The Characterization and Evolution of Strategies about Vector Equations in the game *Vector Unknown*

Matthew Mauntel (mmauntel@fsu.edu)

Florida State University, Tallahassee, FL, USA

ORCID 0000-0002-2168-8006

Benjamin Levine (blevine3@asu.edu)

Arizona State University, Mesa, AZ, USA

David Plaxco (davidplaxco@clayton.edu)

ORCID 0000-0002-9491-9995

Clayton State University, Morrow, GA, USA

Michelle Zandieh (zandieh@asu.edu)

Arizona State University, Mesa, AZ, USA

ORCID 0000-0002-2279-8231

Abstract

We present results of a grounded analysis of individual interviews in which students play *Vector Unknown* - a digital game designed to introduce visualizing vectors, scaling vectors, vector addition, and vector equations attending to the geometric and algebraic representations of vectors. The game was designed to be used at the beginning of a linear algebra course, and participants in the study were students who had not taken a linear algebra course. We categorize the strategies students employed while playing the game and analyze how these strategies evolved throughout the students' gameplay. These strategies range from less-anticipatory button-pushing to more sophisticated strategies based on approximating solutions and choosing vectors based on their direction. We find that student focus alternated between numeric and geometric aspects of the game interface, which provides additional insight into their strategies. These results have informed revisions to the game and also inform our team's plans for incorporating the game into classroom instruction.

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Linear algebra is an important course for students in the STEM disciplines because of its unifying power within mathematics and its applicability to areas outside of mathematics. The goal of our project is to explore linear algebra in the context of a video game developed from an inquiry-oriented curriculum. An essential piece of learning linear algebra is the ability to connect different modes of description or reasoning which incorporate different representations (Hillel, 2000; Sierpinska, 2000; Larson & Zandieh, 2013). Well-designed, theoretically grounded games with multiple representations have been shown to improve student's problem-solving capabilities (Ke & Clark, 2019). Because of this, we see video games as an avenue for students to connect different representations in linear algebra. We created the game *Vector Unknown* by drawing on theories from Realistic Mathematics Education (RME), Inquiry Oriented Linear Algebra (IOLA)(Wawro et al., 2013), and Game-Based Learning (GBL) (Zandieh, Plaxco, Williams-Pierce, & Amresh, 2018; Mauntel, Sipes, Zandieh, Plaxco, & Levine, 2019). In this paper we describe strategies students use when progressing through different levels of *Vector Unknown* by connecting multiple representations during gameplay.

Literature Review

GBL

Game–based learning (GBL) is the use of games (including video games) for educational purposes. While GBL has gained popularity in recent years (Coleman & Money, 2020; Gresalfi & Barnes, 2016; Foster & Shah, 2015) there are few games for higher levels of mathematics. Gee (2005) notes that good games present players with well-ordered problems that allow the players to learn crucial skills necessary for gameplay. Players also engage in cycles of expertise where they encounter difficult problems, solve them, practice, and master them using appropriately-timed feedback (Gee, 2005; Williams-Pierce, 2019). Players are then confronted with new problems that will challenge the assumptions of the initial problems, requiring them to apply their skills in a new context. Good games also allow players to be creative agents taking an active role in the learning process. GBL does have some limitations, for example Jorgenstin and Lowrie (2012) observed Gee's principles of good gaming (Gee, 2003), but found that players had problems moving beyond trial-and-error to more complex theories. They hypothesized that this was because the game lacked supports for more advanced tactics. This highlights the importance of domain-specific learning theories in games that present pathways for more advanced thinking.

Despite the recent growth in game-based research for K-12 education (Byun & Joung, 2018), there are relatively few games that address post-secondary mathematics particularly in linear algebra. A review of the literature found only one game that deals with linear algebra directly and involves a three-dimensional vector representing different statistics of an avatar (Nishizawa, 2013). As the players picked the stats of their avatars a vector displaying the stats in three dimensions was visualized. The avatars did battle with each other and other monsters where the result was determined based upon an operation on the two vectors for each of the avatars. As players won more battles, they gained points which allowed them to upgrade their avatars. The goal of the game was to visualize the vectors and pairs of vectors in three-dimensional space. The game *Vector Unknown,* while only in two-dimensions, was designed to aid players in visualizing not just individual vectors, but coordinating a vector equation with its geometric representations.

Linear Algebra and IOLA

Research in linear algebra has evolved over the last twenty years from the pioneering work at the turn of the century (e.g., Dorier & Sierpinska, 2001; Harel, 1999; Hillel, 2000) to more recent work on improvements to teaching and learning using Tall's three worlds (e.g., Stewart & Thomas, 2010), models (e.g., Trigueros, 2018), dynamic geometry systems (e.g., Sinclair & Tabaghi, 2010) and everyday examples (e.g., Adiredja, Bélanger-Rioux, & Zandieh, 2019). Our work is most closely aligns with the curriculum design work of the Inquiry Oriented Linear Algebra (IOLA) group (e.g., Andrews-Larson, Wawro, & Zandieh, 2017; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Zandieh, Wawro, & Rasmussen, 2017).

This project leverages the instructional activities of the IOLA curriculum (http://iola.math.vt.edu; Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013), which is a set of research-based curricular and instructional materials based on RME design principles and designed to move students through the different levels of mathematical activity from an experientially real situation to a formal mathematical understanding (Wawro, Rasmussen, Zandieh, & Larson, 2013). The curriculum begins with the Magic Carpet Ride (MCR) task, which asks students to use two forms of transportation – a hoverboard and magic carpet (standing for the vectors <3,1> and <1,2> respectively) – to reach Old Man Gauss's house located at the coordinates (107, 64) (Wawro et al., 2012). The goal of this lesson is for students to investigate the general properties of vectors including scaling and adding vectors. In particular, the task associates vector addition with the act of moving along one form of transportation followed by another. Scaling vectors is translated into traveling a certain amount of time on a particular form of transportation. This metaphor in this task encourages students to visualize the journey, providing the opportunity for them to connect the geometric and algebraic representations of taking a linear combination of vectors. The curriculum continues to develop the notion of vector spaces into higher dimensions and explore the concepts of linear independence and span. The goal of this project was to develop a game based upon the MCR task.

Vector Unknown

Zandieh, Plaxco, Williams-Pierce, and Amresh (2018) note an alignment between principles of GBL and the design of IOLA curricular materials in terms of the nature of tasks, the structure of tasks, the teacher's role in providing feedback, and the student's role in being a producer of knowledge. The game *Vector Unknown* is the product of taking the principles of GBL and coordinating them with design principles behind the MCR in IOLA. The game maintains the travel metaphor of the MCR task by having the rabbit travel along a path given by the two vectors. This allows the player to visually connect the numeric aspect of adjusting the scalar in the vector equation with the geometric notion of scaling or stretching a vector in addition to geometrically visualizing the sum of two vectors. There are multiple places the players can observe the geometric and numeric representations of the vectors including a Vector Equation and a Log that tracks the history of the rabbit's journey (Figure 1). The geometric representation is given in two different map locations, one of which the player can manipulate to change the camera's perspective. The goal of playing *Vector Unknown* is to familiarize students with two-dimensional vector spaces and vector equations and to understand the connections between the two.

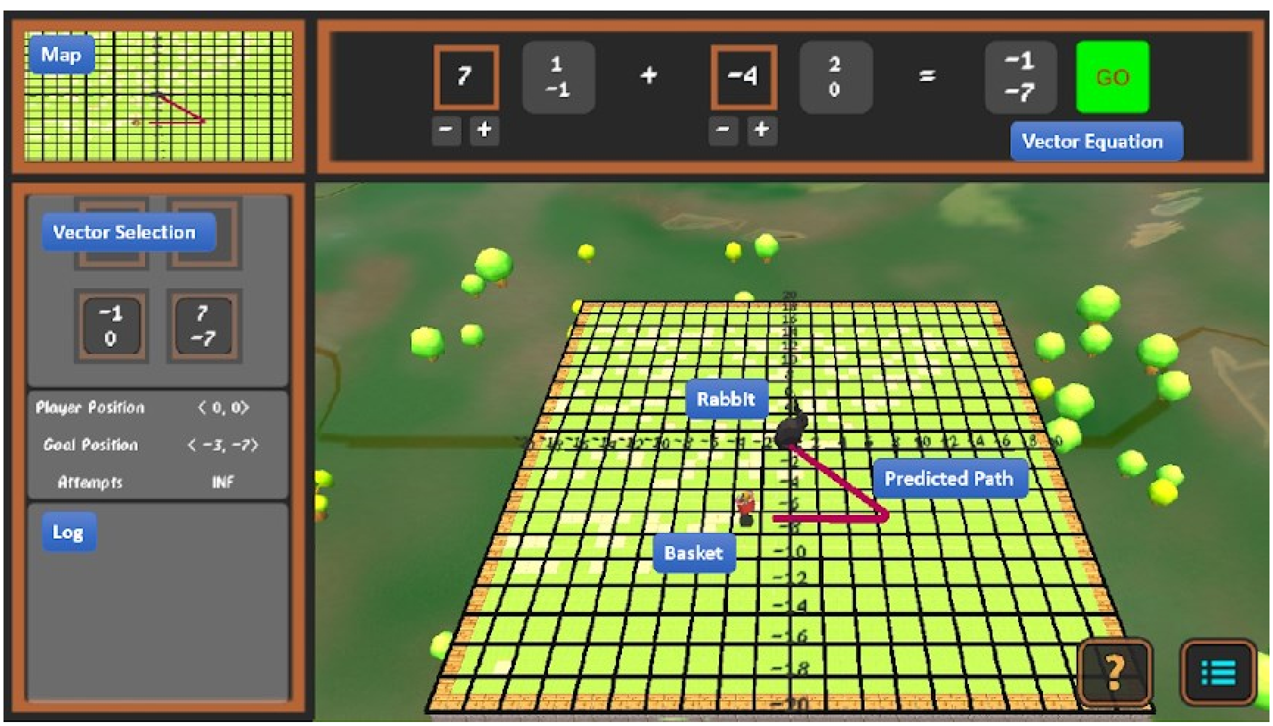

Figure 1. Sample Screen from the Game Vector Unknown.

Vector Unknown Gameplay

In this section we describe the different levels and gameplay of the game *Vector Unknown*. The goal of the game is to guide the Rabbit to the basket (Figure 2). At the time of data collection for this investigation, the game consisted of five levels. Two of the levels were the same as earlier levels, except they limited the number of attempts of the player. No player exhausted their attempts during gameplay for these levels so they were equivalent for the purposes of our investigation. For the purpose of this paper we shall name these three levels: Level A which has the Predicted Path; Level B which removes the Predicted Path, and Level C which has Keys the players must collect and includes the Predicted Path. In each level, the player is presented with a choice of four vectors in the Vector Selection. The player selects one or two vectors from the Vector Selection and places them into the Vector Equation. The player adjusts the scalars in the Vector Equation using the +/- keys on the screen and presses GO. The Rabbit then moves from its current location along the component to Vectors to arrive at the result. For example, in Figure 1 the Vector Equation is 7<1,-1> + -4 <2,0> = <-1,-7>. Here the Rabbit moves from (0,0) to (7,-7) in the direction of <1,-1> and then from (7,-7) to (-1,-7) in the direction of <2,0>. The goal is to guide the Rabbit to the Basket by repeating this sequence of choosing vectors from the Vector Selection, scaling them by pressing the +/- keys, and pressing GO.

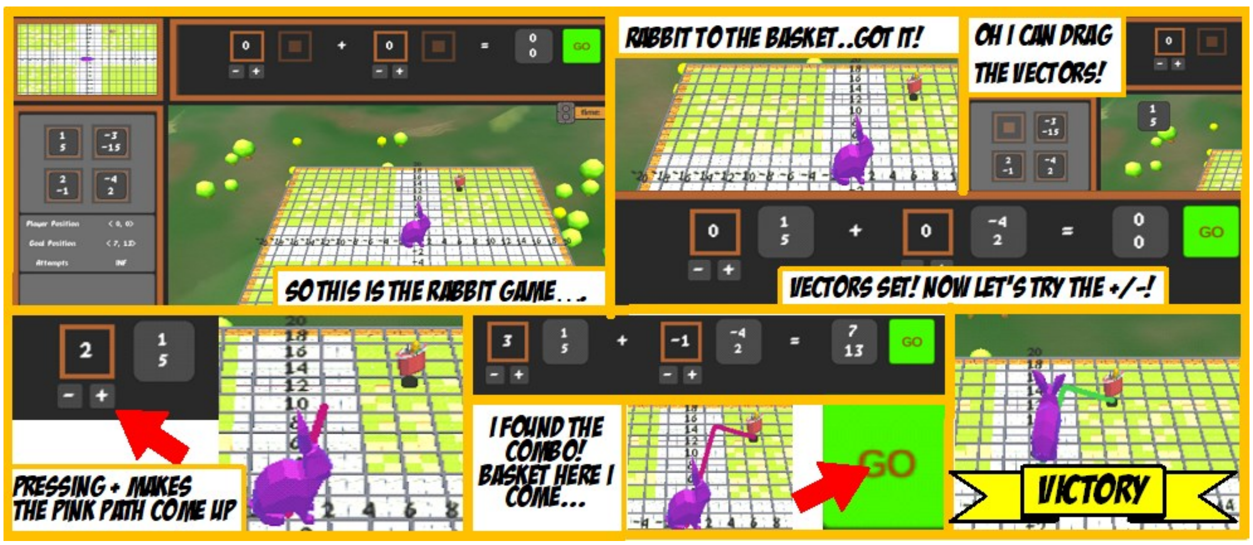

Figure 2. Comic strip of the first level of play.

The levels are designed to encourage players to engage with both the Vector Equation and its geometric representation present in the grid. During Level A, players have a pink Predicted Path which highlights the path of the Rabbit before the player presses GO, providing players with a real-time, dynamic geometric representation of the linear combination in the Vector Equation. They are also provided with a real-time calculation of the result of the Vector Equation. This allows players to focus more on the geometric representation of the Vector Equation present in the pink path or the more numeric/calculational aspect of the Vector Equation present in the controls. Level B removes the Predicted Path and encourages the players to engage directly with the Vector Equation. In Level C, the player is again provided a Predicted Path, but players must now obtain several Keys to unlock a Lock that covers the Basket. The location of the Keys is only present on the graph and thus requires the player to engage with the geometric component of the game. It also requires players to reach the goal from a point other than the origin, forcing them to solve multiple linear algebra problems. This is important because in Levels A and B the player could choose vectors and scale them until the result of the Vector Equation matched the location of the Basket.

Theoretical Framework: Defining Strategies

For the purposes of this paper, we view learning in the context of *Vector Unknown* as the evolution and adaptation of student play characterized by their shifts in strategies. A strategy is a choice that dictates how to move from one game state to another. This perspective is consistent with Williams-Pierce and Thevenow-Harrison (2021) who utilized Brousseau's Theory of Didactical Situations (2006) to frame how students learn using video games. Williams-Pierce and Thevenow-Harrison (2021) describe five different zones (based on Brousseau's theory) that players transition through while playing and learning from mathematical play. These zones present an evolution where the players start from general play and evolve into a situation that is conducive for mathematical analysis and learning. Assessing which zone a player is in could be crucial for seeing how a player's gameplay evolves. The first zone Williams-Pierce and Thevenow-Harrison (2021) identify is that of pure play. During pure play, players react to feedback provided by the game, but "they have not decided that certain actions (and matching reactions) are more desirable than others" (p. 9). The second zone, developing a preference, consists of players identifying the desired effects even if they do not know how those effects are generated. This is achieved when "players experience failure in whatever guise it takes … [and] they begin to seek to avoid failure" (p. 10). The first and second zones often occur at the start of gameplay and do not consume a large amount of the player's playtime. The third zone, determining responsibility and causality, is where "players understand that their actions--such as activating a button--directly cause a consistent reaction from the game" (p. 12). According to Williams-Pierce and Thevenow-Harrison (2021), it is at this stage that mathematical reasoning can begin. In the fourth zone, developing anticipation, players begin "developing, testing, and nuancing mathematical hypotheses about their actions, the game's reactions, and the underlying mathematical design" (p. 13). This is also the stage where play becomes more mathematical in implementation. In the fifth and final zone, developing the adidactical situation, "they have created a mental model of the game based on their previous experiences, and are using that model to guide their gameplay" (p. 16) resulting in learning and generalization.

Hollebrands (2007) observed two different ways students engaged with the digital software Geometer's Sketchpad (GSP): proactive and reactive. Students who engaged in a proactive way used GSP to test theories, while reactive users reacted to the information presented to them on-screen but did not anticipate how their actions would change the on-screen environment. Hollebrands found that whether a student employed a proactive or reactive strategy "appeared to be influenced by their understandings of geometric relations and their perceived affordances of the tool" (p. 188). This means that how a student progresses through William-Pierce and Thevenow-Harrison's (2021) zones for a particular game might be dependent upon how willing the student is to engage with the game and their knowledge of the mathematical material surrounding the game as the fourth zone involves using the game to test hypothesis which is like proactive reasoning. Hollebrands indicates that she believes that what is needed is specific instruction that helps the student reflect upon the technology they are using in a way that is conducive to mathematical relationships. This indicates that a game should be designed to elicit reflection on mathematical relationships whenever possible.

This study was the first stage of a design-based research study (Cobb et al., 2003). The goal was to investigate how the participants interacted with different game components to provide insight into how the game could be revised. In addition, we want to see how student strategies progressed through the game.. Also, we want to see how player's strategies could change as they progressed through the levels of *Vector Unknown* leading to our research questions:

1. What are students' strategies for completing the game *Vector Unknown*?
2. How can student strategies evolve while playing the game *Vector Unknown*?

Methods

Data Collection

We interviewed five participants (Gwen, Lance, Latia, Mouse, and Zo[1]) from a multi-purpose regional university in the southeastern United States. These participants were selected from a total of 11 initial interviews with participants who had varying experience with linear algebra. The five participants chosen for this study were selected because they had the least amount of experience with linear algebra. This was done because the game was initially designed to be utilized at the start of a linear algebra course, and these five participants best represented a starting linear algebra student. The participants included two black men, two black women, and one Asian woman (self-reported). Each participant participated in a semi-structured interview for approximately one hour, during which they were asked to complete the three levels of the *Vector Unknown* game and discuss their strategies as they progressed through the game.

Because the goal of the interview was to inquire into how the participants were making sense of the game's design and gameplay, the protocol questions were limited to inquiring into participants' rationale for their actions and how they were making sense of the gameplay. That is to say that we bracketed interviewer bias by explicitly including instructions in the protocol to minimize interfering with the participants' gameplay experience. The protocol also included instructions for the interviewer to provide support only when the participant asked for help or after multiple unsuccessful attempts to navigate the game's tools.

A camera filmed the participant playing the game to capture all relevant gestures or actions. In addition, the participant's gameplay was captured by screen-capture recording software that allowed us to further analyze the student's gameplay. We also provided the participants with scratch paper to write any equations or notes they needed, which allowed us to determine what external mathematics they used and allowed us to triangulate the data. Combined, these three methods allowed us to see and hear the entirety of what the participant did over the course of the interview. As needed, the interviewer provided help on how to navigate the game's screens and use the controls. The interviewer asked scripted questions along with impromptu follow-up questions to understand how the student was interacting with different elements within the game to determine what mathematical insights or strategies the participant developed during gameplay. The impromptu follow-up questions were used to encourage the participants to express their strategies and thoughts, as well as what connections they were making to their previous mathematics coursework.

Data Analysis

The goal of our analysis is to identify students' strategies developed while playing the game *Vector Unknown*. We operationalize student strategies as an action or collection of actions executed while attending to some elements of the game. The analysis began by creating an expanded transcript of the interviews that included the starting vectors, goal positions, and the actions the participant performed while interacting with the game's interface (such as replacing a vector or scaling a vector) in addition to the dialogue between the participant and interviewer. Then each researcher viewed the videos and added notes in a separate column in the transcript. Consistent with grounded theory (Strauss & Corbin, 1994), we conducted an iterative grounded analysis of the expanded transcripts with notes of the interviews. This process began with an open coding process on the data of each individual participant. Codes were determined from the actions and dialogue of the participant to highlight key elements of the interview. After a participant's data was coded and a list of codes was produced, the next participant's data was analyzed using the same codes and often expanding the initial codes from the previous participant. Occasionally, this coding process meant returning to an earlier interview to further clarify and discuss a previously defined code. For example, we noticed that one participant referred to quadrants geometrically by discussing the geometric representation of the vector, while another participant reasoned about quadrants using the signs of the vector in the vector equation despite both instances being coded as quadrants in their strategies. Gestures

[1] Pseudonyms were self-selected by students.

were utilized as a confirmatory analysis primarily for the geometric components of the game such as tracing a path on the screen.

To establish a uniform nomenclature for our codes and better view patterns seen among participants, we organized similar codes and simplified the content of each code. We grouped codes with similar attributes (such as graphical components or specific game features) and simplified their reference names. The nomenclature created focused on six primary categories (Table 1) encompassing the different aspects needed to define what the participant was utilizing or referencing. Table 1 shows the six categories with some examples from each category. Example codes were selected either because they were utilized to define strategies or because they exemplified the category.

Table 1: Examples of codes from the first round of coding

Categories	Examples		
Focus on the vector equation (**E**)	**Erp**: replaces a vector	**E1**: Uses one vector	**Es**: manipulates scalars
	E2: Uses two vectors	**Erv**: Removes a vector	
Features of the game (**F**)	**Fdl**: Data Log	**Fpp**: Player Position	**Fca**: Camera Angle
Focus on gameplay (P)	**Pg**: Presses Go	**Pcomp**: Anticipates gameplay complexity	**P1**: Press "GO" to get to a location closer to one coordinate of goal
Conceptualizing a Vector	**Vd**: Testing the direction of a vector	**Vp**: Visualization and trace of vector path	**Vq**: Use of scalars or vectors components to reach quadrants
Student-expressed strategy	**Sc**: Connecting between representations	**S1**: Describes a strategy to reach one component of the goal.	**Ste**: Student utilized trial and error to find the goal.
Focus on graphical game components	**Gdir**: Displacement (Direction)	**Gdis**: Displacement (direction and displacement)	**Gpp**: Refers to the Predicted Path

The first category - Focus on the vector equation - includes participant actions within the digital game such as the introduction and removal of a vector or the manipulation of a scalar in the game interface. A second category - Features of the game - refers to instances in which the participant mentions using various features included in the game's interface, such as referencing the Data Log or the locations of the player or goal. The third category - Focus on gameplay - contains codes of participant actions as they play the game, such as pressing GO or undoing a move. The fourth category - Conceptualizing a vector - includes codes about how the participant expresses the ways in which they are conceptualizing a vector, such as defining what the components of a vector are or testing its direction. The fifth category - Student-expressed strategies - contains codes such as focusing on one coordinate at a time or reusing a previous strategy. The final category - Focus on the graphical game components - contains codes such as the direction of the Predicted Path or the graphical displacement from the goal.

There are some nuances that need to be explained from these categories. For example, "Features of the game" and "Focus on gameplay" may appear similar on a cursory glance, but the former focuses on "static" features of the game that do not result in the continuance of the Rabbit to the Basket. The example codes shown in Table 1 reference the Player Position (which updates in accordance with the Rabbit), the Data Log (which is a history of what moves the player has done), and the Camera Angle (which allows for the player's point of view to change. One important piece to note is that none of these codes directly impact the player's ability to complete the level. In contrast, the category "Focus on gameplay" is about dynamic elements that lead towards the game's progression and completion. For instance, the player cannot complete the level without pressing GO.

These categories were important because they allowed for a deeper analysis of the participant's actions within the game. After the transcripts were coded and categories assigned to the codes (by coloring each category with the same color), each participant's transcript was reviewed for patterns in the coding. During this process, we wanted to pair participants' actions and words to deduce what the participant was attending to in the game. These emerging patterns of codes we called strategies. After looking at these patterns for each participant individually, the strategies were compared across all the participants in the study. This analysis resulted in four different types of strategies. Each type of strategy involves a coordination between what the participant says, what they attend to in the game, and their actions in game.

Results

In this section we review the results of the coding process and describe the core strategies that we saw utilized by participants. Then, we present several examples of how these strategies can be used to describe how student gameplay evolved as they progressed with each level. For each of the strategies, we identified similar evidence across multiple participants' interview data.

Four Strategies of Participants' Activity

In this section, we first introduce and describe four broad types of participants' strategies and then use excerpts from five participants' gameplay to demonstrate the strategies. The four broad types of strategies are button-pushing, quadrant-based reasoning, focus on one coordinate, and focus on one vector. A summary of these strategies can be seen in Table 2.

Table 2: Summary of Main Strategies

Theme	Numeric	Geometric	Notes
Button-Pushing	Player presses buttons while attending to how the vector equation changes	Player presses buttons while attending to how the geometric Predicted Path changes.	Button-pushing is less-anticipatory if the participant expressed surprise and more-anticipatory if the participant anticipated how their button-pushing would affect the game.
Quadrant-based Reasoning	Player chooses vector with respect to matching the signs with the goal, but does not reference the Predicted Path.	Player references the direction of the Predicted Path on the graph or a quadrant on the graph to make sense of the direction of a vector. Alternately, students understand vectors as slopes.	Advanced versions of this strategy involve the coordination of the numeric and geometric aspects of the strategy allowing the player to visualize the Predicted Path without its presence.
Focus on one coordinate	Player reduces the aim of the goal to one coordinate and attempts to reach that one coordinate without the Predicted Path.	Player utilizes the Predicted Path or graph to get close to the Goal aligning it with the x-coordinate or y-coordinate	

Focus on one Vector	Player focuses getting as close to the result of the Vector Equation as possible with one vector and then utilizes another vector to reach the goal and/or alternates between the two.	Player focuses on getting as close to the goal as possible with one vector utilizing the Predicted Path or graph and then utilizes another vector to reach the goal and/or alternates between the two.	When multiples of standard basis vectors were present, students interpret getting close to the goal as matching one coordinate aligning the two strategies of Focus on One Coordinate and Focus on One Vector.

Across all the types of strategies, we recognized that students were focusing to varying degrees on the ***geometric*** and ***numeric*** representations in the game's interface (Figure 3). We see ***geometric*** and ***numeric*** as a type of spectrum. The *Vector Unknown* gameplay encourages both uses by its implementation of a graphical representation (the grid and trajectory lines), numerical representation (vector equation), and its progression of levels (Level C does not numerically state key locations and Level B does not visualize expected trajectory). While students might use both geometric and numeric elements during gameplay, we found that during some stages students' strategies tend to exhibit a reliance on one representation more than the other. We chronicled this while recording student gameplay strategies.

When a participant's strategy attended primarily on the vector equation without mentioning the Predicted Path or the graphical component of the interface, we called the strategy ***numeric***. These strategies were typically centered around trying to manipulate the numbers in the Vector Equation and finding some combination that equaled the numbers in the Goal Position. When the participant's strategy attended primarily on either the Predicted Path or the graph, we called the strategy ***geometric***. These strategies centered around producing a linear combination of the Predicted Path vectors that emanated from the rabbit's position and terminated at the goal position as viewed on the game board. Additionally, participants sometimes described geometric relationships and gestured toward the board on the computer screen before manipulating the vector equation, especially in Level B. We also characterized these exchanges as ***geometric*** because they indicate a focus on the geometric artifact in the game's interface, rather than a focus on the Vector Equation.

Evidence for Geometric Strategies	Evidence for Numeric Strategies
• Mentioning the Predicted Path while executing a strategy. • Mentioning the absence of the Predicted Path as they transition from Level A to Level B and B to C. • Gesturing that is directed at the graphical portion of the interface. This was relevant during Level B when the Predicted path was missing. • Discussing geometric properties such as direction between the current location of the rabbit and the goal.	• Mentioning relationships between the numbers of the vector equations (such as wanting an even or odd number for the result). • Gesturing that is directed at the vector equation (i.e., pointing at that portion of the screen). • Translating the vector equation to paper in solve using techniques such as solving system of equations.

Figure 3. Evidence for Geometric and Numeric Strategies

Strategy 1: Button Pushing.

There are several types of strategies we found while analyzing the participant's play. We characterized the first type, ***button-pushing***, as instances in which the participant rapidly adjusted the scalars and switched vectors. Generally, this strategy is consistent with multiple sequential replaces a vector **(Erp)**, manipulates a scalar **(Es)**, and removes a vector **(Erv)** codes, which indicate that the participant was rapidly adjusting the components of the vector equation.

The following participant actions are consistent with a button-pushing strategy: participant removes vectors and/or replaces them without using them (multiple codes of **Erp** or **Erv** without pressing "Go"); participant presses +/- buttons several times, shifting between the two in order to adjust or un-do their preceding action (repeated codes of **Es**); made a statement clearly indicating that their actions are somewhat random or are intended to gain insight toward how to make sense of the game (coded as **Ste**). Not all participants performed all these actions. Rather, they usually performed one or two of these actions around the same time. In our later iterations of analysis and focused coding, we identified these actions as consistent with gameplay in which the participant was developing a sense of how their actions affects the elements in the game board.

This activity is consistent with (but distinct from) existing constructs in the gaming literature, namely "button mashing" (Gingold, 2006), during which players quickly and rapidly press as many buttons as possible with the hopes of serendipitously achieving success. We distinguish ***button-pushing*** from button-mashing by noting that, while initially this was used as a guess-and-check-type strategy, the players in our interviews were typically quick to develop a sense of some idea of the effect of their actions, though they were not always certain, causing them to need to adjust after their initial actions. Earlier in this developmental process, we view this ***button-pushing*** activity as ***less-anticipatory*** because the players often expressed surprise when reacting to a consequence of their actions or articulated that they were performing actions without necessarily knowing what to expect because of those actions. For example, Mouse, after inserting a vector (**Erp**) and quickly alternating the scalar keys (**Es**) made the following comment "what if I press the button that's ... the rabbit's going to go down. ohhh, yeah." Here, the tone of Mouse's "ohhh, yeah" comment indicates his surprise which leads us to describe his activity as ***less-anticipatory button-pushing***. Further, we notice that his reference to the Rabbit "going to go down" indicates that he was attending to the ***geometric*** aspects of the game. Another student, Gwen, employed a less-anticipatory button-pushing strategy as evidenced from her describing her gameplay as "messing around", "mindlessly clicking", and "throwing in numbers to get the answer" while quickly adjusting vectors and scalars in Levels A and B. We classify Gwen's strategy as ***button-pushing*** because of her words and actions of rapidly adjusting scalars and switching vectors. Throughout her discussion, Gwen did not discuss the Predicted Path, but instead frequently referenced the Vector Equation and its components, indicating that her ***button-pushing*** was ***less-anticipatory*** and ***numeric***.

As these participants played the game more, they tended to engage in what we are calling ***more-anticipatory button-pushing***, which involved switching out vectors and scalars, as before, but not as quickly or frequently and typically articulating some hypothesis about the results of an action prior to carrying it out, rather than expressing surprise, as with the ***less-anticipatory button-pushing***. For instance, throughout the course of the interview, Mouse's gameplay shifted from less-anticipatory to more-anticipatory. This is evidenced by a decrease in the number and frequency of times Mouse switched between increasing and decreasing the scalar for a given vector (**Es**) when he was button-pushing. This is further supported through his own discussion of his gameplay. For instance, in Level A, Mouse expressed surprise when the Predicted Path of the linear combination moved toward the goal. In contrast, while playing Level C, Mouse put in the first vector he clicked on the "+" scalar and then uttered "Oh minus" before clicking the "-" scalar. We see this as a ***more-anticipatory*** button-pushing because his actions indicate he was expecting one of the scalars (+ or -) to move the Predicted Path closer to the goal position, although he was not yet certain of which one he should choose. Further, we posit that Mouse anticipated that one of the two scalar directions would be viable and chose one (perhaps at random) to try first, further anticipating that he could adjust his selection.

Strategy 2: Quadrant-based Strategies.

We characterize ***quadrant-based*** strategies as strategies that involve choosing vectors based on which quadrant a vector would extend into, often based on the signs of the components in the vector. This strategy typically manifested when participants focused on which quadrant a vector reached (**Vq**). Typically, participants reasoning with quadrants chose a starting vector in the same quadrant as the goal, or more generally a vector that could be scaled in a direction heading towards the goal. For example, Latia noted the following:

> So I know that the basket is on (-6,6), and my initial thought was to use, to start with either my <-3,9> or my <-1,3> because it is in the second quadrant and then, since I already know I'm at (-6,6), I was trying to think of what I could use so that if I multiplied those two numbers I could get to my basket.

Here, Latia made her initial vector choices by matching the signs of the goal position with that of the potential vector choices, indicating that she is focused primarily on the ***numeric*** aspects of the game interface. In contrast, Latia at a later point (and Mouse) referred to the vectors as "slopes" or slope-like indicating they connected to prior understanding that emphasized the thinking of the vectors as change in x over change in y. This form of geometric quadrant-based reasoning can also be seen in Zo's reasoning explicitly. Zo was given a goal position of (-14,-6) on Level B (which does not have the Predicted Path) and then noted the following:

> So this is the start [hovers over "player position"] and this is where you're supposed to be [hovers over "goal position"] and this is the x and this is the y [left and right positions in carrot notation] and that's x over y [vector <-10,0> in first vector spot]. So you try to get to this point [goal position] by putting these together [puts <-2,-3> in second slot while talking] which is, I was kind of confused by this whole plus stuff at first. I didn't really understand it so that's why I kept doing just one.

During this explanation, Zo was tracing out the down and to the left pattern going from (-10,0) to the goal at (-14,-6) with her finger. This indicates that she was thinking about the direction of the vector <-2,-3> and iterating the vector until she reached the goal. This is an important example of ***geometric quadrant-based*** reasoning that illustrates that geometric reasoning can be present without the Predicted Path.

Strategy 3: Focus on One Coordinate.

The third type of strategy is ***focus on one coordinate*** where the participant focused on trying to match one coordinate of the result of the vector equation with one coordinate of the goal position (**S1**). Lance expressed this strategy on Level B with a goal position of (-8,-2) as follows:

> Uhhmm, so when I started, when I saw my vector choices, I started to see the, um, almost all connections of like how certain, um, how um, I was manipulating them individually based off of like y's first and then the x's. And so I was trying to find paths and where, where I get my first, um, my first um, coordinate and my y. This one was focusing on my y and I noticed that <1,0>.

Here Lance indicated his general strategy was to focus on the y-coordinate first, and then x-coordinate. Matching the y coordinate (**S1**) was accomplished by choosing the vector <-6,-2>. Lance continued:

> So if I can just minimize the effects of one and have the other one just, um, the what gives me, um, my other, um, my other coordinate and one or the other part of the y, which was this one, fine. Then I saw that I had 2, I focused on that. and then I saw how this 2, I know, I know that this one both by 2 or maybe go to 3. Then I used my y-coordinate in it, so I know that one was failed by two. Okay. Yeah, I went over to this one. I know that this zero multiplied by anything was zero. So I know that I don't have to focus on that.

Here Lance tried to reach the x-coordinate by using the vector <-2,0>, but could not because he had a difference of 3. He realized that he needed a vector that would not alter the y-coordinate, so he changed the <-2,0> to <-1,0> and was able to reach the goal. Lance's overall strategy was to focus on matching the y-coordinate and then use a multiple of the standard basis to reach the x-coordinate through a process of guessing and checking the elements within the Vector Equation. The strategy is ***numeric*** as he did not have the Predicted Path when playing this level and did not refer to any geometric components.

Another participant, Zo, noted that when she was trying to get to the basket at (-18,-1) from (-18,-9) on Level B: "I have to go up eight but I don't know how I'm about to do that. There's no, like, straight line in these numbers so I can't do that. But, uh, I'm just going to trial and error it 'cause I don't know how else to do this." Zo's strategy was to match the -18 and then go up 8, indicating she was attending to the ***numeric*** aspects of the game. This also indicates that some participants experienced problems when trying to employ this strategy without a multiple of a standard basis vector that allowed them to alter one coordinate without altering the other. This led some participants to develop an alternate strategy which we call ***focus on one vector***.

Strategy 4: Focus on One Vector.

The fourth strategy observed is a ***focus on one vector***. In this strategy, participants chose one vector (**E1**), scaled it (**Es**) as close to the goal as possible (oftentimes going past the goal in one direction and then reducing the scalar), and then chose a second vector (**E2**) to reach the goal. Applications of this strategy varied, including alternating between adjusting the scalars for the two vectors (after the second vector was introduced) and adjusting one vector after fixing the other with a non-zero scalar.

Distinctions between ***geometric*** and ***numeric*** instances are primarily seen between Levels B and C, respectively, after participants learned how the game operates. For the purposes of defining this strategy, the erratic uses of one vector will not be taken into account. Further, the ***focus on one vector*** is best defined as the participant tries to solve the level utilizing one vector, independent of a quadrant-based or coordinate-based approach, though the choice of vector is often based on one of those factors. We distinguish ***focus on one vector*** from ***quadrant-based reasoning*** based on how the selected vector is utilized. For ***quadrant-based reasoning***, we look primarily at vector selection, while ***focus on one vector*** explains the way in which the vector is used, as discussed in Table 2 above. Although ***focus on one coordinate*** is often used in conjunction with ***focus on one vector,*** there are instances when the participants utilized both vector slots to reach one coordinate, so we are providing some separation here. As explained, when a participant uses one vector, they travel as far as they can with one vector, often going past the goal and back-tracking before adding a second vector to the equation.

An instance of ***geometric focus on one vector*** occurs when Latia went through Level A. She was given a goal position at (-3,-10) and initially used the vector <-3,-2>. After getting a scalar of 2, she then added the vector <3,-2> to the equation and put the scalar to 2, and then increased the first scalar by 1. This adjustment and alternation of scalars is still a ***focus on one vector*** because of Latia's initial use of the first vector and having a set scalar before introducing the second vector to the equation.

A ***numeric*** instance of the ***focus on one vector*** can be seen when Gwen went through Level B. Level B does not contain the Predicted Path, so Gwen focused specifically on the Vector Equation to solve the level. Gwen fixed one vector with a scalar of 2 before including the second vector into the equation. The distinction between here and our previous examples arises from the level definitions. Because of the lack of the Predicted Path, Gwen used the Vector Equation (particularly the right-hand-side) to determine her proximity to the goal.

The ***focus on one vector*** strategy was employed when players were beginning to coordinate the scaling of each vector with the goal (either the basket geometrically or the coordinates of the goal numerically). This strategy was primarily a strategy of approximation where one vector was chosen, scaled until it was approximately close to the goal location or goal vector. After this, another vector was often chosen and the ***focus on one vector*** strategy was applied again to reach the goal and/or get closer to its location.

Overview of the Evolution of Student Strategies

After categorizing the strategies, we began looking at how they evolved throughout gameplay. In Figure 4, we illustrate the progression of strategies previously identified through student gameplay and categorize them as ***geometric*** or ***numeric***. Gwen did not utilize a geometric strategy until she had played five levels (Figure 4). She focused on matching the result of the Vector Equation with the Goal Position until she reached Level C. During Level C she began investigating the Predicted Path with button-pushing. Lance and Mouse began by focusing on the Predicted Path, but when encountering Level B were forced to utilize more numeric strategies (Figure 4). This was made approachable by the presence of Standard Basis Vectors. When the Predicted Path returned, they primarily relied on the Predicted Path, but started the process of coordinating this geometric interpretation with the ***numeric focus on one vector*** strategy. These examples illustrate that initially it is possible to play the game from a primarily geometric or numeric lens, but the Levels B and C provided incentive to play the game from a different lens [***numeric*** in the case of Level B and ***geometric*** in case of Level C].

Looking at the strategies utilized by Zo and Latia, we note that both participants utilized numeric and geometric strategies on every level (Figure 4). Furthermore, the quadrant-based reasoning strategy was utilized by both participants in both a numeric and geometric lens through multiple levels (Figure 4) indicating at least an attempt at coordination. During her second playthrough of Level A Zo noticed that the vectors were “x over y” instead of y over x. Formulating this connection allowed her to employ ***quadrant-based reasoning*** that were both geometric- and numeric-focused on Level B (which did not have the Predicted Path) by tracing paths on the screen with her mouse. Similarly, Latia began coordinating the numeric and geometric aspects of the vectors early in gameplay. This included not only understanding the relationship between the vectors and their geometric representations but also the effect that scalars have on vectors. After playing several levels, Latia began layering and reusing strategies, as well as flexibly switching between geometric and numeric lenses of the game, as needed.

		Gwen							Mouse					Lance					Zo							Latia				
Level		A	A	A	B	C	C	B	A	B	B	C	C	A	A	B	C	B	A	A	B	A	C	B	B	A	B	B	B	C
SBV		*			*					**	**			*		*	*	*	*			*		*	*			**		
Geometric	BP						♦	♦	♦	♦		♦	♦	♦	♦		♦		♦		♦					♦				
	FOV						♦										♦									♦	♦	♦	♦	♦
	FOC																													
	QB																			♦	♦	♦	♦	♦	♦	♦	♦			♦
Numeric	BP	♦	♦	♦		♦		♦					♦						♦		♦					♦				
	FOV		♦		♦		♦				♦																		♦	♦
	FOC			♦						♦	♦	♦			♦	♦	♦	♦			♦			♦	♦			♦		
	QB							♦												♦	♦		♦	♦	♦	♦	♦			♦

SBV **= *standard basis vectors*** (* = One Pair of SBV; ** = Two Pair of SBV); QB = ***reasoning about quadrants***; FOC = ***focus on one coordinate***; FOV = ***focus on one vector***; BP = ***button-pushing***

Figure 4. Student Strategies by Level and Type

Evolution of Strategies as Seen in Latia’s Interview

In this section we discuss the evolution of Latia, who employed the widest range of strategies (both geometric and numeric), and employed multiple strategies per level reaching Zone 5 (Williams-Pierce & Thevinow-Harrison, 2021). For this reason, we utilize Latia to illustrate how strategies can evolve during gameplay. Latia began with ***less-anticipatory button-pushing*** and described her gameplay as “playing with numbers.” She quickly figured out that there was a link between the scalars and the geometric support noting that the Predicted Path noting that “so this doubles it [hovers over the 2 in the scalar box] and this, ok so they're multiplying the um these coordinates by three [referring to the vector coordinates], by whatever number is in that box on the left [hovers over scalar box].” During the first level she made use of the geometric support as noted by her use of the camera position and her ***numeric*** reasoning to find the goal and began to connect the two stating that she made vector and scalar choices based upon the quadrant of the goal expressing ***reasoning about quadrants***. Also, of note, she mentioned that she eliminated a vector choice because of its size and she did not want to go past the goal. Latia also ***focused on one vector*** as indicated by choosing one vector, <-3,-2>, scaling it to what she believed was an appropriate distance to the goal, and then putting in a second vector, <3,2> and adjusting it to get closer to the goal. This is followed by an adjustment to the first vector to reach the goal.

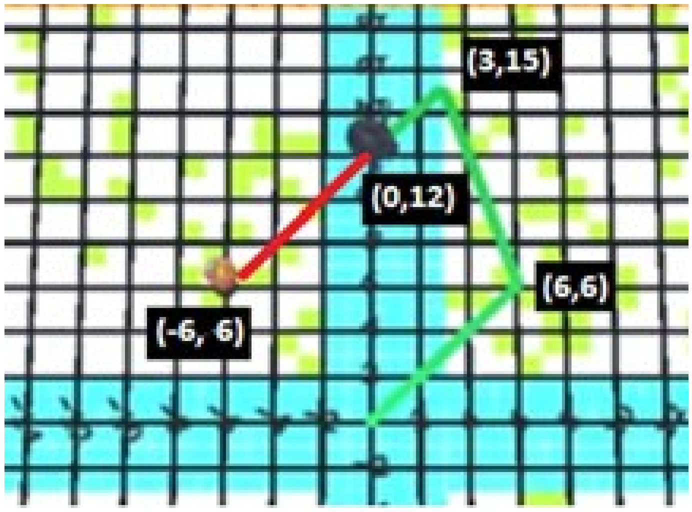

Figure 5. Latia Gameplay during Level B.

During the second level (see Figure 5), Latia chose the vector <-3,-3> and the scalar -2 and pressed GO to take her to (6,6). Having seen the location of the path in the first and not the second quadrant (the goal was at (-6,6)), she used the ***reasoning about quadrants*** strategy to choose the vector <-1,3> and scaled the vector by 3, attempting to ***focus on one vector*** to get close to the goal. She assumed that the Rabbit would always start from zero and was surprised to find out that it did not return to the origin after pressing GO. Latia then stated that "the Rabbit's at (0,12), so I know I need to come down six and go over to the left six, so I know I need <-6,-6>." She tested the vector <-1,-1> with the scalar 3 to "make sure he is going in the direction that I wanted." During this level, Latia had expanded her ***reasoning about quadrants*** to not only include vectors starting from the origin, but also vectors starting from other places. In addition, she started to use the game to make and test conjectures about the direction of vectors. She was also able to reason geometrically without the Predicted Path. This reasoning continued when she replayed Level B, where she noted that:

> I'm thinking, what I want to use instead is the <-6,4> and I was thinking that if I multiply it by--go over six and up four and then went up two and over three it would get me further to the right. But then I was thinking.... so what I'm thinking I want to do is go past the basket and work my way back down. That's what I think I'm going to do.

Here in addition to the reasoning about quadrants, she ***focused on one vector*** trying to use <-6,4> to get close to the goal. The goal for this level was at (-2,10) which was in a similar location to (-6,6) quadrant-wise from her first playthrough of Level B. This may be what motivated her strategy of going past the basket and coming back down which she used in her first playthrough. It is important to note that going past a goal went from a rule with Latia to strategy for solving a certain type of level. This strategy ended up not working out and she eventually took out a piece of paper and wrote a system of equations (Figure 6). She then noted:

> I just realized that I've been thinking about this all wrong. Ok. So what would actually be easier to do is I already know the goal is (-2,10), so I plugged in my (-2,10), you know and added them here and I already know the answer choices I can work with …so I have four and the six and the negative six and four would give me exactly what I want. I didn't see that. That's cool!

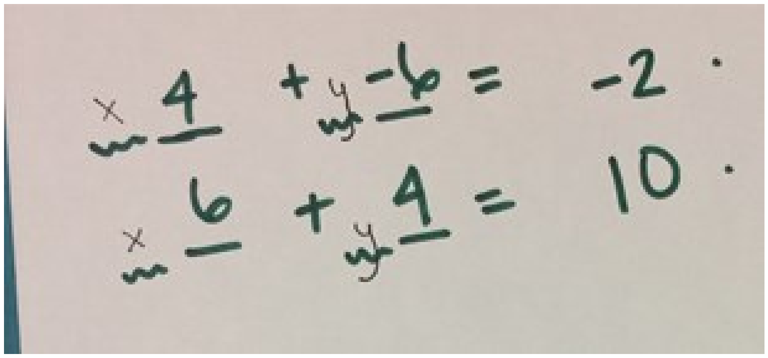

Figure 6. Latia's written work. The xs and ys were provided by the interviewer.

Here Latia switched from a ***geometric*** interpretation of the vector equation to a ***numeric*** (and algebraic) one. ***Reasoning about quadrants*** provided Latia with a way of finding vectors to reach the goal with the <-6,4> representing the quadrant of the goal, but when she was not able to find a geometric solution that worked, she saw a numeric relationship through the vector equation. This indicates that the game could be an invaluable tool for students like Latia to connect the geometric and numeric properties of a vector equation and be able to link them as well as fluidly switch between the different representations of the vector equation.

The Effect of Standard Basis Vectors on Student Strategies

There were several instances where a participant received at least one multiple of a Standard Basis Vector (SBV) (i.e. <1,0>, <0,-5>, or <3,0>) but only a few instances where students received a pair of SBVs. Only two participants received two pairs of SBVs. What makes this sample even more interesting is that all instances of two pairs of SBVs occurred on Level B, when the Predicted Paths are removed. Although the path each student took to get to Level B was different by how often they attempted Level A (and Level B), both Mouse and Latia had an attempt at Level B when they received two pairs of SBVs in their vector pool.

On Level A, Mouse relied more on the Predicted Path (***geometric*** approach) that resulted from ***button-pushing*** and looked at (and solved) the vector equation produced in the Data Log upon completion of the level (***numeric*** interpretation). When he moved to Level B, he sought to further learn how the vector equation worked and focused on how the scalars related to the direction the rabbit traveled. The inclusion of SBVs allowed Mouse to focus more on the numerical components and see how the scalars affect the resultant vector. When asked about how he solved the level, he responded: "I saw the position which is three and the other position which is -5. So, I kinda made it to where I'm trying to get it to where the vector has three and five […]. I'm guessing that if I multiply negative one to negative three it will get three and negative one again to negative five." The incorporation of SBVs allowed Mouse to ***focus on one coordinate*** by allowing each vector slot to align with one coordinate of the goal position. Having just learned of the Vector Equation from the Data Log in Level A, Mouse used the one coordinate approach as a tool to learn how the vector equation operates and see how the scalars affect individual vectors and the resultant vector.

Latia completed Level A and one attempt at Level B before she encountered SBVs. Upon opening Level A, she noticed the vector equation and began to make sense of how the scalars and the equation relate to the graphical component of the game. On Level B, she continued to explore this connection and gained a working understanding of how the resultant vector relates to the rabbit's motion, adjusting her strategy when starting away from the origin. When given SBVs, she ***focused on one coordinate*** and explained that the orientation of the vector does not matter because the negative scalar can flip directions. Latia said of her strategy:

> This time, instead of trial-and-error steps, I was looking for the numbers that would multiply to get me to where I wanted to be on the x-axis and where I wanted to be on the y-axis and since I noticed that these were all zero something or something zero I was able to use both values on this equation.

The inclusion of SBVs allowed Latia to focus more directly on how the scalars affect the vector equation. Latia explicitly stated her intention to ***focus on one coordinate*** and how the inclusion of zeros in the SBVs allows her to do so. In previous levels, Latia gained an understanding of the Vector Equation and how the scalars affected vector length, so the inclusion of SBVs allowed her to confirm her assumptions. Initially, Latia had a ***focus on one vector*** approach and the distinct directions provided by the SBVs allowed her to include both vectors in the vector equation. Both Mouse and Latia used the SBVs as a way to validate their understanding garnered from previous levels. Mouse solved the equation produced by the Data Log upon completion of Level A and used Level B as an application of what he learned. Latia used the SBVs to reaffirm her knowledge of the Vector Equation and scalars by incorporating two vectors in the vector equation. Although this is the first time Latia used both vector slots, she began to gain the understanding in her first attempt at Level B when she had a starting position off the origin and needed only one vector to reach the goal from her second position.

An interesting note of SBVs is that both participants used a strategy that was a ***focus on one coordinate***. Due to the zeros inherent to SBVs, the ***focus on one vector*** and ***focus on one coordinate*** become synonymous with each other because each vector is a directional movement along one coordinate. Latia explains this combination as she describes how the zeros allow her to move along a coordinate line (either the x- or y-axis) to reach the goal.

Discussion

Throughout our analysis we found that participants began with less-anticipatory gameplay. This early gameplay resembled reactive strategies (Hollebrands, 2007) or Zones 1 and 2 gameplay (Williams-Pierce & Thevenow-Harrison, 2021) which focus on participants understanding how the controls of the game function. This was true of all participants, but was perhaps most pronounced in the gameplay of Mouse and Gwen who spent prolonged periods of time trying to understand how their actions impacted the objects in the game. Over time, most participants shifted to more-anticipatory gameplay allowing them to develop complex strategies aligning with a shift to more proactive strategies (Hollebrands, 2007). For example, Mouse began to anticipate how changing the vectors or adjusting the scalar would alter the Predicted Path the more he played the game. He began making conjectures about the game and even creating his own rules. In Level C, he wondered if it was possible to gather all the keys in one move. He used the game to test his hypothesis and found that it was not possible. He also connected the mathematics in the game to solving systems of equations indicating his ability to connect the mathematics of the game to the mathematics he had already known. In line with Hollebrands thinking surrounding willingness to engage with the machine and to connect gameplay to mathematics, Mouse moved from ***less-anticipatory*** to ***more-anticipatory button pushing*** with occasional advanced strategies being presented in certain cases like with standard basis vectors. In contrast, Gwen remained for the most part at a less anticipatory stage of development. This could be a result of her willingness to engage with technology and/or her ability to see the mathematics in the game.

Eventually, most participants began to enter Williams-Pierce and Thevenow-Harrison (2021) Zone 3 where players anticipated what is desirable in their gameplay. They began to coordinate their emerging understanding of how their actions affect the objects in the game with their understanding of the goal of the game. This understanding resulted in several different strategies emerging, including a ***quadrant-based*** strategy which was developed from coordinating the direction from the rabbit's position to the goal position. Mouse, Latia, and Zo all utilized the ***focus on one vector*** and ***focus on one coordinate*** strategies which could be explained by participants coordinating their stopped position with some aspect of the goal such as its coordinate or being close to the goal. This indicates that the participants were progressing from less-anticipatory reasoning to more anticipatory reasoning and eventually to discerning how to reach the goal. Additionally, designing a game with William-Pierce's zones in mind might lead students to progress from reactive strategies to proactive strategies.

We found that at least one participant, Latia, eventually made a hypothesis about the direction of a vector and utilized the game to test the hypothesis. This aligns with students using GSP to test hypotheses (Hollebrands, 2007) as a tenet of proactive strategies and of the characterization of Zone 4 gameplay (Williams-Pierce & Thevenow-Harrison, 2021). Finally, Latia's reuse of a previous strategy is a key indicator that a player has moved to Zone 5 of mathematical gameplay which is ripe for mathematical discovery and advancement. A key result of our analysis is that as most of the participants spent more time playing the game they were able to anticipate what would happen in the game, allowing for a greater variety of strategies to form which could be valuable points of departure for educators using the game either inside or outside of class. While playing the game, Latia frequently switched between ***numeric*** and ***geometric*** strategies which indicates that one of the goals of having the students who play the game be able to readily switch between viewing a vector equation from an algebraic lens and viewing it with a geometric lens was accomplished. Mouse provides some insight into this as well when looking at the ease of calculation with standard basis vectors helped him engage with more numeric strategies of the vector equation when he was previously using more geometric strategies.

The strategies presented can be used to categorize student gameplay of the game *Vector Unknown* - button pushing (less and more anticipatory), quadrantal reasoning, focusing on one vector, and focusing on one coordinate. For each of these strategies, we witnessed students relying on the ***geometric*** or ***numeric*** features of the game to implement the strategy toward reaching the goal (sometimes both). Further, consistent with current literature, we have seen that the participants appear to go through several different zones of gameplay that begin with less-anticipatory reasoning and culminate in more diverse strategies. While our analysis only consists of gameplay from *Vector Unknown*, it provides some insight into how students can initially reason about vector equations, scaling, and linear combinations of vectors. Finally, it takes aspects of the work of Williams-Pierce and Thevenow-Harrison's (2021) theory about elementary education games and Hollebrands' theories related to students' reasoning about GSP and shows that aspects of their theories can be extended to game-based learning in a game designed for college students.

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