}, 1)_h = (I_h(p_{wg}(S_{h,\tau})), 1)_h$$

owing to (1.23). Then the properties of p_{wg} and the boundedness of $S_{h,\tau}$ imply that

$$\left|(I_h(p_{wg}(S_{h,\tau})), 1)_h\right| \leqslant C.$$

Similarly,

$$\int_\Omega U_{o,h,\tau} = (I_h(p_{wg}(S_{h,\tau})) + p_c(0), 1)_h$$

a bounded quantity. Then we infer from (3.26) that

$$\|U_{\alpha,h,\tau}\|_{L^2(Q)} \leqslant C, \quad \alpha = w, o. \tag{3.27}$$

With this, (2.1) implies that there exist functions $\overline{W}_\alpha \in L^2(0,T;H^1(\Omega))$, $\alpha = w, o$, and a subsequence of h and τ (not indicated) such that,

$$\lim_{(h,\tau)\to(0,0)} U_{\alpha,h,\tau} = \overline{W}_\alpha \quad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{3.28}$$

Likewise, the function $I_h(g(S_{h,\tau}))$ is bounded in $L^2(Q)$ and it follows from this and (2.2) that there exists a function $\overline{K} \in L^2(0,T;H^1(\Omega))$ such that, up to a subsequence,

$$\lim_{(h,\tau)\to(0,0)} I_h(g(S_{h,\tau})) = \overline{K} \quad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{3.29}$$

This implies in particular that for almost every time t, $I_h(g(S_{h,\tau}))$ converges strongly in $L^2(\Omega)$. But as is well-known, these convergences are not sufficient to pass to the limit in the nonlinear terms: they must be supplemented by a bound for a fractional derivative in time of $S_{h,\tau}$ that yields compactness in time. This will stem via a bound for a fractional derivative in time of $g(S_{h,\tau})$.

3.3 Compactness in time

Following the argument introduced by Kazhikhov (see [9]) and recalling that $\|\cdot\|_h^\varphi$ is equivalent to the L^2 norm in finite dimension, we want to derive first a fractional estimate in time for $I_h(g(S_{h,\tau}))$ and next for $g(S_{h,\tau})$. The following lemma is a preliminary bound written in terms of sums of the pointwise values in time.

Lemma 3.4. *Under the assumptions of Theorem* 2.1, *there exist constants C, independent of h and τ, such that for all integers* $1 \leqslant \ell \leqslant N-1$,

$$\sum_{m=1}^{N-\ell} \tau\big(\|g(S_h^{m+\ell}) - g(S_h^m)\|_h^\varphi\big)^2 \leqslant C(\ell\tau), \qquad \sum_{m=1}^{N-\ell} \tau \|g(S_h^{m+\ell}) - g(S_h^m)\|_{L^2(\Omega)}^2 \leqslant C(\ell\tau). \tag{3.30}$$

Proof. The starting point is the inequality

$$\sum_{m=1}^{N-\ell} \tau\Big(\|g(S_h^{m+\ell}) - g(S_h^m)\|_h^\varphi\Big)^2 \leqslant L \sum_{m=1}^{N-\ell} \tau\Big(g(S_h^{m+\ell}) - g(S_h^m), S_h^{m+\ell} - S_h^m\Big)_h^\varphi \tag{3.31}$$

owing that g is Lipschitz continuous and increasing. Thus, by writing

$$S_h^{m+\ell} - S_h^m = \sum_{k=1}^{\ell} (S_h^{m+k} - S_h^{m+k-1})$$

testing each line of (1.20) taken at level $m+k$ with $I_h(g(S_h^{m+\ell}) - g(S_h^m))$, and applying (3.31), we obtain

$$\begin{aligned} \sum_{m=1}^{N-\ell} \tau(\|g(S_h^{m+\ell}) - g(S_h^m)\|_h^\varphi)^2 \leqslant L \sum_{m=1}^{N-\ell} \tau \sum_{k=1}^{\ell} \tau \Big| \big(f_w(s_{\text{in},h}^{m+k})\overline{q}_h^{m+k} - f_w(S_h^{m+k})\underline{q}_h^{m+k}, g(S_h^{m+\ell}) - g(S_h^m)\big)_h \\ + [P_{w,h}^{m+k}, I_h(\eta_w(S_h^{m+k})); P_{w,h}^{m+k}, I_h(g(S_h^{m+\ell}) - g(S_h^m))]_h\Big|. \end{aligned} \tag{3.32}$$

It is easy to check that, on one hand, with $r = \ell$ or $r = 0$,

$$\begin{aligned} &\Big|[P_{w,h}^{m+k}, I_h(\eta_w(S_h^{m+k})); P_{w,h}^{m+k}, I_h(g(S_h^{m+r}))]_h\Big| \\ &= \frac{1}{2}\left| \sum_{i,j=1}^{M} (g(S^{m+r,j}) - g(S^{m+r,i})) c_{ij} \eta_w(S_w^{m+k,ij})(P_w^{m+k,j} - P_w^{m+k,i}) \right| \\ &\leqslant \frac{1}{4} \sum_{i,j=1}^{M} c_{ij}\Big(\eta_w(1)|g(S^{m+r,j}) - g(S^{m+r,i})|^2 + \eta_w(S_w^{m+k,ij})|P_w^{m+k,j} - P_w^{m+k,i}|^2\Big) \end{aligned}$$

since η_w is increasing and $S_{h,\tau}$ is bounded by one. On the other hand,

$$\big|\big(f_w(s_{\text{in},h}^{m+k})\overline{q}_h^{m+k} - f_w(S_h^{m+k})\underline{q}_h^{m+k}, g(S_h^{m+\ell}) - g(S_h^m)\big)_h\big| \leqslant C(\|\overline{q}^{m+k}\|_{L^1(\Omega)} + \|\underline{q}^{m+k}\|_{L^1(\Omega)})$$

where here and below, C denotes constants that are independent of ℓ, h, and τ. Therefore, recalling [7]:

$$\forall V_h \in X_h, \quad \|\nabla V_h\|_{L^2(\Omega)} = \frac{1}{\sqrt{2}}\Big(\sum_{i,j=1}^{M} c_{ij}|V^j - V^i|^2\Big)^{1/2} \tag{3.33}$$

we have

$$\begin{aligned}
\sum_{m=1}^{N-\ell}\tau(\|g(S_h^{m+\ell}) - g(S_h^m)\|_h^{\varphi})^2 &\leqslant L\sum_{m=1}^{N-\ell}\tau\bigg(\bigg[\frac{1}{2}\eta_w(1)(\ell\tau)\sum_{r=\ell,0}\|\nabla I_h(g(S_h^{m+r}))\|^2_{L^2(\Omega)}\\
&\quad + \frac{1}{2}\sum_{k=1}^{\ell}\tau\sum_{i,j=1}^{M} c_{ij}\eta_w(S_w^{m+k,ij})\Big|P_w^{m+k,j} - P_w^{m+k,i}\Big|^2\bigg] + C\sum_{k=1}^{\ell}\tau(\|\overline{q}^{m+k}\|_{L^1(\Omega)} + \|\underline{q}^{m+k}\|_{L^1(\Omega)})\bigg)\\
&\leqslant \frac{1}{2}\eta_w(1)L(\ell\tau)\bigg[\sum_{m=1+\ell}^{N}\tau\|\nabla I_h(g(S_h^m))\|^2_{L^2(\Omega)} + \sum_{m=1}^{N-\ell}\tau\|\nabla I_h(g(S_h^m))\|^2_{L^2(\Omega)}\bigg]\\
&\quad + \frac{1}{2}L\sum_{m=1}^{N-\ell}\tau\sum_{k=1}^{\ell}\tau\bigg(\sum_{i,j=1}^{M} c_{ij}\eta_w(S_w^{m+k,ij})|P_w^{m+k,j} - P_w^{m+k,i}|^2 + C(\|\overline{q}^{m+k}\|_{L^1(\Omega)} + \|\underline{q}^{m+k}\|_{L^1(\Omega)})\bigg).
\end{aligned} \tag{3.34}$$

By (2.2), it suffices to bound the terms in the last line above. This is achieved by interchanging the sums over m and k. Let $n = m + k$; n runs from 2 to N and m runs from $\max(1, n - \ell)$ to $\min(n - 1, N - \ell)$. Thus

$$\sum_{m=1}^{N-\ell}\tau\sum_{k=1}^{\ell}\tau\sum_{i,j=1}^{M} c_{ij}\eta_w(S_w^{m+k,ij})|P_w^{m+k,j} - P_w^{m+k,i}|^2 = \sum_{n=2}^{N}\tau\Big(\sum_{m=\max(1,n-\ell)}^{\min(n-1,N-\ell)}\tau\Big)\sum_{i,j=1}^{M} c_{ij}\eta_w(S_w^{n,ij})|P_w^{n,j} - P_w^{n,i}|^2.$$

But $\min(n - 1, N - \ell) - \max(1, n - \ell) \leqslant \ell - 1$. Hence

$$\sum_{m=1}^{N-\ell}\tau\sum_{k=1}^{\ell}\tau\sum_{i,j=1}^{M} c_{ij}\eta_w(S_w^{m+k,ij})|P_w^{m+k,j} - P_w^{m+k,i}|^2 \leqslant (\ell\tau)\sum_{n=2}^{N}\tau\sum_{i,j=1}^{M} c_{ij}\eta_w(S_w^{n,ij})|P_w^{n,j} - P_w^{n,i}|^2 \tag{3.35}$$

and we know from [7] that this last sum over n is bounded. In the same fashion,

$$\sum_{m=1}^{N-\ell}\tau\sum_{k=1}^{\ell}\tau(\|\overline{q}^{m+k}\|_{L^1(\Omega)} + \|\underline{q}^{m+k}\|_{L^1(\Omega)}) \leqslant (\ell\tau)(\|\overline{q}\|_{L^1(Q)} + \|\underline{q}\|_{L^1(Q)}). \tag{3.36}$$

Then, under the assumptions of Theorem 2.1, (3.30) follows by substituting (2.2), (3.35), and (3.36) into (3.34). The second inequality stems from the first one and (3.1). □

The next theorem transforms (3.30) into integrals. The proof is skipped because the argument is not new, see for instance [10].

Theorem 3.1. *Under the assumptions of Theorem 2.1, there exists a constant C, independent of h, and τ, such that for all real numbers δ, $0 < \delta < T$,*

$$\int_0^{T-\delta}\|g(S_{h,\tau}(t+\delta)) - g(S_{h,\tau}(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \leqslant C\delta. \tag{3.37}$$

Similarly,

$$\int_0^{T-\delta}\|I_h(g(S_{h,\tau}(t+\delta)) - g(S_{h,\tau}(t))\|^2_{L^2(\Omega)}\,\mathrm{d}t \leqslant C\delta \tag{3.38}$$

with another constant C, independent of h, and τ.

3.4 Strong convergence

With Theorem 3.1, it follows from Kolmogorov's theorem that the sequence $I_h(g(S_{h,\tau}))$ is compact in $L^2(Q)$, see [9]. Thus, again up to a subsequence, $I_h(g(S_{h,\tau}))$ converges strongly in $L^2(Q)$. Since it converges weakly to $\overline{K}$ in $L^2(0, T; H^1(\Omega))$ ($\overline{K}$ belongs also to $L^\infty(Q)$), uniqueness of the limit implies

$$\lim_{(h,\tau)\to(0,0)} I_h(g(S_{h,\tau})) = \overline{K} \quad \text{strongly in } L^2(Q). \tag{3.39}$$

By Lemma 3.1, this also implies

$$\lim_{(h,\tau)\to(0,0)} g(S_{h,\tau}) = \overline{K} \quad \text{strongly in } L^2(Q). \tag{3.40}$$

From here, let us prove the strong convergence of $S_{h,\tau}$. Recall that g is invertible with range $]0,\beta[$ and inverse $g^{-1} \in W^{1,\infty}(]0,\beta[)$. Let $F_{h,\tau} = g(S_{h,\tau})$; then

$$S_{h,\tau} = g^{-1}(F_{h,\tau}).$$

The strong convergence of $F_{h,\tau}$ and the continuity of g^{-1} imply the strong convergence of $S_{h,\tau}$ to $g^{-1}(\overline{K})$ in $L^2(Q)$, and since $S_{h,\tau}$ converges weakly to $\overline{s}$, uniqueness of the limit implies that $\overline{s} = g^{-1}(\overline{K})$, i.e.,

$$\lim_{(h,\tau)\to(0,0)} S_{h,\tau} = \overline{s} = g^{-1}(\overline{K}) \quad \text{strongly in } L^2(Q). \tag{3.41}$$

This strong convergence and the continuity of g, $p_{\alpha g}$, $\alpha = w, o$, and p_c, also imply that

$$\lim_{(h,\tau)\to(0,0)} g(S_{h,\tau}) = g(\overline{s}), \qquad \lim_{(h,\tau)\to(0,0)} p_{\alpha g}(S_{h,\tau}) = p_{\alpha g}(\overline{s}),\ \alpha = w, o, \qquad \lim_{(h,\tau)\to(0,0)} p_c(S_{h,\tau}) = p_c(\overline{s}) \tag{3.42}$$

all strongly in $L^2(Q)$. Furthermore Lemma 3.3 and (3.22) yield

$$\lim_{(h,\tau)\to(0,0)} I_h(p_{\alpha g}(S_{h,\tau})) = p_{\alpha g}(\overline{s}), \qquad \lim_{(h,\tau)\to(0,0)} I_h(p_c(S_{h,\tau})) = p_c(\overline{s}), \quad \text{strongly in } L^2(Q). \tag{3.43}$$

In view of (3.28), this convergence implies that $P_{\alpha,h,\tau}$ converges weakly in $L^2(Q)$ to some function $\overline{p}_\alpha \in L^2(Q)$, $\alpha = w, o$. Furthermore, uniqueness of the limit implies that $\overline{W}_\alpha$, the limit function of $U_{\alpha,h,\tau}$ has the form

$$\overline{W}_w = \overline{p}_w + p_{wg}(\overline{s}), \qquad \overline{W}_o = \overline{p}_o - p_{og}(\overline{s}). \tag{3.44}$$

Note that, on the one hand, the uniform boundedness of $g(S_{h,\tau})$, $p_{\alpha g}(S_{h,\tau})$, $I_h(p_{\alpha g}(S_{h,\tau}))$, $p_c(S_{h,\tau})$, and $I_h(p_c(S_{h,\tau}))$ and their strong convergences in $L^2(Q)$ imply their strong convergence in $L^r(Q)$ for any finite r. On the other hand, the weak convergence of $U_{\alpha,h,\tau}$ in $L^2(0,T;H^1(\Omega))$ implies its strong convergence in $L^r(\Omega)$ for $r < 6$ (and any finite r when $d = 2$) for almost every t. These two results yield the strong convergence of $P_{\alpha,h,\tau}$, $\alpha = w, o$, in $L^r(\Omega)$ for $r < 6$, any finite r when $d = 2$, for almost every t.

4 Identification of the limit

Let us pass to the limit in the equations of the scheme. This is done in several steps because we do not have convergence of the pressure gradient.

4.1 The upwind terms

Since the discrete auxiliary pressures $U_{\alpha,h,\tau}$ converge weakly to $\overline{W}_\alpha$ in $L^2(0,T;H^1(\Omega))$, instead of treating directly the upwind terms $[P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); P_{\alpha,h,\tau}, \vartheta_h]_h$, we begin with $[P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); U_{\alpha,h,\tau}, \vartheta_h]_h$.

4.1.1 Discrete auxiliary pressure

Let us start with the wetting phase, the treatment of the non-wetting phase being much the same. Let v be a smooth function, say $v \in \mathcal{C}^1(\overline{Q})$ and let $V_{h,\tau} = \rho_\tau(I_h(v))$. Assume for the moment that $\overline{s}$, the limit of $S_{h,\tau}$, is sufficiently smooth, say $\overline{s} \in W^{1,\infty}(Q)$ and let $\overline{s}_\tau = \overline{s}(t_n)$ in $]t_{n-1}, t_n]$. Then assumption (1.6) implies

$$\left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \eta_w(\overline{s})\,\mathrm{d}t - \eta_w(\overline{s}_\tau^n) \right\|_{L^\infty(\Omega)} \leqslant C\tau \|\eta_w'\|_{L^\infty(0,1)} \|\partial_t \overline{s}\|_{L^\infty(Q)}. \tag{4.1}$$

We treat the upwinding in several steps and consider first

$$\int_Q \eta_w(\overline{s})\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau} = \int_Q \nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau}\big(\rho_\tau(\eta_w(\overline{s})) - \eta_w(\overline{s}_\tau) + \eta_w(\overline{s}_\tau)\big). \tag{4.2}$$

But in view of (4.1),

$$\left|\int_Q \nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau}\big(\rho_\tau(\eta_w(\overline{s})) - \eta_w(\overline{s}_\tau)\big)\right| \leqslant C\tau\|\eta_w'\|_{L^\infty(0,1)}\|\partial_t\overline{s}\|_{L^\infty(Q)}\,\|U_{w,h,\tau}\|_{L^2(0,T;H^1(\Omega))}\|V_{h,\tau}\|_{L^2(0,T;H^1(\Omega))}$$

and the boundedness of all factors of τ, owing to (2.1) and the regularity of v, implies

$$\lim_{(h,\tau)\to(0,0)}\int_Q \nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau}\big(\rho_\tau(\eta_w(\overline{s})) - \eta_w(\overline{s}_\tau)\big) = 0. \tag{4.3}$$

Next the weak convergence of $U_{w,h,\tau}$ to $\overline{W}_w$ in $L^2(0,T;H^1(\Omega))$, the strong convergence of $V_{h,\tau}$ to v in $L^\infty(0,T;W^{1,\infty}(\Omega))$, the continuity of η_w, the regularity of $\overline{s}$, and (4.3) imply

$$\lim_{(h,\tau)\to(0,0)}\int_Q \eta_w(\overline{s}_\tau)\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau} = \int_Q \eta_w(\overline{s})\nabla\overline{W}_w\cdot\nabla v.$$

Let us expand the expression in the above left-hand side.

Setting $c_{ij,K} = \int_K |\nabla\varphi_i\cdot\nabla\varphi_j|$ and $w_K = |K|^{-1}\int_K w$, we have the following proposition [7].

Proposition 4.1. Let (1.16) hold. The following identity holds for all w in $L^1(\Omega)$:

$$\forall Z_h, V_h\in X_h,\qquad \int_\Omega w\nabla Z_h\cdot\nabla V_h = -\sum_{i=1}^M Z^i\sum_{j=1}^M\Big(\sum_{K\subset\Delta_i\cap\Delta_j} c_{ij,K}w_K\Big)(V^j - V^i). \tag{4.4}$$

Therefore, in view of Proposition 4.1 we have

$$\int_Q \eta_w(\overline{s}_\tau)\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau} = \frac{1}{2}\sum_{n=1}^N\tau\sum_{i,j=1}^M\Big(\sum_{K\subset\Delta_i\cap\Delta_j} c_{ij,K}(\eta_w(\overline{s}_\tau^n))_K\Big)(U_w^{n,j} - U_w^{n,i})(V^{n,j} - V^{n,i}).$$

Hence

$$\lim_{(h,\tau)\to(0,0)}\frac{1}{2}\sum_{n=1}^N\tau\sum_{i,j=1}^M\Big(\sum_{K\subset\Delta_i\cap\Delta_j} c_{ij,K}(\eta_w(\overline{s}_\tau^n))_K\Big)(U_w^{n,j} - U_w^{n,i})(V^{n,j} - V^{n,i}) = \int_Q \eta_w(\overline{s})\nabla\overline{W}_w\cdot\nabla v. \tag{4.5}$$

Now, if w is in $W^{1,\infty}(\Omega)$, then again, standard finite element approximation shows that there exists a constant C, independent of h, $K\subset\Delta_i\cap\Delta_j$, and w, such that

$$\|w_K - w\|_{L^\infty(K)} \leqslant C\,h\,|w|_{W^{1,\infty}(K)} \leqslant C\,h\,|w|_{W^{1,\infty}(\Omega)}. \tag{4.6}$$

According to (1.6) and the regularity of $\overline{s}$, $\eta_w(\overline{s})$ belongs to $L^\infty(0,T;W^{1,\infty}(\Omega))$, and (4.6) gives

$$\|(\eta_w(\overline{s}_\tau^n))_K - \eta_w(\overline{s}_\tau^n)\|_{L^\infty(K)} \leqslant C\,h\,\|\eta_w'\|_{L^\infty(0,1)}\|\nabla\overline{s}\|_{L^\infty(Q)}$$

that allows to replace $(\eta_w(\overline{s}_\tau^n))_K$ by any value of $\eta_w(\overline{s}_\tau^n)$ in K. Let us choose the upwind value of $\overline{s}_\tau^n$ as in (1.19), i.e.,

$$\overline{s}_{w,\tau}^{n,ij} = \begin{cases} (\overline{s}_\tau^n)^i, & P_w^{n,i} > P_w^{n,j}\\ (\overline{s}_\tau^n)^j, & P_w^{n,i} < P_w^{n,j}\\ \max((\overline{s}_\tau^n)^i, (\overline{s}_\tau^n)^j), & P_w^{n,i} = P_w^{n,j}\end{cases} \tag{4.7}$$

and set

$$R_{ij} = \sum_{K\subset\Delta_i\cap\Delta_j} c_{ij,K}\big((\eta_w(\overline{s}_\tau^n))_K - \eta_w(\overline{s}_{w,\tau}^{n,ij})\big).$$

By proceeding as in Theorem 2.4 in [7] and applying (2.1), the regularity of v, and the approximation properties of I_h, we obtain

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{N}\tau\sum_{i,j=1}^{M}R_{ij}(U_w^{n,j}-U_w^{n,i})(V^{n,j}-V^{n,i}) &\leqslant \frac{1}{2}\sum_{n=1}^{N}\tau\Big(\sum_{i,j=1}^{M}|R_{ij}|(U_w^{n,j}-U_w^{n,i})^2\Big)^{1/2}\Big(\sum_{i,j=1}^{M}|R_{ij}|(V^{n,j}-V^{n,i})^2\Big)^{1/2}\\ &\leqslant C\,h\,\|\eta_w'\|_{L^\infty(0,1)}\|\nabla\overline{s}\|_{L^\infty(Q)}\|\nabla U_{w,h,\tau}\|_{L^2(Q)}\|\nabla V_{h,\tau}\|_{L^2(Q)}\\ &\leqslant C\,h\,\|\eta_w'\|_{L^\infty(0,1)}\|\nabla\overline{s}\|_{L^\infty(Q)}|v|_{H^1(0,T;H^2(\Omega))}.\end{aligned}$$

With (4.5), this implies

$$\lim_{(h,\tau)\to(0,0)}\frac{1}{2}\sum_{n=1}^{N}\tau\sum_{i,j=1}^{M}c_{ij}\eta_w(\overline{s}_{w,\tau}^{n,ij})(U_w^{n,j}-U_w^{n,i})(V^{n,j}-V^{n,i})=\int_Q\eta_w(\overline{s})\nabla\overline{W}_w\cdot\nabla v. \tag{4.8}$$

To recover $\int_0^T[P_{w,h,\tau},I_h(\eta_w(S_{h,\tau}));U_{w,h,\tau},V_{h,\tau}]_h$, we write

$$\eta_w(\overline{s}_{w,\tau}^{n,ij})=\eta_w(\overline{s}_{w,\tau}^{n,ij})-\eta_w(S_w^{n,ij})+\eta_w(S_w^{n,ij})$$

and we must examine the convergence of

$$X:=\frac{1}{2}\sum_{n=1}^{N}\tau\sum_{i,j=1}^{M}c_{ij}(\eta_w(\overline{s}_{w,\tau}^{n,ij})-\eta_w(S_w^{n,ij}))(U_w^{n,j}-U_w^{n,i})(V^{n,j}-V^{n,i}).$$

On the one hand, owing to the smoothness of v, we have

$$|V^{n,j}-V^{n,i}|\leqslant Ch_{ij}\|\nabla v\|_{L^\infty(Q)} \tag{4.9}$$

where h_{ij} is the length of the edge whose endpoints are the vertices i and j. On the other hand,

$$|\eta_w(\overline{s}_{w,\tau}^{n,ij})-\eta_w(S_w^{n,ij})|\leqslant C\|\eta_w'\|_{L^\infty(0,1)}|\overline{s}_{w,\tau}^{n,ij}-S_w^{n,ij}|.$$

Hence

$$|X|\leqslant C\|\nabla v\|_{L^\infty(Q)}\|\nabla U_{w,h,\tau}\|_{L^2(Q)}\Big(\sum_{n=1}^{N}\tau\sum_{i,j=1}^{M}c_{ij}h_{ij}^2|\overline{s}_{w,\tau}^{n,ij}-S_w^{n,ij}|^2\Big)^{1/2}.$$

It is easy to check that

$$\sum_{i,j=1}^{M}c_{ij}h_{ij}^2|\overline{s}_{w,\tau}^{n,ij}-S_w^{n,ij}|^2\leqslant C\sum_{i=1}^{M}m_i|\overline{s}_\tau^{n,i}-S^{n,i}|^2.$$

Therefore

$$\begin{aligned}|X|&\leqslant C\|\nabla v\|_{L^\infty(Q)}\|\nabla U_{w,h,\tau}\|_{L^2(Q)}\Big(\sum_{n=1}^{N}\tau\big\|I_h(\overline{s}_\tau^n)-S_{h,\tau}^n\big\|_{L^2(\Omega)}^2\Big)^{1/2}\\ &=C\|\nabla v\|_{L^\infty(Q)}\|\nabla U_{w,h,\tau}\|_{L^2(Q)}\|I_h(\overline{s}_\tau)-S_{h,\tau}\|_{L^2(Q)}\end{aligned}$$

where we have used the equivalence of norms. Then, we write

$$\|I_h(\overline{s}_\tau)-S_{h,\tau}\|_{L^2(Q)}\leqslant\|I_h(\overline{s}_\tau)-\overline{s}_\tau\|_{L^2(Q)}+\|\overline{s}_\tau-\overline{s}\|_{L^2(Q)}+\|\overline{s}-S_{h,\tau}\|_{L^2(Q)}$$

and the approximation properties of I_h, the strong convergence of $\overline{s}_\tau$ to $\overline{s}$ and of $S_{h,\tau}$ to $\overline{s}$, all in $L^2(Q)$ imply that

$$\lim_{(h,\tau)\to(0,0)}\sum_{n=1}^{N}\tau\sum_{i,j=1}^{M}c_{ij}(\eta_w(\overline{s}_{w,\tau}^{n,ij})-\eta_w(S_w^{n,ij}))(U_w^{n,j}-U_w^{n,i})(V^{n,j}-V^{n,i})=0. \tag{4.10}$$

A combination of (4.10) and (4.8) yields the intermediate convergence result when the limit function $\overline{s}$ is smooth,

$$\lim_{(h,\tau)\to(0,0)}-\sum_{n=1}^{N}\tau[P_{w,h,\tau},I_h(\eta_w(S_{h,\tau}));U_{w,h,\tau},V_{h,\tau}]_h=\int_Q\eta_w(\overline{s})\nabla\overline{W}_w\cdot\nabla v. \tag{4.11}$$

It remains to lift the regularity restriction on $\overline{s}$. Let $(S_m)_{m\geqslant 1}$ be a sequence of smooth functions that tend to $\overline{s}$ in $L^2(Q)$. Then for each $\varepsilon > 0$, there exists an integer M_0 such that

$$\|S_{M_0} - \overline{s}\|_{L^2(Q)} \leqslant \varepsilon. \tag{4.12}$$

From (4.12), the projection properties, and the fact that M_0 is fixed, we infer

$$\begin{aligned}\|\rho_\tau(\eta_w(\overline{s})) - \eta_w(\overline{s})\|_{L^2(Q)} &\leqslant \|\rho_\tau(\eta_w(\overline{s}) - \eta_w(S_{M_0}))\|_{L^2(Q)} + \|\rho_\tau(\eta_w(S_{M_0})) - \eta_w(S_{M_0})\|_{L^2(Q)}\\ &\quad + \|\eta_w(S_{M_0}) - \eta_w(\overline{s})\|_{L^2(Q)} \leqslant (2\,\varepsilon + C\,\tau)\|\eta_w'\|_{L^\infty(0,1)}.\end{aligned} \tag{4.13}$$

Now, we replace (4.2) by

$$\begin{aligned}\int_Q \eta_w(\overline{s})\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau} &= \int_Q \rho_\tau(\eta_w(\overline{s}) - \eta_w(S_{M_0}))\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau} + \int_Q \rho_\tau(\eta_w(S_{M_0}))\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau}\\ &= \int_Q \rho_\tau(\eta_w(\overline{s}) - \eta_w(S_{M_0}))\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau} + \int_Q \eta_w(S_{M_0})\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau}.\end{aligned} \tag{4.14}$$

For the first term, owing to (4.12), the projection properties, and (1.6), we have

$$\begin{aligned}\left|\int_Q \rho_\tau(\eta_w(\overline{s}) - \eta_w(S_{M_0}))\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau}\right| &\leqslant \|\nabla V_{h,\tau}\|_{L^\infty(Q)}\|\nabla U_{w,h,\tau}\|_{L^2(Q)}\|\eta_w(\overline{s}) - \eta_w(S_{M_0})\|_{L^2(Q)}\\ &\leqslant \varepsilon\,\|\eta_w'\|_{L^\infty(0,1)}\|\nabla V_{h,\tau}\|_{L^\infty(Q)}\|\nabla U_{w,h,\tau}\|_{L^2(Q)}.\end{aligned}$$

Then the uniform boundedness of $U_{w,h,\tau}$ and $V_{h,\tau}$ yield

$$\left|\int_Q \rho_\tau(\eta_w(\overline{s}) - \eta_w(S_{M_0}))\nabla U_{w,h,\tau}\cdot\nabla V_{h,\tau}\right| \leqslant C\,\varepsilon \tag{4.15}$$

with a constant C independent of h and τ. Thus, we must examine the limit of the second term. Since M_0 is fixed and S_{M_0} is smooth, by reproducing the previous steps, we derive the analogue of (4.8) for the function S_{M_0},

$$\begin{aligned}\lim_{(h,\tau)\to(0,0)} \frac{1}{2}\sum_{n=1}^{N}\tau\sum_{i,j=1}^{M} c_{ij}\eta_w((S_{M_0})_{w,\tau}^{n,ij})(U_w^{n,j} - U_w^{n,i})(V^{n,j} - V^{n,i})\\ = \int_Q \eta_w(S_{M_0})\nabla\overline{W}_w\cdot\nabla v = \int_Q \eta_w(\overline{s})\nabla\overline{W}_w\cdot\nabla v + R\end{aligned} \tag{4.16}$$

where

$$|R| = \left|\int_Q (\eta_w(S_{M_0}) - \eta_w(\overline{s}))\nabla\overline{W}_w\cdot\nabla v\right| \leqslant \|\eta_w'\|_{L^\infty(0,1)}\|S_{M_0} - \overline{s}\|_{L^2(Q)}\|\nabla\overline{W}_w\|_{L^2(Q)}\|\nabla v\|_{L^\infty(Q)} \leqslant C\,\varepsilon. \tag{4.17}$$

To relate the left-hand side of (4.16) to $[P_{w,h,\tau}, I_h(\eta_w(S_{h,\tau})); U_{w,h,\tau}, V_h]_h$, we split

$$\eta_w((S_{M_0})_{w,\tau}^{n,ij}) = \eta_w(S_w^{n,ij}) + \eta_w((S_{M_0})_{w,\tau}^{n,ij}) - \eta_w(S_w^{n,ij})$$

and examine the convergence of

$$Y := \frac{1}{2}\sum_{n=1}^{N}\tau\sum_{i,j=1}^{M} c_{ij}(\eta_w((S_{M_0})_{w,\tau}^{n,ij}) - \eta_w(S_w^{n,ij}))(U_w^{n,j} - U_w^{n,i})(V^{n,j} - V^{n,i}).$$

By arguing as above and using the interpolant I_h, we derive

$$|Y| \leqslant C\|\eta_w'\|_{L^\infty(0,1)}\|\nabla v\|_{L^\infty(Q)}\|\nabla U_{w,h,\tau}\|_{L^2(Q)}\|I_h((S_{M_0})_\tau) - S_{h,\tau}\|_{L^2(Q)}.$$

Finally, we write

$$\begin{aligned}\|I_h((S_{M_0})_\tau) - S_{h,\tau}\|_{L^2(Q)} &\leqslant \|I_h((S_{M_0})_\tau) - (S_{M_0})_\tau\|_{L^2(Q)} + \|(S_{M_0})_\tau - S_{M_0}\|_{L^2(Q)} + \|S_{M_0} - \overline{s}\|_{L^2(Q)} + \|\overline{s} - S_{h,\tau}\|_{L^2(Q)}\\ &\leqslant C\,h\|S_{M_0}\|_{L^\infty(0,T;H^2(\Omega))} + C\tau\|S_{M_0}\|_{H^1(0,T;L^2(\Omega))} + \varepsilon + \|\overline{s} - S_{h,\tau}\|_{L^2(Q)}\end{aligned}$$

so that

$$|Y| \leqslant C(h+\tau+\varepsilon) + \|\overline{s} - S_{h,\tau}\|_{L^2(Q)}. \tag{4.18}$$

In the next theorem, the limit (4.11) when $\overline{s}$ is only in $L^2(Q)$ follows by combining (4.14)–(4.18). The same argument holds when w is replaced by o.

Theorem 4.1. *Let $v \in \mathcal{C}^1(\overline{Q})$ be a smooth function and let $V_{h,\tau} = I_h(v)(t_n)$ in $]t_{n-1}, t_n]$;*

$$\lim_{(h,\tau)\to(0,0)} -\int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); U_{\alpha,h,\tau}, V_{h,\tau}]_h = \int_Q \eta_\alpha(\overline{s})\nabla \overline{W}_\alpha \cdot \nabla v \tag{4.19}$$

where $\overline{s}$ is the strong limit of $S_{h,\tau}$ and $\overline{W}_\alpha$ the weak limit of $U_{\alpha,h,\tau}$, $\alpha = w, o$.

4.1.2 The term with $p_{\alpha g}$

This paragraph is dedicated to the limit of

$$\int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); I_h(p_{\alpha g}(S_{h,\tau})), V_{h,\tau}]_h, \quad \alpha = w, o.$$

It shall be split below, as suggested by the following observation, derived from (1.12) and (1.11):

$$\eta_w(S_w^{ij})p_{wg}(S^j) + g(S^j) = \int_0^{S^j} f_o(x)(\eta_w(S_w^{ij}) - \eta_w(x))p_c'(x)\,\mathrm{d}x$$

$$\eta_o(S_o^{ij})p_{og}(S^j) + g(S^j) = \int_0^{S^j} f_w(x)(\eta_o(S_o^{ij}) - \eta_o(x))p_c'(x)\,\mathrm{d}x.$$

Thus, we add and subtract g and write by applying Proposition 2.1 of [7],

$$\begin{aligned}\int_0^T &[P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); I_h(p_{\alpha g}(S_{h,\tau})), V_{h,\tau}]_h \\ &= \sum_{n=1}^N \tau \sum_{i,j=1}^M V^{n,i} c_{ij}\Big[\eta_\alpha(S_\alpha^{n,ij})(p_{\alpha g}(S^{n,j}) - p_{\alpha g}(S^{n,i})) + g(S^{n,j}) - g(S^{n,i})\Big] \\ &\quad + \int_Q \nabla g(S_{h,\tau}) \cdot \nabla V_{h,\tau} = T_1 + T_2.\end{aligned}$$

Since

$$\lim_{(h,\tau)\to(0,0)} T_2 = \int_Q \nabla g(\overline{s}) \cdot \nabla v \tag{4.20}$$

we must prove that the first term tends to zero. When $\alpha = w$, it has the form

$$T_1 = -\frac{1}{2}\sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij}\bigg(\int_{S^{n,i}}^{S^{n,j}} f_o(x)(\eta_w(S_w^{n,ij}) - \eta_w(x))p_c'(x)\,\mathrm{d}x\bigg)(V^{n,j} - V^{n,i}) \tag{4.21}$$

with an analogous expression in the non-wetting phase. Then (4.9) yields,

$$|T_1| \leqslant \frac{C}{2}\|\nabla v\|_{L^\infty(Q)} \sum_{n=1}^N \tau \sum_{i,j=1}^M h_{ij} c_{ij} \bigg|\int_{S^{n,i}}^{S^{n,j}} f_o(x)(\eta_w(S_w^{n,ij}) - \eta_w(x))p_c'(x)\,\mathrm{d}x\bigg|. \tag{4.22}$$

Showing that T_1 is small requires a technical argument that we split into several steps.

Proposition 4.2. For the wetting phase, we have

$$\bigg|\int_{S^i}^{S^j} f_o(x)(\eta_w(S_w^{ij}) - \eta_w(x))p_c'(x)\,\mathrm{d}x\bigg| \leqslant -(\eta_w(S^j) - \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)). \tag{4.23}$$

For the non-wetting phase, the corresponding expression is bounded by

$$\bigg|\int_{S^i}^{S^j} f_w(x)(\eta_o(S_o^{ij}) - \eta_o(x))p_c'(x)\,\mathrm{d}x\bigg| \leqslant (\eta_o(S^j) - \eta_o(S^i))(p_{og}(S^j) - p_{og}(S^i)). \tag{4.24}$$

Proof. Let us prove (4.23), the proof of (4.24) being similar. The discussion depends on the respective values of S^j and S^i. There are two cases: $S^i < S^j$ or $S^i > S^j$. Of course $S^i = S^j$ brings nothing.

1. If $S^i < S^j$ and $S_w^{ij} = S^i$, then $\eta_w(S_w^{ij}) - \eta_w(x) = \eta_w(S^i) - \eta_w(x)$, and, as p_{wg} is decreasing,

$$0 \leqslant \int_{S^i}^{S^j} f_o(x)(-p_c'(x))(\eta_w(x) - \eta_w(S^i))\,dx \leqslant -(\eta_w(S^j) - \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)).$$

If $S_w^{ij} = S^j$, then $\eta_w(S_w^{ij}) - \eta_w(x) = \eta_w(S^j) - \eta_w(x)$, and

$$0 \leqslant \int_{S^i}^{S^j} f_o(x)(-p_c'(x))(\eta_w(S^j) - \eta_w(x))\,dx \leqslant -(\eta_w(S^j) - \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)).$$

2. If $S^i > S^j$ and $S_w^{ij} = S^i$, then

$$0 \leqslant \int_{S^j}^{S^i} f_o(x)(-p_c'(x))(\eta_w(S^i) - \eta_w(x))\,dx \leqslant -(\eta_w(S^i) - \eta_w(S^j))(p_{wg}(S^i) - p_{wg}(S^j)).$$

Finally, suppose that $S_w^{ij} = S^j$. Then

$$0 \leqslant \int_{S^j}^{S^i} f_o(x)\,p_c'(x)(\eta_w(S^j) - \eta_w(x))\,dx \leqslant -(\eta_w(S^i) - \eta_w(S^j))(p_{wg}(S^i) - p_{wg}(S^j)).$$

This proves (4.23). □

By substituting (4.23) into (4.22), we arrive at

$$|T_1| \leqslant \frac{C}{2}\|\nabla v\|_{L^\infty(Q)} \sum_{n=1}^{N} \tau \sum_{i,j=1}^{M} h_{ij} c_{ij}\Big(- (\eta_w(S^{n,j}) - \eta_w(S^{n,i}))(p_{wg}(S^{n,j}) - p_{wg}(S^{n,i}))\Big) \tag{4.25}$$

with an analogous bound in the non-wetting phase. Up to the factor h_{ij}, they behave like $\int_Q \nabla(I_h(\eta_\alpha(S_{h,\tau}))) \times \nabla(I_h(p_{\alpha g}(S_{h,\tau})))$, $\alpha = w, o$. Thus T_1 tends to zero if this quantity is bounded or is of the order of a negative power of h that is larger than -1. We have no direct bound for it, but as we do have a bound for $\int_Q \nabla(I_h(f_\alpha(S_{h,\tau}))) \cdot \nabla(I_h(g(S_{h,\tau})))$, see (2.43), we can gain some insight by relating the two integrands. Again, we examine the wetting phase, the treatment of the non-wetting phase being the same. The proposition below will be applied to $x_1 = S^{n,i}$ and $x_2 = S^{n,j}$. The condition $x_1 < x_2$ is not a restriction because if it does not hold, the indices i and j can be interchanged without changing the value of the two integrands.

Proposition 4.3. We have for all pairs x_1, x_2 with $0 \leqslant x_1 < x_2 \leqslant 3/4$,

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C(x_2^{\vartheta_w} - x_1^{\vartheta_w})(x_2^{\beta_3} - x_1^{\beta_3}) \tag{4.26}$$

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geqslant C(x_2^{\vartheta_w} - x_1^{\vartheta_w})(x_2^{\vartheta_w+\beta_3} - x_1^{\vartheta_w+\beta_3}). \tag{4.27}$$

Similarly, we have for all pairs x_1, x_2 with $1/4 \leqslant x_1 < x_2 \leqslant 1$,

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C(x_2 - x_1)((1 - x_1)^{\vartheta_o+\beta_4} - (1 - x_2)^{\vartheta_o+\beta_4}) \tag{4.28}$$

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geqslant C((1 - x_1)^{\vartheta_o} - (1 - x_2)^{\vartheta_o})((1 - x_1)^{\vartheta_o+\beta_4} - (1 - x_2)^{\vartheta_o+\beta_4}). \tag{4.29}$$

Finally, we have for all pairs x_1, x_2 with $0 \leqslant x_1 \leqslant 1/4$ and $3/4 \leqslant x_2 \leqslant 1$,

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)). \tag{4.30}$$

All constants C above are independent of x_1 and x_2.

Proof. According to (1.6),

$$\eta_w(x_2) - \eta_w(x_1) \leqslant \frac{1}{\alpha_w \vartheta_w}(x_2^{\vartheta_w} - x_1^{\vartheta_w}).$$

Next, recalling that $p'_{wg}(x) = f_o(x)p'_c(x)$, we have, owing to (1.8), (1.9), and (1.5),

$$p_{wg}(x_1) - p_{wg}(x_2) = \int_{x_1}^{x_2} f_o(x)(-p'_c(x))\,dx \leqslant \frac{1}{\eta_*}\frac{1}{\alpha_3}\frac{1}{\alpha_o\vartheta_o}\int_{x_1}^{x_2} x^{\beta_3-1}(1-x)^{\vartheta_o+\beta_4-1}\,dx$$
$$\leqslant \frac{1}{\eta_*}\frac{1}{\alpha_3}\frac{1}{\alpha_o\vartheta_o}\int_{x_1}^{x_2} x^{\beta_3-1}\,dx \leqslant \frac{1}{\eta_*}\frac{1}{\alpha_3\beta_3}\frac{1}{\alpha_o\vartheta_o}(x_2^{\beta_3} - x_1^{\beta_3}) \tag{4.31}$$

and (4.26), valid on [0, 1], follows from these two inequalities.

For (4.27), we use (2.50) that gives

$$f_w(x_2) - f_w(x_1) \geqslant \frac{\alpha_o\alpha_w}{C_{\max}^2}\frac{1}{\vartheta_o\vartheta_w}\left(\frac{1}{4}\right)^{\vartheta_o}(x_2^{\vartheta_w} - x_1^{\vartheta_w}) \tag{4.32}$$

and we use (2.48) that gives

$$g(x_2) - g(x_1) \geqslant \frac{\alpha_3}{C_{\max}}\frac{\alpha_w}{\vartheta_w}\frac{\alpha_o}{\vartheta_o}\frac{1}{\vartheta_w+\beta_3}\left(\frac{1}{4}\right)^{\vartheta_o+\beta_4-1}(x_2^{\vartheta_w+\beta_3} - x_1^{\vartheta_w+\beta_3}).$$

The product of the two leads to (4.27).

Regarding (4.28), (4.26), albeit valid for all $x \in [0,1]$, is not adequate for the comparison we have in mind, and instead we use that

$$\eta'_w(x) \leqslant \frac{1}{\alpha_w}$$

which implies that

$$\eta_w(x_2) - \eta_w(x_1) \leqslant \frac{1}{\alpha_w}(x_2 - x_1).$$

Similarly, we use

$$-p'_{wg}(x) \leqslant \frac{1}{\eta_*}\frac{1}{\alpha_3}\frac{1}{\alpha_o\vartheta_o}4^{1-\beta_3}(1-x)^{\vartheta_o+\beta_4-1}$$

so that

$$p_{wg}(x_1) - p_{wg}(x_2) \leqslant \frac{1}{\eta_*}\frac{1}{\alpha_3}\frac{1}{\alpha_o\vartheta_o}\frac{1}{\vartheta_o+\beta_4}4^{1-\beta_3}((1-x_1)^{\vartheta_o+\beta_4} - (1-x_2)^{\vartheta_o+\beta_4})$$

thus proving (4.28). Next, by applying (2.51), we have

$$f_w(x_2) - f_w(x_1) \geqslant \frac{1}{C_{\max}^2}\frac{\alpha_o\alpha_w}{\vartheta_w\vartheta_o}\left(\frac{1}{4}\right)^{\vartheta_w}((1-x_1)^{\vartheta_o} - (1-x_2)^{\vartheta_o}).$$

Likewise, by applying (2.48), we obtain

$$g(x_2) - g(x_1) \geqslant \frac{\alpha_3}{C_{\max}}\frac{\alpha_w}{\vartheta_w}\frac{\alpha_o}{\vartheta_o}\frac{1}{\vartheta_o+\beta_4}\left(\frac{1}{4}\right)^{\vartheta_w-1+\beta_3}((1-x_1)^{\vartheta_o+\beta_4} - (1-x_2)^{\vartheta_o+\beta_4}).$$

The product of the two yields (4.29).

Finally, when $0 \leqslant x_1 \leqslant 1/4$ and $3/4 \leqslant x_2 \leqslant 1$, since both η_w and $-p_{wg}$ are both increasing, they satisfy

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant \eta_w(1)(-p_{wg}(1)) > 0.$$

Likewise, as both f_w and g are increasing, they satisfy

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geqslant \left(f_w\left(\frac{3}{4}\right) - f_w\left(\frac{1}{4}\right)\right)\left(g\left(\frac{3}{4}\right) - g\left(\frac{1}{4}\right)\right) =: D > 0.$$

Hence

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant -\frac{1}{D}(\eta_w\,p_{wg})(1)(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1))$$

whence (4.30). Clearly all constants involved are independent of x_1 and x_2. □

It stems from (4.26) and (4.27), that the two left-hand sides cannot be compared when x_1 and x_2 are too small. The same observation applies to (4.28) and (4.29) when $1 - x_1$ and $1 - x_2$ are too small. But in this case, there is no need for comparison because the expression we want to bound is sufficiently small, as is shown in the next proposition where again, $x_1 = S^{n,i}$ and $x_2 = S^{n,j}$.

Proposition 4.4. Suppose that $x_1 < x_2 \leqslant h_{ij}^{\gamma_1}$ for some exponent $\gamma_1 > 0$ such that

$$\gamma_1 > \frac{1}{\vartheta_w + \beta_3}. \tag{4.33}$$

Then

$$h_{ij}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C\, h_{ij}^{2+\delta_1} \tag{4.34}$$

where

$$0 < \delta_1 \leqslant \gamma_1(\vartheta_w + \beta_3) - 1. \tag{4.35}$$

Similarly, suppose that $1 - x_2 < 1 - x_1 \leqslant h_{ij}^{\gamma_2}$ for some exponent $\gamma_2 > 0$ such that

$$\gamma_2 > \frac{1}{1 + \vartheta_o + \beta_4}. \tag{4.36}$$

Then

$$h_{ij}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C\, h_{ij}^{2+\delta_2} \tag{4.37}$$

where

$$0 < \delta_2 \leqslant \gamma_2(1 + \vartheta_o + \beta_4) - 1. \tag{4.38}$$

In both cases, the constants C are independent of x_1, x_2, and h_{ij}.

Proof. In the first case, according to (4.26), the choice (4.35) and (4.33) on γ_1, we have

$$h_{ij}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C\, h_{ij}^{1+\gamma_1(\vartheta_w+\beta_3)}$$

with the constant C of (4.26), which gives (4.34). In the second case, the same argument leads to

$$h_{ij}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant h_{ij}^{1+\gamma_2(1+\vartheta_o+\beta_4)}$$

with the constant C of (4.28), thus implying (4.37) with the choice (4.38) for δ_2 and the condition (4.36) on γ_2. □

Now, we turn to the case when x_2 is not too small.

Proposition 4.5. In addition to (4.33), suppose that the exponent γ_1 of Proposition 4.4 satisfies

$$\gamma_1 < \frac{1}{\vartheta_w}. \tag{4.39}$$

Suppose that $x_1 < x_2$ and $3/4 \geqslant x_2 > h_{ij}^{\gamma_1}$. Then

$$h_{ij}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C h_{ij}^{\delta_1'}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \tag{4.40}$$

where

$$0 < \delta_1' = \min(1 - \gamma_1\vartheta_w, \delta_1). \tag{4.41}$$

Again, the constant C is independent of x_1, x_2, and h_{ij}.

Proof. Either $x_1 \leqslant x_2/2$ or $x_1 > x_2/2$, and we examine each case.

1. When $x_1 \leqslant x_2/2$, formula (4.27) leads to
$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geqslant \left(1 - \left(\frac{1}{2}\right)^{\vartheta_w}\right)\left(1 - \left(\frac{1}{2}\right)^{\vartheta_w+\beta_3}\right) C x_2^{2\vartheta_w+\beta_3}$$
with the constant C of (4.27), whereas
$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C x_2^{\vartheta_w+\beta_3}$$
with the constant C of (4.26). Hence
$$h_{ij}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C \frac{h_{ij}}{x_2^{\vartheta_w}}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1))$$
with another constant C independent of x_1, x_2, and h_{ij}. Now, we use the assumption that $x_2 > h_{ij}^{\gamma_1}$. Then, owing to (4.41),
$$\frac{h_{ij}}{x_2^{\vartheta_w}} \leqslant h_{ij}^{1-\gamma_1\vartheta_w} \leqslant h_{ij}^{\delta_1'}$$
and we recover (4.40).
2. When $x_1 > x_2/2$, we infer from the next to last inequality in (4.31) that
$$p_{wg}(x_1) - p_{wg}(x_2) \leqslant \frac{1}{\eta_*}\frac{1}{\alpha_3}\frac{1}{\alpha_o\vartheta_o}(x_2 - x_1)x_1^{\beta_3-1} \leqslant \frac{1}{\eta_*}\frac{1}{\alpha_3}\frac{1}{\alpha_o\vartheta_o}2^{1-\beta_3}\frac{1}{x_2^{1-\beta_3}}(x_2 - x_1).$$
Thus, on the one hand,
$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C\frac{1}{x_2^{1-\beta_3}}(x_2 - x_1)(x_2^{\vartheta_w} - x_1^{\vartheta_w}) \tag{4.42}$$
where C is the above constant divided by $\alpha_w\vartheta_w$. On the other hand, we use the lower bound (4.32) for the difference in f_w and we need a lower bound for the difference in g. It is derived from (2.48),
$$\begin{aligned} g(x_2) - g(x_1) &\geqslant \frac{\alpha_3}{C_{\max}}\frac{\alpha_w}{\vartheta_w}\frac{\alpha_o}{\vartheta_o}\left(\frac{1}{4}\right)^{\vartheta_o+\beta_4-1} x_1^{\vartheta_w+\beta_3-1}(x_2 - x_1) \\ &\geqslant \frac{\alpha_3}{C_{\max}}\frac{\alpha_w}{\vartheta_w}\frac{\alpha_o}{\vartheta_o}\left(\frac{1}{4}\right)^{\vartheta_o+\beta_4-1}\left(\frac{1}{2}\right)^{\vartheta_w+\beta_3-1} x_2^{\vartheta_w+\beta_3-1}(x_2 - x_1). \end{aligned} \tag{4.43}$$
Hence (4.32) and (4.43) yield
$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geqslant C x_2^{\vartheta_w+\beta_3-1}(x_2^{\vartheta_w} - x_1^{\vartheta_w})(x_2 - x_1) \tag{4.44}$$
with the product of the constants of (4.32) and (4.43). Then by combining (4.42) and (4.44), we deduce that
$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant \frac{C}{h_{ij}^{\gamma_1\vartheta_w}}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1))$$
which is (4.40) when δ_1' satisfies (4.41).

The proof is completed. □

The case when $1 - x_1$ is not too small is handled by the next proposition.

Proposition 4.6. In addition to (4.36), suppose that the exponent γ_2 of Proposition 4.4 satisfies
$$\gamma_2 < \frac{1}{\vartheta_o - 1}. \tag{4.45}$$
Suppose that $1/4 < x_1 < x_2 \leqslant 1$ and $1 - x_1 > h_{ij}^{\gamma_2}$. Then
$$h_{ij}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C h_{ij}^{\delta_2'}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \tag{4.46}$$

where

$$0 < \delta_2' = \min(\delta_2, 1 - \gamma_2(\vartheta_o - 1)). \tag{4.47}$$

Again, the constant C is independent of x_1, x_2, and h_{ij}.

Proof. The proof is analogous to that of Proposition 4.5, but we sketch the steps for the reader's convenience. We skip the constants' details, but stress that they are independent of x_1, x_2, and h_{ij}. Again, there are two possibilities, either $1 - x_2 \leqslant (1 - x_1)/2$ or $1 - x_2 > (1 - x_1)/2$, and we examine each case.

1. In the first case,

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C(1 - x_1)^{1+\vartheta_0+\beta_4}$$

and

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geqslant C(1 - x_1)^{2\vartheta_0+\beta_4}.$$

Hence

$$\begin{aligned}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) &\leqslant C\frac{1}{(1 - x_1)^{\vartheta_o-1}}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1))\\ &\leqslant C\frac{1}{h_{ij}^{\gamma_2(\vartheta_o-1)}}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)).\end{aligned}$$

With (4.45) and (4.47), this implies (4.46).

2. In the second case, we have on the one hand,

$$p_{wg}(x_1) - p_{wg}(x_2) \leqslant C(x_2 - x_1)(1 - x_1)^{\vartheta_o+\beta_4-1}$$

so that

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leqslant C(x_2 - x_1)^2(1 - x_1)^{\vartheta_o+\beta_4-1}.$$

On the other hand,

$$f_w(x_2) - f_w(x_1) \geqslant C(x_2 - x_1)(1 - x_1)^{\vartheta_o-1}$$

and

$$g(x_2) - g(x_1) \geqslant C(x_2 - x_1)(1 - x_1)^{\vartheta_o+\beta_4-1}$$

and thus

$$\begin{aligned}(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) &\leqslant C\frac{1}{(1 - x_1)^{\vartheta_o-1}}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1))\\ &\leqslant C\frac{1}{h_{ij}^{\gamma_2(\vartheta_o-1)}}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1))\end{aligned}$$

whence (4.46).

This completes the proof. □

In view of (4.33), (4.35), (4.39), and (4.41), let us choose

$$\delta_1 = \delta_1' = \frac{\beta_3}{2\vartheta_w + \beta_3}, \qquad \gamma_1 = \frac{2}{2\vartheta_w + \beta_3}. \tag{4.48}$$

Then (4.33) and (4.35) are satisfied, as well as (4.39) and (4.41). Likewise, in view of (4.36), (4.38), (4.45), and (4.47), the choice

$$\delta_2 = \delta_2' = \frac{2 + \beta_4}{2\vartheta_o + \beta_4}, \qquad \gamma_2 = \frac{2}{2\vartheta_o + \beta_4} \tag{4.49}$$

satisfies (4.36), (4.38), (4.45), (4.47). Then the desired limit follows by collecting these results.

Lemma 4.1. *The term T_1 defined in* (4.21) *tends to zero, with a similar limit in the non-wetting phase, i.e.,*

$$\lim_{(h,\tau)\to(0,0)} \sum_{n=1}^{N} \tau \sum_{i,j=1}^{M} c_{ij} \bigg(\int_{S^{n,i}}^{S^{n,j}} f_o(x)(\eta_w(S_w^{n,ij}) - \eta_w(x)) p_c'(x)\, dx \bigg)(V^{n,j} - V^{n,i}) = 0$$
$$\lim_{(h,\tau)\to(0,0)} \sum_{n=1}^{N} \tau \sum_{i,j=1}^{M} c_{ij} \bigg(\int_{S^{n,i}}^{S^{n,j}} f_w(x)(\eta_o(S_o^{n,ij}) - \eta_o(x)) p_c'(x)\, dx \bigg)(V^{n,j} - V^{n,i}) = 0. \tag{4.50}$$

Proof. We prove the first limit. Here the parameters of Propositions 4.4 and 4.5 are chosen by (4.48) and (4.49). It stems from the above considerations that, for each index n, the set of all indices (i,j) from 1 to M can be grouped into three subsets,

$$\mathcal{O}_1 = \Big\{(i,j)\,;\, 0 \leqslant S^{n,i} < S^{n,j} \leqslant \frac{3}{4}\Big\}, \qquad \mathcal{O}_2 = \Big\{(i,j)\,;\, \frac{1}{4} \leqslant S^{n,i} < S^{n,j} \leqslant 1\Big\}$$
$$\mathcal{O}_3 = \Big\{(i,j)\,;\; 0 \leqslant S^{n,i} \leqslant \frac{1}{4},\; \frac{3}{4} \leqslant S^{n,j} \leqslant 1\Big\}.$$

In turn, $\mathcal{O}_1$ and $\mathcal{O}_2$ can each be partitioned into two subsets

$$\mathcal{O}_{1,1} = \{(i,j) \in \mathcal{O}_1\,;\, S^{n,j} \leqslant h_{ij}^{\gamma_1}\}, \qquad \mathcal{O}_{1,2} = \{(i,j) \in \mathcal{O}_1\,;\, S^{n,j} > h_{ij}^{\gamma_1}\}$$
$$\mathcal{O}_{2,1} = \{(i,j) \in \mathcal{O}_2\,;\, 1 - S^{n,i} \leqslant h_{ij}^{\gamma_2}\}, \qquad \mathcal{O}_{2,2} = \{(i,j) \in \mathcal{O}_2\,;\, 1 - S^{n,i} > h_{ij}^{\gamma_2}\}.$$

To simplify, let

$$A_{i,j} = c_{ij} \bigg(\int_{S^{n,i}}^{S^{n,j}} f_o(x)(\eta_w(S_w^{n,ij}) - \eta_w(x)) p_c'(x)\, dx \bigg)(V^{n,j} - V^{n,i}).$$

In view of (4.34) and (4.37), for all pairs (i,j) in $\mathcal{O}_{\ell,1}$, $\ell = 1, 2$, $A_{i,j}$ satisfies

$$|A_{i,j}| \leqslant C\|\nabla v\|_{L^\infty(Q)} h_{ij}^{2+\delta_\ell} c_{ij}.$$

Owing to (4.40) and (4.46), for all pairs (i,j) in $\mathcal{O}_{\ell,2}$, $\ell = 1, 2$, we have

$$|A_{i,j}| \leqslant C\|\nabla v\|_{L^\infty(Q)} h_{ij}^{\delta_\ell} c_{ij} \big(f_w(S^{n,j}) - f_w(S^{n,i})\big)\big(g(S^{n,j}) - g(S^{n,i})\big).$$

Finally, for all pairs (i,j) in $\mathcal{O}_3$,

$$|A_{i,j}| \leqslant C\|\nabla v\|_{L^\infty(Q)} h_{ij} c_{ij} \big(f_w(S^{n,j}) - f_w(S^{n,i})\big)\big(g(S^{n,j}) - g(S^{n,i})\big).$$

According to (2.43), the sum of the terms over all (i,j) in $\mathcal{O}_{\ell,2}$ and $\mathcal{O}_3$ tends to zero. For the remaining terms, observe that by definition,

$$h_{ij}^2 c_{ij} \leqslant C|\Delta_i \cap \Delta_j|$$

so that the sum over all (i,j) in $\mathcal{O}_{\ell,1}$ is bounded by $Ch_{ij}^{\delta_\ell}$ that also tends to zero, whence the first part of the limit (4.50). The same limit to zero holds for the non-wetting phase. □

With (4.20), this lemma leads to the desired limit of the term with the auxiliary pressures.

Theorem 4.2. *Let $v \in \mathcal{C}^1(\overline{Q})$ be a smooth function and let $V_{h,\tau}(t) = I_h(v)(t_n)$ in $]t_{n-1}, t_n]$;*

$$\lim_{(h,\tau)\to(0,0)} \int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); I_h(p_{\alpha g}(S_{h,\tau})), V_{h,\tau}]_h = \int_Q \nabla g(\overline{s}) \cdot \nabla v, \quad \alpha = w, o \tag{4.51}$$

where $\overline{s}$ is the limit of $S_{h,\tau}$.

We remark that the derivative of g satisfies formally the identities

$$\forall x \in [0,1], \quad \eta_\alpha(x) p_{\alpha g}'(x) + g'(x) = 0, \quad \alpha = w, o. \tag{4.52}$$

Finally, Theorems 4.1 and 4.2, together with (4.52) and (3.44), give the desired convergence of the upwind terms.

Theorem 4.3. *With the notation and assumptions of Theorem* 4.1, *we have for all functions* $v \in \mathcal{C}^1(\overline{Q})$,

$$\lim_{(h,\tau)\to(0,0)} -\int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); P_{\alpha,h,\tau}, V_{h,\tau}]_h$$
$$= \int_Q (\eta_w(\overline{s})\nabla(\overline{p}_w + p_{wg}(\overline{s})) + \nabla g(\overline{s})) \cdot \nabla v \quad \text{if } \alpha = w$$
$$= \int_Q (\eta_o(\overline{s})\nabla(\overline{p}_o - p_{og}(\overline{s})) - \nabla g(\overline{s})) \cdot \nabla v \quad \text{if } \alpha = o. \tag{4.53}$$

4.2 Convergence of the right-hand sides

In order to pass to the limit in the right-hand sides of (1.20)–(1.21) it is convenient to replace the quadrature formulas by integrals. Since the quadrature formulas are exact for polynomials of degree one, this is achieved by approximating some functions with the operator ρ_h, see (1.26). As s_{in} belongs to $L^\infty(Q)$, standard approximation properties of ρ_τ and r_h and a density argument imply

$$\lim_{(h,\tau)\to(0,0)} \rho_\tau(\rho_K(s_{\text{in}})) = s_{\text{in}} \quad \text{in } L^\infty(K\times]0,T[). \tag{4.54}$$

Then the continuity of f_α, for $\alpha = w, o$, yields

$$\lim_{(h,\tau)\to(0,0)} f_\alpha(\rho_\tau(\rho_K(s_{\text{in}}))) = f_\alpha(s_{\text{in}}) \quad \text{in } L^\infty(K\times]0,T[). \tag{4.55}$$

Similarly, since $\overline{q}$ belongs to $L^2(Q)$,

$$\lim_{(h,\tau)\to(0,0)} \rho_\tau(\rho_K(\overline{q})) = \overline{q} \quad \text{in } L^2(K\times]0,T[).$$

Also the (constant in space) correction added to $\rho_\tau(r_h(\overline{q}))$ satisfies

$$\lim_{(h,\tau)\to(0,0)} \rho_\tau\left(\frac{1}{|\Omega|}\int_\Omega (r_h(\overline{q}) - \overline{q})\right) = 0 \quad \text{in } L^2(Q).$$

Therefore

$$\lim_{(h,\tau)\to(0,0)} \overline{q}_{h,\tau} = \overline{q} \quad \text{in } L^2(Q). \tag{4.56}$$

With the same function $V_{h,\tau}$, consider the first term in the right-hand sides of (1.20)–(1.21):

$$X := \sum_{n=1}^N \tau(I_h(f_\alpha(s^n_{\text{in},h}))\overline{q}^n_h, V^n_h)_h = \int_0^T (I_h(f_\alpha(s_{\text{in},h,\tau}))\overline{q}_{h,\tau}, V_{h,\tau})_h.$$

By definition of the quadrature formula, X has the following expression:

$$X = \sum_{n=1}^N \tau \sum_{K\in\mathcal{T}_h} \frac{|K|}{d+1} \sum_{\ell=1}^{d+1} f_\alpha(s^{n,\ell_i}_{\text{in,h},\tau})\overline{q}^{n,\ell_i}_{h,\tau} V^{n,\ell_i}_{h,\tau}.$$

By inserting $f_\alpha(\rho_\tau(\rho_K(s_{\text{in}})))$ and $\rho_\tau(\rho_K(\overline{q}))$, this becomes

$$X = \sum_{n=1}^N \tau \sum_{K\in\mathcal{T}_h} \frac{|K|}{d+1} \sum_{\ell=1}^{d+1} (f_\alpha(s^{n,\ell_i}_{\text{in,h},\tau}) - f_\alpha(\rho_\tau(\rho_K(s_{\text{in}}))))\overline{q}^{n,\ell_i}_{h,\tau} V^{n,\ell_i}_{h,\tau}$$
$$+ \sum_{n=1}^N \tau \sum_{K\in\mathcal{T}_h} \frac{|K|}{d+1} \sum_{\ell=1}^{d+1} f_\alpha(\rho_\tau(\rho_K(s_{\text{in}})))(\overline{q}^{n,\ell_i}_{h,\tau} - \rho_\tau(\rho_K(\overline{q})))V^{n,\ell_i}_{h,\tau}$$
$$+ \int_Q f_\alpha(\rho_\tau(\rho_K(s_{\text{in}})))\rho_\tau(\rho_K(\overline{q}))V_{h,\tau} = X_1 + X_2 + X_3$$

since the last summand is a polynomial of degree one. We have

$$\lim_{(h,\tau)\to(0,0)} X_3 = \int_Q f_o(s_{\text{in}})\,\overline{q}\,v.$$

It remains to show that X_1 and X_2 tend to zero. For X_1, since f_o and f_w have the same derivative (up to the sign), we deduce from (2.49), (1.6), (1.7), (1.9), and (1.10) that f'_α is bounded in $[0,1]$; hence

$$\left|f_\alpha(s^{n,\ell_i}_{\text{in,h},\tau}) - f_\alpha(\rho_\tau(\rho_K(s_{\text{in}})))\right| \leqslant C\left|s^{n,\ell_i}_{\text{in,h},\tau} - \rho_\tau(\rho_K(s_{\text{in}}))\right|.$$

Thus, the summand is bounded by polynomials and the equivalence of norms yields

$$|X_1| \leqslant C\|v\|_{L^\infty(Q)}\|s_{\text{in,h},\tau} - \rho_\tau(\rho_K(s_{\text{in}}))\|_{L^2(Q)}\|\overline{q}_{h,\tau}\|_{L^2(Q)}$$

that tends to zero with (h,τ). It is easy to check that the same holds for X_2. Hence

$$\lim_{(h,\tau)\to(0,0)} \int_0^T (I_h(f_\alpha(s_{\text{in},h,\tau}))\overline{q}_{h,\tau}, V_{h,\tau})_h = \int_Q f_\alpha(s_{\text{in}})\,\overline{q}\,v. \tag{4.57}$$

The argument for the second term in the right-hand side of (1.20) is much the same; we insert $\rho_\tau(\rho_K(\overline{s}))$ and we use the fact that

$$\lim_{(h,\tau)\to(0,0)} \|S_{h,\tau} - \rho_\tau(\rho_K(\overline{s}))\|_{L^2(Q)} = 0.$$

Then the argument used for the first term readily gives

$$\lim_{(h,\tau)\to(0,0)} \int_0^T (I_h(f_\alpha(S_{h,\tau}))\underline{q}_{h_\tau}, V_{h,\tau})_h = \int_Q f_\alpha(\overline{s})\,\overline{q}\,v. \tag{4.58}$$

By combining (4.57) and (4.58), we obtain convergence of the right-hand sides,

$$\lim_{(h,\tau)\to(0,0)} \int_0^T (I_h(f_\alpha(s_{\text{in},h,\tau}))\overline{q}_{h,\tau} - I_h(f_\alpha(S_{h,\tau}))\underline{q}_{h_\tau}, V_{h,\tau})_h = \int_Q (f_\alpha(s_{\text{in}})\,\overline{q} - f_\alpha(\overline{s})\,\overline{q})v. \tag{4.59}$$

4.3 The full scheme

It remains to pass to the limit in the time derivative, say in (1.20), summed over n, and tested with the same $V_{h,\tau}$ as previously, except that here we take $v(T) = 0$. After summation by parts, this term reads

$$\sum_{n=1}^{N}(S^n_h - S^{n-1}_h, V^n_h)^\varphi_h = -\sum_{n=1}^{N-1}(V^{n+1}_h - V^n_h, S^n_h)^\varphi_h - (V^1_h, S^0_h)^\varphi_h. \tag{4.60}$$

By definition,

$$(V^{n+1}_h - V^n_h, S^n_h)^\varphi_h = \sum_{K\in\mathcal{T}_h}\frac{|K|}{d+1}\varphi|_K\sum_{\ell=1}^{d+1}(V^{n+1,i_\ell} - V^{n,i_\ell})S^{n,i_\ell}.$$

By inserting $\rho_K(V^{n+1,i_\ell} - V^{n,i_\ell})$ in each element, this becomes

$$(V^{n+1}_h - V^n_h, S^n_h)^\varphi_h = (V^{n+1}_h - V^n_h - \rho_h(V^{n+1} - V^n), S^n_h)^\varphi_h + \int_\Omega \varphi\rho_h(V^{n+1} - V^n)S^n_h.$$

The first term has the bound

$$\left|(V^{n+1}_h - V^n_h - \rho_h(V^{n+1} - V^n), S^n_h)^\varphi_h\right| \leqslant \|\varphi\|_{L^\infty(\Omega)}\|V^{n+1}_h - V^n_h - \rho_h(V^{n+1} - V^n)\|_h\|S^n_h\|_h.$$

Since the functions are piecewise polynomials, the equivalence of norms yields

$$\left|\sum_{n=1}^{N-1}(V^{n+1}_h - V^n_h - \rho_h(V^{n+1} - V^n), S^n_h)^\varphi_h\right| \leqslant C\,\|\varphi\|_{L^\infty(\Omega)}$$
$$\times\left(\sum_{n=1}^{N-1}\tau\|\frac{1}{\tau}(I_h(v^{n+1} - v^n) - \rho_h(V^{n+1} - V^n))\|^2_{L^2(\Omega)}\right)^{1/2}\left(\sum_{n=1}^{N-1}\tau\|S^n_h\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Then the regularity of v, the approximation properties of I_h and ρ_h and the boundedness of $S_{h,\tau}$ imply that

$$\lim_{(h,\tau)\to(0,0)} \left| \sum_{n=1}^{N-1} (V_h^{n+1} - V_h^n - \rho_h(V^{n+1} - V^n), S_h^n)_h^{\varphi} \right| = 0.$$

Similarly, it is easy to check from the convergence of $S_{h,\tau}$ that

$$-\lim_{(h,\tau)\to(0,0)} \sum_{n=1}^{N-1} \int_\Omega \varphi \rho_h (V^{n+1} - V^n) S_h^n = -\int_Q \varphi(\partial_t v)\overline{s}.$$

The treatment of the initial term is the same. Hence

$$\lim_{(h,\tau)\to(0,0)} \sum_{n=1}^{N} (S_h^n - S_h^{n-1}, V_h^n)_h^{\varphi} = -\int_Q \varphi(\partial_t v)\overline{s} - \int_\Omega \varphi s^0 v. \tag{4.61}$$

By combining (4.61), with Theorem 4.3 and (4.59), we readily see that the limit functions $\overline{s}$, $\overline{p}_\alpha$ and $p_{\alpha g}(\overline{s})$ satisfy the weak formulation (1.13). This proves Theorem 1.1.

5 Conclusions

This paper complete the analysis of a $\mathbb{P}_1$ finite element method to solve the immiscible two-phase flow problem in porous media. The unknowns are physical, namely the phase pressure and saturation, and they are continuous piecewise linear polynomials. Thanks to mass lumping, the scheme directly solves for the nodal values of the unknowns. The method is general, in the sense that the mobilities are allowed to vanish at the endpoints of the saturation interval and the derivative of the capillary pressure is unbounded. In this work, we show that the discrete approximations of pressure and saturation converge to the weak solution as the time step and mesh sizes tend to zero.

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