HOMOGENIZATION OF QUASI-CRYSTALLINE FUNCTIONALS VIA TWO-SCALE-CUT-AND-PROJECT CONVERGENCE

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Abstract. We consider a homogenization problem associated with quasi-crystalline multiple integrals of the form

$$u_\varepsilon \in L^p(\Omega;\mathbb{R}^{\mathfrak{d}}) \mapsto \int_\Omega f_R\left(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\right) \mathrm{d}x,$$

where u_ε is subject to constant-coefficient linear partial differential constraints. The quasi-crystalline structure of the underlying composite is encoded in the dependence on the second variable of the Lagrangian, f_R, and is modeled via the *cut-and-project* scheme that interprets the heterogeneous microstructure to be homogenized as an irrational subspace of a higher-dimensional space. A key step in our analysis is the characterization of the quasi-crystalline two-scale limits of sequences of the vector fields u_ε that are in the kernel of a given constant-coefficient linear partial differential operator, $\mathcal{A}$, that is, $\mathcal{A}u_\varepsilon = 0$. Our results provide a generalization of related ones in the literature concerning the $\mathcal{A} = \mathrm{curl}$ case to more general differential operators $\mathcal{A}$ with constant coefficients, and without coercivity assumptions on the Lagrangian f_R.

Keywords: homogenization, quasi-crystalline composites, multi-scale variational problems, PDE constraints, two-scale-cut-and-project convergence, Γ*-convergence*

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Contents

1 Introduction

The theory of homogenization addresses the description of the macroscopic or effective behavior of a microscopically heterogeneous system. There are multiple applications in the fields of physics, mechanics, materials science and other areas of engineering, including problems aimed at the modeling of composites, stratified or porous media, finely damaged materials, or materials with many holes or cracks.

From the mathematical viewpoint, homogenization is often associated with the study of the asymptotic behavior of oscillating partial differential equations, or of minimization problems deriving from certain oscillating functionals, depending on one or more small-scale parameters that represent the length scales of the heterogeneities.

A common assumption in the literature is based on the premise that the heterogeneities are evenly distributed, leading to the mathematical assumption of periodicity in the so-called fast variable, which encodes the heterogeneities in the mathematical problem. Even though the study of the effective behavior of periodically structured heterogeneous media has enabled the study of more complex ones, it is commonly accepted that periodicity is often not the most suited structural hypothesis. This fact is at the basis of many recent works devoted to the study of the effective behavior of random heterogenous materials whose small-length-scale properties are described at a statistical level only.

Here, we are interested in materials with a quasi-crystalline microstructure characterized by small-length-scale properties that are neither periodic nor random. Quasicrystals, also known as quasiperiodic crystals, are ordered structures that do not share the translational symmetry of traditional crystals [56, 57]. A quasi-crystalline pattern can continuously fill an $\mathbb{n}$-dimensional space, but will never be translational symmetric in more than $\mathbb{n}-1$ linearly independent directions.

The discovery of quasicrystals was announced in the early 1980s by two groups of crystallographers, Schechtman, Blech, Gratias, and Cahn [56] and Levine and Steinhardt [44]. At first, this was received with scepticism, and even hostility, by the the scientific community as quasicrystals violate the foundations of classical crystallography. However, in 2011, Shechtman was awarded the Nobel Prize in Chemistry for this discovery. A striking feature of quasicrystals is that their Bragg diffraction displays peculiar five-, ten-, or twelve-fold symmetry orders in contrast with the rigid crystallography of periodic crystals. Moreover, the assembly of quasi-crystalline tiling patterns is nonlocal and exhibit similar patterns at different scales (self-similarity).

There has been a rich discussion and extensive efforts in various mathematical communities to model quasicrystals; see [7, 16, 43, 47] and related references. A well-established mathematical approach to study quasicrystals is based on aperiodic tilings of hyperplanes, in which one aims at finding a set of geometric shapes, called tiles, paving the Euclidean plane without gaps or overlaps, in a non-periodic manner only (see Figure 1). A systematic, but not exhaustive, scheme to derive such tilings is via the cut-and-project method, introduced by de Bruijn [27] and further developed by Duneau and Katz [28], which extends Penrose's ideas of aperiodic tilings of the plane [54] to higher dimensions (see [8] for a more detailed description).

Roughly speaking, $\mathbb{n}$-dimensional quasi-crystalline patterns can be modeled by cutting periodic tilings in an $\mathbb{m}$-dimensional space, with $\mathbb{m} > \mathbb{n}$, through an $\mathbb{n}$-dimensional subspace with *irrational slope*. To be precise, given an $\mathbb{n}$-dimensional quasicrystal R and representing by $\sigma_R : \mathbb{R}^{\mathbb{n}} \to \mathbb{R}$ a constitutive property of R, we can find $\mathbb{m} \in \mathbb{N}$, with $\mathbb{m} > \mathbb{n}$, a $Y^{\mathbb{m}}$-periodic function $\sigma : \mathbb{R}^{\mathbb{m}} \to \mathbb{R}$ with $Y^{\mathbb{m}} \subset \mathbb{R}^{\mathbb{m}}$ a parallelotope, and a linear map $\boldsymbol{R} : \mathbb{R}^{\mathbb{n}} \to \mathbb{R}^{\mathbb{m}}$ such that

$$\sigma_R(x) = \sigma(\boldsymbol{R}x). \tag{1.1}$$

In the homogenization literature, the structural condition (1.1) is referred to as quasi-periodicity [16, 41]. Here, and in the sequel, we do not distinguish the linear map from its associated matrix in $\mathbb{R}^{\mathbb{m}\times\mathbb{n}}$, and denote both by $\boldsymbol{R}$. For example (see [36]), setting $\tau = \frac{1+\sqrt{5}}{2}$, the matrices

$$\boldsymbol{R} = \boldsymbol{R}_{\bar{5}m\text{-}sym} := \frac{1}{\sqrt{10}}\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 1 & \frac{\sqrt{5}}{\sqrt{2+\tau}}\tau & 1-\tau \\ 1 & \frac{\sqrt{5}}{\sqrt{2+\tau}} & \tau \\ 1 & -\frac{\sqrt{5}}{\sqrt{2+\tau}} & \tau \\ 1 & 0 & -2 \\ -1 & \frac{\sqrt{5}}{\sqrt{2+\tau}}\tau & \tau-1 \end{bmatrix}, \quad \boldsymbol{R} = \boldsymbol{R}_{m\bar{3}} := \frac{1}{\sqrt{2(\tau+2)}}\begin{bmatrix} 1 & \tau & 0 \\ \tau & 0 & 1 \\ 0 & 1 & \tau \\ -1 & \tau & 0 \\ \tau & 0 & -1 \\ 0 & -1 & \tau \end{bmatrix}, \quad \text{and}$$

$$\boldsymbol{R} = \boldsymbol{R}_{\bar{3}m} := \begin{bmatrix} \frac{2\tau+1}{\sqrt{6(4\tau+3)}} & \frac{-1}{\sqrt{2(\tau+2)}} & \frac{1}{\sqrt{6(\tau+2)}} \\ \frac{2\tau+1}{\sqrt{6(4\tau+3)}} & 0 & \frac{-2}{\sqrt{6(\tau+2)}} \\ \frac{2\tau+1}{\sqrt{6(4\tau+3)}} & \frac{1}{\sqrt{2(\tau+2)}} & \frac{1}{\sqrt{6(\tau+2)}} \\ \frac{1}{\sqrt{6(4\tau+3)}} & 0 & \frac{2\tau}{\sqrt{6(\tau+2)}} \\ \frac{1}{\sqrt{6(4\tau+3)}} & \frac{-\tau}{\sqrt{2(\tau+2)}} & \frac{-\tau}{\sqrt{6(\tau+2)}} \\ \frac{1}{\sqrt{6(4\tau+3)}} & \frac{\tau}{\sqrt{2(\tau+2)}} & \frac{-\tau}{\sqrt{6(\tau+2)}} \end{bmatrix}$$

are associated with icosahedral quasi-crystalline patterns exhibiting a pentagonal $\bar{5}m$-symmetry phase, a cubic $m\bar{3}$-symmetry phase, and a rhombohedral trigonal $\bar{3}m$-symmetry phase, respectively, where we adopted the terminology commonly used for the classification of quasicrystals' symmetry space groups (see, for instance, [59]). These icosahedral quasi-crystalline patterns are found in alloys within, for instance, the Al-Mn-Si, the Al-Li-Cu, and the Al-Cu-Fe ternary systems (see [36]).

In general, there are multiple choices for $\mathbb{m}$, σ, and $\boldsymbol{R}$, which could lead to some ambiguity in our asymptotic analysis. However, as proved in [8], the homogenization analysis does not depend on $\boldsymbol{R}$ provided it satisfies the following diophantine condition

$$\boldsymbol{R}^* k \neq 0 \ \text{ for all } k \in \mathbb{Z}^{\mathbb{m}}\backslash\{0\}, \tag{1.2}$$

where $\boldsymbol{R}^*$ denotes the transpose of $\boldsymbol{R}$. This condition implies that some entries of $\boldsymbol{R}$ must be irrational, which justifies the expression *irrational slope* used above.

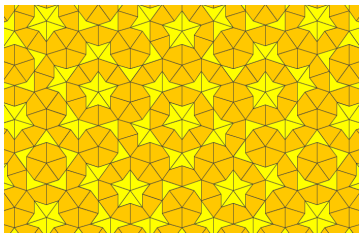

FIGURE 1. A quasi-crystalline heterogeneous microstructure corresponding to the so-called "kite and dart" tiling of the plane with five-fold symmetry. Image source: wikipedia.

Quasi-crystalline composites and alloys have played a central role in materials science and other areas of engineering [4, 6, 29, 37, 39, 40, 49, 61]. Indeed, Al-Cu-Fe quasi-crystalline materials in polymer-based composites have significantly shown to improve wear-resistance to volume loss, and a two-fold increase in the elastic moduli. As we mentioned before, the mathematical study of such quasi-crystalline composites does not fit within the *classical* periodic homogenization theory. More appropriate in the context of quasicrystal composites are almost-periodic and stochastic homogenization, which were initiated with the works of Papanicolaou and Varadhan [53], Kozlov [41], and Oleĭnik and Zhikov [52] for partial differential equations, and Dal Maso and Modica [23] and Braides [9, 10, 11] within a variational framework; see also [12, 17, 26] and the references therein. However, such approaches do not take full advantage of the quasi-crystalline feature of the problem, often leading to asymptotic formulas that pose computational difficulties and are not stable under perturbations. Instead, we adopt and further develop a homogenization procedure based on the two-scale-cut-and-project convergence introduced in [8], and recently revisited in [60], which leads to a more tractable (even if higher-dimensional) cell problem.

In this paper, we initiate a research program devoted to the study of quasi-crystalline homogenization problems involving oscillating integral energies under quasi-crystalline oscillating differential constraints, in the framework of $\mathcal{A}$-quasiconvexity. To be precise, we aim at characterizing the asymptotic behavior of integral energies of the form

$$F_\varepsilon(u_\varepsilon) := \int_\Omega f_R\Big(x, \frac{x}{\varepsilon^\alpha}, u_\varepsilon(x)\Big)\,\mathrm{d}x \tag{1.3}$$

as $\varepsilon \to 0^+$, where $\varepsilon^\alpha > 0$, with $\alpha \geqslant 0$, represents the length-scale of the tiles featuring the quasi-crystalline composite. Moreover, $\Omega \subset \mathbb{R}^{\mathfrak{n}}$ with, $\mathfrak{n} \in \mathbb{N}$, is an open and bounded set that represents the container occupied by the composite, and f_R is the Lagrangian of the system whose dependence in the second variable, the fast variable, encodes the quasi-crystalline structure of the composite, highlighted with the subscript R as in (1.1). Finally, u_ε is an abstract vector-valued order-parameter whose physical interpretation might depend on the problem in question. A typical case is that in which u_ε is curl-free, $u_\varepsilon = \nabla v_\varepsilon$ for some potential deformation v_ε. However, many applications require that u_ε instead satisfies other linear partial differential constraints, such as Maxwell's equations in the case of electromagnetism, or, in the case of linear elasticity, u_ε is the symmetric part of a gradient. A unified abstract approach to several of these constraints is that of $\mathcal{A}$-free fields, as pioneered by Fonseca and Müller [33] (see also [20, 21, 55]). To be precise, $u_\varepsilon \in L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$ is subject to quasi-crystalline oscillating differential constraints such as

$$\mathcal{A}_\varepsilon u_\varepsilon := \sum_{i=1}^{\mathfrak{n}} A_R^i\Big(\frac{\cdot}{\varepsilon^\beta}\Big)\frac{\partial u_\varepsilon}{\partial x_i}(\cdot) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^{\mathfrak{l}})$$

or, in divergence form,

$$\mathcal{A}_\varepsilon u_\varepsilon := \sum_{i=1}^{\mathfrak{n}} \frac{\partial}{\partial x_i}\Big(A_R^i\Big(\frac{\cdot}{\varepsilon^\beta}\Big)u_\varepsilon(\cdot)\Big) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^{\mathfrak{l}})$$

with $\mathfrak{d}, \mathfrak{l} \in \mathbb{N}$ and $1 < p < \infty$, where for every $x \in \mathbb{R}^{\mathfrak{n}}$, $A_R^i(x) \in \mathrm{Lin}(\mathbb{R}^{\mathfrak{d}};\mathbb{R}^{\mathfrak{l}})$ features a quasi-crystalline pattern, and $\beta \geqslant 0$ is a parameter. For the study of homogenization of integral energies with periodic

energy densities and under periodically oscillating $\mathcal{A}$-free differential constraints, we refer the reader to [13, 24, 25, 32, 42, 46].

As in the periodic setting [24, 25], we expect different asymptotic regimes according the ratio between α and β. As a starting point to this extensive research project, we first focus here on the case where $\beta = 0$ and A_R^i is independent of x, in which case u_ε is subjected to homogeneous first-order linear partial differential constraints. Precisely, in this manuscript we address the problem of characterizing the asymptotic behavior as $\varepsilon \to 0^+$ of integral energies of the form

$$F_\varepsilon(u) := \int_\Omega f_R\Big(x, \frac{x}{\varepsilon}, u(x)\Big)\,\mathrm{d}x \tag{1.4}$$

for $u \in L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$ satisfying $\mathcal{A}u = 0$, where

$$\mathcal{A}u := \sum_{i=1}^{\mathfrak{n}} A^{(i)} \frac{\partial u}{\partial x_i} \quad \text{with } A^{(i)} \in \mathbb{R}^{\mathfrak{l}\times\mathfrak{d}} \text{ for all } i \in \{1,...,\mathfrak{n}\}. \tag{1.5}$$

We refer to Section 2.2 for a rigorous definition of the identity $\mathcal{A}u = 0$, in which case we say that the vector field u is $\mathcal{A}$-free (see Definition 2.1). A common assumption within studies involving $\mathcal{A}$-free vector fields is the constant-rank property, which states that there exists $\boldsymbol{r} \in \mathbb{N}$ such that for all $w \in \mathbb{R}^{\mathfrak{n}} \setminus \{0\}$, we have

$$\operatorname{rank} \mathbb{A}(w) = \boldsymbol{r}, \tag{1.6}$$

where $\mathbb{A} : \mathbb{R}^{\mathfrak{n}} \to \mathbb{R}^{\mathfrak{l}\times\mathfrak{d}}$ denotes the symbol of $\mathcal{A}$, and is defined by

$$\mathbb{A}(w) := \sum_{i=1}^{\mathfrak{n}} A^{(i)} w_i \tag{1.7}$$

for $w \in \mathbb{R}^{\mathfrak{n}}$. We assume that our operator $\mathcal{A}$ satisfies the constant-rank property, and we refer the reader to [33, 48, 58] for further insights on this property and on $\mathcal{A}$-free fields.

Our asymptotic analysis of the energy integrals in (1.4) under the constraint (1.5) is based on Γ-convergence techniques, whose key point is to find an integral representation to

$$F_{\mathrm{hom}}(u) := \inf\Big\{ \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \colon\ u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^{\mathfrak{d}}),\ \ \mathcal{A}u_\varepsilon = 0\Big\}. \tag{1.8}$$

To state our main theorem regarding this integral representation, we first introduce the hypotheses on the Lagrangian, $f_R : \Omega \times \mathbb{R}^{\mathfrak{n}} \times \mathbb{R}^{\mathfrak{d}} \to [0,\infty)$:

(H1) (Quasi-crystallinity:) there exist $\mathfrak{m} \in \mathbb{N}$, with $\mathfrak{m} > \mathfrak{n}$, a matrix $\boldsymbol{R} \in \mathbb{R}^{\mathfrak{m}\times\mathfrak{n}}$ satisfying (1.2), and a continuous function $f : \Omega \times \mathbb{R}^{\mathfrak{m}} \times \mathbb{R}^{\mathfrak{d}} \to [0,\infty)$ such that the function $f(x,\cdot,\xi)$ is $Y^{\mathfrak{m}}$-periodic for each $(x,\xi) \in \Omega \times \mathbb{R}^{\mathfrak{d}}$, with $Y^{\mathfrak{m}}$ denoting a paralleletope in $\mathbb{R}^{\mathfrak{m}}$, and

$$f_R(x,z,\xi) = f(x,\boldsymbol{R}z,\xi)$$

for all $(x,z,\xi) \in \Omega \times \mathbb{R}^{\mathfrak{n}} \times \mathbb{R}^{\mathfrak{d}}$.

(H2) (Growth:) there exist $p \in (1,\infty)$ and $C > 0$ such that

$$0 \leqslant f_R(x,z,\xi) \leqslant C(1 + |\xi|^p)$$

for all $(x,z,\xi) \in \Omega \times \mathbb{R}^{\mathfrak{n}} \times \mathbb{R}^{\mathfrak{d}}$.

In the proof of the lower bound for the integral representation of F_{hom}, we will require, in addition,

(H3) (Convexity:) for all $(x,y) \in \Omega \times \mathbb{R}^{\mathfrak{m}}$, the function $\xi \mapsto f(x,y,\xi)$ is convex and C^1.

We refer the reader to Section 2 for a list of the main notations we use in this manuscript. However, for the readability of our main results, we clarify upfront that $L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}})$ denotes the space of $Y^{\mathfrak{m}}$-periodic functions belonging to $L^p_{\mathrm{loc}}(\mathbb{R}^{\mathfrak{m}})$. Moreover, given a Lebesgue measurable set $B \subset \mathbb{R}^{\mathfrak{k}}$, with $\mathfrak{k} \in \mathbb{N}$, we use the notation $⨍_B \cdot$ in place of $\frac{1}{\mathcal{L}^{\mathfrak{k}}(B)} \int_B \cdot$, where $\mathcal{L}^{\mathfrak{k}}(B)$ denotes the $\mathfrak{k}$-dimensional Lebesgue measure of B.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{\mathfrak{n}}$ be an open and bounded set, let $f_R : \Omega \times \mathbb{R}^{\mathfrak{n}} \times \mathbb{R}^{\mathfrak{d}} \to [0,\infty)$ be a function satisfying (H1)–(H3), let F_{hom} be the functional introduced in (1.8), and assume that (1.6) holds. Then, for all*

$$u \in \mathcal{U}_{\mathcal{A}} := \big\{u \in L^p(\Omega;\mathbb{R}^{\mathfrak{d}}) : \mathcal{A}u = 0\big\}, \tag{1.9}$$

we have

$$F_{\mathrm{hom}}(u) = \int_\Omega f_{\mathrm{hom}}(x,u(x))\,\mathrm{d}x,$$

where

$$f_{\hom}(x,\xi) := \inf_{v\in\mathcal{V}_{\mathcal{A}}} \fint_{Y^{\mathfrak{m}}} f(x,y,\xi+v(y))\,\mathrm{d}y$$

with

$$\mathcal{V}_{\mathcal{A}} := \left\{ v \in L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}) \colon v \text{ is } \mathcal{A}_{\boldsymbol{R}^*}\text{-free in the sense of Definition 3.11 and } \int_{Y^{\mathfrak{m}}} v(y)\,\mathrm{d}y = 0 \right\}. \tag{1.10}$$

Remark 1.2 (On the hypotheses of Theorem 1.1). (i) In the homogenization literature, measurability of f with respect to the fast-variable is often preferred over continuity. As we further discuss in Section 2.1, measurability of f_R requires, in general, Borel-measurability of f. A common approach to deal with lack of continuity is to combine periodicity with Scorza–Dragoni's type results that, up to a set of small measure, allow to reduce the problem to the continuity setting. Here, however, we cannot use such an argument because a set of small $\mathfrak{m}$-dimensional Lebesgue measure, the ambient space for the fast variable in terms of (the periodic function) f, may not have small $\mathfrak{n}$-dimensional Lebesgue, the ambient space for the fast variable in terms of (the quasi-crystalline function) f_R. (ii) The non-convex case raises non-trivial difficulties in the quasi-crystalline setting, and will be the subject of a forthcoming work.

In the Sobolev setting, homogenization of integral energies of the form (1.4) under *non-periodic* assumptions was undertaken in [12, 23, 38] in the $\mathcal{A}=\operatorname{curl}$ case, assuming coercivity. Within the quasi-crystalline framework, Theorem 1.1 extends these results to the general $\mathcal{A}$-free setting and without coercivity. We prove Theorem 1.1 in Section 4; the main tools we use here are based on Γ-convergence and on two-scale convergence adapted to the quasi-crystalline setting, also called two-scale-cut-and-project convergence. For brevity, and having in mind the relation (1.1), we refer to the two-scale-cut-and-project convergence as $\boldsymbol{R}$-two-scale convergence. This notion was introduced in [8] (also see [60]) as an extension of the usual notion of two-scale convergence [1, 51] to enable the study of composites whose underlying microstructure has a quasi-crystalline feature.

Here, we further extend the study of $\boldsymbol{R}$-two-scale convergence in two different ways. In [8, 60], the authors consider sequences in L^2 and their arguments are based on Fourier analysis relying heavily on Parseval's and Plancherel's identities. Instead, we consider the more general case of L^p with $p\in(1,\infty)$. Moreover, in [8] the authors characterize the limit, with respect to the $\boldsymbol{R}$-two-scale convergence, of bounded sequences in $W^{1,2}$, while in [60] the authors characterize the limit associated with bounded sequences in L^2 that are divergence-free or curl-free. Here, besides generalizing these results to the L^p case, we provide a unified approach to all these cases by considering bounded sequences in L^p that are $\mathcal{A}$-free, in the spirit of [32] concerning the periodic case.

Next, we state our main result regarding the characterization of the limits of bounded sequences in L^p that are $\mathcal{A}$-free. We refer the reader to Sections 2.2 and 3.1, where we give a precise meaning to the expressions "$\mathcal{A}$-free" and "$(\mathcal{A},\mathcal{A}^y_{\boldsymbol{R}^*})$-free" that we make use in this statement.

Theorem 1.3. *Let $\boldsymbol{R}\in\mathbb{R}^{\mathfrak{m}\times\mathfrak{n}}$ satisfy (1.2). A function $u\in L^p(\Omega\times Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})$ is the $\boldsymbol{R}$-two-scale limit of an $\mathcal{A}$-free sequence $\{u_\varepsilon\}_\varepsilon\subset L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$ if and only if u is $(\mathcal{A},\mathcal{A}^y_{\boldsymbol{R}^*})$-free in the sense of Definition 3.13; that is,*

$$\mathcal{A}\bar{u}_0 = 0 \quad \text{and} \quad \mathcal{A}^y_{\boldsymbol{R}^*}\bar{u}_1 = 0 \tag{1.11}$$

in the sense of Definition 2.1 and Definition 3.11, respectively, where $\bar{u}_0 := \fint_{Y^{\mathfrak{m}}} u(\cdot,y)\,\mathrm{d}y$ and $\bar{u}_1 := u-\bar{u}_0$.

We prove Theorem 1.3 in Section 3.1, where we use similar arguments to those in [32] concerning the periodic case (see [32, Theorem 2.12]). We observe that the sufficient part in Theorem 1.3, which guarantees that (1.11) fully characterizes the $\boldsymbol{R}$-two-scale limits, is new in the literature even for $p=2$ and $\mathcal{A}=\operatorname{curl}$ or $\mathcal{A}=\operatorname{div}$ treated in [8, 60]. Furthermore, in Section 5 we give an alternative proof of Theorem 1.3 for the $\mathcal{A}=\operatorname{curl}$ case using arguments based on Fourier analysis that differ from those in [8, 60] because Parseval's and Plancherel's identities do not hold for $p\neq 2$. This alternative proof provides an equivalent alterative characterization for the $\boldsymbol{R}$-two-scale limit of bounded sequences in $W^{1,p}$, and may provide useful arguments to study homogenization problems involving quasi-crystalline functionals in the $\mathcal{A}=\operatorname{curl}$ case. This alternative characterization can be stated as follows.

Theorem 1.4. *Let $\boldsymbol{R}\in\mathbb{R}^{\mathfrak{m}\times\mathfrak{n}}$ satisfy (1.2) and let $Y^{\mathfrak{m}}\subset\mathbb{R}^{\mathfrak{m}}$ be a parallelotope. Then, a function $v\in L^p(\Omega\times Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}})$ is the $\boldsymbol{R}$-two-scale limit of a sequence $\{\nabla v_\varepsilon\}_\varepsilon$ with $\{v_\varepsilon\}_\varepsilon$ bounded in $W^{1,p}(\Omega)$ if*

and only if there exist $v_0 \in W^{1,p}(\Omega)$ *and* $v_1 \in L^p(\Omega; \mathcal{G}^p_{\boldsymbol{R}})$ *such that*

$$v = \nabla v_0 + v_1,$$

where

$$\mathcal{G}^p_{\boldsymbol{R}} := \Big\{ w \in L^p_{\#}(Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{n}}) \colon \hat{w}_k = \lambda_k \boldsymbol{R}^* k \text{ for some } \{\lambda_k\}_{k \in \mathbb{Z}^{\mathbb{m}}} \subset \mathbb{C} \text{ with } \lambda_0 = 0 \Big\} \tag{1.12}$$

with $\hat{w}_k := \fint_{Y^{\mathbb{m}}} w(y) e^{-2\pi i k \cdot y} \, \mathrm{d}y$, $k \in \mathbb{Z}^{\mathbb{m}}$, *denoting the Fourier coefficients of* w.

Remark 1.5. We recall that if $u_\varepsilon \in L^p(\Omega; \mathbb{R}^{\mathbb{n}})$ is curl-free in $\mathbb{R}^{\mathbb{n}}$ with Ω simply connected, then there exists $v_\varepsilon \in W^{1,p}(\Omega)$ such that $u_\varepsilon = \nabla v_\varepsilon$. Thus, in terms of the notations in the two previous results with $\mathbb{d} = \mathbb{n}$, we have $\bar{u}_0 = \nabla v_0$ and $\bar{u}_1 = v_1$. In particular, (1.12) provides an alternative characterization of $\mathcal{A}_{\boldsymbol{R}^*}$- and $\mathcal{A}^y_{\boldsymbol{R}^*}$-free vector fields introduced in Definition 3.11 in the $\mathcal{A} =$ curl case (also see Remark 5.7 for a detailed argumentation).

2 Notation and Preliminaries

Throughout this manuscript, $\mathbb{m}, \mathbb{n} \in \mathbb{N}$ are such that $\mathbb{m} > \mathbb{n}$, $\Omega \subset \mathbb{R}^{\mathbb{n}}$ is an open and bounded set, $Y^{\mathbb{m}}$ is a parallelotope in $\mathbb{R}^{\mathbb{m}}$, $\Pi \subset \mathbb{R}^{\mathbb{n}}$ is a parallelotope in $\mathbb{R}^{\mathbb{n}}$, $\mathbb{d}, \mathbb{l} \in \mathbb{N}$, and $p, p' \in (1, \infty)$ are such that $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, we assume that ε takes values on an arbitrary sequence of positive numbers that converges to zero.

We use the subscript # within function spaces to highlight an underlying periodicity, in which case the domain indicates the periodicity cell. For instance, $C_{\#}(Y^{\mathbb{m}}) = \{u \in C(\mathbb{R}^{\mathbb{m}}) \colon u \text{ is } Y^{\mathbb{m}}\text{-periodic}\}$ and $L^p_{\#}(\Pi) = \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^{\mathbb{n}}) \colon u \text{ is } \Pi\text{-periodic}\}$.

Next, we compile the notation and main properties of the cut-and-project maps $\boldsymbol{R}$ and differential operators $\mathcal{A}$ introduced in the Introduction, and that we make use in the sequel.

2.1 Cut-and-project maps $\boldsymbol{R}$. In this paper, $\boldsymbol{R} : \mathbb{R}^{\mathbb{n}} \to \mathbb{R}^{\mathbb{m}}$ is a linear map, whose associated matrix in $\mathbb{R}^{\mathbb{m} \times \mathbb{n}}$ is also denoted by $\boldsymbol{R}$. We do not distinguish between the transpose matrix and the adjoint of $\boldsymbol{R}$, and denote both by $\boldsymbol{R}^*$. We often assume that the criterion (1.2) on $\boldsymbol{R}$ holds, in which case we refer to it explicitly.

As shown in [8], if $g : \mathbb{R}^{\mathbb{m}} \to \mathbb{R}$ is a $Y^{\mathbb{m}}$-periodic trigonometric polynomial, with $Y^{\mathbb{m}}$ a parallelotope in $\mathbb{R}^{\mathbb{m}}$, then the ergodic mean of $g \circ \boldsymbol{R}$, $\mathscr{M}(g \circ \boldsymbol{R})$, is uniquely defined provided that $\boldsymbol{R}$ satisfies (1.2), in which case we have

$$\mathscr{M}(g \circ \boldsymbol{R}) := \lim_{\tau \to \infty} \frac{1}{(2\tau)^{\mathbb{n}}} \int_{(-\tau, \tau)^{\mathbb{n}}} g(\boldsymbol{R}x) \, \mathrm{d}x = \fint_{Y^{\mathbb{m}}} g(y) \, \mathrm{d}y.$$

Throughout this manuscript, we consider functions σ_R as in (1.1). We observe that such definition raises measurability issues. In fact, we can only guarantee that σ_R in (1.1) is measurable provided that σ is Borel-measurable. We conjecture that there are functions $\sigma \in L^\infty(\mathbb{R}^{\mathbb{m}})$ for which the corresponding function σ_R in (1.1) is not measurable. This conjecture is based upon the observation that the pre-image of a measurable set $B \subset \mathbb{R}^{\mathbb{m}}$ through R, $R^{-1}(B)$, acts as a projection of the set B onto the lower-dimensional space $\mathbb{R}^{\mathbb{n}}$; moreover, as it is well-know, the projection of a measurable set may not be measurable. To overcome this issue, we take in (1.1) the Borel representative of σ.

2.2 Differential operators $\mathcal{A}$ with constant coefficients. We consider homogeneous first-order linear partial differential operators with constant coefficients, $\mathcal{A}$, that map $u : \Omega \to \mathbb{R}^{\mathbb{d}}$ into $\mathcal{A}u : \Omega \to \mathbb{R}^{\mathbb{l}}$, of the form

$$\mathcal{A}u := \sum_{i=1}^{\mathbb{n}} A^{(i)} \frac{\partial u}{\partial x_i} \quad \text{with } A^{(i)} \in \mathbb{R}^{\mathbb{l} \times \mathbb{d}} \text{ for all } i \in \{1, ..., \mathbb{n}\}.$$

The formal adjoint of $\mathcal{A}$, which we denote by $\mathcal{A}^*$, maps $v : \Omega \to \mathbb{R}^{\mathbb{l}}$ into $\mathcal{A}^* v : \Omega \to \mathbb{R}^{\mathbb{d}}$ and is defined by

$$\mathcal{A}^* v := -\sum_{i=1}^{\mathbb{n}} \big(A^{(i)}\big)^T \frac{\partial v}{\partial x_i}.$$

We observe that $\mathcal{A}$ can be viewed as a bounded, linear operator $\mathcal{A} : L^p(\Omega; \mathbb{R}^{\mathbb{d}}) \to W^{-1,p}(\Omega; \mathbb{R}^{\mathbb{l}})$ by setting

$$\langle \mathcal{A}u, v \rangle := \int_\Omega u \cdot \mathcal{A}^* v \, \mathrm{d}x$$

for all $u \in L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$ and $v \in W_0^{1,p'}(\Omega;\mathbb{R}^{\mathfrak{l}})$. We observe further that if $u \in C_c^1(\Omega;\mathbb{R}^{\mathfrak{d}})$ and $v \in C_c^1(\Omega;\mathbb{R}^{\mathfrak{l}})$, then

$$\int_\Omega \mathcal{A}u \cdot v \,\mathrm{d}x = \int_\Omega u \cdot \mathcal{A}^* v \,\mathrm{d}x$$

by integration by parts. Similarly, if $u \in C_\#^1(\Pi;\mathbb{R}^{\mathfrak{d}})$ and $v \in C_\#^1(\Pi;\mathbb{R}^{\mathfrak{l}})$, then

$$\int_\Pi \mathcal{A}u \cdot v \,\mathrm{d}x = \int_\Pi u \cdot \mathcal{A}^* v \,\mathrm{d}x.$$

We assume that $\mathcal{A}$ satisfies the constant-rank property, that is, there exists $\mathfrak{r} \in \mathbb{N}$ such that for all $w \in \mathbb{R}^{\mathfrak{n}} \setminus \{0\}$, we have $\operatorname{rank} \mathbb{A}(w) = \mathfrak{r}$, where $\mathbb{A} : \mathbb{R}^{\mathfrak{n}} \to \mathbb{R}^{\mathfrak{l}\times\mathfrak{d}}$ denotes the symbol of $\mathcal{A}$, and is defined by (1.7). As we mentioned in the Introduction, the constant-rank property is a common assumption within studies involving $\mathcal{A}$-free vector fields. We refer the reader to [33, 48, 58] for further insights on this property and on $\mathcal{A}$-free fields, whose notion we recall next.

Definition 2.1 ($\mathcal{A}$-free fields). *(i) Given $u \in L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$, we say that $\mathcal{A}u$ exists in $L^p(\Omega;\mathbb{R}^{\mathfrak{l}})$ if there exists a function $U \in L^p(\Omega;\mathbb{R}^{\mathfrak{l}})$ such that, for every $\phi \in C_c^1(\Omega;\mathbb{R}^{\mathfrak{l}})$, we have*

$$\int_\Omega u \cdot \mathcal{A}^*\phi \,\mathrm{d}x = \int_\Omega U \cdot \phi \,\mathrm{d}x. \tag{2.1}$$

In this case, we write $\mathcal{A}u := U$. We say that u is $\mathcal{A}$-free, and write $\mathcal{A}u = 0$, if (2.1) holds with $U = 0$.

(ii) Given $v \in L^p_\#(\Pi;\mathbb{R}^{\mathfrak{d}})$, we say that $\mathcal{A}v$ exists in $L^p_\#(\Pi;\mathbb{R}^{\mathfrak{l}})$ if there exists a function $V \in L^p_\#(\Pi;\mathbb{R}^{\mathfrak{l}})$ such that, for every $\varphi \in C_\#^1(\Pi;\mathbb{R}^{\mathfrak{l}})$, we have

$$\int_\Pi u \cdot \mathcal{A}^*\varphi \,\mathrm{d}y = \int_\Pi V \cdot \varphi \,\mathrm{d}y. \tag{2.2}$$

In this case, we write $\mathcal{A}v := V$. We say that v is $\mathcal{A}$-free, and write $\mathcal{A}v = 0$, if (2.2) holds with $V = 0$.

Remark 2.2 ($\mathcal{A}$ applied to vector fields depending on several variables). Whenever a vector field depends on two or more variables, we index $\mathcal{A}$ with the underlying variable to which $\mathcal{A}$ is being applied to the vector field. For instance, if $u = u(x, y)$, then $\mathcal{A}_x u$ refers to $\mathcal{A}$ applied to u as a function of x with y regarded as a fixed parameter. Similarly, $\mathcal{A}_y u$ refers to $\mathcal{A}$ applied to u as a function of y with x regarded as a fixed parameter.

A crucial result in the variational theory of $\mathcal{A}$-free fields is the following $\mathcal{A}$-free periodic extension lemma, established in [33, Lemma 2.15]. We make repeated use of a similar statement, also proved in [32, Lemma 2.8], and hence record it here for the readers' convenience.

Lemma 2.3 ($\mathcal{A}$-free periodic extension). *Let $\Pi \subset \mathbb{R}^{\mathfrak{n}}$ be a parallelotope, let $O \subset \Pi$ be an open set, let $1 < p < \infty$, and assume that $\mathcal{A}$ satisfies (1.6). Let $\{v_n\} \subset L^p(O;\mathbb{R}^{\mathfrak{d}})$ be a p-equiintegrable sequence in O, with $v_n \rightharpoonup 0$ in $L^p(O;\mathbb{R}^{\mathfrak{d}})$ and $\mathcal{A}v_n \to 0$ in $W^{-1,p}(O;\mathbb{R}^{\mathfrak{l}})$. Then, there exist an $\mathcal{A}$-free sequence $\{u_n\} \subset L^p_\#(\Pi;\mathbb{R}^{\mathfrak{d}})$, that is p-equiintegrable in Π, and a positive constat $C = C(\mathcal{A})$ such that*

$$\begin{aligned} &u_n - v_n \to 0 \text{ in } L^p(O;\mathbb{R}^{\mathfrak{d}}), \quad u_n \to 0 \text{ in } L^p(\Pi\setminus O;\mathbb{R}^{\mathfrak{d}}), \quad \fint_\Pi u_n \,\mathrm{d}y = 0, \\ &\|u_n\|_{L^p(\Pi;\mathbb{R}^{\mathfrak{d}})} \leqslant C\|v_n\|_{L^p(O;\mathbb{R}^{\mathfrak{d}})} \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Proof. The proof of this lemma with $\Pi = (0,1)^{\mathfrak{n}}$ can be found in [33, Lemma 2.15] and [32, Lemma 2.8]. The case in which Π is an arbitrary parallelotope follows by an affine change of variables. □

3 Cut-and-project-two-scale convergence

The notion of two-scale convergence was first introduced in the L^2 setting by Nguetseng [51], and further developed by Allaire [1]. Initially, it was used to provide a mathematical rigorous justification of the formal asymptotic expansions that are commonly adopted in the study of homogenization problems. Posteriorly, the notion of two-scale convergence was extended, in particular, to L^p, L^1, BV, and Besicovitch spaces [3, 15, 17, 31, 45], and also to the multiple-scales case [2, 30, 34], that enhanced several variational homogenization studies hinged on a Γ-convergence approach, such as [5, 19, 31, 34, 50].

In this section, we first address the study of the notion of two-scale convergence in the quasi-crystalline setting, which we refer to as cut-and-project-two-scale convergence (or, for brevity, $\boldsymbol{R}$-two-scale convergence), with $\boldsymbol{R}$ as in Section 2.1. We then prove Theorem 1.3.

As we mentioned in the Introduction, the $\boldsymbol{R}$-two-scale convergence was introduced in [8] (also see [60]) as an extension of the usual notion of two-scale convergence to enable the study of composites whose underlying microstructure has a quasi-crystalline feature. Using arguments based on Fourier analysis, the authors in [8] characterize the limit, with respect to the $\boldsymbol{R}$-two-scale convergence, of bounded sequences in $W^{1,2}$, while the authors in [60] characterize the limit associated with bounded sequences in L^2 that are divergence-free or curl-free. Here, besides generalizing these results to the L^p setting, with $1 < p < \infty$, we provide a unified approach to all these cases by considering bounded sequences in L^p that are $\mathcal{A}$-free, with $\mathcal{A}$ as in Section 2.2. Our arguments are close to those in [32] concerning the periodic case, and are hinged on properties of $\mathcal{A}$-free vector fields.

We first introduce the definition of $\boldsymbol{R}$-two-scale convergence in $L^p(\Omega;\mathbb{R}^{\Bbbk})$. We make use of the results in this section with $\Bbbk$ equal to either 1, $\mathbb{d}$, $\mathbb{l}$, or $\mathbb{n}$.

Definition 3.1 ($\boldsymbol{R}$-two-scale convergence). *We say that a sequence $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega;\mathbb{R}^{\Bbbk})$ $\boldsymbol{R}$-two-scale converges to a function $u \in L^p(\Omega \times Y^{\mathbb{m}};\mathbb{R}^{\Bbbk})$ if for all $\varphi \in L^{p'}(\Omega; C_\#(Y^{\mathbb{m}};\mathbb{R}^{\Bbbk}))$ we have*

$$\lim_{\varepsilon\to 0^+} \int_\Omega u_\varepsilon(x) \cdot \varphi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \fint_{Y^{\mathbb{m}}} u(x,y)\cdot\varphi(x,y)\,\mathrm{d}x\mathrm{d}y \tag{3.1}$$

and we write $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$.

Remark 3.2 (Uniqueness of $\boldsymbol{R}$-two-scale limits). There is uniqueness of the $\boldsymbol{R}$-two-scale limit. In fact, if $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega;\mathbb{R}^{\Bbbk})$ and u, $\tilde{u} \in L^p(\Omega\times Y^{\mathbb{m}};\mathbb{R}^{\Bbbk})$ are such that $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$ and $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} \tilde{u}$, then

$$\int_\Omega \fint_{Y^{\mathbb{m}}} (u(x,y)-\tilde{u}(x,y))\cdot\varphi(x,y)\,\mathrm{d}x\mathrm{d}y = 0$$

for all $\varphi \in L^{p'}(\Omega; C_\#(Y^{\mathbb{m}};\mathbb{R}^{\Bbbk}))$. Hence, $u = \tilde{u}$ a.e. in $\Omega\times Y^{\mathbb{m}}$.

Remark 3.3 (On the test functions for $\boldsymbol{R}$-two-scale convergence). Assume that $\{u_\varepsilon\}_\varepsilon$ is a bounded sequence in $L^p(\Omega;\mathbb{R}^{\Bbbk})$. Then, $\{u_\varepsilon\}_\varepsilon$ $\boldsymbol{R}$-two-scale converges to a function $u \in L^p(\Omega\times Y^{\mathbb{m}};\mathbb{R}^{\Bbbk})$ if and only if (3.1) holds for all $\varphi \in C_c^\infty(\Omega; C_\#^\infty(Y^{\mathbb{m}};\mathbb{R}^{\Bbbk}))$. To prove this statement, it suffices to use the density of $C_c^\infty(\Omega; C_\#^\infty(Y^{\mathbb{m}};\mathbb{R}^{\Bbbk}))$ in $L^{p'}(\Omega; C_\#(Y^{\mathbb{m}};\mathbb{R}^{\Bbbk}))$ and the boundedness of $\{u_\varepsilon\}_\varepsilon$ in $L^p(\Omega;\mathbb{R}^{\Bbbk})$.

The next two propositions characterize the relationship between the $\boldsymbol{R}$-two-scale limit and the usual weak and strong limits in $L^p(\Omega;\mathbb{R}^{\Bbbk})$. These two results are simple adaptations of [45, Theorems 9 and 10] concerning the periodic case (also see [8, Proposition 2.10] concerning the $p = 2$ case), which we include here for completeness.

Proposition 3.4. *Assume that $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega;\mathbb{R}^{\Bbbk})$ is a sequence that $\boldsymbol{R}$-two-scale converges to a function $u \in L^p(\Omega\times Y^{\mathbb{m}};\mathbb{R}^{\Bbbk})$. Then, $u_\varepsilon \rightharpoonup \bar{u}_0$ weakly in $L^p(\Omega;\mathbb{R}^{\Bbbk})$, where $\bar{u}_0(\cdot) := \fint_{Y^{\mathbb{m}}} u(\cdot,y)\,\mathrm{d}y$. In particular, $\{u_\varepsilon\}_\varepsilon$ is bounded in $L^p(\Omega;\mathbb{R}^{\Bbbk})$.*

Proof. Let $\phi \in L^{p'}(\Omega;\mathbb{R}^{\Bbbk})$, and set $\varphi(x,y) := \phi(x)$ for $(x,y) \in \Omega\times Y^{\mathbb{m}}$. Then, $\varphi \in L^{p'}(\Omega; C_\#(Y^{\mathbb{m}};\mathbb{R}^{\Bbbk}))$, and by (3.1) we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_\Omega u_\varepsilon(x)\cdot\phi(x)\,\mathrm{d}x &= \lim_{\varepsilon\to 0^+}\int_\Omega u_\varepsilon(x)\cdot\varphi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x\\ &= \int_\Omega\fint_{Y^{\mathbb{m}}} u(x,y)\cdot\varphi(x,y)\,\mathrm{d}x\mathrm{d}y = \int_\Omega\Big(\fint_{Y^{\mathbb{m}}} u(x,y)\,\mathrm{d}y\Big)\cdot\phi(x)\,\mathrm{d}x,\end{aligned}$$

and this concludes the proof. □

Proposition 3.5. *Let $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega;\mathbb{R}^{\Bbbk})$ and $u \in L^p(\Omega;\mathbb{R}^{\Bbbk})$ be such that $u_\varepsilon \to u$ in $L^p(\Omega;\mathbb{R}^{\Bbbk})$ as $\varepsilon \to 0^+$. Then, $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$.*

Proof. Let $\varphi \in L^{p'}(\Omega; C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk}))$. Using Hölder's inequality, the convergence $u_\varepsilon \to u$ in $L^p(\Omega; \mathbb{R}^{\Bbbk})$, and Proposition 3.7 applied to $\psi(x,y) := u(x) \cdot \varphi(x,y)$, we get

$$\begin{aligned}
&\limsup_{\varepsilon \to 0^+} \left| \int_\Omega u_\varepsilon(x) \cdot \varphi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \,\mathrm{d}x - \int_\Omega \fint_{Y^{\mathfrak{m}}} u(x) \cdot \varphi(x,y) \,\mathrm{d}x\mathrm{d}y \right| \\
&\quad \leqslant \limsup_{\varepsilon \to 0^+} \Bigg(\|u_\varepsilon - u\|_{L^p(\Omega;\mathbb{R}^{\Bbbk})} \|\varphi\|_{L^{p'}(\Omega; C_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\Bbbk}))} \\
&\qquad\qquad + \left| \int_\Omega u(x) \cdot \varphi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \,\mathrm{d}x - \int_\Omega \fint_{Y^{\mathfrak{m}}} u(x) \cdot \varphi(x,y) \,\mathrm{d}x\mathrm{d}y \right| \Bigg) \\
&\quad = 0.
\end{aligned}$$

□

The proof of the following version of Riemann–Lebesgue's lemma may be found in [8, Lemma 2.4]. This lemma will be used in the subsequent proposition, which encodes non-trivial examples of sequences that $\boldsymbol{R}$-two-scale converge.

Lemma 3.6 (c.f. [8, Lemma 2.4]). *Let $\phi \in C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk})$, and assume that $\boldsymbol{R}$ satisfies (1.2). Then, the sequence $\{\phi_\varepsilon\}_\varepsilon \subset L^\infty(\Omega; \mathbb{R}^{\Bbbk})$ defined by $\phi_\varepsilon(x) := \phi\big(\frac{\boldsymbol{R}x}{\varepsilon}\big)$, $x \in \Omega$, converges weakly-$\star$ in $L^\infty(\Omega; \mathbb{R}^{\Bbbk})$ to the constant function $\bar{\phi} := \fint_{Y^{\mathfrak{m}}} \phi(y) \,\mathrm{d}y$.*

As we mentioned above, the next proposition provides non-trivial examples of sequences that $\boldsymbol{R}$-two-scale converge; moreover, it will be useful to prove compactness of bounded sequences in $L^p(\Omega; \mathbb{R}^{\Bbbk})$ with respect to the $\boldsymbol{R}$-two-scale convergence. The proof is similar to that of [45, Lemma 2.5] concerning the periodic case, which we include here for completeness.

Proposition 3.7. *Let $\psi \in L^1(\Omega; C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk}))$, and assume that $\boldsymbol{R}$ satisfies (1.2). Then $\big\{\psi\big(\cdot, \frac{\boldsymbol{R}\cdot}{\varepsilon}\big)\big\}_\varepsilon$ is an equiintegrable sequence in $L^1(\Omega; \mathbb{R}^{\Bbbk})$ such that*

$$\left\| \psi\Big(\cdot, \frac{\boldsymbol{R}\cdot}{\varepsilon}\Big) \right\|_{L^1(\Omega;\mathbb{R}^{\Bbbk})} \leqslant \|\psi\|_{L^1(\Omega; C_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\Bbbk}))} = \int_\Omega \sup_{y \in Y^{\mathfrak{m}}} |\psi(x,y)| \,\mathrm{d}x \tag{3.2}$$

and

$$\lim_{\varepsilon \to 0^+} \int_\Omega \psi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \,\mathrm{d}x = \int_\Omega \fint_{Y^{\mathfrak{m}}} \psi(x,y) \,\mathrm{d}x\mathrm{d}y. \tag{3.3}$$

Proof. The proof of (3.2) is immediate. Using this estimate (that holds with Ω replaced by any measurable set) and the integrability of the map $x \in \Omega \mapsto \sup_{y \in Y^{\mathfrak{m}}} |\psi(x,y)|$, we conclude that $\big\{\psi\big(\cdot, \frac{\boldsymbol{R}(\cdot)}{\varepsilon}\big)\big\}_\varepsilon$ is equiintegrable in $L^1(\Omega; \mathbb{R}^{\Bbbk})$. Finally, the proof of (3.3) follows along the lines of that of [45, Lemma 2.5], which we detail next.

Step 1. Assume that ψ is of the form $\psi(x,y) = \varphi(x)\phi(y)$ with $\varphi \in L^1(\Omega)$ and $\phi \in C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk})$. Then, (3.3) follows from Lemma 3.6.

Step 2. Assume that ψ is of the form $\psi(x,y) = \sum_{k=1}^j c_k \chi_{A_k}(x) \phi_k(y)$, where $j \in \mathbb{N}$, c_k are distinct real numbers, A_k are mutually disjoint measurable subsets of Ω, and $\phi_k \in C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk})$. Then, (3.3) follows from Step 1.

Step 3. Let $\psi \in L^1(\Omega; C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk}))$. We can find a sequence $\{\psi_j\}_{j \in \mathbb{N}}$ of step functions as in Step 2 such that $\psi_j \to \psi$ in $L^1(\Omega; C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk}))$ as $j \to \infty$. Fix $j \in \mathbb{N}$; in view of (3.2), we have

$$\begin{aligned}
&\left| \int_\Omega \psi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \,\mathrm{d}x - \int_\Omega \fint_{Y^{\mathfrak{m}}} \psi(x,y) \,\mathrm{d}x\mathrm{d}y \right| \\
&\quad \leqslant \int_\Omega \left| \psi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) - \psi_j\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \right| \mathrm{d}x + \left| \int_\Omega \psi_j\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \,\mathrm{d}x - \int_\Omega \fint_{Y^{\mathfrak{m}}} \psi_j(x,y) \,\mathrm{d}x\mathrm{d}y \right| \\
&\qquad + \int_\Omega \fint_{Y^{\mathfrak{m}}} \big|\psi_j(x,y) - \psi(x,y)\big| \,\mathrm{d}x\mathrm{d}y \\
&\quad \leqslant \big(1 + [\mathcal{L}^{\mathfrak{m}}(Y^{\mathfrak{m}})]^{-1}\big) \|\psi - \psi_j\|_{L^1(\Omega; C_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\Bbbk}))} + \left| \int_\Omega \psi_j\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \,\mathrm{d}x - \int_\Omega \fint_{Y^{\mathfrak{m}}} \psi_j(x,y) \,\mathrm{d}x\mathrm{d}y \right|.
\end{aligned}$$

Letting $\varepsilon \to 0^+$ and using Step 2 first, and then letting $j \to \infty$, we obtain (3.3) from the convergence $\psi_j \to \psi$ in $L^1(\Omega; C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk}))$ as $j \to \infty$. □

Corollary 3.8. *Let $\psi \in L^p(\Omega; C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\Bbbk}))$, and assume that $\boldsymbol{R}$ satisfies (1.2). Then, $\big\{\psi\big(\cdot, \frac{\boldsymbol{R}\cdot}{\varepsilon}\big)\big\}_\varepsilon$ is a p-equiintegrable sequence in $L^p(\Omega; \mathbb{R}^{\Bbbk})$ that $\boldsymbol{R}$-two-scale converges to ψ.*

Proof. The p-equiintegrability assertion follows from Proposition 3.7 applied to $|\psi|^p$. The $\boldsymbol{R}$-two-scale convergence assertion follows from (3.3) with ψ replaced by $\psi\varphi$, where $\varphi \in L^{p'}(\Omega; C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{k}}))$ is an arbitrary function. □

Using the previous proposition, we establish next a compactness property with respect to the $\boldsymbol{R}$-two-scale convergence. The proof follows along the lines of that of [45, Theorem 14] in the context of the periodic case, which we include here for completeness..

Proposition 3.9. *Let $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega; \mathbb{R}^{\mathbb{k}})$ be a bounded sequence, and assume that $\boldsymbol{R}$ satisfies (1.2). Then, there exist a subsequence $\varepsilon' \preceq \varepsilon$ and a function $u \in L^p(\Omega \times Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{k}})$ such that $u_{\varepsilon'} \xrightarrow{\boldsymbol{R}\text{-}2sc} u$.*

Proof. To simplify the notation, set $X := L^{p'}(\Omega; C_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{k}}))$, and denote by X' the dual of X. Let $L_\varepsilon : X \to \mathbb{R}$ be the linear map defined, for $\varphi \in X$, by

$$L_\varepsilon(\varphi) := \int_\Omega u_\varepsilon(x) \cdot \varphi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)\, \mathrm{d}x.$$

By Hölder's inequality, we have $|L_\varepsilon(\varphi)| \leqslant c\|\varphi\|_X$, where $c := \sup_\varepsilon \|u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^{\mathbb{k}})}$. Thus, by the Riesz representation theorem, there exists $U_\varepsilon \in X'$ such that $\langle U_\varepsilon, \varphi\rangle_{X',X} = L_\varepsilon(\varphi)$ for all $\varphi \in X$. Next, we observe that X is separable and $\|U_\varepsilon\|_{X'} = \sup_{\varphi\in X, \|\varphi\|_X \leqslant 1} |\langle U_\varepsilon, \varphi\rangle_{X',X}| \leqslant c$. Hence, by the Alaoglu theorem, there exist a subsequence $\varepsilon' \preceq \varepsilon$ and a function $U \in X'$ such that $\lim_{\varepsilon'\to 0^+} \langle U_{\varepsilon'}, \varphi\rangle_{X',X} = \langle U, \varphi\rangle_{X',X}$ for all $\varphi \in X$. Passing the inequality

$$|\langle U_{\varepsilon'}, \varphi\rangle_{X',X}| = |L_{\varepsilon'}(\varphi)| \leqslant c\bigg(\int_\Omega \Big|\varphi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon'}\Big)\Big|^{p'} \,\mathrm{d}x\bigg)^{\frac{1}{p'}}$$

to the limit as $\varepsilon' \to 0^+$, and invoking Proposition 3.7 applied to $\psi(x,y) := |\varphi(x,y)|^{p'}$, we obtain

$$|\langle U, \varphi\rangle_{X',X}| \leqslant c\bigg(\int_\Omega \fint_{Y^{\mathfrak{m}}} |\varphi(x,y)|^{p'}\,\mathrm{d}x\mathrm{d}y\bigg)^{\frac{1}{p'}} = c\big[\mathcal{L}^{\mathfrak{m}}(Y^{\mathfrak{m}})\big]^{-\frac{1}{p'}} \|\varphi\|_{L^{p'}(\Omega\times Y^{\mathfrak{m}};\mathbb{R}^{\mathbb{k}})} \tag{3.4}$$

for all $\varphi \in X$. Finally, using the density of X in $L^{p'}(\Omega \times Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{k}})$, U can be continuously extended to $L^{p'}(\Omega\times Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{k}})$ with (3.4) valid for all $\varphi \in L^{p'}(\Omega\times Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{k}})$. Consequently, by the Riesz representation theorem there exists $\bar{u} \in L^p(\Omega \times Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{k}})$ such that, for all $\varphi \in L^{p'}(\Omega \times Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{k}})$,

$$\langle U, \varphi\rangle_{X',X} = \int_\Omega \int_{Y^{\mathfrak{m}}} \bar{u}(x,y) \cdot \varphi(x,y)\,\mathrm{d}x\mathrm{d}y.$$

In particular, this last identity holds for all $\varphi \in X$, from which we conclude the proof by taking $u := \mathcal{L}^{\mathfrak{m}}(Y^{\mathfrak{m}})\,\bar{u}$. □

Remark 3.10. As shown in [8, Remark 2.8], Proposition 3.9 may fail if there exists $k \in \mathbb{Z}^{\mathfrak{m}}\backslash\{0\}$ such that $\boldsymbol{R}^* k = 0$.

3.1 $\boldsymbol{R}$-two-scale limits of $\mathcal{A}$-free sequences.

In this subsection, we characterize the $\boldsymbol{R}$-two-scale limits associated with L^p-bounded sequences of $\mathcal{A}$-free vector fields, as stated in Theorem 1.3. As we will show, this characterization is intimately related to the notion of $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free vector fields introduced below.

Definition 3.11 ($\mathcal{A}_{\boldsymbol{R}^*}$- and $\mathcal{A}^y_{\boldsymbol{R}^*}$-free fields). *We say that $v \in L^p_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{d}})$ is $\mathcal{A}_{\boldsymbol{R}^*}$-free, and write $\mathcal{A}_{\boldsymbol{R}^*} v = 0$, if for all $\psi \in C^1_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{l}})$, we have*

$$\int_{Y^{\mathfrak{m}}} v(y) \cdot \mathcal{A}^*_{\boldsymbol{R}}\psi(y)\,\mathrm{d}y = 0, \tag{3.5}$$

where

$$\mathcal{A}^*_{\boldsymbol{R}} := -\sum_{i=1}^{\mathfrak{n}} \sum_{m=1}^{\mathfrak{m}} (A^{(i)})^T \boldsymbol{R}_{mi} \frac{\partial}{\partial y_m}.$$

We say that $w \in L^p(\Omega; L^p_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathbb{d}}))$, with $w = w(x,y)$, is $\mathcal{A}^y_{\boldsymbol{R}^*}$-free, and write $\mathcal{A}^y_{\boldsymbol{R}^*} w = 0$, if $\mathcal{A}_{\boldsymbol{R}^*} w(x,\cdot) = 0$ for a.e. $x \in \Omega$.

Remark 3.12 (On the notion of $\mathcal{A}_{\boldsymbol{R}^*}$-free). If $v \in C^1_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})$ satisfies (3.5), then integration by parts yields

$$\begin{aligned} 0 &= \int_{Y^{\mathfrak{m}}} v(y)\cdot \mathcal{A}^*_{\boldsymbol{R}}\psi(y)\,\mathrm{d}y = -\int_{Y^{\mathfrak{m}}} v(y)\cdot \sum_{i=1}^{\mathfrak{n}}\sum_{m=1}^{\mathfrak{m}} (A^{(i)})^T \boldsymbol{R}_{mi}\frac{\partial \psi}{\partial y_m}(y)\,\mathrm{d}y \\ &= -\int_{Y^{\mathfrak{m}}} \sum_{i=1}^{\mathfrak{n}}\sum_{m=1}^{\mathfrak{m}} \boldsymbol{R}^*_{im}A^{(i)}v(y)\cdot \frac{\partial \psi}{\partial y_m}(y)\,\mathrm{d}y = \int_{Y^{\mathfrak{m}}} \sum_{i=1}^{\mathfrak{n}}\sum_{m=1}^{\mathfrak{m}} \boldsymbol{R}^*_{im}A^{(i)}\frac{\partial v}{\partial y_m}(y)\cdot\psi(y)\,\mathrm{d}y \end{aligned}$$

for all $\psi \in C^1_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{l}})$. Thus, $\mathcal{A}_{\boldsymbol{R}^*}v = 0$ pointwise in $\mathbb{R}^{\mathfrak{m}}$, where $\mathcal{A}_{\boldsymbol{R}^*} := -\sum_{i=1}^{\mathfrak{n}}\sum_{m=1}^{\mathfrak{m}} \boldsymbol{R}^*_{im}A^{(i)}\frac{\partial}{\partial y_m}$.

We observe further that, as a consequence of our analysis in the Appendix (see Remark 5.7), in the $\mathcal{A}=\operatorname{curl}$ case, in $\mathbb{R}^{\mathfrak{n}}$, we have that $v \in L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}})$ is $\mathcal{A}_{\boldsymbol{R}^*}$-free if and only if $v \in \mathcal{G}^p_{\boldsymbol{R}}$, where $\mathcal{G}^p_{\boldsymbol{R}}$ is given by (1.12).

Definition 3.13 ($(\mathcal{A},\mathcal{A}^y_{\boldsymbol{R}^*})$-free fields). *Let $w \in L^p(\Omega;L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))$, and define $\bar{w}_0 \in L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$ and $\bar{w}_1 \in L^p(\Omega;L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))$ by setting $\bar{w}_0 := \fint_{Y^{\mathfrak{m}}} w(\cdot,y)\,\mathrm{d}y$ and $\bar{w}_1 := w - \bar{w}_0$. We say that w is $(\mathcal{A},\mathcal{A}^y_{\boldsymbol{R}^*})$-free if*

$$\mathcal{A}\bar{w}_0 = 0 \quad \text{and} \quad \mathcal{A}^y_{\boldsymbol{R}^*}\bar{w}_1 = 0$$

in the sense of Definition 2.1 and Definition 3.11, respectively.

The next proposition shows that the $\boldsymbol{R}$-two-scale limit of an L^p-bounded sequence of $\mathcal{A}$-free vector fields is necessarily $(\mathcal{A},\mathcal{A}^y_{\boldsymbol{R}^*})$-free. This result is the quasi-periodic counterpart of [32, Proposition 2.11] concerning the periodic case.

Proposition 3.14. *Let $\{u_\varepsilon\}_\varepsilon$ be a bounded and $\mathcal{A}$-free sequence in $L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$. Assume that there exists a function $u \in L^p(\Omega\times Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})$ such that $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$. Then, u is $(\mathcal{A},\mathcal{A}^y_{\boldsymbol{R}^*})$-free in the sense of Definition 3.13.*

Proof. Let $\phi \in C^1_c(\Omega;\mathbb{R}^{\mathfrak{l}})$. Using the fact that each u_ε is $\mathcal{A}$-free first, and invoking (3.1) applied to $\varphi := \mathcal{A}^*\phi$, we get

$$0 = \lim_{\varepsilon\to 0^+}\int_\Omega u_\varepsilon(x)\cdot\mathcal{A}^*\phi(x)\,\mathrm{d}x = \int_\Omega \fint_{Y^{\mathfrak{m}}} u(x,y)\cdot\mathcal{A}^*\phi(x)\,\mathrm{d}x\mathrm{d}y = \int_\Omega \bar{u}_0(x)\cdot\mathcal{A}^*\phi(x)\,\mathrm{d}x, \tag{3.6}$$

where $\bar{u}_0 := \fint_{Y^{\mathfrak{m}}} u(\cdot,y)\,\mathrm{d}y$. Recalling Definition 2.1, (3.6) shows that $\mathcal{A}\bar{u}_0 = 0$ in $L^p(\Omega;\mathbb{R}^{\mathfrak{l}})$.

Next, we prove that $\mathcal{A}^y_{\boldsymbol{R}^*}\bar{u}_1 = 0$ with $\bar{u}_1 := u - \bar{u}_0$. Let $\phi \in C^1_c(\Omega)$ and $\psi \in C^1_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{l}})$, and set $\varphi_\varepsilon(x) := \varepsilon\phi(x)\psi\big(\frac{\boldsymbol{R}x}{\varepsilon}\big)$ for $x\in\Omega$. Then $\varphi_\varepsilon \in C^1_c(\Omega;\mathbb{R}^{\mathfrak{l}})$ with

$$\begin{aligned} \mathcal{A}^*\varphi_\varepsilon(x) &= -\sum_{i=1}^{\mathfrak{n}}(A^{(i)})^T\frac{\partial\varphi_\varepsilon}{\partial x_i}(x) \\ &= -\varepsilon\sum_{i=1}^{\mathfrak{n}}\frac{\partial\phi}{\partial x_i}(x)(A^{(i)})^T\psi\Big(\frac{\boldsymbol{R}x}{\varepsilon}\Big) - \phi(x)\sum_{i=1}^{\mathfrak{n}}\Big[(A^{(i)})^T\sum_{m=1}^{\mathfrak{m}}\frac{\partial\psi}{\partial y_m}\Big(\frac{\boldsymbol{R}x}{\varepsilon}\Big)\boldsymbol{R}_{mi}\Big] \\ &= -\varepsilon\sum_{i=1}^{\mathfrak{n}}\frac{\partial\phi}{\partial x_i}(x)(A^{(i)})^T\psi\Big(\frac{\boldsymbol{R}x}{\varepsilon}\Big) - \phi(x)\sum_{i=1}^{\mathfrak{n}}\sum_{m=1}^{\mathfrak{m}}(A^{(i)})^T\boldsymbol{R}_{mi}\frac{\partial\psi}{\partial y_m}\Big(\frac{\boldsymbol{R}x}{\varepsilon}\Big). \end{aligned}$$

Hence, arguing as above, we have

$$\begin{aligned} 0 &= \lim_{\varepsilon\to0^+}\int_\Omega u_\varepsilon(x)\mathcal{A}^*\varphi_\varepsilon(x)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to0^+}\int_\Omega u_\varepsilon(x)\cdot\Big(-\varepsilon\sum_{i=1}^{\mathfrak{n}}\frac{\partial\phi}{\partial x_i}(x)(A^{(i)})^T\psi\Big(\frac{\boldsymbol{R}x}{\varepsilon}\Big) - \phi(x)\sum_{i=1}^{\mathfrak{n}}\sum_{m=1}^{\mathfrak{m}}(A^{(i)})^T\boldsymbol{R}_{mi}\frac{\partial\psi}{\partial y_m}\Big(\frac{\boldsymbol{R}x}{\varepsilon}\Big)\Big)\,\mathrm{d}x \\ &= -\int_\Omega\fint_{Y^{\mathfrak{m}}} u(x,y)\cdot\Big(\phi(x)\sum_{i=1}^{\mathfrak{n}}\sum_{m=1}^{\mathfrak{m}}(A^{(i)})^T\boldsymbol{R}_{mi}\frac{\partial\psi}{\partial y_m}(y)\Big)\,\mathrm{d}x\mathrm{d}y \\ &= -\int_\Omega\fint_{Y^{\mathfrak{m}}} u(x,y)\cdot\phi(x)\mathcal{A}^*_{\boldsymbol{R}}\psi(y)\,\mathrm{d}x\mathrm{d}y = -\int_\Omega\fint_{Y^{\mathfrak{m}}}\bar{u}_1(x,y)\cdot\phi(x)\mathcal{A}^*_{\boldsymbol{R}}\psi(y)\,\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{3.7}$$

where in the last equality we used the fact that $\bar{u}_0$ depends only on x and $\int_{Y^{\mathfrak{m}}}\mathcal{A}^*_{\boldsymbol{R}}\psi(y)\,\mathrm{d}y = 0$ by the periodicity of ψ.

Because (3.7) holds for all $\phi \in C_c^1(\Omega)$ and $\psi \in C^1_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{l}})$ and $C^1_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{l}})$ is separable, we conclude that $\mathcal{A}^y_{\boldsymbol{R}^*}\bar{u}_1 = 0$ in the sense of Definition 3.11. □

The next proposition shows that Proposition 3.14 fully characterizes the $\boldsymbol{R}$-two-scale limit of an L^p-bounded sequence of $\mathcal{A}$-free vector fields, as we prove that any $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free vector field is attained as the $\boldsymbol{R}$-two-scale limit of such a sequence. This result is the quasi-periodic counterpart of [32, Proposition 2.11] concerning the periodic case. Moreover, as we mentioned in the Introduction, this result is new in the literature even for $p = 2$ and $\mathcal{A} = \operatorname{curl}$ or $\mathcal{A} = \operatorname{div}$ which were treated in [8, 60].

Proposition 3.15. *Let $u \in L^p(\Omega; L^p_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))$ be a $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free vector field in the sense of Definition 3.13, and assume that $\boldsymbol{R}$ satisfies (1.2). Then, there exists a bounded and $\mathcal{A}$-free sequence, $\{u_\varepsilon\}_\varepsilon$, in $L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$ such that $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$.*

Proof. Fix $u \in L^p(\Omega; L^p_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))$, a $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free vector field. We have

$$\mathcal{A}\bar{u}_0 = 0 \quad \text{and} \quad \mathcal{A}^y_{\boldsymbol{R}^*}\bar{u}_1 = 0 \tag{3.8}$$

in the sense of Definition 2.1 and Definition 3.11, respectively, where $\bar{u}_0 := \fint_{Y^{\mathfrak{m}}} u(\cdot, y)\,dy$ and $\bar{u}_1 := u - \bar{u}_0$. Note that for a.e. $x \in \Omega$, it holds

$$\int_{Y^{\mathfrak{m}}} \bar{u}_1(x, y)\,dy = 0. \tag{3.9}$$

We will proceed in three steps.

Step 1. Assume that $\bar{u}_0 = 0$ and $\bar{u}_1 \in C^1_c(\mathbb{R}^{\mathfrak{n}}; C^1_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))$. In this case, (3.9) holds for all $x \in \Omega$ and, as observed in Remark 3.12, we have

$$\mathcal{A}_{\boldsymbol{R}^*}\bar{u}_1 = 0 \text{ pointwise in } \Omega \times \mathbb{R}^{\mathfrak{m}}, \text{ where } \mathcal{A}_{\boldsymbol{R}^*} = -\sum_{i=1}^{\mathfrak{n}}\sum_{m=1}^{\mathfrak{m}} \boldsymbol{R}^*_{im} A^{(i)} \frac{\partial}{\partial y_m}. \tag{3.10}$$

For each $\varepsilon > 0$, define $v_\varepsilon \in C^1_c(\mathbb{R}^{\mathfrak{n}})$ by setting

$$v_\varepsilon(x) := \bar{u}_1\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \quad \text{for } x \in \mathbb{R}^{\mathfrak{n}}. \tag{3.11}$$

By Corollary 3.8 and Proposition 3.4, together with (3.9), we obtain

$$\begin{aligned} &\{v_\varepsilon\}_\varepsilon \text{ is a } p\text{-equiintegrable sequence in } L^p(\Omega;\mathbb{R}^{\mathfrak{d}}),\\ &v_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} \bar{u}_1,\\ &v_\varepsilon \rightharpoonup 0 \text{ weakly in } L^p(\Omega;\mathbb{R}^{\mathfrak{d}}). \end{aligned} \tag{3.12}$$

On the other hand, in view of (3.10) and recalling Remark 2.2, we have

$$\mathcal{A}v_\varepsilon(x) = (\mathcal{A}_x\bar{u}_1)\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)$$

for all $x \in \Omega$. Because $\mathcal{A}_x\bar{u}_1 \in C_c(\mathbb{R}^{\mathfrak{n}}; C^1_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{l}}))$, we may invoke Proposition 3.4 and (3.9) once more to conclude that

$$\mathcal{A}v_\varepsilon \rightharpoonup \fint_{Y^{\mathfrak{m}}} (\mathcal{A}_x\bar{u}_1)(x, y)\,dy = \mathcal{A}_x\Big(\fint_{Y^{\mathfrak{m}}} \bar{u}_1(x, y)\,dy\Big) = 0 \text{ weakly in } L^p(\Omega;\mathbb{R}^{\mathfrak{l}}).$$

Hence,

$$\mathcal{A}v_\varepsilon \to 0 \text{ in } W^{-1,p}(\Omega;\mathbb{R}^{\mathfrak{l}}). \tag{3.13}$$

Let $\Pi \subset \mathbb{R}^{\mathfrak{n}}$ be a parallelotope containing Ω. By Lemma 2.3, we can find a p-equiintegrable sequence in Π, $\{u_\varepsilon\}_\varepsilon \subset L^p(\Pi;\mathbb{R}^{\mathfrak{d}})$, and a positive constant depending only on $\mathcal{A}$, $C = C(\mathcal{A})$, such that

$$\begin{aligned} &\mathcal{A}u_\varepsilon = 0 \text{ for all } \varepsilon > 0,\\ &u_\varepsilon - v_\varepsilon \to 0 \text{ in } L^p(\Omega;\mathbb{R}^{\mathfrak{d}}),\\ &\|u_\varepsilon\|_{L^p(\Pi;\mathbb{R}^{\mathfrak{d}})} \leqslant C\|v_\varepsilon\|_{L^p(\Omega;\mathbb{R}^{\mathfrak{d}})} \text{ for all } \varepsilon > 0. \end{aligned} \tag{3.14}$$

To conclude Step 1, we observe that the second condition in (3.12) and (3.14), together with Proposition 3.5, yield

$$u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} \bar{u}_1. \tag{3.15}$$

Step 2. Assume that $\bar{u}_0 = 0$ and $\bar{u}_1 \in L^p(\Omega; L^p_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))$.

For all $y \in \mathbb{R}^{\mathfrak{m}}$, we extend $\bar{u}_1(\cdot, y)$ by zero outside Ω, which we still denote by $\bar{u}_1$. Let $\{\rho_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^{\mathfrak{n}})$ and $\{\rho_j^{\#}\}_{j\in\mathbb{N}} \subset C_{\#}^\infty(Y^{\mathfrak{m}})$ be sequences of standard, symmetric, mollifiers. For each $j \in \mathbb{N}$, we define

$$\begin{aligned}\tilde{u}_j(x,y) &:= \int_{\mathbb{R}^{\mathfrak{n}}}\int_{Y^{\mathfrak{m}}} \bar{u}_1(x',y')\rho_j(x-x')\rho_j^{\#}(y-y')\,\mathrm{d}x'\mathrm{d}y' \\ &= \int_{\mathbb{R}^{\mathfrak{n}}}\int_{Y^{\mathfrak{m}}} \bar{u}_1(x',y'+y)\rho_j(x-x')\rho_j^{\#}(y')\,\mathrm{d}x'\mathrm{d}y,\end{aligned}$$

where in the last equality we used the $Y^{\mathfrak{m}}$-periodicity of $\bar{u}_1$ along with the symmetry and $Y^{\mathfrak{m}}$-periodicity of $\rho_j^{\#}$. By standard mollification arguments, we have $\tilde{u}_j \in C_c^\infty(\mathbb{R}^{\mathfrak{n}}; C_{\#}^\infty(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))$ with

$$\|\tilde{u}_j\|_{L^p(\mathbb{R}^{\mathfrak{n}};L^p(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))} \leqslant \|\bar{u}_1\|_{L^p(\Omega;L^p(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))}.$$

Moreover,

$$\mathcal{A}_{\boldsymbol{R}^*}\tilde{u}_j = 0 \text{ pointwise in } \Omega\times\mathbb{R}^{\mathfrak{m}} \quad \text{and} \quad \int_{Y^{\mathfrak{m}}} \tilde{u}_j(\cdot,y)\,\mathrm{d}y = 0$$

by (3.8) and (3.9), together with the $Y^{\mathfrak{m}}$-periodicity of $\bar{u}_1$ and Fubini's theorem.

By Step 1, for each $j \in \mathbb{N}$, we can find a p-equiintegrable sequence in Π, $\{u_\varepsilon^{(j)}\} \subset L^p(\Pi;\mathbb{R}^{\mathfrak{d}})$, satisfying (3.14)–(3.15) with u_ε replaced by $u_\varepsilon^{(j)}$ and, recalling (3.11), v_ε replaced by

$$v_\varepsilon^{(j)}(x) := \tilde{u}_j\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \quad \text{for } x \in \mathbb{R}^{\mathfrak{n}}.$$

In particular, we have

$$\begin{aligned}\limsup_{j\to\infty}\limsup_{\varepsilon\to0^+}\int_\Omega \big|u_\varepsilon^{(j)}(x)\big|^p\,\mathrm{d}x &\leqslant C^p \limsup_{j\to\infty}\limsup_{\varepsilon\to0^+}\int_\Omega \Big|\tilde{u}_j\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\Big|^p\,\mathrm{d}x \\ &= C^p\limsup_{j\to\infty}\int_\Omega \fint_{Y^{\mathfrak{m}}} |\tilde{u}_j(x,y)|^p\,\mathrm{d}x\mathrm{d}y \\ &\leqslant \mathcal{L}^{\mathfrak{m}}(Y^{\mathfrak{m}})C^p\|\bar{u}_1\|^p_{L^p(\Omega;L^p(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))}.\end{aligned}$$

This estimate and the separability of $L^{p'}(\Omega; C_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}))$ allow us to use a diagonalization argument as in [30, proof of Proposition 1.11 (p.449)] to find a sequence $(j_\varepsilon)_\varepsilon$ such that $j_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ and $u_\varepsilon := u_\varepsilon^{(j_\varepsilon)}$ satisfies the required properties.

Step 3. We treat the general case.

By Step 2, there exists a bounded and $\mathcal{A}$-free sequence, $\{u_\varepsilon\}_\varepsilon$, in $L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$ such that $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u_1$. Defining $\hat{u}_\varepsilon := u_0 + u_\varepsilon$, we have $\mathcal{A}\hat{u}_\varepsilon = 0$ and $\hat{u}_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u_0 + u_1 = u$, using (3.8) and Proposition 3.5. □

Proof of Theorem 1.3. The statement in Theorem 1.3 in an immediate consequence of Propositions 3.14 and 3.15. □

4 Γ-convergence homogenization

In this section, we prove Theorem 1.1. To this end, we first show in Theorem 4.1 below that the sequence $\{F_\varepsilon\}_\varepsilon$, with F_ε given by (1.4), Γ-converges to a certain functional, $\mathcal{F}_{\mathrm{hom}}$, with respect to the weak topology in $L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$, as $\varepsilon \to 0^+$. Then, in Proposition 4.6 below, we establish the integral representation for this Γ-limit as stated in Theorem 1.1.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^{\mathfrak{n}}$ be an open and bounded set, let $f_R : \Omega\times\mathbb{R}^{\mathfrak{n}}\times\mathbb{R}^{\mathfrak{d}} \to [0,\infty)$ be a function satisfying (H1)–(H3), let F_ε be the functional introduced in (1.4), and assume that (1.6) holds. Then, the sequence $\{F_\varepsilon\}_\varepsilon$ Γ-converges on $\mathcal{U}_{\mathcal{A}} = \{u \in L^p(\Omega;\mathbb{R}^{\mathfrak{d}}) : \mathcal{A}u = 0\}$ as $\varepsilon \to 0^+$, with respect to the weak topology in $L^p(\Omega;\mathbb{R}^{\mathfrak{d}})$, to the functional $\mathcal{F}_{\mathrm{hom}}$ defined, for $u \in \mathcal{U}_{\mathcal{A}}$, by*

$$\mathcal{F}_{\mathrm{hom}}(u) := \inf_{w\in\mathcal{W}_{\mathcal{A}}}\int_\Omega \fint_{Y^{\mathfrak{m}}} f(x,y,u(x)+w(x,y))\,\mathrm{d}x\mathrm{d}y,$$

where

$$\mathcal{W}_{\mathcal{A}} := \Big\{ w \in L^p(\Omega; L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})) : w \text{ is } (\mathcal{A},\mathcal{A}^y_{\boldsymbol{R}^*})\text{-free in the sense of Definition 3.13, with } \int_{Y^{\mathfrak{m}}} w(\cdot,y)\,\mathrm{d}y = 0\Big\}. \tag{4.1}$$

Precisely, given an arbitrary sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ converging to 0, the following pair of statements holds:

1. *(Γ-liminf inequality) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\mathcal{A}}$ be a sequence such that $u_n \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^{\mathtt{d}})$ for some $u \in L^p(\Omega;\mathbb{R}^{\mathtt{d}})$. Then, $u \in \mathcal{U}_{\mathcal{A}}$ and*
$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) \geqslant \mathcal{F}_{\rm hom}(u).$$
2. *(recovery sequence) For every $u \in \mathcal{U}_{\mathcal{A}}$, there exists sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\mathcal{A}}$ such that $u_n \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^{\mathtt{d}})$ and*
$$\limsup_{n\to\infty} F_{\varepsilon_n}(u_n) \leqslant \mathcal{F}_{\rm hom}(u).$$

The proof of Theorem 4.1 is obtained as a consequence of Propositions 4.3 and 4.4 below. We begin with a lemma that will be used in the subsequent proposition, where we establish the *recovery sequence* property, and is a simple adaptation of [32, Proposition 3.5-(i)].

Lemma 4.2. *Assume that hypotheses (H1)–(H2) hold. Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be a sequence converging to 0, and let $\{u_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{\mathtt{d}})$ be two p-equiintegrable sequences such that $\lim_{n\to\infty} \|u_n - w_n\|_{L^p(\Omega;\mathbb{R}^{\mathtt{d}})} = 0$. Then,*
$$\lim_{n\to\infty} \int_\Omega \left[f_R\Big(x, \frac{x}{\varepsilon_n}, u_n(x)\Big)\,\mathrm{d}x - f_R\Big(x, \frac{x}{\varepsilon_n}, w_n(x)\Big)\right] \mathrm{d}x = 0.$$

Proof. Fix $\tau > 0$. We want to show that there exists $n_0 = n_0(\tau) \in \mathbb{N}$ such that if $n \geqslant n_0$, then
$$\left| \int_\Omega \left[f_R\Big(x, \frac{x}{\varepsilon_n}, u_n(x)\Big)\,\mathrm{d}x - f_R\Big(x, \frac{x}{\varepsilon_n}, w_n(x)\Big)\right] \mathrm{d}x \right| \leqslant \tau.$$

Using the p-equiintegrability of $\{u_n\}$ and $\{w_n\}$, there exists $\delta = \delta(\tau) > 0$ such that if $E \subset \Omega$ is a measurable set with $|E| < \delta$, then
$$\sup_{n\in\mathbb{N}} \int_E C\big(2 + |u_n(x)|^p + |w_n(x)|^p\big)\,\mathrm{d}x < \frac{\tau}{8}. \tag{4.2}$$
Moreover, there exists $r_\delta > 0$ such that
$$\sup_{n\in\mathbb{N}} \Big[\big|\{|u_n| \geqslant r_\delta\}\big| + \big|\{|w_n| \geqslant r_\delta\}\big| \Big] \leqslant \delta. \tag{4.3}$$

Let $\Omega_\delta \Subset \Omega$ be such that $|\Omega \setminus \Omega_\delta| \leqslant \delta$. Using the continuity assumption on f and the $Y^{\mathtt{m}}$-periodicity of f with respect to its second variable, we conclude that f is uniformly continuous on $\overline{\Omega}_\delta \times \mathbb{R}^{\mathtt{m}} \times \overline{B}_{r_\delta}(0)$. Thus, we can find $0 < \bar{\delta} \leqslant \delta$ such that, for all $x \in \Omega_\delta$, $y \in \mathbb{R}^{\mathtt{m}}$, and ξ_1, $\xi_2 \in B_{r_\delta}(0)$ with $|\xi_1 - \xi_2| \leqslant \bar{\delta}$, we have
$$\big|f(x,y,\xi_1) - f(x,y,\xi_2)\big| \leqslant \frac{\tau}{2|\Omega_\delta|}. \tag{4.4}$$

Finally, by Chebyshev's inequality, there exists $0 < \tilde{\delta} < \bar{\delta}$ such that if $\|v\|_{L^p(\Omega;\mathbb{R}^{\mathtt{d}})} < \tilde{\delta}$, then
$$\big|\{|v| \geqslant \bar{\delta}\}\big| \leqslant \delta. \tag{4.5}$$

We observe further that because $\lim_{n\to\infty} \|u_n - w_n\|_{L^p(\Omega;\mathbb{R}^{\mathtt{d}})} = 0$, we can find $n_0 = n_0(\tau) \in \mathbb{N}$ such that $\|u_n - w_n\|_{L^p(\Omega;\mathbb{R}^{\mathtt{d}})} < \tilde{\delta}$ for all $n \geqslant n_0$.

Thus, for each $n \geqslant n_0$ and for $A := (\Omega \setminus \Omega_\delta) \cup \{|u_n| \geqslant r_\delta\} \cup \{|w_n| \geqslant r_\delta\} \cup \{|u_n - w_n| \geqslant \bar{\delta}\}$, we conclude from (H2), (H3), and (4.2)–(4.5) that
$$\begin{aligned}
&\left| \int_\Omega \left[f_R\Big(x, \frac{x}{\varepsilon_n}, u_n(x)\Big)\,\mathrm{d}x - f_R\Big(x, \frac{x}{\varepsilon_n}, w_n(x)\Big)\right] \mathrm{d}x \right| \\
&\quad\leqslant \int_A C\big(2 + |u_n(x)|^p + \ |w_n(x)|^p\big)\,\mathrm{d}x + \left| \int_{\Omega\setminus A} \left[f\Big(x, \frac{\boldsymbol{R}x}{\varepsilon_n}, u_n(x)\Big)\,\mathrm{d}x - f\Big(x, \frac{\boldsymbol{R}x}{\varepsilon_n}, w_n(x)\Big)\right] \mathrm{d}x \right| \\
&\quad\leqslant \frac{\tau}{2} + \frac{\tau}{2|\Omega_\delta|}|\Omega \setminus A| \leqslant \tau.
\end{aligned}$$
□

The *recovery sequence* property in Theorem 4.1 is a simple consequence of the following proposition. We observe that this result does not require assumption (H3) to hold, and is the quasi-periodic counterpart of [32, Proposition 2.7] concerning the periodic case.

Proposition 4.3. *Assume that hypotheses (H1)–(H2) hold, and let $\mathcal{U}_{\mathcal{A}}$ be the set introduced in (1.9). Then, for each $\delta > 0$, $u \in \mathcal{U}_{\mathcal{A}}$, and $w \in \overline{\mathcal{W}}_{\mathcal{A}} := \big\{w \in L^p(\Omega; L^p_{\#}(Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{d}})) : w \text{ is } (\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})\text{-free in the sense of Definition 3.13}\big\}$, there exists a sequence $\{u_\varepsilon\} \subset \mathcal{U}_{\mathcal{A}}$ such that $u_\varepsilon \rightharpoonup u + \bar{w}_0$ weakly in $L^p(\Omega; \mathbb{R}^{\mathbb{d}})$ as $\varepsilon \to 0^+$ and, for all $\kappa \in \mathbb{N}$,*

$$\lim_{\varepsilon \to 0^+} \int_\Omega f_R\Big(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\Big)\,\mathrm{d}x \leqslant \int_\Omega \fint_{Y^{\mathbb{m}}} f(x, \kappa y, u(x) + w(x,y))\,\mathrm{d}y\mathrm{d}x + \delta, \tag{4.6}$$

where, recalling Definition 3.13, $\bar{w}_0 := \fint_{Y^{\mathbb{m}}} w(\cdot, y)\,\mathrm{d}y$.

Proof. Fix $\delta > 0$, $u \in \mathcal{U}_{\mathcal{A}}$, and $w \in \overline{\mathcal{W}}_{\mathcal{A}}$. We will proceed in two steps, first assuming extra regularity on w, and then treating the general case.

Step 1. Recalling the decomposition $w = \bar{w}_0 + \bar{w}_1$ introduced in Definition 3.13, assume that $\bar{w}_1 \in C^1(\overline{\Omega}; C^1_{\#}(Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{d}}))$.

For $\kappa \in \mathbb{N}$ and $(x, y) \in \Omega \times Y^{\mathbb{m}}$, define

$$\psi(x,y) := f(x, \kappa y, u(x) + w(x,y)) = f(x, \kappa y, u(x) + \bar{w}_0(x) + \bar{w}_1(x,y)).$$

Using (H1), (H2), the continuity of f, and the regularity of $\bar{w}_1$, we conclude that $\psi \in L^1(\Omega; C_{\#}(Y^{\mathbb{m}}))$. Then, by Proposition 3.7, we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega \psi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \fint_{Y^{\mathbb{m}}} \psi(x,y)\,\mathrm{d}x\mathrm{d}y;$$

i.e.,

$$\lim_{\varepsilon \to 0^+} \int_\Omega f_R\Big(x, \frac{x}{\varepsilon}, w_\varepsilon(x)\Big)\,\mathrm{d}x = \int_\Omega \fint_{Y^{\mathbb{m}}} f(x, \kappa y, u(x) + w(x,y))\,\mathrm{d}y\mathrm{d}x, \tag{4.7}$$

where, for $x \in \Omega$,

$$w_\varepsilon(x) := u(x) + \bar{w}_0(x) + \bar{w}_1\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big).$$

Arguing as in Step 1 of the proof of Proposition 3.15 with $\bar{u}_1$ replaced by $\bar{w}_1$ in (3.11), and using the fact that $\mathcal{A}u + \mathcal{A}\bar{w}_0 = 0$ by the definition of $\mathcal{U}_{\mathcal{A}}$ and $\overline{\mathcal{W}}_{\mathcal{A}}$, we conclude that (see (3.12)–(3.13))

$$\begin{aligned}
&\{w_\varepsilon\}_\varepsilon \text{ is a } p\text{-equiintegrable sequence in } L^p(\Omega; \mathbb{R}^{\mathbb{d}}),\\
&w_\varepsilon \rightharpoonup u + \bar{w}_0 \text{ weakly in } L^p(\Omega; \mathbb{R}^{\mathbb{d}}),\\
&\mathcal{A}w_\varepsilon \to 0 \text{ in } W^{-1,p}(\Omega; \mathbb{R}^{\mathbb{l}}).
\end{aligned}$$

Then, by Lemma 2.3, we can find a sequence $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega; \mathbb{R}^{\mathbb{d}})$ such that

$$\begin{aligned}
&\{u_\varepsilon\}_\varepsilon \text{ is } p\text{-equiintegrable},\\
&\mathcal{A}u_\varepsilon = 0 \text{ in } L^p(\Omega; \mathbb{R}^{\mathbb{l}}),\\
&u_\varepsilon - w_\varepsilon \to 0 \text{ in } L^p(\Omega; \mathbb{R}^{\mathbb{d}}).
\end{aligned}$$

In particular, $u_\varepsilon \rightharpoonup u + \bar{w}_0$ weakly in $L^p(\Omega; \mathbb{R}^{\mathbb{d}})$. Moreover, by Lemma 4.2, we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega f_R\Big(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\Big)\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} \int_\Omega f_R\Big(x, \frac{x}{\varepsilon}, w_\varepsilon(x)\Big)\,\mathrm{d}x,$$

which, together with (4.7), concludes Step 1.

Step 2. We treat the general case.

Fix $j \in \mathbb{N}$. Arguing as in Step 1 of the proof of Proposition 3.15 with $\bar{u}_1$ replaced by $\bar{w}_1$, we can find $\tilde{w}_j \in C^1(\overline{\Omega}; C^1_{\#}(Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{d}}))$ such that $\bar{w}_0 + \tilde{w}_j \in \overline{\mathcal{W}}_{\mathcal{A}}$ and $\|\tilde{w}_j - \bar{w}_1\|_{L^p(\Omega; L^p_{\#}(Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{d}}))} \leqslant \frac{1}{j}$. Then, extracting a subsequence of $\{\tilde{w}_j\}_{j \in \mathbb{N}}$ if necessary, Vitali–Lebesgue theorem and (H1)–(H2) yield

$$\begin{aligned}
&\lim_{j \to \infty} \int_\Omega \fint_{Y^{\mathbb{m}}} f(x, \kappa y, u(x) + \bar{w}_0(x) + \tilde{w}_j(x,y))\,\mathrm{d}y\mathrm{d}x\\
&\quad = \int_\Omega \fint_{Y^{\mathbb{m}}} f(x, \kappa y, u(x) + \bar{w}_0(x) + \bar{w}_1(x,y))\,\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Hence, we can find $j_\delta \in \mathbb{N}$ such that

$$\begin{aligned}
&\int_\Omega \fint_{Y^{\mathbb{m}}} f(x, \kappa y, u(x) + \bar{w}_0(x) + \tilde{w}_{j_\delta}(x,y))\,\mathrm{d}y\mathrm{d}x\\
&\quad \leqslant \int_\Omega \fint_{Y^{\mathbb{m}}} f(x, \kappa y, u(x) + \bar{w}_0(x) + \bar{w}_1(x,y))\,\mathrm{d}y\mathrm{d}x + \delta.
\end{aligned}$$

To conclude, we invoke Step 1 to find a sequence $\{u_\varepsilon\} \subset \mathcal{U}_{\mathcal{A}}$, that depends on δ, u, and w, such that $u_\varepsilon \rightharpoonup u + \bar{w}_0$ weakly in $L^p(\Omega;\mathbb{R}^{\mathtt{d}})$ as $\varepsilon \to 0^+$ and, for all $\kappa \in \mathbb{N}$,

$$\lim_{\varepsilon\to 0^+} \int_\Omega f_R\Big(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\Big)\,\mathrm{d}x = \int_\Omega \fint_{Y^{\mathtt{m}}} f(x, \kappa y, u(x) + \bar{w}_0(x) + \tilde{w}_{j_\delta}(x,y))\,\mathrm{d}y\mathrm{d}x. \qquad \square$$

Next, we establish the Γ-liminf inequality property stated in Theorem 4.1.

Proposition 4.4. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be a sequence converging to 0, and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\mathcal{A}}$ be a sequence such that $u_n \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^{\mathtt{d}})$ for some $u \in L^p(\Omega;\mathbb{R}^{\mathtt{d}})$. Then, under the assumptions of Theorem 4.1, we have $u \in \mathcal{U}_{\mathcal{A}}$ and*

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) \geqslant \mathcal{F}_{\mathrm{hom}}(u). \tag{4.8}$$

Proof. The condition $u \in \mathcal{U}_{\mathcal{A}}$ follows from the fact that $u_n \in \mathcal{U}_{\mathcal{A}}$ for all $n \in \mathbb{N}$ together with the convergence $u_n \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^{\mathtt{d}})$. Moreover, by Propositions 3.14 and 3.4, and the uniqueness of $\boldsymbol{R}$-two-scale limits (see Remark 3.2), we have $u_n \xrightarrow{\boldsymbol{R}\text{-}2sc} v$ for a vector-field v that is $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free in the sense of Definition 3.13, with $\int_{Y^{\mathtt{m}}} v(\cdot,y)\,\mathrm{d}y = u(\cdot)$. In particular, we have the decomposition

$$v = u + v_1, \quad v_1 \in L^p(\Omega; L^p_{\#}(Y^{\mathtt{m}};\mathbb{R}^{\mathtt{d}})), \quad \mathcal{A}^y_{\boldsymbol{R}^*} v_1 = 0, \quad \int_{Y^{\mathtt{m}}} v_1(\cdot,y)\,\mathrm{d}y = 0.$$

Let $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c(\Omega; C_{\#}(Y^{\mathtt{m}};\mathbb{R}^{\mathtt{l}}))$ be a sequence converging to v in $L^p(\Omega\times Y^{\mathtt{m}};\mathbb{R}^{\mathtt{d}})$ and pointwise in $\Omega \times Y^{\mathtt{m}}$. By (H3), we have, for all n, $j \in \mathbb{N}$,

$$f\Big(x, \frac{\boldsymbol{R}x}{\varepsilon_n}, u_n(x)\Big) \geqslant f\Big(x, \frac{\boldsymbol{R}x}{\varepsilon_n}, \psi_j\Big(x, \frac{\boldsymbol{R}x}{\varepsilon_n}\Big)\Big) + \frac{\partial f}{\partial \xi}\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}, \psi_j\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)\Big) \cdot \Big(u_n(x) - \psi_j\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)\Big).$$

Integrating this estimate over Ω and passing to the limit as $n \to \infty$, we invoke Proposition 3.7 and (H2)–(H3) to infer that

$$\begin{aligned}\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) &= \liminf_{n\to\infty} \int_\Omega f\Big(x, \frac{\boldsymbol{R}x}{\varepsilon_n}, u_n(x)\Big)\,\mathrm{d}x \\ &\geqslant \int_\Omega \fint_{Y^{\mathtt{m}}} f(x,y,\psi_j(x,y))\,\mathrm{d}x\mathrm{d}y + \int_\Omega \fint_{Y^{\mathtt{m}}} \frac{\partial f}{\partial \xi}(x,y,\psi_j(x,y)) \cdot \big(v(x,y) - \psi_j(x,y)\big)\,\mathrm{d}x\mathrm{d}y\end{aligned} \tag{4.9}$$

for all $j \in \mathbb{N}$. Letting $j \to \infty$ in this inequality, Fatou's lemma and (H1) yield

$$\begin{aligned}\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) &\geqslant \int_\Omega \fint_{Y^{\mathtt{m}}} f(x,y,v(x,y))\,\mathrm{d}y\mathrm{d}x = \int_\Omega \fint_{Y^{\mathtt{m}}} f(x,y,u(x)+v_1(x,y))\,\mathrm{d}y\mathrm{d}x \\ &\geqslant \inf_{w\in\mathcal{W}_{\mathcal{A}}} \int_\Omega \fint_{Y^{\mathtt{m}}} f(x,y,u(x)+w(x,y))\,\mathrm{d}x\mathrm{d}y = \mathcal{F}_{\mathrm{hom}}(u). \qquad \square\end{aligned}$$

Proof of Theorem 4.1. Proving that both the Γ-*liminf inequality* and the *recovery sequence* properties in Theorem 4.1 hold is equivalent to proving that (see [22]) for all $u \in \mathcal{U}_{\mathcal{A}}$, we have

$$\mathcal{F}_{\mathrm{hom}}(u) = \Gamma\text{-}\liminf_{n\to\infty} F_{\varepsilon_n}(u) = \Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u), \tag{4.10}$$

where $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ is an arbitrary sequence converging to 0 and

$$\Gamma\text{-}\liminf_{n\to\infty} F_{\varepsilon_n}(u) := \inf\Big\{\liminf_{n\to\infty} F_{\varepsilon_n}(u_n)\colon\ u_n \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^{\mathtt{d}}) \text{ as } n\to\infty,\ \ \mathcal{A}u_n = 0 \text{ for all } n\in\mathbb{N}\Big\},$$
$$\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u) := \inf\Big\{\limsup_{n\to\infty} F_{\varepsilon_n}(u_n)\colon\ u_n \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^{\mathtt{d}}) \text{ as } n\to\infty,\ \ \mathcal{A}u_n = 0 \text{ for all } n\in\mathbb{N}\Big\}.$$

Taking the infimum over all admissible sequences on (4.8), we conclude from Proposition 4.4 that

$$\Gamma\text{-}\liminf_{n\to\infty} F_{\varepsilon_n}(u) \geqslant \mathcal{F}_{\mathrm{hom}}(u). \tag{4.11}$$

On the other hand, Proposition 4.3 with $\kappa = 1$ and $\bar{w}_0 = 0$ yields

$$\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u) \leqslant \int_\Omega \fint_{Y^{\mathtt{m}}} f(x,\kappa y, u(x) + \bar{w}_1(x,y))\,\mathrm{d}y\mathrm{d}x + \delta$$

for all $\delta > 0$ and $\bar{w}_1 \in \mathcal{W}_{\mathcal{A}}$. Hence, taking the infimum over $\bar{w}_1 \in \mathcal{W}_{\mathcal{A}}$, and then letting $\delta \to 0$, we get

$$\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u) \leqslant \mathcal{F}_{\mathrm{hom}}(u). \tag{4.12}$$

Because $\Gamma\text{-}\liminf_{n\to\infty} F_{\varepsilon_n}(u) \leqslant \Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u)$, we obtain (4.10) from (4.11) and (4.12). $\square$

Next, we establish an integral representation for the functional $\mathcal{F}_{\rm hom}$ introduced in Theorem 4.1. This integral representation is the quasi-periodic counterpart of [32, Corollary 3.2] concerning the periodic case, and its proof make of the following measurable selection criterion, proved in [32, Lemma 3.10] (also see [18]).

Lemma 4.5. *Let Z be a separable metric space, let T be a measurable space, and let $\Upsilon : T \to 2^Z$ be a multi-valued function such that (i) for every $t \in T$, $\Upsilon(t) \subset Z$ is nonempty and open, and (ii) for every $z \in Z$, $\{t \in T\colon z \in \Upsilon(t)\} \subset T$ is measurable. Then, Υ admits a measurable selection; that is, there exists a measurable function, $v : T \to Z$, such that $v(t) \in \Upsilon(t)$ for all $t \in T$.*

Proposition 4.6. *Under the assumptions of Theorem 4.1, for all $u \in \mathcal{U}_{\mathcal{A}}$, we have*

$$\mathcal{F}_{\rm hom}(u) = \int_\Omega f_{\rm hom}(x, u(x))\,{\rm d}x, \tag{4.13}$$

where

$$f_{\rm hom}(x,\xi) = \inf_{v\in\mathcal{V}_{\mathcal{A}}} \fint_{Y^{\mathfrak{m}}} f(x, y, \xi + v(y))\,{\rm d}y$$

with $\mathcal{V}_{\mathcal{A}}$ given by (1.10).

Proof. Let $u \in \mathcal{U}_{\mathcal{A}}$. Note that, by (H2), we have

$$0 \leqslant f_{\rm hom}(x,\xi) \leqslant C(1 + |\xi|^p) \tag{4.14}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}^{\mathfrak{d}}$. Moreover,

$$x \in \Omega \mapsto f_{\rm hom}(x, u(x)) \tag{4.15}$$

is a measurable map. In fact, let $V_{\mathcal{A}}$ be a countable and dense subset of $\mathcal{V}_{\mathcal{A}}$ with respect to the (strong) topology of $L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})$. We observe that such set $V_{\mathcal{A}}$ exists because $\mathcal{V}_{\mathcal{A}}$ is a subset of the separable metric space $L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})$. Then, the continuity of f (see (H1)), Vitali–Lebesgue's theorem, and (H2) yield

$$\inf_{v\in\mathcal{V}_{\mathcal{A}}} \fint_{Y^{\mathfrak{m}}} f(x, y, u(x) + v(y))\,{\rm d}y = \inf_{v\in V_{\mathcal{A}}} \fint_{Y^{\mathfrak{m}}} f(x, y, u(x) + v(y))\,{\rm d}y,$$

from which we conclude the measurability of the map in (4.15).

Fix $w \in \mathcal{W}_{\mathcal{A}}$. For a.e. $x \in \Omega$, we have $w(x,\cdot) \in \mathcal{V}_{\mathcal{A}}$; hence, for a.e. $x \in \Omega$,

$$\inf_{v\in\mathcal{V}_{\mathcal{A}}} \fint_{Y^{\mathfrak{m}}} f(x, y, u(x) + v(y))\,{\rm d}y \leqslant \fint_{Y^{\mathfrak{m}}} f(x, y, u(x) + w(x, y))\,{\rm d}y.$$

Integrating this estimate over Ω, and then taking the infimum over $w \in \mathcal{W}_{\mathcal{A}}$, we conclude that

$$\int_\Omega f_{\rm hom}(x, u(x))\,{\rm d}x \leqslant \mathcal{F}_{\rm hom}(u).$$

To prove the converse inequality, we first observe that, by (4.14), we may assume that

$$f_{\rm hom}(x, u(x)) \in \mathbb{R} \text{ for all } x \in \Omega, \tag{4.16}$$

without loss of generality. Fix $\delta > 0$, and consider the multi-valued function $\Upsilon_\delta : \Omega \to 2^{L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})}$ defined, for $x \in \Omega$, by

$$\Upsilon_\delta(x) := \left\{ v \in \mathcal{V}_{\mathcal{A}}\colon \fint_{Y^{\mathfrak{m}}} f(x, y, u(x) + v(y))\,{\rm d}y < f_{\rm hom}(x, u(x)) + \delta \right\}.$$

Also, let $\bar\delta \in (0,\delta)$ be such that

$$\int_E C(1 + |u(x)|^p)\,{\rm d}x < \delta \tag{4.17}$$

whenever $E \subset \Omega$ is a measurable set with $\mathcal{L}^{\mathfrak{m}}(E) < \bar\delta$, where C is given by (H2).

By (4.16), we have $\Upsilon_\delta(x) \neq \emptyset$ for all $x \in \Omega$. Furthermore, arguing as above, using the continuity of f, Vitali–Lebesgue's theorem, and (H2), it can be checked that for each $x \in \Omega$, $L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}}) \setminus \Upsilon_\delta(x)$ is a closed subset of $L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})$. On the other hand, recalling the measurability of the map in (4.15), we have that

$$x \mapsto h(x) := \fint_{Y^{\mathfrak{m}}} f(x, y, u(x) + v(y))\,{\rm d}y - f_{\rm hom}(x, u(x)) - \delta$$

defines a measurable map for each $v \in L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})$. Thus, $\{x \in \Omega\colon v \in \Upsilon_\delta(x)\} = h^{-1}((-\infty, 0))$ is a measurable set for each $v \in L^p_{\#}(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{d}})$. Consequently, by Lemma 4.5, there exists a measurable

selection, $w_\delta : \Omega \to L^p_\#(Y^{\mathbb{m}};\mathbb{R}^{\mathbb{d}})$, of Υ_δ. Moreover, by Lusin's theorem, $w_\delta \in L^p(\Omega_\delta; L^p_\#(Y^{\mathbb{m}};\mathbb{R}^{\mathbb{d}}))$ for a suitable measurable set Ω_δ such that $\mathcal{L}^{\mathbb{n}}(\Omega \setminus \Omega_\delta) < \bar{\delta}$.

Finally, we define $\bar{w}_\delta \in \mathcal{W}_{\mathcal{A}}$ by setting $\bar{w}_\delta(x) := w_\delta(x)$ if $x \in \Omega_\delta$, and $\bar{w}_\delta(x) := 0$ if $x \in \Omega \setminus \Omega_\delta$. Then, using the definition of $\mathcal{F}_{\rm hom}(u)$, (H2), (4.17), and (4.14), we get

$$\begin{aligned}\mathcal{F}_{\rm hom}(u) &\leqslant \int_\Omega \fint_{Y^{\mathbb{m}}} f(x,y,u(x)+\bar{w}_\delta(x,y))\,{\rm d}x{\rm d}y \leqslant \delta + \int_{\Omega_\delta} \fint_{Y^{\mathbb{m}}} f(x,y,u(x)+w_\delta(x,y))\,{\rm d}x{\rm d}y\\ &\leqslant \delta(1+\mathcal{L}^{\mathbb{n}}(\Omega)) + \int_\Omega f_{\rm hom}(x,u(x))\,{\rm d}x,\end{aligned}$$

from which we conclude the desired inequality by letting $\delta \to 0$. □

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1. Theorem 1.1 is an immediate consequence of Theorem 4.1 (also see (4.10)) and Proposition 4.6. □

5 The curl case

In this section, we prove Theorem 1.4 that provides an equivalent alterative characterization for the $\boldsymbol{R}$-two-scale limit of bounded sequences in $W^{1,p}$, corresponding to $\mathcal{A} = \operatorname{curl}$ in Theorem 1.3. In this case, by Proposition 3.9 and extracting a subsequence if necessary, we have $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u_0$ and $\nabla u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} U_0$ for some $u_0 \in L^p(\Omega \times Y^{\mathbb{m}})$ and $U_0 \in L^p(\Omega \times Y^{\mathbb{m}};\mathbb{R}^{\mathbb{n}})$. Next, to study the relationship between u_0 and U_0, we use Fourier analysis. As we mentioned before, this is the approach adopted in [8]; however, the arguments in [8] hinge on the Parseval and Plancherel identities, which are valid in $L^2(\Omega)$ only. Instead, our main tool here relies on the following theorem, which may be found in [35]. For simplicity, we take

$$Y^{\mathbb{m}} = [0,1)^{\mathbb{m}},$$

and we use the Einstein convention on repeated indices.

Theorem 5.1. *Let $w \in L^p_\#(Y^{\mathbb{m}})$, and define*[1]

$$w_N(y) := \sum_{\substack{k\in\mathbb{Z}^{\mathbb{m}}\\ |k|_\infty \leqslant N}} \hat{w}_k e^{2\pi i k\cdot y},\quad y \in Y^{\mathbb{m}},\quad N \in \mathbb{N},$$

where $\hat{w}_k := \int_{Y^{\mathbb{m}}} w(y)e^{-2\pi i k\cdot y}\,{\rm d}y$, $k \in \mathbb{Z}^{\mathbb{m}}$, are the Fourier coefficients of w. Then,

$$\begin{aligned}&\Big\| \sup_{N\in\mathbb{N}} |w_N| \Big\|_{L^p(Y^{\mathbb{m}})} \leqslant C_{p,m}\|w\|_{L^p(Y^{\mathbb{m}})},\\ &\lim_{N\to\infty} \|w_N - w\|_{L^p(Y^{\mathbb{m}})} = 0,\\ &\lim_{N\to\infty} w_N(y) = w(y) \text{ for a.e. } y \in Y^{\mathbb{m}},\end{aligned} \tag{5.1}$$

where $C_{p,m}$ is a positive constant depending on p and m only. Moreover, for all $k \in \mathbb{Z}^{\mathbb{m}}$,

$$|\hat{w}_k|^p \leqslant \|w\|^p_{L^p(Y^{\mathbb{m}})}. \tag{5.2}$$

Proof. The proof of (5.1) may be found in [35, Thm. 4.1.8 and Thm. 4.3.16]) (see also [35, Def. 3.2.3]).

To prove (5.2), we use Jensen's inequality and the equality $|e^{-2\pi i k\cdot y}| = 1$, $k \in \mathbb{Z}^{\mathbb{m}}$, to obtain

$$|\hat{w}_k|^p \leqslant \int_{Y^{\mathbb{m}}} \left|w(y)e^{-2\pi i k\cdot y}\right|^p {\rm d}y = \|w\|^p_{L^p(Y^{\mathbb{m}})}.$$ □

Remark 5.2. Let $w \in L^p(\Omega; L^p_\#(Y^{\mathbb{m}}))$, and define

$$v_N(x) := \int_{Y^{\mathbb{m}}} |w_N(x,y) - w(x,y)|^p\,{\rm d}y,\quad x \in \Omega,\quad N \in \mathbb{N},$$

where $w_N(x,y) := \sum_{\substack{k\in\mathbb{Z}^{\mathbb{m}}\\ |k|_\infty\leqslant N}} \hat{w}_k(x)e^{2\pi i k\cdot y}$ with $\hat{w}_k(x) := \int_{Y^{\mathbb{m}}} w(x,y)e^{-2\pi i k\cdot y}\,{\rm d}y$, $k \in \mathbb{Z}^{\mathbb{m}}$. By (5.1), for a.e.-$x \in \Omega$, we have

$$\sup_{N\in\mathbb{N}} |v_N(\cdot)| \leqslant \tilde{C}_{p,m}\int_{Y^{\mathbb{m}}} |w(\cdot,y)|^p\,{\rm d}y \in L^1(\Omega) \text{ and } \lim_{N\to\infty} v_N(x) = 0.$$

[1] In the literature, w_N are called the square partial sums of the Fourier series of w.

Thus, by the Lebesgue dominated convergence theorem, it follows that $w_N \to w$ in $L^p(\Omega \times Y^{\mathfrak{m}})$ as $N \to \infty$.

Next, we study some properties of the space $\mathcal{G}^p_{\boldsymbol{R}}$ introduced in (1.12) that will be useful in the sequel. We first observe that if $p = 2$, then it can be checked that

$$\mathcal{G}^2_{\boldsymbol{R}} = \left\{ w \in L^2_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}}) \colon\ w(y) = \sum_{k \in \mathbb{Z}^{\mathfrak{m}} \setminus \{0\}} \lambda_k \boldsymbol{R}^* k e^{2\pi i k \cdot y} \text{ for some } \{\lambda_k\}_{k \in \mathbb{Z}^{\mathfrak{m}} \setminus \{0\}} \subset \mathbb{C} \right\},$$

and we recover the space introduced in [8].

Lemma 5.3. *Assume that $\boldsymbol{R}$ satisfies (1.2). Then, the vector space $\mathcal{G}^p_{\boldsymbol{R}}$ introduced in (1.12) is a closed subspace of $L^p_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})$.*

Proof. Let $\{w_j\}_{j \in \mathbb{N}} \subset \mathcal{G}^p_{\boldsymbol{R}}$ and $w \in L^p_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})$ be such that $\lim_{j \to \infty} \|w_j - w\|_{L^p(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})} = 0$. We want to show that $w \in \mathcal{G}^p_{\boldsymbol{R}}$.

For each $j \in \mathbb{N}$, let $\hat{w}^j_k$ and $\hat{w}_k$ denote the Fourier coefficients of w_j and w, respectively. By (5.2), we have $\lim_{j \to \infty} |\hat{w}^j_k - \hat{w}_k| = 0$.

On the other hand, by definition of $\mathcal{G}^p_{\boldsymbol{R}}$, for each $k \in \mathbb{Z}^{\mathfrak{m}}$ and $j \in \mathbb{N}$, there exists $\lambda^j_k \in \mathbb{C}$ such that $\hat{w}^j_k = \lambda^j_k \boldsymbol{R}^* k$ and $\lambda^j_0 = 0$. In particular, $\hat{w}_0 = 0$. Fix $k_0 \in \mathbb{Z}^{\mathfrak{m}} \setminus \{0\}$; by (5.2), for all $j, j' \in \mathbb{N}$, we have

$$\left|\lambda^j_{k_0} - \lambda^{j'}_{k_0}\right| \left|\boldsymbol{R}^* k_0\right| = \left|(\lambda^j_{k_0} - \lambda^{j'}_{k_0}) \boldsymbol{R}^* k_0\right| = \left|\hat{w}^j_{k_0} - \hat{w}^{j'}_{k_0}\right| \leqslant \|w_j - w_{j'}\|_{L^p(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})}.$$

Because $\boldsymbol{R}^* k_0 \neq 0$ by (1.2), we conclude that $\{\lambda^j_{k_0}\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. Thus, there exists $\lambda_{k_0} \in \mathbb{C}$ such that $\lim_{j \to \infty} |\lambda^j_{k_0} - \lambda_{k_0}| = 0$. Consequently, passing the equality $\hat{w}^j_{k_0} = \lambda^j_{k_0} \boldsymbol{R}^* k_0$ to the limit as $j \to \infty$, we obtain $\hat{w}_{k_0} = \lambda_{k_0} \boldsymbol{R}^* k_0$. □

Lemma 5.4. *Assume that $\boldsymbol{R}$ satisfies (1.2), and let $w_0 \in L^p_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})$ be such that*

$$\int_{Y^{\mathfrak{m}}} w_0(y) \cdot \psi(y) \, \mathrm{d}y = 0 \tag{5.3}$$

for all $\psi \in C^\infty_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})$ with $\frac{\partial \psi_l}{\partial y_\tau} \boldsymbol{R}_{\tau l} = 0$ in $Y^{\mathfrak{m}}$, where $l \in \{1, \dots, \mathfrak{n}\}$ and $\tau \in \{1, \dots, \mathfrak{m}\}$. Then, $w_0 \in \mathcal{G}^p_{\boldsymbol{R}}$.

Proof. By contradiction, assume that $w_0 \notin \mathcal{G}^p_{\boldsymbol{R}}$. Then, using Lemma 5.3, together with (a corollary to) the Hahn–Banach theorem (see, for instance, [14, Cor. I.8]), there exists $v \in L^{p'}_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})$ such that

$$\int_{Y^{\mathfrak{m}}} v(y) \cdot w(y) \, \mathrm{d}y = 0 \tag{5.4}$$

for all $w \in \mathcal{G}^p_{\boldsymbol{R}}$, and

$$\int_{Y^{\mathfrak{m}}} v(y) \cdot w_0(y) \, \mathrm{d}y \neq 0. \tag{5.5}$$

We claim that $\frac{\partial v_l}{\partial y_\tau} \boldsymbol{R}_{\tau l} = 0$ in the sense of distributions. In fact, let $\phi \in C^\infty_{\#}(Y^{\mathfrak{m}})$, and set $w := \boldsymbol{R}^* \nabla_y \phi$. Then, $w \in L^p_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})$, $\hat{w}_0 = 0$, and $\hat{w}_k = 2\pi i \hat{\phi}_k \boldsymbol{R}^* k$ for all $k \in \mathbb{Z}^{\mathfrak{m}} \setminus \{0\}$. Thus, $w \in \mathcal{G}^p_{\boldsymbol{R}}$ and so, by (5.4),

$$0 = \int_{Y^{\mathfrak{m}}} v(y) \cdot w(y) \, \mathrm{d}y = \int_{Y^{\mathfrak{m}}} \boldsymbol{R} v(y) \cdot \nabla_y \phi(y) \, \mathrm{d}y,$$

which shows that $0 = \operatorname{div}_y(\boldsymbol{R} v(y)) = \operatorname{div}_y(v(y) \boldsymbol{R}^*) = \frac{\partial v_l}{\partial y_\tau} \boldsymbol{R}_{\tau l}$ in the sense of distributions because $\phi \in C^\infty_{\#}(Y^{\mathfrak{m}})$ is arbitrary.

Using standard mollification techniques with a $Y^{\mathfrak{m}}$-periodic, smooth kernel, we may construct a sequence $\{v_h\}_{h \in \mathbb{N}} \subset C^\infty_{\#}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})$ such that $\operatorname{div}_y(v_h(y) \boldsymbol{R}^*) = 0$ in $Y^{\mathfrak{m}}$ and $\lim_{h \to \infty} \|v_h - v\|_{L^{p'}(Y^{\mathfrak{m}}; \mathbb{R}^{\mathfrak{n}})} = 0$. Then, by (5.3) with $\psi = v_h$ and Lebesgue dominated convergence theorem, we obtain $\int_{Y^{\mathfrak{m}}} w_0(y) \cdot v(y) \, \mathrm{d}y = 0$, which contradicts (5.5). Thus, $w_0 \in \mathcal{G}^p_{\boldsymbol{R}}$. □

Proposition 5.5. *Let $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega)$ be a bounded sequence, and assume that $\boldsymbol{R}$ satisfies (1.2). Then, there exist a subsequence $\varepsilon' \preceq \varepsilon$ and functions $u \in W^{1,p}(\Omega)$ and $w \in L^p(\Omega; \mathcal{G}^p_{\boldsymbol{R}})$ such that*

$$u_{\varepsilon'} \xrightarrow{\boldsymbol{R}\text{-}2sc} u \quad \text{and} \quad \nabla u_{\varepsilon'} \xrightarrow{\boldsymbol{R}\text{-}2sc} \nabla u + w.$$

Proof. By the reflexibility of $W^{1,p}(\Omega)$ and Proposition 3.9, there exist $u \in W^{1,p}(\Omega)$, $u_0 \in L^p(\Omega \times Y^{\mathbb{m}})$, and $U_0 \in L^p(\Omega \times Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{n}})$ such that, extracting a subsequence if necessary,

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega), \quad u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u_0, \quad \text{and} \quad \nabla u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} U_0. \tag{5.6}$$

By the Rellich–Kondrachov theorem, $u_\varepsilon \to u$ in $L^p(\Omega)$. Hence, Proposition 3.5 and the uniqueness of the $\boldsymbol{R}$-two-scale limit (see Remark 3.2) yield

$$u \equiv u_0.$$

We are left to prove that U_0 in (5.6) is of the form $U_0(x,y) = \nabla u(x) + w(x,y)$ for some $w \in L^p(\Omega; \mathcal{G}_{\boldsymbol{R}})$. Let $\Phi \in C_c^\infty(\Omega; C_\#^\infty(Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{n}}))$ be such that $\frac{\partial \Phi_l}{\partial y_\tau} \boldsymbol{R}_{\tau l} = 0$ in $\Omega \times Y^{\mathbb{m}}$, $l \in \{1, ..., \mathbb{n}\}$, $\tau \in \{1, ..., \mathbb{m}\}$. Then, using integration by parts, the second convergence in (5.6), and the fact that $u_0 \equiv u \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned}
&\int_{\Omega \times Y^{\mathbb{m}}} U_0(x,y) \cdot \Phi(x,y)\, \mathrm{d}x\mathrm{d}y \\
&= \lim_{\varepsilon \to 0^+} \int_\Omega \nabla u_\varepsilon(x) \cdot \Phi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)\, \mathrm{d}x \\
&= -\lim_{\varepsilon \to 0^+} \left(\int_\Omega u_\varepsilon(x) \operatorname{div}_x \Phi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)\, \mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega u_\varepsilon(x) \frac{\partial \Phi_l}{\partial y_\tau}\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big) \boldsymbol{R}_{\tau l}\, \mathrm{d}x \right) \\
&= -\lim_{\varepsilon \to 0^+} \int_\Omega u_\varepsilon(x) \operatorname{div}_x \Phi\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big)\, \mathrm{d}x = -\int_{\Omega \times Y^{\mathbb{m}}} u(x) \operatorname{div}_x \Phi(x,y)\, \mathrm{d}x\mathrm{d}y \\
&= \int_{\Omega \times Y^{\mathbb{m}}} \nabla u(x) \cdot \Phi(x,y)\, \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Hence,

$$\int_{\Omega \times Y^{\mathbb{m}}} \big(U_0(x,y) - \nabla u(x)\big) \cdot \Phi(x,y)\, \mathrm{d}x\mathrm{d}y = 0$$

for all $\Phi \in C_c^\infty(\Omega; C_\#^\infty(Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{n}}))$ such that $\frac{\partial \Phi_l}{\partial y_\tau} \boldsymbol{R}_{\tau l} = 0$ in $\Omega \times Y^{\mathbb{m}}$. Invoking Lemma 5.4, we conclude for a.e.-$x \in \Omega$, $U_0(x,\cdot) - \nabla u(x) \in \mathcal{G}_{\boldsymbol{R}}^p$, which, together with the fact that the map $(x,y) \mapsto U_0(x,y) - \nabla u(x)$ belongs to $L^p(\Omega \times Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{n}})$, concludes the proof. □

The next proposition shows that Proposition 5.5 fully characterizes the $\boldsymbol{R}$-two-scale limit of bounded sequences in $W^{1,p}(\Omega)$. To the best of our knowledge, in the framework of $\boldsymbol{R}$-two-scale convergence, this result is new in the literature even for $p = 2$.

Proposition 5.6. *Let $u \in W^{1,p}(\Omega)$ and $w \in L^p(\Omega; \mathcal{G}_{\boldsymbol{R}}^p)$, and assume that $\boldsymbol{R}$ satisfies (1.2). Then, there exists a bounded sequence $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega)$ such that*

$$u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u \quad \text{and} \quad \nabla u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} \nabla u + w.$$

Proof. We first consider the case in which $w \in W^{1,p}(\Omega; \mathcal{G}_{\boldsymbol{R}}^p)$.

For each $k \in \mathbb{Z}^{\mathbb{m}}$, define

$$\hat{w}_k(x) := \int_{Y^{\mathbb{m}}} w(x,y) e^{-2\pi i k \cdot y}\, \mathrm{d}y, \quad x \in \Omega.$$

Because $w \in W^{1,p}(\Omega; \mathcal{G}_{\boldsymbol{R}}^p)$, we have $\hat{w}_k \in W^{1,p}(\Omega; \mathbb{C}^n)$ with $\hat{w}_0 \equiv 0$; moreover, for a.e.-$x \in \Omega$, we have $\hat{w}_k(x) = \lambda_k(x) \boldsymbol{R}^* k$ for some $\lambda_k \in W^{1,p}(\Omega; \mathbb{C})$ with $\lambda_0 \equiv 0$.

For each $N \in \mathbb{N}$, let $\tilde{w}_N \in W^{1,p}(\Omega; C_\#^\infty(Y^{\mathbb{m}}; \mathbb{C}))$ be the function defined by

$$\tilde{w}_N(x,y) := \sum_{\substack{k \in \mathbb{Z}^{\mathbb{m}} \\ |k|_\infty \leqslant N}} \frac{1}{2\pi i} \lambda_k(x) e^{2\pi i k \cdot y}, \quad (x,y) \in \Omega \times \mathbb{R}^{\mathbb{m}}.$$

Note that the function

$$w_N(x,y) := \boldsymbol{R}^* \nabla_y \tilde{w}_N(x,y) = \sum_{\substack{k \in \mathbb{Z}^{\mathbb{m}} \\ |k|_\infty \leqslant N}} \lambda_k(x) \boldsymbol{R}^* k e^{2\pi i k \cdot y} = \sum_{\substack{k \in \mathbb{Z}^{\mathbb{m}} \\ |k|_\infty \leqslant N}} \hat{w}_k(x) e^{2\pi i k \cdot y}, \quad (x,y) \in \Omega \times \mathbb{R}^{\mathbb{m}},$$

belongs to $W^{1,p}(\Omega; C_\#^\infty(Y^{\mathbb{m}}; \mathbb{R}^{\mathbb{n}}))$ and, by Remark 5.2, satisfies

$$\lim_{N \to \infty} \int_{\Omega \times Y^{\mathbb{m}}} |w_N(x,y) - w(x,y)|^p\, \mathrm{d}x\mathrm{d}y. \tag{5.7}$$

Finally, for each ε, we define $\bar{w}_N := Re(\tilde{w}_N)$ and

$$u_{\varepsilon,N}(x) := u(x) + \varepsilon \bar{w}_N\Big(x, \frac{\boldsymbol{R}x}{\varepsilon}\Big), \quad x \in \Omega.$$

Then, $u_{\varepsilon,N} \in W^{1,p}(\Omega)$ with

$$\|u_{\varepsilon,N}\|_{L^p(\Omega)} \leqslant \|u\|_{L^p(\Omega)} + \varepsilon \|\bar{w}_N\|_{L^p(\Omega; C_\#(Y^{\mathfrak{m}}))}$$

and

$$\|\nabla u_{\varepsilon,N}\|_{L^p(\Omega;\mathbb{R}^{\mathfrak{n}})} \leqslant \|\nabla u\|_{L^p(\Omega;\mathbb{R}^{\mathfrak{n}})} + \varepsilon \|\nabla_x \bar{w}_N\|_{L^p(\Omega; C_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}}))} + \|w_N\|_{L^p(\Omega; C_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}}))}.$$

Let $\varphi \in L^{p'}(\Omega; C_\#(Y^{\mathfrak{m}}))$ and $\Phi \in L^{p'}(\Omega; C_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}}))$. By Proposition 3.7, we have

$$\begin{aligned}\lim_{N\to\infty}\lim_{\varepsilon\to 0^+}\int_\Omega u_{\varepsilon,N}(x)\varphi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x &= \lim_{N\to\infty}\lim_{\varepsilon\to 0^+}\int_\Omega \Big(u(x)+\varepsilon\bar{w}_N\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\Big)\varphi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x \\ &= \int_{\Omega\times Y^{\mathfrak{m}}} u(x)\varphi(x,y)\,\mathrm{d}x\mathrm{d}y\end{aligned} \tag{5.8}$$

and, also using (5.7),

$$\begin{aligned}&\lim_{N\to\infty}\lim_{\varepsilon\to 0^+}\int_\Omega \nabla u_{\varepsilon,N}(x)\cdot\Phi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x \\ &= \lim_{N\to\infty}\lim_{\varepsilon\to 0^+}\int_\Omega \Big(\nabla u(x)+\varepsilon\nabla_x\bar{w}\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)+w_N\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\Big)\cdot\Phi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x \\ &= \int_{\Omega\times Y^{\mathfrak{m}}}(\nabla u(x)+w(x,y))\cdot\Phi(x,y)\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{5.9}$$

Due to the separability of $L^{p'}(\Omega; C_\#(Y^{\mathfrak{m}}))$ and $L^{p'}(\Omega; C_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}}))$ and (5.8)–(5.9), we can proceed as in [30, proof of Prop. 1.11 (p.449)] to find a sequence $\{N_\varepsilon\}_\varepsilon$ such that $N_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ and $\tilde{u}_\varepsilon := u_{\varepsilon,N_\varepsilon} \in W^{1,p}(\Omega)$ satisfies

$$\lim_{\varepsilon\to 0^+}\int_\Omega \tilde{u}_\varepsilon(x)\varphi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x = \int_{\Omega\times Y^{\mathfrak{m}}} u(x)\varphi(x,y)\,\mathrm{d}x\mathrm{d}y$$

and

$$\lim_{\varepsilon\to 0^+}\int_\Omega \nabla\tilde{u}_\varepsilon(x)\cdot\Phi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x = \int_{\Omega\times Y^{\mathfrak{m}}}(\nabla u(x)+w(x,y))\cdot\Phi(x,y)\,\mathrm{d}x\mathrm{d}y$$

for all $\varphi \in L^{p'}(\Omega; C_\#(Y^{\mathfrak{m}}))$ and $\Phi \in L^{p'}(\Omega; C_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}}))$; that is,

$$\tilde{u}_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u \quad \text{and} \quad \nabla\tilde{u}_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} \nabla u + w.$$

The boundedness of $\tilde{u}_\varepsilon$ in $W^{1,p}(\Omega)$ follows from Proposition 3.4.

To conclude, we treat the general case in which $w \in L^p(\Omega; \mathcal{G}^p_{\boldsymbol{R}})$. We claim that there exists a sequence $\{\bar{w}_N\}_{N\in\mathbb{N}} \subset W^{1,p}(\Omega; \mathcal{G}^p_{\boldsymbol{R}})$ such that

$$\lim_{N\to\infty}\|\bar{w}_N - w\|_{L^p(\Omega\times Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}})} = 0. \tag{5.10}$$

Assume that the claim holds. Then, by the previous case, for each $N \in \mathbb{N}$, there exists a bounded sequence $\{u^N_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega)$ such that

$$u^N_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u \quad \text{and} \quad \nabla u^N_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} \nabla u + \bar{w}_N \tag{5.11}$$

as $\varepsilon \to 0$.

Let $\varphi \in L^{p'}(\Omega; C_\#(Y^{\mathfrak{m}}))$ and $\Phi \in L^{p'}(\Omega; C_\#(Y^{\mathfrak{m}};\mathbb{R}^{\mathfrak{n}}))$. Using (5.11) first, and then (5.10), we obtain

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0^+}\int_\Omega u^N_\varepsilon(x)\varphi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x = \int_{\Omega\times Y^{\mathfrak{m}}} u(x)\varphi(x,y)\,\mathrm{d}x\mathrm{d}y$$

and

$$\begin{aligned}\lim_{N\to\infty}\lim_{\varepsilon\to 0^+}\int_\Omega \nabla u^N_\varepsilon(x)\cdot\Phi\Big(x,\frac{\boldsymbol{R}x}{\varepsilon}\Big)\,\mathrm{d}x &= \lim_{N\to\infty}\int_{\Omega\times Y^{\mathfrak{m}}}(\nabla u(x)+\bar{w}_N(x,y))\cdot\Phi(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= \int_{\Omega\times Y^{\mathfrak{m}}}(\nabla u(x)+w(x,y))\cdot\Phi(x,y)\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Finally, arguing as in the previous case, we can find a sequence $\{N_\varepsilon\}_\varepsilon$ such that $N_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ and $\tilde{u}_\varepsilon := u^{N_\varepsilon}_\varepsilon \in W^{1,p}(\Omega)$ satisfies the requirements.

We are left to prove (5.10). As before, for each $k \in \mathbb{Z}^{\mathbb{m}}$, define

$$\hat{w}_k(x) := \int_{Y^{\mathbb{m}}} w(x,y) e^{-2\pi i k\cdot y}\,\mathrm{d}y, \quad x \in \Omega.$$

Because $w \in L^p(\Omega; \mathcal{G}^p_{\boldsymbol{R}})$, we have $\hat{w}_k \in L^p(\Omega;\mathbb{C}^{\mathbb{n}})$ with $\hat{w}_0 \equiv 0$; moreover, for a.e.-$x \in \Omega$, we have $\hat{w}_k(x) = \lambda_k(x)\boldsymbol{R}^* k$ for some $\lambda_k \in L^p(\Omega;\mathbb{C})$ with $\lambda_0 \equiv 0$. Then, for each $k \in \mathbb{Z}^{\mathbb{m}}$, we can find a sequence $\{\lambda^j_k\}_{j\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{C})$, with $\lambda^j_0 = 0$, such that $\lambda^j_k \to \lambda_k$ in $L^p(\Omega;\mathbb{C})$ as $j \to \infty$. In particular, we have

$$\lim_{j\to\infty}\left(\int_\Omega |\lambda^j_k(x)\boldsymbol{R}^*k - \lambda_k(x)\boldsymbol{R}^*k|^p\,\mathrm{d}x\right)^{\frac{1}{p}} = \lim_{j\to\infty}|\boldsymbol{R}^*k|\left(\int_\Omega |\lambda^j_k(x)-\lambda_k(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} = 0. \tag{5.12}$$

Fix $N \in \mathbb{N}$; by (5.12), there exits $j_N \in \mathbb{N}$ such that

$$\sum_{\substack{k\in\mathbb{Z}^{\mathbb{m}}\\ |k|_\infty \leqslant N}}\left(\int_\Omega |\lambda^{j_N}_k(x)\boldsymbol{R}^*k - \lambda_k(x)\boldsymbol{R}^*k|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \leqslant \frac{1}{N}. \tag{5.13}$$

Defining

$$\tilde{w}_N(x,y) := \sum_{\substack{k\in\mathbb{Z}^{\mathbb{m}}\\ |k|_\infty\leqslant N}} \lambda^{j_N}_k(x)\boldsymbol{R}^*k e^{2\pi i k\cdot y} \quad \text{and} \quad \bar{w}_N := Re(\tilde{w}_N),$$

we have $\bar{w}_N \in W^{1,p}(\Omega;\mathcal{G}^p_{\boldsymbol{R}})$; moreover, invoking Remark 5.2 and (5.13), we have

$$\begin{aligned}
\limsup_{N\to\infty}\left(\int_{\Omega\times Y^{\mathbb{m}}}|\bar{w}_N(x,y)-w(x,y)|^p\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} &\leqslant \limsup_{N\to\infty}\left(\int_{\Omega\times Y^{\mathbb{m}}}|\tilde{w}_N(x,y)-w(x,y)|^p\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\\
&\leqslant \limsup_{N\to\infty}\left[\left(\int_{\Omega\times Y^{\mathbb{m}}}|\tilde{w}_N(x,y)-w_N(x,y)|^p\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} + \left(\int_{\Omega\times Y^{\mathbb{m}}}|w_N(x,y)-w(x,y)|^p\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\right]\\
&= \limsup_{N\to\infty}\left(\int_{\Omega\times Y^{\mathbb{m}}}\left|\sum_{\substack{k\in\mathbb{Z}^{\mathbb{m}}\\ |k|_\infty\leqslant N}}(\lambda^{j_N}_k(x)-\lambda_k(x))\boldsymbol{R}^*k e^{2\pi i k\cdot y}\right|^p\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} \leqslant \limsup_{N\to\infty}\frac{1}{N} = 0,
\end{aligned}$$

which concludes the proof of (5.10). □

Remark 5.7. Let $\Omega \subset \mathbb{R}^{\mathbb{n}}$ be a simply connected, bounded, and open set. Applying Proposition 5.6 to $u = 0$ and $w \in \mathcal{G}^p_{\boldsymbol{R}}$, we can find a bounded sequence $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega)$ such that

$$u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} 0 \quad \text{and} \quad \nabla u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} w.$$

Then, by Proposition 3.4, we have $u_\varepsilon \rightharpoonup 0$ in $W^{1,p}(\Omega)$ and $\int_{Y^{\mathbb{m}}} w(y)\,\mathrm{d}y = 0$. On the other hand, using the uniqueness of the $\boldsymbol{R}$-two-scale limit (see Remark 3.2) and Proposition 3.14 with $\mathbb{d} = \mathbb{n}$ and $\mathcal{A} = \mathrm{curl}$ in $\mathbb{R}^{\mathbb{n}}$, we conclude that that $w \in L^p_\#(Y^{\mathbb{m}};\mathbb{R}^{\mathbb{n}})$ is $\mathcal{A}_{\boldsymbol{R}^*}$-free in the sense of Definition 3.11.

Conversely, if $w \in L^p_\#(Y^{\mathbb{m}};\mathbb{R}^{\mathbb{n}})$ is $\mathcal{A}_{\boldsymbol{R}^*}$-free with $\int_{Y^{\mathbb{m}}} w(y)\,\mathrm{d}y = 0$, then by Proposition 3.15 there exists a bounded and $\mathcal{A}$-free sequence, $\{u_\varepsilon\}_\varepsilon$, in $L^p(\Omega;\mathbb{R}^{\mathbb{n}})$ such that $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$. As we are in the $\mathcal{A} = \mathrm{curl}$ case and Ω is simply connected, we can find a bounded sequence, $\{v_\varepsilon\}_\varepsilon$, in $W^{1,p}(\Omega)$ such that $\int_\Omega v_\varepsilon(x)\,\mathrm{d}x = 0$ and $\nabla v_\varepsilon = u_\varepsilon$. Then, by Proposition 3.4, we deduce that $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(\Omega)$. Finally, Remark 3.2 and Proposition 5.5 yield $w \in \mathcal{G}^p_{\boldsymbol{R}}$.

Thus, in the $\mathcal{A} = \mathrm{curl}$ case in $\mathbb{R}^{\mathbb{n}}$, we have that $w \in L^p_\#(Y^{\mathbb{m}};\mathbb{R}^{\mathbb{n}})$ is $\mathcal{A}_{\boldsymbol{R}^*}$-free in the sense of Definition 3.11 if and only if $w \in \mathcal{G}^p_{\boldsymbol{R}}$.

To conclude, we observe that Theorem 1.4 is an immediate consequence of the previous results.

Proof of Theorem 1.4. The claim in Theorem 1.4 follows from Propositions 5.5 and 5.6. □

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