

GLOBAL DYNAMICS OF THE EULER-ALIGNMENT SYSTEM WITH WEAKLY SINGULAR KERNEL

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ABSTRACT. This letter studies the Euler-alignment system with weakly singular influence functions by introducing a novel technique to bound the density. Instead of resorting to a nonlinear maximum principle used in [C. Tan, Nonlinearity, 33: 1907–1924, 2020] to bound the interaction term $\psi * \rho$ by ρ^s with $s \in (0, 1)$ for ψ with an algebraic singularity at origin, we bound $\psi * \rho$ by a relaxed constant for any $\psi \in L^1$. We thus establish the global-in-time existence results with weaker assumptions on ψ and refined solution bounds, characterized by the structure of ψ .

1. INTRODUCTION

The Euler-alignment system

$$(1.1a) \quad \rho_t + (\rho u)_x = 0, \quad t > 0, x \in X,$$

$$(1.1b) \quad u_t + uu_x = \psi * (\rho u) - u\psi * \rho,$$

as a hydrodynamic model characterizes the self-organized collective behaviors, in particular alignment and flocking [5, 6], originating from the Cucker-Smale agent-based models [2, 3]. Here, ρ represents the density of the group, and u is the associated velocity. The term that appears on the right of (1.1b) is the *alignment force*, where we use the notation

$$\psi * \rho = \int_X \psi(|x - y|) \rho(t, y) dy.$$

The spatial domain X can be all of \mathbb{R} or a periodic domain and the initial data is $(\rho(0, x) \geq 0, u(0, x))$. Since $M := \int_X \rho dx$ is conserved and a simple maximum principle on u indicates that $\int_X \rho u$ is well-defined, we have that the right hand side of (1.1b) is well-defined.

For system (1.1) the global dynamics has been studied for all three scenarios: bounded Lipschitz interaction [1, 9], the strongly singular interaction [4, 7], and the weakly singular interaction [8]. In this letter we make an attempt to extend the results in [8] on the weakly singular interaction, with focus on a new technique circumventing the use of the nonlinear maximum principle in [8], and also recovering the result in [1].

In order to compare the results in [8], we recall some conventions here. A sharp critical threshold condition is obtained in [1] for bounded interaction with the help of an important quantity

$$G := u_x + \psi * \rho,$$

the dynamics of which then becomes

$$\partial_t G + \partial_x(Gu) = 0.$$

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This together with the continuity equation for ρ can serve as an alternative representation of (1.1). The velocity field u can be recovered by the relation for G and the conservation of momentum. We rewrite this new equivalent system here,

$$(1.2a) \quad \rho_t + (\rho u)_x = 0, \quad t > 0,$$

$$(1.2b) \quad G_t + (uG)_x = 0,$$

$$(1.2c) \quad u_x = G - \psi * \rho.$$

For bounded Lipschitz interaction, the critical threshold results in [1] may be stated as: *system (1.2) admits global regular solution if and only if $\inf_x G(0, x) \geq 0$* . In sharp contrast, the so-called *strongly singular interaction*, say $\psi(r)$ is not integrable at $r = 0$, has been shown to have a regularization effect in the sense that global-in-time regular solution is always ensured, see [4, 7]. This is true for Euler-alignment, see [4], as well as Euler-Poisson-alignment, see [7], systems. For the *weakly singular interaction*, the critical thresholds obtained in [8] are about similar to those for the bounded interaction, while the principal global existence states the following.

Theorem 1.1. [8] *Consider (1.2) with smooth initial data $(\rho(0, x) \geq 0, G(0, x))$ and weakly singular interaction $\psi \in L^1(X)$, nonnegative and satisfying,*

$$\lambda r^{-s} \leq \psi(r) \leq \Lambda r^{-s}, \Lambda \geq \lambda > 0, s \in (0, 1)$$

uniformly in a neighborhood of the origin. If

$$\inf_x G(0, x) > 0,$$

then there exists a globally regular solution to the system (1.2).

In this letter, we extend that to the following.

Theorem 1.2. *Let $\psi \in L^1(X)$ be non-negative. Consider (1.2) with smooth initial data $(\rho(0, x) \geq 0, G(0, x))$. If*

$$\inf_x G(0, x) > 0,$$

then there exists globally regular solution to the system (1.2).

Moreover, for this solution we have the global bound,

$$\begin{aligned} \|\rho(t, \cdot)\|_\infty &\leq \max \{ \|\rho(0, \cdot)\|_\infty, \beta \}, \\ \|G(t, \cdot)\|_\infty &\leq \max \{ \|G(0, \cdot)\|_\infty, \gamma \}, \end{aligned}$$

where $C_0 := \inf_x \frac{G(0, x)}{\rho(0, x)} > 0$, and

$$\begin{aligned} \beta &= \begin{cases} 0 & \text{if } C_0 > \|\psi\|_{L^1}, \\ \frac{M\|\psi\|_\infty}{C_0} & \text{if } \psi \in L^\infty, \\ \inf_k \left\{ \frac{Mk}{C_0 - \int_{\psi \geq k} \psi dx} : C_0 > \int_{\psi \geq k} \psi dx \right\} & \text{otherwise,} \end{cases} \\ \gamma &= \begin{cases} M\|\psi\|_\infty & \text{if } \psi \in L^\infty(X), \\ \|\psi\|_1 \max \{ \|\rho(0, \cdot)\|_\infty, \beta \} & \text{otherwise.} \end{cases} \end{aligned}$$

Some remarks are in order.

Remark 1.3. For the finite time blow-up, both bounded interaction and weakly singular interaction feature the similar behavior: If $\inf_x G(0, x) < 0$, then the solution admits a finite time blow up [1, 8]. For the global-in-time existence, our method unifies the two scenarios as well. More precisely, we recover the global regular solution result in [1].

Remark 1.4. A remarkable feature of our approach is that apart from the assumption that $\psi \in L^1$, we only require the nonnegativity of ψ . Our analysis neither requires the need of an explicit singularity condition (2.6) nor any restriction on the number or position of singularities, and hence, is applicable on the larger class of L^1 as well as bounded influence functions. In contrast, the main technique used by the author in [8] is to derive a nonlinear maximum principle, which requires that ψ only has the one singularity at the origin and obeys (2.6).

The main contribution of this letter is two fold: (i) A novel technique to bound the interaction term is introduced, leading to a global control of the density; (ii) we obtain refined solution bounds and characterize how they are determined by the structure of ψ . Here we not only give an alternative proof to the existence of global-in-time solutions, but also do so for more general influence functions. In addition, we are able to establish sharp solution bounds which indicate the effect from the profile of $\psi \in L^1(X)$.

In Section 2, we review the existing method, and the proof of Theorem 1.2 is presented in Section 3. In Section 4, we give a particular example of our general result to showcase our technique and the fineness of the bounds obtained.

2. REVIEW OF THE EXISTING METHOD [8]

It is known from [8] that the local-in-time classical solutions to (1.2), and in turn (1.1), can be extended for further times if the following holds,

$$(2.1) \quad \int_0^T \|G(t, \cdot)\|_\infty + \|\rho(t, \cdot)\|_\infty dt < \infty.$$

Hence, the persistence of smooth solutions upto any time $T > 0$ is guaranteed if there are a priori bounds on the infinity norms of $G(T, \cdot), \rho(T, \cdot)$.

Writing (1.2) as ODEs along the characteristic path,

$$(2.2) \quad \Gamma = \left\{ (t, \mathcal{X}(t)) : \frac{d\mathcal{X}}{dt} = u(t, \mathcal{X}(t)), \mathcal{X}(0) = x_0 \right\},$$

we have,

$$(2.3a) \quad \frac{d}{dt} \rho(t, \mathcal{X}(t)) = -\rho(G - \psi * \rho),$$

$$(2.3b) \quad \frac{d}{dt} G(t, \mathcal{X}(t)) = -G(G - \psi * \rho),$$

with initial point $(\rho_0, G_0) := (\rho(0, x_0), G(0, x_0))$. If $\rho_0 = 0$, then $\rho(t, \mathcal{X}(t)) \equiv 0$ along the characteristic. Observing (2.3b), the bounds on G depend on the global behavior of ρ rather than along a single path. If one is able to bound ρ, G along characteristic paths for which $\rho_0 > 0$, then the solutions are indeed bounded along all other characteristic paths as well. For $\rho_0 > 0$, we have,

$$G = \frac{\rho}{\rho_0} G_0,$$

since along Γ one has,

$$\frac{d}{dt} \left(\frac{G(t, \mathcal{X}(t))}{\rho(t, \mathcal{X}(t))} \right) = 0.$$

Plugging this in (2.3a),

$$(2.4) \quad \frac{d}{dt} \rho(t, \mathcal{X}(t)) = -\frac{G_0}{\rho_0} \rho^2 + \rho \psi * \rho.$$

From this we see that if $G_0 < 0$, then ρ will blow up at finite time since $\frac{d}{dt} \rho(t, \mathcal{X}(t)) > \frac{|G_0|}{\rho_0} \rho^2$, also $G \rightarrow -\infty$ at a finite time. For $G_0 > 0$, from (2.3b) we see that

$$(2.5) \quad 0 \leq G \leq \max\{G_0, \gamma\}$$

for all $t > 0$, if $\psi * \rho \leq \gamma$ for some $\gamma > 0$.

It is left to control ρ in the case $G_0 > 0$ and the problem boils down to ensuring a bound on the term $\psi * \rho$. A bounded kernel would immediately result in the needful since the mass is conserved in time. However, a singular kernel could aggravate this term. The author in [8] bounded this term by proving a nonlinear maximum principle on ρ . It states that if

$$(2.6) \quad \lambda|x|^{-s} \leq \psi(x) \leq \Lambda|x|^{-s},$$

close to the origin, then for a function $f \in L^1$

$$(\psi * f)(x_*) \leq C_1 f^s(x_*),$$

where $s \in (0, 1)$ and x_* is any point where f attains maximum value. Evaluating $\partial_t \rho + u \partial_x \rho = -\rho(G - \psi * \rho)$ at (t, x^*) and using the above result, one obtains

$$\partial_t \rho(t, x_*) \leq -C_0 \rho^2(t, x_*) + C_1 \rho^{1+s}(t, x_*),$$

where $C_0 = \inf \frac{G(0, x)}{\rho(0, x)} > 0$. Hence, ρ cannot become very large resulting in being upper-bounded. We can further use this bound in (2.5) to obtain that G remains bounded from above if $G_0 > 0$, thereby obtaining a priori bounds on $\|\rho(t, \cdot)\|_\infty, \|G(t, \cdot)\|_\infty$ for all time. The condition (2.1) then guarantees global-in-time classical solutions.

3. PROOF OF THEOREM 1.2

Step 1: (Global bound on ρ) We use system (2.3) along the characteristic path (2.2) and some preliminary notations from Section 2 in this Section. Recall $C_0 = \inf \frac{G(0, x)}{\rho(0, x)}$ which is strictly positive from the hypothesis of the Theorem.

For a number $k \geq 0$,

$$(3.1) \quad B = \{x : \psi(x) \geq k\},$$

and B^c be its complement on which $\psi \leq k$. Also, a set B is admissible if and only if $4 \int_B \psi(x) dx < C_0$. Since $\psi \in L^1$, there exists a $k \geq 0$ for which the set B is admissible. In particular, for all sufficiently large k , the corresponding sets B are admissible. However, note that the set B need not be admissible for an arbitrary $k \geq 0$. In particular, if $C_0 < \|\psi\|_1$, then for $k = 0$, we have $B = X$ and $C_0 \not\geq 4 \int_B \psi(x) dx$.

We can obtain an implicit bound on $\psi * \rho$,

$$\psi * \rho = \int_B \psi(y) \rho(t, x - y) dy + \int_{B^c} \psi(y) \rho(t, x - y) dy$$

$$\begin{aligned}
&\leq \|\rho(t, \cdot)\|_\infty \int_B \psi(x) dx + k \int \rho(t, x) dx \\
(3.2) \quad &\leq \|\rho(t, \cdot)\|_\infty \int_B \psi(x) dx + Mk.
\end{aligned}$$

We shall identify a uniform bound for ρ , which is valid for all time $t > 0$. Suppose $\|\rho(0, \cdot)\|_\infty \leq a$, then there exists $T = T(a)$ such that $\|\rho(t, \cdot)\|_\infty \leq 2a$ for $t \in [0, T]$. For any k for which the corresponding set B is admissible and for $t \in [0, T]$, we use (3.2) to get,

$$\psi * \rho \leq 2a \int_B \psi(x) dx + Mk \leq \frac{aC_0}{2} + Mk =: b.$$

On the other hand, equation (2.4) for ρ leads to the following differential inequality

$$\frac{d}{dt} \rho(t, \mathcal{X}(t)) \leq -C_0 \rho^2 + b\rho,$$

for $t \in [0, T]$. This inequality ensures that along the path Γ as in (2.2),

$$\rho(t, \mathcal{X}(t)) \leq \max \left\{ \rho_0, \frac{b}{C_0} \right\}, \quad t \in [0, T].$$

Thus, $\rho(t, \mathcal{X}(t)) \leq a$ as long as $a \geq \frac{b}{C_0}$. Therefore, it suffices to take a so that $a \geq \frac{2Mk}{C_0}$. By induction we have,

$$(3.3) \quad \rho(t, \mathcal{X}(t)) \leq \max \left\{ \rho_0, \frac{2Mk}{C_0} \right\}.$$

Step 2: (Refined bound on ρ) After we know ρ is globally bounded, we can further improve the bound in two steps. Again assume this bound to be a , then we have

$$\psi * \rho \leq a \int_A \psi(x) dx + Mk.$$

For $k \geq 0$, let

$$(3.4) \quad A = \{x : \psi(x) \geq k\}.$$

Also, a set A is admissible if and only if $\int_A \psi(x) dx < C_0$. Note that by relaxing the admissibility condition on A compared to B in (3.1), we allow for more, and in particular, smaller values of k , which will result in an even sharper bound. We have,

$$\frac{d}{dt} \rho(t, \mathcal{X}(t)) \leq -C_0 \rho^2 + \rho \left(a \int_A \psi(x) dx + Mk \right),$$

where a is a bound on $\rho(t, \mathcal{X}(t))$ which we know exists from Step 1. Consequently,

$$\frac{d}{dt} \rho(t, \mathcal{X}(t)) \leq -C_0 \rho \left(\rho - \frac{a \int_A \psi(x) dx + Mk}{C_0} \right)$$

Therefore, for any k for which the set A is admissible, it suffices to take an a such that,

$$a \geq \max \left\{ \rho_0, \frac{a \int_A \psi(x) dx + Mk}{C_0} \right\}.$$

Hence,

$$a \geq \max \left\{ \rho_0, \frac{Mk}{C_0 - \int_A \psi(x) dx} \right\}.$$

Consequently, we have a global bound on density,

$$(3.5) \quad \|\rho(t, \cdot)\|_\infty \leq \max \left\{ \|\rho(0, \cdot)\|_\infty, \frac{Mk}{C_0 - \int_A \psi(x) dx} \right\}.$$

For fixed k , (3.5) is a better bound than (3.3).

Step 3: (Further narrowing down by minimization in k) Next we examine how the profile of ψ can affect this bound, and how to even further sharpen the bound. Firstly, observe that if $C_0 > \|\psi\|_1$, then k can be zero with A as the complete space being an admissible set from Step 2. This results in a maximum principle on ρ ,

$$\|\rho(t, \cdot)\|_\infty \leq \|\rho(0, \cdot)\|_\infty.$$

Also, if ψ is bounded, then for any $\epsilon > 0$ we can have $k = \max \psi + \epsilon$ with $A = \emptyset$. By taking limit as $\epsilon \rightarrow 0$, we obtain,

$$\|\rho(t, \cdot)\|_\infty \leq \max \left\{ \|\rho(0, \cdot)\|_\infty, \frac{M \max \psi}{C_0} \right\}.$$

This points to the fact that the quantity $\frac{Mk}{C_0 - \int_A \psi(x) dx}$ adjusts itself as and when ψ changes. This motivates us to treat the following optimization problem to further sharpen the bounds on density and see how ψ affects these bounds,

$$(3.6) \quad \beta := \inf \left\{ \frac{Mk}{C_0 - \int_{A_k} \psi(x) dx} : C_0 > \int_{A_k} \psi(x) dx \right\}.$$

Note that we now explicitly show the dependence of the set A on k by the subscript. We consider the case where $C_0 \leq \|\psi\|_1$ for otherwise, $\beta = 0$ (minimum in this case) trivially. We set,

$$g(k) := \frac{Mk}{C_0 - \int_{A_k} \psi(x) dx},$$

with domain such that $C_0 > \int_{A_k} \psi(x) dx$. In particular, let

$$k_0 := \sup \left\{ k : C_0 = \int_{A_k} \psi(x) dx \right\} \geq 0.$$

Then $g : (k_0, \infty) \rightarrow \mathbb{R}^+$ is a well-defined function. Also, $\lim_{k \rightarrow \infty} g(k) = \infty$. This combined with the fact that g is bounded from below, we have that (3.6) is finite and the minimizer if exists is contained in the set $[k_0, \infty)$. Consider the sequence $\{k_n\}_{n=1}^\infty$ such that $g(k_n) \searrow \beta$. We have,

$$\|\rho(t, \cdot)\|_\infty \leq \max \{ \|\rho(0, \cdot)\|_\infty, g(k_n) \}, \quad n \geq 1.$$

Taking limit as $n \rightarrow \infty$, we obtain our final optimized bound,

$$\|\rho(t, \cdot)\|_\infty \leq \max \{ \|\rho(0, \cdot)\|_\infty, \beta \}, \quad t > 0.$$

With this bound we have

$$\psi * \rho \leq \|\rho(t, \cdot)\|_\infty \|\psi\|_1 \leq \max \{ \|\rho(0, \cdot)\|_\infty, \beta \} \|\psi\|_1 =: \gamma.$$

This when combined with (2.5) gives the bound for G . Note that for $\psi \in L^\infty(X)$, we simply take $\gamma = M\|\psi\|_\infty$. In summary, the uniform bounds for all time $t > 0$ read as,

$$(3.7a) \quad \|\rho(t, \cdot)\|_\infty \leq \max \{ \|\rho(0, \cdot)\|_\infty, \beta \},$$

$$(3.7b) \quad \|G(t, \cdot)\|_\infty \leq \max \{ \|G(0, \cdot)\|_\infty, \gamma \},$$

where

$$\gamma = \begin{cases} M\|\psi\|_\infty & \text{if } \psi \in L^\infty(X), \\ \|\psi\|_1 \max \{ \|\rho(0, \cdot)\|_\infty, \beta \} & \text{otherwise.} \end{cases}$$

This finishes the proof of Theorem 1.2. \square

4. AN EXAMPLE

In this section, we show the importance of optimization with level set $\psi = k$ by explicitly calculating β for some specific choices of alignment kernels with finite weight. It allows us to get an insight into the optimized bounds on density.

We consider an example: $\psi = |x|^{-\alpha}$ with $\alpha \in (0, 1)$ on the periodic domain $X = \mathbb{T} = (-1/2, 1/2]$, hence $\|\psi\|_1 = \frac{2^\alpha}{1-\alpha}$. We have two cases relative to the weight of ψ .

Case 1: If $C_0 > \frac{2^\alpha}{1-\alpha}$, then corresponding to $k = 0$, we have $A = \mathbb{T}$ is admissible. Therefore, the minimizer is $k = 0$ and consequently, $\beta = 0$. We then have the maximum principle on ρ .

Case 2: If $0 < C_0 \leq \frac{2^\alpha}{1-\alpha}$, then we need to minimize the function,

$$\beta(k) = \frac{Mk}{C_0 - \int_{A_k} \psi(x) dx},$$

with the domain being all $k \in \mathbb{R}^+$ such that $C_0 > \int_{A_k} \psi(x) dx$. Note that

$$A_k = \{x : |x|^{-\alpha} \geq k\} = \{x, |x| \leq k^{-1/\alpha}\}.$$

Therefore,

$$\int_{A_k} \psi(x) dx = 2 \int_0^{k^{-1/\alpha}} x^{-\alpha} dx = \frac{2k^{1-1/\alpha}}{1-\alpha}.$$

Let k_0 be such that $\int_{A_{k_0}} \psi(x) dx = C_0$, then the minimization problem reduces to the following:

$$\min_{(k_0, \infty)} \beta(k) := \min_{(k_0, \infty)} \frac{Mk}{C_0 - \frac{2k^{1-1/\alpha}}{1-\alpha}},$$

where $k_0 = \left(\frac{2}{C_0(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}$. Let the minimizer of $\beta(k)$ be k^* . We have $\frac{d\beta}{dk}(k^*) = 0$. Hence,

$$k^* = \left(\frac{2}{C_0\alpha(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}.$$

Note that since $\alpha \in (0, 1)$, we have $k^* > k_0$. Consequently, the minimizer is unique. Consequently,

$$\beta := \beta(k^*) = \left(\frac{2}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} \frac{M}{(C_0(1-\alpha))^{\frac{1}{1-\alpha}}}.$$

In the special case when $\alpha = 1/2$, we have $\beta = 16M/C_0^2$. Hence the bound on density would be,

$$\|\rho(t, \cdot)\|_\infty \leq \max \left\{ \|\rho(0, \cdot)\|_\infty, \frac{16M}{C_0^2} \right\}, \quad t > 0.$$

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