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Zonotopes whose cellular strings are all coherent

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ABSTRACT

A cellular string of a polytope is a sequence of faces stacked on top of each other in a given direction. The poset of cellular strings, ordered by refinement, is known to be homotopy equivalent to a sphere. The subposet of coherent cellular strings is the face lattice of the fiber polytope, hence is homeomorphic to a sphere. In some special cases, every cellular string is coherent. Such polytopes are said to be all-coherent. We give a complete classification of zonotopes with the all-coherence property in terms of their oriented matroid structure. Although the face lattice of the fiber polytope in this case is not an oriented matroid invariant, we prove that the all-coherence property is invariant.

1. Introduction

We consider the problem of realizing spaces of polytopal subdivisions as the boundary complex of a convex polytope. For example, the set of triangulations of a convex polygon forms the vertices of a polytope known as the associahedron. The facets of an associahedron correspond to diagonals where a facet and a vertex are incident exactly when the triangulation includes the diagonal. In this situation, we are given the combinatorics of the polytope, namely the face lattice, and we need to provide a geometric realization.

Among the many polytopal realizations of the associahedron (see [7] for a survey), the secondary polytope construction of Gel'fand, Kapranov, and Zelevinsky is particularly elegant [13, Chapter 7]. Given a finite set of points $\mathcal{A}$ in $\mathbb{R}^d$, the *secondary polytope* $\Sigma(\mathcal{A})$ is the Newton polytope of the

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$\mathcal{A}$-determinant. Its vertices correspond to *regular triangulations* of $\mathcal{A}$, where a triangulation $\mathcal{T}$ is *regular* (or *coherent*) if there exists a function $f : \mathcal{A} \to \mathbb{R}$ such that $\mathcal{T}$ is the set of lower faces of the convex hull of $\{(x, f(x)) : x \in \mathcal{A}\} \subseteq \mathbb{R}^{d+1}$. If $\mathcal{A}$ is the set of vertices of a convex polygon, then every triangulation is regular, so the secondary polytope of $\mathcal{A}$ realizes the associahedron.

Given a linear surjection of polytopes $\pi : P \to Q$, the *Baues poset* $\omega(P, Q)$ is the set of polytopal subdivisions of Q by images of faces of P, ordered by refinement; see Section 2.1 for a more precise definition. We frequently make the assumption that π is *generic*, meaning that whenever F is a face of P of dimension at most $\dim Q$, we have $\dim \pi(F) = \dim F$. For a generic function π, Baues observed that if P is a d-simplex and Q is 1-dimensional, then $\omega(P, Q)$ is the face lattice of a $(d-1)$-cube [3, Chapter 3]. He also showed that if P is a d-cube and Q is 1-dimensional, then $\omega(P, Q)$ is the face lattice of a $(d-1)$-dimensional permutahedron. When P is a d-dimensional permutahedron, the maximally refined subdivisions of Q may be identified with reduced words for the longest element of a type A_d Coxeter system. Baues noticed that the order complex of $\omega(P, Q)$ is not always a simplicial sphere, but he conjectured that it is homotopy-equivalent to a sphere. This was proved for any polytopes P, Q with $\dim Q = 1$ by Billera, Kapranov, and Sturmfels [4]. We refer to the survey article by Reiner for many other results of this type [16].

As with triangulations, there is a distinguished subset of polytopal subdivisions in $\omega(P, Q)$ called *coherent* subdivisions. Generalizing the secondary polytope construction, the *fiber polytope* introduced by Billera and Sturmfels is a polytope of dimension $\dim P - \dim Q$ whose faces correspond to coherent subdivisions of Q [5]. Hence, the Baues poset $\omega(P, Q)$ contains a canonical spherical subspace of dimension $\dim P - \dim Q - 1$, and Baues' original question is whether this subspace is homotopy-equivalent to the full poset $\omega(P, Q)$. We consider the stronger question:

Question 1.1. For which projections $f : P \to Q$ is $\omega(P, Q)$ isomorphic to the face lattice of a polytope of dimension $\dim P - \dim Q$?

Using the fiber polytope construction, the poset $\omega(P, Q)$ is the face lattice of a $(\dim P - \dim Q)$-dimensional polytope exactly when every element of $\omega(P, Q)$ is coherent; see Section 2.2 for details. Hence, we may rephrase Question 1.1 as asking whether every subdivision of Q by faces of P is coherent. If $\pi : P \to Q$ has this property, we say the projection is *all-coherent*. The all-coherence property was previously studied for projections $P \to Q$ where P is a cube and Q is 2-dimensional [9] or 3-dimensional [2], or where P and Q are cyclic polytopes [1].

In this work, we examine the special case of Question 1.1 where P is a zonotope and Q is 1-dimensional. A *zonotope with n zones* is the image of an n-dimensional cube under a linear map. To any zonotope with a distinguished linear functional, one may associate an (acyclic) *oriented matroid*, an abstract combinatorial object that records the faces of the zonotope. Some background on oriented matroids is given in Section 3. Given P and Q as above, it is known that the facial structure of the fiber polytope of $\pi : P \to Q$ is not an oriented matroid invariant. That is, there exists a zonotope with two distinct linear functionals defining the same oriented matroid whose fiber polytopes are not combinatorially equivalent; see Remark 3.3. However, we prove that the all-coherence property *is* invariant.

Our main result is a characterization of the all-coherence property for projections in which P is a zonotope and Q is 1-dimensional. Monotone paths and oriented matroids are introduced in Section 3.

Theorem 1.2. *Let Z be a zonotope with a generic linear functional $\pi : Z \to \mathbb{R}$. The following are equivalent.*

1. *The pair (Z, π) is all-coherent.*
2. *The Baues poset $\omega(Z, \pi)$ is the face lattice of a polytope.*
3. *The order complex $\Delta(\omega(Z, \pi))$ is a simplicial sphere.*
4. *Every monotone path of (Z, π) is coherent.*
5. *The acyclic oriented matroid $\mathcal{M}$ associated to (Z, π) is in the list given in Section 5.*

The equivalence of (1)–(3) is proved for all polytopes in Corollary 2.2. Since the order complex $\Delta(\omega(Z,\pi))$ only depends on the oriented matroid corresponding to (Z,π), we may take Statement (3) as the definition of the all-coherence property for an oriented matroid. We prove the equivalence of (1) and (4) in Corollary 4.2. Our proof of this equivalence works for zonotopes only. Finally, we give a complete classification of zonotopes with the all-coherence property in Section 5 using the techniques developed in Section 4.

This paper continues the work by the first author [10], which contains several techniques to detect incoherent monotone paths and has a partial classification of all-coherent zonotopes. However, our proof of the full classification primarily relies on Proposition 4.5, which is new to this paper.

2. All-coherence for polytopes

In this section, we recall coherence and the Baues poset of polyhedral subdivisions induced by a projection of polytopes. In Section 2.2, we prove that every element of the Baues poset is coherent exactly when its order complex is homeomorphic to a sphere of a certain dimension.

2.1. Fiber polytope

We first review the basics of fiber polytopes. We refer to Ziegler [17, Chapter 9] for proofs of the statements in this subsection. In this paper, the term *polytope* always means a convex polytope.

Given a polytope P and a linear functional ψ, let P^ψ be the face

$$P^\psi = \{x \in P : \ \psi(x) = \min_{y\in P} \psi(y)\}.$$

That is, P^ψ is the face of P at which ψ is minimized. We observe that if $\psi = 0$, then P^ψ is the entire polytope. For our purposes, it will be convenient to consider P to be a face of itself. The *normal fan* of $P \subseteq \mathbb{R}^d$ is a subdivision of $(\mathbb{R}^d)^*$ into relatively open cones where ψ_1 and ψ_2 lie in the same cone if $P^{\psi_1} = P^{\psi_2}$.

We define a *projection* of polytopes P, Q to be a surjective affine map $P \to Q$. Let $\pi : P \to Q$ be a projection of polytopes. A collection Δ of faces of P is called a π*-induced subdivision* of Q if

- $\{\pi(F) : \ F \in \Delta\}$ is a polyhedral complex subdividing Q with all $\pi(F)$ distinct, and
- for all $F, F' \in \Delta$, if $\pi(F) \subseteq \pi(F')$ then $F = F' \cap \pi^{-1}(\pi(F))$.

A π-induced subdivision Δ is *coherent* if there exists a linear functional $\psi : P \to \mathbb{R}$ such that for any $x \in Q$ and for any $F \in \Delta$ with $x \in \pi(F)$,

$$(\pi^{-1}(x))^\psi = F \cap \pi^{-1}(x).$$

Conversely, if ψ is any linear functional, there exists a unique π-induced subdivision Δ^ψ whose coherence is witnessed by ψ. The π-induced subdivision Δ^0 is called the *trivial subdivision*.

The *Baues poset* $\omega(P \xrightarrow{\pi} Q)$ is the set of π-induced subdivisions, partially ordered by refinement: $\Delta \leq \Delta'$ if for any face F in Δ there exists F' in Δ' such that $F \subseteq F'$. The subposet of coherent subdivisions is denoted $\omega_{\mathrm{coh}}(P \xrightarrow{\pi} Q)$. When the map π between the polytopes P and Q is understood, we usually use the notations $\omega(P, Q)$ and $\omega_{\mathrm{coh}}(P, Q)$.

The *fiber polytope* $\Sigma(P, Q)$ was defined by Billera and Sturmfels as the Minkowski integral $\frac{1}{\mathrm{Vol}(Q)} \int_Q \pi^{-1}(x)dx$; see [5] for details. This polytope lives in the ambient space of P and is of dimension $\dim P - \dim Q$. The faces of the fiber polytope encode coherent subdivisions of Q in the following sense. For linear functionals $\psi_1, \psi_2 : P \to \mathbb{R}$, $\Delta^{\psi_1} = \Delta^{\psi_2}$ if and only if ψ_1 and ψ_2 lie in the relative interior of the same cone of the normal fan of $\Sigma(P, Q)$. For the most part, we will only need results about the normal fan of a fiber polytope rather than the polytope itself.

2.2. All-coherence property

Given a poset X, the *order complex* $\Delta(X)$ is the abstract simplicial complex whose faces are chains $x_0 < \cdots < x_d$ of X. If X has a unique maximum or minimum element, we let $\overline{X}$ be the same poset with these elements removed. In particular, $\overline{\omega}(P, Q)$ is the poset of *nontrivial* π-induced subdivisions of Q by faces of P.

We say that a projection $\pi : P \to Q$ is *all-coherent* if every π-induced subdivision of Q is coherent. The following theorem gives a useful characterization of the all-coherence property.

Theorem 2.1. *Given a projection $\pi : P \to Q$, the following statements are equivalent.*

1 *The map π is all-coherent.*
2 *The Baues poset $\omega(P, Q)$ is the face lattice of a polytope of dimension* $\dim P - \dim Q$.
3 *The order complex $\Delta(\overline{\omega}(P, Q))$ is a simplicial sphere of dimension* $\dim P - \dim Q - 1$.

Proof. We prove (1)⇒(2)⇒(3)⇒(1).

If $\pi : P \to Q$ is all-coherent, then $\omega(P, Q) = \omega_{\text{coh}}(P, Q)$. The latter poset is isomorphic to the face lattice of the fiber polytope $\Sigma(P, Q)$, which is a polytope of dimension $\dim P - \dim Q$.

If $\omega(P, Q)$ is the face lattice of a polytope R, then its proper part $\overline{\omega}(P, Q)$ is the face poset of its boundary complex ∂R. The order complex $\Delta(\overline{\omega}(P, Q))$ is realized by the barycentric subdivision of ∂R, so it is a simplicial sphere. If $\dim R = \dim P - \dim Q$, then its boundary complex has dimension $\dim P - \dim Q - 1$.

Now assume $\Delta(\overline{\omega}(P, Q))$ is a simplicial sphere of dimension $d = \dim P - \dim Q - 1$. The inclusion $\omega_{\text{coh}}(P, Q) \hookrightarrow \omega(P, Q)$ induces an injective simplicial map

$$\Delta(\overline{\omega}_{\text{coh}}(P, Q)) \hookrightarrow \Delta(\overline{\omega}(P, Q)).$$

The geometric realization of this map is equivalent to a continuous map $\mathbb{S}^d \to \mathbb{S}^d$, where $\mathbb{S}^d$ is the unit d–sphere in $\mathbb{R}^{d+1}$. This map is a homeomorphism onto its image. If it is not surjective, then we obtain a subspace of $\mathbb{S}^d$ which is homeomorphic to $\mathbb{S}^d$. This is well-known to be impossible, e.g. by an application of the Borsuk–Ulam Theorem; see [15, Theorem 2.1.1]. Hence, $\omega(P, Q) = \omega_{\text{coh}}(P, Q)$ holds, which means that $\pi : P \to Q$ is all-coherent. □

If $\dim Q = 1$, we can drop the dimension condition to get Corollary 2.2. This corollary proves the equivalence of Theorem 1.2(1)–(3). Its proof follows from the fact that if P is any polytope and $\dim Q = 1$, then $\Delta(\overline{\omega}(P, Q))$ is homotopy equivalent to a sphere of dimension $\dim P - 2$ by [4, Theorem 1.2].

Corollary 2.2. *If* $\dim Q = 1$, *then the following are equivalent.*

1. *The map $\pi : P \to Q$ is all-coherent.*
2. *The Baues poset $\omega(P, Q)$ is the face lattice of a polytope.*
3. *The order complex $\Delta(\overline{\omega}(P, Q))$ is a simplicial sphere.*

3. All-coherence for zonotopes

Starting in this section, we restrict our attention to projections $\pi : P \to Q$ for which P is a *zonotope*, i.e. when P is an affine linear projection of a hypercube. Section 3.1 introduces some notation for zonotopes and hyperplane arrangements and features an interesting example for which π is not all-coherent but the order complex of the Baues poset is a homotopy sphere. In the remaining subsections, we restrict further to the case that Q is 1-dimensional. In this case, the elements of the Baues poset $\omega(P, Q)$ may be identified with cellular strings, certain sequences of faces of P as defined in Section 3.2. In Section 3.3, we present a 2-parameter family of examples with the all-coherence property. Finally, we recall notation for oriented matroids in Section 3.4, which are combinatorial abstractions of zonotopes.

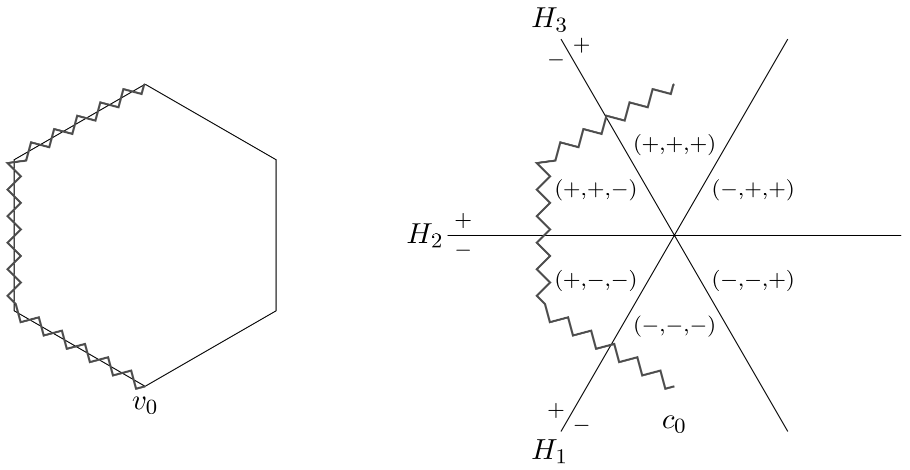

Fig. 1. (left) A zonogon generated by three vectors in the plane (right) A hyperplane arrangement whose faces form the normal fan to the zonogon.

3.1. Hyperplane arrangements

The normal fan of a zonotope is the set of cones cut out by a real, central hyperplane arrangement. In this subsection, we recall some notation for hyperplane arrangements.

A *real hyperplane arrangement* (or *arrangement*) is a finite set of hyperplanes in a real finite-dimensional vector space V. We will assume that the arrangement is *central*, which means that every hyperplane contains the origin. An arrangement is *essential* if the origin is the only point in the intersection of all of its hyperplanes. Each hyperplane partitions V into two open half-spaces and the hyperplane itself. We typically assume that an arrangement comes with an *orientation*; that is, for each hyperplane H, we set $H^0 = H$ and assign the label H^+ and H^- to the two open half-spaces defined by H. A *face* of an arrangement $\mathcal{A}$ is any nonempty cone of the form $\bigcap_{H\in\mathcal{A}} H^{x(H)}$ where $x \in \{0, +, -\}^{\mathcal{A}}$. Faces that are maximal under inclusion are called *chambers*. A *sign vector* is an element of $\{0, +, -\}^E$ for some finite set E. A sign vector $x \in \{0, +, -\}^{\mathcal{A}}$ is a *covector* of $\mathcal{A}$ if there is a face F of $\mathcal{A}$ such that $F = \bigcap_{H\in\mathcal{A}} H^{x(H)}$. The set of covectors of $\mathcal{A}$ will be denoted $\mathcal{L}(\mathcal{A})$.

A *cocircuit* is the covector of a nonzero face of $\mathcal{A}$ that is minimal by inclusion. So if $\mathcal{A}$ is essential, then a cocircuit is the covector of a 1-dimensional face. The *support* of a sign vector x is the set of $H \in \mathcal{A}$ such that $x(H) \neq 0$. A *circuit* is a nonzero sign vector x whose support is minimal by inclusion such that

$$\bigcap_{x(H)=+} H^+ \cap \bigcap_{x(H)=-} H^- = \emptyset.$$

Example 3.1. The right side of Fig. 1 shows a central arrangement of three hyperplanes $\{H_1, H_2, H_3\}$ in $\mathbb{R}^2$, i.e. an arrangement of three lines through the origin. With the orientation as shown, the chambers correspond to the six displayed covectors. The arrangement also has six cocircuits corresponding to the 1-dimensional faces of $\mathcal{A}$. The only circuits are $(+, -, +)$ and $(-, +, -)$.

We say a zonotope $Z = Z(\boldsymbol{\rho})$ is generated by a set of vectors $\boldsymbol{\rho} = \{\rho_1, \ldots, \rho_n\} \subseteq V$ if it is an affine translation of the image of $[0, 1]^n$ under the linear transformation $R : \mathbb{R}^n \to V$ where $R(e_i) = \rho_i$. For a real matrix A, we also use the notation $Z(A)$ to denote the zonotope generated by the set of column vectors of A. The normal fan of Z is the set of faces of the hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in V^* where H_i is the set of linear functionals annihilating ρ_i. In this situation, we say that Z is dual to $\mathcal{A}$.

Example 3.2. A zonogon generated by three vectors

$$\rho_1 = \left(\frac{-\sqrt{3}}{2}, \ \frac{1}{2} \right), \quad \rho_2 = (0, 1), \quad \rho_3 = \left(\frac{\sqrt{3}}{2}, \ \frac{1}{2} \right)$$

is shown on the left side of Fig. 1. We view the zonogon as a subset of a vector space $V \cong \mathbb{R}^2$ and its associated hyperplane arrangement as a subset of the dual space $V^* \cong \mathbb{R}^2$. In this way, the hyperplane H_i is identified with the set of linear functionals annihilating ρ_i.

Remark 3.3. The all-coherence property for a projection of the form $\pi : [0, 1]^n \to Z, \ Z \subseteq \mathbb{R}^2$ was completely classified by Edelman and Reiner [9]. In that paper, the authors observed that the number of rhombic tilings of Z admits a nice product formula when they are all coherent, as given by MacMahon when Z is a hexagon [14] or Elnitsky when Z is an octagon [11].

3.2. Cellular strings

Fix an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^n$. Designate one of the chambers of $\mathcal{A}$ to be the *fundamental chamber* c_0. Given two chambers c and c' of $\mathcal{A}$, let $S(c, c')$ denote the set of hyperplanes of $\mathcal{A}$ for which c and c' lie on opposite sides of the hyperplane. A *gallery* is a sequence of chambers $c_0, \ldots, c_l$ such that $|S(c_{i-1}, c_i)| = 1$ for all i. This gallery may be drawn as a path in $\mathbb{R}^n$ connecting generic points in each chamber. It is *reduced* if $|S(c_0, c_l)| = l$, which means that it may be represented by a path that crosses each hyperplane at most once. For our purposes, a gallery is always assumed to be reduced and connecting a pair of antipodal chambers $c_0, \ -c_0$. Such a gallery defines a total order on the arrangement $\mathcal{A}$ where $H < H'$ if H is crossed before H'.

Let Z be a zonotope with a generic linear functional π. A *monotone path* of (Z, π) is a sequence of vertices $v_0, \ldots, v_l$ of Z such that the line segment $[v_i, v_{i+1}]$ is an edge of Z and $\pi(v_i) < \pi(v_{i+1})$ for all i, where v_0 is the π-minimizing vertex and v_l is the π-maximizing vertex of Z. If Z is a zonotope dual to $\mathcal{A}$, then a gallery on $\mathcal{A}$ corresponds to a monotone path on Z. For this correspondence, one chooses a linear functional π that is minimized on Z at v_0, the vertex whose normal cone is c_0.

Example 3.4. We continue Example 3.2. On the right side of Fig. 1, a reduced gallery is marked by a zigzag path from c_0 to the antipodal chamber. This gallery determines the total order $H_1 < H_2 < H_3$ on the arrangement.

Choose an element π in the interior of c_0. The displayed vertex v_0 has normal cone c_0, so π is minimized over the zonogon at v_0. Each step in a monotone path now corresponds to crossing from the negative to the positive side of one of the three lines. The zigzag path on the left side of Fig. 1 traces the monotone path corresponding to the gallery indicated on the right side.

The cellular strings of $\mathcal{A}$ may be defined in several equivalent ways, e.g. as sequences of faces, covectors (Section 3.4), or chambers. Given a fundamental chamber c_0, we orient each $H \in \mathcal{A}$ so that c_0 is on the negative side of H. We define a *cellular string* as a sequence of faces $(F_1, \ldots, F_l)$ of $\mathcal{A}$ such that for each $H \in \mathcal{A}$, there exists $i \in [l]$ such that F_i is supported by H, $F_j \subseteq H^-$ if $j < i$ and $F_j \subseteq H^+$ if $j > i$. In particular, a cellular string determines an ordered set partition $(\mathcal{A}_1, \ldots, \mathcal{A}_l)$ of the set $\mathcal{A}$, where $\mathcal{A}_i$ is the set of hyperplanes supporting F_i. We remark that by definition each F_i in a cellular string is *not* a chamber, since it has to be supported by a hyperplane. The *trivial cellular string* is the singleton (F) where F is the intersection of all of the hyperplanes.

An equivalent way to define the cellular strings is in terms of a zonotope Z with a generic functional π. Here, when we say that π is generic (with respect to Z), we mean that π is not constant on any face of Z of dimension at least 1. In terms of (Z, π), a *cellular string* is a sequence of faces $(F_1, \ldots, F_l)$ where, each F_i is not a vertex, F_1 contains the bottom vertex (i.e. the π-minimizing vertex of Z), F_l contains the top vertex (i.e. the π-maximizing vertex of Z), and adjacent faces F_i, F_{i+1} meet at a vertex v_i such that $\pi(x) \leq \pi(v_i) \leq \pi(y)$ for all $x \in F_i$ and $y \in F_{i+1}$ (cf. [16]). In this case, the *trivial cellular string* is the sequence (Z).

In what follows, we may identify a monotone path by the sequence of edges between its adjacent pairs of vertices in order to view it as a kind of cellular string. Dually, we may consider a gallery as a special kind of cellular string of a hyperplane arrangement.

Example 3.5. The gallery from Example 3.4 may be interpreted as a cellular string (F_1, F_2, F_3) of faces of the arrangement whose covectors are

$$x_{F_1} = (0, -, -), \quad x_{F_2} = (+, 0, -), \quad x_{F_3} = (+, +, 0).$$

The corresponding ordered set partition of the arrangement of three lines is $(\{H_1\}, \{H_2\}, \{H_3\})$.

The monotone path from Example 3.4 may also be interpreted as a cellular string whose elements are the three edges between adjacent pairs of vertices of the path.

We note that the definition of a cellular string agrees with that of a π-induced subdivision given in Section 2.1 in the following manner. When $P = Z$ is a zonotope and the projection $\pi : Z \twoheadrightarrow \pi(Z) \subseteq \mathbb{R}^1$ is generic, we have a correspondence between the cellular strings on Z and the π-induced subdivisions as follows. A cellular string $(F_1, \dots, F_l)$ corresponds to the π-induced subdivision consisting of (i) $F_1, \dots, F_l$, (ii) the π-minimizing faces of F_i for all i, and (iii) the π-maximizing faces of F_i for all i. Conversely, a collection Δ of faces of Z which is a π-induced subdivision corresponds to the tuple of maximal faces in Δ arranged by their values under π. One may easily check that these maps are well-defined and that they are inverses (cf. [16, pp 305–306]). Note that from this correspondence, we can now define the notion of *π-coherence* for a cellular string: we say that a cellular string is π-coherent when its corresponding π-induced subdivision is coherent.

Lemma 3.6. *Given a real vector space V, let $\mathcal{A}$ be a set of hyperplanes in V^* and let π be a generic point in V^*. For each $H \in \mathcal{A}$, let v_H be a vector in V whose annihilator is H and such that $\pi(v_H) > 0$. Let Z be the zonotope generated by $\{v_H : H \in \mathcal{A}\}$. Then an ordered set partition $(\mathcal{A}_1, \dots, \mathcal{A}_l)$ of $\mathcal{A}$ defines a π-coherent cellular string of Z if and only if there exists $\psi \in V^*$ such that*

$$\frac{\psi(v_H)}{\pi(v_H)} = \frac{\psi(v_{H'})}{\pi(v_{H'})}$$

for H, H' in the same block, and

$$\frac{\psi(v_H)}{\pi(v_H)} < \frac{\psi(v_{H'})}{\pi(v_{H'})}$$

if $H \in \mathcal{A}_i,\ H' \in \mathcal{A}_j,\ i < j$. Moreover, if $(\mathcal{A}_1, \dots, \mathcal{A}_l)$ is π-coherent, then it is generated by any linear functional ψ satisfying the above equalities and inequalities.

Proof. First assume that the ordered set partition $(\mathcal{A}_1, \dots, \mathcal{A}_l)$ of $\mathcal{A}$ defines a π-coherent cellular string. Let $\mathbf{F} = (F_1, \dots, F_l)$ denote the sequence of faces of Z and $(x_1, \dots, x_l)$ the sequence of covectors associated to this cellular string. Since this string is π-coherent, there exists a linear functional ψ such that the coherent π-induced subdivision Δ^{ψ} corresponds to the cellular string $\mathbf{F}$.

For an index i, let $p_i = \sum_{j:\ j<i} \sum_{H \in \mathcal{A}_j} v_H$. The face F_i is the Minkowski sum of the point p_i with the image of $[0, 1]^{|\mathcal{A}_i|}$ under a matrix with column set $\{v_H : H \in \mathcal{A}_i\}$.

Fix some $q \in \mathbb{R}$ such that $q > \pi(p_i)$ and $q < \pi(p_i + v_H)$ for all $H \in \mathcal{A}$. By the coherence assumption, the face $(\pi^{-1}(q))^{\psi}$ is equal to $\pi^{-1}(q) \cap F_i$. Fix $H, H' \in \mathcal{A}$, and set $v = v_H,\ v' = v_{H'}$. Pick $\lambda, \lambda' \in (0, 1)$ such that $\pi(p_i + \lambda v) = q = \pi(p_i + \lambda' v')$. Then $\lambda \pi(v) = \lambda' \pi(v')$.

Assume H and H' are both in $\mathcal{A}_i$. Then $\psi(p_i + \lambda v) = \psi(p_i + \lambda' v')$, so $\lambda \psi(v) = \lambda' \psi(v')$. Consequently, $\frac{\psi(v)}{\pi(v)} = \frac{\psi(v')}{\pi(v')}$.

On the other hand, suppose $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$ with $i < j$. Then $p_i + \lambda' v'$ is in Z but not in F_i. Hence, $\psi(p_i + \lambda v) < \psi(p_i + \lambda' v')$ holds, which implies $\frac{\psi(v)}{\pi(v)} < \frac{\psi(v')}{\pi(v')}$.

Now let $(\mathcal{A}_1, \dots, \mathcal{A}_l)$ be an ordered set partition of $\mathcal{A}$ and assume that there exists $\psi \in V^*$ such that

$$\frac{\psi(v_H)}{\pi(v_H)} = \frac{\psi(v_{H'})}{\pi(v_{H'})}$$

for H, H' in the same block, and

$$\frac{\psi(v_H)}{\pi(v_H)} < \frac{\psi(v_{H'})}{\pi(v_{H'})}$$

if $H \in \mathcal{A}_i,\ H' \in \mathcal{A}_j,\ i < j$.

For each i, let λ_i denote the ratio $\frac{\psi(v_H)}{\pi(v_H)}$ for some $H \in \mathcal{A}_i$. Let $p_i = \sum_{j:\ j<i} \sum_{H \in \mathcal{A}_j} v_H$. Let F_i be the zonotope generated by $\{v_H :\ H \in \mathcal{A}_i\}$, translated by p_i.

We claim that F_i is a face of Z, namely that $F_i = Z^{\psi_i}$ where $\psi_i = \psi - \lambda_i \pi$. For $H \in \mathcal{A}$,

$$\psi_i(v_H) = \psi(v_H) - \lambda_i \pi(v_H) \begin{cases} = 0 \text{ if } H \in \mathcal{A}_i \\ < 0 \text{ if } H \in \mathcal{A}_j,\ j < i \\ > 0 \text{ if } H \in \mathcal{A}_j,\ j > i. \end{cases}$$

Using the case $H \in \mathcal{A}_i$, we see that ψ_i takes a common value $\psi_i(p_i)$ on the zonotope F_i. The zonotope Z is the set of points

$$Z = \{\sum_{H \in \mathcal{A}} \mu_H v_H :\ (\mu_H) \in [0, 1]^{\mathcal{A}}\}.$$

For $p \in Z$, there exists a representation $p - p_i = \sum_{H \in \mathcal{A}} \mu_H v_H$ where $\mu_H \leq 0$ for $H \in \mathcal{A}_j,\ j < i$ and $\mu_H \geq 0$ for $H \in \mathcal{A}_j,\ j > i$. By the previous calculation, we have $\psi_i(p - p_i) \geq 0$. Hence, ψ_i is minimized on Z at F_i.

The image $\pi(F_i)$ is the interval $[\pi(p_i), \pi(p_{i+1})]$. Hence, the images of the faces F_i partition the image of Z into intervals. For $q \in \pi(F_i)$, ψ_i is minimized on $\pi^{-1}(q)$ at $F_i \cap \pi^{-1}(q)$. But $\psi_i = \psi - \lambda_i \pi$ and π is constant on $\pi^{-1}(q)$, so ψ is minimized on $\pi^{-1}(q)$ at $F_i \cap \pi^{-1}(q)$ as well. Therefore, $(F_1, \ldots, F_l)$ is a π-coherent cellular string induced by ψ. $\square$

Since the set of cellular strings of $\mathcal{A}$ starting from the chamber c_0 does not depend on the choice of $\pi \in c_0$, we may define the pair $(\mathcal{A}, c_0)$ to be *all-coherent* if there exists some π in the interior of c_0 such that $\pi : Z \to \mathbb{R}^1$ is all-coherent. If π is replaced by some other π' in c_0, it would still be all-coherent by Corollary 2.2.

3.3. An all-coherent family

In this subsection, we consider an explicit 2-parameter family of zonotopes Z with linear functional π such that (Z, π) is all-coherent. We present this example in detail as it will be useful for the classification in Section 5.

Fix positive integers r, m. Let $e_1, \ldots, e_r$ be the elementary basis vectors in $\mathbb{R}^r$, and define vectors $a_1, \ldots, a_m$ where $a_i = ie_r + \sum_{j=1}^r e_j$. Let Z be the zonotope generated by $E = \{a_1, \ldots, a_m, e_1, \ldots, e_r\}$. Let $\pi : \mathbb{R}^r \to \mathbb{R}$ be the sum of coordinates function. In particular, $\pi(e_i) = 1$ and $\pi(a_j) = j + r$.

We claim that (Z, π) is all-coherent. To prove this, we first determine some necessary conditions for a permutation of E to define a monotone path. Then we show that each such permutation may be realized as a coherent monotone path by a suitable linear functional. Finally, we invoke Lemma 4.1 to complete the proof.

Let p be a monotone path on Z, and let $\prec$ be the induced total order on E. For $1 \leq k < l \leq m$, we have a relation $(l-k)e_r - a_l + a_k = 0$, so either $e_r \prec a_l \prec a_k$ or $a_k \prec a_l \prec e_r$ holds. Combining these relations, we have either $e_r \prec a_m \prec \cdots \prec a_1$ or $a_1 \prec \cdots \prec a_m \prec e_r$. By the relation $a_i = ie_r + \sum_{j=1}^r e_j$, the maximum and minimum elements under $\prec$ must be elementary basis vectors.

Now assume $\prec$ is a total ordering on E whose maximum and minimum elements are elementary basis vectors, and either $e_r \prec a_m \prec \cdots \prec a_1$ or $a_1 \prec \cdots \prec a_m \prec e_r$ holds. We construct a linear functional ψ realizing $\prec$ as a coherent monotone path. To do so, we distinguish three cases:

1. e_r is maximal with respect to $\prec$,
2. e_r is minimal with respect to $\prec$, or
3. e_r is neither minimal nor maximal.

We let $\psi_j = \psi(e_j)$ and $\alpha = \sum_{j=1}^r \psi_j$. We observe that $\psi(e_j)/\pi(e_j) = \psi_j$ and $\psi(a_i)/\pi(a_i) = (\alpha + i\psi_r)/(i + r)$. We assume throughout that ψ is chosen so that if $e_i \prec e_j$ then $\psi_i < \psi_j$. We impose additional constraints on ψ separately for the three cases above.

Suppose we are in Case 1, and let e_k be the minimum element under $\prec$. Set $\psi_r = 1$, and for $j \in [r-1] \setminus \{k\}$, let ψ_j be a positive number such that $0 < \psi_j < 1$ and

- if $a_i \prec e_j$ then $\frac{i}{i+r} < \psi_j$, and
- if $e_j \prec a_i$ then $\psi_j < \frac{i}{i+r}$.

Finally, choose $\psi_k < 0$ such that $\alpha = 0$. Then $\psi(a_i)/\pi(a_i) = \frac{i}{i+r}$. By the preceding conditions, we conclude that $\prec$ is realized as a coherent monotone path by ψ.

Case 2 follows by similar reasoning as Case 1. Assume we are in Case 3, and let e_k and e_l be the minimum and maximum elements of E under $\prec$, respectively. Furthermore, assume that $a_1 \prec \cdots \prec a_m \prec e_r$ holds. Set $\psi_r = 0$, and for $j \in [r-1] \setminus \{k, l\}$, let ψ_j be a real number such that

- if $a_i \prec e_j$ then $\frac{-1}{i+r} < \psi_j$, and
- if $e_j \prec a_i$ then $\psi_j < \frac{-1}{i+r}$.

Finally, choose $\psi_k < 0$, $\psi_l > 0$ such that $\psi_k < \psi_i < \psi_l$ for all i and $\alpha = -1$. Then $\psi(a_i)/\pi(a_i) = \frac{-1}{i+r}$, and we again conclude that $\prec$ is realized as a coherent monotone path by ψ. A similar argument exists when $e_r \prec a_m \prec \cdots \prec a_1$ holds. In that case, we fix $\alpha = 1$ and choose ψ accordingly.

We have now proven that every monotone path is coherent. By Corollary 4.2, this implies that every cellular string is coherent.

3.4. Covector axioms

Fix a finite set E. Recall a *sign vector* is an element of $\{+, -, 0\}^E$. Sign vectors form a monoid under composition, where

$$(x \circ y)(e) = \begin{cases} x(e) & \text{if } x(e) \neq 0 \\ y(e) & \text{otherwise,} \end{cases}$$

for $x, y \in \{+, -, 0\}^E$. We note that the identity element of the monoid is the 0-vector. The set $\{+, -, 0\}$ is partially ordered where $+ < 0$, $- < 0$, and $+$ and $-$ are incomparable. This ordering extends to $\{+, -, 0\}^E$ by the product ordering. We will also consider an alternate partial order $- \prec 0 \prec +$ on $\{+, -, 0\}$ and its extension to $\{+, -, 0\}^E$.

The *negation* $-x$ is the vector whose signs are reversed from x. Given sign vectors x, y, the *separation set* $S(x, y)$ is the set of elements $e \in E$ such that $x(e) = -y(e)$ and $x(e) \neq 0$. The *support* of a sign vector x is the set $\operatorname{supp}(x) = \{e \in E : x(e) \neq 0\}$.

A submonoid $\mathcal{L}$ of sign vectors is said to be the set of *covectors of an oriented matroid*, or just an *oriented matroid*, if it is closed under negation and satisfies the *elimination axiom*: For $x, y \in \mathcal{L}$, $e \in S(x, y)$, there exists $z \in \mathcal{L}$ such that $z(e) = 0$ and $z(f) = (x \circ y)(f) = (y \circ x)(f)$ for $f \in E \setminus S(x, y)$.

As an example, the set of covectors of a hyperplane arrangement defined in Section 3.1 satisfies these properties. An oriented matroid that comes from a hyperplane arrangement is said to be *realizable*. While not every oriented matroid is realizable, a theorem of Folkman and Lawrence states that oriented matroids are "close" to being realizable [12]. For our purposes, we use the following consequence: the proper part of the poset of covectors of any oriented matroid is isomorphic to the poset of faces of a regular CW-sphere. In particular, the poset of covectors is graded. The *rank* of a covector x, denoted $\operatorname{rk} x$, is the length of any maximal chain whose maximal element is x. The *rank* of a matroid $\mathcal{M}$ is the length of any maximal chain of covectors. The *corank* of a covector x is $\operatorname{rk} \mathcal{M} - \operatorname{rk} x$.

We use the notation $-^E$ or $+^E$ to represent the all-negative sign vector or all positive sign vector, respectively. An oriented matroid is *acyclic* if the all-negative sign vector $-^E$ is a covector.

The set of covectors completely determines the rest of the data associated with an oriented matroid, such as its set of circuits. Hence, we say that two oriented matroids are equal if they have the same set of covectors. For a discussion of other axiomatizations of oriented matroids, we refer to [6, Chapter 3].

Cellular strings may be defined at the level of oriented matroids as they were for zonotopes in Section 3.2. If $(\mathcal{M}, E)$ is an acyclic oriented matroid with $c_0 = -^E$ being the all-negative covector, then a cellular string of $\mathcal{M}$ is a sequence of covectors $(x_1, \dots, x_l)$ such that $x_1 \circ c_0 = c_0,\ x_l \circ (-c_0) = -c_0$ and $x_i \circ (-c_0) = x_{i+1} \circ c_0$ for all i. Observe that if $(x_1, \dots, x_l)$ is a cellular string, then $x_1 \prec \cdots \prec x_l$. We let $\omega(\mathcal{M})$ denote the set of cellular strings of $\mathcal{M}$.

The main difference between cellular strings of zonotopes and oriented matroids is that we do not know of a notion of "coherence" for oriented matroids. In particular, given two realizations $(Z, \pi), (Z', \pi')$ of an acyclic oriented matroid, a cellular string may be coherent for (Z, π) but incoherent for (Z', π'). However, we may define a notion of all-coherence for oriented matroids. Namely, we say that an acyclic oriented matroid $\mathcal{M}$ is *all-coherent* if $\omega(\mathcal{M})$ is homeomorphic to a sphere of dimension $\operatorname{rk} \mathcal{M} - 2$. This definition agrees with the equivalence in Theorem 2.1 in case $\mathcal{M}$ originates from an all-coherent pair (Z, π).

4. All-coherence property for zonotopes

4.1. Coherence of monotone paths

The main result in this subsection is Corollary 4.2, which says that if all monotone paths of a zonotope are coherent, then so is every cellular string. This result appeared previously as [10, Lemma 4.16].

For this subsection, we fix a zonotope $Z = Z(A)$ for some matrix A and generic linear functional π. We recall that the notation $Z = Z(A)$ means that Z is generated by the column vectors of A. We let E denote the set of column vectors of A. We will assume that E contains no zero vectors or any parallel or antiparallel pairs of vectors. Furthermore, we scale the elements of E so that $\pi(e) = 1$ for each $e \in E$. This results in no loss of generality as the poset of cellular strings is invariant under these changes.

Lemma 4.1 (*[10]*)**.** *Let* $\mathbf{F}$ *be a cellular string. If every cellular string properly refining* $\mathbf{F}$ *is coherent, then* $\mathbf{F}$ *is coherent.*

Proof. Assume that every cellular string properly refining $\mathbf{F}$ is coherent. Let $(E_1, \dots, E_l)$ be the ordered set partition of E induced by $\mathbf{F}$. Let $\mathbf{F}^{(1)}$ be a maximal proper cellular string of $\mathbf{F}$. Then $\mathbf{F}^{(1)}$ corresponds to the ordered set partition

$$(E_1, \dots, E_{k-1}, E'_1, \dots, E'_m, E_{k+1}, \dots, E_l)$$

which differs from $\mathbf{F}$ by breaking up a unique block E_k. Let F be the element of $\mathbf{F}$ generated by E_k. Then F is a zonotope containing a cellular string generated by $(E'_1, \dots, E'_m)$. Consequently, F admits an opposite cellular string generated by $(E'_m, \dots, E'_1)$. Hence, there is a cellular string $\mathbf{F}^{(2)}$ of Z generated by

$$(E_1, \dots, E_{k-1}, E'_m, \dots, E'_1, E_{k+1}, \dots, E_l).$$

By assumption, both $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ are coherent cellular strings. Hence, there exist linear functionals $\psi^{(1)}, \psi^{(2)}$ such that $\mathbf{F}^{(i)} = \Delta^{\psi^{(i)}}$ for $i = 1, 2$. By Lemma 3.6 and the normalization assumption ($\pi(e) = 1$ for all $e \in E$), there is a unique value $\psi^{(i)}_{E_j} = \psi^{(i)}(e)$ for $e \in E_j$. We may similarly define $\psi^{(i)}_{E'_j}$ for blocks E'_j.

For all $j \in \{1, 2, \dots, m-1\}$, since $\psi^{(1)}_{E'_j} < \psi^{(1)}_{E'_{j+1}}$ and $\psi^{(2)}_{E'_j} > \psi^{(2)}_{E'_{j+1}}$, there exists $t \in [0, 1]$ such that

$$(1-t)\psi^{(1)}_{E'_j} + t\psi^{(2)}_{E'_j} = (1-t)\psi^{(1)}_{E'_{j+1}} + t\psi^{(2)}_{E'_{j+1}}.$$

Let t be the smallest number in the interval $[0, 1]$ such that there exists $j \in \{1, \dots, m-1\}$ for which the above equation holds.

Let $\psi = (1-t)\psi^{(1)} + t\psi^{(2)}$. It is clear that ψ is constant on each block of $\mathbf{F}^{(1)}$. As before, we let $\psi_{E_j} = \psi(e)$ for $e \in E_j$ and $\psi_{E'_j} = \psi(e)$ for $e \in E'_j$. Then for $e \in E_k$, we have

$$\psi_{E_1} < \cdots < \psi_{E_{k-1}} < \psi_{E'_1} \leq \cdots \leq \psi_{E'_m} < \psi_{E_{k+1}} < \cdots < \psi_{E_l}$$

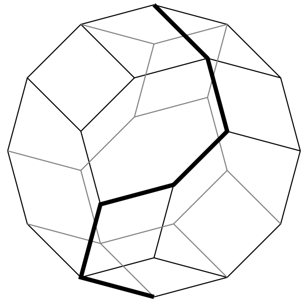

Fig. 2. A zonotope with the all-coherence property.

and $\psi_{E'_j} = \psi_{E'_{j+1}}$. Hence Δ^ψ is a cellular string that refines $\mathbf{F}$ and is properly refined by $\mathbf{F}^{(1)}$. By the maximality of $\mathbf{F}^{(1)}$, we deduce $\mathbf{F} = \Delta^\psi$, as desired. $\square$

Take $\mathbf{F} = (Z)$ to be the trivial cellular string. As every other cellular string refines $\mathbf{F}$, we deduce the following corollary.

Corollary 4.2. *If every monotone path is coherent, then (Z, π) is all-coherent.*

Remark 4.3. The analogue of Lemma 4.1 for general polytope projections fails to hold. For example, there exists a point configuration with two coherent triangulations connected by an incoherent bistellar flip [8, Example 5.3.4].

Example 4.4. Consider the zonotope in Fig. 2 with a linear functional π such that the bolded path is monotone. This path can be flipped to the left through the middle hexagon face or to the right through one of the rhombic faces. Indeed, one can check that every monotone path is adjacent to exactly two monotone paths. Hence, the graph of monotone paths is a cycle. The subgraph of coherent paths is also a cycle since it is combinatorially equivalent to the 1-skeleton of a 2-dimensional fiber polytope. We conclude that every monotone path is coherent. Corollary 4.2 then implies that the zonotope is all-coherent.

4.2. Gradedness

The face lattice of a polytope is graded by dimension. Thus, if $(\mathcal{A}, c_0)$ is all-coherent, then $\omega(\mathcal{A}, c_0)$ is a graded lattice. In practice, the poset of cellular strings is typically not graded, which makes this a useful marker for the all-coherence property. In the following proposition, which we state at the generality of oriented matroids, we give a useful refinement to this marker.

Proposition 4.5. *Let $(\mathcal{M}, E)$ be an all-coherent, acyclic oriented matroid. Let $r = \operatorname{rk} \mathcal{M}$.*

1 *The poset $\omega(\mathcal{M})$ is graded and every maximal chain contains exactly r elements.*
2 *For every cellular string $\mathbf{F}$, the inequality $\sum_{x\in\mathbf{F}} (\operatorname{rk} x - 1) \le r - 2$ holds.*

Proof. The first claim is immediate from the definition of all-coherence of oriented matroids.

Let $\mathbf{F} = (x_1, \dots, x_l)$ be a cellular string of $\mathcal{M}$, and suppose $\operatorname{rk} x_k > 1$ for some k. Let y be any covector of $\mathcal{M}$ such that $y \le x_k$ and $\operatorname{rk} x = \operatorname{rk} y + 1$. We claim that there exists a cellular string $\mathbf{F}'$ of the form $\mathbf{F}' = (x_1, \dots, x_{k-1}, y_1, \dots, y_m, x_{k+1}, \dots, x_l)$ where $y = y_i$ for some i.

Let $c_0 = -^E$ be the all-negative covector. Observe that $x_{k-1} \prec x_k \circ c_0 \prec y \prec x_k \circ (-c_0) \prec x_{k+1}$. Now suppose $(y_1, \dots, y_m)$ is a sequence of covectors containing y such that $x_{k-1} \circ (-c_0) \preceq y_1 \circ c_0$, $y_m \circ (-c_0) \preceq x_{k+1} \circ c_0$, and $y_i \circ (-c_0) \preceq y_{i+1} \circ c_0$ for all i. If $(x_1, \dots, x_{k-1}, y_1, \dots, y_m, x_{k+1}, \dots, x_l)$

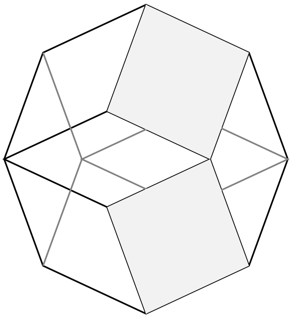

Fig. 3. A cellular string in the zonotope from Example 4.6.

is not a cellular string, then either $y_1 \circ c_0 \neq x_{k-1} \circ (-c_0)$, $y_m \circ (-c_0) \neq x_{k+1} \circ c_0$ or there exists some i with $y_i \circ (-c_0) \neq y_{i+1} \circ c_0$. We consider only the third case as the other two cases are similar.

Let $c = y_i \circ (-c_0)$ and $c' = y_{i+1} \circ c_0$, and suppose $c \neq c'$. Since $y_i \prec y_{i+1}$, we have $c \prec c'$. Let $e \in E$ such that $c(e) = -$ and $c'(e) = +$. By the elimination axiom, there exists a covector z such that $z(e) = 0$ and $z(f) = c(f)$ whenever $c(f) = c'(f)$. Hence, $y_i \prec c \prec z \prec c' \prec y_{i+1}$, so we may extend the sequence of covectors to $(y_1, \dots, y_i, z, y_{i+1}, \dots, y_m)$. Since the number of covectors is finite, we must eventually reach some sequence that we cannot refine in this manner, which gives a cellular string $\mathbf{F}' = (x_1, \dots, x_{k-1}, y'_1, \dots, y'_p, x_{k+1}, \dots, x_l)$ such that $y = y'_j$ for some j. We find

$$1 + \sum_{w \in \mathbf{F}'} (\operatorname{rk} w - 1) \geq \sum_{w \in \mathbf{F}} (\operatorname{rk} w - 1).$$

Suppose $\mathcal{M}$ has a proper cellular string $\mathbf{F}$ such that $\sum_{x \in \mathbf{F}} (\operatorname{rk} x - 1) \geq r - 1$. Then we may apply the above construction repeatedly to obtain a chain of cellular strings with at least r elements. By including the trivial cellular string, we find a chain in $\omega(\mathcal{M})$ with at least $r + 1$ elements, contradicting (1). □

Proposition 4.5(2) gives a useful test for incoherence, as in the following example.

Example 4.6. Let Z be the image of the 4-cube under the map

$$\pi = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and define $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f(x, y, z) = x + y + z$. This zonotope is a rhombic dodecahedron, which is shown in Fig. 3. Let $\mathcal{M}$ be the acyclic oriented matroid associated to (Z, f) with ground set $\{a, b, c, d\}$ corresponding to the four columns of π. We order the ground set $a < b < c < d$ to agree with the ordering of the columns of π. Then $\mathcal{M}$ has a unique circuit $(+, +, -, -)$, along with its negative $(-, -, +, +)$. The other 14 elements of $\{+, -\}^{\{a,b,c,d\}}$ are covectors of $\mathcal{M}$.

The oriented matroid $\mathcal{M}$ has a cellular string consisting of two cocircuits $(0, -, -, 0)$ and $(+, 0, 0, +)$, which correspond to the filled in faces of the zonotope in Fig. 3. But $(\operatorname{rk}(0, -, -, 0) - 1) + (\operatorname{rk}(+, 0, 0, +) - 1) = 2 = \operatorname{rk} \mathcal{M} - 1$, so (Z, f) is not all-coherent by Proposition 4.5(2).

4.3. Matroid operations

Let V be a real vector space of dimension r. Fix an $r \times n$ real matrix A, and let E be the set of column vectors of A. We let $Z = Z(A)$ and let π be a linear functional on $\mathbb{R}^r$ that is generic with respect to Z. This defines an acyclic oriented matroid $\mathcal{M}$. Since the coherence of cellular strings is invariant under scaling the columns of A, we will assume throughout this subsection that $\pi(e) = 1$ for all $e \in E$. Let $\mathcal{A}$ be the set of hyperplanes in V^* for which $\mathcal{L}(\mathcal{A})$ is the normal fan of Z. Let c_0 be the chamber of $\mathcal{A}$ containing π. For simplicity, we will assume that the set E does not contain any zero vectors or pairs of parallel vectors.

Lemma 4.7 (*cf.* [10](*Corollary 5.2*))**.** *Let $\widetilde{Z}$ be the direct product of Z with an m-cube. We realize this zonotope by a matrix*

$$\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix}$$

where I_m is the rank m identity matrix. If $\widetilde{\pi}$ is a generic linear functional on $\mathbb{R}^{r+m}$ whose restriction to Z is equal to π, then (Z, π) is all-coherent if and only if $(\widetilde{Z}, \widetilde{\pi})$ is all-coherent.

Proof. It suffices to prove the lemma for $m = 1$ since if $m > 1$, the zonotope $\widetilde{Z}$ decomposes as $(Z \times [0, 1]^{m-1}) \times [0, 1]$, and the result follows by induction. Hence, we will assume $m = 1$ for the remainder of the proof.

Let e be the unit vector in the last coordinate of $\mathbb{R}^{r+1}$. Let $\widetilde{\pi}$ be a generic linear functional on $\mathbb{R}^{r+1}$ whose restriction to Z is π. We may assume that $\widetilde{\pi}(e) = 1$ without affecting coherence of a cellular string on $\widetilde{Z}$.

Let $\widetilde{\gamma}$ be a cellular string on $\widetilde{Z}$. Let γ be the cellular string on Z obtained by deleting e from $\widetilde{\gamma}$. That is, we obtain γ from $\widetilde{\gamma}$ by projecting along the last coordinate. Suppose γ is π-coherent. Then there exists a linear functional ψ on $\mathbb{R}^r$ for which γ is the cellular string corresponding to the π-induced subdivision Δ^{ψ}. Let $(E_1, \ldots, E_l)$ be the ordered set partition of $\{a_1, \ldots, a_n\}$ induced by γ. This means for $a \in E_i,\ b \in E_j$, $\psi(a) \leq \psi(b)$ if $i \leq j$.

We define a linear functional $\widetilde{\psi}$ that determines the string $\widetilde{\gamma}$ as follows. For each a_i, set $\widetilde{\psi}(a_i) = \psi(a_i)$. If e is added to a block E_i to form $\widetilde{\gamma}$, then set $\widetilde{\psi}(e) = \psi(a)$ for some $a \in E_i$. If e is in its own block between E_i and E_{i+1}, then set $\widetilde{\psi}(e)$ to be a value strictly between $\psi(a)$ and $\psi(b)$ where $a \in E_i,\ b \in E_{i+1}$. In either case, we have produced the desired linear functional.

Conversely, if we start with a $\widetilde{\pi}$-coherent string $\widetilde{\gamma}$ induced by $\widetilde{\psi}$, then the restriction of $\widetilde{\psi}$ to the first r coordinates induces γ on Z. This completes the proof. □

For oriented matroids $(\mathcal{M}, E), (\mathcal{M}', E')$ on disjoint ground sets E, E', the *direct sum* $\mathcal{M} \oplus \mathcal{M}'$ of $\mathcal{M}$ and $\mathcal{M}'$ is an oriented matroid with covectors $\mathcal{L}(\mathcal{M}) \times \mathcal{L}(\mathcal{M}')$. An oriented matroid is *indecomposable* if it cannot be expressed as a direct sum of two nontrivial oriented matroids. The set of circuits of $\mathcal{M} \oplus \mathcal{M}'$ is the union of $\mathcal{C}(\mathcal{M}) \times \{0\}$ and $\{0\} \times \mathcal{C}(\mathcal{M}')$ [6, Proposition 7.6.1].

Lemma 4.8. *Let $\mathcal{M}$ be an acyclic oriented matroid that decomposes as $\mathcal{M}_1 \oplus \mathcal{M}_2$. If both $\mathcal{M}_1$ and $\mathcal{M}_2$ have a circuit supported by at least 3 elements, then $\mathcal{M}$ is not all-coherent.*

Proof. Let E_i be the ground set of $\mathcal{M}_i$ for $i = 1, 2$. Assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ each have a circuit supported by at least 3 elements. Then for $i = 1, 2$ there exists $x_i \in \mathcal{L}(\mathcal{M}_i)$ such that x_i is a corank 1 covector and $\operatorname{rk}\left(\mathcal{M}_i \setminus x_i^{-1}(0)\right) \geq 2$.

Let $\mathbf{F}_i = (y_1^{(i)}, y_2^{(i)}, \ldots, y_{l_i}^{(i)})$ be a cellular string of $\mathcal{M}_i$ containing x_i such that each cell $y_j^{(i)}$ is rank 1 unless $y_j^{(i)} = x_i$ for $i = 1, 2$. Without loss of generality we may assume that $l_1 \leq l_2$. By the assumption on x_i, we have $l_1 \geq 3$. We define a new string $\mathbf{F} = (y_1, \ldots, y_m)$ for $\mathcal{M}$ where $y_j = y_j^{(1)} \oplus y_j^{(2)}$ if $j \leq l_1$ and $y_j = +^{E_1} \oplus y_j^{(2)}$ if $l_1 < j \leq l_2$. Then

$$\begin{aligned} \sum (\operatorname{rk} y_j - 1) &= \sum_{j=1}^{l_1} \operatorname{rk} y_j^{(1)} + \sum_{j=1}^{l_2} (\operatorname{rk} y_j^{(2)} - 1) \\ &\geq (\operatorname{rk} \mathcal{M}_1 + 1) + (\operatorname{rk} \mathcal{M}_2 - 1) \\ &= \operatorname{rk} \mathcal{M}. \end{aligned}$$

By Proposition 4.5, we conclude that $\mathcal{M}$ is not all-coherent. □

Example 4.9. Let Z be the image of the 6-cube under the map

$$\begin{pmatrix} 1 & 0 & .5 & 0 & 0 & 0 \\ 0 & 1 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 & .5 \end{pmatrix}$$

and define $\pi : \mathbb{R}^4 \to \mathbb{R}$ such that $\pi(w, x, y, z) = w + x + y + z$. We note that $\pi(v) = 1$ for each column vector v. Let $\mathcal{M}$ be the acyclic oriented matroid associated to (Z, π) with ground set $\{a, b, c, d, e, f\}$ corresponding to the six columns of the matrix. Then $\mathcal{M}$ decomposes into two rank 2 matroids, each of corank 1. Hence, (Z, π) is not all-coherent.

In particular, we claim that the cellular string

$$\{(0, -, -, 0, -, -),\ (+, -, -, +, -, 0),\ (+, -, 0, +, -, +),\ (+, 0, +, +, 0, +)\}$$

is not coherent. Suppose to the contrary that it is picked out by a linear functional $\psi : \mathbb{R}^4 \to \mathbb{R}$ in the sense of Lemma 3.6. By definition, this means

$$\psi(a) = \psi(d) < \psi(f) < \psi(c) < \psi(b) = \psi(e).$$

But the first and last equalities imply

$$2\psi(c) = \psi(a) + \psi(b) = \psi(d) + \psi(e) = 2\psi(f),$$

contradicting the inequality $\psi(f) < \psi(c)$.

5. Classification

5.1. Oriented matroids of point sets

Let W be an affine space and $\mathcal{P}$ a set of points in W. A vector $x \in \{0, +, -\}^{\mathcal{P}}$ is a *covector* of $\mathcal{P}$ if there is an affine functional $\psi : W \to \mathbb{R}$ such that $x(p)$ is the sign of $\psi(p)$ for all $p \in \mathcal{P}$. The set of covectors of $\mathcal{P}$ is an oriented matroid, which we call the oriented matroid of $\mathcal{P}$.

Suppose that W is the hyperplane in $\mathbb{R}^{d+1}$ given by the equation $x_{d+1} = 1$. Then the oriented matroid of $\mathcal{P}$ is the same as the oriented matroid of the zonotope generated by $\mathcal{P}$.

A *coloop* of $\mathcal{P}$ is an element of $\mathcal{P}$ which is not contained in the affine span of the other points of $\mathcal{P}$.

5.2. Affine models and monotone families

Let Z be a zonotope generated by $v_1, \ldots, v_n \in V$ and let $\pi \in V^*$ be a generic linear functional. Let H_π be the hyperplane $\{x \in V : \pi(x) = 1\}$. We define

$$\mathcal{P}(Z, \pi) := \bigcup_{i=1}^{n} (\{tv_i : t \in \mathbb{R}\} \cap H_\pi).$$

Thus, $\mathcal{P}(Z, \pi)$ is a set of n points in H_π. Note that the oriented matroid of Z depends only on the oriented matroid of $\mathcal{P}(Z, \pi)$ as an affine point set. In particular, the all-coherence property depends only on this oriented matroid. We will call any set of points in affine space with the same oriented matroid as $\mathcal{P}(Z, \pi)$ an *affine model* for (Z, π).

Given a finite-dimensional affine space W, we define a *(closed) half-space* of W to be a subset of the form $\{x \in W : \psi(x) \geq 0\}$ for some affine map $\psi : W \to \mathbb{R}$. This differs from the usual definition of half-space in that we count $\emptyset$ and W as half-spaces of W in this definition. We denote the set of all half-spaces of W by $HS(W)$. For $K \in HS(W)$, we denote by ∂K and K° the boundary and the interior, respectively, of K as a subspace of W.

We give the set $HS(W)$ a topology in the following description. Suppose W is a d-dimensional affine space. By picking coordinates, we may assume that W is the hyperplane in $\mathbb{R}^{d+1}$ given by the equation $x_{d+1} = 1$, so that a point $x \in W$ is of the form $(x_1, \ldots, x_d, 1)$. Then for every half-space

$K \in HS(W)$, there is a *non-zero* linear functional $\psi : \mathbb{R}^{d+1} \to \mathbb{R}$ such that $K = \{x \in W : \psi(x) \geq 0\}$. In this case we say that K is defined by ψ. (Note that the whole space W is defined by $x \mapsto cx_{d+1}$ where $c > 0$, and $\emptyset$ is defined by $x \mapsto cx_{d+1}$ where $c < 0$.) Moreover, two non-zero linear functionals ψ, $\psi' \in (\mathbb{R}^{d+1})^*$ define the same half-space if and only if $\psi' = a\psi$ for some constant $a > 0$. This observation leads to the identification of $HS(W)$ with a sphere:

$$HS(W) \approx \frac{(\mathbb{R}^{d+1})^* - \{0\}}{\psi \sim \psi' \iff \exists a > 0, \psi' = a\psi} \approx \{\psi \in (\mathbb{R}^{d+1})^* : \|\psi\| = 1\} \approx \mathbb{S}^d. \qquad (*)$$

(Here $\|\cdot\|$ is the dual Euclidean norm.) Using this identification, we endow $HS(W)$ with the Euclidean topology of the sphere $\mathbb{S}^d$.

If $\mathcal{P}$ is a finite set of points in W, we define a *monotone family* with respect to $\mathcal{P}$ to be a continuous path $\{K_t\}_{t\in[0,1]}$ in $HS(W)$ such that

1. $K_0 = \emptyset$
2. $K_1 = W$
3. For every $p \in \mathcal{P}$, there exists $t_p \in (0, 1)$ such that $p \in \partial K_{t_p}$, $p \in W \setminus K_t$ for $t < t_p$, and $p \in K_t^\circ$ for $t > t_p$.

To each monotone family $\{K_t\}_{t\in[0,1]}$ with respect to $\mathcal{P}$ we associate an ordered set partition of $\mathcal{P}$ as follows: Let $t_1 < \cdots < t_l$ be all the numbers in $(0, 1)$ such that $\partial K_{t_i} \cap \mathcal{P} \neq \emptyset$. (Note that by property (3) above, for each $p \in \mathcal{P}$ there is exactly one t for which $p \in \partial K_t \cap \mathcal{P}$, so the list $t_1, t_2, \ldots$ is finite.) Then the ordered set partition associated to $\{K_t\}_{t\in[0,1]}$ is

$$(\partial K_{t_1} \cap \mathcal{P}, \partial K_{t_2} \cap \mathcal{P}, \ldots, \partial K_{t_l} \cap \mathcal{P}).$$

We call any partition of $\mathcal{P}$ obtained in this way a *monotone partition*. As the following proposition states, monotone partitions of affine models are in bijection with proper cellular strings.

Proposition 5.1. *Let $\mathcal{P}$ be an affine model for (Z, π). Then there is a bijection from the set of all monotone partitions of $\mathcal{P}$ to the set of all proper cellular strings of (Z, π), and such that the image of $(E_1, \ldots, E_l)$ under this bijection is $(F_1, \ldots, F_l)$ where $\dim(F_i) = |E_i|$.*

Proof. We may assume $\mathcal{P} = \mathcal{P}(Z, \pi)$. In Eq. $(*)$ we defined a homeomorphism from $S(V^*) := \{\psi \in V^* : \|\psi\| = 1\}$ to $HS(H_\pi)$ given by $\psi \mapsto \{x \in H_\pi : \psi(x) \geq 0\}$. Recall that each proper cellular string of (Z, π) can be specified by a path $\gamma : [0, 1] \to S(V^*)$ such that $\gamma(0) = -\pi/\|\pi\|$, $\gamma(1) = \pi/\|\pi\|$, and for each i, γ intersects the hyperplane $\{\psi \in V^* : \psi(v_i) = 0\}$ exactly once. The image of the set of all such paths under the above homeomorphism is precisely the set of monotone families with respect to $\mathcal{P}(Z, \pi)$. It is easy to check that this homeomorphism induces a bijection between the proper cellular strings of (Z, π) and the monotone partitions of $\mathcal{P}(Z, \pi)$, and this bijection satisfies the desired property. Explicitly, each proper cellular string of (Z, π) is associated to a unique ordered set partition $(A_1, \ldots, A_l)$ of the set E of generating vectors of Z, and the map which sends E to $\mathcal{P}$ sends $(A_1, \ldots, A_l)$ to a monotone partition of $\mathcal{P}$. This map is the desired bijection. □

In the proof of classification we will need the following lemma.

Lemma 5.2. *Let $\mathcal{P}$ be a finite set of points in an affine space W. Suppose that $K_1, \ldots, K_m \in HS(W)$ are such that for all $1 \leq i < m$, we have $K_i \cap \mathcal{P} \subseteq K_{i+1}^\circ \cap \mathcal{P}$. Then there exists a monotone family with respect to $\mathcal{P}$ containing $K_1, \ldots, K_m$.*

Proof. We may assume that W is the hyperplane in $V := \mathbb{R}^{d+1}$ given by the equation $x_{d+1} = 1$. Let $S(V^*)$ be the unit sphere in V^*. Recall that we have a homeomorphism $\Phi : S(V^*) \to HS(W)$ given by $\psi \mapsto \{x \in W : \psi(x) \geq 0\}$. Let $\psi_i = \Phi^{-1}(K_i)$ for each $i = 1, \ldots, m$. Let $\psi_0 = \Phi^{-1}(\emptyset)$ and $\psi_{m+1} = \Phi^{-1}(W)$.

Let $\gamma : [0, 1] \to S(V^*)$ be a path in $S(V^*)$ which is the composition of paths $\gamma_0, \ldots, \gamma_m$, where γ_i is any geodesic in $S(V^*)$ from ψ_i to ψ_{i+1}. We claim that $\Phi \circ \gamma$ is a monotone family with respect to $\mathcal{P}$. It suffices to show that for every $p \in \mathcal{P}$, the path γ intersects the hyperplane

$$H_p := \{\psi \in V^* : \psi(p) = 0\}$$

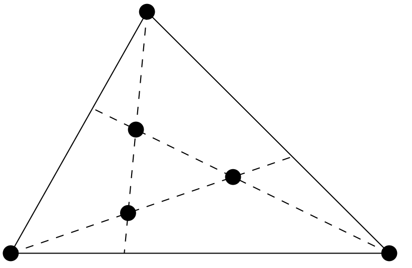

Fig. 4. The affine model R_3.

exactly once. Since $\psi_0(p) < 0$, $\psi_{m+1}(p) > 0$, and each γ_i is a geodesic from ψ_i to ψ_{i+1}, it suffices to prove the following: For each $1 \leq i \leq m$ such that $\psi_i(p) \geq 0$, we have $\psi_{i+1}(p) > 0$. Suppose that $\psi_i(p) \geq 0$ and $\psi_{i+1}(p) \leq 0$. Then $p \in K_i$ and $p \notin K_i^\circ$, This contradicts the assumptions on $K_1, \ldots, K_m$, so the claim holds. Hence $\Phi \circ \gamma$ is a monotone family with respect to $\mathcal{P}$. By definition it contains $K_1, \ldots, K_m$, as desired. $\square$

5.3. Classification theorem

To classify the all-coherent zonotopes, it suffices to give their affine models. Here are three such affine models:

1. For $r \geq 2$, the set $E_{r,m} = \{e_1, \ldots, e_r, a_1, \ldots, a_m\}$, where $e_1, \ldots, e_r$ are the vertices of an $(r-1)$-dimensional simplex and $a_1, \ldots, a_m$ are points in the interior of the simplex such that $a_1, \ldots, a_m, e_r$ are collinear. This is the affine model for the all-coherent family in Section 3.3.
2. For $r \geq 3$, the set $\tilde{E}_{r,m}$, which is defined in the same way as $E_{r,m}$ except a_1 is on the boundary of the simplex.
3. The set R_3, the set of 6 points in $\mathbb{R}^2$ shown in Fig. 4. The corresponding zonotope Z for R_3 is shown in Fig. 2, with π the vertical direction in the figure. One can check that the poset of proper cellular strings of (Z, π) is homeomorphic to a circle, which implies the all-coherence property for R_3.

We now show that, up to the addition of coloops, these are the only all-coherent affine models.

Theorem 5.3. *Let Z be a zonotope of dimension r and π a generic linear functional. The following are equivalent.*

1. *(Z, π) is all-coherent.*
2. *There does not exist a proper cellular string $(F_1, \ldots, F_l)$ of (Z, π) with*

$$\sum_{i=1}^{l} (\dim(F_i) - 1) \geq r - 1.$$

3. *The affine model for (Z, π) is $E \cup L$, where E is of the form $\emptyset$, $E_{r,m}$, $\tilde{E}_{r,m}$, or R_3 and L is a (possibly empty) set of coloops of $E \cup L$.*

Proof. (1)$\Rightarrow$(2) follows from Proposition 4.5. The implication (3)$\Rightarrow$(1) was been proven for $E_{r,m}$ in Section 3.3, and the proof for $\tilde{E}_{r,m}$ is the same. As noted before, the proof for R_3 is a straightforward check. Finally, Lemma 4.7 implies that the addition of coloops preserves all-coherence. This proves (3)$\Rightarrow$(1).

It remains to prove (2)$\Rightarrow$(3). Assume (2) holds. Let $\mathcal{P}$ be an affine model for (Z, π). We may assume $\mathcal{P}$ is full-dimensional in $\mathbb{R}^{r-1}$. For any polytope Q, let $V(Q)$ denote its vertex set. We first show the following.

Claim 5.4. *The convex hull of $\mathcal{P}$ is a simplex.*

Proof. Suppose the contrary, and let Q be the convex hull of $\mathcal{P}$. Let F be any facet of Q. If there are at least two vertices of Q not in F, then there is an edge e of Q disjoint from F. Otherwise, Q is a pyramid over F, and since Q is not a simplex, if we replace F with any other facet of Q then there will be two vertices of Q not in F. So we may assume there is an edge e of Q disjoint from F.

Let K_1 be the half-space of $\mathbb{R}^{r-1}$ which does not contain Q and whose boundary supports F. Let K_2 be any half-space of $\mathbb{R}^{r-1}$ which contains Q and whose boundary supports e. Then $K_1 \cap Q \subset K_2^\circ \cap Q$, so by Lemma 5.2, there is a monotone family with respect to $\mathcal{P}$ containing K_1, K_2. In particular, the associated monotone partition of $\mathcal{P}$ contains $V(F)$ and $V(e)$. Thus, by Proposition 5.1, there is a cellular string $(F_1, \ldots, F_l)$ of (Z, π) with

$$\sum_{i=1}^{l}(\dim(F_i) - 1) \geq (|V(F)| - 1) + (|V(e)| - 1) \geq (r - 2) + 1 = r - 1.$$

This contradicts (2), proving the claim. □ □

Now, let σ denote the convex hull of $\mathcal{P}$.

Claim 5.5. *If a_1, a_2 are distinct points in $\mathcal{P} \setminus V(\sigma)$, then there exists $b \in V(\sigma)$ such that a_1, a_2, and b are collinear.*

Proof. Suppose the contrary. Let ℓ be the line through a_1 and a_2, and let p_1, p_2 be the endpoints of the segment $\ell \cap \sigma$. Let C_1 and C_2 be the vertex sets of the minimal faces of σ containing p_1 and p_2, respectively. By assumption, C_1 and C_2 have at least two elements. It is thus possible to choose a partition (B_1, B_2) of $V(\sigma)$ such that $C_i \not\subseteq B_j$ for all i, j.

Let σ_1, σ_2 be the faces of σ with vertex sets B_1, B_2 respectively. Let K_1 be a half-space of $\mathbb{R}^{r-1}$ which does not contain σ and whose boundary supports σ_1. Let K_2 be a half-space of $\mathbb{R}^{r-1}$ whose boundary contains ℓ and such that $\sigma_1 \subset K_2^\circ$ and $\sigma_2 \cap K_2 = \emptyset$. (This is possible by the definitions of σ_1 and σ_2.) Finally, let K_3 be a half-space of $\mathbb{R}^{r-1}$ which contains σ and whose boundary supports σ_3. Then $K_1 \cap \sigma \subset K_2^\circ \cap \sigma$ and $K_2 \cap \sigma \subset K_3^\circ \cap \sigma$, so by Lemma 5.2 there is a monotone family with respect to $\mathcal{P}$ containing K_1, K_2, K_3. Hence, there is a cellular string $(F_1, \ldots, F_l)$ of (Z, π) with

$$\sum_{i=1}^{l}(\dim(F_i) - 1) \geq (|V(\sigma_1)| - 1) + (|\{a_1, a_2\}| - 1) + (|V(\sigma_2)| - 1) = r - 1$$

which contradicts (2). □

We can now complete the proof of Theorem 5.3. If $\mathcal{P} \setminus V(\sigma)$ has at most one point, then it is easy to see that $\mathcal{P}$ is of one of the forms described in (3). Assume $\mathcal{P} \setminus V(\sigma)$ has at least two distinct points a_1, a_2, and let ℓ_{12} be the line through these points. By Claim 5.5, ℓ_{12} contains some vertex b_{12} of σ. If every point of $\mathcal{P} \setminus V(\sigma)$ is on ℓ_{12}, then $\mathcal{P}$ is of one of the forms described in (3), as desired.

Assume there is some $a_3 \in \mathcal{P} \setminus V(\sigma)$ not on ℓ_{12}. Then by Claim 5.5, there are vertices b_{13} and b_{23} of σ such that a_1, a_3, b_{13} lie on a line ℓ_{13} and a_2, a_3, b_{23} lie on a line ℓ_{23}. Moreover, since $a_3 \notin \ell_{12}$, $b_{12} \neq b_{13} \neq b_{23}$. Since ℓ_{12} and ℓ_{13} intersect at a_1, there is a 2-plane Γ containing ℓ_{12} and ℓ_{13}. Since $a_2, a_3 \in \Gamma$, we have $\ell_{23} \subset \Gamma$, and hence $b_{23} \in \Gamma$. Hence $a_1, a_2, a_3, b_{12}, b_{13}, b_{23}$ all lie in Γ, and thus a_1, a_2, a_3 are all in the face F of σ with vertices b_{12}, b_{13}, b_{23}. It is easy to check that these six points must be in the configuration R_3.

Now suppose that there is some $a_4 \in \mathcal{P} \setminus (V(\sigma) \cup \{a_1, a_2, a_3\})$. The above argument works with a_4 instead of a_3. However, because a_1 is in the interior of F, F is the only 2-face of σ containing a_1, and hence $a_4 \in F$. However, it is easy to check that one cannot have 4 distinct points in a triangle, not all on the same line, such that the line through any two of them passes through a vertex of the triangle. Thus there is no such a_4. Hence, $\mathcal{P} = R_3 \cup L$, where L is a set of coloops of $\mathcal{P}$. This concludes the proof. □

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