

THE INDUCTIVE MCKAY–NAVARRO CONDITIONS FOR THE PRIME 2 AND SOME GROUPS OF LIE TYPE

L. RUHSTORFER AND A. A. SCHAEFFER FRY

(Communicated by Martin Liebeck)

ABSTRACT. For a prime ℓ , the McKay conjecture suggests a bijection between the set of irreducible characters of a finite group with ℓ' -degree and the corresponding set for the normalizer of a Sylow ℓ -subgroup. Navarro’s refinement suggests that the values of the characters on either side of this bijection should also be related, proposing that the bijection commutes with certain Galois automorphisms. Recently, Navarro–Späth–Vallejo have reduced the McKay–Navarro conjecture to certain “inductive” conditions on finite simple groups. We prove that these inductive McKay–Navarro (also called the inductive Galois–McKay) conditions hold for the prime $\ell = 2$ for several groups of Lie type, namely the untwisted groups without non-trivial graph automorphisms.

1. INTRODUCTION

Many of the current problems in the character theory of finite groups are motivated by the celebrated McKay conjecture. This conjecture states that for a prime ℓ dividing the size of a finite group G , there should be a bijection between the set $\text{Irr}_{\ell'}(G)$ of irreducible characters of degree relatively prime to ℓ and the corresponding set $\text{Irr}_{\ell'}(N_G(P))$ for the normalizer of a Sylow ℓ -subgroup P of G . Navarro later posited that this bijection should moreover be compatible with the action of certain Galois automorphisms.

We write $\mathbb{Q}^{\text{ab}}$ for the field generated by all roots of unity in $\mathbb{C}$, and let $\mathcal{G} := \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ be the corresponding Galois group. For a prime ℓ , let $\mathcal{H}_{\ell}$ be the subgroup of $\mathcal{G}$ consisting of those $\tau \in \mathcal{G}$ such that there is an integer f such that $\tau(\xi) = \xi^{\ell^f}$ for every root of unity ξ of order not divisible by ℓ . The following is Navarro’s refinement of the McKay conjecture, sometimes also referred to as the Galois–McKay conjecture.

Received by the editors June 28, 2021, and, in revised form, January 24, 2022, and January 25, 2022.

2020 *Mathematics Subject Classification.* Primary 20C15, 20C33.

Key words and phrases. Local-global conjectures, characters, McKay conjecture, Galois–McKay, McKay–Navarro conjecture, finite simple groups, Lie type, Galois automorphisms.

The authors were supported by the Isaac Newton Institute for Mathematical Sciences in Cambridge and the organizers of the Spring 2020 INI program Groups, Representations, and Applications: New Perspectives, EPSRC grant EP/R014604/1, where this work began.

The second author was supported by the National Science Foundation (Award Nos. DMS-1801156 and DMS-2100912).

McKay–Navarro conjecture ([Nav04]). Let G be a finite group and let ℓ be a prime. Let $P \in \text{Syl}_\ell(G)$. Then there exists an $\mathcal{H}_\ell$ -equivariant bijection between the sets $\text{Irr}_{\ell'}(G)$ and $\text{Irr}_{\ell'}(\text{N}_G(P))$.

A huge step toward the proof of this conjecture was recently accomplished in [NSV20], where Navarro–Späth–Vallejo proved a reduction theorem for the McKay–Navarro conjecture. There, they show that the conjecture holds for all finite groups if every finite simple group satisfies certain “inductive” conditions.

These inductive McKay–Navarro conditions build upon the inductive conditions for the usual McKay conjecture, which was reduced to simple groups in [IMN07]. Much work has been done toward proving the original inductive McKay conditions, and they were even completed for the prime $\ell = 2$ in [MS16]. The focus of the current paper is to work toward expanding that result to the case of the inductive McKay–Navarro conditions.

In [Ruh21], the first author proves that the simple groups of Lie type satisfy the inductive McKay–Navarro conditions for their defining prime, with a few low-rank exceptions that were settled in [Joh20]. In [SF21], the second author shows that the maps used in [Mal07, Mal08, MS16] to prove the original inductive McKay conditions for the prime $\ell = 2$ are further $\mathcal{H}_\ell$ -equivariant, making them promising candidates for the inductive McKay–Navarro conditions.

Here we expand on the work of [SF21] to complete the proof that when $\ell = 2$, the inductive McKay–Navarro (also called inductive Galois–McKay) conditions from [NSV20] are satisfied for the untwisted groups of Lie type that do not admit graph automorphisms.

Theorem A. *The inductive McKay–Navarro conditions [NSV20, Definition 3.1] hold for the prime $\ell = 2$ for simple groups $\text{PSp}_{2n}(q) = \text{C}_n(q)$ with $n \geq 1$; $\text{B}_n(q)$ with $n \geq 3$ and $(n, q) \neq (3, 3)$; $\text{G}_2(q)$ with $q \neq 3^f$; $\text{F}_4(q)$; $\text{E}_7(q)$; and $\text{E}_8(q)$, where q is a power of an odd prime.*

We remark that Theorem A gives the first result in which a series of groups of Lie type is shown to satisfy the inductive McKay–Navarro conditions for a non-defining prime. Given the results of [SF21], the bulk of what remains to prove Theorem A is to study the behavior of extensions of odd-degree characters of groups of Lie type to their inertia group in the automorphism group, and similarly for the local subgroup used in the inductive conditions. To do this, we utilize work by Digne and Michel [DM94] on Harish-Chandra theory for disconnected groups and work of Malle and Späth [MS16] that tells us that, in our situation, odd-degree characters lie in very specific Harish-Chandra series.

The paper is structured as follows. In Section 2, we discuss a generalization of some aspects of Harish-Chandra theory to the case of disconnected reductive groups, which will help us understand the extensions of the relevant characters in our situation. In Section 3, we use this to make some observations specific to the principal series representations of groups of Lie type that will be useful in our extensions. In Section 4, we make some additional observations regarding extensions and the inductive McKay–Navarro conditions. In Section 5, we complete the proof of Theorem A in the case of type C, and in Section 6, we finish the proof in the remaining cases.

1.1. Notation. We introduce some standard notation that we will use throughout. For finite groups $X \triangleleft Y$, if $\psi \in \text{Irr}(X)$, we use the notation $\text{Irr}(Y|\psi)$ to denote the

set of $\chi \in \text{Irr}(Y)$ such that $\langle \psi, \text{Res}_X^Y \chi \rangle \neq 0$. If $\chi \in \text{Irr}(Y)$, we use $\text{Irr}(X|\chi)$ to denote the set of $\psi \in \text{Irr}(X)$ such that $\langle \psi, \text{Res}_X^Y \chi \rangle \neq 0$. Further, if a group A acts on $\text{Irr}(X)$ and $\psi \in \text{Irr}(X)$, we write A_ψ for the stabilizer in A of ψ .

2. HARISH-CHANDRA THEORY AND DISCONNECTED GROUPS

Let $\mathbf{G}^\circ$ be a connected reductive group defined over an algebraic closure of $\mathbb{F}_p$ and let $F : \mathbf{G}^\circ \rightarrow \mathbf{G}^\circ$ be a Frobenius endomorphism defining an $\mathbb{F}_q$ -structure. Consider a quasi-central automorphism $\sigma : \mathbf{G}^\circ \rightarrow \mathbf{G}^\circ$ commuting with the action of the Frobenius F as in [DM94, Definition-Theorem 1.15]. We form the reductive group $\mathbf{G} := \mathbf{G}^\circ \rtimes \langle \sigma \rangle$. Let $\mathbf{P}^\circ$ be a σ -stable parabolic subgroup of $\mathbf{G}^\circ$ with σ -stable Levi complement $\mathbf{L}^\circ$ and unipotent radical $\mathbf{U}$. We can then consider the parabolic subgroup $\mathbf{P} := \mathbf{P}^\circ \langle \sigma \rangle$ with Levi subgroup $\mathbf{L} := \mathbf{L}^\circ \langle \sigma \rangle$ in $\mathbf{G}$. In the following, we assume that $(\mathbf{L}, \mathbf{P})$ is F -stable such that we can consider the generalized Harish-Chandra induction

$$\mathbf{R}_{\mathbf{L}}^{\mathbf{G}} := \text{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} \text{Infl}_{\mathbf{L}^F}^{\mathbf{P}^F} : \mathbb{Z}\text{Irr}(\mathbf{L}^F) \rightarrow \mathbb{Z}\text{Irr}(\mathbf{G}^F).$$

We note that, according to [DM94, Theorem 3.2], this map satisfies a Mackey formula in the sense of [DM94, Definition 3.1]. We call a character $\delta \in \text{Irr}(\mathbf{L}^F)$ cuspidal if any (hence every) irreducible constituent $\delta^\circ \in \text{Irr}((\mathbf{L}^\circ)^F)$ of $\delta|_{(\mathbf{L}^\circ)^F}$ is cuspidal.

Lemma 2.1. *Let $\mathbf{L}$ and $\mathbf{M}$ be two F -stable Levi subgroups of F -stable parabolic subgroups of $\mathbf{G}$ as above. If $\delta \in \text{Irr}(\mathbf{L}^F)$ is cuspidal and $\mathbf{M}$ is a proper Levi subgroup of $\mathbf{L}$, then we have ${}^*\mathbf{R}_{\mathbf{M}}^{\mathbf{L}}\delta = 0$.*

Proof. By [DM94, Corollary 2.4] we have

$$\text{Res}_{(\mathbf{M}^\circ)^F}^{\mathbf{M}^F} {}^*\mathbf{R}_{\mathbf{M}}^{\mathbf{L}}\delta = {}^*\mathbf{R}_{\mathbf{M}^\circ}^{\mathbf{L}^\circ} \text{Res}_{(\mathbf{L}^\circ)^F}^{\mathbf{L}^F} \delta = 0,$$

where the last equality follows from the fact that all irreducible constituents of $\text{Res}_{(\mathbf{L}^\circ)^F}^{\mathbf{L}^F} \delta$ are cuspidal. Since ${}^*\mathbf{R}_{\mathbf{M}}^{\mathbf{L}}\delta \in \mathbb{N}\text{Irr}(\mathbf{M}^F)$ and $\text{Res}_{(\mathbf{M}^\circ)^F}^{\mathbf{M}^F} {}^*\mathbf{R}_{\mathbf{M}}^{\mathbf{L}}\delta = 0$ it follows that ${}^*\mathbf{R}_{\mathbf{M}}^{\mathbf{L}}\delta = 0$. $\square$

Proposition 2.2. *Let $\delta \in \text{Irr}(\mathbf{L}^F)$ and $\eta \in \text{Irr}(\mathbf{M}^F)$ be two cuspidal characters of F -stable Levi subgroups of F -stable parabolic subgroups of $\mathbf{G}$ as above. Then we have*

$$\langle \mathbf{R}_{\mathbf{L}}^{\mathbf{G}}\delta, \mathbf{R}_{\mathbf{M}}^{\mathbf{G}}\eta \rangle = |\{x \in ((\mathbf{G}^\sigma)^\circ)^F \mid {}^x\mathbf{L} = \mathbf{M} \text{ and } {}^x\delta = \mu\}| / |((\mathbf{L}^\sigma)^\circ)^F|.$$

Proof. We follow the proof in the case of connected reductive groups, see [DM91, Lemma 6.5]. Denote

$$\mathcal{S} := \{x \in ((\mathbf{G}^\sigma)^\circ)^F \mid {}^x\mathbf{M} \cap \mathbf{L} \text{ contains a maximal torus of } \mathbf{G}^\circ\}.$$

Then according to the Mackey formula from [DM94, Theorem 3.2] we obtain

$$\langle \mathbf{R}_{\mathbf{L}}^{\mathbf{G}}\delta, \mathbf{R}_{\mathbf{M}}^{\mathbf{G}}\eta \rangle = \langle \delta, {}^*\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}\mathbf{R}_{\mathbf{M}}^{\mathbf{G}}\eta \rangle = \langle \delta, \sum_{x \in ((\mathbf{L}^\sigma)^\circ)^F \setminus \mathcal{S} / ((\mathbf{M}^\sigma)^\circ)^F} \text{ad}(x^{-1}) \mathbf{R}_x^{\mathbf{L}} \cap \mathbf{M} {}^*\mathbf{R}_{\mathbf{L} \cap \mathbf{M}}^{\mathbf{M}}\eta \rangle.$$

As η is cuspidal, we can apply Lemma 2.1 and deduce that ${}^*\mathbf{R}_{\mathbf{L} \cap \mathbf{M}}^{\mathbf{M}} \eta = 0$ unless ${}^x\mathbf{L} \cap \mathbf{M} = \mathbf{M}$ which is equivalent to ${}^x\mathbf{L} \supset \mathbf{M}$. We therefore obtain

$$\begin{aligned} \langle \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta, \mathbf{R}_{\mathbf{M}}^{\mathbf{G}} \eta \rangle &= \langle \delta, \sum_{x \in ((\mathbf{L}^\sigma)^\circ)^F \backslash \mathcal{S} / ((\mathbf{M}^\sigma)^\circ)^F | {}^x\mathbf{L} \supset \mathbf{M}} \text{ad}(x^{-1}) \mathbf{R}_{\mathbf{L} \cap \mathbf{M}}^{\mathbf{L}} \eta \rangle \\ &= \langle \sum_{x \in ((\mathbf{L}^\sigma)^\circ)^F \backslash \mathcal{S} / ((\mathbf{M}^\sigma)^\circ)^F | {}^x\mathbf{L} \supset \mathbf{M}} {}^*\mathbf{R}_{\mathbf{L} \cap \mathbf{M}}^{\mathbf{L}} {}^x\delta, \eta \rangle. \end{aligned}$$

Applying the same argument as before to the cuspidal character δ now yields

$$\langle \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta, \mathbf{R}_{\mathbf{M}}^{\mathbf{G}} \eta \rangle = \sum_{x \in ((\mathbf{L}^\sigma)^\circ)^F \backslash \mathcal{S} / ((\mathbf{M}^\sigma)^\circ)^F | {}^x\mathbf{L} = \mathbf{M}} \langle {}^x\delta, \mu \rangle.$$

The claim of the proposition now easily follows from this. $\square$

Proposition 2.3. *Suppose that $\delta^\circ \in \text{Irr}((\mathbf{L}^\circ)^F)$ is a σ -invariant cuspidal character that extends to a character of M_{δ° , where $M = \mathbf{L}^F \mathbf{N}_{((\mathbf{G}^\sigma)^\circ)^F}(\mathbf{L})$ and let $\chi^\circ \in \text{Irr}((\mathbf{G}^\sigma)^\circ)^F$ be a character in the Harish-Chandra series of δ° . Then for $\chi \in \text{Irr}(\mathbf{G}^F | \chi^\circ)$, there exists a unique $\delta \in \text{Irr}(\mathbf{L}^F | \delta^\circ)$ such that $\langle \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta, \chi \rangle \neq 0$. In particular, we have*

$$\langle \chi, \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta \rangle = \langle \text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi, \mathbf{R}_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \delta^\circ \rangle = \langle \text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi, \text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi \rangle \langle \chi^\circ, \mathbf{R}_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \delta^\circ \rangle.$$

Proof. Since Harish-Chandra induction commutes with induction we have

$$\langle \chi, \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \text{Ind}_{(\mathbf{L}^\circ)^F}^{\mathbf{L}^F} \delta^\circ \rangle = \langle \chi, \text{Ind}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \mathbf{R}_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \delta^\circ \rangle = \langle \text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi, \mathbf{R}_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \delta^\circ \rangle,$$

where the last equality follows from Frobenius reciprocity. From this it follows that there exists some $\delta \in \text{Irr}(\mathbf{L}^F | \delta^\circ)$ such that $\langle \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta, \chi \rangle \neq 0$. It remains to show that δ is unique. By assumption δ° extends to a σ -stable character $\hat{\delta}'$ of $(\mathbf{L}^\circ)^F \mathbf{N}_{((\mathbf{G}^\sigma)^\circ)^F}(\mathbf{L})_{\delta^\circ}$. Thus, there exists by [Spä12, Lemma 2.11] a unique character $\hat{\delta} \in \text{Irr}(M_{\delta^\circ})$ such that $\delta = \text{Res}_{\mathbf{L}^F}^{M_{\delta^\circ}} \hat{\delta}$ and $\hat{\delta}' = \text{Res}_{(\mathbf{L}^\circ)^F \mathbf{N}_{((\mathbf{G}^\sigma)^\circ)^F}(\mathbf{L})_{\delta^\circ}}^{M_{\delta^\circ}} \hat{\delta}$.

Assume now that $\delta' \in \text{Irr}(\mathbf{L}^F | \delta^\circ)$ is a second character with the property $\langle \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta', \chi \rangle \neq 0$. Proposition 2.2 therefore shows that $\delta' = {}^x\delta$ for some $x \in \mathbf{N}_{((\mathbf{G}^\sigma)^\circ)^F}(\mathbf{L})$. Note that both δ and δ' must be extensions of δ° since δ° is σ -invariant, and hence we see $\delta^\circ = {}^x\delta^\circ$. Since δ is the restriction of the character $\hat{\delta}$ it is invariant under the action of $\mathbf{N}_{((\mathbf{G}^\sigma)^\circ)^F}(\mathbf{L})_{\delta^\circ}$, we therefore must necessarily have $\delta' = \delta$.

Finally, observe that the irreducible constituents of $\text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi$ are all $\mathbf{G}^F$ -conjugate. Since δ° is $\mathbf{L}^F$ -stable we deduce that $\mathbf{R}_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \delta^\circ$ is $\mathbf{G}^F = (\mathbf{G}^\circ)^F \mathbf{L}^F$ -stable. Thus, every irreducible constituent of $\text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi$ occurs with the same multiplicity in $\mathbf{R}_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \delta^\circ$ and so

$$\langle \text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi, \mathbf{R}_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \delta^\circ \rangle = \langle \text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi, \text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \chi \rangle \langle \chi^\circ, \mathbf{R}_{\mathbf{L}^\circ}^{\mathbf{G}^\circ} \delta^\circ \rangle. \quad \square$$

If δ is the trivial character of the torus $\mathbf{T}^F$ then the result of the previous proposition is a consequence of [Mal91, Satz 1.5]. In this case, a more precise description of the constituents of $\mathbf{R}_{\mathbf{T}}^{\mathbf{G}} 1_{\mathbf{T}^F}$ can be given by analyzing the action of σ on W .

Corollary 2.4. *Assume that we are in the situation of the previous proposition. Then we have an $\text{Irr}(\mathbf{G}^F/(\mathbf{G}^\circ)^F) \times \mathcal{G}_{\delta^\circ}$ -equivariant surjection*

$$\text{Irr}(\mathbf{G}^F \mid \chi^\circ) \rightarrow \text{Irr}(\mathbf{L}^F \mid \delta^\circ), \chi \mapsto \delta.$$

Further, this is a bijection if χ° is σ -invariant.

Proof. The map itself follows from Proposition 2.3, since the fact that $\mathbf{G}^F = (\mathbf{G}^\circ)^F \mathbf{L}^F$ and $(\mathbf{G}^\circ)^F \cap \mathbf{L}^F = (\mathbf{L}^\circ)^F$ yields that $\text{Res}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \text{Ind}_{\mathbf{L}^F}^{\mathbf{G}^F} = \text{Ind}_{(\mathbf{L}^\circ)^F}^{(\mathbf{G}^\circ)^F} \text{Res}_{(\mathbf{L}^\circ)^F}^{\mathbf{L}^F}$ (see e.g. [Isa06, Problem (5.2)]), implying that given $\delta \in \text{Irr}(\mathbf{L}^F \mid \delta^\circ)$, there exists $\chi \in \text{Irr}(\mathbf{G}^F \mid \chi^\circ)$ with $\langle \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta, \chi \rangle \neq 0$. For $\lambda \in \text{Irr}(\mathbf{G}^F/(\mathbf{G}^\circ)^F)$ we have $\lambda \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta = \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \lambda \delta$, by e.g. [Isa06, Problem 5.3], so the compatibility with linear characters follows from the uniqueness statement in Proposition 2.3. As $\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}$ is equivariant with respect to Galois-automorphisms we similarly obtain that the map is $\mathcal{G}_{\delta^\circ}$ -equivariant. If further χ° is σ -invariant, then χ° extends to $\mathbf{G}^F$ and the map is a bijection thanks to Gallagher's theorem [Isa06, Corollary 6.17] and Proposition 2.3. $\square$

The previous corollary allows us in principle to compute the action of the Galois group $\mathcal{G}_{\delta^\circ}$ on constituents of $\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}(\delta)$. However, it could happen that $\mathcal{G}_{\chi^\circ}$ is not contained in $\mathcal{G}_{\delta^\circ}$. Lemma 2.5 solves this problem under the assumption that σ centralizes the relative Weyl group of $\mathbf{L}$.

Lemma 2.5. *Keep the notation and assumption from Proposition 2.3.*

- (1) *We have $N_{\mathbf{G}^F}(\mathbf{L}) = \{g \in N_{(\mathbf{G}^\circ)^F}(\mathbf{L}^\circ) \mid \sigma(g)g^{-1} \in (\mathbf{L}^\circ)^F\} \langle \sigma \rangle$.*
- (2) *Suppose that $\tau \in \mathcal{G}$ stabilizes χ° . If σ centralizes $N_{(\mathbf{G}^\circ)^F}(\mathbf{L}^\circ)/(\mathbf{L}^\circ)^F$ and stabilizes χ° , then we have $\chi^\tau = \chi\lambda$ for some $\lambda \in \text{Irr}(\mathbf{G}^F)$.*

Proof. Part (1) is an easy computation. Let us prove part (2). Since $\tau \in \mathcal{G}$ stabilizes χ° we have $(\delta^\circ)^{\tau x} = \delta^\circ$ for some $x \in N_{(\mathbf{G}^\circ)^F}(\mathbf{L}^\circ)$ by [DM91, Lemma 6.5]. By assumption, σ centralizes $N_{(\mathbf{G}^\circ)^F}(\mathbf{L}^\circ)/(\mathbf{L}^\circ)^F$ and so we have $\sigma(x)x^{-1} \in \mathbf{L}^{\circ F}$. From this we deduce that x normalizes $\mathbf{L}^F$ and thus there exists some linear character λ of $\mathbf{L}^F/(\mathbf{L}^\circ)^F \cong \mathbf{G}^F/(\mathbf{G}^\circ)^F$ such that $\delta^{\tau x} = \delta\lambda$. We deduce that

$$0 \neq \langle \chi^{\tau x}, \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta^{\tau x} \rangle = \langle \chi^{\tau x}, \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta \lambda \rangle = \langle \chi^{\tau x} \lambda^{-1}, \mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \delta \rangle.$$

Since both $\chi^{\tau x} \lambda^{-1}$ and χ cover χ° , we obtain $\chi^{\tau x} \lambda^{-1} = \chi^\tau \lambda^{-1} = \chi$ by Corollary 2.4. $\square$

2.1. Descent of scalars. Let us from now on assume that $\mathbf{G}$ is a connected reductive group with Frobenius endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$. If $\mathbf{H}$ is an F -stable subgroup of $\mathbf{G}$ then we write H for the set $\mathbf{H}^F$ of F -fixed points.

We assume that $F_0 : \mathbf{G} \rightarrow \mathbf{G}$ is a Frobenius endomorphism satisfying $F_0^r = F$ for some positive integer r . For an F_0 -stable subgroup $\mathbf{H}$ of $\mathbf{G}$, we consider $\underline{\mathbf{H}} = \mathbf{H}^r$ with Frobenius endomorphism $F_0 \times \cdots \times F_0 : \underline{\mathbf{H}} \rightarrow \underline{\mathbf{H}}$ which we also denote by F_0 and the permutation

$$\pi : \underline{\mathbf{H}} \rightarrow \underline{\mathbf{H}}, (g_1, \dots, g_r) \mapsto (g_2, \dots, g_r, g_1).$$

The projection map $\text{pr} : \underline{\mathbf{H}} \rightarrow \mathbf{H}$ onto the first coordinate then yields an isomorphism $\underline{\mathbf{H}}^{\pi F_0} \cong \mathbf{H}^F$ and the automorphism $\pi \in \text{Aut}(\underline{\mathbf{G}}^{\pi F_0})$ corresponds to $F_0^{-1} \in \text{Aut}(\mathbf{G}^F)$ under this isomorphism. Moreover, pr induces a bijection $\text{pr}^\vee : \text{Irr}(\mathbf{H}^F) \rightarrow \text{Irr}(\underline{\mathbf{H}}^{\pi F_0})$ on the level of characters.

Note that π is a quasi-central automorphism of $\mathbf{G}$. We fix an F_0 -stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with F_0 -stable Levi subgroup $\mathbf{L}$. We assume for the remainder of this section that F_0 centralizes the relative Weyl group $N_G(\mathbf{L})/L$. This implies that the projection map pr maps $N_{\mathbf{G}\langle\pi\rangle}(\mathbf{L}\langle\pi\rangle)$ onto $N_G(\mathbf{L})$.

For any F_0 -stable subgroup H of G we set $\hat{H} := H\langle F_0 \rangle$. We define a generalized Harish-Chandra induction

$$R_L^{\hat{G}} := \text{Ind}_P^{\hat{G}} \circ \text{Inf}_L^{\hat{P}},$$

and by construction we obtain $\text{pr}^\vee R_L^{\hat{G}} = R_{\mathbf{L}\langle\pi\rangle}^{\mathbf{G}\langle\pi\rangle} \text{pr}^\vee$. With this notation, we can reformulate Proposition 2.3:

Proposition 2.6. *Suppose that $\delta \in \text{Irr}(L)$ is an F_0 -invariant cuspidal character that extends to a character of $N_G(\mathbf{L})_\delta\langle F_0 \rangle$, and let $\chi \in \text{Irr}(G)$ be a character in the Harish–Chandra series of δ . Then for $\hat{\chi} \in \text{Irr}(\hat{G} \mid \chi)$, there exists a unique $\hat{\delta} \in \text{Irr}(\hat{L} \mid \delta)$ such that $\langle R_L^{\hat{G}} \hat{\delta}, \hat{\chi} \rangle \neq 0$. If χ is F_0 -invariant, this yields a bijection $\text{Irr}(\hat{G} \mid \chi) \rightarrow \text{Irr}(\hat{L} \mid \delta)$.*

Proof. To prove this, consider Proposition 2.3 in the case of the π -invariant character $\delta_0 := \text{pr}^\vee(\delta) \in \text{Irr}(\mathbf{L}^{\pi F_0})$ which is a cuspidal character of $\mathbf{L}^{\pi F_0} \cong \mathbf{L}^F$. Using the fact that $N_{(\mathbf{G}\langle\pi\rangle)^{\pi F_0}}(\mathbf{L}\langle\pi\rangle) \cong N_G(\mathbf{L})\langle F_0 \rangle$ via pr we see that the assumptions of Proposition 2.3 are all satisfied. The commutation of Harish-Chandra induction with $\text{pr}^\vee$ then yields the result. $\square$

In particular, we obtain the following consequence of Lemma 2.5.

Corollary 2.7. *Keep the notation and assumption from Proposition 2.6 and assume that χ is F_0 -invariant. Suppose that $\tau \in \mathcal{G}$ stabilizes χ . Then we have $\hat{\chi}^\tau = \hat{\chi}\lambda$ for some $\lambda \in \text{Irr}(G\langle F_0 \rangle/G)$. In particular, λ is such that $\delta^{\tau x} = \delta$ and $\hat{\delta}^{\tau x} = \hat{\delta}\lambda$ for some $x \in N_G(\mathbf{L})$.*

3. PRINCIPAL SERIES

In the situation of Theorem A, most of the characters that we are concerned with will lie in the principal series, so we present here some consequences of the previous section in that situation. We begin with the following, which will be useful on the “local” side.

Lemma 3.1. *Let $\hat{Y} = YA$ be a finite group with $Y \triangleleft \hat{Y}$, $A \leq Y$, and $Y \cap A = \{1\}$. Let $X \triangleleft Y$ be an A -stable normal subgroup and let $\theta \in \text{Irr}(X)$ be A -invariant and extend to $\hat{Y}_\theta = Y_\theta A$. Let $\psi \in \text{Irr}(Y|\theta)$ extend to $\hat{Y}$. Then the following hold.*

- (1) *Let $\chi \in \text{Irr}(Y_\theta|\theta)$ such that $\psi = \text{Ind}_{Y_\theta}^Y(\chi)$ via Clifford correspondence. Then χ extends to $\hat{Y}_\theta$ and $\text{Ind}_{\hat{Y}_\theta}^{\hat{Y}}$ induces a bijection $\text{Irr}(\hat{Y}_\theta|\chi) \rightarrow \text{Irr}(\hat{Y}|\psi)$ such that extensions of χ are mapped to extensions of ψ .*
- (2) *There is an extension $\hat{\theta}$ of θ to $\hat{X} := XA \triangleleft \hat{Y}_\theta$, and for any such extension, $\text{Res}_{\hat{Y}_\theta}^{\hat{Y}} \hat{\theta}$ induces a bijection $\text{Irr}(\hat{Y}_\theta|\hat{\theta}) \rightarrow \text{Irr}(Y_\theta|\theta)$.*
- (3) *If $\theta(1) = 1$, then there is an extension $\hat{\psi} \in \text{Irr}(\hat{Y}|\psi)$ such that $\mathcal{G}_{\hat{\psi}} \cap \mathcal{G}_\theta = \mathcal{G}_\psi \cap \mathcal{G}_\theta$.*
- (4) *Continue to assume that $\theta(1) = 1$, and suppose further that $|A|$ is relatively prime to the order $o(\theta)$ of θ . Let $\Gamma \leq \text{Aut}(\hat{Y})$ preserve $\hat{X}$ and Y (and hence also $X = \hat{X} \cap Y$). Then the extension $\hat{\psi}$ from (3) satisfies $\Gamma_{\hat{\psi}} \cap \Gamma_\theta = \Gamma_\psi \cap \Gamma_\theta$.*

Proof. (1) Let $\hat{\psi} \in \text{Irr}(\hat{Y})$ be an extension of ψ . In particular, note that $\hat{\psi} \in \text{Irr}(\hat{Y}|\theta)$, and hence by Clifford correspondence, there is a unique $\hat{\chi} \in \text{Irr}(\hat{Y}_\theta|\theta)$ such that $\hat{\psi} = \text{Ind}_{\hat{Y}_\theta}^{\hat{Y}}(\hat{\chi})$. Now, since $Y \cap \hat{Y}_\theta = Y_\theta$ and $\hat{Y} = \hat{Y}_\theta Y$, we have $\psi = \text{Res}_Y^{\hat{Y}}(\hat{\psi}) = \text{Res}_Y^{\hat{Y}} \text{Ind}_{\hat{Y}_\theta}^{\hat{Y}}(\hat{\chi}) = \text{Ind}_{Y_\theta}^Y \text{Res}_{\hat{Y}_\theta}^{\hat{Y}}(\hat{\chi})$ (see e.g. [Isa06, Problem (5.2)]). This forces $\text{Res}_{Y_\theta}^{\hat{Y}}(\hat{\chi})$ to be irreducible and $\chi = \text{Res}_{Y_\theta}^{\hat{Y}}(\hat{\chi})$ by the uniqueness of χ under Clifford correspondence, since both lie over θ . Now, identifying $A \cong \hat{Y}_\theta/Y_\theta \cong \hat{Y}/Y$, we have by Gallagher's theorem that $\text{Irr}(\hat{Y}_\theta|\chi) = \{\hat{\chi}\gamma \mid \gamma \in \text{Irr}(A)\}$ and $\text{Irr}(\hat{Y}|\psi) = \{\hat{\psi}\gamma \mid \gamma \in \text{Irr}(A)\}$. Note that under our identification, we have $\text{Ind}_{Y_\theta}^{\hat{Y}}(\hat{\chi}\gamma) = \hat{\psi}\gamma$, as the identification $\text{Irr}(\hat{Y}/Y) = \text{Irr}(\hat{Y}_\theta/Y_\theta)$ is via restriction (see e.g. [Isa06, Problem (5.3)]).

(2) Let $\hat{\theta} := \text{Res}_{\hat{X}}^{\hat{Y}_\theta}(\hat{\Lambda}(\theta))$, where $\hat{\Lambda}(\theta)$ is an extension of θ to $\hat{Y}_\theta$, which exists by assumption. Then $\hat{\theta}$ is an extension of θ to $\hat{X}$. Again we use Gallagher's theorem to achieve the stated bijection, noting that $\hat{Y}_\theta/\hat{X} \cong Y_\theta/X$, with their characters identified via restriction.

(3) Certainly, we have $\mathcal{G}_{\hat{\psi}} \cap \mathcal{G}_\theta \subseteq \mathcal{G}_\psi \cap \mathcal{G}_\theta$, and we wish to prove that there exists such a $\hat{\psi}$ satisfying the converse inclusion. Let $\tau \in \mathcal{G}_\psi \cap \mathcal{G}_\theta$. Note that the action of τ commutes with induction, so that letting χ be as in (1), we have $\chi^\tau = \chi$ by the uniqueness of Clifford correspondence and it suffices by (1) to find an extension $\hat{\chi} \in \text{Irr}(\hat{Y}_\theta|\chi)$ such that $\hat{\chi}^\tau = \hat{\chi}$. Now, let $\hat{\theta} \in \text{Irr}(\hat{X})$ be the unique extension of θ to $\hat{X}$ such that $A \in \ker(\hat{\theta})$. (Indeed, note that such a character exists since θ is now assumed to be linear.) Then $\hat{\theta}^\tau = \hat{\theta}$, and by (2) there is a unique $\hat{\chi} \in \text{Irr}(\hat{Y}_\theta|\hat{\theta})$ extending χ . But since $\hat{\chi}^\tau$ also extends $\chi = \chi^\tau$ and lies above $\hat{\theta} = \hat{\theta}^\tau$, this forces $\hat{\chi}^\tau = \hat{\chi}$.

(4) We have $\hat{\theta}$ from the proof of (3) which is the unique extension of θ to $\hat{X}$ of order $o(\theta)$, by [Isa06, Corollary (8.16)], and therefore $\hat{\theta}^\beta = \hat{\theta}$ for any $\beta \in \Gamma_\theta$. Arguing similarly to part (3) now yields the last statement. $\square$

Remark 3.2. We remark that the conclusion of (4) from Lemma 3.1 would also hold under the assumption that A is Γ -invariant, but this will not necessarily be the situation for our application.

Let $\mathbf{G}$, F , and F_0 be as in Section 2.1 so that $G = \mathbf{G}^F$ is defined over $\mathbb{F}_q$, and let $\mathbf{T} \leq \mathbf{B}$ be an F -stable maximal torus and Borel subgroup, respectively, in $\mathbf{G}$ such that F_0 centralizes N/T , where $T := \mathbf{T}^F$ and $N := N_G(\mathbf{T})$. We will write $\hat{T} := T\langle F_0 \rangle$, $\hat{G} := G\langle F_0 \rangle$, and $\hat{N} := N\langle F_0 \rangle$, as before.

Let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding, as in [CE04, (15.1)], and let $\tilde{\mathbf{T}}$ be an F -stable maximally split torus of $\tilde{\mathbf{G}}$ such that $\mathbf{T} = \tilde{\mathbf{T}} \cap \mathbf{G}$. Write $\tilde{T} := \tilde{\mathbf{T}}^F$ and $\tilde{G} := \tilde{\mathbf{G}}^F$. Then $\tilde{G}$, and hence $\tilde{T}$, induces the so-called diagonal automorphisms on G .

For $\delta \in \text{Irr}(T)$, the principal series $\mathcal{E}(G, (T, \delta))$ is the Harish-Chandra series of G corresponding to (T, δ) , and is in bijection with the set of irreducible characters of N_δ/T . We write $\mathbf{R}_{\mathbf{T}}^{\mathbf{G}}(\delta)_\eta$ as in [MS16, 4.D] for the character in $\mathcal{E}(G, (T, \delta))$ corresponding to $\eta \in \text{Irr}(N_\delta/T)$.

For finite groups $X \triangleleft Y$, if every character $\theta \in \text{Irr}(X)$ extends to its inertia group Y_θ , then by an extension map with respect to $X \triangleleft Y$, we mean a map

$\Lambda: \text{Irr}(X) \rightarrow \bigcup_{\theta \in \text{Irr}(X)} \text{Irr}(Y_\theta)$ such that $\Lambda(\theta) \in \text{Irr}(Y_\theta)$ is an extension of θ for each $\theta \in \text{Irr}(X)$.

Lemma 3.3. *Let $\ell \nmid q$ be a prime and write $\mathcal{H} := \mathcal{H}_\ell$. Let $\tau \in \mathcal{H}$ and suppose δ and $\chi \in \mathcal{E}(G, (T, \delta))$ are each invariant under both F_0 and τ . Then:*

- (1) *There exists an extension $\hat{\chi}$ of χ to $\hat{G}$ such that $\hat{\chi}^\tau = \hat{\chi}$. If further F_0 has order prime to $o(\delta)$, then $\hat{\chi}^a = \hat{\chi}$ for any $a \in \tilde{T}_\chi$.*
- (2) *If Λ is an extension map with respect to $T \triangleleft N$ such that the map*

$$\Omega: \bigcup_{\delta \in \text{Irr}(T)} \mathcal{E}(G, (T, \delta)) \rightarrow \text{Irr}(N)$$

defined by $\text{R}_T^G(\delta)_\eta \mapsto \text{Ind}_{N_\delta}^N(\Lambda(\delta)\eta)$ is $\mathcal{H} \times \langle F_0 \rangle$ -equivariant, then there also exists an extension $\hat{\psi}$ of $\psi := \Omega(\chi)$ to $\hat{N}$ such that $\hat{\psi}^\tau = \hat{\psi}$.

Proof. Let $\hat{\delta} \in \text{Irr}(\hat{T})$ be the extension of δ to $\hat{T}$ such that $F_0 \in \ker(\hat{\delta})$. Then in the notation of Corollary 2.7, we have $\delta^{\tau x} = \delta = \delta^\tau$, so $x \in N_G(T)$ may be chosen to be trivial. Then $\hat{\delta}^{\tau x} = \hat{\delta}^\tau = \hat{\delta}$ by our choice of $\hat{\delta}$. That is, $\lambda = 1$ in the notation of Corollary 2.7. Now, let $\hat{\chi} \in \text{Irr}(\hat{G}|\chi)$ correspond to $\hat{\delta}$ as in Proposition 2.6. Then Corollary 2.7 yields $\hat{\chi}^\tau = \hat{\chi}$, completing the proof of the first statement of (1). If F_0 has order prime to $o(\delta)$ and $a \in \tilde{T}_\chi$, we have $\hat{\delta}^a = \hat{\delta}$ since it is the unique extension of δ to $\hat{T}$ of order $o(\delta)$, by [Isa06, Corollary (8.16)], and δ is necessarily $\hat{T}$ -invariant. Then $\langle \text{R}_T^G \hat{\delta}, \hat{\chi}^a \rangle \neq 0$, forcing $\hat{\chi}^a = \hat{\chi}$ by Proposition 2.6.

Now, in the situation of (2), we have ψ is τ - and F_0 -invariant by the equivariance of Ω . Hence the statement follows from Lemma 3.1 applied to $(X, Y, \theta) = (T, N, \delta)$. $\square$

4. EXTENSIONS AND THE INDUCTIVE MCKAY–NAVARRO CONDITIONS

Let S be a non-abelian finite simple group and ℓ a prime dividing $|S|$. Let H be a universal covering group of S and $P \in \text{Syl}_\ell(H)$. Write $\mathcal{H} := \mathcal{H}_\ell$. The inductive McKay–Navarro conditions [NSV20, Definition 3.1] (referred to there as inductive Galois–McKay conditions) require an $\text{Aut}(H)_P$ -stable subgroup $N_H(P) \leq M < H$ and an $\mathcal{H} \times \text{Aut}(H)_P$ -equivariant bijection $\Omega: \text{Irr}_{\ell'}(H) \rightarrow \text{Irr}_{p'}(M)$ such that, roughly speaking, the projective representations extending any $\chi \in \text{Irr}_{\ell'}(H)$ to its inertia subgroup in $H \rtimes \text{Aut}(H)_{P, \chi^\mathcal{H}}$ and $\Omega(\chi)$ to its inertia subgroup in $M \rtimes \text{Aut}(H)_{P, \chi^\mathcal{H}}$ behave similarly when twisted by an element of $(M \rtimes \text{Aut}(H)_{P, \chi^\mathcal{H}} \times \mathcal{H})_\chi$. (See [NSV20, Definitions 1.5 and 3.1] for the precise definition.) Here $\chi^\mathcal{H}$ is the $\mathcal{H}$ -orbit of χ .

In our situation of Theorem A, the second author has proven in [SF21] that the groups M and bijections Ω from [Mal07, Mal08, MS16] for the inductive McKay conditions are indeed $\mathcal{H}$ -equivariant. Hence here we must study the behavior of the projective representations under the appropriate twists. We recall the statement of [NSV20, Lemma 1.4], which makes this idea more precise:

Lemma 4.1. *Let Y be a finite group and $X \triangleleft Y$. Let $\theta \in \text{Irr}(X)$ and assume that $\theta^{g\tau} = \theta$ for some $g \in Y$ and $\tau \in \mathcal{G}$. Let $\mathcal{P}$ be a projective representation of Y_θ associated with θ with values in $\mathbb{Q}^{\text{ab}}$ and factor set α . Then $\mathcal{P}^{g\tau}$ is a projective representation associated with θ , with factor set $\alpha^{g\tau}(x, y) = \alpha^g(x, y)^\tau$ for $x, y \in Y_\theta$.*

In particular, there exists a unique function

$$\mu_{g\tau} : Y_\theta \rightarrow K^\times$$

with $\mu_{g\tau}(1) = 1$, constant on cosets of X such that the projective representation $\mathcal{P}^{g\tau}$ is similar to $\mu_{g\tau}\mathcal{P}$.

Given the main results of [MS16, SF21], our primary remaining obstruction to proving Theorem A is in working with the characters $\mu_{g\tau}$ as in Lemma 4.1. If θ extends to Y_θ , then $\mu_{g\tau}$ is just the linear character guaranteed by Gallagher's theorem [Isa06, Corollary 6.17]. In this situation, our main task will be to prove part (iv) of [NSV20, Definition 1.5] for certain triples, which, roughly speaking, will require that the characters $\mu_{g\tau}$ for $g\tau \in (M \rtimes \text{Aut}(H)_{P,\chi^\mathcal{H}} \times \mathcal{H})_\chi$ are the same on either side of the bijection Ω when χ and $\Omega(\chi)$ (or their corresponding projective representations) are extended to their inertia groups in $M \rtimes \text{Aut}(H)_{P,\chi^\mathcal{H}}$ and $H \rtimes \text{Aut}(H)_{P,\chi^\mathcal{H}}$, respectively. We will assume throughout, without loss of generality, that all linear representations are realized over the field $\mathbb{Q}^{\text{ab}}$.

4.1. Further notation. Keep the notation for $T = \mathbf{T}^F$, $G = \mathbf{G}^F$, and N , from Section 3 and further assume that $\mathbf{G}$ is simple of simply connected type. If $Z(G) = 1$, write $\tilde{\mathbf{G}} = \mathbf{G}$, $\tilde{G} = G$, $\tilde{\mathbf{T}} = \mathbf{T}$, and $\tilde{T} := T$, and if $Z(G) \neq 1$, let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding and let $\tilde{G} := \tilde{\mathbf{G}}^F$ and $\tilde{T} := \tilde{\mathbf{T}}^F$ as before. Note that for an appropriate group D generated by field and graph automorphisms, we have $\tilde{G} \rtimes D$ induces $\text{Aut}(G)$.

Let $d := d_\ell(q)$ be the order of q modulo ℓ for ℓ odd, or the order of q modulo 4 for $\ell = 2$, and suppose that $d \in \{1, 2\}$. Let $\mathbf{S}_0$ be a Sylow d -torus for $(\mathbf{G}, F)$ and write $N_0 := N_G(\mathbf{S}_0)$ and $\tilde{N}_0 := N_{\tilde{G}}(\mathbf{S}_0)$.

4.2. The characters $\mu_{g\tau}$ in our situation. Let $\tilde{X} \in \{\tilde{G}, \tilde{N}_0\}$ and $X := G \cap \tilde{X} \in \{G, N_0\}$. Suppose that $\psi \in \text{Irr}(X)$ is a character such that $(\tilde{X}D)_\psi = \tilde{X}_\psi D_\psi$ and such that ψ extends to XD_ψ . Note that since restrictions of irreducible characters from $\tilde{G}$ to G are multiplicity-free by the work of Lusztig and the same is true for $\tilde{N}_0$ to N_0 by [MS16, Corollary 3.21], we also have ψ extends to $\tilde{X}_\psi$.

Let $\mathcal{D}$ be a representation affording ψ and let $\mathcal{D}_1$, respectively $\mathcal{D}_2$, be the representation of $\tilde{X}_\psi$, respectively XD_ψ , extending $\mathcal{D}$. For $i = 1, 2$ we let ψ_i be the character of the representation $\mathcal{D}_i$. Then we define $\mathcal{P}$ to be the projective representation of $(\tilde{X}D)_\psi$ above ψ given by

$$\mathcal{P}(\tilde{g}d) := \mathcal{D}_1(\tilde{g})\mathcal{D}_2(d)$$

for $\tilde{g} \in \tilde{X}_\psi$ and $d \in XD_\psi$, as in [Spä12, Lemma 2.11].

In this situation, we can now state Lemma 4.2, whose proof is exactly the same as [Ruh21, Lemma 7.2].

Lemma 4.2. *Assume we are in the situation above. Let $y \in \mathcal{H} \times N_{\tilde{N}_0 D}(XD_\psi)$ with $\psi^y = \psi$. Suppose that $\mu_1 \in \text{Irr}(\tilde{X}_\psi/X)$ and $\mu_2 \in \text{Irr}(XD_\psi/X)$ are such that $\psi_i^y = \mu_i \psi_i$ for $i = 1, 2$. Then there exists an invertible matrix M such that*

$$\mathcal{P}^y(\tilde{g}d) = \mu_1(\tilde{g})\mu_2(d)M\mathcal{P}(\tilde{g}d)M^{-1}$$

for all $\tilde{g} \in \tilde{X}_\psi$ and $d \in XD_\psi$.

5. PROOF OF THEOREM A FOR SYMPLECTIC GROUPS

We fix some notation to be used throughout our proof of Theorem A in the case of the symplectic groups. Let q be a power of an odd prime p and let $S := \mathrm{PSp}_{2n}(q)$ with $n \geq 1$ be one of the simple symplectic groups. Let $G := \mathrm{Sp}_{2n}(q)$ and $\tilde{G} := \mathrm{CSp}_{2n}(q)$, so that G is a universal covering group for S unless $(n, q) = (1, 9)$, and $\tilde{G} \rtimes D$ with $D = \langle F_p \rangle$ induces all automorphisms on G . Here F_p is the field automorphism induced by the map $x \mapsto x^p$ in $\mathbb{F}_q$.

Note that for $n = 1$, we have $S \cong \mathrm{PSL}_2(q)$, $G \cong \mathrm{SL}_2(q)$, and $\tilde{G} \cong \mathrm{GL}_2(q)$. We begin with a lemma to help in this case.

Lemma 5.1. *Let $G = \mathrm{SL}_2(q)$ and $\tilde{\chi} \in \mathrm{Irr}(\tilde{G})$. Then there exists a character $\chi \in \mathrm{Irr}(\tilde{G} \mid \tilde{\chi})$ which has an extension $\hat{\chi}$ to GD_χ such that $\tilde{G}(\tilde{G}\mathcal{G})_\chi = \tilde{G}(\tilde{G}\mathcal{G})_{\hat{\chi}}$.*

Proof. There exists a field automorphism F_0 such that $D_\chi = \langle F_0 \rangle$. The statement is true for the unipotent characters by Corollary 2.7 as both of them lie in the principal series. So we may assume χ lies in a rational Lusztig series $\mathcal{E}(G, s)$ for some semisimple element $1 \neq s$ of the dual group G^* . It follows that $C_{G^*}^\circ(s)$ is a maximal torus. In particular, χ is a regular-semisimple character. As such a character, it is a constituent of a Gelfand–Graev character. Let $\phi_0 \in \mathrm{Irr}(U)$ be a non-trivial F_0 -stable character, where U is the subgroup of upper unitriangular matrices in G . It follows that some $\tilde{G}$ -conjugate of χ is a constituent of multiplicity one of the Gelfand–Graev character $\Gamma := \mathrm{Ind}_U^{\tilde{G}}(\phi_0)$. As $(2, p-1) = (2, q-1)$ all diagonal automorphisms are induced by elements of $\tilde{T}^{F_0}$. Therefore, we can assume by possibly replacing χ by its $\tilde{G}$ -conjugate character that χ is a constituent of Γ .

Let $\hat{\phi}_0 \in \mathrm{Irr}(U\langle F_0 \rangle)$ be the trivial extension of ϕ_0 and set $\hat{\Gamma} := \mathrm{Ind}_{U\langle F_0 \rangle}^{G\langle F_0 \rangle}(\hat{\phi}_0)$. By Mackey’s formula $\mathrm{Res}_G^{G\langle F_0 \rangle}(\hat{\Gamma}) = \Gamma$ and Γ is multiplicity free. By Frobenius reciprocity, there exists a constituent $\hat{\chi}$ of $\hat{\Gamma}$ of multiplicity one, which is an extension of χ . The group $\tilde{G}$ stabilizes $\mathcal{E}(G, s)$ and acts transitively on it. Hence, $\tilde{G}(\tilde{G}\mathcal{G})_\chi = \tilde{G}\mathcal{G}_{\mathcal{E}(G, s)}$. Let $\tau \in \mathcal{G}$ be a Galois automorphism stabilizing the set $\mathcal{E}(G, s)$. Then we find $\tilde{t} \in \tilde{T}^{F_0}$ such that $\phi_0^{\tilde{t}\tau} = \phi_0$, see e.g. [Ruh21, Lemma 6.3] or by direct calculation. Thus, $\chi^{\tilde{t}\tau} \in \mathcal{E}(G, s)$ and $\chi \in \mathcal{E}(G, s)$ are both constituents of Γ . From this we deduce that $\chi^{\tilde{t}\tau} = \chi$. Moreover, as $\tilde{t}$ is F_0 -stable the character $\hat{\phi}_0$, and therefore also $\hat{\Gamma}$, is $\tilde{t}\tau$ -stable. It follows that $\hat{\chi}$ is $\tilde{t}\tau$ -stable as well. Thus, $\tilde{G}(\tilde{G}\mathcal{G})_{\hat{\chi}} = \tilde{G}\mathcal{G}_{\mathcal{E}(G, s)}$. $\square$

Now, letting $P \in \mathrm{Syl}_2(G)$, there is an appropriate $\mathrm{Aut}(G)_P$ -stable subgroup M satisfying $N_G(P) \leq M < G$, defined as in [IMN07, Theorem 15.3] if $n = 1$, [Mal07, Theorem 7.8] if $q \equiv 1 \pmod{8}$ with $n \geq 2$, and as in [Mal08, Section 4.4] otherwise. Throughout the next proof, we let M be this group.

Proof of Theorem A for $S = \mathrm{PSp}_{2n}(q)$. Keep the notation from before. Let $P \in \mathrm{Syl}_2(G)$ and write $\mathcal{H} := \mathcal{H}_2$. Then for $n \geq 2$, by [SF21, Theorem 6.2 and Proposition 7.3], there is an $(\mathrm{Aut}(G)_P \times \mathcal{H})$ -equivariant bijection

$$\Omega: \mathrm{Irr}_{2'}(G) \rightarrow \mathrm{Irr}_{2'}(M).$$

When $n = 1$, note that the assumption S is simple implies $q \geq 5$. Further, the case $S = \mathrm{PSL}_2(9) \cong \mathfrak{A}_6 \cong \mathrm{B}_2(2)'$ has been handled in [Joh20, Proposition 5.3]. Hence we may assume G is the universal covering group for S . The odd-degree characters of G are the two unipotent characters $1_G, \mathrm{St}_G$, and the two characters of

degree $\frac{1}{2}(q + \epsilon)$, where $q \equiv \epsilon \pmod{4}$ with $\epsilon \in \{\pm 1\}$. The two unipotent characters are rational-valued, and the remaining two odd-degree characters are stabilized by $\mathcal{H}$ when $q \equiv \pm 1 \pmod{8}$ and are fixed by $\tau \in \mathcal{H}$ if and only if τ fixes $\sqrt{\epsilon p}$ when $q \equiv \pm 3 \pmod{8}$. When $q \equiv \pm 1 \pmod{8}$, the group M is the normalizer N_0 of a Sylow $d_2(q)$ -torus, which is of the form $C_{q-\epsilon}.2$ with the $.2$ acting via inversion and induced by an element c such that c^2 has order 2. Here $C_{q-\epsilon}.2/\langle c^2 \rangle$ is a semidirect product $C_{q-\epsilon} \rtimes C_2$. In this case, the odd-degree characters of M are the extensions to M of the trivial character and the unique character of order 2 of $C_{q-\epsilon}$, both of which are trivial on c^2 . It follows that the members of $\text{Irr}_{2'}(M)$ here are then rational-valued. When $q \equiv \pm 3 \pmod{8}$, the group M is isomorphic to $\text{SL}_2(3)$, so the set $\text{Irr}_{2'}(M)$ is analogous to $\text{Irr}_{2'}(G)$, comprised of $\{1_M, \text{St}_M\}$ and two characters of degree 1 whose behavior under $\mathcal{H}$ is the same as the non-unipotent characters in $\text{Irr}_{2'}(G)$. From this, we can see that the $\text{Aut}(G)_P$ -equivariant bijection $\Omega: \text{Irr}_{2'}(G) \rightarrow \text{Irr}_{2'}(M)$ in [IMN07, Theorem 15.3] is also $\mathcal{H}$ -equivariant.

In all cases, it therefore suffices to show that

$$(G \rtimes \text{Aut}(G)_{P, \chi^{\mathcal{H}}}, G, \chi)_{\mathcal{H}} \geq_c (M \rtimes \text{Aut}(G)_{P, \chi^{\mathcal{H}}}, M, \Omega(\chi))_{\mathcal{H}}$$

for all $\chi \in \text{Irr}_{2'}(G)$, in the notation of [NSV20, Definition 1.5]. Let $\widetilde{M} := MN_{\widetilde{G}}(P)$. Since $\widetilde{G}D$ induces all automorphisms of G , and M can be chosen to be D -stable, [NSV20, Theorem 2.9] further implies that it suffices to show

$$((\widetilde{G}D)_{\chi^{\mathcal{H}}}, G, \chi)_{\mathcal{H}} \geq_c ((\widetilde{M}D)_{\chi^{\mathcal{H}}}, M, \Omega(\chi))_{\mathcal{H}}$$

for all $\chi \in \text{Irr}_{2'}(G)$. We remark that the groups involved satisfy part (i) of [NSV20, Definition 1.5] and that since Ω is $\widetilde{M}D \times \mathcal{H}$ -equivariant, part (ii) is also satisfied.

Let $\chi \in \text{Irr}_{2'}(G)$. By [Mal08, Theorem 2.5 and Proposition 4.9], we have $(\widetilde{G}D)_{\chi} = \widetilde{G}D$ if χ is a unipotent character and $(\widetilde{G}D)_{\chi} = GZ(\widetilde{G}) \rtimes D_{\chi}$ otherwise.

Note that since $Z(G)$ is a 2-group, χ is trivial on $Z(G)$ (since χ has odd degree and G is perfect), so through deflation and inflation, we may view χ as a character of $GZ(\widetilde{G})$ trivial on $Z(\widetilde{G})$. Throughout, we will sometimes make this identification. In particular, since the maps Ω are constructed so that χ and $\Omega(\chi)$ lie over the same character of $Z(G)$, their extensions to $(\widetilde{G}D)_{\chi}$ and $(\widetilde{M}D)_{\chi}$, which exist by [Mal08, Proposition 4.9] (see also the proofs of [Mal08, Theorems 4.10 and 4.11]), may be taken to lie over the same (namely, trivial) character of $Z(\widetilde{G})$, giving part (iii) of [NSV20, Definition 1.5]. Hence, it suffices to find extensions of χ and $\Omega(\chi)$ to $(\widetilde{G}D)_{\chi}$ and $(\widetilde{M}D)_{\chi}$, respectively, satisfying part (iv) of [NSV20, Definition 1.5]. We also note that by [Mal08, Proposition 4.5], χ lies in a rational Lusztig series $\mathcal{E}(G, s)$ with $s^2 = 1$.

(0) Let $n = 1$, so $G = \text{SL}_2(q)$. Let $\tau \in \mathcal{H}$ and let $\chi \in \text{Irr}_{2'}(G)$. Note that χ is either τ -invariant or $\chi^{\tau} = \chi^{\tilde{t}}$ for $\tilde{t}$ inducing the non-trivial diagonal automorphism in $\text{Out}(G)$. First suppose that $q \equiv \pm 1 \pmod{8}$. Then in the proof of Lemma 5.1, we may take $\tilde{t} = 1$, since all members of $\text{Irr}_{2'}(G)$ are $\mathcal{H}$ -invariant (see the discussion in the second paragraph of the proof). Hence, there is a τ -invariant extension $\hat{\chi}$ to $G\langle F_0 \rangle$. Recalling that the odd-degree characters of M are linear and rational-valued here, we may extend them trivially to $M\langle F_0 \rangle$ as well, to obtain the desired extensions when χ is not unipotent. When $\chi \in \{1_G, \text{St}_G\}$, there is a rational extension to $\widetilde{G}D$. Here $\Omega(\chi)$ is trivial on $C_{q-\epsilon}$ and we have $\widetilde{M} = \widetilde{C}.2$, where $\widetilde{C}$ is an abelian group containing $C_{q-\epsilon}$. Then taking the two extensions of

the trivial character of $\tilde{C}$, we may argue as before to get extensions of $\Omega(\chi)$ to $\widetilde{M}\langle F_0 \rangle$.

Now let $q \equiv \pm 3 \pmod{8}$. Then since $M \cong \mathrm{SL}_2(3)$, the required extensions exist if χ (and hence $\Omega(\chi)$) are unipotent, as before. So assume that χ is one of the non-unipotent members of $\mathrm{Irr}_{2'}(G)$. Since F_0 centralizes M , $\Omega(\chi)$ extends to a character of $M\langle F_0 \rangle$ with the same field of values as $\Omega(\chi)$. Let $\tilde{t}$ be as in the proof of Lemma 5.1. Recall that χ is τ -invariant if and only if $\Omega(\chi)$ is, in which case we may take $\tilde{t} = 1$. If $\chi^\tau \neq \chi$, we have $\chi^{\tilde{t}\tau} = \chi$ with $\tilde{t}$ inducing the non-trivial diagonal automorphism, and the same is true for $\Omega(\chi)$. (Note that such a $\tilde{t}$ also induces the non-trivial diagonal automorphism on $M \cong \mathrm{SL}_2(3)$.) By Lemma 5.1, we have an extension $\hat{\chi}$ to $G\langle F_0 \rangle$ that also satisfies $\hat{\chi}^{\tilde{t}\tau} = \hat{\chi}$. Since the same holds for $\Omega(\chi)$, we have obtained the desired extensions.

(1) Now, suppose that $n \geq 2$ and $q \equiv 1 \pmod{8}$, and let $\chi \in \mathrm{Irr}_{2'}(G)$. Then by [SFT20, Lemma 4.10 and Theorem B], $\mathbb{Q}(\chi) = \mathbb{Q}$ and hence $\chi^{\mathcal{H}} = \chi$, so we aim to show

$$((\tilde{G}D)_\chi, G, \chi)_{\mathcal{H}} \geq_c ((\tilde{M}D)_\chi, M, \Omega(\chi))_{\mathcal{H}}$$

in this case. Further, by [MS16, Lemma 7.5 and Theorem 7.7], χ lies in a principal series $\mathcal{E}(G, (T, \delta))$ with $\delta^2 = 1$. Note that δ is $D \times \mathcal{H}$ -invariant, so $\mathcal{H} = \mathcal{H}_\chi = \mathcal{H}_\delta$. In this case, the group M is $N_G(\mathbf{T})$, where $\mathbf{T}$ is a maximally split torus in $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{F}_q)$, and the map Ω is induced by one as in Lemma 3.3(2).

Now, Lemma 3.3 yields an extension $\hat{\chi}$ of χ to $G\langle F_0 \rangle$ and $\hat{\psi}$ of $\Omega(\chi)$ to $M\langle F_0 \rangle$ that are each $\mathcal{H}$ -invariant. If χ is not unipotent, this implies that there are $\mathcal{H}$ -invariant extensions to $(\tilde{G}D)_\chi = GZ(\tilde{G}) \rtimes D_\chi$ and $(\tilde{M}D)_\chi = MZ(\tilde{G}) \rtimes D_\chi$, giving part (iv) of [NSV20, Definition 1.5] in this case.

Now assume χ is unipotent. Since χ is unipotent and lies in a principal series, we have $\chi \in \mathcal{E}(G, (T, 1))$ and extends to a unipotent character $\tilde{\chi}$ in the principal series $\mathcal{E}(\tilde{G}, (\tilde{T}, 1))$ for $\tilde{G}$. The character $\tilde{\chi}$ is also rational-valued and satisfies $D_{\tilde{\chi}} = D_\chi$. On the other hand, $\Omega(\chi)$ may be identified with an odd-degree character of $W = M/T \cong \tilde{M}/\tilde{T}$. Then the same arguments as in Lemma 3.3 yields a rational-valued character of $\tilde{G}\langle F_0 \rangle$ extending $\tilde{\chi}$, and hence χ , as well as a rational-valued character of $\tilde{M}\langle F_0 \rangle$ extending $\Omega(\chi)$, again giving part (iv) of [NSV20, Definition 1.5].

We may therefore assume that $n \geq 2$ and q is an odd power of an odd prime.

(2) Assume $n = 2^k \geq 2$ is a power of 2. Here M is of the form $\mathrm{Sp}_n(q) \wr C_2$ and the non-unipotent members of $\mathrm{Irr}_{2'}(X)$ for $X \in \{G, \mathrm{Sp}_n(q)\}$ lie in a single principal series $\mathcal{E}(X, (T_X, \delta_X))$ with T_X a maximally split torus of X and $\delta_X \in \mathrm{Irr}(T_X)$ with $\delta_X^2 = 1$. The map Ω sends unipotent characters to the extensions in M of characters $\mu \otimes \mu \in \mathrm{Irr}(\mathrm{Sp}_n(q) \times \mathrm{Sp}_n(q))$ for unipotent μ and sends non-unipotent characters to corresponding extensions for non-unipotent μ . (See [SF21, Lemma 7.2] for details.)

The same arguments as above yield extensions of χ to $\tilde{G}_\chi\langle F_0 \rangle$, and of μ to $\mathrm{CSp}_n(q)_\mu\langle F_0 \rangle$ that are invariant under any $\tau \in \mathcal{H}_\chi$. Here since F_0 induces the action of F_p on each $\mathrm{Sp}_n(q)$ component, we by an abuse of notation write F_0 for the corresponding field automorphism of $\mathrm{Sp}_n(q)$. Further, note that the action of C_2 on $\mathrm{Sp}_n(q) \times \mathrm{Sp}_n(q)$ (viewed as being embedded into G block-diagonally) is induced by the matrix $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, so we see that F_0 commutes with this element.

With this, we have an extension of $\Omega(\chi)$ to $\widetilde{M}_\chi\langle F_0 \rangle$ invariant under such a τ ,

applying [SF21, Lemma 5.5]. If $q \equiv 7 \pmod{8}$, $\mathcal{H}_\chi = \mathcal{H}$ by [SFT20, Lemma 4.10 and Theorem B] and this completes part (iv) of [NSV20, Definition 1.5].

So, suppose $q \equiv \pm 3 \pmod{8}$ and $\tau \in \mathcal{H}$ does not fix χ . Then χ is non-unipotent and we have $\chi \in \mathcal{E}(G, (T, \delta))$ with $\delta^2 = 1$, as above. Suppose that $a \in \widetilde{MD}$ satisfies $\chi^{a\tau} = \chi$. Then since χ is MD -invariant, we may without loss assume that $a \in \widetilde{T}$ induces the non-trivial diagonal automorphism of order 2 in $\text{Out}(G)$ and that $[a, F_0] = 1$. Here $\widetilde{T}$ is a maximally split torus of $\widetilde{G}$ containing T . Note then that a preserves $\mathbf{T}$ and $\delta^{a\tau} = \delta$.

As in Lemma 3.3, extending δ to $\widehat{\delta}$ in $\text{Irr}(T\langle F_0 \rangle)$ via $\widehat{\delta}(F_0) = 1$ yields $\widehat{\delta}^{a\tau} = \widehat{\delta}$. Let $\widehat{\chi} \in \text{Irr}(G\langle F_0 \rangle | \chi)$ be such that $\langle \mathbf{R}_{T\langle F_0 \rangle}^{G\langle F_0 \rangle} \widehat{\delta}, \widehat{\chi} \rangle \neq 0$ as in Proposition 2.6. Note that $\langle \mathbf{R}_{T\langle F_0 \rangle}^{G\langle F_0 \rangle} \widehat{\delta}, \widehat{\chi}^{a\tau} \rangle \neq 0$ and that $\widehat{\chi}^{a\tau}$ lies above χ . This forces $\widehat{\chi}^{a\tau} = \widehat{\chi}$ by Proposition 2.6. This then implies that there is an $a\tau$ -invariant extension of χ to $(\widetilde{GD})_\chi$.

Arguing analogously for the characters of $\text{Sp}_n(q)$ and noting that a induces the corresponding diagonal automorphism on the two $\text{Sp}_n(q)$ components, we obtain an $a\tau$ -invariant extension of $\mu \otimes \mu$ to $(\text{Sp}_n(q) \times \text{Sp}_n(q))\langle F_0 \rangle$, and hence an $a\tau$ -invariant extension of $\Omega(\chi)$ to $M\langle F_0 \rangle$, and hence $(\widetilde{MD})_\chi$. (Indeed, since a may be chosen to commute with the C_2 -action, arguing exactly as in [SF21, Lemma 5.5] for $a\tau$ yields such an extension.) This completes part (iv) of [NSV20, Definition 1.5] in this case.

(3) Now assume the 2-adic decomposition of n is $n = \sum_{j \in J} 2^j$ with $|J| \geq 2$, and write $k := \max\{j \in J\}$ and $m := 2^k$. Then in this case, $M = \text{Sp}_{2(n-m)}(q) \times \text{Sp}_{2m}(q)$. Write $M_1 := \text{Sp}_{2(n-m)}(q)$ and $M_2 := \text{Sp}_{2m}(q)$. The map Ω sends unipotent characters in $\text{Irr}_{2'}(G)$ to products of unipotent characters in $\text{Irr}_{2'}(M)$. Again, every odd-degree unipotent character of $\text{Irr}_{2'}(X)$ for $X \in \{G, M_1, M_2\}$ lies in the principal series, and hence arguing as before yields part (iv) of [NSV20, Definition 1.5] if χ is unipotent.

We therefore assume χ is not unipotent, so $\Omega(\chi) = \chi_1 \otimes \chi_2 \in \text{Irr}_{2'}(M_1) \otimes \text{Irr}_{2'}(M_2)$ with at least one of χ_1 or χ_2 non-unipotent. Note that the diagonal and field automorphisms of G induce the corresponding automorphism on M_1 and M_2 . Also, recall that non-unipotent members of $\text{Irr}_{2'}(M_2)$ lie in a single principal series as in part (2). Further, if both χ and χ_1 also lie in a principal series, then arguing as in part (2) gives the desired extensions for χ, χ_1 , and χ_2 , and gives part (iv) of [NSV20, Definition 1.5].

Now assume χ does not lie in a principal series. Then $q \equiv 3 \pmod{4}$, n is odd, and χ lies in a Harish-Chandra series $\mathcal{E}(G, (L, \delta))$ where $L \cong \text{Sp}_2(q) \times T_0$ with $T_0 = \mathbf{T}_0^F$ a maximally split torus of $\text{Sp}_{2n-2}(q)$ and $\delta = \psi \otimes \delta_0$ with $\delta_0 \in \text{Irr}(T_0)$ satisfying $\delta_0^2 = 1$ and ψ one of the two characters of $\text{Sp}_2(q)$ of degree $\frac{q-1}{2}$. (See [MS16, Theorem 7.7].) Analogous odd-degree characters exist for M_1 , and the map Ω is constructed so that χ_1 is also of this form. Say $\chi_1 \in \mathcal{E}(M_1, (L_1, \delta_1))$ with $\delta_1 := \psi \otimes \delta_{0,1} \in \text{Irr}(\text{Sp}_2(q)) \otimes \text{Irr}(T_{0,1})$. (Here ψ is the same as for χ .)

Now, [SFT20, Theorem B], together with the explicit knowledge of the values of ψ , yields that χ is fixed by $\tau \in \mathcal{H}$ if and only if ψ is, and that both are fixed by τ^2 . Further, recall that [Mal08, Theorem 4.9] yields that χ and ψ have the same stabilizer in $\text{Out}(G)$. Note that F_0 induces the corresponding field automorphism on the $\text{Sp}_2(q)$ and T_0 components of L and that $N_G(\mathbf{L}) = \text{Sp}_2(q) \times N_{\text{Sp}_{2n-2}(q)}(\mathbf{T}_0)$. Then since ψ is F_0 -invariant, we see δ satisfies the assumptions of Proposition 2.6 since δ_0 does in $\text{Sp}_{2n-2}(q)$, being of the same form as the case of principal series characters in $\text{Sp}_{2n-2}(q)$.

Assume that $\chi^{a\tau} = \chi$ for $a \in \widetilde{MD}$ and $\tau \in \mathcal{H}$. If a is non-trivial, we may again assume that $a \in \widetilde{T}$ induces the non-trivial diagonal automorphism and that it further induces a non-inner diagonal automorphism on $\mathrm{Sp}_2(q)$ and T_0 . As in part (2), δ_0 is invariant under τ, F_0 , and a .

As in part (2), δ_0 may be extended to a character $\widehat{\delta}_0$ of $T_0\langle F_0 \rangle$ satisfying $\widehat{\delta}_0^{a\tau} = \widehat{\delta}_0$. Let $\widehat{\psi}$ be some extension of ψ to a character of $\mathrm{Sp}_2(q)\langle F_0 \rangle$ with $\widehat{\psi}^{a\tau} = \widehat{\psi}\lambda$ for $\lambda \in \mathrm{Irr}(\langle F_0 \rangle)$. Then the extension $\widehat{\delta} := (\widehat{\psi} \otimes \widehat{\delta}_0)_{L\langle F_0 \rangle}$ of δ to $L\langle F_0 \rangle$ satisfies $\widehat{\delta}^{a\tau} = \widehat{\delta}\lambda$. Let $\widehat{\chi} \in \mathrm{Irr}(G\langle F_0 \rangle|\chi)$ be such that $\langle R_{L\langle F_0 \rangle}^{G\langle F_0 \rangle} \widehat{\delta}, \widehat{\chi} \rangle \neq 0$ as in Proposition 2.6. Note that $0 \neq \langle R_{L\langle F_0 \rangle}^{G\langle F_0 \rangle} \widehat{\delta}^{a\tau}, \widehat{\chi}^{a\tau} \rangle = \langle R_{L\langle F_0 \rangle}^{G\langle F_0 \rangle} \widehat{\delta}\lambda, \widehat{\chi}^{a\tau} \rangle = \langle R_{L\langle F_0 \rangle}^{G\langle F_0 \rangle} \widehat{\delta}, \widehat{\chi}^{a\tau}\lambda^{-1} \rangle$ and that $\widehat{\chi}^{a\tau}\lambda^{-1}$ lies above χ . This forces $\widehat{\chi}^{a\tau}\lambda^{-1} = \widehat{\chi}$ by Proposition 2.6, and hence $\widehat{\chi}^{a\tau} = \widehat{\chi}\lambda$.

Now, since Ω is $\widetilde{MD}$ -equivariant, we have $\chi_1^{a\tau} = \chi_1$ and $\chi_2^{a\tau} = \chi_2$. Recalling that δ_1 is defined using the same ψ as for δ , the exact same arguments can be made for δ_1 and χ_1 , so there is an extension $\widehat{\chi}_1$ of χ_1 to $M_1\langle F_0 \rangle$ such that $\widehat{\chi}_1^{a\tau} = \widehat{\chi}_1\lambda$. The argument in part (2) yields an extension $\widehat{\chi}_2$ of χ_2 to $M_2\langle F_0 \rangle$ such that $\widehat{\chi}_2^{a\tau} = \widehat{\chi}_2$. Together, the extension $\widehat{\Omega}(\chi) := (\widehat{\chi}_1 \otimes \widehat{\chi}_2)_{M\langle F_0 \rangle}$ of $\Omega(\chi)$ to $M\langle F_0 \rangle$ satisfies $(\widehat{\Omega}(\chi))^{a\tau} = \widehat{\Omega}(\chi)\lambda$, giving part (iv) of [NSV20, Definition 1.5] in this final case and completing the proof. $\square$

6. THE PROOF OF THEOREM A: REMAINING CASES

We keep the notation of Section 4.1. We now assume that $S = G/Z(G)$ is one of the simple groups listed in Theorem A. Then $\widetilde{G}D$, where D is a well-chosen group of field and graph automorphisms, induces all automorphisms of S .

It will also be useful to keep the notation of [SF21, Section 2.2], so that $v \in \mathbf{G}$ is the canonical representative in the extended Weyl group of G of the longest element in the Weyl group $\mathbf{W} := \mathbf{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, which induces an isomorphism $G \cong \mathbf{G}^{vF}$; $\mathbf{T} = \mathbf{N}_{\mathbf{G}}(\mathbf{S})$ for a Sylow d -torus $\mathbf{S}$ of $(\mathbf{G}, vF)$; $T_1 := \mathbf{T}^{vF}$; and $N_1 := \mathbf{N}_{\mathbf{G}^{vF}}(\mathbf{S})$. Further let $\widetilde{N}_1 := \mathbf{N}_{\widetilde{\mathbf{G}}^{vF}}(\mathbf{S})$. (We remark that when $d = 1$, we have $T_1 = T$, $N_1 = N$, and $\widetilde{N}_1 = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{T})$.)

Proof of Theorem A. From Section 5, we may assume that $S \neq \mathrm{PSp}_{2n}(q)$ is one of the remaining groups in Theorem A. Further, recall that we assume that G is the universal covering group of S . (That is, $S \neq \mathrm{B}_3(3)$.) Let $P \in \mathrm{Syl}_2(G)$, $\mathcal{H} := \mathcal{H}_2$, and let $d \in \{1, 2\}$ be the order of q modulo 4.

By [SF21, Theorem 6.2 and Proposition 8.1], there is an $\mathrm{Aut}(G)_P \times \mathcal{H}$ -equivariant bijection

$$\Omega: \mathrm{Irr}_{2'}(G) \rightarrow \mathrm{Irr}_{2'}(M),$$

where $M := \mathbf{N}_G(\mathbf{S}_0)$ for a Sylow d -torus $\mathbf{S}_0$ of $(\mathbf{G}, F)$. Let $\chi \in \mathrm{Irr}_{2'}(G)$. By [SF21, Proposition 8.1], χ is rational-valued, so that $\chi^{\mathcal{H}} = \chi$. By [MS16, Lemma 7.5 and Theorem 7.7] and [SF21, Lemma 2.6], χ lies in a principal series $\mathcal{E}(G, (T, \delta))$ with $\delta^2 = 1$. Note also that Ω is as in Lemma 3.3(2) if $q \equiv 1 \pmod{4}$.

Write $\psi := \Omega(\chi)$ and $\widetilde{M} := \mathbf{N}_{\widetilde{G}}(\mathbf{S}_0)$. Note that, as in the proof for symplectic groups, it again suffices to show

$$((\widetilde{G}D)_{\chi}, G, \chi)_{\mathcal{H}} \geq_c ((\widetilde{M}D)_{\chi}, M, \psi)_{\mathcal{H}}$$

for all $\chi \in \text{Irr}_{2'}(G)$, and that we again have parts (i) and (ii) of [NSV20, Definition 1.5] are satisfied. Note that $D = \langle F_p \rangle$ for a generating field automorphism F_p . When $q \equiv 3 \pmod{4}$, by applying [NSV20, Lemma 2.1], it will be useful to identify M with N_1 , G with $\mathbf{G}^{vF}$, and $\widetilde{M}$ with $\widetilde{N}_1$, in which case there is a field automorphism $\hat{F}_p$ that acts on $\mathbf{G}^{vF}$ as F_p and so that T_1 and N_1 are stabilized by $\hat{F}_p$ (see Notation 3.3 and the proof of Proposition 3.4 of [MS16]).

Now, note that $|\widetilde{G}/GZ(\widetilde{G})|$ divides 2 and that $D_\chi = \langle F_0 \rangle = D_\psi$ for some field automorphism F_0 . By [GLS98, Theorem 2.5.12(g)], we have F_0 commutes with the action of $\widetilde{G}$ up to conjugation by G . Write $\hat{G} = G\langle F_0 \rangle$ and $\hat{M} = M\langle F_0 \rangle$.

By [MS16, Corollary 5.3], we have $(\widetilde{G}D)_{\chi_0} = \widetilde{G}_{\chi_0}D_{\chi_0}$ for some $\widetilde{G}$ -conjugate χ_0 of χ . However, this gives $(\widetilde{G}D)_{\underline{\chi}} = \widetilde{G}_{\underline{\chi}}D_{\underline{\chi}}$ since D and $\widetilde{G}$ commute up to the action of G . We further have $(\widetilde{M}D)_{\underline{\psi}} = \widetilde{M}_{\underline{\psi}}D_{\underline{\psi}}$, arguing similarly and using [MS16, Theorems 3.1 and 3.18]. We have part (iii) of [NSV20, Definition 1.5] by [Spä12, Lemma 2.11] and [Spä12, Lemma 2.13] and its proof. We are therefore left to prove that part (iv) of [NSV20, Definition 1.5] holds.

Let $a\tau \in \widetilde{M}D \times \mathcal{H}$ such that $\chi^{a\tau} = \chi$. Then $a \in (\widetilde{M}D)_{\underline{\chi}}$ since χ is rational-valued. By Lemma 4.2, it suffices to show that there are extensions $\widetilde{\chi}$ and $\hat{\chi}$, respectively $\widetilde{\psi}$ and $\hat{\psi}$, of χ to $\widetilde{G}_{\underline{\chi}}$ and $\hat{G}$, resp. of ψ to $\widetilde{M}_{\underline{\psi}}$ and $\hat{M}$, such that if $\widetilde{\chi}^{a\tau} = \mu_1\widetilde{\chi}$, $\widetilde{\psi}^{a\tau} = \mu'_1\widetilde{\psi}$, $\hat{\chi}^{a\tau} = \mu_2\hat{\chi}$, $\hat{\psi}^{a\tau} = \mu'_2\hat{\psi}$, then $\mu_i = \mu'_i$ for $i = 1, 2$. (Throughout, we will identify the characters of $\widetilde{T}/T$, $\widetilde{G}/G$, and $\widetilde{M}/M$, and similar for $\hat{T}/T$, $\hat{G}/G$ and $\hat{M}/M$.)

We first show that there are appropriate extensions $\hat{\chi}$ and $\hat{\psi}$ as above such that $\mu_2 = \mu'_2$. We may assume that $a \in \widetilde{M}_{\underline{\chi}}$ since certainly extensions $\hat{\chi}$ and $\hat{\psi}$ to $\hat{G}$ and $\hat{M}$ will be F_0 -invariant. Hence we may assume $a \in \widetilde{T}_{\underline{\chi}}$. Note that a preserves $\hat{T}$. If $q \equiv 1 \pmod{4}$, then Lemma 3.3 yields extensions $\hat{\chi}$ and $\hat{\psi}$ such that $\hat{\chi}^\tau = \hat{\chi}$ and $\hat{\psi}^\tau = \hat{\psi}$. Let $\mu_2 \in \text{Irr}(\hat{G}/G)$ such that $\hat{\chi}^a = \mu_2\hat{\chi}$. In the notation of the proof of Lemma 3.3, we have $\langle R_{\hat{T}}^{\hat{G}}\hat{\delta}^a, \hat{\chi}^a \rangle \neq 0$. This forces $\hat{\delta}^a = \mu_2\hat{\delta}$ by Proposition 2.6, since $R_{\hat{T}}^{\hat{G}}(\mu_2\hat{\delta}) = \mu_2 R_{\hat{T}}^{\hat{G}}\hat{\delta}$ by [Isa06, Problem (5.3)]. Now, recall that $\psi = \text{Ind}_{N_\delta}^N(\Lambda(\delta)\gamma)$ and $\hat{\psi} = \text{Ind}_{\hat{N}_\delta}^{\hat{N}}(\hat{\Lambda}(\hat{\delta})\gamma)$, where Λ is the extension map with respect to $T \triangleleft N$ studied in [SF21, Section 4], γ is a linear character of $N_\delta/T \cong \hat{N}_\delta/\hat{T}$, and $\hat{\Lambda}(\hat{\delta})$ is the unique common extension of $\Lambda(\delta)$ and $\hat{\delta}$ to $\hat{N}_\delta$. Let $\nu \in N_\delta/T$ be such that $\Lambda(\delta)^a = \Lambda(\delta)\nu$. Since $\psi^a = \psi$, the uniqueness of Clifford correspondence and [MS16, Proposition 3.15] yields that $\Lambda(\delta)\nu\gamma^a = \Lambda(\delta)\nu\gamma = \Lambda(\delta)\gamma$. That is, $\Lambda(\delta)^a = \Lambda(\delta)$ and $\gamma^a = \gamma$. Then it follows that $\hat{\Lambda}(\hat{\delta})^a = \mu_2\hat{\Lambda}(\hat{\delta})$ and $\hat{\psi}^a = \text{Ind}_{\hat{N}_\delta}^{\hat{N}}(\mu_2\hat{\Lambda}(\hat{\delta})\gamma) = \mu_2\hat{\psi}$, completing the claim in this case.

If $q \equiv 3 \pmod{4}$, note that q is not a square, so F_0 has odd order. Here $\text{Irr}_{2'}(N_1)$ is in bijection with pairs (δ_1, η_1) where $\delta_1 \in \text{Irr}(T_1)$ satisfies $[N_1 : (N_1)_{\delta_1}]$ is odd, $\delta_1^2 = 1$, and $\eta_1 \in \text{Irr}_{2'}(W_1(\delta_1))$. Here we define $W_1(\delta_1) := (N_1)_{\delta_1}/T_1$. Now, the member of $\text{Irr}_{2'}(N_1)$ corresponding to (δ_1, η_1) is of the form $\text{Ind}_{(N_1)_{\delta_1}}^{N_1}(\Lambda_1(\delta_1)\eta_1)$, where Λ_1 is an extension map with respect to $T_1 \triangleleft N_1$. (See [SF21, Section 8].) Then in this case, since the order of F_0 is relatively prime to $o(\delta)$ and $o(\delta_1)$, we may appeal to Lemma 3.3(1) and Lemma 3.1 with $(X, Y, \alpha, \beta) = (T_1, N_1, \hat{F}_0, a)$, to obtain the extensions such that $\mu_2 = \mu'_2$.

Finally, it remains to show that we may obtain extensions $\tilde{\chi}$ and $\tilde{\psi}$ as described above such that $\mu_1 = \mu'_1$ when $\tilde{G}_\chi = \tilde{G}$ in the cases of $B_n(q)$ or $E_7(q)$. It suffices here to assume that $a \in D_\chi = \langle F_0 \rangle$, since certainly an extension of χ or ψ to $\tilde{G}$ or $\tilde{M}$ is fixed by $\tilde{M}$. Further, note that $|Z(G)| = 2$, so $\chi \in \text{Irr}_{2'}(G)$ is necessarily trivial on the center. Hence we may trivially extend χ to $\chi' \in \text{Irr}_{2'}(GZ(\tilde{G}))$, and we may similarly extend ψ to $\psi' \in \text{Irr}_{2'}(MZ(\tilde{G}))$. Then since $|\tilde{G}/GZ(\tilde{G})| = 2$, we wish to show that $\tilde{\chi} \in \text{Irr}(\tilde{G}|\chi')$ is fixed by $a\tau$ if and only if $\tilde{\psi} \in \text{Irr}(\tilde{M}|\psi')$ is. Now, from [MS16, Theorem 6.3], we see there is a D_χ -equivariant map $\text{Irr}(\tilde{G}|\chi') \rightarrow \text{Irr}(\tilde{M}|\psi')$, so it suffices to show that $\tilde{\chi}$ is fixed by τ if and only if $\tilde{\psi}$ is.

Now, recall that χ lies in a principal series $\mathcal{E}(G, (T, \delta))$ with $\delta^2 = 1$. Let $\tilde{T} = \tilde{\mathbf{T}}^F$, where $\tilde{\mathbf{T}} := \mathbf{T}Z(\tilde{\mathbf{G}})$ is a maximally split torus of $\tilde{\mathbf{G}}$ containing $\mathbf{T}$. Then an application of Mackey's theorem yields that $\tilde{\chi} \in \mathcal{E}(\tilde{G}, (\tilde{T}, \tilde{\delta}))$ for some $\tilde{\delta} \in \text{Irr}(\tilde{T}|\delta)$. Since the corresponding relative Weyl group $N_{\tilde{G}}(\tilde{\mathbf{T}})_{\tilde{\delta}}/\tilde{T}$ is a Weyl group, [SF19, Theorem 3.8] (see also [Gec03, Proposition 5.5]) implies that $\tilde{\chi}^\tau = \tilde{\chi}$ if and only if $\tilde{\delta}^\tau = \tilde{\delta}$.

On the other hand, recall that $\psi = \text{Ind}_{(N_1)_{\delta_1}}^{N_1}(\Lambda_1(\delta_1)\eta_1)$. Note that

$$(1) \quad \Lambda_1(\delta_1)^\tau = \Lambda_1(\delta_1) = \Lambda_1(\delta_1^\tau)$$

by [SF21, Proposition 4.6 and the last paragraph of Proposition 8.1]. Let $\tilde{\delta}_1$ be an extension of δ_1 to $\tilde{\mathbf{T}}^{vF}$ and let $\tilde{\Lambda}(\tilde{\delta}_1) \in \text{Irr}((\tilde{N}_1)_{\tilde{\delta}_1})$ be the unique common extension of $\Lambda_1(\delta_1)|_{(N_1)_{\tilde{\delta}_1}}$ and $\tilde{\delta}_1$ as in the proof of [MS16, Proposition 3.20]. Similarly, let $\tilde{\Lambda}(\tilde{\delta}_1^\tau)$ be the unique common extension of $\Lambda_1(\delta_1)|_{(N_1)_{\tilde{\delta}_1}}$ and $\tilde{\delta}_1^\tau$. Then using (1) and by uniqueness, this forces $\tilde{\Lambda}(\tilde{\delta}_1^\tau) = \tilde{\Lambda}(\tilde{\delta}_1)^\tau$.

Now, [Isa06, Problems (5.1)-(5.3)] imply that $\tilde{\psi}$ is of the form $\text{Ind}_{(\tilde{N}_1)_{\tilde{\delta}_1}}^{\tilde{N}_1}(\tilde{\Lambda}(\tilde{\delta}_1)\tilde{\eta}_1)$ for some linear $\tilde{\eta}_1 \in \text{Irr}((\tilde{N}_1)_{\tilde{\delta}_1}/\tilde{T}_1)$. Note that $\tilde{\eta}_1^\tau = \tilde{\eta}_1$ since it is a linear character of a real reflection group. Hence $\tilde{\psi}^\tau = \tilde{\psi}$ if and only if $\tilde{\delta}_1^\tau = \tilde{\delta}_1$. In the case $q \equiv 1 \pmod{4}$ (and hence $N = N_1 = M$), this completes the proof. Now, note that $\tilde{\delta}_1$ is in duality with $(\tilde{\mathbf{T}}^{vF}, \tilde{s}_1)$, where $\tilde{\chi}$ lies in the Lusztig series of $\tilde{s}_1$ when viewed as a character of $\tilde{\mathbf{G}}^{vF}$. But we also have $\tilde{\delta}$ is in duality with $(T, \tilde{s})$, where $\tilde{\chi}$ lies in the series $\mathcal{E}(\tilde{G}, \tilde{s})$. Hence when $q \equiv 3 \pmod{4}$, it remains to note that $\tilde{\delta}^\tau = \tilde{\delta}$ if and only if $\tilde{\delta}_1^\tau = \tilde{\delta}_1$, since they are linear and must have the same order. $\square$

ACKNOWLEDGMENTS

The authors thank G. Malle for helpful comments on an early preprint, as well as the anonymous referee for their careful reading and comments that improved the exposition.

REFERENCES

- [CE04] Marc Cabanes and Michel Enguehard, *Representation theory of finite reductive groups*, New Mathematical Monographs, vol. 1, Cambridge University Press, Cambridge, 2004, DOI 10.1017/CBO9780511542763. MR2057756
- [DM91] François Digne and Jean Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, vol. 21, Cambridge University Press, Cambridge, 1991, DOI 10.1017/CBO9781139172417. MR1118841

- [DM94] François Digne and Jean Michel, *Groupes réductifs non connexes* (French, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 3, 345–406. MR1272294
- [Gec03] Meinolf Geck, *Character values, Schur indices and character sheaves*, Represent. Theory **7** (2003), 19–55, DOI 10.1090/S1088-4165-03-00170-5. MR1973366
- [GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 2. Part I. Chapter G*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1996. General group theory, DOI 10.1090/surv/040.2. MR1358135
- [IMN07] I. M. Isaacs, Gunter Malle, and Gabriel Navarro, *A reduction theorem for the McKay conjecture*, Invent. Math. **170** (2007), no. 1, 33–101, DOI 10.1007/s00222-007-0057-y. MR2336079
- [Isa06] I. Martin Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423], DOI 10.1090/chel/359. MR2270898
- [Joh20] Birte Johansson, *On the inductive Galois–McKay condition for finite groups of Lie type in their defining characteristic*, Preprint, arXiv:2010.14837, 2020.
- [Mal91] Gunter Malle, *Darstellungstheoretische Methoden bei der Realisierung einfacher Gruppen vom Lie Typ als Galoisgruppen* (German), Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), Progr. Math., vol. 95, Birkhäuser, Basel, 1991, pp. 443–459, DOI 10.1007/978-3-0348-8658-1_20. MR1112174
- [Mal07] Gunter Malle, *Height 0 characters of finite groups of Lie type*, Represent. Theory **11** (2007), 192–220, DOI 10.1090/S1088-4165-07-00312-3. MR2365640
- [Mal08] Gunter Malle, *Extensions of unipotent characters and the inductive McKay condition*, J. Algebra **320** (2008), no. 7, 2963–2980, DOI 10.1016/j.jalgebra.2008.06.033. MR2442005
- [MS16] Gunter Malle and Britta Späth, *Characters of odd degree*, Ann. of Math. (2) **184** (2016), no. 3, 869–908, DOI 10.4007/annals.2016.184.3.6. MR3549625
- [Nav04] Gabriel Navarro, *The McKay conjecture and Galois automorphisms*, Ann. of Math. (2) **160** (2004), no. 3, 1129–1140, DOI 10.4007/annals.2004.160.1129. MR2144975
- [NSV20] Gabriel Navarro, Britta Späth, and Carolina Vallejo, *A reduction theorem for the Galois–McKay conjecture*, Trans. Amer. Math. Soc. **373** (2020), no. 9, 6157–6183, DOI 10.1090/tran/8111. MR4155175
- [Ruh21] Lucas Ruhstorfer, *The Navarro refinement of the McKay conjecture for finite groups of Lie type in defining characteristic*, J. Algebra **582** (2021), 177–205, DOI 10.1016/j.jalgebra.2021.04.025. MR4259262
- [SF19] A. A. Schaeffer Fry, *Galois automorphisms on Harish-Chandra series and Navarro’s self-normalizing Sylow 2-subgroup conjecture*, Trans. Amer. Math. Soc. **372** (2019), no. 1, 457–483, DOI 10.1090/tran/7590. MR3968776
- [SF21] A. A. Schaeffer Fry, *Galois-equivariant McKay bijections for primes dividing $q - 1$* , Israel J. Math., 2021, Online first, <https://doi.org/10.1007/s11856-021-2266-2>.
- [SFT20] A. A. Schaeffer Fry and Jay Taylor, *Galois automorphisms and classical groups*, Preprint, arXiv:2005.14088, 2020.
- [Spä12] Britta Späth, *Inductive McKay condition in defining characteristic*, Bull. Lond. Math. Soc. **44** (2012), no. 3, 426–438, DOI 10.1112/blms/bdr100. MR2966987

FACHBEREICH MATHEMATIK, TU KAISERSLAUTERN, 67653 KAISERSLAUTERN, GERMANY
 Email address: ruhstorf@mathematik.uni-kl.de

DEPARTMENT OF MATHEMATICS AND STATISTICS, METROPOLITAN STATE UNIVERSITY OF DENVER, DENVER, COLORADO 80217
 Email address: aschae6@msudenver.edu