



# On Global-in- $x$ Stability of Blasius Profiles

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Communicated by P. CONSTANTIN

## Abstract

We characterize the well known self-similar Blasius profiles,  $[\bar{u}, \bar{v}]$ , as downstream attractors to solutions  $[u, v]$  to the 2D, stationary Prandtl system. It was established in Serrin (Proc R Soc Lond A 299:491–507, 1967) using maximum principle techniques that  $\|u - \bar{u}\|_{L_y^\infty} \rightarrow 0$  as  $x \rightarrow \infty$ . In the case of localized data near Blasius, this paper provides an energy based proof of asymptotic stability. Central to our analysis is a new weighted “quotient estimate” which couples with a higher order, nonlinear energy cascade. Similar quotient estimates have played a crucial role in establishing the validity of the inviscid Prandtl layer expansion in Guo and Iyer (Validity of steady Prandtl layer expansions. [arXiv:1805.05891](https://arxiv.org/abs/1805.05891) 2018).

## 1. Introduction

The 2D, stationary, homogeneous Prandtl equations are given by

$$uu_x + vu_y - u_{yy} = 0, \quad u_x + v_y = 0, \quad (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+. \quad (1)$$

The system is typically supplemented with initial data at  $\{x = 0\}$  and boundary data at  $\{y = 0\}$ , and  $y \uparrow \infty$ :

$$u|_{x=0} = u_0(y), \quad [u, v]|_{y=0} = 0, \quad u|_{y \uparrow \infty} = u_E(x). \quad (2)$$

For simplicity, we will take  $u_E(x) = 1$ , but any constant will also work. The  $x$  direction is considered a time-like direction, while the  $y$ -direction is considered a space-like direction, and the Equation (1) is considered as an evolution in the  $x$

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S. Iyer Partially supported by NSF Grant DMS-1611695, DMS-1802940.

variable. Correspondingly,  $u_0(y)$  is called the “initial data” and as a general matter of terminology, in this paper the words “global” and “local” refer to the  $x$ -direction.

The following is a classical result due to Oleinik (see [5], P. 21, Theorem 2.1.1):

**Theorem 1.** [5] *Assume that*

$$u_0(y) > 0 \text{ for } y > 0,$$

$$u'_0(0) > 0,$$

$$u_0 \in C^\infty(y \geq 0),$$

$$u''_0(y) \sim y^2 \text{ near } y = 0,$$

$$|u_0(y) - 1| \text{ and } \partial_y^k u_0(y) \text{ decay exponentially for } k \geq 1.$$

Then there exists a global solution,  $[u, v]$  to (1) satisfying, for some  $y_0, m > 0$ ,

$$\sup_x \sup_{y \in (0, y_0)} |u, v, u_y, u_{yy}, u_x| \lesssim 1, \quad (3)$$

$$u_y(x, 0) > 0 \text{ and } u > 0. \quad (4)$$

Given the global existence of a solution to (1), the next point is to describe more precisely the asymptotics of the evolution as  $x \rightarrow \infty$ . In order to do this, let us introduce the self-similar Blasius solutions:

$$[\bar{u}, \bar{v}] = \left[ f'(\eta), \frac{1}{\sqrt{x+x_0}} \{ \eta f'(\eta) - f(\eta) \} \right], \quad \text{where } \eta = \frac{y}{\sqrt{x+x_0}}, \quad (5)$$

where  $f$  satisfies

$$ff'' + f''' = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \quad \frac{f(\eta)}{\eta} \xrightarrow{\eta \rightarrow \infty} 1. \quad (6)$$

Here,  $x_0 > 0$  is a free parameter. The following hold:

$$0 \leq f' \leq 1, \quad f''(\eta) \geq 0, \quad f''(0) > 0, \quad f'''(\eta) \leq 0. \quad (7)$$

We now recall the following result of Serrin:

**Theorem 2.** [6] *Let  $u$  be a solution to (1), (2) such that  $\partial_y u_0(y)$  is continuous. Then the following asymptotics hold:*

$$\|u - \bar{u}\|_{L_y^\infty} \rightarrow 0 \text{ as } x \rightarrow \infty. \quad (8)$$

First, let us mention that the results in [6] are more general than the theorem stated above in the sense that  $u_E(x)$  in (2) is allowed to have polynomial growth in  $x$ , whereas in the present paper we are only concerned with constant  $u_E$  (which corresponds to shear flow).

The purpose of the present work is to provide a new interpretation of [6] under the assumption of small, localized perturbations of the Blasius profile. More specifically, the point is to introduce energy methods as opposed to the maximum principle methods used in [6], which are more robust and have a hope of finding application to other realms, particularly in the study of the inviscid limit.

### 1.1. Asymptotic Stability and Change of Coordinates

When studying the asymptotic stability of  $\bar{u}$ , we need an appropriate notion of *the difference* between the two functions,  $u, \bar{u}$ , which, in our case, does not coincide with pointwise subtraction  $u(x, y) - \bar{u}(x, y)$ , but rather is a version of *modulated subtraction*. To define what we mean, we first introduce a change of coordinates called the von-Mise coordinate system (also used in [6]).

First, we introduce the stream function,  $\psi$ , associated to  $u$ , a solution to the system (1):

$$\psi = \int_0^y u(x, y') dy'. \quad (9)$$

A classical idea ([5]) is to write the Prandtl system, (1) in the variables  $(x, \psi)$ :

$$\partial_x(u^2) - u \partial_\psi \psi (u^2) = 0.$$

Define the difference unknown:

$$\varepsilon \phi(x, \psi) := u^2(x, \psi) - \bar{u}^2(x, \psi). \quad (10)$$

Above, the parameter  $\varepsilon$  is introduced based on the size of the initial data (this is made clear in the statement of Theorem 3, but for now, one can ignore it as it is just a rescaling).

The stream function  $\psi$  is considered an independent variable, which takes values in  $\mathbb{R}_+$ . In order to recover the value of the quantity  $u$  appearing in the diffusive term of (13), one uses the relation (9) (which is invertible). Similarly, in order to recover the value of  $\bar{u}$ , one again uses the relation, but this time (9) *with the  $u$  replaced with  $\bar{u}$* . Hence, the difference shown in (10), which we shall adopt as our notion of difference, means in the original coordinate system, that we compare  $u$  and  $\bar{u}$  at *different values of  $y$*  determined nonlinearly:

$$u^2(x, y_1) - \bar{u}^2(x, y_2), \quad \text{where } \int_0^{y_1} u(x, y) dy = \int_0^{y_2} \bar{u}(x, y) dy.$$

**Theorem 3.** *Fix any  $0 < \varepsilon \ll 1$  and  $K_0 \gg 1$  relative only to universal constants. Assume the function  $\phi_0(\psi)$  satisfies*

$$|\partial_\psi^l \phi_0(\psi) \langle \psi \rangle^{10}| \leq 1 \quad \text{for } 0 \leq l \leq 2K_0. \quad (11)$$

*Assume also standard parabolic compatibility conditions up to order  $K_0$  on the data  $\phi_0$  (these are described in Definition 7). Define the difference,  $\phi$ , according to (10). Then there exists a unique solution,  $[u, v]$ , to the Prandtl equations, (1), so that  $\phi$ , as defined by (10), realizes as initial data  $\phi|_{x=1} = \phi_0$ , and this  $\phi$  exists globally in the space  $X$  (defined precisely in (15)) and satisfies the global estimate*

$$\|\phi\|_X \leq C(\|\phi_0\|_{X_{in}}) \quad (12)$$

*for a constant depending on the norm of the initial data (see (26) for a precise definition of the norm  $X_{in}$ ).*

**Remark 4.** The parameter  $\varepsilon$ , fixed as a hypothesis in the theorem, has entered in the definition of  $\phi$  from (10), when coupled with the prescription of order 1 initial data, (11). Thus, prescribing that  $\phi_0$  is order 1 through (11) has the effect of fixing  $u^2 - \bar{u}^2$  to be size  $\varepsilon$  (small) initially.

**Remark 5.** An inspection of the norm  $X$ , defined in (15), shows that this norm encodes global decay information for  $\phi$ , and this is the precise sense in which  $u$  converges to  $\bar{u}$ . In fact, enhanced decay as compared to what is encoded in  $\|\cdot\|_X$  is available due to our Nash-type argument provided in (132), but we do not state this as part of the main theorem.

Note that we require the small  $\kappa_0 > 0$  in (12) to avoid logarithmic singularities at  $x = \infty$ . The fundamental difficulties in establishing (12) are that the equation we analyze, shown below in (20), is *degenerate* at the top (diffusive) order, and *quasilinear*.

One of the motivations for establishing quantitative estimates of the type (12) is due to recent advances in the validity theory for steady Navier–Stokes flows, for instance the works of [2,3]. In particular, using the estimates (12) we can generalize the class of data treated by [2].

**Corollary 6.** *Consider initial data,  $u_0(y)$ , that is a small perturbation of Blasius in the sense of Theorem 3. Then for  $x_0 \gg 1$ , we may take  $[u(x_0, \cdot), v(x_0, \cdot)]$  as the  $\{x = 0\}$  data in Theorem 1 of [2].*

**Proof.** This follows immediately upon applying the estimates (12) above in the proof of Lemma 9 of [2].  $\square$

A second motivation for this work is that in order to prove the global validity of steady Prandtl expansions, a work currently underway, one needs a precise understanding of the decay mechanism in the Prandtl equations, which is established in the present work.

Let us also point the reader towards the related work of [1], which studies the formation of singularities (in this context called “separation”) for the inhomogeneous version of (1) (with adverse pressure gradient).

## 1.2. Main Objects

It is shown in [6], Equation (12), that  $\phi$  satisfies the equation

$$\begin{aligned} \phi_x - u\phi_{\psi\psi} + A\phi = 0, \quad A = -2\frac{\bar{u}_{yy}}{\bar{u}(\bar{u} + u)}, \\ \phi|_{x=1} = \phi_0(\psi), \quad \phi|_{\psi=0} = 0, \quad \phi|_{\psi \uparrow \infty} = 0. \end{aligned} \tag{13}$$

First, note that we have introduced the parameter  $\varepsilon$  in the definition of  $\phi$ , (10). This is due to the fact that the perturbation,  $u_0 - \bar{u}_0$  is initially small (precisely, (11)), and thus it is convenient to rescale to order one quantities according to (20).

Recall the self-similar variable  $\eta$  as defined in (5). For simplicity, we will set the parameter  $x_0 = 1$ . We define a new self-similar variable, which reflects the diffusive scaling in (13) via

$$\xi := \frac{\psi}{\sqrt{x+1}}.$$

It is instructive to compare the variables  $\xi, \psi$  with the self-similar variable  $\eta = \frac{y}{\sqrt{x}}$ . It is well known (for instance from the result of Oleinik, Theorem 1) that both  $\bar{u}$  and  $u$  behave like  $\eta$  in the region where  $\eta \leq 1$ . More precisely, there exist universal constants  $C_0, C_1$  such that

$$C_0\eta \leq \bar{u}, \quad u \leq C_1\eta \quad \text{for } 0 \leq \eta \leq 1. \quad (14)$$

For the purpose of easing notation, we will denote (14) by  $\bar{u} \sim \eta$  and  $u \sim \eta$  for  $0 \leq \eta \leq 1$ . In order to compare  $\psi$  with  $\eta$ , we thus use (9) via

$$\begin{aligned} \psi = \psi(x, y) &= \int_0^y u(x, y') dy' \sim \int_0^y \eta' dy' = \frac{1}{\sqrt{x}} \int_0^y y' dy' \\ &= \frac{y^2}{2\sqrt{x}} = \frac{1}{2} \left( \frac{y}{\sqrt{x}} \right)^2 \sqrt{x} = \frac{1}{2} \eta^2 \sqrt{x}, \end{aligned}$$

and we thus obtain the relation

$$\sqrt{\xi} \sim \eta \quad \text{for } \eta \leq 1.$$

The basic object of study throughout the paper will be  $\phi$ , which satisfies the Equation (13), in the variables  $(x, \psi)$  and correspondingly the self-similar variable  $\xi$ .

Let us now give a brief review of the properties of  $\bar{u}$  and  $u$ . First, as we have already mentioned, Oleinik's global existence result, Theorem 1, gives that  $u \sim \eta$  near  $\eta \leq 1$ . Regarding  $\bar{u}$ , the main properties are summarized in (6). Of particular note is the concavity of  $\bar{u}$ , guaranteed by  $f''' < 0$ . In particular this implies that  $A \geq 0$  in (13).

We will now introduce the norms in which we measure the solution  $\phi$ . First, we simplify notation throughout the paper by putting  $\phi^{(k)} := \partial_x^k \phi$ . Let  $I \subset (0, \infty)$  be an interval of  $x$ . Then we define

$$\begin{aligned} \|\phi\|_{X(I)} &:= \sum_{k=0}^{K_0} \|\phi\|_{X_k(I)}, \\ \|\phi\|_{X_k(I)} &:= \sup_{x \in I} \left( \left\| \phi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L^2_\psi} + \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\sigma_k} \right\|_{L^2_\psi} \right. \\ &\quad \left. + \left\| \phi_\psi^{(k-1)} \langle x \rangle^{k-1-\omega_k} \right\|_{L^2_\psi}^2 + \left\| \phi_\psi^{(k-1)} \langle x \rangle^{k-1-\omega_k} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \right\|_{L^2_\psi} \right) \end{aligned} \quad (15)$$

$$\begin{aligned}
& + \left\| \left\| \sqrt{u} \phi_\psi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \right\|_{L_x^2(I)} + \left\| \left\| \phi_\psi^{(k)} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \right\|_{L_x^2(I)} \\
& + \left\| \left\| \phi^{(k)} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \right\|_{L_x^2(I)} + \left\| \left\| \phi^{(k)} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \right\|_{L_\psi^2} \right\|_{L_x^2(I)},
\end{aligned} \tag{16}$$

for  $1 \leq k \leq K_0$ . In the particular case when  $k = 0$ , we have

$$\begin{aligned}
\|\phi\|_{X_0(I)} &:= \sup_{x \in I} \left( \|\phi\|_{L_\psi^2} + \|\phi \frac{\langle \psi \rangle^{\frac{1}{2}}}{\sqrt{u}}\|_{L_\psi^2} \right) \\
&+ \left\| \left\| \sqrt{u} \phi_\psi \right\|_{L_\psi^2} \right\|_{L_x^2(I)} + \left\| \left\| \phi_\psi \langle \psi \rangle^{\frac{1}{2}} \right\|_{L_\psi^2} \right\|_{L_x^2(I)}.
\end{aligned} \tag{17}$$

Above, we let  $\{\sigma_k\}$ ,  $\{\omega_k\}$  be sequences whose precise requirements are given in (23).  $K_0$  will be a fixed, large number.  $\kappa$  appearing above in (16) is a fixed, small number. In the event that  $I = (0, \infty)$ , we simply drop the  $I$  from (15), and denote the corresponding norms by  $\|\cdot\|_X$ ,  $\|\cdot\|_{X_k}$ .

We will denote by  $E_k(x)$  and  $I_k(x)$  arbitrary quantity satisfying respectively

$$\sup_x |E_k(x)| \leq \sum_{j=0}^k \|\phi\|_{X_j}, \quad \int_0^\infty |I_k(x)| \, dx \leq \sum_{j=0}^k \|\phi\|_{X_j}. \tag{18}$$

We now introduce the function  $\rho$  via

$$\varepsilon \rho(x, \psi) := u(x, \psi) - \bar{u}(x, \psi). \tag{19}$$

Inserting this into the Equation (13) generates the equation that we will study

$$\begin{aligned}
\phi_x - (\bar{u} + \varepsilon \rho) \phi_{\psi\psi} + A \phi &= 0, \quad A = -2 \frac{\bar{u}_{yy}}{\bar{u}(2\bar{u} + \varepsilon \rho)}, \\
\phi|_{x=1} &= \phi_0(\psi), \quad \phi|_{\psi=0} = 0, \quad \phi|_{\psi \uparrow \infty} = 0,
\end{aligned} \tag{20}$$

where the unknowns are  $\rho, \phi$  (which clearly can be expressed in terms of one another according to (10), (19)). Note that this has the effect of clearly distinguishing between the linearized operator and the quadratic terms, the latter being those having an  $\varepsilon$  in front of them in (20).

### 1.3. Main Ideas

The main mechanisms can be summarized in four steps listed below. Overall, at each order of  $x$  regularity up to  $\partial_x^{K_0}$  for a fixed  $K_0$  large, there are two estimates that are performed. We call these the “Energy estimate” and the “Quotient estimate”. This results in the control of the norm  $\|\phi\|_X$  as shown above.

### Step 1: $L^2$ level

At the  $L^2$  level, we may center our discussion around the linearized operator from (20), which reads as

$$\phi_x - \bar{u}\phi_{\psi\psi} + \bar{A}\phi, \quad \bar{A} = -\frac{\bar{u}_{yy}}{\bar{u}^2}. \quad (21)$$

The standard energy estimate performed on (21) gives a bound on  $\sup_x \|\phi\|_{L_\psi^2}^2 + \|\sqrt{\bar{u}}\phi_\psi\|_{L_x^2 L_\psi^2}^2$ . The crucial point here is that  $\sqrt{\bar{u}}$  enters the  $\phi_\psi$  term, which creates difficulties due to the degeneracy of  $\bar{u}$  near  $\psi = 0$ . Due to the structure of the Blasius profiles,  $\bar{u}_{yy} \leq 0$ , and so  $\bar{A} \geq 0$  and thus has a favorable (but not too powerful) contribution at this stage.

The second estimate at the  $L^2$  level is the “Quotient estimate”, which can be found in Lemma 25. There are two distinguished features of the quantities that are controlled (see the estimate (83)). First, there is a far-field weight of  $\langle\psi\rangle$ . Second, there is a nonlinear weight  $\frac{1}{u}$  which gives additional control near the boundary  $\{\psi = 0\}$ .

The reason we can close this Quotient estimate is due to the precise structure of Blasius solutions. Indeed, the choice of weight  $\frac{\langle\psi\rangle}{u}$  is specially designed so that the interaction with the linearized equation, (21), produces the quantity

$$\int \phi^2 \times \text{positive quantities} \times \Omega, \quad \text{where } \Omega = -\bar{u}_{yy} + \frac{1}{2}\bar{u}\bar{u}_x.$$

This type of quantity would be out of reach of the norm  $X$ . This is because this is a linear term, and thus does not come with a small-parameter of  $\varepsilon$ , so it's contribution cannot be absorbed to the left-hand side of estimate (83). However, by using the convexity of  $\bar{u}$ , a Blasius solution, we are able to show that  $\Omega(x, y)$  is globally positive.

The reason we need the quotient estimate is two-fold, corresponding to the two weights appearing on  $\phi$  term on the left-hand side of (83). The weight  $\langle\psi\rangle$  comes in for Step 4, whereas the boundary weight  $\frac{1}{u}$  comes in for Step 3.

### Step 2: $H^k$ for $1 \leq k \leq K_1$

We now fix  $K_1$  so that  $1 \ll K_1 \ll K_0$ . The tier of derivatives between 1 and  $K_1$  we call the “middle tier”. The middle tier is distinguished from the top tier because we are able to expend derivatives. More precisely, since the derivative count is less than  $K_1$  which is substantially smaller than  $K_0$ , we can invoke the estimates (64) which lose one derivative. The middle tier is distinguished from the bottom ( $L^2$ ) tier because the linearized operator is no longer (21), but rather  $(\phi^{(1)} := \partial_x \phi)$

$$\phi_x^{(1)} - \bar{u}\phi_{\psi\psi}^{(1)} + \bar{A}\phi^{(1)} - \frac{\partial_x \bar{u}}{\bar{u}}\phi^{(1)}. \quad (22)$$

We arrive here by substituting the Equation (20) upon differentiating it in  $x$ . This is shown in Equation (86) (the linearized operator in (22) comes from the linear contributions of the first four terms in (22)). The reason the linearized equation has changed is due to the quasilinearity present in (21). At this stage we repeat the

process of Step 1, taking advantage of the further property of Blasius solutions that  $\partial_x \bar{u} < 0$ .

*Step 3:  $H^k$  for  $K_1 + 1 \leq k \leq K_0$*

We now arrive at the top tier of derivative in the norm  $X$ . The top tier is distinguished because we do not have derivatives to expend. Precisely, this means we must take care to invoke only those estimates from Lemma 16 with indices  $l, m$ , which have feature of applying in particular when  $l = m = K_0$  (the top order derivative). First of all, we select  $K_0$  large enough so that the “tame principle” kicks in. For instance, terms like  $\partial_x^j f \times \partial_x^{K_0-j} g \times \partial_x^{K_0} h$  are bound to have either  $j$  or  $K_0 - j$  to be much smaller than  $K_0 - 3$ . This is standard in quasilinear problems.

The important part, however, is that the crucial weight of  $\frac{1}{u}$  available due to the quotient estimate, is used to “save derivatives”. This is most easily seen in a term such as (“high” and “low” refer to order of  $x$  derivative)

$$\int \phi^{\text{high}} \phi_{\psi\psi}^{\text{low}} \phi^{\text{high}} \chi(\xi \lesssim 1).$$

For a term such as this, we are forced to put  $\phi_{\psi\psi}^{\text{low}}$  in an  $L^\infty$  type norm in order to conserve the high derivatives. To do this, with the weights of  $u$  distributed as optimally as we are allowed with the  $X$  norm, we *must* invoke the additional  $\frac{1}{\sqrt{u}}$  weight available due to the quotient estimate. This is quantified by proving a localized, optimal weight, uniform estimate on  $\phi_{\psi\psi}^{\text{low}}$  (see (75)).

*Step 4: Optimal decay*

Using Steps 1–3 we are able to show global existence of  $\phi$  in the space  $X$ . The space  $X$  certainly encodes decay information regarding the solution  $\phi$  – this is evident by consulting (15). However, one notices that the quantity  $\|\phi\|_{L_\psi^2}$  is only shown to be bounded from the specification of the norm  $X$ . For the heat equation set on  $\mathbb{R}$ , the quantity  $\|\phi\|_{L_\psi^2}$  is expected to decay at rate  $\langle x \rangle^{-\frac{1}{4}}$ .

To explain how we obtain this “optimal” decay, the reader should now recall the classical Nash inequality, [4], which states that  $\|\phi\|_{L_\psi^2}^2 \lesssim \|\phi_\psi\|_{L_\psi^2}^{\frac{2}{3}} \|\phi\|_{L_\psi^2}^{\frac{4}{3}}$ . Typically, one uses this by saying  $\|\phi\|_{L_\psi^1}$  is conserved (say) and thus one inserts the Nash inequality to the basic energy bound to obtain an ODE of the form  $\dot{\eta} + \eta^3 = 0$ , for  $\eta = \|\phi\|_{L_\psi^2}^2$ , which immediately results in  $\langle x \rangle^{-\frac{1}{4}}$  decay of  $\|\phi\|_{L_\psi^2}$ .

In our case, two difficulties are present in order to carry out this procedure to optimize the decay. First, we only have the degenerate weighted quantity  $\|\sqrt{u} \phi_\psi\|_{L_\psi^2}$  appearing in the energy. Second, we cannot control  $\|\phi\|_{L^1}$  by integrating the equation.

To contend with these difficulties, we establish a new Nash-type inequality in Lemma 37 which (1) accounts for the degenerate weight of  $\sqrt{u}$  and (2) replaces the  $L^1$  norm by  $L^2(\langle \psi \rangle^{\frac{1}{2}})$  (which scales the same way). The type of inequality we are able to establish is piecewise (as is seen from Lemma 37). Remarkably, both upper bounds in estimate (129) yield the same, optimal, decay rate of  $\langle x \rangle^{-\frac{1}{4}}$ .

#### 1.4. Preliminaries and Notations

There are various parameters and notations appearing in our analysis. We include here an explanation of what they all mean. The bracket is defined via  $\langle x \rangle^p := 1 + x^p$ .  $K_0, K_1$  will be fixed, large universal numbers which refer to the regularity indices in  $x$ .  $K_0$  appears in (15), whereas  $K_1$  is selected to be  $1 \ll K_1 \ll K_0$ . For concreteness, one can take  $K_0 = 100, K_1 = 50$ .

$\kappa$  is reserved for the exponent of the  $\psi$  weight appearing in (16).  $\{\sigma_k\}_{k=0}^{K_0}$  and  $\{\omega_k\}_{k=1}^{K_0}$  are reserved for the  $x$  weights appearing in (16). These exponents satisfy the following criteria:

$$\begin{aligned} \sigma_{k-1} < \omega_k < \sigma_k, \quad 2\omega_k > \sigma_k + \sigma_{k-1}, \quad \sigma_k, \omega_k \text{ increasing,} \\ \sigma_{K_0} < \frac{1}{100}, \quad \sigma_0 = 0. \end{aligned} \quad (23)$$

It is clearly possible to find sequences satisfying the conditions in (23).

We will often localize to regions of  $\xi$  using  $\chi(\cdot)$ , where  $\chi$  is a smooth cut-off function:

$$\chi = \begin{cases} 1 & \text{on } (0, 1) \\ 0 & \text{on } (2, \infty) \end{cases}, \quad \chi \leq 0. \quad (24)$$

For  $L^2$  norms of functions of two variables, say  $f(x, \psi)$ , we use the notation

$$\|f\|_{L_\psi^2}^2 := \int f(x, \psi)^2 d\psi.$$

We will also use the notation  $\|f\|_{L^2(\xi \leq 1)}^2 := \int f^2 \chi(\xi) d\psi$ .

The parameter  $\delta$  will play the role of a small parameter introduced by Young's inequality for products:  $ab \leq \delta a^2 + C_\delta b^2$ .  $C_\delta$  will refer to a constant that grows as  $\delta \downarrow 0$ . As Young's inequality needs to be applied several times, we will use  $\delta$  each time with the understanding that it refers to different values, as opposed to indexing the  $\delta$ 's per use of Young's inequality.

## 2. The Space $X$

### 2.1. Bootstrap Assumptions

Global existence in the space  $X$  will be obtained by a standard continuity argument which relies on the following bootstrap assumption:

$$x \in I_* := (0, x_*), \quad \text{where } \|\phi\|_{X(I_*)} \leq 2\|\phi_0\|_{X_{in}}. \quad (25)$$

Above,  $\phi_0$  is the initial data,  $\phi|_{x=0}$ , according to (20). Motivated by (16), we use the notation  $X(\{0\})$  to denote the norm of the initial data (which is (16) formally with  $I = \{0\}$  to eliminate the integrations in  $X$ ):

$$\|\phi_0\|_{X_{\{0\}}^{(0)}} := \|\phi_0\|_{L_\psi^2} + \left\| \phi_0 \frac{\langle \psi \rangle^{\frac{1}{2}}}{\sqrt{u|_{x=1}}} \right\|_{L_\psi^2},$$

$$\begin{aligned} \|\phi_0\|_{X_{\{0\}}^k} &:= \left\| \phi_0^{(k)} \right\|_{L_\psi^2} + \left\| \phi_0^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-k}}{\sqrt{u|_{x=1}}} \right\|_{L_\psi^2} + \left\| \partial_\psi \phi_0^{(k-1)} \right\|_{L_\psi^2}^2 + \left\| \partial_\psi \phi_0^{(k-1)} \frac{\langle \psi \rangle^{\frac{1}{2}-k}}{\sqrt{u|_{x=1}}} \right\|_{L_\psi^2}, \\ \|\phi_0\|_{X_{in}} &:= \sum_{k=0}^{K_0} \|\phi_0\|_{X_{\{0\}}^k}, \end{aligned} \quad (26)$$

for  $1 \leq k \leq K_0$ . Above, we have used  $\phi_0^{(k)} = \partial_x^k \phi|_{x=0}$ . As usual in evolution problems, one obtains this initial data iteratively from the equation, (20).

**Definition 7.** (*Compatibility Conditions*) We evaluate the Equation (20) at  $x = 1$  iteratively (as is standard in parabolic problems). For instance,  $\phi_0^{(1)} = u_0 \phi_0 \psi_\psi - A|_{x=0} \phi_0$ . We now need to ensure that  $\phi_0^{(1)}|_{\psi=0} = 0$ , which is ensured so long as  $(u_0 \phi_0 \psi_\psi - A|_{x=0} \phi_0)|_{\psi=0} = 0$ . In this way, higher order conditions are also derived (we do not write them down explicitly, as they are cumbersome).

**Lemma 8.** (*Initial Data*) *Assume (11) and the compatibility conditions as in Definition 7. Then the initial data satisfies*

$$\|\phi_0\|_{X_{in}} \leq C_{IN} \quad (27)$$

for a universal constant  $C_{IN}$  independent of  $\varepsilon$ .

**Proof.** The assumption of (11) ensures that  $\|\phi\|_{X_{\{0\}}^{(0)}} \lesssim 1$ . Specifically, the term  $\|\phi_0\|_{L_\psi^2}$  is clearly controlled by the assumption (11). For the latter term in  $\|\phi\|_{X_{\{0\}}^{(0)}}$ , we split into the region  $\psi \leq 1$  and  $\psi \geq 1$ . The region  $\psi \geq 1$  is clearly controlled by (11) due to the fact  $u \gtrsim 1$  in that region. For the region  $\psi \leq 1$ , by using the Hardy inequality, admissible because  $\phi_0(0) = 0$ , we can estimate  $\|\phi_0 \frac{1}{\sqrt{u_0}}\|_{L_\psi^2} \|\phi_0 \frac{1}{u_0}\|_{L_\psi^2} \lesssim \|\partial_\psi \phi_0\|_{L_\psi^2}$ , which again is controlled by (11). From here, one observes that the assumptions on (11) due to the  $\psi$  derivative going up to order  $2K_0$ , and the compatibility conditions, Definition 7, are sufficient to control  $\phi_0^{(k)}$  in the norm  $X_{\{0\}}^{(k)}$  for  $k \leq K_0$ .  $\square$

In particular, this means that quantities denoted by  $E_k(x)$  and  $I_k(x)$  as in (18) satisfy

$$\sup_{x \in I_*} |E_{K_0}(x)| \leq 2\|\phi_0\|_{X_{in}}, \quad \int_1^{x_*} |I_{K_0}(x)| \, dx \leq 2\|\phi_0\|_{X_{in}}. \quad (28)$$

## 2.2. Embeddings

The reader should recall the specification of the  $X(I)$  norm given in (15).

**2.2.1.  $\omega$  and  $\alpha$  Estimates** The proofs of the estimates of this sub-section are inter-connected due to their nonlinear nature, and so it will be convenient to give a parameter assignment to the value:

$$\alpha := \sup_{x \in I} \left\| \frac{\rho}{u} \right\|_{L_\psi^\infty}, \quad \omega := \sup_{x \in I} \left( \left\| \frac{u}{\bar{u}} \right\|_{L_\psi^\infty} + \left\| \frac{\bar{u}}{u} \right\|_{L_\psi^\infty} \right). \quad (29)$$

**Lemma 9.** *For  $\alpha, \omega$  defined as in (29), under the bootstrap assumption (25), the following inequalities are valid:*

$$\omega \leq \frac{1 - \varepsilon\alpha}{1 - 2\varepsilon\alpha}. \quad (30)$$

**Proof.** To prove the boundedness of  $|\frac{u}{\bar{u}}|$ , we estimate

$$\begin{aligned} \left\| \frac{u}{\bar{u}} \right\|_{L_\psi^\infty} &= \left\| \frac{\bar{u} + \varepsilon\rho}{\bar{u}} \right\|_{L_\psi^\infty} \leq 1 + \varepsilon \left\| \frac{\rho}{\bar{u}} \right\|_{L_\psi^\infty} \leq 1 + \varepsilon \left\| \frac{\rho}{u} \right\|_{L_\psi^\infty} \left\| \frac{u}{\bar{u}} \right\|_{L_\psi^\infty}, \\ &\leq 1 + \varepsilon\alpha \left\| \frac{u}{\bar{u}} \right\|_{L_\psi^\infty}, \end{aligned}$$

from which we get

$$\left\| \frac{u}{\bar{u}} \right\|_{L_\psi^\infty} \leq \frac{1}{1 - \varepsilon\alpha}. \quad (31)$$

For the second inequality, we appeal to the fact that  $\rho \leq \alpha u \leq \frac{\alpha}{1 - \varepsilon\alpha} \bar{u}$  to estimate

$$\begin{aligned} \left\| \frac{\bar{u}}{u} \right\|_{L_\psi^\infty} &= \left\| \frac{\bar{u}}{\bar{u} + \varepsilon\rho} \right\|_{L_\psi^\infty} \leq \left\| \frac{\bar{u}}{\bar{u} - \varepsilon|\rho|} \right\|_{L_\psi^\infty} \\ &\leq \left\| \frac{\bar{u}}{\left(1 - \frac{\varepsilon\alpha}{1 - \varepsilon\alpha}\right) \bar{u}} \right\|_{L_\psi^\infty} \leq \frac{1}{1 - \frac{\varepsilon\alpha}{1 - \varepsilon\alpha}} = \frac{1 - \varepsilon\alpha}{1 - 2\varepsilon\alpha}. \end{aligned} \quad (32)$$

□

**Lemma 10.** *Fix any subinterval  $I \subset I_* \subset (0, \infty)$ , the following estimate is valid:*

$$\sup_{x \in I} \left( \|\phi_\psi \langle x \rangle^{\frac{3}{4} - \sigma_1}\|_{L_\psi^\infty} + \left\| \sqrt{u} \phi_{\psi\psi} \langle \psi \rangle^{\frac{1}{2} - \kappa} \langle x \rangle^{1 - \sigma_1} \right\|_{L_\psi^2} \right) \lesssim \frac{\langle \omega \rangle}{1 - \varepsilon\alpha} \|\phi\|_{X_1(I)}, \quad (33)$$

for a universal constant, independent of the interval  $I$ .

**Proof.** We estimate using the Equation (20)

$$\begin{aligned}
& \left\| \sqrt{u} \phi_{\psi} \psi \langle x \rangle^{1-\sigma_1} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \\
& \leq \left\| \sqrt{u} \frac{\phi^{(1)}}{u} \langle x \rangle^{1-\sigma_1} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} + \left\| \frac{1}{\sqrt{u}} A \phi \langle x \rangle^{1-\sigma_1} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \\
& \leq \left\| \sqrt{u} \frac{\phi^{(1)}}{u} \langle x \rangle^{1-\sigma_1} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} + \left\| \frac{1}{\sqrt{u}} A \phi \langle x \rangle^{1-\sigma_1} \chi(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \\
& + \left\| \frac{1}{\sqrt{u}} A \phi \langle x \rangle^{1-\sigma_1} (1 - \chi(\xi)) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}}. \tag{34}
\end{aligned}$$

Above,  $\chi$  is the function fixed in (24). It is clear from the definition of (16) that the first term on the right-hand side of (34) is bounded by  $\|\phi\|_{X_1}$ . For the third term, we use that the Blasius profile,  $\bar{u}$ , is non-degenerate in the far-field:

$$(1 - \chi(\xi))u \geq \omega(1 - \chi(\xi))\bar{u} \gtrsim \omega.$$

We now estimate  $A$  via

$$|A| = 2 \left| \frac{\bar{u}_{yy}}{\bar{u} (2\bar{u} + \varepsilon \bar{u} \frac{\rho}{u} \frac{u}{\bar{u}})} \right| \lesssim \frac{1}{1 - \varepsilon \alpha} \langle x \rangle^{-1}.$$

Thus, the third term is controlled by a factor of  $\frac{\omega}{1 - \varepsilon \alpha} \|\phi\|_{X_1}$ .

For the second term, we estimate using first (30), second the self-similarity of the Blasius profile:  $C_0 \xi \leq \bar{u} \leq C_1 \sqrt{\xi}$  when  $\xi \leq 1$ , and third the Hardy inequality:

$$\begin{aligned}
& \left\| \frac{\phi}{\sqrt{u}} \chi(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \lesssim \omega^{\frac{1}{2}} \left\| \frac{\phi}{\sqrt{u}} \chi(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \\
& \lesssim \omega^{\frac{1}{2}} \left\| \frac{\phi}{\psi^{\frac{1}{4}}} \chi(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \langle x \rangle^{\frac{1}{8}} \\
& \lesssim \omega^{\frac{1}{2}} \langle x \rangle^{\frac{1}{8}} \left( \left\| \psi^{\frac{3}{4}} \phi \chi(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \right. \\
& \quad \left. + \left\| \psi^{\frac{3}{4}} \phi \langle x \rangle^{\frac{1}{8}} \langle x \rangle^{-\frac{1}{2}} \chi'(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \right) \\
& \lesssim \omega^{\frac{1}{2}} \left( \left\| \langle x \rangle^{\frac{1}{2}} \phi \chi(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} + \left\| \phi \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \right).
\end{aligned}$$

Above, we have used in the support of  $\chi$  and  $\chi'$  that  $\psi \lesssim \sqrt{x}$ . Upon inserting into the second term of (34), we obtain

$$\begin{aligned}
& \left\| \frac{1}{\sqrt{u}} A \phi \langle x \rangle^{1-\sigma_1} \chi(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \\
& \leq \|A\langle x \rangle\|_{\infty} \langle x \rangle^{-\sigma_1} \omega^{\frac{1}{2}} \left( \left\| \langle x \rangle^{\frac{1}{2}} \phi \chi(\xi) \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} + \left\| \phi \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_{\psi}} \right) \\
& \lesssim \omega^{\frac{1}{2}} \|\phi\|_{X_1(I)}.
\end{aligned}$$

Above, we have used that  $\sigma_1 > \omega_1$  in the estimation of the  $\phi_\psi$  term, which is required according to the specification of  $\|\cdot\|_{X_1}$  in (16).

To obtain the estimate on the  $\phi_\psi$  quantity in (33), we first localize based on the location of  $\xi$ . Letting  $\chi = 1$  on  $(0, 1)$ , and 0 on  $(2, \infty)$ , be a smooth, decreasing cut-off function, we first have

$$\begin{aligned} |\phi_\psi|^2 \chi(\xi) &= - \int_\psi^\infty \partial_\psi (\phi_\psi^2 \chi(\xi)) = - \int_\psi^\infty 2\phi_\psi \phi_{\psi\psi} \chi(\xi) - \int_\psi^\infty \phi_\psi^2 \chi'(\xi) \frac{1}{\sqrt{x}} \\ &\lesssim \left\| \frac{\phi_\psi}{\psi^{\frac{1}{4}}} \chi \right\|_{L_\psi^2} \left\| \psi^{\frac{1}{4}} \phi_{\psi\psi} \chi \right\|_{L_\psi^2} + x^{-\frac{1}{2}} \|\phi_\psi\|_{L_\psi^2}^2 \\ &\lesssim \left( \left\| \psi^{\frac{3}{4}} \phi_{\psi\psi} \chi \right\|_{L_\psi^2} + \left\| \psi^{\frac{3}{4}} x^{-\frac{1}{2}} \phi_\psi \chi' \right\|_{L_\psi^2} \right) \|\sqrt{u} \phi_{\psi\psi}\|_{L_\psi^2} x^{\frac{1}{8}} + x^{-\frac{1}{2}} \|\phi_\psi\|_{L_\psi^2}^2 \\ &\lesssim x^{-\frac{3}{2}+2\sigma_1} \left( \|\phi\|_{X_1}^2 + \|\sqrt{u} \phi_{\psi\psi} \langle x \rangle^{1-\sigma_1}\|^2 \right). \end{aligned} \quad (35)$$

The far-field contribution is estimated easily upon using that  $u \gtrsim 1$  in the support of  $1 - \chi$ .  $\square$

**Lemma 11.** *The following estimate is valid, for a constant independent of  $I$ :*

$$\alpha + \sup_{x \in I} \left\| \frac{\phi}{u^2} \right\|_{L_\psi^\infty} \lesssim \omega^3 \langle x \rangle^{-\frac{1}{4}+\sigma_1} \|\phi\|_{X_1(I)}. \quad (36)$$

**Proof.** Clearly we may restrict to the region  $\xi \lesssim 1$ , in which case we relate  $u$  to the self-similar variable,  $\eta$ , via  $u^2 \geq \omega^2 \bar{u}^2 \gtrsim \omega^2 \xi$ . Now, since  $\phi|_{\psi=0} = 0$ , we may use the standard Hardy inequality via

$$\begin{aligned} \left| \frac{\phi}{u^2} \right| &\lesssim \omega^2 \frac{1}{\xi} |\phi(x, \psi)| \lesssim \omega^2 \frac{\sqrt{x}}{\psi} \left| \int_0^\psi \phi_\psi(x, \psi') d\psi' \right| \\ &\lesssim \omega^2 \sqrt{x} \|\phi_\psi\|_{L_\psi^\infty} \lesssim \omega^3 \langle x \rangle^{-\frac{1}{4}+\sigma_1} \|\phi\|_{X_1(I)}, \end{aligned}$$

from which we obtain

$$\sup_{x \in I} \left\| \frac{\phi}{u^2} \right\|_{L_\psi^\infty} \lesssim \omega^3 \langle x \rangle^{-\frac{1}{4}+\sigma_1} \|\phi\|_{X_1(I)}. \quad (37)$$

By using the identity  $\rho = \frac{\phi}{u+\bar{u}}$ , we estimate

$$\begin{aligned} \alpha &= \left\| \frac{\rho}{u} \right\|_{L_\psi^\infty} \leq \left\| \frac{\phi}{u(u+\bar{u})} \right\|_{L_\psi^\infty} = \left\| \frac{\phi}{u^2 (2 - \varepsilon \frac{\rho}{u})} \right\|_{L_\psi^\infty} \leq \left\| \frac{1}{2 - \varepsilon \frac{\rho}{u}} \right\|_{L_\psi^\infty} \left\| \frac{\phi}{u^2} \right\|_{L_\psi^\infty} \\ &\lesssim \frac{1}{2 - \varepsilon \alpha} \langle x \rangle^{-\frac{1-\sigma_1}{4}} \omega^3 \|\phi\|_{X_1(I)}. \end{aligned} \quad (38)$$

$\square$

**Corollary 12.** *The following estimate is valid, for a constant independent of  $I$*

$$\omega + \alpha \lesssim \|\phi\|_{X_1(I)}. \quad (39)$$

From here on, due to (39), and our bootstrap assumption, (25), we are able to drop the dependence on  $\alpha$  and  $\omega$  in forthcoming estimates.

**2.2.2. Higher Order Embeddings** We will again need to proceed in a nonlinear fashion. Thus, we give parameter names to the following quantities:

$$\begin{aligned}\beta^{(k)} &:= \sup_{x \in I} \left\| \frac{\rho^{(k)}}{u} \langle x \rangle^{k+\frac{1}{4}-\sigma_{k+1}} \right\|_{L_\psi^\infty}, \\ \gamma^{(k)} &:= \sup_{x \in I} \left\| \frac{\rho^{(k)}}{u} \langle x \rangle^{k-\sigma_{k+1}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2}, \\ \Upsilon^{(k)} &:= \sup_{x \in I} \left\| \sqrt{u} \phi_{\psi\psi}^{(k)} \langle x \rangle^{k+1-\sigma_{l+1}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2}, \\ \iota^{(k)} &:= \sup_{x \in I} \left\| \sqrt{u} \phi_{\psi\psi}^{(k)} \langle x \rangle^{k+\frac{5}{4}-\sigma_{k+1}} \right\|_{L_\psi^\infty}\end{aligned}\quad (40)$$

**Lemma 13.** *Let  $P(\cdot, \dots, \cdot)$  be a polynomial in its arguments. The following estimates are valid, for constants independent of  $I$ :*

$$\sup_{x \in I} \|A \langle x \rangle\|_{L_\psi^\infty} + \|A \langle x \rangle^{\frac{3}{4}}\|_{L_\psi^2} \lesssim \|\phi\|_{X_1(I)}, \quad (41)$$

$$\sup_{x \in I} \left\| \partial_x^k A \langle x \rangle^{k+1} \right\|_\infty \lesssim 1 + \varepsilon \beta^{(k)} + \varepsilon P(\beta^{(0)}, \dots, \beta^{(k-1)}), \quad (42)$$

$$\sup_{x \in I} \left\| \partial_x^k A \langle x \rangle^{k+\frac{3}{4}} \right\|_{L_\psi^2} \lesssim 1 + \varepsilon \gamma^{(k)} + \varepsilon P(\gamma^{(0)}, \dots, \gamma^{(k-1)}, \beta^{(0)}, \dots, \beta^{k-1}) \quad (43)$$

There exists a decomposition  $\partial_x^j A = A_0^{(j)} + \varepsilon A_1^{(j)}$  such that

$$\sup_{x \in I} \left\| A_0^{(k)} \langle x \rangle^{k+1} \right\|_{L_\psi^\infty} \lesssim 1, \quad (44)$$

$$\sup_{x \in I} \left\| A_1^{(k)} \langle x \rangle^{k+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \lesssim \gamma^{(j)} + P(\gamma^{(0)}, \dots, \gamma^{(k-1)}, \beta^{(0)}, \dots, \beta^{(k-1)}). \quad (45)$$

**Proof.** We use the expression for  $A$  to generate the identity

$$A \left( 2 + \varepsilon \frac{\rho}{\bar{u}} \right) = -2 \frac{\bar{u}_{yy}}{\bar{u}^2}. \quad (46)$$

We first use the self-similarity of  $\bar{u}$  from (7) to evaluate

$$\frac{\bar{u}_{yy}}{\bar{u}^2} = \frac{1}{x} g(\eta), \text{ smooth, bounded, and decaying } g(\cdot). \quad (47)$$

From here, the base case of  $k = 0$  follows upon estimating  $|2 + \varepsilon \frac{\rho}{\bar{u}}| \geq \frac{3}{2}$ , according to (39) because it holds true for the function on the right-hand side of (46). We thus assume (42), (43) hold true for indices  $0, \dots, k-1$ . We apply  $\partial_x^k$  to (46) to obtain

$$\partial_x^k A \left( 2 + \varepsilon \frac{\rho}{\bar{u}} \right) = -2 \partial_x^k \frac{\bar{u}_{yy}}{\bar{u}^2} - \varepsilon \sum_{l=0}^{k-1} c_l^k \partial_x^l A \partial_x^{k-l} \frac{\rho}{\bar{u}} =: A_0^{(k)} + \varepsilon A_1^{(k)}. \quad (48)$$

By (47),

$$\sup_{x \in I} \|A_0^{(k)} \langle x \rangle^{k+1}\|_{L_\psi^\infty} + \sup_{x \in I} \|A_0^{(k)} \langle x \rangle^{k+\frac{3}{4}}\|_{L_\psi^2} \lesssim 1.$$

For the  $A_1$  term, we first note that (eventually we will set the index  $m = k - l$  from (48))

$$\partial_x^m \frac{\rho}{\bar{u}} = \frac{u}{\bar{u}} \frac{\rho^{(m)}}{u} + \sum_{l=0}^{m-1} c_l^m \left( \bar{u} \partial_x^l \frac{1}{\bar{u}} \right) \frac{u}{\bar{u}} \frac{\rho^{(m-l)}}{u}. \quad (49)$$

We now note that  $\frac{u}{\bar{u}}$  is bounded from above using (39), and that  $\|(\bar{u} \partial_x^l \frac{1}{\bar{u}}) \langle x \rangle^l\|_{L^\infty} \lesssim 1$ . From here, we estimate

$$\left\| \partial_x^m \frac{\rho}{\bar{u}} \langle x \rangle^m \right\|_{L_\psi^\infty} \lesssim \beta^{(m)} + \dots + \beta^{(0)}, \quad \left\| \partial_x^m \frac{\rho}{\bar{u}} \langle x \rangle^{m-\sigma_{m+1}} \right\|_{L_\psi^2} \lesssim \gamma^{(m)} + \dots + \gamma^{(0)}. \quad (50)$$

From here, we can estimate

$$\begin{aligned} \sup_{x \in I} \|A_1^{(k)} \langle x \rangle^{k+1}\|_{L_\psi^\infty} &\lesssim \sup_{x \in I} \sum_{l=0}^{k-l} \left\| \partial_x^l A \langle x \rangle^{l+1} \right\|_{L_\psi^\infty} \left\| \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k-l} \right\|_{L_\psi^\infty} \\ &\lesssim P(\beta^{(0)}, \dots, \beta^{(k-1)}) + \beta^{(k)}, \end{aligned}$$

which establishes (42).

Next, we estimate

$$\begin{aligned} \|A_1^{(k)} \langle x \rangle^{k+\frac{3}{4}}\|_{L_\psi^2} &\lesssim \sum_{l=0}^{\lfloor \frac{k-1}{2} \rfloor} \left\| \partial_x^l A \langle x \rangle^{l+1} \right\|_{L_\psi^\infty} \left\| \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k-l-\frac{1}{4}} \right\|_{L_\psi^2} \\ &\quad + \sum_{l=\lfloor \frac{k-1}{2} \rfloor}^{k-l} \left\| \partial_x^l A \langle x \rangle^{l+\frac{3}{4}} \right\|_{L_\psi^2} \left\| \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k-l} \right\|_{L_\psi^\infty}, \end{aligned}$$

from which (43) follows upon using (42), (43) inductively, coupled with (50), and using that  $\sigma_m < \frac{1}{4}$ , for all  $m$ , according to our assumption, (23). Estimates (44), (45) follow in essentially the same manner.  $\square$

**Lemma 14.** *Let  $\phi$  solve (20), and assume the bootstrap (25). The following estimates are valid, for constants independent of  $I$ :*

$$\sup_{x \in I} \left\| \frac{\phi^{(k)}}{u^2} \right\|_{L_\psi^\infty} \langle x \rangle^{k+(\frac{1}{4}-\sigma_{k+1})} \lesssim \Upsilon^{(k)} + \|\phi\|_{X_{k+1}(I)}, \text{ for } k \geq 0 \quad (51)$$

$$\beta^{(k)} \lesssim \Upsilon^{(k)} + \|\phi\|_{X_{k+1}(I)} + (1 + \varepsilon \beta^{(k-1)}) \beta^{(k-1)}, \text{ for } k \geq 1$$

$$\beta^{(0)} \lesssim \langle x \rangle^{-(\frac{1}{4}-\sigma_1)} \|\phi\|_{X_1(I)}. \quad (52)$$

**Proof.** We first restrict the  $\phi$  quantity to the region,  $\xi \leq 1$ , and control this contribution via the same Hardy type inequality we established in (35),

$$\begin{aligned} \left| \frac{\phi^{(k)}}{u^2} \right| (1 - \chi(\xi)) &\lesssim \sqrt{x} \left\| \phi_\psi^{(k)} \right\|_{L_\psi^\infty} \\ &\lesssim \sqrt{x} \langle x \rangle^{-k - \frac{3}{4} + \sigma_{k+1}} \left( \left\| \sqrt{u} \phi_\psi^{(k)} \langle x \rangle^{k+1 - \sigma_{k+1}} \right\|_{L_\psi^2} + \left\| \phi \right\|_{X_{k+1}(I)} \right) \\ &\leq \langle x \rangle^{-k - \frac{1}{4} + \sigma_{k+1}} (\Upsilon^{(k)} + \|\phi\|_{X_{k+1}(I)}). \end{aligned} \quad (53)$$

The far-field contribution is controlled by integrating from  $\infty$  and using that  $\frac{1}{u^2} \gtrsim 1$  in the region  $\xi \geq 1$ , via

$$\begin{aligned} \left| \phi^{(k)} \right|^2 (1 - \chi(\xi)) &= - \int_\psi^\infty \partial_\psi \left( \left| \phi^{(k)} \right|^2 (1 - \chi(\xi)) \right) \\ &= - \int_\psi^\infty 2\phi^{(k)} \phi_\psi^{(k)} (1 - \chi(\xi)) + \int \left| \phi^{(k)} \right|^2 \frac{1}{\sqrt{x}} \chi'(\xi) \\ &\lesssim \left\| \phi^{(k)} \right\|_{L_\psi^2} \left\| \phi_\psi^{(k)} \right\|_{L_\psi^2} + \frac{1}{\sqrt{x}} \left\| \phi^{(k)} \right\|_{L_\psi^2}^2. \end{aligned}$$

Thus,

$$\begin{aligned} \left| \phi^{(k)} \right| (1 - \chi(\xi)) \langle x \rangle^{k + \frac{1}{4} - \sigma_{k+1}} &\lesssim \langle x \rangle^{k + \frac{1}{4} - \sigma_{k+1}} \left( \left\| \phi^{(k)} \right\|_{L_\psi^2}^{\frac{1}{2}} \left\| \phi_\psi^{(k)} \right\|_{L_\psi^2}^{\frac{1}{2}} + \frac{1}{\langle x \rangle^{\frac{1}{4}}} \left\| \phi^{(k)} \right\|_{L_\psi^2} \right) \\ &\lesssim \|\phi\|_{X_{k+1}(I)}, \end{aligned}$$

upon using that  $\sigma_{k+1} > \sigma_k$  and  $\sigma_{k+1} > \omega_{k+1}$ .

By using the identity  $\rho = \frac{\phi}{u + \bar{u}}$ , we obtain

$$\begin{aligned} \rho^{(N)} &= \frac{\phi^{(N)}}{u + \bar{u}} - \frac{1}{u + \bar{u}} \sum_{j=0}^{N-1} c_{j,N} \partial_x^{N-j} (\bar{u} + u) \partial_x^j \rho \quad \text{for any } N, \\ \rho^{(0)} &= \frac{\phi}{u + \bar{u}}. \end{aligned} \quad (54)$$

According to (52), we will set  $N = k$ . We proceed inductively, the base case being already established in (36). Assume that (52) is known for indices  $0, \dots, k-1$ . Set  $N = k$  in the expression (54), and estimate via

$$\begin{aligned} \left\| \frac{\rho^{(k)}}{u} \langle x \rangle^{(k + \frac{1}{4} - \sigma_{k+1})} \right\|_{L_\psi^\infty} &\lesssim \left\| \frac{\phi^{(k)}}{u^2} \langle x \rangle^{(k + \frac{1}{4} - \sigma_{k+1})} \right\|_{L_\psi^\infty} \\ &\quad + \sum_{j=0}^{k-1} \left\| \frac{\partial_x^{k-j} (2\bar{u} + \varepsilon\rho)}{u} \right\|_{L_\psi^\infty} \left\| \frac{\partial_x^j \rho}{u} \right\|_{L_\psi^\infty} \langle x \rangle^{(k + \frac{1}{4} - \sigma_{k+1})} \end{aligned}$$

$$\lesssim \Upsilon^{(k)} + \varepsilon \left\| \frac{\rho^{(k)}}{u} \langle x \rangle^{(k+\frac{1}{4}-\sigma_{k+1})} \right\|_{L_\psi^\infty} + \left(1 + \varepsilon \beta^{(k-1)}\right) \beta^{(k-1)}, \quad (55)$$

upon invoking the induction hypothesis and estimate (51), and that  $\sigma_{k+1} > \sigma_{j+1}$  when  $0 < j < k$ . This concludes the proof.  $\square$

**Lemma 15.** *Let  $\phi$  solve (20), and assume the bootstrap (25). The following estimates are valid, for constants independent of  $I$ :*

$$\sup_{x \in I} \left\| \frac{\phi^{(k)}}{u^2} \langle x \rangle^{k-\sigma_{k+1}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \lesssim \|\phi\|_{X_{k+1}(I)} \quad (56)$$

$$\gamma^{(k)} \lesssim \|\phi\|_{X_{k+1}(I)} + \varepsilon \gamma^{(k-1)} \beta^{\lfloor \frac{k-1}{2} \rfloor} + \gamma^{(k-1)}, \quad \gamma^{(0)} \lesssim \langle x \rangle^{\sigma_1} \|\phi\|_{X_1(I)}. \quad (57)$$

**Proof.** First, consider (56). In the region  $\xi \geq 1$ , we have  $u \gtrsim 1$ , in which case the result follows from the definition of the norm. Thus we may restrict to  $\xi \leq 1$ , in which case we use that  $u^2 \gtrsim \eta^2 \gtrsim \xi = \frac{\psi}{\sqrt{x}}$  on the region where  $\xi \leq 1$ :

$$\begin{aligned} \left\| \frac{\phi^{(k)}}{u^2} \right\|_{L_\psi^2(\xi \lesssim 1)} &\lesssim \left\| \frac{\phi^{(k)}}{\xi} \right\|_{L_\psi^2(\xi \lesssim 1)} = \sqrt{x} \left\| \frac{\phi^{(k)}}{\psi} \right\|_{L_\psi^2(\xi \lesssim 1)} \\ &\lesssim \sqrt{x} \left\| \phi_\psi^{(k)} \right\|_{L_\psi^2(\xi \lesssim 1)} + \left\| \phi^{(k)} \chi(\xi \sim 1) \right\|_{L_\psi^2} \\ &\lesssim \langle x \rangle^{-(k-\sigma_{k+1})} \left\| \phi \right\|_{X_{k+1}(I)}. \end{aligned}$$

Above, we have used the Hardy inequality in the  $\psi$  direction, admissible because  $\phi^{(k)}|_{\psi=0} = 0$ . The same proof works when the weight of  $\psi^{\frac{1}{2}-\kappa}$  is added.

We now move to the second line. First, for the base case of  $k = 0$ ,

$$\left\| \frac{\rho}{u} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \leq \left\| \frac{\phi}{u(u+\bar{u})} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \lesssim \left\| \frac{\phi}{u^2} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \lesssim \langle x \rangle^{\sigma_1} \left\| \phi \right\|_{X(I)}. \quad (58)$$

This gives the  $\gamma^{(0)}$  estimate in (57).

We now fix  $k$  in the range  $1 \leq k \leq K_0 - 1$  (so, in particular, (25) is valid), and assume inductively that

$$\left\| \frac{\rho^{(r)}}{u+\bar{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \lesssim \langle x \rangle^{-r+\sigma_{r+1}}, \quad \text{for } r = 0, \dots, k-1. \quad (59)$$

Using (54), we obtain

$$\left\| \frac{\rho^{(k)}}{u} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2}$$

$$\begin{aligned}
&\lesssim \left\| \frac{\phi^{(k)}}{u(u+\bar{u})} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} + \sum_{j=0}^k c_{j,k} \left\| \frac{\partial_x^{k-j}(\bar{u}+u)}{u(u+\bar{u})} \partial_x^j \rho \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\
&\lesssim \left\| \frac{\phi^{(k)}}{\bar{u}^2} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} c_{j,k} \left\| \frac{\partial_x^{k-j}(\bar{u}+u)}{u} \frac{\partial_x^j \rho}{(u+\bar{u})} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\
&+ \sum_{j=\lfloor \frac{k-1}{2} \rfloor}^{k-1} c_{j,k} \left\| \frac{\partial_x^{k-j}(\bar{u}+u)}{u(u+\bar{u})} \right\|_{L^\infty_\psi} \left\| \partial_x^j \rho \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\
&\lesssim \langle x \rangle^{-(k-\sigma_{k+1})} \left\| \phi \right\|_{X_{k+1}(I)} + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} c_{j,k} \left\| \frac{\varepsilon \partial_x^{k-j} \rho}{u} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \left\| \frac{\partial_x^j \rho}{\bar{u}} \right\|_{L^\infty_\psi} \\
&+ \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} c_{j,k} \left\| \frac{\partial_x^{k-j} \bar{u}}{u} \right\|_{L^\infty_\psi} \left\| \frac{\partial_x^j \rho}{\bar{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\
&+ \sum_{j=\lfloor \frac{k-1}{2} \rfloor}^{k-1} c_{j,k} \left\| \frac{\partial_x^{k-j}(2\bar{u}+\varepsilon\rho)}{\bar{u}} \right\|_{L^\infty_\psi} \left\| \frac{\partial_x^j \rho}{u} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\
&\lesssim \langle x \rangle^{-(k-\sigma_{k+1})} \left( \left\| \phi \right\|_{X_{k+1}(I)} + \varepsilon \left\| \frac{\rho^{(k)}}{u} \right\|_{L^2_\psi} \left\| \phi \right\|_{X_1(I)} + \gamma^{(k-1)} + \varepsilon \gamma^{(k-1)} \beta^{\lfloor \frac{k-1}{2} \rfloor} \right), \tag{60}
\end{aligned}$$

which thus gives the desired result.  $\square$

**Lemma 16.** Fix any subinterval  $I \subset I_* \subset (0, \infty)$ . For  $k \geq 1$ ,

$$\begin{aligned}
\Upsilon^{(k)} &\lesssim \left\| \phi \right\|_{X_{k+1}(I)} + \Upsilon^{(k-1)} + \varepsilon \Upsilon^{(k-1)} \beta^{\lfloor \frac{k-1}{2} \rfloor} + \varepsilon \gamma^{(k-1)} \iota^{\lfloor \frac{k-1}{2} \rfloor} \\
&\quad + \varepsilon \gamma^{(k)} \Upsilon^{(k-1)} + (1 + \varepsilon \beta^{(k-1)}) \left\| \phi \right\|_{X_k}, \\
\Upsilon^{(0)} &\lesssim \left\| \phi \right\|_{X_1(I)} \tag{61}
\end{aligned}$$

for a universal constant, independent of the interval  $I$ .

**Proof.** The base case has been treated in estimate (33). By taking  $\partial_x^{k-1}$  of Equation (20), we have

$$u \phi_{\psi\psi}^{(k)} = \phi^{(k+1)} - \underbrace{\sum_{j=1}^k c_j \partial_x^j u \partial_x^{k-j} \phi_{\psi\psi}}_{L^{(k)}} + \sum_{j=0}^k c_j \partial_x^j A \partial_x^{k-j} \phi. \tag{62}$$

Dividing through by  $\sqrt{u}$ ,

$$\left\| \sqrt{u} \phi_{\psi\psi}^{(k)} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi}$$

$$\begin{aligned}
&\leq \left\| \frac{\phi^{(k+1)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \left\| \frac{\partial_x^j (\bar{u} + \varepsilon \rho)}{u} \right\|_{L_\psi^\infty} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-j)} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \\
&\quad + \sum_{j=\lfloor \frac{k}{2} \rfloor}^k \left\| \frac{\varepsilon \partial_x^j \rho}{u} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-j)} \right\|_{L_\psi^\infty} + \sum_{j=\lfloor \frac{k}{2} \rfloor}^k \left\| \frac{\partial_x^j \bar{u}}{u} \right\|_{L_\psi^\infty} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-j)} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \\
&\quad + \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \left\| \partial_x^j A \right\|_{L_\psi^\infty} \left\| \frac{\phi^{(k-j)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} + \sum_{j=\lfloor \frac{k}{2} \rfloor}^k \left\| \partial_x^j A_0 \right\|_{L_\psi^\infty} \left\| \frac{\phi^{(k-j)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \\
&\quad + \sum_{j=\lfloor \frac{k}{2} \rfloor}^k \varepsilon \left\| \partial_x^j A_1 \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \left\| \frac{\phi^{(k-j)}}{\sqrt{u}} \right\|_{L_\psi^\infty} \\
&\lesssim \langle x \rangle^{-(k+1-\sigma_{k+1})} \left( \left\| \phi \right\|_{X_{k+1}} + \Upsilon^{(k-1)} + \varepsilon \beta^{\lfloor \frac{k}{2} \rfloor} \Upsilon^{(k-1)} + \varepsilon \gamma^{(k)} \iota^{\lfloor \frac{k}{2} \rfloor} \right. \\
&\quad \left. + (1 + \varepsilon \gamma^{(k)} + \varepsilon P(\gamma^{(0)}, \dots, \gamma^{(k-1)}, \beta^{(0)}, \dots, \beta^{(k-1)})) \left( \Upsilon^{\lfloor \frac{k}{2} \rfloor} + \left\| \phi \right\|_{X_{\lfloor \frac{k}{2} \rfloor + 1}} \right) \right. \\
&\quad \left. + (1 + \varepsilon \beta^{\lfloor \frac{k}{2} \rfloor} + \varepsilon P(\beta^{(0)}, \dots, \beta^{\lfloor \frac{k}{2} \rfloor - 1})) \left\| \phi \right\|_{X_k} \right).
\end{aligned}$$

Above, we have invoked that  $\sigma_{k+1} > \sigma_j$  when  $j < k+1$ . We have also invoked the decomposition of  $\partial_x^j A = A_0^{(j)} + \varepsilon A_1^{(j)}$  and applied estimates for each piece from (44), (45).  $\square$

**Lemma 17.** Fix any subinterval  $I \subset I_* \subset (0, \infty)$ . For  $k \geq 1$ ,

$$\begin{aligned}
\iota^{(k)} &\lesssim \Upsilon^{(k+1)} + \left\| \phi \right\|_{X_{k+1}(I)} + (1 + \varepsilon \beta^{(k)}) \iota^{(k-1)} + (1 + \varepsilon \beta^{(k)}) \Upsilon^{(k)}, \\
\iota^{(0)} &\lesssim \left\| \phi \right\|_{X_1(I)}
\end{aligned} \tag{63}$$

for a universal constant, independent of the interval  $I$ .

**Proof.** We again estimate using (5) via

$$\begin{aligned}
\left\| \sqrt{u} \phi_{\psi\psi}^{(k)} \right\|_{L_\psi^\infty} &\leq \left\| \frac{\phi^{(k+1)}}{\sqrt{u}} \right\|_{L_\psi^\infty} + \sum_{j=1}^k \left\| \frac{\partial_x^j (\bar{u} + \varepsilon \rho)}{u} \right\|_{L_\psi^\infty} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-j)} \right\|_{L_\psi^\infty} \\
&\quad + \sum_{j=0}^k \left\| \partial_x^j A \right\|_{L_\psi^\infty} \left\| \frac{\phi^{(k-j)}}{\sqrt{u}} \right\|_{L_\psi^2}.
\end{aligned}$$

From here the result follows upon invoking (51) for the first term, estimate (42) for the final term, and recalling the definitions of (40), (16).  $\square$

By combining the estimates (52), (57), (61) and (63), we have established

**Corollary 18.** The following estimates are valid, independent of the interval  $I$ :

$$\iota^{(k-1)} + \Upsilon^{(k)} + \beta^{(k)} + \gamma^{(k)} \lesssim \left\| \phi \right\|_{X_{k+1}(I)}, \tag{64}$$

$$\sup_{x \in I} \left\| L^{(k-1)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \lesssim \varepsilon \left\| \phi \right\|_{X_k(I)} + \left\| \phi \right\|_{X_{k-1}(I)}. \tag{65}$$

### 2.2.3. Miscellaneous Estimates

**Lemma 19.** *Let  $\phi$  solve (20), and  $\rho$  be given by (19). Then the following estimates are valid:*

$$\sup_{x \in I} \left( \left\| \sqrt{u} \rho^{(k)} \langle \psi \rangle^{\frac{1}{2} - \kappa} \langle x \rangle^{k - \sigma_k} \right\|_{L_\psi^2} + \left\| u^{\frac{3}{2}} \partial_x^k \frac{\rho}{\bar{u}} \langle \psi \rangle^{\frac{1}{2} - \kappa} \langle x \rangle^{k - \sigma_k} \right\|_{L_\psi^2} \right) \lesssim \|\phi\|_{X_k(I)}. \quad (66)$$

**Proof.** We directly estimate using (54), which gives

$$\begin{aligned} & \left\| \sqrt{u} \rho^{(k)} \langle \psi \rangle^{\frac{1}{2} - \kappa} \right\|_{L_\psi^2} \\ & \leq \left\| \sqrt{u} \frac{\phi^{(k)}}{u + \bar{u}} \langle \psi \rangle^{\frac{1}{2} - \kappa} \right\|_{L_\psi^2} + \sum_{j=0}^m c_{j,k} \left\| \frac{1}{\bar{u}} \partial_x^{k-j} (\bar{u} + u) \partial_x^j \rho \sqrt{u} \langle \psi \rangle^{\frac{1}{2} - \kappa} \right\|_{L_\psi^2} \\ & \lesssim \langle x \rangle^{-(k - \sigma_k)} \|\phi\|_{X_k(I)} + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} c_{j,k} \left\| \frac{1}{\bar{u}} \partial_x^{k-j} (2\bar{u} + \varepsilon\rho) \partial_x^j \rho \sqrt{u} \langle \psi \rangle^{\frac{1}{2} - \kappa} \right\|_{L_\psi^2} \\ & \quad + \sum_{j=\lfloor \frac{k-1}{2} \rfloor}^{k-1} c_{j,k} \left\| \frac{1}{\bar{u}} \partial_x^{k-j} (2\bar{u} + \varepsilon\rho) \partial_x^j \rho \sqrt{u} \langle \psi \rangle^{\frac{1}{2} - \kappa} \right\|_{L_\psi^2} \\ & \lesssim \langle x \rangle^{-(k - \sigma_k)} \|\phi\|_{X(I)} + \varepsilon \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} c_{j,k} \left\| \partial_x^{k-j} \rho \sqrt{u} \langle \psi \rangle^{\frac{1}{2} - \kappa} \right\|_{L_\psi^2} \left\| \frac{\partial_x^j \rho}{u} \right\|_{L_\psi^\infty} \\ & \quad + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} c_{j,k} \left\| \frac{\partial_x^{k-j} \bar{u}}{u} \right\|_{L_\psi^\infty} \left\| \partial_x^j \rho \sqrt{u} \langle \psi \rangle^{\frac{1}{2} - \kappa} \right\|_{L_\psi^2} \\ & \quad + \sum_{j=\lfloor \frac{k-1}{2} \rfloor}^{k-1} c_{j,k} \left\| \frac{1}{\bar{u}} \partial_x^{k-j} (2\bar{u} + \varepsilon\rho) \right\|_{L_\psi^\infty} \left\| \partial_x^j \rho \sqrt{u} \langle \psi \rangle^{\frac{1}{2} - \kappa} \right\|_{L_\psi^2} \\ & \lesssim \langle x \rangle^{-(k - \sigma_k)} \left( \|\phi\|_{X_k(I)} + \varepsilon \left\| \sqrt{u} \rho^{(k)} \langle \psi \rangle^{\frac{1}{2} - \kappa} \langle x \rangle^{k - \sigma_k} \right\|_{L_\psi^2} \|\phi\|_{X_{k-1}(I)} \right). \end{aligned}$$

This concludes the estimate of the first term in (66). For the second term, we simply expand using the product rule, as done in (49). The result then follows from the first estimate in (66), upon using that  $\|\frac{u}{\bar{u}}\|_{L^\infty} \lesssim 1$ , and  $\|\bar{u} \partial_x^l \frac{1}{\bar{u}} \langle x \rangle^l\|_{L^\infty} \lesssim 1$ .  $\square$

We will also invoke the following lemma about  $A$ , defined in (20):

**Lemma 20.** *Assume (25). For  $k \geq 1$ , there exists a decomposition  $\partial_x^k A = A_0^{(k)} + \varepsilon A_1^{(k)}$  where*

$$\sup_{x \in I} \left\| A_0^{(k)} \langle x \rangle^k \right\|_{L^\infty} \leq C_k, \quad (67)$$

$$\sup_{x \in I} \left\| u^{\frac{3}{2}} A_1^{(k)} \langle x \rangle^{k+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \lesssim \varepsilon \|\phi\|_{X_k(I)} + \|\phi\|_{X_{k-1}(I)}, \quad (68)$$

$$\sup_{x \in I} \left\| u^{\frac{3}{2}} \partial_x^k A \langle x \rangle^{k+\frac{3}{4}} \right\|_{L^2_\psi} \lesssim \varepsilon \|\phi\|_{X_k(I)} + \|\phi\|_{X_{k-1}(I)} + 1 \quad (69)$$

for a universal constant  $C_k$  and an implicit constant in (68) that are independent of  $I$ .

**Proof.** We first recall the decomposition from (48). The first estimate, (67), is clear from the definition of  $A_0$  and by using the property (47).

For estimate (68), we recall the definition of  $A_1$  from (48). We first establish the result for  $A_1^{(1)}$ , which will serve as our base case:

$$\begin{aligned} \left\| u^{\frac{3}{2}} A_1^{(1)} \langle x \rangle^{1+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} &\lesssim \left\| u^{\frac{3}{2}} A \partial_x \frac{\rho}{\bar{u}} \langle x \rangle^{1+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\ &\lesssim \|A \langle x \rangle\|_\infty \left\| u^{\frac{3}{2}} \partial_x \frac{\rho}{\bar{u}} \langle x \rangle^{\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\ &\lesssim \|\phi\|_{X_1}, \end{aligned}$$

upon invoking (41) to estimate the  $A$  term, and (66) to estimate the  $\rho$  term (and the bootstrap, (25)).

We thus assume (68) holds for all indices  $0, \dots, k-1$ , and further decompose it via

$$\begin{aligned} &\left\| u^{\frac{3}{2}} A_1^{(k)} \langle x \rangle^{k+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\ &\lesssim \sum_{l=0}^{k-1} \left\| u^{\frac{3}{2}} \partial_x^l A_0 \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} + \varepsilon \sum_{l=0}^{k-1} \left\| u^{\frac{3}{2}} \partial_x^l A_1 \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\ &\lesssim \sum_{l=0}^{k-1} \left\| u^{\frac{3}{2}} \partial_x^l A_0 \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} + \varepsilon \sum_{l=0}^{\lfloor \frac{k-1}{2} \rfloor} \left\| u^{\frac{3}{2}} \partial_x^l A_1 \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\ &\quad + \varepsilon \sum_{l=\lfloor \frac{k-1}{2} \rfloor}^{k-1} \left\| u^{\frac{3}{2}} \partial_x^l A_1 \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\ &\lesssim \sum_{l=0}^{k-1} \left\| \partial_x^l A_0 \langle x \rangle^{l+1} \right\|_{L^\infty} \left\| u^{\frac{3}{2}} \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k-l+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\ &\quad + \varepsilon \sum_{l=0}^{\lfloor \frac{k-1}{2} \rfloor} \left\| \partial_x^l A_1 \langle x \rangle^{l+1} \right\|_{L^\infty} \left\| u^{\frac{3}{2}} \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k-l+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \\ &\quad + \varepsilon \sum_{l=\lfloor \frac{k-1}{2} \rfloor}^{k-1} \left\| u^{\frac{3}{2}} \partial_x^l A_1 \langle x \rangle^{l+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi} \left\| \partial_x^{k-l} \frac{\rho}{\bar{u}} \langle x \rangle^{k-l} \right\|_{L^\infty_\psi} \\ &\lesssim \|\phi\|_{X_k}. \end{aligned}$$

The  $\rho$  terms in  $L^2_\psi$  in the first two sums above is estimated upon invoking (66), the  $A_0$  term in  $L^\infty$  in the first sum above is estimated using (47), the  $A_1$  term in  $L^\infty$  in the second sum above is estimated using (42). For the third sum, we use that  $l \leq k-1$  to invoke the induction hypothesis, and finally the  $L^\infty$  term of  $\rho$  we expand as in (49) to see it is controlled by  $\|\phi\|_{X_{k-l+1}}$ .

For the final inequality, (69), we have

$$\left\| u^{\frac{3}{2}} \partial_x^k A \langle x \rangle^{k+\frac{3}{4}} \right\|_{L^2_\psi} \leq \left\| u^{\frac{3}{2}} A_0^{(k)} \langle x \rangle^{k+\frac{3}{4}} \right\|_{L^2_\psi} + \varepsilon \left\| u^{\frac{3}{2}} A_1^{(k)} \langle x \rangle^{k+\frac{3}{4}} \right\|_{L^2_\psi};$$

the  $A_0^{(k)}$  term is clearly seen to be bounded by a universal constant upon invoking (47), whereas the  $A_1^{(k)}$  term is bounded according to (68).  $\square$

A key feature we take advantage of is that decay is enhanced in the region  $\xi \leq 1$  due to controlling the weight  $\psi^{\frac{1}{2}-\kappa}$  in the norm, (16).

**Lemma 21.** ( $L^2(\xi \lesssim 1)$  Estimates) *Let  $\phi$  solve (20), and assume the bootstrap (25). Then, for  $\alpha = 0, 1, 2$ ,*

$$\sup_{x \in I} \left\| \partial_\psi^\alpha \phi^{(k)} \langle x \rangle^{k+\frac{\alpha}{2}+\frac{1}{4}-\sigma_{k+2}-\frac{\kappa}{2}} \right\|_{L^2_\psi(\xi \leq 1)} \lesssim \|\phi\|_{X_{k+2}(I)} \quad (70)$$

for a constant independent of  $I$ .

**Proof.** We first address the  $j = 0$  case by rearranging the Equation (20) to obtain

$$\begin{aligned} \|\phi_{\psi\psi}\|_{L^2_\psi(\xi \lesssim 1)} &= \left\| \frac{1}{u} \phi_x \right\|_{L^2_\psi(\xi \leq 1)} + \left\| \frac{1}{u} A \phi \right\|_{L^2_\psi(\xi \leq 1)} \\ &\lesssim \langle x \rangle^{\frac{1}{4}} \left\| \frac{\phi^{(1)}}{\sqrt{\psi}} \right\|_{L^2_\psi(\xi \leq 1)} + \|A \langle x \rangle\|_\infty \langle x \rangle^{-\frac{3}{4}} \left\| \frac{\phi}{\sqrt{\psi}} \right\|_{L^2(\xi \leq 1)} \\ &\lesssim \langle x \rangle^{\frac{1}{4}} \left\| \frac{\phi^{(1)}}{\sqrt{\psi}} \right\|_{L^2_\psi(\xi \leq 1)} + \|A \langle x \rangle\|_\infty \langle x \rangle^{-\frac{3}{4}} \left\| \frac{\phi}{\sqrt{\psi}} \right\|_{L^2(\xi \leq 1)} \\ &\lesssim \langle x \rangle^{\frac{1}{4}+\frac{\kappa}{2}} \left\| \frac{\phi^{(1)}}{\psi^{\frac{1}{2}+\kappa}} \right\|_{L^2_\psi(\xi \leq 1)} + \|A \langle x \rangle\|_\infty \langle x \rangle^{-\frac{3}{4}+\frac{\kappa}{2}} \left\| \frac{\phi}{\psi^{\frac{1}{2}+\kappa}} \right\|_{L^2(\xi \leq 1)} \\ &\lesssim \langle x \rangle^{\frac{1}{4}+\frac{\kappa}{2}} \left\| \phi_\psi^{(1)} \psi^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi(\xi \leq 1)} + \langle x \rangle^{\frac{1}{4}+\frac{\kappa}{2}} \left\| \phi^{(1)} \psi^{\frac{1}{2}-\kappa} \langle x \rangle^{-\frac{1}{2}} \chi'(\xi) \right\|_{L^2_\psi} \\ &\quad + \langle x \rangle^{-\frac{3}{4}+\frac{\kappa}{2}} \left\| \psi^{\frac{1}{2}-\kappa} \phi_\psi \right\|_{L^2(\xi \leq 1)} + \langle x \rangle^{-\frac{3}{4}+\frac{\kappa}{2}} \left\| \psi^{\frac{1}{2}-\kappa} \phi \chi'(\xi) \langle x \rangle^{-\frac{1}{2}} \right\|_{L^2} \\ &\lesssim \langle x \rangle^{-\frac{5}{4}+\sigma_2+\frac{\kappa}{2}} \left\| \phi \right\|_{X_2(I)}. \end{aligned} \quad (71)$$

Above, we have used the localization to the region  $\{\xi \leq 1\}$  in several ways. First, we have bounded  $u$  below by a factor of  $\sqrt{\xi}$ . Second, we use the inequality  $1 \leq \frac{x^{\frac{\kappa}{2}}}{\psi^\kappa}$ . The second to last inequality is the Hardy inequality, admissible because both

$\phi, \phi^{(1)}|_{\psi=0} = 0$ . In the final estimate, we simply appeal to the definition of the norm, (16).

We now integrate via

$$\begin{aligned} \chi(\xi)|\phi| &= \chi(\xi) \left| \int_0^\psi \phi_\psi \right| \lesssim \chi(\xi) \sqrt{\psi} \left\| \phi_\psi \right\|_{L_\psi^2(\xi \lesssim 1)} \lesssim \chi(\xi) \langle x \rangle^{\frac{1}{4}} \left\| \phi_\psi \right\|_{L_\psi^2(\xi \lesssim 1)} \\ &\lesssim \langle x \rangle^{-\frac{1}{2}} \left( \langle x \rangle^{\frac{3}{4}-\sigma_{j+2}-\frac{\kappa}{2}} \left\| \phi_\psi \right\|_{L_\psi^2(\xi \lesssim 1)} \right), \end{aligned} \quad (72)$$

Above, we have used in the support of  $\chi(\xi)$  that  $\sqrt{\psi} \leq \langle x \rangle^{\frac{1}{4}}$ . In turn, (72) implies

$$\left\| \phi \right\|_{L_\psi^2(\xi \lesssim 1)} \lesssim \langle x \rangle^{-\frac{1}{4}} \left( \langle x \rangle^{\frac{3}{4}-\sigma_{j+2}-\frac{\kappa}{2}} \left\| \phi_\psi \right\|_{L_\psi^2(\xi \lesssim 1)} \right). \quad (73)$$

We can make the above estimate global in  $\xi$  by simply noting that

$$\begin{aligned} \left\| \phi \right\|_{L_\psi^2(\xi \geq 1)} &= \left\| \phi \psi^{\frac{1}{2}-\kappa} \psi^{-\frac{1}{2}+\kappa} \right\|_{L_\psi^2(\xi \geq 1)} \\ &\lesssim \langle x \rangle^{-\left(\frac{1}{4}-\frac{\kappa}{2}\right)} \left\| \phi \psi^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2(\xi \geq 1)} \lesssim \langle x \rangle^{-\left(\frac{1}{4}-\frac{\kappa}{2}\right)} \left\| \phi \right\|_{X_0(I)}. \end{aligned} \quad (74)$$

For the enhanced localized  $\phi_\psi$  estimate, we have first, by a standard interpolation, and second by inserting (73) for the  $\left\| \phi \right\|_{L_\psi^2}$  term, (71) for the  $\left\| \phi_\psi \right\|_{L_\psi^2}$  term, and (74) for the final  $\phi$  term as follows

$$\begin{aligned} \left\| \phi_\psi \right\|_{L_\psi^2(\xi \leq 1)} &\lesssim \left\| \phi \right\|_{L_\psi^2(\xi \leq 1)}^{\frac{1}{2}} \left\| \phi_\psi \right\|_{L_\psi^2(\xi \leq 1)}^{\frac{1}{2}} + \langle x \rangle^{-\frac{1}{2}} \left\| \phi \right\|_{L_\psi^2(\xi \geq 1)} \\ &\lesssim \langle x \rangle^{-\frac{1}{8}} \left( \langle x \rangle^{\frac{3}{4}-\sigma_{j+2}-\frac{\kappa}{2}} \left\| \phi_\psi \right\|_{L_\psi^2(\xi \lesssim 1)} \right)^{\frac{1}{2}} \langle x \rangle^{-\left(\frac{5}{8}-\frac{\sigma_2}{2}-\frac{\kappa}{4}\right)} \left\| \phi \right\|_{X_2}^{\frac{1}{2}} \\ &\quad + \langle x \rangle^{-\left(\frac{3}{4}-\frac{\kappa}{2}\right)} \left\| \phi \right\|_{X_0(I)} \\ &\leq \delta \left\| \phi_\psi \right\|_{L_\psi^2(\xi \leq 1)} + C_\delta \langle x \rangle^{-\frac{3}{4}} \left\| \phi \right\|_{X_2(I)} + C \langle x \rangle^{-\left(\frac{3}{4}-\frac{\kappa}{2}\right)} \left\| \phi \right\|_{X_0(I)}. \end{aligned}$$

For any  $\delta > 0$ . By absorbing the  $\delta$  term to the left-hand side, we conclude the proof for  $k = 0$ . The general  $k$  case follows in an analogous manner, upon invoking the expression (5) for the  $\phi_{\psi\psi}$  estimate.  $\square$

**Lemma 22.** ( $L^\infty$  Estimates) *Let  $\phi$  solve Equation (20), and assume the bootstrap (25). Then the following estimate is valid:*

$$\left\| \sqrt{u} \phi_{\psi\psi}^{(k)} \right\|_{L_\psi^\infty(\xi \leq 1)} \langle x \rangle^{k+\frac{3}{2}-\sigma_{j+3}-\frac{\kappa}{2}} \lesssim \left\| \phi \right\|_{X_{k+3}(I)}, \quad (75)$$

for a constant independent of  $I$ .

**Proof.** We again consider the  $k = 0$  case. We use the Equation (20) to write

$$\left\| \sqrt{u} \phi_{\psi\psi} \right\|_{L_{\psi}^{\infty}(\xi \leq 1)} = \left\| \frac{1}{\sqrt{u}} \phi^{(1)} \right\|_{L_{\psi}^{\infty}(\xi \leq 1)} + \left\| \frac{A}{\sqrt{u}} \phi \right\|_{L_{\psi}^{\infty}(\xi \leq 1)}. \quad (76)$$

For the first term, we estimate

$$\begin{aligned} \chi(\xi) \left| \frac{1}{u} \phi^{(1)} \right| &\lesssim \chi(\xi) \langle x \rangle^{\frac{1}{8}} \psi^{-\frac{1}{4}} \left| \int_0^{\psi} \phi_{\psi}^{(1)} \right| \lesssim \chi(\xi) \langle x \rangle^{\frac{1}{8}} \psi^{-\frac{1}{4}} \psi^{\frac{1}{2}} \left\| \phi_{\psi}^{(1)} \right\|_{L_{\psi}^2(\xi \leq 1)} \\ &\lesssim \left\| \phi_{\psi}^{(1)} \right\|_{L_{\psi}^2(\xi \lesssim 1)} \lesssim \langle x \rangle^{-\left(\frac{3}{2} - \sigma_j + 3 - \frac{\kappa}{2}\right)} \left\| \phi \right\|_{X_3(I)}, \end{aligned}$$

upon invoking (70). The second term in (76) follows similarly, upon invoking that  $\|A\langle x \rangle\|_{\infty} \lesssim 1$ . The general  $k$  case follows from invoking expression (5). This concludes the proof.  $\square$

### 3. Baseline Tier: $L^2$ Estimates

In this section, we obtain two estimates at the  $L^2$  level - the energy estimate and the quotient estimate. The reader is urged to keep in mind the linearized structure which is present at the  $L^2$  level, Equation (21).

We urge the reader to keep in mind the bootstrap assumption, (25), which will be in implicit use throughout the estimates of this section. In particular, we will repeatedly use the following inequalities to bound nonlinear quantities, which has been rigorously established in (30) - (51):

$$u \lesssim \bar{u} \lesssim u, \quad |\rho| \lesssim u, \quad |\rho| \lesssim \bar{u}. \quad (77)$$

Before coming to our energy estimate, we first prove the following lemma:

**Lemma 23.** *Let  $\bar{u}$  be the Blasius solution. Then denoting  $\bar{u}_x$  the  $\partial_x$  in  $(x, \psi)$  coordinates,*

$$\bar{u}_x(x, \psi) = \bar{u}_x(x, y) + \frac{\bar{v}}{\bar{u}} \bar{u}_y(x, y) = \frac{\bar{u}_{yy}}{\bar{u}}. \quad (78)$$

**Proof.** This follows from the chain rule. Specifically, differentiate the equality  $\bar{u}(x, y) = \bar{u}(x, \psi)$  to obtain the identities

$$\bar{u}_y(x, y) = \bar{u} \bar{u}_{\psi}(x, \psi), \quad \bar{u}_{yy}(x, y) = \bar{u} \bar{u}_{\psi}^2(x, \psi) + \bar{u}^2 \bar{u}_{\psi\psi}(x, \psi), \quad (79)$$

and

$$\bar{u}_x(x, y) = \bar{u}_x(x, \psi) + \bar{u}_{\psi} \psi_x(x, \psi) = \bar{u}_x(x, \psi) - \bar{v} \bar{u}_{\psi}(x, \psi). \quad (80)$$

Rearranging (80) and invoking the first identity in (79) yields  $\bar{u}_x(x, \psi) = \bar{u}_x(x, y) + \bar{v} \frac{\bar{u}_y}{\bar{u}}(x, y)$ , which is the first equality of (78). The second equality of (78) follows from  $\bar{u}$  solving the Prandtl equation, (1).  $\square$

**Lemma 24.** (Energy Estimate) *Let  $\phi$  solve (20). Assume the bootstrap assumption (25). Then, for  $K_0 \gg 1$ ,*

$$\frac{\partial_x}{2} \int \phi^2 + \int \bar{u} |\phi_\psi|^2 \lesssim \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x). \quad (81)$$

**Proof.** We take inner product of the Equation (20) against  $\phi$  to obtain

$$\frac{\partial_x}{2} \int \phi^2 + \int \bar{u} |\phi_\psi|^2 - \frac{1}{2} \int \bar{u}_{\psi\psi} |\phi|^2 + \int \varepsilon \rho \phi_{\psi\psi} \phi + \int A |\phi|^2 = 0. \quad (82)$$

We now use two properties:

$$\bar{u} \geq 0, \quad \bar{u}^2 \bar{u}_{\psi\psi} = \bar{u}_{yy} - \bar{u} |\bar{u}_\psi|^2 \leq 0,$$

which hold by properties of the Blasius profile. We also use that  $A \geq 0$ , which again holds by the concavity of the Blasius profile, (7). Finally, we estimate

$$\begin{aligned} \left| \int \varepsilon \rho \phi_{\psi\psi} \phi \right| &\lesssim \varepsilon \langle x \rangle^{-(\frac{5}{4} - 2\sigma_1)} \left\| \frac{\rho}{u} \langle x \rangle^{\frac{1}{4} - \sigma_1} \right\|_{L_\psi^\infty} \|u \phi_{\psi\psi} \langle x \rangle^{1 - \sigma_1}\|_{L_\psi^2} \|\phi\|_{L_\psi^2} \\ &\lesssim \varepsilon \langle x \rangle^{-(\frac{5}{4} - 2\sigma_1)} \beta^{(0)} \Upsilon^{(0)} E_0(x) \\ &\lesssim \varepsilon \langle x \rangle^{-(\frac{5}{4} - 2\sigma_1)} E_{K_0}(x)^3 \leq \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x), \end{aligned}$$

where we have used the definitions (40), inequalities (64) to control the  $\beta$ ,  $\Upsilon$  terms, and that  $2\sigma_1 < \frac{1}{4}$ . This concludes the proof.  $\square$

**Lemma 25.** (Quotient Estimate) *Let  $\phi$  be a solution to (20). Assume the bootstrap assumption, (25). Then the following estimate is valid:*

$$\frac{\partial_x}{2} \int \phi^2 \frac{1}{u} \langle \psi \rangle + \int \phi_\psi^2 \langle \psi \rangle \lesssim \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x). \quad (83)$$

**Proof.** We have the identity

$$\frac{\partial_x}{2} \int \phi^2 \frac{1}{u} \langle \psi \rangle + \int \phi_\psi^2 \langle \psi \rangle + \int \phi^2 \langle \psi \rangle \frac{1}{2} \frac{u_x}{u^2} + \int A \phi^2 \frac{\langle \psi \rangle}{u} = 0. \quad (84)$$

We group the latter two terms together via

$$(84.3) + (84.4) =: \int \phi^2 \langle \psi \rangle \frac{1}{u^3} \mathring{\Omega} = \int \phi^2 \langle \psi \rangle \frac{1}{u^3} [\Omega + \Omega^R],$$

where

$$\begin{aligned} \mathring{\Omega} &= \frac{u u_x}{2} + u^2 A = \frac{u u_x}{2} - 2u^2 \frac{\bar{u}_{yy}}{\bar{u}(\bar{u} + u)}, \\ \Omega &= -\bar{u}_{yy} + \frac{1}{2} \bar{u} \bar{u}_x, \\ \Omega^R &:= A \varepsilon \phi + \frac{\bar{u}_{yy} \varepsilon \rho}{2\bar{u} + \varepsilon \rho} + \frac{\varepsilon \rho \bar{u}_x + \varepsilon \bar{u} \rho_x + \varepsilon^2 \rho \rho_x}{2}. \end{aligned}$$

For the reader's convenience, we derive the equality above upon substituting (10):

$$\begin{aligned}
\dot{\Omega} &:= \frac{uu_x}{2} - 2u^2 \frac{\bar{u}_{yy}}{\bar{u}(\bar{u} + u)} \\
&= \frac{1}{2}(\bar{u} + \varepsilon\rho)(\bar{u}_x + \varepsilon\rho_x) - 2\bar{u}^2 \frac{\bar{u}_{yy}}{\bar{u}(2\bar{u} + \varepsilon\rho)} - \varepsilon\phi 2 \frac{\bar{u}_{yy}}{\bar{u}(\bar{u} + u)} \\
&= \frac{1}{2}\bar{u}\bar{u}_x + \frac{1}{2}\left(\rho\bar{u}_x + \bar{u}\rho_x + \varepsilon\rho\rho_x\right) - \bar{u}_{yy} + \varepsilon\rho \frac{\bar{u}_{yy}}{2\bar{u} + \varepsilon\rho} + \varepsilon\phi A \\
&= \Omega + \Omega^R.
\end{aligned}$$

First, by (78) and by convexity of the Blasius profile,  $\Omega = -\frac{1}{2}\bar{u}_{yy} > 0$ . We thus need to estimate the nonlinear part in  $\Omega^R$ :

$$\begin{aligned}
\left| \int \phi^2 \langle \psi \rangle \frac{1}{u^3} \Omega^R \right| &\leq \varepsilon \|\phi \langle \psi \rangle^{\frac{1}{2}-\kappa} \frac{1}{\sqrt{u}}\|_{L_\psi^2}^2 \left[ \left\| \frac{A\phi}{u^2} \right\|_{L_\psi^\infty} + \left\| \frac{\bar{u}_{yy}\rho}{u^2(2\bar{u} + \varepsilon\rho)} \right\|_{L_\psi^\infty} \right. \\
&\quad \left. + \frac{1}{2} \left\| \frac{\rho\bar{u}_x}{u^2} \right\|_{L_\psi^\infty} + \frac{1}{2} \left\| \frac{\rho_x\bar{u}}{u^2} \right\|_{L_\psi^\infty} + \frac{\varepsilon}{2} \left\| \frac{\rho\rho_x}{u^2} \right\|_{L_\psi^\infty} \right] \\
&\lesssim \varepsilon \langle x \rangle^{-(1+)} E_{K_0}(x)^3 \leq \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x). \tag{85}
\end{aligned}$$

We now proceed to prove the final inequality above after Equation (85) by estimating all five of the  $L_\psi^\infty$  terms above. First, upon invoking (41), (51), and (64), we obtain

$$\left\| \frac{A\phi}{u^2} \right\|_{L_\psi^\infty} \leq \|A\|_{L_\psi^\infty} \left\| \frac{\phi}{u^2} \right\|_{L_\psi^\infty} \lesssim \langle x \rangle^{-1} \langle x \rangle^{-(0+)} E_{K_0}(x)^2.$$

Above, we have also estimated  $A$  via invoking (41).

Second, upon invoking (39),  $u \sim \eta$ ,  $\bar{u}_{yy} \sim \eta^2 \langle x \rangle^{-1}$  on  $\eta \leq 1$ , we estimate

$$\begin{aligned}
\left\| \frac{\bar{u}_{yy}\rho}{u^2(2\bar{u} + \varepsilon\rho)} \right\|_{L_\psi^\infty} &\leq \left\| \frac{\bar{u}_{yy}}{u^2} \right\|_{L_\psi^\infty} \left\| \frac{\rho}{2\bar{u} + \varepsilon\rho} \right\|_{L_\psi^\infty} \lesssim \langle x \rangle^{-1} \left\| \frac{\rho}{2u - \varepsilon u \frac{\rho}{u}} \right\|_{L_\psi^\infty} \\
&\lesssim \langle x \rangle^{-1} \left\| \frac{1}{2 - \varepsilon \frac{\rho}{u}} \right\|_{L_\psi^\infty} \left\| \frac{\rho}{u} \right\|_{L_\psi^\infty} \\
&\lesssim \langle x \rangle^{-1} \left\| \frac{1}{2 - \varepsilon \frac{\rho}{u}} \right\|_{L_\psi^\infty} \beta^{(0)} \lesssim \langle x \rangle^{-(0+)} I_{K_0}(x),
\end{aligned}$$

where we have invoked (64) to  $\beta^{(0)}$ .

Third, again upon invoking (40), (64) and (39),

$$\begin{aligned}
\left\| \frac{\rho\bar{u}_x}{u^2} \right\|_{L_\psi^\infty} &\lesssim \left\| \frac{\rho}{u} \right\|_{L_\psi^\infty} \left\| \frac{\bar{u}_x}{u} \right\|_{L_\psi^\infty} \lesssim \left\| \frac{\rho}{u} \right\|_{L_\psi^\infty} \left\| \frac{\bar{u}_x}{\bar{u}} \right\|_{L_\psi^\infty} \left\| \frac{\bar{u}}{u} \right\|_{L_\psi^\infty} \\
&\lesssim \langle x \rangle^{-(0+)} \beta^{(0)} \langle x \rangle^{-1} \omega \lesssim \langle x \rangle^{-(0+)} I_{K_0}(x).
\end{aligned}$$

Fourth, again upon invoking (40), (64) and (39),

$$\left\| \frac{\rho_x\bar{u}}{u^2} \right\|_{L_\psi^\infty} \lesssim \left\| \frac{\bar{u}}{u} \right\|_{L_\psi^\infty} \left\| \frac{\rho_x}{u} \right\|_{L_\psi^\infty} \lesssim \langle x \rangle^{-(1+)} \omega \beta^{(1)} \lesssim \langle x \rangle^{-(0+)} I_{K_0}(x).$$

Fifth, again upon invoking (40) and (64),

$$\left\| \frac{\rho \rho_x}{u^2} \right\|_{L_\psi^\infty} \leq \left\| \frac{\rho}{u} \right\|_{L_\psi^\infty} \left\| \frac{\rho_x}{u} \right\|_{L_\psi^\infty} \lesssim \langle x \rangle^{-(0+)} \langle x \rangle^{-(1+)} \beta^{(0)} \beta^{(1)} \lesssim \langle x \rangle^{-(0+)} I_{K_0}(x).$$

Inserting these estimates into (85) yields the estimate shown beneath (85). This concludes the proof.  $\square$

#### 4. Middle Tier: $H^k$ for $1 \leq k \leq K_1$

At the  $H^1$  level, the linearized equation changes and so requires a new treatment (compare (21) versus (22)). Taking one  $x$  derivative of (20), we obtain

$$\phi_x^{(1)} - u \phi_{\psi\psi}^{(1)} + A \phi^{(1)} - u^{(1)} \phi_{\psi\psi} + A_x \phi = 0.$$

The point here is that  $u^{(1)}$  can be separated into  $u^{(1)} =: \bar{u}^{(1)} + \varepsilon \rho^{(1)}$ . While the  $\rho^{(1)}$  contribution is quadratic (it carries an  $\varepsilon$  factor), the  $\bar{u}^{(1)}$  contribution is linear and highest order in  $\phi$ . To see this, we use the equation to rewrite  $\phi_{\psi\psi}$  via

$$\phi_x^{(1)} - u \phi_{\psi\psi}^{(1)} + A \phi^{(1)} - u^{(1)} \left[ \frac{\phi^{(1)}}{u} + \frac{A \phi}{u} \right] + A_x \phi = 0. \quad (86)$$

**Remark 26.** (Notational Convention) We introduce the following notation, convenient for majorizing terms appearing in the energy estimates:

$$R_{k-1}(x) := \langle x \rangle^{-(0+)} I_{k-1}(x) + \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x). \quad (87)$$

**Lemma 27.** Let  $\phi$  solve the equation (20). Assume the bootstrap assumption, (25). Then the following inequality is valid:

$$\frac{\partial_x}{2} \int |\phi_\psi|^2 \langle x \rangle^{1-2\omega_1} + \int |\phi_x|^2 \frac{1}{u} \langle x \rangle^{1-2\omega_1} \lesssim R_0(x). \quad (88)$$

**Proof.** To obtain this estimate, we take the inner-product of (20) against  $\phi^{(1)} \frac{1}{u} \langle x \rangle^{1-2\omega_1}$ , which produces the identity

$$\begin{aligned} & \int |\phi_x|^2 \frac{1}{u} \langle x \rangle^{1-2\omega_1} + \frac{\partial_x}{2} \int |\phi_\psi|^2 \langle x \rangle^{1-2\omega_1} \\ &= \frac{1-2\omega_1}{2} \int |\phi_\psi|^2 \langle x \rangle^{-2\omega_1} - \int A \frac{1}{u} \langle x \rangle^{1-2\omega_1} \phi \phi_x. \end{aligned} \quad (89)$$

For the first term on the right-hand side above, we invoke from (83) that  $\|\phi_\psi\|_{L_\psi^2}^2$  is an integrable quantity in  $x$ , and therefore upon using  $\omega_1 > 0$ , it can be bounded by  $\langle x \rangle^{-(0+)} I_0(x)$  (recall our convention for  $I_k(x)$  defined in (18)). For the second term on the right-hand side of (89), we estimate via

$$\left| \int A \frac{1}{u} \langle x \rangle^{1-2\omega_1} \phi \phi_x \right| \lesssim \|A(x)\|_\infty \left\| \frac{\phi}{\sqrt{u}} \langle x \rangle^{-\frac{1}{2}-} \right\|_{L_\psi^2} \left\| \phi_x \langle x \rangle^{\frac{1}{2}-\omega_1} \frac{1}{\sqrt{u}} \right\|_{L_\psi^2}$$

$$\begin{aligned}
&\lesssim \|A(x)\|_\infty \left\| \frac{\phi}{\sqrt{u}} \langle x \rangle^{-\frac{1}{2}} \right\|_{L^2_\psi} \left\| \phi_x \langle x \rangle^{\frac{1}{2}-\omega_1} \frac{1}{\sqrt{u}} \right\|_{L^2_\psi} \\
&\leq \delta \left\| \phi_x \langle x \rangle^{\frac{1}{2}-\omega_1} \frac{1}{\sqrt{u}} \right\|_{L^2_\psi} + C_\delta \langle x \rangle^{-(1+)} E_0(x) \\
&\leq \delta \left\| \phi_x \langle x \rangle^{\frac{1}{2}-\omega_1} \frac{1}{\sqrt{u}} \right\|_{L^2_\psi} + C_\delta \langle x \rangle^{-(0+)} I_0(x),
\end{aligned}$$

where we have used Young's inequality for products for any parameter  $\delta$ . We pick  $\delta$  small, universally, so as to absorb the first term on the right-hand side above to the left-hand side of (89). We have used (41) for  $A$  and (83) for the  $\phi$  term. Again, we invoke (83) to establish that the above quantities are integrable in  $x$ .  $\square$

**Lemma 28.** (Energy Estimate) *Let  $\phi$  solve the equation (20). Assume the bootstrap assumption, (25). Then the following inequality is valid:*

$$\frac{\partial_x}{2} \int |\phi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1} + \int u |\phi_\psi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1} \lesssim R_0(x). \quad (90)$$

**Proof.** We apply the multiplier  $\phi^{(1)} \langle x \rangle^{2-2\sigma_1}$  to (86). Denote by  $R(x)$  the right-hand side of (90). This generates the identity

$$\begin{aligned}
&\frac{\partial_x}{2} \int |\phi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1} + \int u |\phi_\psi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1} - \int \frac{u \psi \psi}{2} |\phi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1} \\
&+ \int A |\phi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1} - \int u^{(1)} \frac{|\phi^{(1)}|^2}{u} \langle x \rangle^{2-2\sigma_1} \\
&= \frac{2-2\sigma_1}{2} \int |\phi^{(1)}|^2 \langle x \rangle^{1-2\sigma_1} + \int u^{(1)} \frac{A \phi}{u} \phi^{(1)} \langle x \rangle^{2-2\sigma_1} + \int A_x \phi \phi^{(1)} \langle x \rangle^{2-2\sigma_1}.
\end{aligned} \quad (91)$$

The third and fourth terms on the left-hand side are damping terms, as in the lowest order estimate. The first term on the right-hand side is controlled by  $R_0(x)$  due to the previous estimate, (88), and our choice that  $\sigma_1 > \omega_1$ .

The new leading order contribution is the last term on the left-hand side, which, precisely, is

$$-\int \frac{\bar{u}_x}{u} |\phi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1} - \int \frac{\varepsilon \rho_x}{u} |\phi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1}. \quad (92)$$

The key point is that the first term above is nonnegative, because  $\bar{u}_x < 0$  for Blasius solutions and according to (78),

$$-\int \frac{\bar{u}_x}{u} |\phi^{(1)}|^2 \langle x \rangle^{2-2\sigma_1} > 0.$$

We estimate the  $\rho$  contribution from (92), which enables us to use the smallness of  $\varepsilon$ :

$$\left| (92.2) \right| \lesssim \varepsilon \left\| \frac{\rho^{(1)}}{u} \right\|_{L^\infty} \left\| \phi^{(1)} \langle x \rangle^{1-\sigma_1} \right\|_{L^2_\psi}^2 \lesssim \varepsilon \langle x \rangle^{-(1+)} \beta^{(1)} \left\| \phi^{(1)} \langle x \rangle^{1-\sigma_1} \right\|_{L^2_\psi}^2$$

$$\lesssim \varepsilon \langle x \rangle^{-(1+)} E_{K_0}(x)^3 \lesssim \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x),$$

where we have invoked estimate (64) to estimate the  $\beta^{(1)}$  term, and the specification of the norm (16).

We now need to estimate

$$\begin{aligned} & \left| \int u^{(1)} \frac{1}{u} A \phi \phi^{(1)} \langle x \rangle^{2-2\sigma_1} \right| \\ & \leq \left| \int \frac{\bar{u}^{(1)}}{u} A \phi \phi^{(1)} \langle x \rangle^{2-2\sigma_1} \right| + \varepsilon \left| \int \frac{\rho^{(1)}}{u} A \phi \phi^{(1)} \langle x \rangle^{2-2\sigma_1} \right|. \end{aligned} \quad (93)$$

For the first term in (93), we do not have any factors of  $\varepsilon$ , so we take advantage of the fact that one of the terms,  $\phi$ , is lower order:

$$\begin{aligned} |(93.1)| & \lesssim \langle x \rangle^{-(\frac{1}{2}+)} \left\| \frac{\bar{u}^{(1)}}{u} \langle x \rangle \right\|_{L_\psi^\infty} \|A \langle x \rangle\|_{L_\psi^\infty} \|\phi\|_{L_\psi^2} \|\phi^{(1)} \langle x \rangle^{\frac{1}{2}-\omega_1}\|_{L_\psi^2} \\ & \lesssim \langle x \rangle^{-(\frac{1}{2}+)} \omega \left\| \frac{\bar{u}^{(1)}}{\bar{u}} \langle x \rangle \right\|_{L_\psi^\infty} \|A \langle x \rangle\|_{L_\psi^\infty} E_0(x) R_0(x)^{\frac{1}{2}} \\ & \lesssim \langle x \rangle^{-(1+)} E_0(x) + R_0(x) \lesssim R_0(x). \end{aligned}$$

Above, we have used (39) to estimate  $\omega$ , (41) to estimate the  $A$  term, (16) to estimate the  $\phi$  term by  $E_0(x)$ , and (88) to estimate the  $\phi^{(1)}$  term. We have also used that  $\sigma_1 > \omega_1$ . We used Young's inequality to go from the second to third inequality, and the definition of  $R_0$  in (87) to ultimately bound from above by  $R_0$ .

For the second term from (93), we use the smallness of  $\varepsilon$  and that  $\sigma_1 > \omega_1$  via

$$\begin{aligned} |(93.2)| & \lesssim \varepsilon \langle x \rangle^{-(0+)} \|A \langle x \rangle\|_\infty \left\| \frac{\rho^{(1)}}{u} \langle x \rangle^{1+} \right\|_{L_\psi^\infty} \|\phi \langle x \rangle^{-\frac{1}{2}-\frac{\sigma_1}{2}}\|_{L_\psi^2} \|\phi^{(1)} \langle x \rangle^{\frac{1}{2}-\omega_1}\|_{L_\psi^2} \\ & \lesssim \varepsilon \langle x \rangle^{-(0+)} \|A \langle x \rangle\|_\infty \beta^{(1)} \|\phi \langle x \rangle^{-\frac{1}{2}-\frac{\sigma_1}{2}}\|_{L_\psi^2} \|\phi^{(1)} \langle x \rangle^{\frac{1}{2}-\omega_1}\|_{L_\psi^2} \\ & \lesssim \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x), \end{aligned}$$

upon invoking (41) for the  $A$  term, (64) for the  $\beta$  term, and just the norm (16) for the  $L_\psi^2$  terms.

For the final term on the right-hand side of (91), we estimate

$$\begin{aligned} \left| \int A_x \phi \phi^{(1)} \langle x \rangle^{2-2\sigma_1} \right| & \lesssim \langle x \rangle^{-(0+)} \|A_x \langle x \rangle^2\|_\infty \|\phi \langle x \rangle^{-\frac{1}{2}-\frac{\sigma_1}{2}}\|_{L_\psi^2} \|\phi^{(1)} \langle x \rangle^{\frac{1}{2}-\sigma_1}\|_{L_\psi^2} \\ & \lesssim \langle x \rangle^{-(0+)} I_0(x) R_0(x)^{\frac{1}{2}} \lesssim R_0(x). \end{aligned}$$

We have used (42) and then (64) to estimate the  $A_x$  term. For the  $\phi$  term, we use the definition of the norm,  $X_0$ . For the  $\phi^{(1)}$  term, we use (88) and that  $\sigma_1 > \omega_1$ .

This concludes the proof.  $\square$

**Lemma 29.** *Let  $\phi$  solve the Equation (20). Assume the bootstrap assumption, (25). Then the following inequality is valid:*

$$\begin{aligned} & \frac{\partial_x}{2} \int \phi_\psi^2 \psi^{1-2\kappa} x^{1-2\omega_1} + \int \frac{1}{u} \phi_x^2 x^{1-2\omega_1} \psi^{1-2\kappa} \\ & \leq C_\delta R_0(x) + \delta \int |\phi_\psi^{(1)}|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2-2\sigma_1}. \end{aligned} \quad (94)$$

**Proof.** We apply the weighted multiplier  $\frac{1}{u} \phi_x x^{1-2\omega_1} \langle \psi \rangle^{1-2\kappa}$ , which produces the identity

$$\begin{aligned} & \frac{\partial_x}{2} \int \phi_\psi^2 \langle \psi \rangle^{1-2\kappa} x^{1-2\omega_1} + \int \frac{1}{u} \phi_x^2 x^{1-2\omega_1} \langle \psi \rangle^{1-2\kappa} \\ & = \frac{1-2\omega_1}{2} \int \phi_\psi^2 \langle \psi \rangle^{1-2\kappa} x^{-2\omega_1} + (1-2\kappa) \int \phi_\psi \phi_x x^{1-2\omega_1} \psi^{-2\kappa} \\ & \quad + \int A \phi \frac{1}{u} \phi_x x^{1-2\omega_1} \langle \psi \rangle^{1-2\kappa}. \end{aligned} \quad (95)$$

We clearly see that the first term on the right-hand side of (95) is bounded by  $\langle x \rangle^{-(0+)} I_0(x)$  upon invoking the second term on the left-hand side of (83). We estimate the second term on the right-hand side of (95) via the Hardy inequality on  $\phi_x$ , admissible because  $\phi_x|_{\psi=0} = 0$ ,

$$\begin{aligned} \left| \int \phi_\psi \phi_x x^{1-2\omega_1} \psi^{-2\kappa} \right| & \lesssim \langle x \rangle^{-(0+)} \left\| \phi_\psi \psi^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \left\| \phi_x \psi^{-\frac{1}{2}-\kappa} \langle x \rangle^{1-\sigma_1} \right\|_{L_\psi^2} \\ & \leq C_\kappa \langle x \rangle^{-(0+)} \left\| \phi_\psi \psi^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \left\| \phi_x \psi^{\frac{1}{2}-\kappa} \langle x \rangle^{1-\sigma_1} \right\|_{L_\psi^2} \\ & \leq C_{\kappa,\delta} \langle x \rangle^{-(0+)} \left\| \phi_\psi \psi^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2}^2 + \delta \left\| \phi_x \psi^{\frac{1}{2}-\kappa} \langle x \rangle^{1-\sigma_1} \right\|_{L_\psi^2}^2 \\ & \leq C_{\kappa,\delta} x^{-(0+)} I_0(x) + \delta \left\| \phi_x \psi^{\frac{1}{2}-\kappa} \langle x \rangle^{1-\sigma_1} \right\|_{L_\psi^2}^2. \end{aligned} \quad (96)$$

Above, we have used that  $2\omega_1 > \sigma_1$ .

Lastly, we estimate

$$\begin{aligned} & \left| \int A \phi \frac{1}{u} \phi_x x^{1-2\omega_1} \langle \psi \rangle^{1-2\kappa} \right| \\ & \lesssim \left\| A \langle x \rangle \right\|_{L_\psi^\infty} \left\| \frac{\phi}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} x^{-\frac{1}{2}-\omega_1} \right\|_{L_\psi^2} \left\| \frac{\phi_x}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} x^{\frac{1}{2}-\omega_1} \right\|_{L_\psi^2} \\ & \lesssim C_\delta \langle x \rangle^{-(0+)} I_0(x) + \delta \left\| \frac{\phi_x}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} x^{\frac{1}{2}-\omega_1} \right\|_{L_\psi^2}^2, \end{aligned} \quad (97)$$

the latter term being absorbed to the left-hand side of (95).  $\square$

**Lemma 30.** (Quotient Estimate) Let  $\phi$  be a solution to (20). Assume the bootstrap assumption, (25). Then the following estimate is valid:

$$\begin{aligned} & \frac{\partial_x}{2} \int \left| \phi^{(1)} \right|^2 \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2-2\sigma_1} + \int \left| \phi_\psi^{(1)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2-2\sigma_1} \\ & \lesssim R_0(x) + \int \left| \phi^{(1)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{1-2\sigma_1}. \end{aligned}$$

**Proof.** We take the inner product of (86) with  $\frac{\phi^{(1)}}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2-2\sigma_1}$ , which produces the identity

$$\begin{aligned} & \frac{\partial_x}{2} \int \left| \phi^{(1)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2-2\sigma_1} + \frac{1}{2} \int \frac{u_x}{u^2} \left| \phi^{(1)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2-2\sigma_1} \\ & + \int \left| \phi_\psi^{(1)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2-2\sigma_1} + \frac{2\kappa(1-2\kappa)}{2} \int \left| \phi^{(1)} \right|^2 \psi^{-2\kappa-1} \langle x \rangle^{2-2\sigma} \\ & + \int A \left| \phi^{(1)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2-2\sigma_1} - \int \frac{\bar{u}_x}{u} \left| \phi^{(1)} \right|^2 \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2-2\sigma_1} \\ & = \frac{2-2\sigma_1}{2} \int \left| \phi^{(1)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{1-2\sigma_1} + \varepsilon \int \frac{\rho^{(1)}}{u} \left| \phi^{(1)} \right|^2 \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2-2\sigma_1} \\ & + \int A_x \phi \phi^{(1)} \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2-2\sigma_1}. \end{aligned} \quad (98)$$

All of the terms on the left-hand side above are identical to that of Lemma 25, with the exception of the sixth, final term which is a further positive contribution due to the sign condition  $\bar{u}_x \leq 0$ . The first term on the left-hand side appears on the right-hand side of the estimate we are proving. We are thus left with controlling the final two terms on the right-hand side, and these are treated identically to the corresponding terms in Lemma 28.  $\square$

A nearly identical sequence of estimates is performed for the 2 through  $K_1$  order of  $\partial_x$ .

**Lemma 31.** Let  $2 \leq k \leq K_1 \ll K_0$ , and let  $\phi$  solve (20). Assume the bootstrap assumption, (25). For any  $0 < \delta \ll 1$ , the following estimates hold:

$$\frac{\partial_x}{2} \int \left| \phi_\psi^{(k-1)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} + \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle x \rangle^{2k-1-2\omega_k} \lesssim R_{k-1}(x), \quad (99)$$

$$\frac{\partial_x}{2} \int \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-2\sigma_k} + \int u \left| \phi_\psi^{(k)} \right|^2 \langle x \rangle^{2k-2\sigma_k} \lesssim R_{k-1}(x) \quad (100)$$

$$\begin{aligned} & \frac{\partial_x}{2} \int \left| \phi_\psi^{(k-1)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\omega_k} + \int \frac{1}{u} \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} \langle \psi \rangle^{1-2\kappa} \\ & \leq C_\delta R_{k-1}(x) + \delta \int \left| \phi_\psi^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-2\sigma_k} \end{aligned} \quad (101)$$

$$\begin{aligned} & \frac{\partial_x}{2} \int \left| \phi^{(k)} \right|^2 \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2k-2\sigma_k} + \int \left| \phi_\psi^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-2\sigma_k} \\ & \lesssim R_{k-1}(x) + \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\omega_k}. \end{aligned} \quad (102)$$

### 5. Highest Tier: $H^k$ for $K_1 < k \leq K_0$

We take  $\partial_x^k$  to the Equation (20) to obtain

$$\partial_x^k \phi_x - u \partial_x^k \phi_{\psi\psi} + A \partial_x^k \phi - \sum_{j=1}^k c_j \partial_x^j u \partial_x^{k-j} \phi_{\psi\psi} + \sum_{j=1}^k c_j \partial_x^j A \partial_x^{k-j} \phi = 0.$$

We will simplify notations by setting  $\phi^{(k)} := \partial_x^k \phi$ , in which case the above equation reads as

$$\phi_x^{(k)} - u \phi_{\psi\psi}^{(k)} + A \phi^{(k)} - \sum_{j=1}^k c_j u^{(j)} \phi_{\psi\psi}^{(k-j)} + \sum_{j=1}^k c_j \partial_x^j A \phi^{(k-j)} = 0. \quad (103)$$

As a preliminary to the energy estimate, we will perform the following estimate:

**Lemma 32.** *Let  $\phi$  be a solution to (20). Assume the bootstrap assumption, (25). Then the following estimate is valid:*

$$\frac{\partial_x}{2} \int |\phi_{\psi}^{(k-1)}|^2 \langle x \rangle^{2k-1-2\omega_k} + \int |\phi^{(k)}|^2 \langle x \rangle^{2k-1-2\omega_k} \lesssim R_{k-1}(x). \quad (104)$$

**Proof.** We consider Equation (103) with  $k$  replaced by  $k-1$ , which we write as follows for the readers' convenience:

$$\phi_x^{(k-1)} - u \phi_{\psi\psi}^{(k-1)} + A \phi^{(k-1)} - \sum_{j=1}^{k-1} c_j u^{(j)} \phi_{\psi\psi}^{(k-1-j)} + \sum_{j=1}^{k-1} c_j \partial_x^j A \phi^{(k-1-j)} = 0. \quad (105)$$

We now apply the multiplier  $\frac{\phi^{(k)}}{u} \langle x \rangle^{2k-1-2\omega_k}$ . This generates the identity

$$\begin{aligned} & \frac{\partial_x}{2} \int \left| \phi_{\psi}^{(k-1)} \right|^2 \langle x \rangle^{2k-1-\omega_k} + \int \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-1-\omega_k} \\ &= - \int A \phi^{(k-1)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} + \sum_{j=1}^{k-1} \int c_j u^{(j)} \phi_{\psi\psi}^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \\ & \quad - \sum_{j=1}^{k-1} \int c_j \partial_x^j A \phi^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \\ &= - \int A \phi^{(k-1)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} + \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \int c_j u^{(j)} \phi_{\psi\psi}^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \\ & \quad + \sum_{j=\lfloor \frac{k-1}{2} \rfloor}^{k-1} \int c_j u^{(j)} \phi_{\psi\psi}^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \int c_j \partial_x^j A \phi^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \\
& - \sum_{j=\lfloor \frac{k-1}{2} \rfloor}^{k-1} \int c_j \partial_x^j A \phi^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k}.
\end{aligned} \tag{106}$$

We now proceed to estimate the five terms appearing on the right-hand side of (106), starting with

$$\begin{aligned}
& \left| \int A \phi^{(k-1)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \right| \\
& \lesssim \langle x \rangle^{-(0+)} \|A \langle x \rangle\|_{\infty} \left\| \frac{\phi^{(k-1)}}{\sqrt{u}} \langle x \rangle^{k-\frac{3}{2}-\omega_{k-1}} \right\|_{L^2_{\psi}} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_{\psi}} \\
& \lesssim \langle x \rangle^{-(0+)} \|A \langle x \rangle\|_{\infty} \left\| \frac{\phi^{(k-1)}}{\sqrt{u}} \langle x \rangle^{k-\frac{3}{2}-\omega_{k-1}} \right\|_{L^2_{\psi}} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_{\psi}} \\
& \leq C_{\delta} \langle x \rangle^{-(0+)} E_1(x) R_{k-1}(x) + \delta \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_{\psi}}^2
\end{aligned}$$

for any  $\delta > 0$ . We select  $\delta \ll 1$  so that the final term above can be absorbed to the left-hand side of (106). Above, we have invoked (41), the definition of the norm, (16), and that  $\omega_k > \omega_{k-1}$ .

Next, we have, when  $1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor$ ,

$$\begin{aligned}
& \left| \int u^{(j)} \phi_{\psi\psi}^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \right| \\
& \lesssim \langle x \rangle^{-\frac{1}{2}} \left\| \frac{\bar{u}^{(j)} + \varepsilon \rho^{(j)}}{u} \langle x \rangle^j \right\|_{L^{\infty}_{\psi}} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-1-j)} \langle x \rangle^{k-j-\sigma_{k-j+1}} \right\|_{L^2_{\psi}} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_{\psi}} \\
& \lesssim \langle x \rangle^{-\frac{1}{2}} (1 + \varepsilon \beta^{(j)}) \Upsilon^{(k-1-j)} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_{\psi}} \\
& \leq C_{\delta} \langle x \rangle^{-(1+)} (1 + \varepsilon E_{j+2}(x)) E_{k-j}(x) + \delta \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_{\psi}}^2 \\
& \leq C_{\delta} \langle x \rangle^{-(0+)} R_{k-1}(x) + \delta \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_{\psi}}^2,
\end{aligned}$$

where above, we have used the definitions (40), estimate (64), and that  $\omega_k > \sigma_{k-1}$ .

Next, we have, when  $\lfloor \frac{k-1}{2} \rfloor \leq j \leq k-1$ ,

$$\left| \int u^{(j)} \phi_{\psi\psi}^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \right|$$

$$\begin{aligned}
&\lesssim \langle x \rangle^{-\frac{1}{2}-} \left\| \frac{\bar{u}^{(j)} + \varepsilon \rho^{(j)}}{u} \langle x \rangle^{j-\frac{1}{4}} \right\|_{L^2_\psi} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-1-j)} \langle x \rangle^{(k-1-j)+\frac{5}{4}-\sigma_{k-j}} \right\|_{L^\infty_\psi} \\
&\quad \times \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi} \\
&\lesssim \langle x \rangle^{-\frac{1}{2}-} (1 + \varepsilon \gamma^{(j)}) \ell^{(k-1-j)} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi} \\
&\leq C_\delta \langle x \rangle^{-(1+)} (1 + \varepsilon E_j(x))^2 E_{k-j+1}(x)^2 + \delta \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi}^2 \\
&\leq C_\delta R_{k-1}(x) + \delta \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi}^2,
\end{aligned}$$

where, above, we have used the definitions (40), and estimate (64) and that  $\omega_k > \sigma_{k-1}$  for  $j$  in the specified range.

We now again consider the range, when  $1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor$ ,

$$\begin{aligned}
&\left| \int \partial_x^j A \phi^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \right| \\
&\lesssim \langle x \rangle^{-(0+)} \left\| \partial_x^j A \langle x \rangle^{j+1} \right\|_{L^\infty} \left\| \frac{\phi^{(k-1-j)}}{\sqrt{u}} \langle x \rangle^{k-1-j-\frac{1}{2}-\omega_{k-j}} \right\|_{L^2_\psi} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi} \\
&\lesssim \langle x \rangle^{-(0+)} (1 + \varepsilon E_{j+1}(x)) I_{k-1-j}(x) \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi} \\
&\lesssim C_\delta R_{k-1}(x) + \delta \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi}^2.
\end{aligned}$$

Above, we have used (42) and (64).

Next, we have when  $\lfloor \frac{k-1}{2} \rfloor \leq j \leq k-1$ . For this, we estimate via

$$\begin{aligned}
&\left| \int \partial_x^j A \phi^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \right| \\
&\lesssim \langle x \rangle^{-(\frac{1}{2}+)} \left\| u^{\frac{3}{2}} \partial_x^j A \langle x \rangle^{j+\frac{3}{4}} \right\|_{L^2_\psi} \left\| \frac{\phi^{(k-1-j)}}{u^2} \langle x \rangle^{k-1-j+\frac{1}{4}-\sigma_{k-j}} \right\|_{L^\infty_\psi} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi} \\
&\lesssim \langle x \rangle^{-(\frac{1}{2}+)} (E_{j-1}(x) + \varepsilon E_j(x)) E_{k-j}(x) \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi} \\
&\leq C_\delta R_{k-1}(x) + \delta \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi}^2.
\end{aligned}$$

We have invoked (69), (51), (64), and that  $\omega_k > \sigma_{k-1} > \sigma_{k-j}$ .

This concludes the proof.  $\square$

**Lemma 33.** (Energy Estimate) *Let  $\phi$  be a solution to (20). Assume the bootstrap assumption, (25). Then the following estimate is valid:*

$$\frac{\partial_x}{2} \int |\phi^{(k)}|^2 \langle x \rangle^{2(k-\sigma_k)} + \int u |\phi_\psi^{(k)}|^2 \langle x \rangle^{2(k-\sigma_k)} \lesssim R_{k-1}(x). \quad (107)$$

**Proof.** We apply the weighted multiplier  $\phi^{(k)} \langle x \rangle^{2(k-\sigma_k)}$  to Equation (103), which produces the identity

$$\begin{aligned} & \frac{\partial_x}{2} \int |\phi^{(k)}|^2 \langle x \rangle^{2k-2\sigma_k} + \int \bar{u} |\phi_\psi^{(k)}|^2 \langle x \rangle^{2k-2\sigma_k} - \int \frac{\bar{u}_\psi \psi}{2} |\phi^{(k)}|^2 \langle x \rangle^{2k-2\sigma_k} \\ & + \int A |\phi^{(k)}|^2 \langle x \rangle^{2k-2\sigma_k} - \int \bar{u}^{(1)} \phi_\psi^{(k-1)} \phi^{(k)} \langle x \rangle^{2k-2\sigma_k} \\ & = \frac{2k-2\sigma_k}{2} \int |\phi^{(k)}|^2 \langle x \rangle^{2k-1-2\sigma_k} + \varepsilon \int \rho^{(1)} \phi_\psi^{(k-1)} \phi^{(k)} \langle x \rangle^{2k-2\sigma_k} \\ & + \int \varepsilon \rho \phi_\psi^{(k)} \phi^{(k)} \langle x \rangle^{2(k-\sigma_k)} + \sum_{j=2}^k c_j \int u^{(j)} \phi_\psi^{(k-j)} \phi^{(k)} \langle x \rangle^{2k-2\sigma_k} \\ & + \sum_{j=1}^k c_j \int \partial_x^j A \phi^{(k-j)} \phi^{(k)} \langle x \rangle^{2k-2\sigma_k}. \end{aligned} \quad (108)$$

The left-hand side is the same as that appearing in the Middle-Tier energy estimate, the only exception being the final term on the left-hand side. For this, we invoke the identity to write

$$- \int \bar{u}^{(1)} \phi_\psi^{(k-1)} \phi^{(k)} \langle x \rangle^{2k-2\sigma_k} = - \int \frac{\bar{u}^{(1)}}{u} |\phi^{(k)}|^2 \langle x \rangle^{2k-2\sigma_k} - \int \frac{\bar{u}^{(1)}}{u} L^{(k-1)} \phi^{(k)} \langle x \rangle^{2k-2\sigma_k}.$$

While the first term above is a damping term, we estimate the latter by

$$\begin{aligned} & \left| \int \frac{\bar{u}^{(1)}}{u} L^{(k-1)} \phi^{(k)} \langle x \rangle^{2k-2\sigma_k} \right| \\ & \lesssim \langle x \rangle^{-\frac{1}{2}} - \left\| \frac{\bar{u}^{(1)}}{u} \langle x \rangle \right\|_\infty \left\| L^{(k-1)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \left\| \phi^{(k)} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ & \lesssim E_3(x) \left( E_{k-1}(x) + \varepsilon E_k(x) \right) R_{k-1}(x)^{\frac{1}{2}} \\ & \lesssim R_{k-1}(x), \end{aligned}$$

where we have invoked estimates (65), (104), and  $\sigma_k > \omega_k$ .

The first term on the right-hand side of (108) is controlled by  $R_k(x)$  already by Lemma 32 upon using that  $\sigma_k > \omega_k$ . We thus begin with

$$\begin{aligned} & \left| \varepsilon \int \rho^{(1)} \phi_{\psi\psi}^{(k-1)} \phi^{(k)} \langle x \rangle^{2k-2\sigma_k} \right| \\ & \lesssim \varepsilon \langle x \rangle^{-(\frac{1}{2}+)} \left\| \frac{\rho^{(1)}}{u} \langle x \rangle \right\|_{L_\psi^\infty} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-1)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \left\| \phi^{(k)} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ & \lesssim \varepsilon \langle x \rangle^{-(\frac{1}{2}+)} \beta^{(1)} \Upsilon^{(k-1)} R_{k-1}(x)^{\frac{1}{2}} \\ & \leq \varepsilon \langle x \rangle^{-(\frac{1}{2}+)} E_2(x) E_k(x) R_{k-1}(x)^{\frac{1}{2}} \leq R_{k-1}(x), \end{aligned}$$

upon invoking (64), (104), and  $\sigma_k > \omega_k$ .

We now move to

$$\begin{aligned} & \int \varepsilon \rho \phi_{\psi\psi}^{(k)} \phi^{(k)} \langle x \rangle^{2(k-\sigma_k)} \\ & = - \int \varepsilon \rho_\psi \phi_\psi^{(k)} \phi^{(k)} \langle x \rangle^{2(k-\sigma_k)} - \int \varepsilon \rho \left| \phi_\psi^{(k)} \right|^2 \langle x \rangle^{2(k-\sigma_k)} \\ & = - \int \varepsilon \frac{\phi_\psi - \frac{\bar{u}_y}{\bar{u}} \rho}{1 - \varepsilon \frac{\rho}{u}} \phi_\psi^{(k)} \frac{\phi^{(k)}}{u} \langle x \rangle^{2(k-\sigma_k)} - \int \varepsilon \rho \left| \phi_\psi^{(k)} \right|^2 \langle x \rangle^{2(k-\sigma_k)}, \quad (109) \end{aligned}$$

where above, we have used the identity

$$u \rho_\psi = \frac{1}{1 - \varepsilon \frac{\rho}{u}} \left( \phi_\psi - \frac{\bar{u}_y}{\bar{u}} \rho \right).$$

The first term above is majorized by invoking (33) via

$$\begin{aligned} |(109.1)| & \lesssim \varepsilon \langle x \rangle^{-\frac{3}{4}+\sigma_1} \left\| u \rho_\psi \langle x \rangle^{\frac{3}{4}-\sigma_1} \right\|_{L_\psi^\infty} \left\| \phi_\psi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \left\| \frac{\phi^{(k)}}{u^2} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \\ & \lesssim \varepsilon \langle x \rangle^{-\left(\frac{3}{4}-\sigma_1\right)} \left\| \phi_\psi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \left( E_k(x) + \langle x \rangle^{\frac{1}{2}} I_k(x) \right) \\ & \lesssim \varepsilon \langle x \rangle^{-\left(\frac{1}{4}-\sigma_1\right)} I_{K_0}(x). \end{aligned}$$

Above, we have localized based on the location of the self-similar variable,  $\xi$ :

$$\left\| \frac{\phi^{(k)}}{u^2} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \leq \left\| \frac{\phi^{(k)}}{u^2} \langle x \rangle^{k-\sigma_k} \chi(\xi) \right\|_{L_\psi^2} + \left\| \frac{\phi^{(k)}}{u^2} \langle x \rangle^{k-\sigma_k} (1 - \chi(\xi)) \right\|_{L_\psi^2}.$$

In the support of  $1 - \chi$ , we use that  $u \gtrsim 1$  according to (30) and the corresponding fact for  $\bar{u}$ . Hence,

$$\left\| \frac{\phi^{(k)}}{u^2} \langle x \rangle^{k-\sigma_k} (1 - \chi(\xi)) \right\|_{L_\psi^2} \lesssim \left\| \phi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \lesssim E_k(x).$$

For the localized contribution, we perform the Hardy inequality. Specifically, we again appeal to (30) to assert that  $u^2 \gtrsim \xi \gtrsim \frac{\sqrt{x}}{\psi}$ , from which we have

$$\begin{aligned} \left\| \frac{\phi^{(k)}}{u^2} \langle x \rangle^{k-\sigma_k} \chi(\xi) \right\|_{L^2_\psi} &\lesssim \left\| \frac{\phi^{(k)}}{\psi} \langle x \rangle^{\frac{1}{2}+k-\sigma_k} \chi(\xi) \right\|_{L^2_\psi} \leq \left\| \frac{\phi^{(k)}}{\psi} \langle x \rangle^{\frac{1}{2}+k-\sigma_k} \right\|_{L^2_\psi} \\ &\lesssim \left\| \phi_\psi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L^2_\psi} \langle x \rangle^{\frac{1}{2}} \leq I_k(x) \langle x \rangle^{\frac{1}{2}}. \end{aligned} \quad (110)$$

The second term above is easily majorized by

$$\begin{aligned} |(109.2)| &\lesssim \varepsilon \langle x \rangle^{-\left(\frac{1}{4}-\sigma_1\right)} \left\| \frac{\rho}{u} \langle x \rangle^{\frac{1}{4}-\sigma_1} \right\|_{L^\infty_\psi} \left\| \sqrt{u} \phi_\psi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L^2_\psi}^2 \\ &\lesssim \varepsilon \langle x \rangle^{-\left(\frac{1}{4}-\right)} \beta^{(0)} \left\| \sqrt{u} \phi_\psi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L^2_\psi}^2 \\ &\lesssim \varepsilon \langle x \rangle^{-\left(\frac{1}{4}-\right)} E_{K_0} I_{K_0}. \end{aligned}$$

Above, we have appealed to estimate (64).

For the fourth term on the right-hand side of (108), we begin by considering the case when  $j = \min\{j, k-j\} \geq 2$ . We estimate directly that

$$\begin{aligned} &\left| \int u^{(j)} \phi_\psi^{(k-j)} \phi^{(k)} \langle x \rangle^{2(k-\sigma_k)} \right| \\ &\lesssim \langle x \rangle^{-\left(\frac{1}{2}+\right)} \left\| \frac{\bar{u}^{(j)} + \varepsilon \rho^{(j)}}{u} \langle x \rangle^j \right\|_{L^\infty_\psi} \left\| u \phi_\psi^{(k-j)} \langle x \rangle^{k-j-\sigma_{k-j+1}} \right\|_{L^2_\psi} \left\| \phi^{(k)} \langle x \rangle^{k-\omega_k} \right\|_{L^2_\psi} \\ &\lesssim \langle x \rangle^{-\left(\frac{1}{2}+\right)} (1 + \varepsilon \beta^{(j)}) \Upsilon^{(k-j)} R_{k-1}(x)^{\frac{1}{2}} \\ &\lesssim \langle x \rangle^{-\left(\frac{1}{2}+\right)} (1 + \varepsilon E_{j+1}) E_{k-j+1} R_{k-1}^{\frac{1}{2}} \\ &\leq R_{k-1}(x), \end{aligned}$$

where we have used estimate (104), and (64),  $\sigma_k > \omega_k > \sigma_{k-j+1}$ ,

We now treat this term for  $j = \max\{j, k-j\} \leq k$ . We split

$$- \int u^{(j)} \phi_\psi^{(k-j)} \phi^{(k)} \langle x \rangle^{2(k-\sigma_k)} \left( \chi(\xi) + \chi(\xi)^C \right). \quad (111)$$

The far-field term is estimated as

$$\begin{aligned} |(111.2)| &\lesssim \langle x \rangle^{-(1+)} \left\| \bar{u} u^{(j)} \langle x \rangle^{j-\frac{1}{4}} \right\|_{L^2_\psi} \\ &\quad \times \left\| \bar{u} \phi_\psi^{(k-j)} \langle x \rangle^{(k-j)+1-\sigma_{k-j+1}+\frac{1}{4}} \right\|_{L^\infty_\psi} \left\| \phi^{(k)} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi} \\ &\lesssim \langle x \rangle^{-\left(\frac{1}{2}+\right)} \left\| \bar{u} \left( \bar{u}^{(j)} + \varepsilon \rho^{(j)} \right) \langle x \rangle^{j-\frac{1}{4}} \right\|_{L^2_\psi} \end{aligned}$$

$$\begin{aligned} & \times \left\| \bar{u} \phi_{\psi\psi}^{(k-j)} \langle x \rangle^{(k-j)+1-\sigma_{k-j+1}+\frac{1}{4}} \right\|_{L_\psi^\infty} \left\| \phi^{(k)} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ & \lesssim \langle x \rangle^{-\left(\frac{1}{2}+\right)} (1 + \varepsilon E_j(x)) E_{k-j+2} R_{k-1}(x)^{\frac{1}{2}} \leq R_{k-1}(x). \end{aligned}$$

Above, we use (64), (66) and (104).

We estimate the localized contribution via

$$\begin{aligned} |(111.1)| &= \left| \int \sqrt{\psi} u^{(j)} \psi^{\frac{1}{4}} \phi_{\psi\psi}^{(k-j)} \frac{\phi^{(k)}}{\psi^{\frac{3}{4}}} \langle x \rangle^{2(k-\sigma_k)} \chi(\xi) \right| \\ &\lesssim \langle x \rangle^{\frac{1}{2}} \langle x \rangle^{-\left(\frac{5}{4}-\right)} \left\| \bar{u} u^{(j)} \langle x \rangle^{j-\frac{1}{4}} \right\|_{L_\psi^2} \left\| \sqrt{\bar{u}} \phi_{\psi\psi}^{(k-j)} \langle x \rangle^{(k-j)+\left(\frac{3}{2}-\sigma_{k-j+1}\right)} \right\|_{L_\psi^\infty(\xi \lesssim 1)} \\ &\quad \times \left( \left\| \sqrt{\bar{u}} \phi_{\psi}^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} + \langle x \rangle^{-\frac{1}{2}} \left\| \phi^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \right) \\ &\lesssim \langle x \rangle^{-\left(\frac{3}{4}-\right)} E_j(x) E_{k-j+2}(x) \left( \left\| \sqrt{\bar{u}} \phi_{\psi}^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} + R_{k-1}(x)^{\frac{1}{2}} \right) \\ &\leq C_\delta \langle x \rangle^{-\frac{3}{2}} E_j(x)^2 E_{k-j+2}(x)^2 + \delta \left\| \sqrt{\bar{u}} \phi_{\psi}^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2}^2 + R_{k-1}(x) \\ &\leq R(x) + \delta \left\| \sqrt{\bar{u}} \phi_{\psi}^{(k)} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2}^2. \end{aligned}$$

We select  $\delta \ll 1$  based only on universal constants, so that the  $\delta$  term above can be absorbed to the left-hand side of (108). Above, we have used that  $u \sim \eta \sim \sqrt{\xi}$  in the region where  $\xi \lesssim 1$ , which is according to (30). We have also used the Hardy inequality in the  $\psi$  direction, which yields

$$\left\| \frac{\phi^{(k)}}{\psi^{\frac{3}{4}}} \chi(\xi) \right\|_{L_\psi^2} \lesssim \left\| \psi^{\frac{1}{4}} \phi_{\psi}^{(k)} \chi(\xi) \right\|_{L_\psi^2} + \left\| \frac{\psi^{\frac{1}{4}}}{x^{\frac{1}{2}}} \phi^{(k)} \chi'(\xi) \right\|_{L_\psi^2}.$$

Finally, we have used the enhanced uniform decay estimate, (75), the third inequality in (56), and the estimate we established in (104).

We now move to the final term. In the case when  $j = \min\{j, k-j\} \geq 1$ , we estimate

$$\begin{aligned} & \left| \int \partial_x^j A \phi^{(k-j)} \phi^{(k)} \langle x \rangle^{2(k-\sigma_k)} \right| \\ & \leq \langle x \rangle^{-(0+)} \left\| \partial_x^j A \langle x \rangle^{j+1} \right\|_\infty \left\| \phi^{(k-j)} \langle x \rangle^{k-j-\frac{1}{2}-\omega_{k-j}} \right\|_{L_\psi^2} \left\| \phi^{(k)} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ & \lesssim \langle x \rangle^{-(0+)} (1 + \varepsilon E_{j+1}(x)) R_{k-j-1}(x)^{\frac{1}{2}} R_{k-1}(x)^{\frac{1}{2}} \leq R_{k-1}(x). \end{aligned}$$

Above, we use (41) for the  $A$  term, (104) for the  $\phi$  terms, and that  $\sigma_k > \omega_k$ ,  $\sigma_k > \omega_{k-j}$ .

In the case when  $j = \max\{j, k-j\} \leq k$ , we estimate

$$\left| \int \partial_x^j A \phi^{(k-j)} \phi^{(k)} \langle x \rangle^{2(k-\sigma_k)} \right|$$

$$\begin{aligned}
&\lesssim \langle x \rangle^{-\frac{1}{2}-} \left\| u^{\frac{3}{2}} \partial_x^j A \langle x \rangle^{j+\frac{3}{4}} \right\|_{L^2_\psi} \left\| \frac{\phi^{(k-j)}}{u} \langle x \rangle^{k-j+\frac{1}{4}-\sigma_{k-j+1}} \right\|_{L^\infty_\psi} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi} \\
&\lesssim \langle x \rangle^{-\left(\frac{1}{2}+\right)} \left( \varepsilon E_j(x) + E_{j-1}(x) \right) E_{k-j+1}(x) R_{k-1}(x)^{\frac{1}{2}} \leq R_{k-1}(x).
\end{aligned}$$

Above, we use (69) for the  $A$  term, (51), (64) for the  $\phi^{(k-j)}$  term, and (104) for the  $\phi^{(k)}$  term. We also use that  $\sigma_k > \omega_k$ ,  $\sigma_k > \sigma_{k-j+1}$ .

This concludes the proof.  $\square$

**Lemma 34.** *Let  $\phi$  be a solution to (20). Assume the bootstrap assumption, (25). Then the following estimate is valid:*

$$\begin{aligned}
&\frac{\partial_x}{2} \int \left| \phi_\psi^{(k-1)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} \langle \psi \rangle^{1-2\kappa} + \int \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} \langle \psi \rangle^{1-2\kappa} \\
&\leq C_\delta R_{k-1}(x) + \delta \left\| \phi_\psi^{(k)} \langle x \rangle^{k-\sigma_k} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L^2_\psi}^2.
\end{aligned} \tag{112}$$

**Proof.** We apply the multiplier  $\frac{\phi^{(k)}}{u} \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{1-2\kappa}$  to (105). This produces the identity

$$\begin{aligned}
&\frac{\partial_x}{2} \int \left| \phi_\psi^{(k-1)} \right| \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{1-2\kappa} + \int |\phi^{(k)}|^2 \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{1-2\kappa} \\
&= - \int A \phi^{(k-1)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{1-2\kappa} \\
&\quad + \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \int c_j u^{(j)} \phi_{\psi\psi}^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{1-2\kappa} \\
&\quad + \sum_{j=\lfloor \frac{k-1}{2} \rfloor}^{k-1} \int c_j u^{(j)} \phi_{\psi\psi}^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{1-2\kappa} \\
&\quad - \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \int c_j \partial_x^j A \phi^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{1-2\kappa} \\
&\quad - \sum_{j=\lfloor \frac{k-1}{2} \rfloor}^{k-1} \int c_j \partial_x^j A \phi^{(k-1-j)} \frac{1}{u} \phi^{(k)} \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{1-2\kappa} \\
&\quad - (1-2\kappa) \int \phi_\psi^{(k-1)} \phi^{(k)} \langle x \rangle^{2k-1-\omega_k} \langle \psi \rangle^{-2\kappa}.
\end{aligned} \tag{113}$$

The estimation of these terms is nearly identical to Lemma 32, with the exception of the final term, for which we invoke the Hardy inequality:

$$\begin{aligned}
& \left| \int \phi_{\psi}^{(k-1)} \phi^{(k)} \langle x \rangle^{2k-1-2\omega_k} \psi^{-2\kappa} \right| \\
& \lesssim \langle x \rangle^{-(0+)} \left\| \phi_{\psi}^{(k-1)} \psi^{\frac{1}{2}-\kappa} \langle x \rangle^{k-1-\sigma_{k-1}} \right\|_{L_{\psi}^2} \left\| \phi_{\psi}^{(k)} \psi^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\sigma_k} \right\|_{L_{\psi}^2} \\
& \leq C_{\delta} I_{k-1}(x) + \delta \left\| \phi_{\psi}^{(k)} \psi^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\sigma_k} \right\|_{L_{\psi}^2}^2.
\end{aligned}$$

We note that this Hardy inequality is admissible because of the defect factor of  $-\kappa$  in the  $\psi$  weight, as well as the fact that  $\phi^{(k)}|_{\psi=0} = 0$ . We also note that we have used the inequality  $2\omega_k > \sigma_k + \sigma_{k-1}$  regarding the  $x$ -defect factors.  $\square$

**Lemma 35.** (Quotient Estimate) *Let  $\phi$  be a solution to (20). Assume the bootstrap assumption, (25). Then the following estimate is valid:*

$$\begin{aligned}
& \frac{\partial_x}{2} \int \left| \phi^{(k)} \right|^2 \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2k-2\sigma_k} + \int \left| \phi_{\psi}^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-2\sigma_k} \\
& \lesssim R_{k-1}(x) + \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\omega_k}.
\end{aligned} \tag{114}$$

**Proof.** We apply the weighted “quotient multiplier”  $\phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)}$ , which produces the identity

$$\begin{aligned}
& \frac{\partial_x}{2} \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} - \frac{1}{2} \int \left| \phi^{(k)} \right|^2 \partial_x \left( \frac{1}{u} \right) \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \\
& + \int \left| \phi_{\psi}^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} + \kappa(1-2\kappa) \int \left| \phi^{(k)} \right|^2 \psi^{-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \\
& + \int A \frac{1}{u} \left| \phi^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} - \int \bar{u}^{(1)} \phi_{\psi\psi}^{(k-1)} \phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \\
& = (k-\sigma_k) \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\sigma_k} \\
& + \sum_{j=2}^k c_j \int u^{(j)} \phi_{\psi\psi}^{(k-j)} \phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \\
& + \sum_{j=1}^k c_j \int \partial_x^j A \phi^{(k-j)} \phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \\
& + \varepsilon \int \rho^{(1)} \phi_{\psi\psi}^{(k-1)} \phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)}.
\end{aligned} \tag{115}$$

The left-hand side is known to be positive, as in the lower order quotient estimates. We thus move to the right-hand side.

First, consider

$$\begin{aligned}
& - \int \bar{u}^{(1)} \phi_{\psi\psi}^{(k-1)} \phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \\
& = - \int \frac{\bar{u}^{(1)}}{u} |\phi^{(k)}|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} - \int \bar{u}^{(1)} L^{(k-1)} \phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)}.
\end{aligned} \tag{116}$$

The lower order terms,  $L^{(k-1)}$  are inserted into (117) to produce

$$\begin{aligned}
& \left| \int \frac{u^{(1)}}{u} L^{(k-1)} \phi^{(k)} \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2(k-\sigma_k)} \chi(\xi) \right| \\
& \lesssim \langle x \rangle^{-(1+)} \left\| \frac{u^{(1)}}{u} \langle x \rangle \right\|_{L_\psi^\infty} \left\| L^{(k-1)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\omega_k} \right\|_{L_\psi^2} \\
& \lesssim \langle x \rangle^{-(1+)} (1 + \varepsilon \beta^{(1)}) \left\| L^{(k-1)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\omega_k} \right\|_{L_\psi^2} \\
& \lesssim \langle x \rangle^{-(1+)} (1 + \varepsilon E_2(x)) (1 + \varepsilon E_k(x)) \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\omega_k} \right\|_{L_\psi^2} \\
& \leq R_{k-1}(x) + C \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\omega_k} \right\|_{L_\psi^2}^2.
\end{aligned}$$

Above, we have used (64) for the  $\beta^{(1)}$  term, (65) for the  $L^{(k-1)}$  term, and we move the  $\phi^{(k)}$  to the right-hand side of the estimate, according to (114) using Young's inequality for products. We have also used that  $\sigma_k > \omega_k$ .

Next, we consider

$$\begin{aligned}
& \left| \varepsilon \int \rho^{(1)} \phi_{\psi\psi}^{(k-1)} \phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \right| \\
& \lesssim \left| \varepsilon \int \frac{\rho^{(1)}}{u} |\phi^{(k)}|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \right| + \left| \varepsilon \int \frac{\rho^{(1)}}{u} L^{(k-1)} \phi^{(k)} \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2(k-\sigma_k)} \right| \\
& \lesssim \varepsilon \left\| \frac{\rho^{(1)}}{u} \langle x \rangle \right\|_{L_\psi^\infty} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2}^2 \\
& \quad + \varepsilon \langle x \rangle^{-(\frac{1}{2}+)} \left\| \frac{\rho^{(1)}}{u} \langle x \rangle \right\|_{L_\psi^\infty} \left\| L^{(k-1)} \frac{\langle \psi \rangle^{1-2\kappa}}{\sqrt{u}} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \left\| \phi^{(k)} \frac{\langle \psi \rangle^{1-2\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\
& \lesssim \varepsilon \langle x \rangle^{-(0+)} \beta^{(1)} I_k(x) + \varepsilon \langle x \rangle^{-(\frac{1}{2}+)} \beta^{(1)} (1 + \varepsilon E_k(x)) I_k(x)^{\frac{1}{2}} \\
& \lesssim R(x).
\end{aligned}$$

Above, we have used that  $\sigma_k > \omega_k$ , estimate (112) for each of the  $\phi^{(k)}$  terms, (65) for the  $L^{(k-1)}$  term, and (64) for the  $\beta^{(1)}$  term.

Next, we consider

$$-\sum_{j=2}^k c_j \int u^{(j)} \phi_{\psi\psi}^{(k-j)} \phi^{(k)} \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2(k-\sigma_k)} \left( \chi(\xi) + \chi(\xi)^C \right). \quad (117)$$

First, assume  $2 \leq j = \min\{j, k - j\}$ . Then,

$$\begin{aligned}
|(117.1)| &\lesssim \langle x \rangle^{-(\frac{1}{2}+)} \left\| \frac{u^{(j)}}{u} \langle x \rangle^j \right\|_{L_\psi^\infty} \left\| \phi_{\psi\psi}^{(k-j)} \sqrt{u} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{(k-j)+1-\sigma_{k-j+1}} \right\|_{L_\psi^2} \\
&\quad \times \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\
&\lesssim \langle x \rangle^{-(\frac{1}{2}+)} (1 + \varepsilon \beta^{(j)}) \Upsilon^{(k-j)} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\omega_k} \right\|_{L_\psi^2} \\
&\lesssim \langle x \rangle^{-(\frac{1}{2}+)} (1 + \varepsilon E_{j+1}) E_{k-j+1} \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\
&\lesssim \langle x \rangle^{-(1+)} (1 + \varepsilon E_{j+1}) E_{k-j+1} + \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2}^2 \\
&\lesssim R_{k-1} + \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2}^2.
\end{aligned}$$

Above, the  $\phi^{(k)}$  term contributes to the right-hand side of our estimate, (114). We have used (64) to estimate the  $\beta, \Upsilon$  terms. We have also used  $\sigma_k > \sigma_{k-j+1}$  and  $\sigma_k > \omega_k$  for  $j \geq 2$ . Note that the contribution from  $\langle x \rangle^{-(1+)} E_{k-j+1} \leq R_{k-1}$  because  $j \geq 2$ .

Second, assume  $j = \max\{j, k - j\}$ . In this case, we estimate the localized contribution from (117) via

$$\begin{aligned}
|(117.1)| &\lesssim \varepsilon \langle x \rangle^{-(1+)} \left\| \sqrt{u} \rho^{(j)} \langle x \rangle^{j-\sigma_j} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-j)} \langle x \rangle^{(k-j)+\left(\frac{5}{4}-\sigma_{k-j+1}\right)} \right\|_{L_\psi^\infty} \\
&\quad \times \left\| \phi_{\psi}^{(k)} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\sigma_k} \right\|_{L_\psi^2} \\
&\quad + \langle x \rangle^{-(1+)} \left\| \sqrt{u} \bar{u}^{(j)} \langle x \rangle^j \right\|_{L_\psi^\infty} \left\| \sqrt{u} \phi_{\psi\psi}^{(k-j)} \langle x \rangle^{k-j+1-\sigma_{k-j}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \\
&\quad \times \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\omega_k} \right\|_{L_\psi^2} \\
&\lesssim \varepsilon \langle x \rangle^{-(1+)} (1 + E_j) I_k^{\frac{1}{2}} + \langle x \rangle^{-\left(\frac{1}{2}+\right)} \Upsilon^{(k-j)} \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\
&\lesssim R_{k-1}(x) + \langle x \rangle^{-(1+)} E_{k-j+1}(x)^2 + \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2}^2 \\
&\lesssim R_{k-1} + \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2}^2.
\end{aligned}$$

Above, we use that  $\sigma_k < \frac{1}{4}$  and that  $\sigma_k > \omega_k$ , and  $\sigma_k > \sigma_{k-j}$  for  $j > 0$ . We have used (64) to estimate the  $\iota$ ,  $\Upsilon$  terms, and (66) to estimate the  $\rho^{(j)}$  term in  $L^2_\psi$ .

We next treat

$$\int \partial_x^j A \phi^{(k-j)} \phi^{(k)} \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2(k-\sigma_k)}. \quad (118)$$

For  $1 \leq j = \min\{j, k-j\}$ , we estimate

$$\begin{aligned} (118) &\lesssim \langle x \rangle^{-(0+)} \left\| \partial_x^j A \langle x \rangle^{j+1} \right\|_{L_\psi^\infty} \left\| \frac{\phi^{(k-j)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{(k-j)-\frac{1}{2}-\omega_{k-j}} \right\|_{L_\psi^2} \\ &\quad \left\| \frac{\phi^{(k)}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ &\lesssim \langle x \rangle^{-(0+)} (1 + \varepsilon E_{j+1}) I_{k-1}(x) \left\| \frac{\phi^k}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ &\lesssim R_{k-1}(x) + \left\| \frac{\phi^{k-j}}{\sqrt{u}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2}^2. \end{aligned}$$

Above, we have used that  $\sigma_k > \omega_k$ ,  $\sigma_k > \sigma_{k-j}$  for  $j > 0$ , we have also used the definition of (16) for the  $\phi^{(k-j)}$  term, and estimate (42) for the  $A$  term.

Next, we consider the case when  $j = \max\{j, k-j\} \leq k$ , in which case we estimate using the decomposition  $\partial_x^j A = A_0^{(j)} + \varepsilon A_1^{(j)}$  as in (67), (68) via

$$\begin{aligned} (118) &\lesssim \varepsilon \langle x \rangle^{-\left(\frac{1}{2}+\right)} \left\| u^{\frac{3}{2}} A_1^{(j)} \langle x \rangle^{j+\frac{3}{4}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \left\| \frac{\phi^{(k-j)}}{u^2} \langle x \rangle^{k-j+\frac{1}{4}-\sigma_{k-j+1}} \right\|_{L_\psi^\infty} \\ &\quad \times \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ &\quad + \langle x \rangle^{-\left(\frac{1}{2}+\right)} \left\| A_0^{(j)} \langle x \rangle^{j+1} \right\|_{L_\psi^\infty} \left\| \frac{\phi^{(k-j)}}{\sqrt{u}} \langle x \rangle^{k-j-\omega_{k-j}} \langle \psi \rangle^{\frac{1}{2}-\kappa} \right\|_{L_\psi^2} \\ &\quad \times \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ &\lesssim \langle x \rangle^{-\left(\frac{1}{2}+\right)} (\varepsilon E_j E_{k-j+1} + R_{k-j-1}^{\frac{1}{2}}) \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2} \\ &\lesssim \langle x \rangle^{-(1+)} \left( \varepsilon E_j E_{k-j+1} + R_{k-j-1}^{\frac{1}{2}} \right)^2 + \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L_\psi^2}^2 \end{aligned}$$

$$\lesssim R_{k-1}(x) + \left\| \phi^{(k)} \frac{\langle \psi \rangle^{\frac{1}{2}-\kappa}}{\sqrt{u}} \langle x \rangle^{k-\frac{1}{2}-\omega_k} \right\|_{L^2_\psi}^2.$$

Above, we use that  $\sigma_k > \omega_k$ ,  $\sigma_k > \sigma_{k-j+1}$  for  $j = \max\{j, k-j\}$ . We use (67), (68) for the  $A$  term, (51), (64) to estimate the  $\phi^{(k-j)}$  term in  $L^\infty$ , and (112) to bound the  $\phi^{(k-j)}$  term in  $L^2_\psi$ , as we have ensured in this range of indices,  $j \geq 1$ .

This concludes the proof.  $\square$

Summarizing the estimates from this section, we have

**Lemma 36.** *Let  $K_1 \leq k \leq K_0$ , and let  $\phi$  solve (20). Assume the bootstrap assumption, (25). For any  $0 < \delta \ll 1$ , the following estimates hold:*

$$\frac{\partial_x}{2} \int \left| \phi_\psi^{(k-1)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} + \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle x \rangle^{2k-1-2\omega_k} \lesssim R_{k-1}(x), \quad (119)$$

$$\frac{\partial_x}{2} \int \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-2\sigma_k} + \int u \left| \phi_\psi^{(k)} \right|^2 \langle x \rangle^{2k-2\sigma_k} \lesssim R_{k-1}(x) \quad (120)$$

$$\begin{aligned} & \frac{\partial_x}{2} \int \left| \phi_\psi^{(k-1)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\omega_k} + \int \frac{1}{u} \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} \langle \psi \rangle^{1-2\kappa} \\ & \leq C_\delta R_{k-1}(x) + \delta \int \left| \phi_\psi^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-2\sigma_k} \end{aligned} \quad (121)$$

$$\begin{aligned} & \frac{\partial_x}{2} \int \left| \phi^{(k)} \right|^2 \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2k-2\sigma_k} + \int \left| \phi_\psi^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-2\sigma_k} \\ & \lesssim R_{k-1}(x) + \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\omega_k}. \end{aligned} \quad (122)$$

This concludes our scheme of *a-priori* estimates.

## 6. Global Existence in $X$

The aim of this section is to close the bootstrap and establish global existence in  $X$ , which proves the main theorem.

**Proof of Theorem 3.** First, combining estimates (81), (83), Lemma 31 and Lemma 36, we obtain the following estimates on the interval  $I_* = (0, x_*)$ , which, as we recall from (25), is the maximal interval on which  $\|\phi\|_{X(I_*)} \leq 2\|\phi_0\|_{X_{in}}$ :

$$\frac{\partial_x}{2} \int \phi^2 + \int \bar{u} |\phi_\psi|^2 \lesssim \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x), \quad (123)$$

$$\frac{\partial_x}{2} \int \phi^2 \frac{1}{u} \langle \psi \rangle + \int \phi_\psi^2 \langle \psi \rangle \lesssim \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x), \quad (124)$$

and for  $1 \leq k \leq K_0$ ,

$$\frac{\partial_x}{2} \int \left| \phi_\psi^{(k-1)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} + \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle x \rangle^{2k-1-2\omega_k} \lesssim R_{k-1}(x), \quad (125)$$

$$\begin{aligned} & \frac{\partial_x}{2} \int \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-2\sigma_k} + \int u \left| \phi_\psi^{(k)} \right|^2 \langle x \rangle^{2k-2\sigma_k} \lesssim R_{k-1}(x) \\ & \frac{\partial_x}{2} \int \left| \phi_\psi^{(k-1)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\omega_k} + \int \frac{1}{u} \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} \langle \psi \rangle^{1-2\kappa} \quad (126) \end{aligned}$$

$$\begin{aligned} & \leq C_\delta R_{k-1}(x) + \delta \int \left| \phi_\psi^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-2\sigma_k} \\ & \frac{\partial_x}{2} \int \left| \phi^{(k)} \right|^2 \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2k-2\sigma_k} + \int \left| \phi_\psi^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-2\sigma_k} \quad (127) \end{aligned}$$

$$\lesssim R_{k-1}(x) + \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\omega_k}. \quad (128)$$

First, we pick  $\delta$  small relative to universal constants to close the two estimates, (127) and (128) in terms of  $C_\delta R_{k-1}(x)$ . Recalling the definition of  $R_{k-1}$  given in (87), we obtain the full estimate

$$\begin{aligned} & \frac{\partial_x}{2} \left( \int \phi^2 + \int \phi^2 \frac{1}{u} \langle \psi \rangle \right) + \frac{\partial_x}{2} \sum_{k=1}^{K_0} \left( \int \left| \phi_\psi^{(k-1)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} \right. \\ & \quad + \int \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-2\sigma_k} + \int \left| \phi_\psi^{(k-1)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-1-2\omega_k} \\ & \quad + \int \left| \phi^{(k)} \right|^2 \frac{\langle \psi \rangle^{1-2\kappa}}{u} \langle x \rangle^{2k-2\sigma_k} \Big) + \int \bar{u} \left| \phi_\psi \right|^2 + \int \phi_\psi^2 \langle \psi \rangle \\ & \quad + \int \left| \phi^{(k)} \right|^2 \frac{1}{u} \langle x \rangle^{2k-1-2\omega_k} + \int u \left| \phi_\psi^{(k)} \right|^2 \langle x \rangle^{2k-2\sigma_k} \\ & \quad + \int \frac{1}{u} \left| \phi^{(k)} \right|^2 \langle x \rangle^{2k-1-2\omega_k} \langle \psi \rangle^{1-2\kappa} + \int \left| \phi_\psi^{(k)} \right|^2 \langle \psi \rangle^{1-2\kappa} \langle x \rangle^{2k-2\sigma_k} \\ & \lesssim \varepsilon \langle x \rangle^{-(0+)} I_{K_0}(x). \end{aligned}$$

Integrating this energy inequality from  $(0, x_*)$ , we obtain, for a constant  $C_0$  determined by the implicit constant above and the initial data (which enters through (28))

$$\| \phi \|_{X(I_*)} \leq \| \phi_0 \|_{X_{in}} + C_0(\| \phi_0 \|_{X_{in}}) \varepsilon \| \phi \|_{X(I_*)}.$$

Thus, by taking  $\varepsilon$  small enough, depending on universal constants and the size of the data  $\| \phi_0 \|_{X_{in}}$  so that  $\varepsilon C_0(\| \phi_0 \|_{X_{in}}) < \frac{1}{3}$ , we obtain that  $\| \phi \|_{X(I_*)} \leq \frac{3}{2} \| \phi_0 \|_{X_{in}}$ , which contradicts the maximality of  $I_*$  in the bootstrap assumption, (25). Thus, no such  $I_*$  can exist.  $\square$

### 6.1. Nash-type Inequalities and Optimal Decay

**Lemma 37.** *Solutions  $\phi \in X$  to the system (20) satisfy the following Nash-type inequality*

$$\left\| \phi \right\|_{L_\psi^2}^2 \lesssim \max \left\{ x^{\frac{1}{10}} \left\| \sqrt{u} \phi_\psi \right\|_{L_\psi^2}^{\frac{4}{5}}, \left\| \sqrt{u} \phi_\psi \right\|_{L_\psi^2}^{\frac{2}{3}} \right\}. \quad (129)$$

**Proof.** We first localize based on  $\xi = \frac{\psi}{\sqrt{x}}$ . Fix a  $\tau$  to be selected later. Then by triangle inequality we split

$$\|\phi\|_{L^2_\psi} \leq \left\| \phi \chi \left( \frac{\xi}{\tau} \right) \right\|_{L^2_\psi} + \left\| \phi \chi \left( \frac{\xi}{\tau} \right)^C \right\|_{L^2_\psi}. \quad (130)$$

For the localized portion, we need to condition on whether or not  $\tau < 1$  or  $\tau > 1$ . We integrate by parts via

$$\left\| \phi \chi \left( \frac{\xi}{\tau} \right) \right\|_{L^2_\psi}^2 = \int \partial_\psi \{\psi\} \phi^2 \chi \left( \frac{\xi}{\tau} \right)^2 = - \int 2\psi \phi \phi_\psi \chi \left( \frac{\xi}{\tau} \right)^2 - \int \psi \phi^2 \frac{1}{\sqrt{x}} \frac{1}{\tau} \chi \chi'. \quad (131)$$

We estimate the former term above term via

$$\left| \int \psi \phi \phi_\psi \chi \left( \frac{\xi}{\tau} \right)^2 \right| \lesssim \begin{cases} \tau^{\frac{3}{2}} x \left\| \sqrt{u} \phi_\psi \right\|_{L^2_\psi}^2 & \text{if } \tau < 1 \\ \rho^2 x \left\| \sqrt{u} \phi_\psi \right\|_{L^2_\psi}^2 & \text{if } \tau \geq 1. \end{cases}$$

More specifically, in the case when  $\rho < 1$ ,

$$\begin{aligned} \left| \int \psi \phi \phi_\psi \chi \left( \frac{\xi}{\tau} \right)^2 \right| &\leq \left\| \phi \chi \right\|_{L^2_\psi} \left\| \psi \phi_\psi \chi \right\|_{L^2_\psi} \lesssim \left\| \phi \chi \right\|_{L^2_\psi} \left\| \frac{\psi}{\sqrt{x}} \phi_\psi \chi \right\|_{L^2_\psi} \sqrt{x} \\ &\lesssim \left\| \phi \chi \right\|_{L^2_\psi} \sqrt{x} \left\| \xi \phi_\psi \chi \right\|_{L^2_\psi} \lesssim \left\| \phi \chi \right\| \sqrt{x} \tau^{\frac{3}{4}} \left\| \xi^{\frac{1}{4}} \phi_\psi \chi \right\|_{L^2_\psi} \\ &\lesssim \left\| \phi \chi \right\|_{L^2_\psi} \sqrt{x} \tau^{\frac{3}{4}} \left\| \sqrt{u} \phi_\psi \chi \right\|_{L^2_\psi} \\ &\leq o(1) \left\| \phi \chi \right\|_{L^2_\psi}^2 + C x \tau^{\frac{3}{2}} \left\| \sqrt{u} \phi_\psi \chi \right\|_{L^2_\psi}^2. \end{aligned}$$

The  $o(1)$  term is absorbed to the left-hand side of (131).

In the case when  $\rho > 1$ , we must estimate  $\xi^{\frac{1}{4}} \leq \tau^{\frac{1}{4}} \sqrt{u}$ . To see that this is true, first assume  $\xi \leq 1$ . Then  $\xi^{\frac{1}{4}} \lesssim \sqrt{u} \lesssim \sqrt{u} \tau^{\frac{1}{4}}$  because  $\tau > 1$  by assumption. Next, suppose  $\xi \geq 1$ . Then  $\xi^{\frac{1}{4}} \leq \tau^{\frac{1}{4}} \lesssim \tau^{\frac{1}{4}} \sqrt{u}$  because  $u \gtrsim 1$  on the region when  $\xi \geq 1$ .

For the second term in (131), we estimate identically to the far-field term from (130), which we now treat.

For the far-field term, we estimate via

$$\left| \int \phi^2 \psi \frac{1}{\psi} \chi \left( \frac{\xi}{\tau} \right)^C \right| \lesssim \frac{1}{\tau \sqrt{x}} \left\| \phi \sqrt{\psi} \right\|_{L^2_\psi}^2.$$

In summary, we have thus established the inequality

$$\left\| \phi \right\|_{L^2_\psi}^2 \lesssim \varphi(\tau) x \left\| \sqrt{u} \phi_\psi \right\|_{L^2_\psi}^2 + \frac{1}{\tau \sqrt{x}} \left\| \phi \sqrt{\psi} \right\|_{L^2_\psi}^2,$$

where  $\varphi(\tau)$  is the piecewise function equal to  $\tau^{\frac{3}{2}}$  on  $\tau < 1$  and  $\tau^2$  on  $\rho \geq 1$ .

We now select

$$\tau = \begin{cases} x^{-\frac{3}{5}} \|\phi\sqrt{\psi}\|_{L_\psi^2}^{\frac{4}{5}} \|\sqrt{u}\phi_\psi\|_{L_\psi^2}^{-\frac{4}{5}} := r^{\frac{6}{5}} & \text{if } r < 1 \\ x^{-\frac{1}{2}} \|\phi\sqrt{\psi}\|_{L_\psi^2}^{\frac{2}{3}} \|\phi_\psi\sqrt{u}\|_{L_\psi^2}^{-\frac{2}{3}} := r \geq 1. \end{cases}$$

The key point is that  $\tau$  is homogeneous in  $r$ , and therefore we may consistently enforce when  $\tau < 1$  and  $\tau > 1$  because these are equivalent to  $r < 1$  and  $r > 1$ .

To conclude, we note that by definition of the  $X$  norm, the weighted quantities  $\|\phi\sqrt{\psi}\|_{L_\psi^2}$  are conserved in  $x$  for solutions to (20). This immediately gives (129).  $\square$

**Corollary 38.** *The solution  $\phi$  to (20) satisfies the following asymptotics for  $j \leq K_0 - 1$ :*

$$\begin{aligned} & \|\phi^{(j)}\|_{L_\psi^2} \langle x \rangle^{j+\frac{1}{4}-} + \|\phi_\psi^{(j)}\|_{L_\psi^2} \langle x \rangle^{j+\frac{3}{4}-} + \|\phi^{(j)}\|_{L_\psi^\infty} \langle x \rangle^{j+\frac{1}{2}-} \\ & + \|\phi_\psi^{(j)}\|_{L_\psi^\infty} \langle x \rangle^{j-} \lesssim \|\phi\|_X. \end{aligned} \quad (132)$$

**Proof.** Using (129) in (81), letting  $\alpha(x) := \|\phi\|_{L_\psi^2}^2$ , we obtain either one of the two ODEs ( $\cdot = \partial_x$ ):

$$\dot{\alpha} + C_2 \alpha^3 \leq 0 \text{ or } \dot{\alpha} + C_3 x^{-\frac{1}{4}} \alpha^{\frac{5}{2}} \leq 0,$$

at each  $x \in \mathbb{R}_+$ . This immediately implies that  $|\alpha| \lesssim \langle x \rangle^{-\frac{1}{2}}$ , which means that  $\|\phi\|_{L_\psi^2} \lesssim \langle x \rangle^{-\frac{1}{4}}$ .

We may  $x$ -differentiate (129) and use them in the higher order energy estimates (90) and (107) in exactly the same fashion, which yields that  $\|\phi^{(j)}\|_{L_\psi^2} \lesssim \langle x \rangle^{-j-\frac{1}{4}-}$ .  $\square$

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(Received December 12, 2018 / Accepted March 26, 2020)

Published online April 15, 2020

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