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Corrigendum

Corrigendum to "Stationary inviscid limit to shear flows" [J. Differ. Equ. 267 (12) (2019) 7135–7153]

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This erratum is meant to correct a mistake in the Appendix of our original article, "Stationary inviscid limit to shear flows". Specifically, the way in which we have constructed our (high order in ε)"boundary layer" terms, $[u_p^1, v_p^1]$, and $[u_p^2, v_p^2]$ in (65) - (66) of that paper needs to be modified. The modification, which we shall first explain, then establish rigorously, has no bearing on the body of the proof since it makes these "boundary layer terms" *weaker* in fact.

The conclusion is that the main theorem, Theorem 1, holds as stated in the original paper, with the modification that the *expansion/ approximate solution* is taken to be (3) - (5) of the present Erratum, as opposed to the one written in equation (6) of the original manuscript.

Let us start by describing the quantity u_p^1. Recall from (99) of the original paper that we first construct $u_p^{1,0,\pm}$, then cut it off to form u_p^1. We focus thus on the quantity $u_p^{1,0,-}$, which satisfies the following equation:

$$\mu \partial_x u_p^{1,0,-} - \partial_{Y_-}^2 u_p^{1,0,-} = 0, \qquad (x, Y_-) \in (0, L) \times (0, \infty). \tag{1}$$

Recalling from Y_- was defined in that paper via $Y_- = \frac{y}{\sqrt{\varepsilon}}$, and $\mu(y) = \mu'(0)y + O(y^2)$ for $y << 1$, this system reads (temporarily omitting the $O(y^2)$ terms from $\mu(y)$, which will be shown rigorously to be higher order in ε)

$$\varepsilon^{\frac{1}{2}} \mu'(0) Y_- \partial_x u_p^{1,0,-} - \partial_{Y_-}^2 u_p^{1,0,-} = 0, \qquad (x, Y_-) \in (0, L) \times (0, \infty). \tag{2}$$

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The estimates stated in (108), (109) of Lemma 12 of the original paper do not, however, apply to this system, due to the unrecognized scaling factor of $\epsilon^{\frac{1}{2}}$ in front of the ∂_x term, above. This was the source of error, which we correct in this erratum.

Let us now explain the reason for this anomaly. One can observe that the leading order of our approximate solution, (65) - (66), only contains a $\mu(y)$, (which, in particular, satisfies $\mu(0) = 0$) and does not require a leading order boundary layer. Therefore, we should not expect the *strong* boundary layer scaling of (68) (even at higher orders in ε). Instead, we should expect a much weaker boundary layer to form at these high orders in ε (u_p^i, v_p^i).

To correct this, we make the change: $Y := \frac{y}{\varepsilon^{\frac{1}{3}}}$ (see below (8) for the precise piecewise definition), and to accompany this change, we modify the approximate solution to progress at order $\varepsilon^{\frac{1}{3}}$, see below (3) - (5). The consequence of this *milder* scaling is that the boundary layer equations become now uniform in ε. These are shown below in (35a) - (35c), and are analyzed in Lemma 2.

The outcome of this modified construction is that we can keep estimates of the type claimed in Lemma 12 of the original paper (stated below in Corollary 4), and the contributed forcing $\mathcal{F}_u, \mathcal{F}_v$ defined in (10) - (11) meets the need for estimate (120) in the original paper altogether.

A.1. Formal asymptotic expansion

We define the asymptotic expansions:

$$u^\varepsilon = \mu + \sum_{i=1}^{M_0} \varepsilon^{1+\frac{i-1}{3}} (u_e^i + u_p^i) + \varepsilon^{\frac{3}{2}+\gamma} u = u_s + \varepsilon^{\frac{3}{2}+\gamma} u \tag{3}$$

$$v^\varepsilon = \sum_{i=1}^{M_0} \varepsilon^{1+\frac{i-1}{3}} (v_e^i + \varepsilon^{\frac{1}{3}} v_p^i) + \varepsilon^{\frac{3}{2}+\gamma} v = v_s + \varepsilon^{\frac{3}{2}+\gamma} v \tag{4}$$

$$P^\varepsilon = \sum_{i=1}^{M_0} \varepsilon^{1+\frac{i-1}{3}} (P_e^i + (P_p^i + \varepsilon^{\frac{2}{3}} P_P^{i,a})) + \varepsilon^{\frac{3}{2}+\gamma} P = P_s + \varepsilon^{\frac{3}{2}+\gamma} P. \tag{5}$$

We will also introduce the notation

$$u_s^E := \mu + \sum_{i=1}^{M_0} \varepsilon^{1+\frac{i-1}{3}} u_e^i, \qquad v_s^E := \sum_{i=1}^{M_0} \varepsilon^{1+\frac{i-1}{3}} v_e^i \tag{6}$$

Denote $\mathbf{u}^\varepsilon = (u^\varepsilon, v^\varepsilon)$, and Let $(\mathbf{u}^\varepsilon, P^\varepsilon)$ solve the Navier-Stokes equations

$$\begin{cases} \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla P^\varepsilon = \varepsilon \Delta \mathbf{u}^\varepsilon \text{ in } \Omega, \\ \nabla \cdot \mathbf{u}^\varepsilon = 0 \text{ in } \Omega, \\ \mathbf{u}^\varepsilon|_{y=0} = 0, \ \mathbf{u}^\varepsilon|_{y=2} = u_b. \end{cases} \tag{7}$$

Above, M_0 generically denotes a large number, which we shall now fix at $M_0 = 6$ for concreteness.

Here the Eulerian profiles are functions of (x, y), whereas the boundary layer profiles are functions of (x, Y), where:

$$Y = \begin{cases} Y_+ := \dfrac{2-y}{\varepsilon^{\frac{1}{3}}} \text{ if } 1 \le y \le 2, \\ Y_- := \dfrac{y}{\varepsilon^{\frac{1}{3}}} \text{ if } 0 \le y \le 1. \end{cases} \tag{8}$$

Due to this, we break up the boundary layer profiles into two components, one supported near $y = 0$ and one supported near $y = 2$ in the following manner:

$$\mathbf{u}_p^i(x, Y) = \begin{cases} \mathbf{u}_p^{i,-}(x, Y_-) \text{ if } 0 \le y \le 1, \\ \mathbf{u}_p^{i,+}(x, Y_+) \text{ if } 1 \le y \le 2. \end{cases} \tag{9}$$

Notice that due to our scaling, we have $\partial_y u_p^i = \frac{1}{\varepsilon^{\frac{1}{3}}} \partial_Y u_p^i$ when $0 \le y \le 1$ and $\partial_y u_p^i = -\frac{1}{\varepsilon^{\frac{1}{3}}} \partial_Y u_p^i$ when $1 \le y \le 2$. We define the error caused by the approximation:

$$\mathcal{F}_u = u_s \partial_x u_s + v_s \partial_y u_s + \partial_x P_s - \varepsilon \Delta u_s \tag{10}$$

$$\mathcal{F}_v = u_s \partial_x v_s + v_s \partial_y v_s + \partial_y P_s - \varepsilon \Delta v_s. \tag{11}$$

Our construction will ensure the estimates (63).

A.2. Euler equations

We remark that this section, the construction of Euler profiles, remains essentially unchanged from the original manuscript. We include it here for the sake of completeness. The equations satisfied by the first Euler layer are obtained by collecting the $\mathcal{O}(\varepsilon)$ order Euler terms from (10) - (11), and are now shown:

$$\left.\begin{aligned} &\mu \partial_x u_e^1 + \mu' v_e^1 + \partial_x P_e^1 = \mu''(y) \\ &\mu \partial_x v_e^1 + \partial_y P_e^1 = 0, \\ &\partial_x u_e^1 + \partial_y v_e^1 = 0, \\ &v_e^1|_{x=0} = V_{0,E}^1(y), v_e^1|_{x=L} = V_{L,E}^1(y), \qquad v_e^1|_{y=0} = v_e^1|_{y=2} = 0. \end{aligned}\right\} \tag{12}$$

We note that we prescribe data on the sides for these Euler profiles through the functions $V_{0,E}^1(y), V_{L,E}^1(y)$. These functions will satisfy standard elliptic compatibility conditions at the corners of our domain $(0, 0), (L, 0), (0, 2), (L, 2)$.

By going to the vorticity formulation, we arrive at the following problem:

$$-\mu \Delta v_e^1 + \mu'' v_e^1 = \mu'''(y), \quad v_e^1|_{y=0} = v_e^1|_{y=2} = 0, \quad u_e^1 := \int_0^x v_{ey}^1. \tag{13}$$

We will make the assumptions that:

$$\frac{\mu''}{\mu}, \frac{\mu'''}{\mu} \text{ vanish at high order at } y = 0, 2. \tag{14}$$

According to (14), we divide (13) by μ to obtain:

$$-\Delta v_e^1 + \frac{\mu''}{\mu} v_e^1 = \frac{\mu'''}{\mu}, \qquad v_e^1|_{y=0,2} = 0. \tag{15}$$

The system satisfied by the rest Euler layers is shown here ($i = 2, ...6$):

$$\left.\begin{aligned}
&\mu\partial_x u_e^i + \mu' v_e^i + \partial_x P_e^i = 0\\
&\mu\partial_x v_e^i + \partial_y P_e^i = 0,\\
&\partial_x u_e^i + \partial_y v_e^i = 0,\\
&v_e^i|_{x=0} = V_{0,E}^i(Y),\ v_e^i|_{x=L} = V_{L,E}^i(Y),\ v_e^i|_{y=2} = -v_p^{i-1}|_{y=2}, \quad v_e^i|_{y=0} = -v_p^{i-1}|_{y=0}.
\end{aligned}\right\} \tag{16}$$

Going to vorticity produces the homogeneous system:

$$-\mu\Delta v_e^i + \mu'' v_e^i = 0. \tag{17}$$

This procedure contributes the error terms to the remainder (analogous to equations (96), (97) in the original paper)

$$\mathcal{C}_{Euler,u} :=(u_s^E - \mu)\partial_x(u_s^E - \mu) + v_s^E\partial_y(u_s^E - \mu) - \varepsilon\Delta(u_s^E - \mu), \tag{18}$$

$$\mathcal{C}_{Euler,v} :=(u_s^E - \mu)\partial_x v_s^E + v_s^E\partial_y v_s^E - \varepsilon\Delta v_s^E \quad . \tag{19}$$

We can observe that $\mathcal{C}_{Euler,u}, \mathcal{C}_{Euler,v}$ are $O(\varepsilon^2)$. The following follow from standard elliptic theory:

Lemma 1. *Assume (14), and that $V_{0,E}^i, V_{L,E}^i,\ i = 1, 2...6$ are prescribed smooth functions satisfying standard elliptic compatibility conditions for arbitrary order. Then there exist unique solutions, $(u_e^i, v_e^i),\ i = 1, 2...6$ to (12) and (16) that are regular:*

$$|\partial_x^l\partial_y^m\{u_e^i, v_e^i\}| \lesssim c_0(\frac{\mu'''}{\mu}) \times C_{l,k} \textit{ for } i = 1, 2...6. \tag{20}$$

A.3. Boundary layer equations

Recalling the fact that $\mu(y) \sim y$ when $y \to 0$, and the boundary layer profiles are broken into two components, one supported near $y = 0$, $\mathbf{u}_p^{i,-}$, and one supported near $y = 2$, $\mathbf{u}_p^{i,+}$. In the following we will only give the construction of $\mathbf{u}_p^{i,-}$ (if $u_b = 0$, the construction of $\mathbf{u}_p^{i,+}$ is the same as $\mathbf{u}_p^{i,-}$, while the case if $u_b > a > 0$ is much simpler, as μ does not vanish at $y = 2$).

Using (12) the leading $\mathcal{O}(\varepsilon^{\frac{4}{3}})$ terms for the boundary layer profiles from (10) are:

$$R_u^1 = \varepsilon^{-\frac{1}{3}}\mu\partial_x u_p^1 + v_p^1\mu' + \varepsilon^{-\frac{1}{3}}\partial_x P_p^1 - \partial_{YY}u_p^1, \tag{21}$$

and the leading terms for (11) are:

$$R_v^1 = \partial_Y P_p^1 = 0. \tag{22}$$

Then to construct $\mathbf{u}_p^{1,-}$, using (14) and the fact that $\mu(y) = \mu'(0)y + O(y^2)$ when $y \to 0$, we first consider the following system:

$$\begin{cases} \mu'(0)Y_-\partial_x u_p^{1,0,-} + v_p^{1,0,-} + \varepsilon^{-\frac{1}{3}}\partial_x P_p^1 - \partial_{Y_-Y_-}u_p^{1,0,-} = 0, \ \ \partial_Y P_p^1 = 0, \\ u_p^{1,0,-}|_{x=0} = 0, \ u_p^{1,0,-}|_{Y_-=0} = -u_e^1(x,0), \ u_p^{1,0,-}|_{Y_-\to\infty} = 0, \\ v_p^{1,0,-} = \int_{Y_-}^{\infty} \partial_x u_p^{1,0,-} \end{cases} \tag{23}$$

First it is easy to obtain $P_p^1 \equiv 0$, and thus

$$\begin{cases} \mu'(0)Y_-\partial_x u_p^{1,0,-} + v_p^{1,0,-} - \partial_{Y_-Y_-}u_p^{1,0,-} = 0, \\ u_p^{1,0,-}|_{x=0} = 0, \ u_p^{1,0,-}|_{Y_-=0} = -u_e^1(x,0), \ u_p^{1,0,-}|_{Y_-\to\infty} = 0, \\ v_p^{1,0,-} = \int_{Y_-}^{\infty} \partial_x u_p^{1,0,-} \end{cases} \tag{24}$$

Note that we construct $u_p^{1,0,-}$, $v_p^{1,0,-}$ on $(0, L) \times (0, \infty)$. We now cut-off these layers and make a higher order error:

$$u_p^{1,-} = \chi(\frac{\varepsilon^{\frac{1}{3}}Y_-}{a_0})u_p^{1,0,-} - \frac{\varepsilon^{\frac{1}{3}}}{a_0}\chi'(\frac{\varepsilon^{\frac{1}{3}}Y_-}{a_0})\int_0^x v_p^{1,0,-}, \quad v_p^{1,-} := \chi(\frac{\varepsilon^{\frac{1}{3}}Y_-}{a_0})v_p^{1,0,-} \tag{25}$$

here $a_0 > 0$ is a fixed constant small enough. $u_p^{1,0,+}$, $v_p^{1,0,+}$, $u_p^{1,+}$, $v_p^{1,+}$ are defined analogously, and we omit these details. We then define:

$$u_p^1(x, Y) = \begin{cases} u_p^{1,-}(x, Y_-) \text{ if } 0 \le y \le 1, \\ u_p^{1,+}(x, Y_+) \text{ if } 1 \le y \le 2, \end{cases} \tag{26}$$

and

$$v_p^1(x, Y) = \begin{cases} v_p^{1,-}(x, Y_-) \text{ if } 0 \le y \le 1, \\ v_p^{1,+}(x, Y_+) \text{ if } 1 \le y \le 2. \end{cases} \tag{27}$$

Note that due to the cut-off in (25), u_p^1, v_p^1 are smooth. After cut-off, the contribution to the next layer are

$$\mathcal{C}_{cut}^1 = \frac{1}{a_0}\varepsilon^{\frac{1}{3}}Y\chi' v_p^{1,0,-} + 3\frac{1}{a_0}\varepsilon^{\frac{1}{3}}\chi'\partial_Y u_p^{1,0,-} + 3\frac{1}{a_0^2}\varepsilon^{\frac{2}{3}}\chi'' u_p^{1,0,-} + \frac{1}{a_0^3}\varepsilon\chi'''\int_0^x v_p^{1,0,-}. \tag{28}$$

We also obtain another error, due to approximating $\varepsilon^{-\frac{1}{3}}\mu$ by Y_- in the support of the cut-off function $\chi(\frac{\varepsilon^{\frac{1}{3}}Y}{a_0})$, and by approximating μ' by 1. This is defined by

$$\mathcal{C}^1_{approx} :=(\varepsilon^{-\frac{1}{3}}\mu(y) - Y)(\chi(\frac{y}{a_0})\partial_x u_p^{1,0,-} + \frac{1}{a_0}\varepsilon^{\frac{1}{3}}\chi' u_p^{1,0,-}) + (\mu' - 1)\chi(\frac{y}{a_0}) v_p^{1,0,-} \quad (29)$$

Finally, we have higher order terms that contribute to the error:

$$\mathcal{C}^1_{quad} := \varepsilon^2(u^1_e + u^1_p)\partial_x u^1_p + \varepsilon^2 u^1_p \partial_x u^1_e + \varepsilon^{\frac{5}{3}} v^1_e u^1_{pY} + \varepsilon^{\frac{7}{3}} v^1_p u^1_{ey} + \varepsilon^2 v^1_p u^1_{pY} - \varepsilon^2 u^1_{pxx}. \quad (30)$$

For the higher order terms in the second equation, we will use our auxiliary pressure to move it to the top equation. This is achieved by defining the first auxiliary pressure, $P_p^{1,a}$ to zero out the terms contributed from

$$\begin{aligned} &\varepsilon^{\frac{4}{3}}(\mu + \varepsilon u^1_e) v^1_{px} + \varepsilon u^1_p(\varepsilon v^1_{ex} + \varepsilon^{\frac{4}{3}} v^1_{px}) + \varepsilon^2 v^1_e v^1_{pY} + \varepsilon^{\frac{7}{3}} v^1_p v^1_{pY} + \varepsilon^{\frac{7}{3}} v^1_p v^1_{ey} \\ &- \varepsilon^{\frac{5}{3}} v^1_{pYY} - \varepsilon^{\frac{7}{3}} v^1_{pxx} + \varepsilon^{\frac{4}{3}} P^{1,a}_{pY} = 0, \end{aligned} \quad (31)$$

which therefore motivates our definition of

$$\begin{aligned} -\varepsilon^{\frac{4}{3}} P_P^{1,a} := \int_Y^\infty \Big(&\varepsilon^{\frac{4}{3}}(\mu + \varepsilon u^1_e) v^1_{px} + \varepsilon u^1_p(\varepsilon v^1_{ex} + \varepsilon^{\frac{4}{3}} v^1_{px}) + \varepsilon v^1_e v^1_{pY} + \varepsilon^{\frac{7}{3}} v^1_p v^1_{pY} \\ &+ \varepsilon^{\frac{7}{3}} v^1_p v^1_{ey} - \varepsilon^{\frac{5}{3}} v^1_{pYY} - \varepsilon^{\frac{7}{3}} v^1_{pxx} \Big) \, \mathrm{d}Y'. \end{aligned} \quad (32)$$

As a result, we can define the forcing for the next order Prandtl layer via

$$F^{(2)} := \varepsilon^{-\frac{5}{3}} \Big(- \varepsilon^{\frac{4}{3}} \mathcal{C}^1_{cut} + \varepsilon^{\frac{4}{3}} \mathcal{C}^1_{approx} + \mathcal{C}^1_{quad} - \varepsilon^{\frac{5}{3}} \partial_x P_P^{1,a} \Big). \quad (33)$$

Computing the higher order forcing terms works in a nearly identical manner, so we do not repeat those formulas here. We thus derive the general equation that we study for the boundary layer profiles for $1 \le i \le M_0 - 1$ (that is, all but the last layer):

$$\mu'(0) Y \partial_x u_p^{i,0,-} + v_p^{i,0,-} - \partial_Y^2 u_p^{i,0,-} = F^{(i)}, \qquad (x, Y) \in (0, L) \times (0, \infty) \quad (34a)$$

$$u_p^{i,0,-}(x, 0) = -u^i_e(x, 0), \qquad \lim_{Y\to\infty} u_p^{i,0,-}(x, Y) = 0, \quad (34b)$$

$$u^{i,0,-}|_{x=0} = U^i_0(Y). \quad (34c)$$

Let us therefore consider the abstract problem (dropping indices):

$$\mu'(0) Y \partial_x u + v - \partial_Y^2 u = F, \qquad (x, Y) \in (0, L) \times (0, \infty) \quad (35a)$$

$$v := \int_Y^\infty \partial_x u \, \mathrm{d}Y', \quad (35b)$$

$$u|_{x=0} = U_0(Y), \qquad u|_{Y=0} = g(x), \qquad u|_{Y\to\infty} = 0. \quad (35c)$$

For this abstract problem, we obtain the following estimates:

Lemma 2. *Assume standard parabolic compatibility conditions on $U_0(Y)$. Fix a large $K >> 1$. Assume also that $U_0(Y)$ decay rapidly at infinity:*

$$|e^Y \partial_Y^l U_0| \lesssim 1 \text{ for } 0 \le l \le K. \tag{36}$$

There exists (u, v) unique solutions to (35a) – (35c) that satisfy the following estimates:

$$|(1+Y)^m \partial_x^k \partial_Y^l \{u, v\}| \le c_0(F, g) \times C_{m,k,l} \text{ for any } 2k + l \le \frac{K}{2}. \tag{37}$$

Proof. First, a standard homogenization enables us to consider the Dirichlet problem, $g = 0$, up to modifying the forcing F. Indeed, fixing a cut-off function $\chi(Y)$ so that $\chi(0) = 1$, $\chi(Y) = 0$, when $Y \ge 2$, and $\int_0^\infty \chi(Y')\,dY' = 0$, we can consider the unknowns

$$\tilde{u} := u - \chi(Y)g(x), \qquad \tilde{v} := v + \partial_x g(x) \int_0^Y \chi(Y')\,dY'. \tag{38}$$

These will satisfy the system

$$\mu'(0)Y\partial_x \tilde{u} + \tilde{v} - \partial_Y^2 \tilde{u} = \tilde{F}, \qquad (x, Y) \in (0, L) \times (0, \infty) \tag{39a}$$

$$\tilde{v} := \int_Y^\infty \partial_x \tilde{u}\,dY', \tag{39b}$$

$$\tilde{u}|_{x=0} = \tilde{U}_0(Y), \qquad \tilde{u}|_{Y=0} = 0, \qquad \tilde{u}|_{Y\to\infty} = 0. \tag{39c}$$

Above, the modified forcing

$$\tilde{F} := F - Y\chi(Y)\partial_x g(x) + \partial_x g(x) \int_0^Y \chi + \partial_Y^2 \chi g. \tag{40}$$

We now drop the $\tilde{u}, \tilde{v}$ notation, and simply consider the homogenized problem above.

We multiply by $-\partial_Y v$ and integrate to get the identity

$$\frac{\partial_x}{2} \int_0^\infty u_Y^2\,dY + \int_0^\infty Y\mu'(0)|\partial_x u|^2\,dY + \frac{1}{2}v(x, 0)^2 = -\int_0^\infty F\partial_Y v\,dY. \tag{41}$$

We estimate the term on the right-hand side by localizing to $\chi(Y > 1)$ and $\chi(Y \le 1)$. First, for $\chi(Y > 1)$, we simply use Cauchy-Schwartz and absorb the $\partial_Y v$ term to the left-hand side as Y is non-degenerate. For $\chi(Y \le 1)$, we proceed as follows. Integrating now in $x \in (0, x_0)$, where $x_0 \le L$,

$$\frac{1}{2}\int\limits_{x=x_0} u_Y^2\,\mathrm{d}Y + \int\limits_0^{x_0}\int\limits_0^{\infty} \mu'(0)Y|\partial_x u|^2\,\mathrm{d}Y + \frac{1}{2}\int\limits_0^{x_0} v(x,0)^2 \tag{42}$$

$$= \int\limits_0^{x_0}\int\limits_0^{\infty} F\partial_x u\chi(Y\le 1) + \frac{1}{2}\int\limits_{x=0} u_Y^2\,\mathrm{d}Y. \tag{43}$$

Integrating the right-hand side by parts in x, we get

$$\begin{aligned}\int\limits_0^{x_0}\int\limits_0^{\infty} F\partial_x u\chi(Y\le 1) = &-\int\limits_0^{x_0}\int\limits_0^{\infty} \partial_x F u\chi(Y\le 1) + \int\limits_{x=x_0} Fu\chi(Y\le 1)\\ &- \int\limits_{x=0} Fu\chi(Y\le 1).\end{aligned} \tag{44}$$

All of the above terms are easily controlled by noticing that $|u| \lesssim \sqrt{Y}\|u_Y\|_{L^2_Y}$, as $u(0)=0$. Consequently we have

$$\int\limits_{\Omega} Yu_x^2 dYdx + \int\limits_0^L v^2(x,0)dx + \sup_{x\in[0,L]}\int\limits_0^{\infty} u_Y^2 dY \le C(\|F\|^2_{H^1} + \|U_0\|^2_{L^2}) \tag{45}$$

Then we differentiate (39a) once in Y to obtain

$$\mu'(0)Yu_{xY} - \partial_Y^2 u_Y = F_Y. \tag{46}$$

As $u_x \to 0$ when $Y\to\infty$, we have $u_x|_{x=0} = \int_\infty^Y \partial_Y u_x dY = \int_\infty^Y \frac{1}{Y}(F_Y(0,Y) + \partial_Y^3 U_0)$. Similarly, we can obtain $\partial_x^k u|_{x=0}$. Then we can clearly repeat the above after commuting ∂_x^k with the equation (it commutes perfectly) to get that, for any $k\ge 0$

$$\int\limits_{\Omega} Y(\partial_x^k u_x)^2 dYdx + \int\limits_0^L (\partial_x^k v)^2(x,0)dx + \sup_{x\in[0,L]}\int\limits_0^{\infty} (\partial_x^k u_Y)^2 dY \le C(\|f\|^2_{H^k} + \|U_0\|^2_{H^k}) \tag{47}$$

To get weighted estimates, we can repeat the exact estimates inductively by applying multipliers of the form $-v_Y(1+Y)^m$ and then repeat the above after commuting ∂_Y^k with the equation. □

For $i = M_0$, we need to slightly modify the abstract problem we are considering. Consider now:

$$\mu'(0)Y\partial_x u + v - \partial_Y^2 u = F, \qquad (x,Y)\in(0,L)\times(0,\infty) \tag{48a}$$

$$v := -\int_0^Y \partial_x u \,\mathrm{d}Y', \tag{48b}$$

$$u|_{x=0} = U_0(Y), \qquad u|_{Y=0} = g(x), \qquad \partial_Y u|_{Y\to\infty} = 0. \tag{48c}$$

Compared to (35a) – (35c), the key difference is that $v(x, 0) = 0$ here.

Lemma 3. *Assume the data U_0 from (48c) satisfies, for some $K >> 1$,*

$$|e^Y \partial_Y^l U_0| \lesssim 1 \text{ for } 0 \le l \le K. \tag{49}$$

There exist unique solutions, (u, v), to (48a) – (48c), satisfying the following estimates:

$$|(1+Y)^m \partial_x^k \partial_Y^l u| \le c_0(F, g) \times C_{m,k,l} \text{ for any } m \ge 0, l \ge 1, 1 \le 2k + l \le \frac{K}{2}, \tag{50}$$

$$|\partial_x^k \{u, \frac{v}{Y}\}| \le c_0(F, g) \times C_k \text{ for any } 0 \le 2k \le \frac{K}{2}, \tag{51}$$

Proof. We cannot proceed as in Lemma 2 due to the lack of decay in v as $Y \to \infty$. Instead, we differentiate once in Y to obtain the simplified system

$$\mu'(0) Y \partial_x (u_Y) - \partial_Y^2 u_Y = F_Y, \qquad (x, Y) \in (0, L) \times (0, \infty), \tag{52}$$

$$\partial_Y u_Y|_{Y=0} = -F(x, 0), \qquad u_Y|_{Y\to\infty} = 0. \tag{53}$$

This system admits smooth solutions for u_Y which obey estimates (50). Once u_Y has been constructed, we recover (u, v) via

$$u = g(x) + \int_0^Y u_Y, \qquad v = -\int_0^Y \partial_x u. \quad \square \tag{54}$$

From here, we can recover the analogue of the estimates of Lemma 12 in the original paper:

Corollary 4. *Define $U_0^{(i)}(Y) := u_p^i|_{x=0}$. Assume $|\partial_Y^l U_0^{(i)}(Y) e^Y| \lesssim 1$ for all $0 \le l \le K$ for some $K >> 1$. Then u_p^i, v_p^i satisfy the following estimates for any $m \ge 0$:*

$$|(1+Y)^m \partial_x^k \partial_Y^l \{u_p^i, v_p^i\}| \le c_0(\frac{\mu'''}{\mu}) \times C_{m,k,l} \text{ for } 1 \le i \le M_0 - 1 \tag{55}$$

$$|(1+Y)^m \partial_x^k \partial_Y^l u_p^{M_0}| \le c_0(\frac{\mu'''}{\mu}) \times C_{m,k,l} \text{ for } l \ge 1, \tag{56}$$

$$|\partial_x^k \{u_p^{M_0}, \frac{v_p^{M_0}}{Y}\}| \le c_0(\frac{\mu'''}{\mu}) \times C_{m,k,l}, \tag{57}$$

so long as $2k + l \le \frac{K}{2}$.

Proof. This follows immediately upon applying Lemma (2) to the system (34a) – (34c), and invoking the formulas (25) to recover u_p^i, v_p^i after cutting them off. $\square$

We now collect the forcing terms which contribute to the remainder equation, namely the expressions for $\mathcal{F}_u, \mathcal{F}_v$. Indeed, we will have the expressions:

$$\mathcal{F}_u :=\varepsilon^{1+\frac{M_0}{3}} \mathcal{C}_{cut}^{M_0} + \varepsilon^{1+\frac{M_0}{3}} \mathcal{C}_{approx}^{M_0} + \mathcal{C}_{quad}^{M_0} + \mathcal{C}_{Euler,u}, \tag{58}$$

$$\begin{aligned}\mathcal{F}_v :=&\varepsilon^{1+\frac{M_0}{3}} u_s \partial_x v_p^{M_0} + \varepsilon^{\frac{2}{3}+\frac{M_0}{3}} v_s v_{pY}^{M_0} + \varepsilon^{\frac{2}{3}+\frac{M_0}{3}} \partial_x (v_s - \varepsilon^{1+\frac{M_0}{3}} v_p^{M_0}) u_p^{M_0} \\ &+ \varepsilon^{1+\frac{M_0}{3}} v_p^{M_0} \partial_Y (v_s - \varepsilon^{1+\frac{M_0}{3}} v_p^{M_0}) - \varepsilon \Delta (\varepsilon^{1+\frac{M_0}{3}} v_p^{M_0}) + \mathcal{C}_{Euler,v}\end{aligned} \tag{59}$$

where we have defined the Euler contributions to the error in (18) and (19).

A corollary to our construction is, by taking M_0 sufficiently large (the choice $M_0 = 6$ will suffice), we can decompose (as in (114) of the original paper)

$$\mathcal{F}_u = \mathcal{T}_{u,\varepsilon^2} + \mathcal{F}_{u,\varepsilon^{\frac{7}{3}}}, \qquad \mathcal{F}_v = \mathcal{T}_{v,\varepsilon^2} + \mathcal{F}_{v,\varepsilon^{\frac{7}{3}}}, \tag{60}$$

where we define

$$\mathcal{T}_{u,\varepsilon^2} :=\varepsilon^2 (u_e^1 \partial_x u_e^1 + v_e^1 \partial_y u_e^1 - \Delta u_e^1) = \varepsilon^2 T_1, \tag{61}$$

$$\mathcal{T}_{v,\varepsilon^2} :=\varepsilon^2 (u_e^1 \partial_x v_e^1 + v_e^1 \partial_y v_e^1 - \Delta v_e^1) = \varepsilon^2 T_2. \tag{62}$$

We subsequently obtain for arbitrary order k,

$$\|\mathcal{F}_{u,\varepsilon^{\frac{7}{3}}}\|_{L^p} + \|\mathcal{F}_{v,\varepsilon^{\frac{7}{3}}}\|_{L^p} \lesssim \varepsilon^{\frac{7}{3}}, \qquad 1 \le p \le \infty, \tag{63}$$

$$\|T_1, \frac{T_2}{\tilde{y}}\|_2 \le c_0(\frac{\mu'''}{\mu}), \qquad \|T_1, T_2\|_{H^k} \lesssim_k c_0(\frac{\mu'''}{\mu}). \tag{64}$$