The Inextricability of Students' Mathematical and Physical Reasoning in Quantum Mechanics Problems

Mathematics and physics have an interconnected, reflexive relationship. Physical problems motivated the origins of several mathematics concepts, and the mathematization of physical phenomena often enables the development of physical theory (e.g., Dirac, 1947). According to Uhden et al. (2012), "the role of mathematics in physics has multiple aspects: it serves as a tool (pragmatic perspective), it acts as a language (communicative function) and it provides a way of logical deductive reasoning (structural function)" (p. 486). Mathematics is not only a tool for computational manipulations; Mathematics is also commonly connected to physics content and can be used to reason about physics concepts and structure physical thought. For instance, differential equations are central to myriad key ideas in physics, such as harmonic oscillation, projectile motion, Newton's Law of Cooling, and the Schrödinger Equation. In multivariate calculus, students learn gradient, divergence, and curl, which are central to electromagnetism. In linear algebra, the notions of change of basis, orthonormality, and inner products are crucial in conceptualizing and computing the probability that a measurement of a particle's spin angular momentum would yield a particular value (in particular, an eigenvalue of a relevant operator). Physics students have to make connections between concepts, notation systems, and procedures they have learned in both mathematics and physics courses, and these sometimes vary between the two disciplinary cultures. Due to the interconnected nature of mathematics and physics, it is essential for undergraduate students to learn how to reason with mathematics as they address physical problems.

In this paper, we examine the complexity of undergraduate physics students' mathematical reasoning used as they solve two probability problems in the context of quantum mechanics. We address the following research question: *How do undergraduate physics students reason with mathematical concepts and procedures as they solve quantum mechanics problems?* In particular, our research goal was to investigate the linear algebra reasoning students leveraged in their solutions and explanations regarding the quantum mechanics problems shown in Figure 1. Our results demonstrate the intricacy of students' problem-solving methods and exhibit how students draw on their understanding of concepts from both mathematics and physics to inform their flexibility in choosing an appropriate problem-solving approach. This paper offers insights into how students reason about mathematics content in ways that are interconnected with and inseparable from physics content. Readers who teach undergraduate mathematics courses in which physics majors (or other STEM-discipline students) are enrolled will benefit from this paper by learning more about how the mathematics commonly taught in a linear algebra course is leveraged in other STEM disciplines.

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Consider the quantum state vector $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$.
a) Calculate the probabilities that the spin component is up or down along the z-axis.
b) Calculate the probabilities that the spin component is up or down along the y-axis.

Fig. 1 The quantum mechanics problems addressed in this study

Brief Physics Background

To assist the reader in following the students' work on the problems in Figure 1, we provide a brief summary of relevant content[1]. Quantum mechanical systems can be assigned to a Hilbert space, every possible state of the physical system is associated with a vector in the Hilbert space, and every possible observable is associated with a Hermitian operator. For example, spin is a measure of a particle's intrinsic angular momentum, which is related to the particle's magnetic moment. This observable is represented mathematically by an operator such as $\hat{S}_z$ (where the z indicates the particle's axis of rotation; the analogous information can be determined for other axes of rotation, such as *y*). In a spin-½ system, there are only two possible results for the S_z measurement: $\pm\frac{\hbar}{2}$, and these are the (necessarily real) eigenvalues of the Hermitian operator $\hat{S}_z$. After the measurement the system will be found in the corresponding eigenstate.

Using the notation system introduced by Dirac (1939), a state vector is denoted as a ket $|\psi\rangle$. The eigenstates corresponding to the possible measurements of an observable create an orthonormal basis for the associated Hilbert space. For example, the eigenstates for the spin-½ operator $\hat{S}_z$ can be expressed as $|+\rangle$ and $|-\rangle$, and they correspond to the measurements $\frac{\hbar}{2}$ and $\frac{-\hbar}{2}$, respectively. Any quantum state $|\psi\rangle$ in this system can be expressed as a linear combination of $|+\rangle$ and $|-\rangle$, namely: $|\psi\rangle = a|+\rangle + b|-\rangle$ for scalars $a, b \in \mathbb{C}$. The complex conjugate transpose of a ket $|\psi\rangle$ is called a bra and is symbolized as $\langle\psi| = a^*\langle+| + b^*\langle-|$. The probabilistic interpretation of the principle of superposition in quantum mechanics means that $|\psi\rangle$ will sometimes have attributes that resemble those of $|+\rangle$ and sometimes those of $|-\rangle$. More precisely, if the particle is in a state $|\psi\rangle$, the measurement of its spin along the *z*-axis will yield one of the eigenvalues $\frac{\hbar}{2}$ and $\frac{-\hbar}{2}$ with probability proportional to the modulus squared of the projection of $|\psi\rangle$ along either the eigenvector $|+\rangle$ or $|-\rangle$, respectively. The state of the system will change from $|\psi\rangle$ to $|+\rangle$ or $|-\rangle$ as a result of the measurement.

Problem A (see Figure 1) asks for the probability of obtaining $\frac{\hbar}{2}$ or $\frac{-\hbar}{2}$ in a measurement of observable S_z on a system in state $|\psi\rangle$. The solution is calculated by $P_\pm = |\langle\pm|\psi\rangle|^2$, where $\langle\pm|\psi\rangle$ is an inner product between one of the *z*-basis kets and psi. The solution for problem B (see Figure 1) is calculated by $P_\pm = |_y\langle\pm|\psi\rangle|^2$, where $_y\langle\pm|\psi\rangle$ is an inner product between one of the *y*-basis kets and psi. To complete problem B, a change of basis is involved because the given state $|\psi\rangle$ is written in terms of the z-basis, but the prompt asks for the probability that the

[1] The summary draws from McIntyre et al. (2012), Shankar (2012), and Wawro et al. (2020).

spin component is up or down along the y-axis. The two main approaches are to either change $|\psi\rangle$ to be written in terms of the y-basis (denoted $|\pm\rangle_y$) and calculate $P_{\pm,y} = |_y\langle\pm|\psi\rangle_y|^2$, or change the y-basis vectors to be written in terms of the z-basis and calculate $P_{\pm,y} = \left|\left(\frac{1}{\sqrt{2}}\langle+| \mp \frac{i}{\sqrt{2}}\langle-|\right)|\psi\rangle\right|^2$. In either approach, one would need to utilize the equations $|\pm\rangle_y = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$.

Literature Review

Several mathematical concepts are involved in the problems shown in Figure 1, including probability, inner products, basis, and change of basis. Some studies have investigated student understanding of probability in quantum mechanics contexts (e.g., Close et al., 2013; Passante et al., 2018). Wan et al. (2019) investigated student understanding of inner products in relation to quantum probabilities. They asserted that students need a functional understanding of inner products and quantum states in order to understand how to determine quantum probabilities. Serbin et al. (2021) analyzed quantum mechanics students' and instructors' discourse in the context of solving probability problems by identifying aspects of their culturally shared social language (Gee, 2005) particular to basis and change of basis. They found that students' and instructors' discourse about change of basis referred to either changing the form of a vector, writing a vector in another form, changing a vector into another vector, or switching bases. They conjectured that these varied forms of discourse could be indicative of different ways of reasoning about change of basis within the quantum mechanics context. Schermerhorn et al. (2019) investigated physics students' reasoning about basis and change of basis in the context of calculating expectation value problems. They claimed that several students "did not attend to the basis representation of vectors or matrices when carrying out matrix multiplication" (p. 020144-17). They found that a challenge for most students was choosing an appropriate basis in which to express the matrices and vectors involved in the calculation. The ubiquitous use of basis and change of basis in solving quantum mechanics problems warrants research about how students reason about these central concepts from linear algebra.

There is a growing body of literature related to student understanding of basis (e.g., Bagley & Rabin, 2016). Focusing on students' productive ways of intuitively reasoning about basis, Adiredja and Zandieh (2017) and Zandieh et al. (2019) conducted interviews to investigate students' conceptual metaphors for basis. Students described real-life examples of basis, including contexts such as recipes, fashion outfit choices, marching band, and religious teachings. In the students' explanations of how the real-life examples related to basis, they described bases as minimal, maximal, representative, essential, different, and non-redundant. Zandieh et al. (2019) found students used real-life examples to illustrate the roles of a basis as generating, structuring, and traveling, and the characteristics different and essential. Stewart and Thomas (2010) found that when students in their study reasoned about basis, they mainly focused on symbolic matrix manipulations such as row-reduction but often did not seem to understand how the matrix-based calculations were related to finding a basis for a vector space.

In addition, the students often did not connect span and linear independence with basis as they created concept maps, nor did they attend to embodied conceptualizations of basis. Schlarmann (2013) found that two students focused on linear independence as they determined a basis for a particular subspace of $\mathbb{R}^n$ and on span as they verified their set actually formed a basis. We found few studies that focused on student understanding of change of basis. One exception is Hillel (2000), who posited challenges students may face with the algebraic notion of change of basis; Hillel stated that students who conceptualize a vector as a string of numbers may not understand how two strings of numbers (i.e., the same vector in two different bases) can be equivalent. These researchers focused on student understanding of linear algebra concepts within undergraduate mathematics, but there is additional complexity associated with students' use of this mathematical understanding in physics contexts.

At a larger grain size, how students reason about the relationship between mathematics and physics is of great interest to educational researchers. Studies have focused on physics students' understanding of calculus (e.g., Bajracharya & Thompson, 2016; Christensen & Thompson, 2012; López-Gay et al., 2015; Schermerhorn & Thompson, 2019), differential equations, (e.g., Wittmann & Cakir, 2008), and linear algebra (e.g., Karakok, 2019; Serbin et al., 2020; Wawro et al., 2020; Wawro et al., 2019). Physics students are often required to take several undergraduate mathematics courses, as the content is often leveraged in their physics courses. Quantum mechanics, in particular, draws on several linear algebra topics, such as matrices, vector spaces, bases, inner products, and eigentheory. Undergraduate physics students thus have to use what they learned in their Linear Algebra courses within the context of quantum mechanics (e.g., Karakok, 2019). As Caballero et al. (2015) noted, researchers have documented that reasoning about undergraduate mathematics in physics contexts can be a difficult endeavor for students.

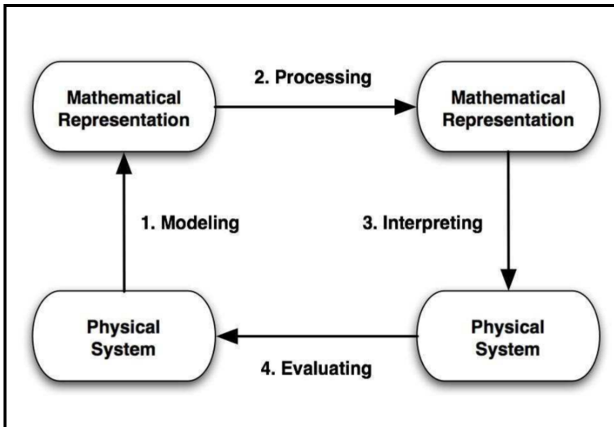
"Math may be the language of science, but math-in-physics is a distinct dialect of that language" (Redish, 2006, p. 1). This presents a potential challenge in that mathematics content can be used differently in physics courses than in what students previously encountered in their undergraduate mathematics courses. Caballero et al. (2015) explained, "While students see many of the mathematical tools and techniques used in upper-division physics in their math courses, the operationalization of these tools in their physics courses can be strikingly different" (p. 5). For instance, bases of vector spaces in quantum mechanics contexts are orthonormal, whereas that is not the case for all vector spaces students encounter in their linear algebra courses. A second example involves the spin component operator matrix $\hat{S}_n = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$ along the general direction $\hat{n}$; determining its eigenvalues and eigenvectors involves utilizing Euler's identity and trigonometric identities, normalizing, and relegating imaginary components to the second term to arrive at $|+\rangle_n = cos\frac{\theta}{2}|+\rangle + sin\frac{\theta}{2}e^{i\phi}|-\rangle$ and $|-\rangle_n = sin\frac{\theta}{2}|+\rangle - e^{i\phi}cos\frac{\theta}{2}|-\rangle$ as eigenvectors for eigenvalues $\frac{\hbar}{2}$ and $\frac{-\hbar}{2}$, respectively.

The previous example, which has eigenvectors written as a superposition of kets in Dirac notation, leads to another potential challenge inherent in this math-in-physics dialect: the different representations and notation systems used in quantum mechanics courses and mathematics courses (e.g., Wagner et al., 2011). One particular example that is very pertinent to quantum mechanics is demonstrating flexibility in using both matrix notation and Dirac notation (e.g., Gire & Price, 2015; Schermerhorn et al., 2019; Wan et al., 2019). Wawro et al. (2020) investigated physics students' metarepresentational competence with these two notation systems. They found that "students' rich understanding of linear algebra and quantum mechanics includes and is aided by their understanding and flexible use of different notational systems" (p. 020112-2). Wan et al. (2019) discussed how structural features of quantum notations can foster or hinder students' reasoning about inner products and quantum probabilities. They found that Dirac notation brackets helped students make sense of inner products of energy eigenstates and state vectors. Overall, familiarity with using both matrix notation and Dirac notation is essential for solving problems in quantum mechanics, but developing fluency within and across both notation systems could be nontrivial for students.

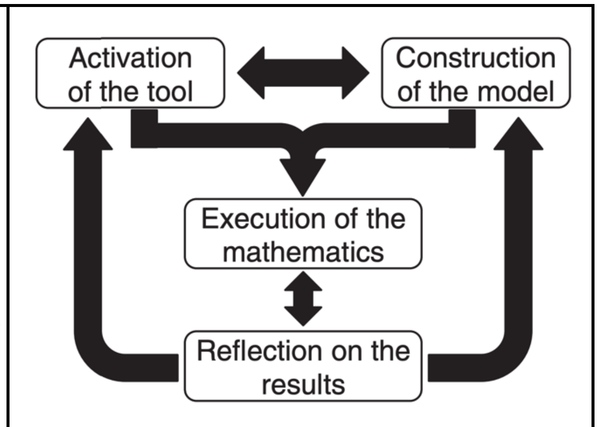
Another potential challenge in reasoning about mathematics in physics contexts is that students have to connect the two domains by interpreting the mathematical symbols in terms of the physical phenomena they symbolize. Redish (2006) suggested that physicists and mathematicians may interpret equations differently due to the meanings they attribute to symbols. Thus, students may reason about equations and symbols differently in mathematics and physics contexts (e.g., Wagner et al., 2011). Caballero et al. (2015) reviewed literature on student reasoning about mathematics in upper division physics courses and found that "fluency with procedural mathematics is often not the primary barrier to student success" (p. 4), rather "students often struggle to interpret/make sense of mathematical expressions in terms of the appropriate physics (i.e., connecting the math and physics)" (p. 4). Her and Loverude (2020) discussed a similar finding that physics students demonstrated fluency in using mathematical procedures but experienced difficulty with interpreting matrix equations in terms of a physical system. Learning to interpret mathematical symbols and structures in terms of physical phenomena is an important, yet nontrivial, endeavor for physics students as they reason about mathematics in physics contexts. We further investigate the complexity of physics students' mathematical reasoning in the current study.

Theoretical Framework

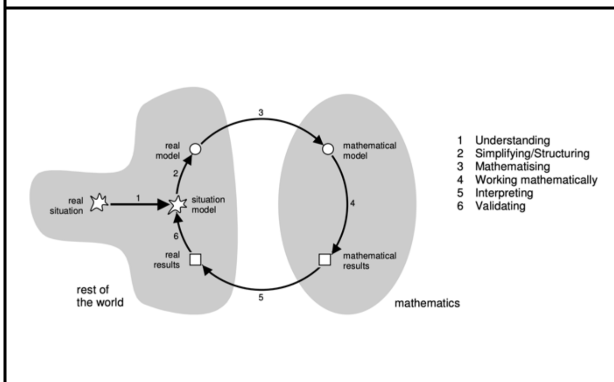
One common theorizing of students' reasoning about and use of mathematics within a physical context leverages the notion of a modeling cycle. For example, such modeling cycles include Redish and Bing's (2009) model (see Figure 2a), Wilcox et al.'s (2013) Activation, Construction, Execution, Reflection (ACER) Framework (see Figure 2b), and Blum and Leiß's (2005) cognitive modelling cycle (see Figure 2c). Despite their prevalence in mathematics and physics education research literature, it has been documented that cycles are not always suitable for tasks in engineering and physics contexts because student reasoning does not always follow a cyclical pattern (e.g., Czocher, 2013). Thus, we draw on the theoretical constructs of mathematization and interpretation that are commonly presented in these models, instead of leveraging their cyclical nature.



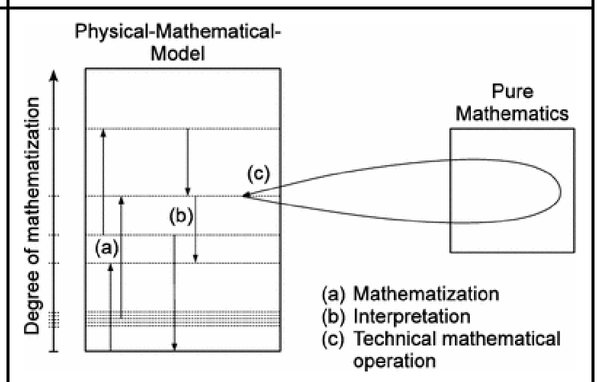
(a) Redish and Bing's (2009) model



(b) Wilcox et al.'s (2013) Framework



(c) Blum and Leiß's (2005) modelling cycle



(d) Uhden et al.'s (2012) model

Fig. 2 Various modeling cycles for students' use of mathematics in physical problems

The aspects of mathematization and interpretation are commonly used in models of student reasoning about mathematics in physics. Redish and Bing (2009) indirectly referred to mapping physical structures into mathematical ones as mathematizing, Blum and Leiß (2005) explained it as what transforms the real model into a mathematical one, and Wilcox et al.'s (2013) "Construction of the model" step is likened by Caballero et al. (2015) to mathematization as described by Karam (2014):

> Mathematizing is understood as the process of constructing a mathematical representation for a physical situation (in the broad sense). This process can be seen as a translation from the physical world (e.g., observations and experiments) into mathematical structures (e.g., numbers, functions, and vectors). (p. 5-6)

Mathematizing is an essential problem-solving skill that students need to develop to structure physical situations in terms of mathematics, which allows them to use mathematical procedures and properties to solve problems. Researchers have investigated students' ability to mathematize and how instructors can support students' development of mathematizing skills (e.g., Caballero et al., 2015; Kanderakis, 2016; Karam, 2014). Researchers have also focused on physics students' ability to interpret mathematical symbols and structures in terms of physical concepts. The notion of *interpreting* involves making sense of mathematics in terms of physics, such as in explaining the physical meaning of mathematical symbols (Uhden et al., 2012).

These reasoning skills of mathematizing and interpreting are central aspects of Uhden et al.'s (2012) mathematical-physical model (Figure 2d), which frames our study. We draw on Uhden, Pietrocola, Karam, and Pospiech's theory proposed in several of their works (Karam, 2014; Karam et al., 2011; Pietrocola, 2008; Uhden et al., 2012). Uhden et al. (2012) used the distinction of technical and structural skills to propose a modelling cycle of how mathematical knowledge is used in modelling physical situations (see Figure 2d). In their modelling cycle, one simplifies and structures phenomena from the world and performs varying degrees of mathematization, denoted by the upward arrows in the figure. One uses technical skills to perform mathematical operations, denoted by the loop to and from the pure mathematics part of the figure. These technical skills involve reasoning with mathematical concepts and procedures. One interprets mathematical structures in terms of the corresponding physical phenomena denoted by the downward arrows in the figure. We chose to draw on Uhden et al.'s model because it highlights the entanglement of mathematics and physics that students navigate. We leverage Uhden et al.'s constructs of technical and structural skills as a framework for analyzing physics students' reasoning about mathematics on a quantum mechanics problem.

Technical skills involve employing properties of mathematical systems in physics contexts, such as in performing algorithms. They are characterized as being "connected to the internal context of mathematical knowledge" (Pietrocola, 2008, p. 7). Technical skills are associated with the instrumental ability to use knowledge of mathematical concepts and procedures as a tool to solve problems in physics; this particular use of mathematical skills is independent of connections to physics. Karam (2014) elucidated two types of technical skills: procedural and conceptual. *Technical-procedural skills* involve using mathematics to perform manipulations or procedures, such as in solving an equation. These skills are related to procedural knowledge (Hiebert & Lefevre, 1986; Star, 2005), which encompasses "knowledge of procedures that is associated with comprehension, flexibility, and critical judgment" (Star, 2005, p. 408). We focus on the aspect of *flexibility*, a central facet of students' decision-making when choosing a particular problem-solving approach. It "incorporates knowledge of multiple ways to solve problems and when to use them" (Rittle-Johnson & Star, 2007, p. 562). Thus, we posit that flexibility in choosing procedures is an important aspect of students' use of technical-procedural skills. *Technical-conceptual skills* involve giving conceptual explanations of mathematical rules and procedures. These skills are akin to Hiebert and Lefevre's (1986) conceptual knowledge, defined as "knowledge that is rich in relationships…a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3–4).

Structural skills incorporate reasoning about the interconnectedness of mathematics and physics and are “based on the capacity of employing the mathematical knowledge for structuring physical situations” (Pietrocola, 2008, p. 7). Karam et al. (2011) suggest that structural skills are related to “the recognition of the deep connection between the physical content and the mathematical formulation of a particular concept” (p. 2), and the authors describe different types of structural skills: mathematizing, interpreting, deriving, and analogizing. Broadly speaking, they associate mathematizing with translating from physics to mathematics and interpreting as translating from mathematics to physics. Karam et al. suggested that the *structural-mathematizing skill* is the process of translating from the physical world to mathematical structures and formulas, and that this “depends on being able to think mathematically, which involves not only a significant understanding of mathematical concepts and theories, but also the ability of abstracting, idealizing and modelling physical phenomena” (p. 2). The *structural-interpreting skill* involves making sense of mathematics in terms of physics, such as in explaining the physical meaning of mathematical expressions or equations, such as interpreting mathematical symbols in a formula in terms of the physical phenomena that they symbolize. Furthermore, interpreting is an important aspect of the advancement of science. As Karam (2014) stated:

> The fact that we are able to find more physics through the interpretation of mathematical expressions testifies that mathematics is not merely a language that offers a precise description of physical phenomena, but that in many cases the mathematical formalism guides the physical thought. (p. 9)

Indeed, in Dirac’s seminal text (1947), after delineating a set of assumptions that completely defined relations between states of a dynamical system at a point in time, Dirac stated that the relations “appear in mathematical form, but they imply physical conditions” (p. 23). He further offered a preliminary example: “if two states are orthogonal, it means at present simply a certain equation in our formalism, but this equation implies a definite physical relationship between the states, which further developments of the theory will enable us to interpret” (p. 23) empirically. Even in the early stages of the development of quantum mechanical theory, interpreting was paramount.

Methods

The participants were 12 undergraduate physics students, of which eight (pseudonyms A#) were enrolled in a junior-level Quantum Mechanics course at University A, a large research institution in the Northwest US. The other four participants (pseudonyms C#) were enrolled in a senior-level Quantum Mechanics course at University C, a medium-sized research institution in the Northeast US. The second author assigned these numeric pseudonyms to identify participants from course rosters. All but one student, A32, had taken a Linear Algebra course prior to enrolling in the Quantum Mechanics course. Based on University A and C’s Linear Algebra course descriptions, the content covered included: linear equations, row echelon form, matrix algebra, determinants, linear independence, orthogonality, vector spaces, matrix representations of linear transformations, eigenvalues, and eigenvectors. At University A, two students, A8 and

A11, were concurrently enrolled in a second Linear Algebra course that covered vector spaces, linear transformations, eigenspaces, diagonalization, singular value decomposition, orthogonality, inner product spaces, and spectral theorems. At University A, the Quantum Mechanics course began with a review of linear algebra content, including determinants, matrix operations, eigenvalues, eigenvectors, linear transformations, and properties of Hermitian matrices. At University C, the students took a separate Mathematical Methods in Physics course that covered pertinent linear algebra content.

Semi-structured interviews (Bernard, 1988) with each participant were conducted with the broad goal of gaining insight into how students reason with linear algebra concepts in quantum mechanics contexts. The interviews were recorded, transcribed, and written work was retained. We analyzed the participants' responses to the problems shown in Figure 1[2]. Follow-up questions were asked as needed during the interview to gain clarity regarding a participant's response.

Our analysis process involved first performing inductive open coding (Miles et al., 2013) on each student's transcript, labeling chunks with a code capturing what knowledge or skill the student implicitly used or explicitly described as they engaged with the problems. The authors independently open coded four students' transcripts, compared, and created a primary code list, which the first author used to code the remaining transcripts. When new codes emerged from interpreting the remaining transcripts, the new codes were added to the original list[3]. To check for internal consistency, the transcripts were coded again to ensure that no transcript segments were missed or miscoded.

We then performed deductive coding by assigning these original codes one of four a priori parent codes: structural-mathematizing, structural-interpreting, technical-conceptual, or technical-procedural. These parent codes derive from Uhden et al.'s (2012) and Karam's (2014) descriptions of structural and technical skills, as well as Hiebert and Lefevre's (1986) and Star's (2005) description of conceptual and procedural knowledge. We coded the students' reasoning as leveraging structural skills whenever students translated between mathematical objects and their corresponding physical entities via interpreting or mathematizing. We assigned the parent code of structural-mathematizing when the student reasoned about physics content in terms of mathematical structures. We used the structural-interpreting code when the student interpreted the mathematical symbols or results in terms of the physics context. These coded structural skills differ from technical skills, which students can perform solely by using mathematical knowledge or procedures that are not tied to the physical context. We thus coded the students' reasoning as leveraging technical skills whenever the students used mathematical conceptual or procedural knowledge on the tasks without having to reason about the mathematical structures or properties in terms of any physical entities. We assigned the technical-conceptual code when the student

[2] Two of the 12 students did not complete Problem a) because of time constraints during the interview.

[3] Our subjectivities influenced our data analysis and code creation. What we noticed in the interview data was influenced by our knowledge of mathematics and quantum mechanics. For instance, we could infer that a student used a mathematical property (e.g., distributivity) in their written work, even if they did not explicitly verbalize that.

used their conceptual knowledge of mathematical concepts (by either explicitly explaining the concept or implicitly using that concept in their solution) and the technical-procedural procedural code when the student used mathematical procedures or procedural flexibility as they solved the interview problems. The original codes, grouped according to their parent codes, are presented in Figure 3. Two of the original codes of recognizing that probabilities sum to one and using the inner products $\langle +|-\rangle = 0$, $\langle +|+\rangle = 1$ were assigned two parent codes of technical-conceptual [■1 and ■3, respectively] and technical-procedural [●12 and ●15, respectively] because students simultaneously demonstrated skills in reasoning about these concepts and using these procedures as they solved the problems. Overall, coding in this manner allowed us to investigate the students' use of structural and technical skills. See the Results section for a discussion of how we analyzed these coded structural and technical skills to inform our claims.

Results

Our analysis revealed two main findings. First, we found that students use intricate, nonuniform problem-solving methods with reasoning that moves fluidly between structural (mathematizing and interpreting) and technical (conceptual and procedural) skills in quick succession, in their solutions for problems A and B (Figure 1). Second, we found that students' technical and structural skills related to reasoning with inner products, orthonormal bases, basis, change of basis, and probability supported their flexibility in choosing an appropriate problem-solving approach on these problems. Our analysis leveraged Uhden et al.'s (2012) model, which facilitates an illumination of the entanglement of mathematics and physics. We discuss these findings and use the symbols ⬟#, ▲#, ●#, and ■# listed in Figure 3 to refer to the coded technical and structural skills throughout the Results.

The Intricacy of Students' Problem-Solving Methods

Our first main finding highlights the intricacy of students' problem-solving methods that were evident in their responses to problems A and B. Students' reasoning moved fluidly between structural (mathematizing and interpreting) and technical (conceptual and procedural) skills in quick succession. They did so in ways that were nonuniform across the students. This highlights the complexity of the students' mathematical and physical work, which could not be adequately captured by a more straightforward modeling cycle such as Redish and Bing's (2009) map-process-interpret-evaluate model or Wilcox et. al's (2013) activate-construct-execute-reflect model.

To determine this finding, we organized the codes of each student's technical and structural skills in chronological order according to when the student used the skill during their work on the problem. We then compared the skills progression across students to try to identify patterns in the sequence of the skills used. We found no apparent patterns, which illustrates the idiosyncratic and intricate problem-solving methods used by the students. Figures 4 and 5 illustrate the progression of each student's use of structural-mathematizing, structural-interpreting, technical-procedural, and technical-conceptual skills on problems A (Figure 4) and

⬣ Structural - Interpreting Skills Codes	
⬣1	Recognize $|\psi\rangle$ is in a different basis
⬣2	Recognize Basis and axis match
⬣3	Interpret probability solution in context
⬣4	Interpret result to be incorrect/irrelevant
⬣5	Recognize a ket is a linear combination of basis vectors or is in a basis
⬣6	Know how to check answer
⬣7	Recognize resulting probabilities are the same
⬣8	Recognize that result of calculation is $|\psi\rangle$ in terms of y- basis
⬣9	Recognize that vectors are in (terms of) same basis

▲ Structural - Mathematizing Skills Codes	
▲1	Recognize $|+\rangle$ and $|-\rangle$ form a basis
▲2	Recognize that having 2 possible results implies there are 2 probabilities
▲3	Acknowledge alternative procedure or approach
▲4	Reason that if basis and axis match, then use Born Rule
▲5	Reason that if basis and axis match, then use probability formula
▲6	Recognize that basis vectors here are orthonormal
▲7	Recognize that change of basis is necessary
▲8	Recognize that changing $|+\rangle_y$ is easier than changing $|\psi\rangle$
▲9	Recognize that solving the problem is easier when the basis and axis match
▲10	Know to use Dirac notation
▲11	Recognize that measuring spin up or down implies one should work with $|+\rangle$ or $|-\rangle$
▲12	Know to use probability formula
▲13	Recognize that measuring spin up or down implies one should square the coefficient of $|+\rangle$ or $|-\rangle$
▲14	Recognizes that approach works for finding probability of both spin up and down
▲15	Recognize quantum state vectors need to be normalized
▲16	Recognize that vectors in inner product do not match, so a change of basis is needed
▲17	Reason that $|\psi\rangle$ being normalized implies probabilities add to one
▲18	Anticipate probability
▲19	Acknowledge that probability is the norm squared of the probability amplitude
▲20	Acknowledge that state is superposition/ LC of eigenvectors
▲21	Acknowledge that terms irrelevant to the measurement drop out of the computation
▲22	Translate between Dirac and column vector/matrix notation

● Technical - Procedural Skills Codes	
●1	Use $1 - P_+ = P_-$ to calculate probability
●2	Add linear combinations of kets
●3	Perform arithmetic with real and complex numbers
●4	Change $|+\rangle_y$ or $|-\rangle_y$ to be in terms of z-basis
●5	Change basis that $|\psi\rangle$ is in terms of
●6	Recognize that coefficients remain after inner product
●7	Use commutativity of scalars with bras & kets
●8	Distribute
●9	Exhibit flexibility in choosing an approach to the problem
●10	Determine Hermitian adjoint/complex conjugate
●11	Recognize that inner product is product of bra and ket
●12	Use the inner products $\langle +|-\rangle = 0, \langle +|+\rangle = 1$
●13	Calculate the norm/modulus
●14	Use order of operations
●15	Reason that probabilities sum to one
●16	Solve a system of equations
●17	Square numbers
●18	Substitute $|+\rangle$ and $|-\rangle$ in terms of y-basis
●19	Substitute $|+\rangle_y$ in terms of z-basis
●20	Substitute $|\psi\rangle$ in terms of z-basis
●21	Use change of basis equations
●22	Acknowledge ${}_y\langle +|$ and ${}_y\langle -|$ have different equations
●23	Conflate coefficients
●24	Forget to conjugate
●25	Use "Freshman's Dream" rule
●26	Know when they are done
●27	Recognize that probability is between 0 and 1
●28	Check that a state is normalized
●29	Acknowledge that changing basis is substituting components
●30	Perform outer product
●31	Perform different products of matrices and vectors

■ Technical - Conceptual Skills Codes	
■1	Recognize that probabilities sum to one
■2	Recognize that change of basis results in vectors being in terms of the same basis
■3	Recognize inner products $\langle +|-\rangle = 0, \langle +|+\rangle = 1$
■4	Recognize that norm of i is 1
■5	Recognize that change of basis is necessary to compute inner product
■6	Recognize that vectors in inner product must be in terms of same basis
■7	Define basis
■8	Acknowledge that conjugation results in other probability result
■9	Acknowledge that inner product is same as dot product
■10	Acknowledge that inner product is same as matrix multiplication
■11	Recognize that neglecting to conjugate the bra will mess up solution
■12	Acknowledge that change of basis does not change norm of vector

Fig. 3 Codes for structural-interpreting, structural-mathematizing, technical-procedural, and technical-conceptual skills

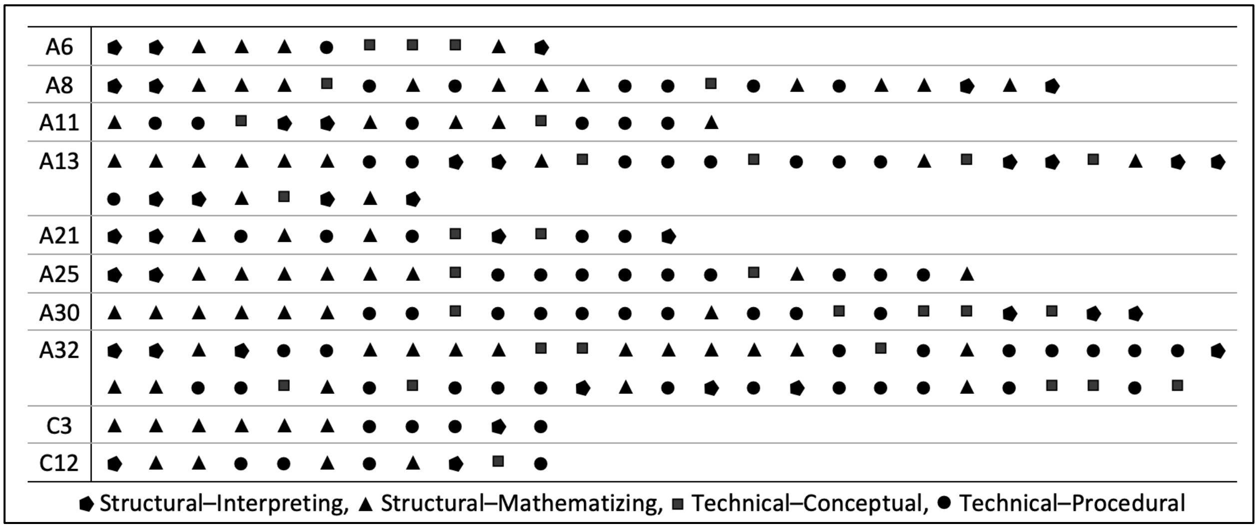


Fig. 4 Progression of students' structural and technical skills on Problem A

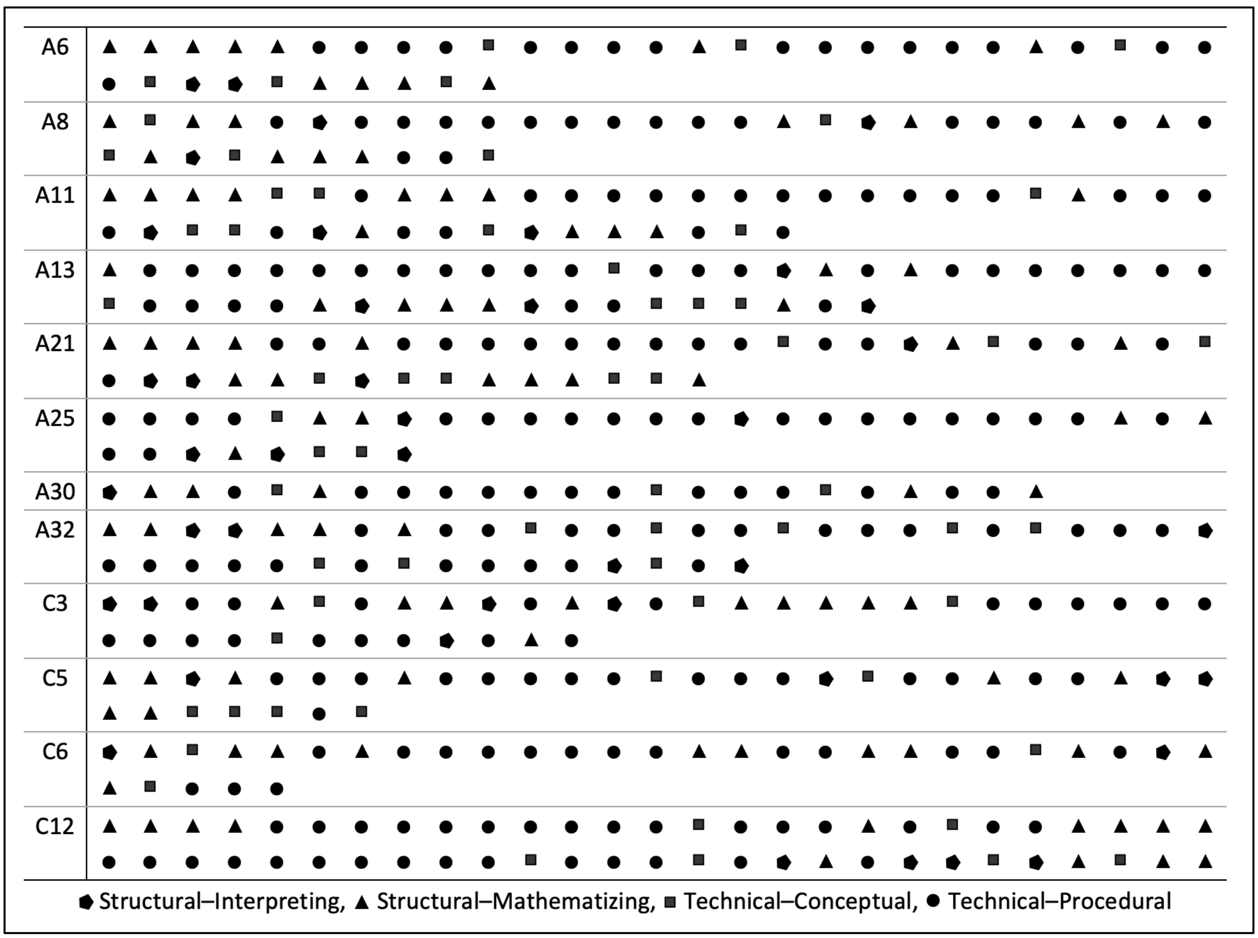


Fig. 5 Progression of students' structural and technical skills on Problem B

problem B (Figure 5). For brevity, we omit some codes in the description of students' reasoning throughout the results section. In the following three subsections, we provide in-depth analysis of three students' problem-solving methods, namely A13 on problem A and C12 and A8 on problem B, to illustrate students' varied and intricate use of structural and technical skills. We chose these students as illustrative examples to exhibit work from both Universities A and C and to highlight A8's unique (within this dataset) problem-solving method.

A13's Use of Structural and Technical Skills on Problem A

A13 used intricate problem-solving methods evident their use of a variety of structural and technical skills throughout their[4] work on problem A. They began by setting up the formula needed for calculating the probability that psi's spin component is up along the *z*-axis. A13 explained their reasoning as:

> "So probability is given by... you take the bra of what you were expecting out versus the ket of what's coming in, and you take the norm of that squared. So, in the case of this, you'd have what's coming out, well you want the spin component up or down. It's in the *z*-axis, so we'll do plus first, plus bra with the psi ket, and you'd square. So, this would, expanding that out, you get the plus, uh, bra for the *z*-axis."

A13 recognized that the basis and axis matched in the problem setting, so they could use the probability formula from the probability postulate [▲5, ▲12]. They also recognized that it was appropriate to use Dirac notation to calculate the inner product [▲10]. A13 also knew that to determine the probability that the spin component was up along the *z*-axis, they needed to use $\langle +|$ as the bra in the inner product [▲11]. Overall, A13 used several structural skills to mathematize the problem situation to the equation $P = |\langle +|\psi\rangle|^2$.

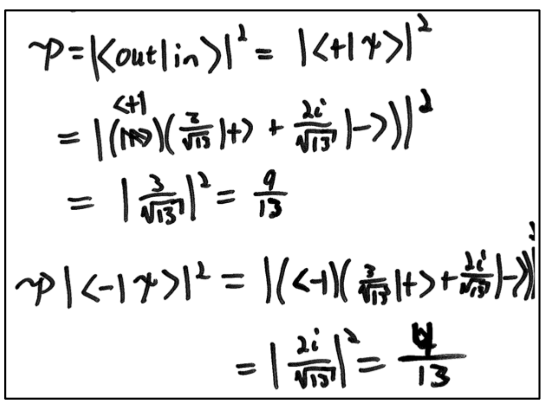


Fig. 6 A13's written work for Problem A

A13 then completed the probability calculation using $P = |\langle +|\psi\rangle|^2$. They performed the technical-procedural skills of substituting $\frac{3}{\sqrt{13}}\,|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ in for $|\psi\rangle$ and writing the inner

[4] We use the gender-neutral singular pronouns "they" and "their" to refer to the students throughout this paper.

product as a product of a bra and a linear combination of kets (see Figure 6) [●11,●20]. A13 then used structural-interpreting, technical-conceptual, and structural-mathematizing skills as they explained their reasoning: "It's good because they're both in the *z* basis, the kets are both in the *z* basis 'cause [if] this was like in the *x* you'd have this longer, more complicated math to do." A13 interpreted the mathematical symbols to acknowledge that the kets in the inner product are both expressed in terms of the *z* basis [⬟5]. In doing so, they interpreted the symbols $\frac{3}{\sqrt{13}}\,|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ and the problem statement to conclude the basis and axis match [⬟2], and they reasoned that kets in an inner product must be expressed in the same basis [■6]. They acknowledged that if $|\psi\rangle$ was expressed in terms of a different basis that did not match the axis, they would have "more complicated math to do," as that would require a change of basis [▲3]. Thus, A13 switched between using various technical and structural skills as they completed calculated the probability calculation using the formula $P = |\langle +|\psi\rangle|^2$, and they then used structural-interpreting, technical-conceptual, and structural-mathematizing skills to explain their reasoning.

Finally, A13 used more technical skills to finish their calculation of the probability $P = |\langle +|\psi\rangle|^2$. After using the distributive property and the commutativity of vector addition and scalar vector multiplication [●7,●8]. A13 then described the skipped steps in their calculation of $\langle +|+\rangle = 1$ and $\langle +|-\rangle = 0$, saying, "A bra times a ket with the same value is just 1, and then a bra times a ket of different values plus one- one would be 0." We coded this as A13 using the technical-conceptual and procedural skill of reasoning about the inner products of the orthonormal *z*-basis vectors being $\langle +|+\rangle = 1$ and$\langle +|-\rangle = 0$ [●12,■3]. A13 then finished the calculation by taking the norm of $\frac{3}{\sqrt{13}}$ and squaring it to find the probability of $\frac{9}{13}$ [●13,●17]. They then interpreted the result of the calculation, explaining "so you get 9 over 13 would be the probability, and it would get spin up" [⬟3].

A13 used a similar approach for calculating the probability that the spin component of angular momentum was down along the *z*-axis. A13 explained, "Then likewise for spin down, you'd have uh, minus and your psi squared," and they wrote $P = |\langle -|\psi\rangle|^2$. We coded this as the structural-mathematizing skill of recognizing that the same approach works for finding both the up and down spin probabilities [▲14]. After using technical-conceptual and technical-procedural skills to explain and complete their calculation, A13 at first reached an incorrect conclusion but quickly self-corrected: "...And so the, uh norm of that would be 2i times negative 2i, so that would go, so it'd just be 2 over 13. Or no, scratch that. 4 over 13. Not 2." A13 used the structural-interpreting skill to interpret their result to be incorrect for the problem [⬟4]. They explained:

> "I forgot to do 2 times 2. Yeah. So, 2 times 2 is actually 4…'cause I was thinking not, 2 plus 9 is not 13, 'cause you'd get a total probability of 1, 'cause the particle has to go up or down, so if that was 2, you'd get a total probability of 11 thirteenths, which is not 1, but looking at it now, 4 thirteenths over 9 thirteenths is 13 over 13, which is 1, so the probability, total probability is 1."

In this explanation, A13 used various structural-mathematizing, structural-interpreting, technical-conceptual, and technical procedural skills in quick succession. A13 recognized that there were two possible results of the measurement that corresponded to two probabilities [▲2]. They then recognized that the two possible probabilities add to one [●15,■1], and they made sense of their probability solution in the problem context, which helped them check their answer [⬟3,⬟6].

Overall, A13 began problem A by using several structural skills to mathematize the situation described in the prompt, set up the probability formula needed for the calculation, and explain their thinking. A13 switched between using structural-interpreting, technical-conceptual, and structural-mathematizing skills primarily related to reasoning about basis and inner products as they explained their reasoning. They used mostly technical-procedural skills to finish their calculation of the probabilities $P = |\langle +|\psi\rangle|^2$ and $P = |\langle -|\psi\rangle|^2$. A13 then used various structural and technical skills related to reasoning about probability in this quantum mechanics context to interpret and correct their results. A13 thus used various structural and technical skills in quick succession, and these skills did not follow a particular sequence. This highlights the intricacy of students' use of technical and structural skills while solving this quantum mechanics problem.

C12's Use of Structural and Technical Skills on Problem B

All of the students in our data set used various structural and technical skills as they worked on and reasoned about problem B, which necessarily involved a change of basis. C12's work serves as an exemplar for the intricacy of the details evident in students' problem-solving methods. C12 first mathematized the situation in the problem prompt using the structural-mathematizing skills of knowing to use Dirac notation for the calculation [▲10], using the probability formula [▲12], using ${}_y\langle +|$ and ${}_y\langle -|$ in the inner product part of the probability formulas $P_{+,y} = \left| {}_y\langle +|\psi\rangle \right|^2$ and $P_{-,y} = \left| {}_y\langle -|\psi\rangle \right|^2$, respectively, when finding the probability that the spin component was up or down along the y-axis [▲11]. C12 then recognized that a change of basis was necessary [▲7] and changed ${}_y\langle +|$ to be written in terms of the z basis using the appropriate change of basis equations [●4,●21]. As they did so, C12 wrote the Hermitian adjoint of $|+\rangle_y$ as ${}_y\langle +| = \frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}} i \langle -|$ and explained, "since this is the complex conjugate, um, I flipped the sign for i" [●10]. C12 then substituted ${}_y\langle +|$ and $|\psi\rangle$ in the inner product ${}_y\langle +|\psi\rangle$ to write it as a product of linear combinations of bras and kets: $(\frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}} i \langle -|)(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$ [●11,●19,●20]. They then used the used the distributivity and commutativity of scalars with bras and kets, arithmetic with real and complex numbers, the appropriate order of operations, and the orthonormality properties of the basis vectors (i.e., $\langle \pm|\pm\rangle = 1$ and $\langle \pm|\mp\rangle = 0$) to compute the inner product to equal $\frac{5}{\sqrt{26}}$ [●3,●7,●8,●12,●14,■3]. In total, C12's problem-solving method to begin their work first utilized structural-mathematizing skills and then mostly technical-procedural skills to compute the inner product.

Next, C12 took the norm squared of this result of the inner product to determine $P_{+,y} = \left|\frac{5}{\sqrt{26}}\right|^2 = \frac{25}{26}$. They explained, “I have to square this. It's 25 over 26. If I didn't make a math error, then to my knowledge, it would be 1 over 26... Okay that's good. It adds up to one.” C12 thus used the technical-procedural skills of finding the norm and squaring the number [●13, ●17] and the structural-mathematizing skill of recognizing that two possible results of the measurement (i.e., up or down along the *y*-axis) yields two probabilities [▲2]. C12 used this along with their technical-conceptual and technical-procedural skills of reasoning that the sum of the probabilities of all possible outcomes is one to subtract $\frac{25}{26}$ from one to find the other probability, that the spin component of angular momentum was down along the *y*-axis, as $\frac{1}{26}$ [●1,●15,■1]. C12 repeated their method to check that their anticipated result for the probability that the spin component of angular momentum was down along the *y*-axis was in fact $\frac{1}{26}$ (see Figure 7). C12 then interpreted their determined probabilities in the context of the problem [⬟3].

In total, C12’s problem solving method integrated all four varieties of structural and technical skills in quick succession. This example of C12’s method on problem B demonstrates the complexity of students’ use of technical and structural skills and the intertwined nature of mathematical and physical reasoning while solving this quantum mechanics problem.

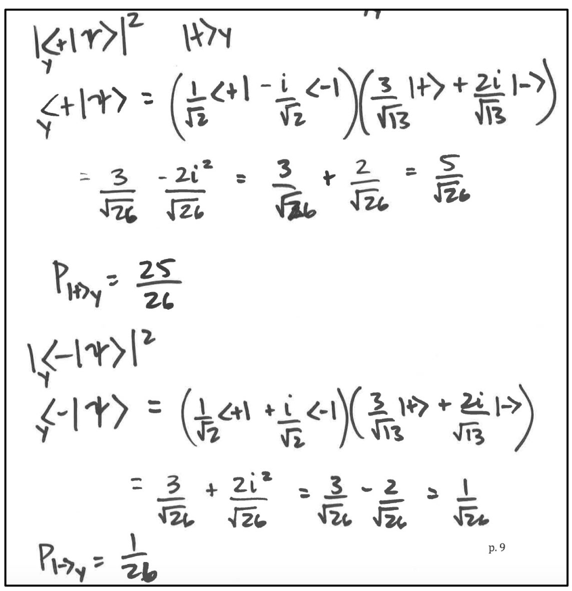


Fig. 7 C12’s written work on Problem B

A8’s Use of Structural and Technical Skills on Problem B

A8 exhibited a complex problem-solving approach (and thus method) that differed from C12’s. A8 began by comparing possible approaches for changing basis (see written work in Figure 8a). They explained:

> There are two ways to go about it, um, one of them is to put this vector in some phi prime that's in the *y* basis, and then just do *y* plus phi prime *y*, cause it makes calculations, and it follows the same rules as this. Um, the other possibility is to do, is to take the spin up *y* and go to whatever it is in the *z*, in the *z* basis, cause we have this in the *z* basis. Um, they're both equivalent.

A8 explained that they could either change $|\psi\rangle$ to be "in the *y*-basis" (i.e., a linear combination of *y*-basis vectors) and use the probability formula $P = \left| {}_y\langle + | \psi\rangle_y \right|^2$, or they could change ${}_y\langle +|$ to "whatever it is in the *z*-basis," which would yield $\left| (\frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}} i\, \langle -|) |\psi\rangle \right|^2$. They used a variety of structural and technical skills just in their initial deliberation of problem B. A8 used structural-mathematizing skills of acknowledging an alternative procedure [▲3], knowing to use the probability formula [▲12], and recognizing that they needed to perform a change of basis [▲7]. A8 also used the technical-conceptual skill of recognizing that a change of basis was necessary to perform the inner product [■5] and the technical-procedural skill of having flexibility in choosing an appropriate problem-solving approach [●9]. A8 then used the structural-interpreting skill of recognizing that the two problem-solving approaches would result in the same probabilities [⬟7].

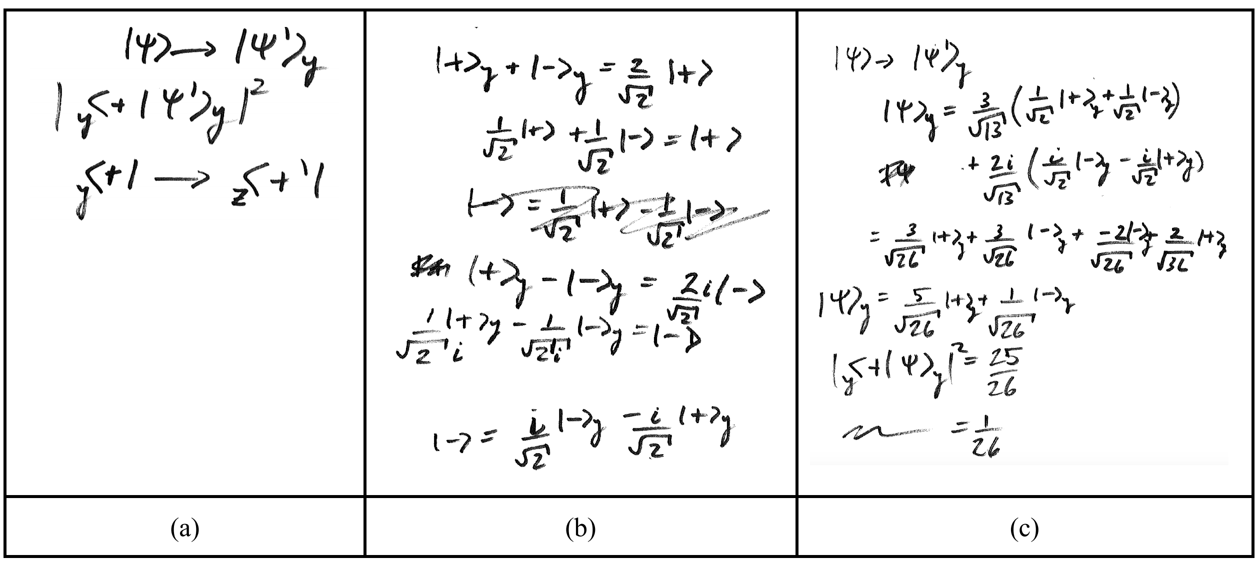


Fig. 8 A8's written work on Problem B

A8 decided to change $|\psi\rangle$ to be a linear combination of *y*-basis vectors, the first option they had mentioned. They then went on to use various technical-procedural skills as they performed the change of basis. A8 added the change of basis equations (given on a reference sheet) $|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle + i\frac{1}{\sqrt{2}}|-\rangle$ and $|-\rangle_y = \frac{1}{\sqrt{2}}|+\rangle - i\frac{1}{\sqrt{2}}|-\rangle$, which yielded $|+\rangle_y + |-\rangle_y = \frac{2}{\sqrt{2}}|+\rangle$ [●2,●5,●21]. They then divided both sides of the equation by $\sqrt{2}$ to get $\frac{1}{\sqrt{2}}|+\rangle_y + \frac{1}{\sqrt{2}}|-\rangle_y = |+\rangle$, which is a *z*-basis vector written as a linear combination of the *y*-basis vectors

(Figure 8b). A8 then subtracted the change of basis equations, which yielded $|+\rangle_y - |-\rangle_y = \frac{2}{\sqrt{2}} i|-\rangle$, and divided both sides by $\sqrt{2}i$ to get $\frac{1}{\sqrt{2}i}|+\rangle_y - \frac{1}{\sqrt{2}i}|-\rangle_y = |-\rangle$ to get the other *z*-basis vector written as a linear combination of the *y*-basis vectors (Figure 8b) [●3,●7,●14,●16]. They substituted $|+\rangle = \frac{1}{\sqrt{2}}|+\rangle_y + \frac{1}{\sqrt{2}}|-\rangle_y$ and $|-\rangle = \frac{1}{\sqrt{2}i}|+\rangle_y - \frac{1}{\sqrt{2}i}|-\rangle_y$ into $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ and simplified the equation to yield $|\psi\rangle_y = \frac{5}{\sqrt{26}}|+\rangle_y + \frac{1}{\sqrt{26}}|-\rangle_y$ (Figure 8c) [●2,●3,●8,●18]. A8 then verified their work by checking that the resulting vector "is normalized because, because twenty, you just double check, 25 and 1 is 26." A8 knew that quantum state vectors need to be normalized [▲15], so they checked that the quantum state $|\psi\rangle$ was indeed normalized [●28]. A8 then used the structural-interpreting skill to recognize that the result of the calculation was a vector in the *y*-basis [⬟8]. This work was all in service of changing $|\psi\rangle$ to be a linear combination of *y*-basis vectors so that the requested probabilities could be calculated.

To calculate the probability that the spin component is up along the *y*-axis, A8 computed $\left|{}_y\langle+|\psi\rangle_y\right|^2 = \frac{25}{26}$ by squaring the coefficient of $|+\rangle_y$, taking for granted that ${}_y\langle+|+\rangle_y = 1$ and ${}_y\langle+|-\rangle_y = 0$. They explained, "now we can really easily pull out the probabilities … we just have that plus *y*, squared is gonna be… 25/26, and for the other one, this is gonna be 1/26". A8 mathematized the situation to conclude that they could square the coefficients when the vectors in the inner product are expressed in terms of the same basis [▲4] and mathematized that because there are two possible results of the measurement (i.e., up or down along the *y*-axis), there are two probabilities [▲2]. A8 also used the technical-procedural skills of taking the norm of the coefficient [●13] and squaring it to find the probability [●17], as well as subtracting that probability from 1 to find the complementary probability [●1]. Overall, A8's method on problem B serves as an exemplar of the observed complexity in the students' problem-solving methods. They used various structural and technical skills in nonuniform ways that did not follow a particular sequence, illustrating the intricacy of students' mathematical and physical reasoning while solving a quantum mechanics problem requiring a change of basis.

Students' Technical and Structural Skills Supported their Flexibility in Choosing an Appropriate Problem-solving Approach

Our second main result that we present from our data analysis is that students' technical and structural skills supported their flexibility in choosing an appropriate approach for the problems. We explored the students' reasoning behind their decisions to use certain problem-solving approaches. To perform this analysis, we identified segments of the students' interview transcripts where the student justified their choice in using a particular approach and assigned those segments the code, "Flexibility in Choosing Approach" [●9]. We aggregated all of the segments labeled with this code and identified which part of the problems the students were working on as they decided which approach to use. We identified three places in the students' work in which they described their choice of problem-solving approach: calculating $|\langle+|\psi\rangle|^2$ and $|\langle-|\psi\rangle|^2$, calculating $\left|{}_y\langle+|\psi\rangle\right|^2$, and calculating $\left|{}_y\langle-|\psi\rangle\right|^2$. To see which technical and structural skills the students used as they chose their problem-solving approach or reflected on their choice, we identified all of the other codes that were assigned within those segments. We then decided which coded technical and structural skills were relevant to the students' choice of problem-solving approach. This helped us identify which technical and structural skills supported the students' flexibility.

Students' technical and structural skills related to inner products and orthonormal bases supported their flexibility in their approach for calculating $|\langle+|\psi\rangle|^2$ and $|\langle-|\psi\rangle|^2$

The students' technical and structural skills supported them in choosing between two possible approaches that could be used to calculate the probability that the spin component of angular momentum was up or down along the *z*-axis (i.e., $|\langle+|\psi\rangle|^2$ and $|\langle-|\psi\rangle|^2$). For Problem A, there are two main approaches that could be used to calculate the probability that the spin component of angular momentum was up or down along the *z*-axis: that is, to compute $P_\pm = |\langle\pm|\psi\rangle|^2$. The first approach involved calculating $P_+ = |\langle+|\psi\rangle|^2$ by substituting $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ into the inner product to get $P_+ = \left|\langle+|(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)\right|^2$, using the distributive and commutative properties to find $P_+ = \left|\frac{3}{\sqrt{13}}\langle+|+\rangle + \frac{2i}{\sqrt{13}}\langle+|-\rangle\right|^2$, using known properties $\langle+|+\rangle = 1$ and $\langle+|-\rangle = 0$ to reduce the equation to $P_+ = \left|\frac{3}{\sqrt{13}}\right|^2$, and simplifying to $P_+ = 9/13$. The procedure for determining the complementary probability $P_- = |\langle-|\psi\rangle|^2$ can be performed similarly to conclude that $P_- = \left|\frac{2i}{\sqrt{13}}\right|^2 = 4/13$. Alternatively, the second approach allows students to skip most of the procedures in the former approach: students could square the norm of the coefficient of $|+\rangle$ or $|-\rangle$, respectively, in $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ to find P_+ and P_-. Most of the students used the latter procedure, as it is simple and efficient. For instance, when the interviewer asked C12 why they chose to use this procedure, they said "because it's the quickest way. Um, yeah. I mean I could go through all the Dirac stuff and all that, which is ultimately just going to lead me here." We include this description of the approaches that the students considered using because their reasoning here includes the acts of considering various options and using their understandings of the mathematical or physical concepts to decide on an approach. In what follows, we discuss how students' technical and structural skills related to inner products [⬟5,■6] and orthonormal bases [▲1,▲6,●12,■3] supported their flexibility [●9] in choosing an appropriate approach to calculating $|\langle+|\psi\rangle|^2$ and $|\langle-|\psi\rangle|^2$.

First, students' technical and structural skills related to *inner products* [⬟5,■6] supported their flexibility in choosing this problem-solving approach. To be able to skip the steps in the first aforementioned approach of evaluating the inner product $\langle+|\,(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$, students first used the structural skill of interpreting the mathematical symbols to recognize that the vectors in the inner product $\langle+|\,(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$ were linear combinations of vectors from the same basis [⬢5]. Their technical-conceptual skill of recognizing that vectors in an inner product must be in terms of the same basis [■6] then allowed them to use the inner products of *z*-basis vectors. For instance, A8 described their reasoning about this as:

> Because this is written along the *z*-axis, I'm assuming that we're working in the *z*-basis here, standard representation. Then you do … norm squared of plus with psi [$|\langle + | \psi \rangle|^2$], and by the same rule I talked about earlier, about how we have this just plus plus minus minus equals 1 [$\langle \pm | \pm \rangle = 1$], then all you get here is plus and minus. Literally just pull out these same coefficients, so you get 3 over root 13 squared and 2 over root 13 squared. So you get 4/13 and 9/13.

A8 acknowledged that "we're working in the *z*-basis here," meaning that the vectors in the inner product were either elements of the *z*-basis or linear combinations of *z*-basis vectors [⬢5]. Their technical-conceptual skill of knowing that the vectors in the inner product were expressed in terms of the same basis [■6] allowed them to take advantage of properties of the *z*-basis. Overall, the students' skills related to reasoning about inner products supported their flexibility in choosing this approach to problem A.

Second, the students' technical and structural skills related to *properties of orthonormal bases* [▲1,▲6,●12,■3] supported their flexibility in choosing to use the more efficient approach to solving problem A. The students used structural-mathematizing skills to recognize that $|+\rangle$ and $|-\rangle$ comprise an orthonormal basis [▲1,▲6] which allowed them to use technical skills involved in using the inner products of orthogonal basis vectors: $\langle \pm | \pm \rangle = 1$, $\langle \pm | \mp \rangle = 0$ [●12,■3]. For example, when asked about how they found their answer, A11 explained:

> Since this is a basis, uh, plus with a plus is equal to 1 [$\langle + | + \rangle = 1$], whereas plus with a minus is equal to 0 [$\langle + | - \rangle = 0$]. So, if I was to distribute a plus [$\langle + |$] out to all of these, this would give us zero automatically because they're orthogonal. This would go to 1, so I square that. Same thing with the other way, because minus plus is equal to 0.

Reasoning about orthonormal bases, namely that $\langle \pm | \pm \rangle = 1$ and $\langle \pm | \mp \rangle = 0$ [●12,■3], allowed students to anticipate that evaluating inner products by distributing [●8] $\langle + |$ to $(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$ and $\langle - |$ to $(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$ would leave only the coefficient of $|+\rangle$ and $|-\rangle$, namely $\frac{3}{\sqrt{13}}$ and $\frac{2i}{\sqrt{13}}$, respectively [●6]. This allowed them to skip these steps and instead calculate the probabilities by squaring the norm of the coefficients of $|+\rangle$ and $|-\rangle$. Overall, the students' technical and structural skills related to reasoning with inner products [⬢5,■6] and orthonormal bases [▲1,▲6,●12,■3] supported their flexibility [●9] for calculating $P_\pm = |\langle \pm | \psi \rangle|^2$.

Students' technical and structural skills related to basis and change of basis supported their flexibility in their approach for calculating $|_y\langle + | \psi \rangle|^2$

To calculate $|_y\langle + | \psi \rangle|^2$, the students recognized a need to perform a change of basis, and they had the choice to either change $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ to be written in terms of the y-basis or change $|+\rangle_y$ to be written in terms of the z-basis. The students' technical and structural skills supported their decision in choosing their problem-solving approach. In particular, the students' technical and structural skills related to reasoning about the *basis that the vectors in the inner product were expressed in terms of* and about the *properties of orthonormal bases supported* their choice to perform a change of basis. Their technical and structural skills also supported their flexibility in deciding which of the aforementioned two possible approaches to use.

The students' technical and structural skills related to reasoning with basis [⬟1] and inner products [■2,■5,■6,▲16] supported their choice to perform a change of basis. For example, A13 described their reasoning on this problem as:

> You'd either have to change this [$|\psi\rangle$] to *y*-basis to fit this, which would not be fun probably, or change your *y*-basis to *z*-basis… psi is in a completely different basis, so you can't just multiply out in- when they're in different bases, so you have to switch bases.

As exemplified in A13's reasoning, the students used their structural skill of interpreting that "psi is in a completely different basis" [⬟1]. They recognized that $|\psi\rangle$ is a linear combination of *z*-basis vectors, which does not match the basis expression of ${}_y\langle+|$, the other vector in the inner product. The students then used their technical-conceptual skills to recognize that the vectors in the inner product needed to be expressed in terms of the same basis for them to be able to perform the inner product [■2,■5,■6]. For instance, C6 claimed, "you can't do anything until you're in the same basis." The students then used the structural-mathematizing skill of recognizing that a change of basis is necessary to be able to perform that inner product [▲16]. Thus, the students' technical and structural skills related to reasoning about the basis of the vectors in the inner product supported their choice to perform a change of basis.

The students' technical and structural skills related to the *orthonormality of the bases* and the associated *inner product values* also supported their choice to perform a change of basis [▲6,●12,■3]. Some students discussed how changing basis made the calculations simpler because of the orthonormality of the bases. For instance, A21 explained:

> I wanna be able to read off those coefficients really easily and do this in bra ket notation if these are in the same uh basis. If I'm expressing plus *y* [${}_y\langle+|$] in the *z*-basis then I can make all those assumptions about one, you know, the pluses and the minuses, the cross terms are gonna be zero. But if I were to do this in the *y*-basis … like write this out as like let's say *y* plus, but against all of this [$\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$], then I can't make any assumptions about that, so I don't really know how to calculate that in bra ket language.

A21 claimed it was necessary to change basis to make assumptions about the inner products of the various (orthonormal) basis elements, such as $\langle\pm|\mp\rangle = 0$ [●12,■3]. A21 suggested that leaving the inner product as ${}_y\langle+|(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$ with the vectors expressed in terms of different bases would not allow them to use $\langle\pm|\mp\rangle = 0$. Thus, the need to take advantage of the orthonormality property [▲6,●12,■3] informed their choice to perform a change of basis. C5 also suggested that a change of basis was necessary for the "inner products to be nice" [■5]:

Because my state vector was given in the z-basis, if I'm doing the inner product of the positive y with that, I need that to be written in the z-basis, or to do those inner products to be nice. So I guess the plus and plus gives you one [$\langle + | + \rangle = 1$]. The plus and minus gives you zero [$\langle + | - \rangle = 0$].

Taking advantage of the orthonormal basis properties motivated the students' selection of the change of basis approach. The students used their structural-mathematizing and technical-conceptual skills to leverage that the y-basis and the z-basis are both orthonormal [▲6], which implies that $\langle \pm | \pm \rangle = 1$, $\langle \pm | \mp \rangle = 0$, ${}_y\langle \pm | \pm \rangle_y = 1$, and ${}_y\langle \pm | \mp \rangle_y = 0$ [●12,■3]. These structural and technical skills informed their choice of approach and therefore their flexibility.

In addition to informing their decision to perform a change of basis, the students' structural and technical skills also supported their flexibility in choosing a change of basis approach: either changing $|\psi\rangle$ to be a linear combination of y-basis vectors or changing ${}_y\langle + |$ to be a linear combination of z-basis vectors. Three students attempted the former approach; A8 did so correctly after acknowledging the two possible approaches:

There are two ways to go about it, um, one of them is to put this vector [$|\psi\rangle$] in some phi prime that's in the y-basis, and then just do y plus phi prime y [${}_y\langle + | \psi' \rangle_y$] … it follows the same rules as this. Um, the other possibility is to do, is to take the spin up y and go to whatever it is in the z, in the z-basis. Um, they're both equivalent.

Expressing $|\psi\rangle$ as a linear combination of y-basis vectors allowed A8 to square the norms of coefficients of y-basis vectors [▲4]. A8 recognized that both methods were "equivalent" and yielded the same probability result [⬟7]. A8 reflected on their choice of approach and compared the efficiency of the two methods: "The other method is probably faster if you think of it. Actually, I don't know if it's really faster. You just save so much time on this side, if you do it this way" [●9]. In summary, A8's structural-mathematizing skill [▲4] and structural-interpreting skill [⬟7] supported their flexibility [●9] in acknowledging the two approaches, comparing their efficiency, and choosing one for solving the problem.

Most students chose to use the approach of changing ${}_y\langle + |$ to be a linear combination of z-basis vectors, and their structural and technical skills [▲8,●19,●21] supported their flexibility [●9] in doing so. Some students chose this one due to computational ease. For instance, A6 explained:

Change of basis was up here, the very first thing, so you can't do anything until you're in the same basis. So, this vector [${}_y\langle + |$] is itself, I mean I can just call this a plus y in the y basis, but I needed plus y in the z basis, because this was in the z basis. If I really wanted to, I could have changed this [$|\psi\rangle$] to the y basis. Um, this [${}_y\langle + |$] is a lot easier because we had the spins sheet, so I changed this from the y basis to the z basis here, so then both of them were in the z basis.

The students had access to a "spins sheet" containing the equation $| + \rangle_y = \frac{1}{\sqrt{2}} | + \rangle + \frac{1}{\sqrt{2}} | - \rangle$. This made the change of basis procedure "a lot easier," only involving substitution [●19] and not the solution of a system of equations. These students' technical-procedural skills of using substitution [●19] and given equations [●21] supported their flexibility by allowing them to compare the efficiency of possible approaches. A13 also acknowledged that changing $|\psi\rangle$ to be in terms of the *y*-basis vectors "would not be fun probably," so they chose to change the basis that ${}_y\langle+|$ was expressed in terms of, instead. A11 similarly recognized that changing $|\psi\rangle$ to be in terms of the *y*-basis vectors would involve more work, explaining, "I didn't really want to have to deal with that math, but like- like I could have done it, because I know that's something you can do now, but I didn't really want to." These students' structural-mathematizing skill of recognizing that changing ${}_y\langle+|$ to be in terms of the *z*-basis vectors was easier for them than changing $|\psi\rangle$ to be in terms of the *y*-basis vectors [▲8] supported their flexibility ●9] in choosing an appropriate way to change basis. Overall, the students' structural and technical skills related to reasoning about change of basis [▲8] via substituting and using equations [●19,●21] supported their flexibility in choosing this method of changing basis.

Students' technical and structural skills related to reasoning with probability supported their flexibility in their approach for calculating $\left|{}_y\langle-|\psi\rangle\right|^2$

To calculate $\left|{}_y\langle-|\psi\rangle\right|^2$, the probability that the spin component of angular momentum was down along the *y*-axis for the given state, the students could either use the same approach they used to calculate $\left|{}_y\langle+|\psi\rangle\right|^2$ or subtract $\left|{}_y\langle+|\psi\rangle\right|^2$ from 1. Over half of the students performed or suggested the latter. For instance, C5 explained, "The probability of minus *y* is equal to one minus 25 over 26, and so it's 1 over 26" (see Figure 9). In general, the students first used their structural skill of mathematizing the scenario of having two possible outcomes (i.e., the spin component of angular momentum being either up or down) to recognize that there are only two possible probabilities, P_+ and P_- [▲2]. The students used their technical conceptual and procedural skills to reason that the probabilities of two possible outcomes sum to 1 [●15, ■1]. Given that the first probability they determined was $P_+ = \frac{25}{26}$, the students used the fact $P_+ + P_- = 1$ to conclude $P_- = 1 - \frac{25}{26} = \frac{1}{26}$ [●1].

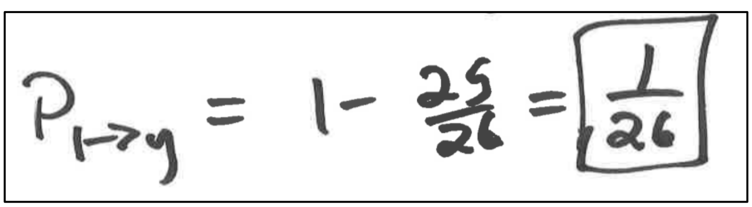


Fig. 9 C5's written work for calculating the probability that the spin component of angular momentum is down along the y-axis on Problem B

Some students acknowledged that this method was more efficient than using the same method used to calculate $\left|{}_y\langle+|\psi\rangle\right|^2$ [▲3,●9]. A11 explained: "if you really want to do the math again, it would be the same thing as just 1 minus the 1 over 26." A11 acknowledged an alternative of "doing the math again" and that it would result in the same answer. This demonstrated that A11 was aware of multiple approaches [▲3] and consciously chose one that helped them avoid "doing the math again" [●9]. A21 similarly acknowledged the alternative approach [▲3]: "The probability of going up or down is gonna be a hundred in this case, but yeah I'm just gonna go with 1/26. But if I was doing this on a test, and I was taking my time, I would just calculate this out by using minus in the y-basis." A21's explanation implied that the alternative approach took more time, so they instead used $P_{-} = 1 - \frac{25}{26} = \frac{1}{26}$ [●1,●9]. Thus, A21's technical-conceptual and technical-procedural skills [●15,■1] of reasoning about the probabilities of two possible outcomes adding to 100% supported them in choosing their problem-solving approach. Overall, drawing on their structural and technical skills related to reasoning with probability [▲2,●1,●15,■1] supported the students' flexibility by enabling them to choose an appropriate approach for calculating $\left| {}_{y}\langle -|\psi\rangle \right|^{2}$.

Discussion

Given the entanglement of mathematics and physics, it is essential for undergraduate physics students to learn how to reason with mathematics as they address physical problems. This is a complex endeavor for students as it involves potentially using mathematical concepts in different ways than in their mathematics courses and connecting their mathematics and physics reasoning via interpreting and mathematizing. The constructs of interpreting and mathematizing are commonly used in researchers' models of student reasoning about mathematics in physics. In this study, we leveraged Uhden et al.'s (2012) and Karam's (2014) framework of students' technical (conceptual and procedural) and structural (mathematizing and interpreting) skills to investigate the intricacy and flexibility of physics students' reasoning about mathematics in relation to physics content addressed in two quantum mechanics problems. We addressed the research question: *How do undergraduate physics students reason with mathematical concepts and procedures as they solve quantum mechanics problems?* Through our qualitative analysis of interview data from twelve physics students, we presented two primary findings: 1) the students used intricate problem-solving methods that leveraged several mathematical concepts with reasoning that moves fluidly between structural and technical skills in quick succession, and 2) the students' technical and structural skills related to reasoning about linear algebra and probability concepts informed their flexibility in choosing a problem-solving approach.

Our results identified that the students used idiosyncratic problem-solving methods that did not follow simplified sequential patterns of reasoning such as mathematize, perform computations, interpret. The students' reasoning relied on and moved fluidly between structural (mathematizing and interpreting) and technical (conceptual and procedural) skills, which illustrates the intricacy of students' reasoning on these problems. No student's reasoning followed patterns or cycles as straightforward as models suggested by previous research in both mathematics education and physics education (Blum & Leiß, 2005; Redish & Bing, 2009; Wilcox et al., 2013). Although these models have merit for various pedagogical or research purposes, they do not adequately capture the complexity of the students' intertwined mathematical and physical work. Using the Uhden et al. (2012) model (Figure 2d) allowed us to tease apart the nuances of students' mathematizing, interpreting, and technical skills as they engaged in problem solving.

Our results also demonstrated how these students reasoned about various linear algebra concepts in their work, particularly: properties of bases, orthonormal bases, inner products, and change of basis. This contributes to what is known about student reasoning with these concepts. The students' work on these quantum mechanics problems illustrates ways that physics students have to leverage their understanding of several linear algebra concepts within that context. Furthermore, how these concepts were leveraged or used in computation in this context often differed from what students would typically encounter in an undergraduate Linear Algebra course. For instance, a basis for a vector space is a linearly independent set that spans the space -- there is no orthogonality or normality condition on a basis in general. However, the bases students used in these quantum mechanics problems, naming the z-basis ($|+\rangle$ and $|-\rangle$) and y-basis ($|+\rangle_y$ and $|-\rangle_y$) for the spin-½ system, are necessarily orthonormal because of quantum mechanical properties[5]. Furthermore, the z-basis and y-basis are two of the most frequently used bases in this system. In fact, the relationship between them was derived in the students' courses and kept track of as an important relationship; it was notated as a linear combination (linear superposition) of the basis kets in Dirac notation: $|\pm\rangle_y = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$. Thus, when the students carried out a change of basis from y to z or vice versa, they carried out algebraic substitutions using these equations. This differs from the approach common in linear algebra courses that utilizes coordinate vectors and change-of-basis matrices.

Our second main result shows that students' technical and structural skills related to reasoning about linear algebra and probability concepts supported their flexibility in choosing a problem-solving approach. Flexibility is an essential aspect of problem-solving, as it involves being aware of multiple approaches to solve a problem and choosing an appropriate one. Other researchers have demonstrated how conceptual knowledge can support procedural flexibility (e.g., Rittle-Johnson et al., 2015). This finding is furthered in our study. In particular, when students draw on their mathematical knowledge to inform their approach for solving these quantum mechanical problems, it relies on their understanding of how the mathematical concepts and physics concepts are intertwined. Students using their mathematical conceptual understanding in their work on these problems has more complexity involved than just reasoning about the mathematical concepts and procedures because students have to reason about them in relation to the physical concepts with which they correspond. Students do not only perform mathematical computations; they also keep track of what those mathematical structures mean in terms of the physics. Thus, it is not just that mathematics and physics are entangled, but rather that the students' reasoning about mathematics and physics is also entangled.

[5] In the spin-½ system, the spin observable has only two possible measurement values, $\pm\hbar/2$. A postulate of quantum mechanics conveys that the only possible measurements of an observable are the eigenvalues of the corresponding operators. Measurements are real-valued, which necessitates using Hermitian operators, which provide an eigenbasis for the corresponding system.

Engaging in this research has raised for us the deliberation of what qualifies as mathematics and what qualifies as physics, particularly at this content level. To what extent are mathematics and physics actually inextricable within quantum mechanics? For instance, the way in which Dirac explicated the benefit of his newly-developed notation conveys a sentiment of inseparability:

> This notation allows a more direct connexion to be made between the formalism in terms of the abstract quantities corresponding to states and observables and the formalism in terms of representatives-*in fact the two formalisms become welded into a single comprehensive scheme*. (Dirac, 1947, page v, emphasis added)

Kets in Dirac notation behave mathematically like vectors, and the mathematics of vector spaces frame the structural behavior of vectors such as scalar multiplication, vector addition, and inner product computation. Thus, when solving a problem in Dirac notation, there isn't a clean line between when students are reasoning about mathematics and when they are reasoning about physics. We hope our research can further the conversation between mathematics and physics education research about productive ways to frame and theorize both learning and our content areas so as to best make sense of student reasoning across and within the disciplines.

With respect to teaching implications, our research contributes to raising the mathematics community's awareness of what concepts from mathematics courses are used and in what way by students in physics courses. Our analysis revealed the centrality of: basis, orthogonality, normality, change of basis, algebraic substitution and simplification of vector equations or system of equations, and inner product in the solution process for a quantum mechanical problem. Linear algebra instructors could integrate problems into their course that not only facilitate the development of conceptual and procedural skills from linear algebra but also demonstrate the mathematization of linear algebra in quantum mechanics. See Figure 10 for such an example.

Linear Algebra is central to much of the mathematical structure of quantum mechanics. States of a physical system are associated with a normalized vector, and observables are associated with Hermitian operators. For example, spin is a measure of a particle's intrinsic angular momentum, which is related to the particle's magnetic moment. The eigenstates of the spin observable create an orthonormal basis for the associated vector space.

In a spin-1/2 system, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ comprise what is known as the "z-basis," $\left\{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}\right\}$ comprise the "x-basis," and $\left\{\begin{bmatrix}1/\sqrt{2}\\i/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\-i/\sqrt{2}\end{bmatrix}\right\}$ comprise the "y-basis" for spin.

1. Suppose $\Psi = \begin{bmatrix}3\\2i\end{bmatrix}$. Normalize Ψ and label the normalized vector as ψ.
2. Determine the coordinate vector of ψ relative to the z, x, and y bases. That is, determine $[\psi]_z$, $[\psi]_x$, and $[\psi]_y$.
3. Suppose you did not need to change the basis for a specific state vector but rather needed to know how to do so for any given state vector. Determine the change of basis matrix $[I]_z^x$ that changes z-coordinates into x-coordinates, and the matrix $[I]_x^y$ that changes x-coordinates into y-coordinates.
4. Prove that the eigenvalues of Hermitian operators are always real-valued. This property is important in quantum mechanics because eigenvalues are associated with the measured values of an observable.

Fig. 10 Instructional example of problems that relate change of basis to quantum mechanics

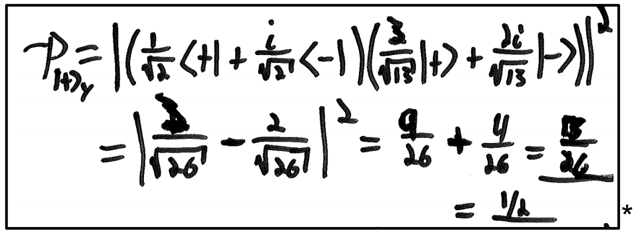


Fig. 11 A13's work on the spin-up aspect of Problem B

While the aforementioned concepts are all core to a linear algebra course, additional mathematical concepts were shown to be central to students' problem-solving approaches, namely the probabilistic relationship of complementary and mutually exclusive events, distribution, and operations with complex numbers. Although not a focus of this paper, competency with algebraic simplifications related to moduli and exponents were essential to this problem. For instance, two students made a computation error similar to what is often colloquially known as "Freshman's Dream" in Problem B. For example, A13 computed $|\frac{3}{\sqrt{26}}-\frac{2}{\sqrt{26}}|^2=\frac{9}{26}+\frac{4}{26}=\frac{1}{2}$, rather than $|\frac{3}{\sqrt{26}}-\frac{2}{\sqrt{26}}|^2=|\frac{1}{\sqrt{26}}|^2=\frac{1}{26}$ (see line 2 in Figure 11)[6]. All of these mathematics concepts are relevant for larger spin systems as well as other quantum mechanical observables; for example, measuring position involves an infinite-dimensional Hilbert space and the inner product for the probability calculation involves integration of complex-valued functions. The analysis in this study focused on one specific type of quantum mechanical problem in a specific physical context. Future research could further address how physics students leverage their understanding of linear algebra concepts in other quantum mechanical contexts.

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[6] There is also a conjugation error in Figure 10. Line 1 should contain -*i* rather than +*i*.

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