Characterizations of student, instructor, and textbook discourse related to basis and change of basis in quantum mechanics

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Communities develop social languages in which utterances take on culturally specific situated meanings. As physics students interact in their classroom, they can learn the broader physics community's social language by co-constructing meanings with their instructors. We provide an exposition of a systematic and productive use of idiosyncratic, socially acquired language in two classroom communities that we consider to be subcultures of the broader community of physicists. We perform a discourse analysis on twelve quantum mechanics students, two instructors, and the course text related to statements about basis and change of basis within a spin-½ probability problem. We classify the utterances' grammatical constructions and situated meanings. Results show that students and instructors' utterances referred to a person, calculation, vector being in, or vector written in a basis. Utterances in these categories had similar situated meanings and were used similarly by the students and instructors. Utterances referred to change of basis as changing the form of a vector, writing the vector in another way, changing the vector into another vector, or switching bases. Utterances in these categories had varying situated meanings and were used similarly by the students and instructors. The students and instructors often switched between different discourse types in quick succession. We found similar utterance types, situated meanings, and grammatical constructions across students and instructors. The textbook's discourse sometimes differed from the discourse of the students and instructors. Within this study, the students and instructors were from two universities, yet they spoke similar utterances when referring to basis and change of basis. This gives evidence to their shared social language with a broader community of physicists. Integrating and leveraging social languages in the classroom could facilitate students' enculturation into the classroom and broader professional community.

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I. INTRODUCTION

Language plays a central role in physics students' learning with respect to their cognition and social participation. Language can serve as a linguistic representation of concepts with which physics students and physicists can reason [1,2]. Language is also integral to students' learning through their participation in a community of practice [3]. As students participate in class and become members of a community, "this entails, above all, the ability to communicate in the language of this community and act according to its particular norms" [4] (p. 6). As students learn to speak the language of the community, they develop their own identity as a member of that community [5]. Language can thereby foster the development of physics students' identities as members in the broader physics community. Given the importance of language in physics students' learning, it is pertinent to study students' language used in physics contexts.

The language used in physics contexts and the language used by physics instructors has been shown to be a source of potential difficulty in students' understanding of physics [6,7]. For instance, physicists sometimes "speak and write about physical systems with a set of one or more systematic metaphors that are well understood in their community" [8] (p. 1). However, physics students might misinterpret or overextend metaphors used by their instructors (e.g., Refs. [1,8,9]). Researchers have suggested physics instructors' seemingly imprecise use of language may contribute to students holding inaccurate conceptions [10,11]. When physicists converse with each other, they may use "vernacular of the laboratory" [12] (p. 163), which involves vague or shorthand language based on shared understanding of interlocutors. Senior [12] claimed, "the laboratory vernacular is successful as a means of communication not because it is accurate and definite, but because it has little need to be" (p. 164). When two interlocutors have common knowledge and experience, there is less need for accuracy in their verbal communication because they have a shared understanding of what phrases mean. This laboratory vernacular is considered useful for physicists when

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communicating with other physicists, who have this shared understanding. However, some researchers posit that using this laboratory vernacular in the classroom may be confusing for physics students (e.g., Ref. [11]).

In response to this potential linguistic confusion students may experience, some researchers have urged physics instructors to use precise explicit language during instruction. Itza-Ortiz *et al.* [6] suggested that instructors should be cognizant of the different meanings students bring to bear for words in introductory physics, especially words with different meanings in everyday language. Romer [7] urged physics teachers and authors to strive for clarity in their word choice used in their teaching and writing. Williams [11] suggested physics instructors should not use laboratory vernacular while teaching, claiming, "We physicists who participate in both research and undergraduate teaching, thus, must be in effect bilingual, using laboratory vernacular for its efficiency in appropriate settings, but using a much more explicit and complete vocabulary in our teaching role" (p. 670). These suggestions were intended to help instructors avoid students' development of inaccurate conceptions on the basis of instructors' language.

Instead of focusing on how ambiguities in experts' vernacular may lead to student confusion, we posit that students and their instructors productively negotiate meanings for language used in their classroom community. It is useful for instructors to speak using the "vernacular of the laboratory" during class because it allows students to learn to speak in ways consistent with the broader community of physicists. Students socially co-construct the meanings of utterances with their instructors, so what may be seemingly imprecise phrases to an observer are in fact sensible to students. Physicists' vernacular of the laboratory is an example of what Gee [13,14] calls a social language. Members of a social group or community of practice develop their own social language in which utterances take on culturally specific, contextually dependent meanings. The situated meanings of interlocutors' utterances are inherently tied to the context in which they were spoken and the social group to which the interlocutors belong. As physics students participate in class, they can speak and co-construct the social language associated with a broader physics community of practice. Learning a social language of a community of physicists may help students become more central members of the community and help them develop and enact their identity as a physicist.

In this qualitative study, we analyzed the use of social language by performing a discourse analysis [13,14] of quantum mechanics students' utterances, their instructors' utterances, and statements from a relevant chapter of their textbook. We used Gee's [14] definition of discourse analysis to mean "the study of language-in-use" (p. (ix) in specific contexts. In particular, we address the following research question[1]: What situated meanings and grammatical constructions for basis and change of basis exist in physics students' discourse as they solve two spin-½ quantum mechanics tasks, and how does their discourse compare with that of their instructor and textbook? Through addressing this research question, we provide an exposition of a systematic and productive use of idiosyncratic, socially acquired language in two classroom communities.

II. BACKGROUND, THEORETICAL FRAMING, AND LITERATURE

Gee [14] defined social languages as "styles or varieties of a language (or a mixture of languages) that enact and are associated with a particular social identity" (p. 156). As one enacts part of their identity as a member of a social group or community of practice [3], they speak the social language associated with that group. Speakers of social languages use distinctive word choices and grammatical patterns particular to their social group. As members of a community communicate, they unconsciously recognize how certain linguistic features pattern together to convey certain meanings. Gee [13] suggested that meanings of words are "integrally linked to and vary across different social and cultural groups" (p. 53). We posit that physics students and instructors have their own social language particular to their community of practice, through which they socially negotiate meanings of phrases and use grammatical patterns in their utterances.

One way to analyze the discourse of a community is to investigate the situated meanings of utterances spoken by members of the social group [13]. Any phrase can have several different potential meanings, yet words and phrases take on more specific meanings in situations of use. Spoken utterances thus have situated meanings; they take on specific interpretations in specific contexts relative to the culture of the people communicating. Gee [14] gave an example of spilled coffee; the situated meaning of "clean it up" would be "get a mop" for liquid coffee but "get a broom" for coffee grounds. Situated meanings of phrases are "very often negotiated between people in and through communicative social interaction" [13] (p. 94). Within any communication, the listener must infer specific meanings of phrases while considering the context in which they were spoken. Listeners must know which inferences they draw are relevant, where relevance "is a matter deeply tied to context, point of view, and culture" [13] (p. 46). Situated meanings are assembled as people communicate in a given context based on their perception of that context. Language and context have a reflexive relationship, for the language used influences how people construe the context, and the

[1]This paper builds from and is an extension of a conference presentation given at the 2020 Conference on Research in Undergraduate Mathematics Education [15].

context influences how people interpret the meaning of the language.

One study of particular interest to our investigation into physics students' and instructors' discourse was that of Ochs, Gonzales, and Jacoby [16], who performed an analysis of grammatical patterns in physicists' spoken utterances as they collaborated in a lab setting. They identified three ways physicists' parlance was grammatically constructed: physicist centered, physics centered, and indeterminate. In physicist-centered constructions, the physicist is either an active agent who affects physical entities or an experiencer of those physical entities. These constructions are characterized by their personal pronominal subject and their clear distinction between the physicist and physical entities. An example is "We did experiments where we brought the system here" (p. 336). Physics-centered grammar is "often characterized by using agentless passive structures or syntactically active structures in which the thematic subject of an utterance is a physical entity or process" (p. 335). Unlike the physicist-centered constructions, the physics-centered constructions highlight physics as the subject of interest and consider physical entities as either active or sentient in some fashion. An example is "This system has no knowledge of that system" (p. 338). Ochs, Gonzales, and Jacoby also identified indeterminate grammatical constructions, which were ubiquitous in the physicists' utterances. These constructions are characterized by a personal pronominal subject with an inanimate predicate (i.e., a clause that grammatically should not apply to a personal pronoun), such as "when I come down I'm in the domain state" (p. 339). Ochs, Gonzales, and Jacoby supported the theory that the pronominal subjects have a "blended identity composed of both the animate physicist and the inanimate physical entity" (p. 348). They further claimed, "By using indeterminate constructions as a linguistic heuristic, scientists constitute an empathy with entities they are struggling to understand" (p. 348). These physicists used language that seemed to sympathize with the inanimate objects they were trying to understand, blurring their identity with that of the physical object by blending properties of animate and inanimate subject and object.

This kind of indeterminate language has been recognized in studies in mathematics and physics education. Moschkovich [17] problematized the compatibility of two recommendations for mathematics education—namely, that classroom activities should parallel both everyday mathematical practice and academic mathematical practices—by exploring what is entailed in everyday and academic mathematics. The work of Ochs, Gonzales, and Jacoby [16] was brought to bear as an ethnographic study examining what physicists do and communicate as they engage in laboratory practices. Schwarz and Baker [18] remarked that ethnographic studies could help "demystify scientific work," in that it "moves away from pure reason and formal logic, the latter appearing as post hoc rationalisations of science" (p. 106). Noting how these results could help illuminate in what ways students act like scientists in the classroom, Moschkovich [17] remarked that without "ethnographic studies of mathematicians' work, it is difficult to know whether we are proposing that students engage in the actual practices of mathematicians or in idealized versions of what mathematicians do" (p. 7). Similar to how physicists have been shown to use indeterminate grammar in their work, mathematicians sometimes use talk, gestures, and diagrams "in ways that blur the distinction between the mathematical and physical world" [19] (p. 223).

Mathematics students have also been shown to use language consistent with Ochs, Gonzales, and Jacoby's [16] characterizations of grammar. Ju and Kwon [20] utilized discourse analysis to investigate changes in differential equations students' beliefs about mathematics. Examining first-person perspective speech (where the speaker attributes the mathematical argument to themself or their small group) and third-person perspective speech (where the speaker attributes the mathematical argument to an external source like a textbook) in classroom discussion, the authors tracked students' transitions from third- to first-person speech over the semester. The authors suggested this change paralleled shifts in students' mathematical beliefs: "what they perceived as a proper way of doing mathematics, the nature of the discipline, the foundation of mathematical certainty, and their views of human agency in the practice of mathematics" (p. 274). Ju and Kwon [20] likened this to the "extreme subjective involvement with science" (p. 275) found in Ochs, Gonzales, and Jacoby's [16] analysis of physicists as active participants in and experiencers of the scientific world, as the distinction between scientist and physics is blurred. In another study of differential equations students, Rasmussen [21] characterized the use of intuition in student reasoning of specific content, such as the stability of differential equations and the relationships among solution functions. One student used language indicative of positioning himself as an actor in a direction field: "If I were unstable, I'd try to get stable as much as I could. So that means if you're stable, you won't want to go away from that line" (p. 82). The student seemed to blur his identity with that of a graphical solution to a differential equation, and Rasmussen credited the work of Ochs, Gonzales, and Jacoby [16] with the notion that learners can leverage their intuition and develop meaning for mathematical concepts through embodied reasoning with graphical representations of the concepts.

For a couple of examples from physics education research, Euler, Rådahl, and Gregorcic [22] illustrated the use of embodiment in physics learning through the case of two students reasoning about binary star dynamics as they communicated in words, gestures, and dance. One investigation culminated with one of the students proclaiming, "I must have a larger orbit!" as she traced an

Consider the quantum state vector $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$.
a) Calculate the probabilities that the spin component is up or down along the z-axis.
b) Calculate the probabilities that the spin component is up or down along the y-axis.

FIG. 1. The interview questions analyzed in this paper.

orbit in the air with her finger. Focusing on the discursive moves of the student, the authors noted how her words "put her in the role of the star grammatically, suggesting a strong conceptual intermingling of the experiential realm of the dance and astronomical realm of the binary stars" (p. 16). Schermerhorn and Thompson [23] investigated student understanding of how differential vector elements are constructed in non-Cartesian coordinate systems. Although students' indeterminate grammatical constructions were not the study's focus, we highlight several notable student utterances from [23]: "So, dl is like you just have some path. So I'm trying to think, like, if I was going to walk in the $\hat{\alpha}$ direction…" (Carol, p. 7); "it's the amount of M for every little dM that I move" (Tyler, p. 10); and "… then you're in $d\varphi$…If I want to get down to φ, I have to go down some θ or in this case α and then I rotate around and that is giving me my φ direction" (Dan, p. 10).

We noticed that the students seemed to blend their identities with that of an actor or entity within a graph. Indeed, students' identification with a physics entity has been shown to be a productive way of making sense of physics concepts [24]. As Ochs, Gonzales, and Jacoby [16] suggested, "referential indeterminacy created through gesture, graphic representation, and talk appears to be a valuable discursive and psychological resource as scientists work through their interpretations" (p. 359). This literature demonstrates the productivity of using indeterminate language to better understand the physical or mathematical entities one is studying.

III. PHYSICS AND MATHEMATICS BACKGROUND

The interview questions that generated the data analyzed in this paper (see Fig. 1) asked students to determine the probability of obtaining the different possible measurements for spin, either $\hbar/2$ or $-\hbar/2$, on a specific system in a given state along both the z axis and y axis of rotation. This probability is determined by calculating the absolute value squared of the inner product of the vector representations of the given state and the desired measured state. For that calculation to be possible, those two vectors (or kets, in Dirac notation) must be expressed in the same mathematical basis.[2] The two possible measurements for spin along the z axis are $\hbar/2$ or $-\hbar/2$, and they correspond to the two orthonormal basis vectors, written in Dirac notation, of $|+\rangle$ and $|-\rangle$ (called "plus z" and "minus z"), respectively. The given state $|\psi\rangle$ in Fig. 1 is written as a linear combination of these plus z and minus z kets, so question (a) in Fig. 1 is often seen as relatively straightforward because one does not need to change basis to compute the inner products $\langle +|\psi\rangle$ and $\langle -|\psi\rangle$. To answer question (a), one can utilize a shortcut for calculating the inner products, the Born rule, which says the probability of obtaining a measurement is the squared norm of the coefficient of the basis vector corresponding to that measurement [thus, the solutions for (a) are $|3/\sqrt{13}|^2 = 9/13$ and $|2i/\sqrt{13}|^2 = 4/13$]. Question (b), however, requires a change of basis because the ket representations of the desired measured states along the y axis are $|\pm\rangle_y$, which does not match the given z-basis representation of the $|\psi\rangle$ state vector. The two main approaches to question (b) are either to change $|\psi\rangle$ to be written in terms of the y-basis kets (denoted $|\pm\rangle_y$), or change the y-basis kets to be written in terms of the z-basis kets. In either change of basis approach, one would need to utilize $|\pm\rangle_y = 1/\sqrt{2}|+\rangle \pm i/\sqrt{2}|-\rangle$, which shows the relationships between the kets in the two basis sets; in other words, given that the z basis contains only $|+\rangle$ and $|-\rangle$, and the y basis contains only $|+\rangle_y$ and $|-\rangle_y$, those equations show how any ket from one basis is expressed as a linear combination of kets from the other basis.

IV. METHODS

We investigate the discourse of two quantum mechanics classroom communities, which we consider to be subcultures of a broader community of physicists. The discourse analysis [13] in this paper examines students' interview responses, instructors' in-class explanations, and a textbook's presentation of material. The second author and a graduate student, both of whom are researchers specializing in undergraduate mathematics education,[3] conducted individual semistructured interviews [25] with 12 students[4] enrolled in a quantum physics course. All interview participants were recruited during class, gave informed consent before being interviewed, and received compensation for their time and effort. Eight were in a junior-level course that

[2]Formally, a basis of a vector space V is a set of linearly independent vectors that generate V. Informally, one may think of it as the "building blocks" for a vector space, from which all other vector space elements can be uniquely created.

[3]We acknowledge that our membership in the mathematics education research community impacts the ways in which we ask questions, interact with students, and analyze data.

[4]We use the gender-neutral singular pronoun "they" to refer to the students throughout this paper.

met for seven hours per week at University A, a large, public, research-active doctoral university in the northwest US; interviews were conducted one month into the course. These students and their instructor were assigned pseudonyms of "A#" and "Instructor A." The other four[5] students were in a senior-level course at University C, a medium, public, research-active doctoral university in the northeast US that met for three hours per week; interviews were conducted two months into the course. These students and their instructor were assigned pseudonyms of "C#" and "Instructor C." The second author assigned these numeric pseudonyms to identify participants from a roster of all students in the courses. Instructor A's course had 33 students, and Instructor C's course had 17 students. Instructor A and Instructor C are seasoned physics education researchers with much experience teaching upper-division quantum mechanics. Instructor A's and Instructor C's classes were video recorded for four weeks and nine weeks, respectively, and the second author attended each of these class sessions. Both courses used McIntyre, Manogue, and Tate [26] as the textbook.

As part of a larger project in which we investigate physics students' understanding of linear algebra concepts utilized in quantum mechanics, the interview aimed to elicit evidence of student understanding of eigentheory, normalization, basis, and change of basis. In this study, we analyze the discourse in students' responses to the questions in Fig. 1, which targets student use of basis and change of basis within a spin-½ context. Follow-up questions were asked as needed during the interview to try to gain clarity regarding a student's response. A follow-up interview question relevant for this analysis was: "How do you see this problem relating to basis or change of basis?" This portion of the interview took ten minutes on average. Students seemed comfortable with the questions in Fig. 1, most likely because they had encountered problems of that nature and wording during their class sessions and in their textbook homework. All students reached a correct solution on question (a), and 7 of the 12 eventually reached a correct solution on question (b) (incorrect solutions were due to algebraic or conjugation errors).

The portion of each student interview involving the questions in Fig. 1 and follow-ups was transcribed. To address the first research question, we performed a grounded analysis of the ways in which students discussed entities in a basis and change of basis by coding each phrase in the interview transcript referring to these concepts [27]. As an author team, two of us performed multiple rounds of coding, in which we produced *in vivo* codes, used the constant-comparative method to identify similarities among the codes across all the transcripts, and grouped codes together to create axial codes that identified the different ways in which students discussed basis and change of basis. Some instances of student discourse necessitated the use of multiple coded phrases in the same sentence. We treated these instances as separate, using italics in Sec. V to focus the reader on the portion of the quote relevant to that category. When applicable, phrases were double coded, using both an entity in a basis category and a change of basis category. We then examined the situated meanings [13] of students' utterances to unpack what the students likely meant. We inferred these meanings by considering the context in which the utterances were spoken (i.e., the part of the task the student was addressing). Some utterances, such as "we're in the z basis," seemed not to have literal meanings (e.g., humans cannot physically be elements of a basis set that is composed of vectors), yet we assumed these utterances held situated meanings that were sensible to the speaker. Because many of these phrases were ubiquitous throughout the interview transcripts, it is consistent with our theoretical framing to assume that the meanings of these phrases are culturally shared among these students. We discussed and resolved any discrepant issues throughout the coding process, with the remaining author confirming interrater reliability by coding a subset of the data. To analyze the grammatical structure of students' utterances, we performed another round of coding the interview transcripts using Ochs, Gonzales, and Jacoby's [16] characterizations of physicist-centered, physics-centered, and indeterminate grammatical constructions. The final step of our analysis with student data was to align, for each utterance, the categories of utterances related to basis and change of basis with the Ochs, Gonzales, and Jacoby characterizations to identify whether utterances within each category tended to have a similar grammatical structure. Two of us conducted this grammatical structure analysis with this *a priori* framework and achieved a high level of interrater reliability.

We performed a similar analysis of the instructors' utterances to address the second portion of our research question. We examined Instructor A and Instructor C's utterances related to basis and change of basis, which were spoken in class as they explained how to solve probabilistic problems for spin-½ systems, similar to those in Fig. 1. These included utterances that either explicitly used the term basis or implicitly referred to basis (e.g., "in the z"). For Instructor A, utterances from days 4, 5, 6, and 8 were analyzed; for Instructor C, utterances from day 4 and 5 were analyzed. For both instructors, we did not consider explanations that focused on basis in general, without regard to the probability setting because we specifically wanted to capture the language they used in classroom settings that were similar to the setting in which the students used that language during the interview. We used the utterance categories pertaining to basis and change of basis that we had developed from the student interview data to code the instructors' transcripts. These codes were

[5]Nine students from University C participated in the interviews, but only four were asked this question because of time constraints.

sufficient for classifying the instructors' discourse referring to entities in a basis and change of basis, so no additional codes were needed for this part of the analysis. We also inferred the situated meanings [13] of the instructors' utterances and analyzed the grammatical structure of the utterances according to Ochs, Gonzales, and Jacoby 's [16] characterizations. We then compared the instructors' discourse to their students' discourse across these characterizations. This allowed us to identify similarities in the students' and their instructors' discourse, which we used to exhibit the systematic use of a social language within each of the classroom communities at University A and University C.

Finally, we examined the first chapter in the course textbook [26] (p. 1-33), which introduced the Stern-Gerlach experiment, quantum state vectors, matrix notation, general quantum systems, and the postulates of quantum mechanics. We analyzed this chapter because it described how to solve probabilistic problems for spin-½ systems, similar to those in Fig. 1. We identified statements in the textbook that referenced basis or change of basis and classified them using the categories we developed, as well as Ochs, Gonzales, and Jacoby's [16] characterizations. We compared these classifications of the discourse in the textbook statements to the classifications of students and instructors' discourse to identify similarities and differences in the statements' syntax (i.e., word usage), grammatical structure, and situated meanings.

V. RESULTS

We present our analysis of students' discourse about basis and change of basis throughout their responses to the interview questions, instructors' utterances spoken during class, and the textbook's statements regarding basis and change of basis. In Sec. V A, we present our findings regarding how students and instructors discussed entities in a basis, along with the utterances' associated situated meanings[6] [13] and grammatical structure. We found that utterances referring to entities in a basis had similar situated meanings that were mutually understood, and these discourse types were used similarly by the students and their instructors. In Sec. V B, we present our findings regarding the ways in which the students and their instructors discussed "change of basis" and the utterances' situated meanings and grammatical structure. We found that utterances referring to change of basis had varied situated meanings that were mutually understood, and these discourse types were used similarly by the students and their instructors. Then, in Sec. V C we use data from two students to exemplify the finding that students often switched between different discourse types and grammatical constructions in quick succession. We conclude the results in Sec. V D where we compare the students and instructors' discourse pertaining to entities in a basis and change of basis to the discourse used in the textbook. We found that the textbook discourse shared some similarities with the instructors and students' discourse, but there were notable differences between the spoken social language and written text.[7]

A. Students and instructors' discourse involving entities in a basis

In our characterization of the students and instructors' utterances referring to entities in a basis, we found four discourse categories: a person is in a basis, a vector is in a basis, a vector is written in a basis, and a calculation is in a basis. Furthermore, for most participants, more than one of these categories was needed to characterize their utterances; the overall assignment of these categories to each student's and instructor's discourse pertaining to entities "in a basis" is summarized in Table I. Throughout this section, we illustrate our claim that the utterances in the discourse categories had similar situated meanings that were mutually understood by and similarly used by the students and their respective instructor. We take the first four subsections, one for each of the aforementioned discourse categories, to describe the ways students discussed entities in a basis and those utterances' situated meanings [13] and types of grammatical constructions [16]. We next present our analysis of the instructors' discourse. Recall from the Sec. IV that utterances were analyzed from four class days for Instructor A and two for Instructor C. We analyzed utterances that either explicitly used or implicitly referred to basis with regard to the probability setting. Our findings show that the instructors' utterances involving entities in a basis were similar to the utterance categories and meanings found in the students' discourse. We give examples of the instructors' utterances in these categories and describe how their discourse was similar to or different from their students' discourse.

We summarize these findings with Table I and Fig. 2 In Table I, we illustrate our assignment of the in a basis categories for each participant. An "X" indicates that a discourse category was assigned to at least one of a participant's utterances; it does not indicate the number of times the category was assigned to that participant's utterances. We refer to this table throughout Sec. V A to indicate how many participants spoke utterances in each discourse category and to demonstrate the common use of these discourse categories across the participants. We use this table to illustrate that these discourse categories were

[6]When we describe the situated meanings of utterances and text throughout the results section, we claim what the phrases "mean." We acknowledge that although we, as researchers, inferred these meanings based on the context in which the utterances were spoken or text was written, it is possible that these meanings differ from what was intended by the speaker or author.

[7]Throughout these sections, we italicize phrases in the students', instructors', and textbook's quotes to direct the reader's attention to the categorized discourse in that particular utterance.

TABLE I. Each participant's overall use of discourse categorized with the entities in a basis categories.

	A person is in a basis	A vector is in a basis	A vector is written in a basis	A calculation is in a basis
Instructor A	X	X	X	X
A6	X	X		X
A8	X	X	X	X
A11		X		
A13	X	X	X	
A21	X	X	X	X
A25	X	X	X	
A30		X		
A32		X	X	
Instructor C	X	X	X	
C3	X	X	X	
C5			X	
C6			X	
C12		X	X	

present in utterances from most of the students, and multiple categories were used to characterize each participant's discourse.

In Fig. 2, we indicate the proportion of utterances coded within each category for both classes and for both instructors. In total, we coded 46 utterances from Class A, 11 from Class C, 17 from Instructor A, and 37 from Instructor C. We use these frequency measures to support our descriptive model [28] of the data set. Student utterances from Class A and Class C are represented by the leftmost top bar and rightmost top bar, respectively, of each of the four categories. The length of each bar communicates the proportion of the utterances within that category, and the shading of that bar (light, middle, or dark) indicates the Ochs, Gonzales, and Jacoby [16] characterization for utterances within that category. This qualitative representation helps to communicate, for instance, that "vector is in a basis" was the most prominent category within student utterances in Class A (over 50% were coded with this category), with the majority of those being physics-centered constructions. Utterances of students in Class C were mostly consistent with the "vector is written in a basis" category. Instructor A's utterances are indicated in the left column, aligned as the second row per utterance category, with students from Class A in the first row per utterance category; Instructor C's information is similarly conveyed in the rightmost column.

1. A person is in a basis

Utterances in this category employed a personal pronoun as the subject of the phrase, such as "we are in a basis" or "I am in a basis." According to Ochs, Gonzales, and Jacoby [16], these phrases are indeterminate grammatical constructions that use a personal pronominal subject (e.g., I, you, we) with a predicate involving the experience of an inanimate object. Because these utterances' meanings may seem linguistically unclear (the speaker cannot literally be in a basis, an abstract mathematical concept), we explore their situated meanings while considering the context in which they were spoken. Five students referred to a person being in a basis (see Table I). For example, A6 and A25 did so as they began question (a) in Fig. 1. A6 explained, "because *we're in the z basis*, I already know that the

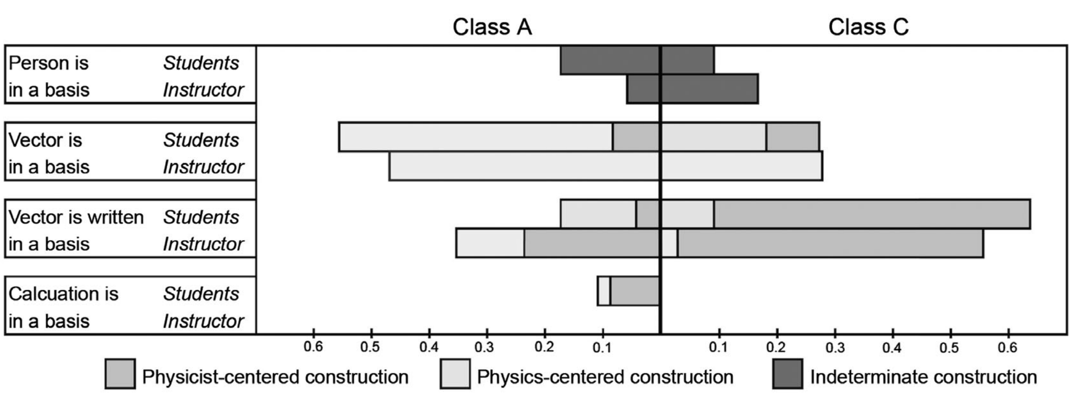

FIG. 2. Proportion of students' and instructors' discourse referring to entities in a basis.

probabilities—my plus z is going to be the square of the norm of my first component." A25 claimed, "*we are in the z basis*, so this makes this pretty easy." This humanizing of being in a basis was used as part of the students' justification for performing the "easy" Born rule procedure. The situated meaning of A6 and A25's utterances of "we are in the z basis" was that the problem's vectors are all expressed in terms of the z basis. Other utterances referring to a person being in a basis were spoken as students justified their change of basis for question (b) in Fig. 1. For instance, A6 claimed change of basis was the first step and said, "you can't do anything until *you're in the same basis*." A8 similarly explained, "You can't actually perform this calculation when *you're… in the different bases*." We inferred the situated meaning of these utterances to be that being in the same basis was a condition necessary for one to be able to perform the inner product. Through their use of these indeterminate grammatical constructions [16], the students each seemed to blend their identity with that of a vector. In this case, the "you" in "you're in the same basis" refers to the vectors in the inner product that are each a linear combination of vectors from the same basis (i.e.,$_y\langle+| = (1/\sqrt{2})\langle+| - (i/\sqrt{2})\langle-|$ and $|\psi\rangle = (3/\sqrt{13})|+\rangle + (2i/\sqrt{13})|-\rangle$). These examples illustrate the situated meanings of utterances within this "person is in a basis" category.

2. A vector is in a basis

Another category of students' discourse that emerged included utterances that referred to a vector being in a basis. In these utterances, the subject was typically a vector or pronoun referring to a vector that was described as being in a basis, such as in the phrase "these are in the z basis" (C3). Some utterances in this category had a human subject having or needing a vector to be in a basis, such as "*I needed plus y in the z basis*" (A6) or "*I have psi in z* " (C3). This category of discourse was the most prevalent in our dataset. Ten of the twelve students in our study referred to a vector in a basis (see Table I), but the meanings of the utterances were varied. Sometimes, we inferred an utterance to mean that a vector is an element of a basis set, whereas other times, an utterance meant that a vector is a linear combination of basis vectors.

Some utterances categorized as "a vector in a basis" were spoken when students explained the task setting. As A13 performed calculations for question (a), they said, "It's good because *they're both in the z basis*, the *kets are both in the z basis*, 'cause if *this was like in the x* [basis], you'd have this longer, more complicated math to do." Other students spoke similar utterances, such as "*they're already in the z basis*, so that's really simple" (A32), and "*these are in the same basis*, so you didn't have to convert anything" (A30). These utterances illustrate both aforementioned situated meanings, given that one of the vectors the students referenced was a z-basis vector $|+\rangle$, while the other vector $|\psi\rangle$ was a linear combination of z-basis vectors.

Four students referred to a vector in a basis as they justified performing a change of basis. For example, A13 did so while discussing the different bases of the vectors in the inner product $_y\langle+|\psi\rangle$. A13 explained, "*Psi* [$|\psi\rangle$] *is in a completely different basis*, so you can't just multiply out in-when *they're in different bases*." A13's utterance meant $|\psi\rangle$ and $_y\langle+|$ were linear combinations of vectors from different bases. Because of this, they could not perform the inner product $_y\langle+|\psi\rangle$, which necessitated a change of basis.

Three students' utterances pertained to a vector being in a basis when they discussed the result of a change of basis. For example, A11 described two possible approaches for changing basis: one involves changing $|\psi\rangle$ to be in the y basis, and the other involves changing $|+\rangle_y$ to be in the z basis. They explained, "in many ways it is better to change the psi because then *you have it in a different basis*." A11's utterance meant $|\psi\rangle$ would be a linear combination of basis vectors from a basis other than the z basis. Furthermore, after A6 performed a change of basis, they explained, "so then *both of them were in the z basis*." Their utterance meant the change of basis resulted in both of the vectors being linear combinations of z-basis vectors.

The students' discourse related to a vector is in a basis was primarily physics centered [16]. Several of the exemplified utterances, such as "*both of them were in the z basis*" (A6) and "*Psi is in a completely different basis*" (A13), had vectors as the subject of the phrase. Although the majority of utterances in this category were physics centered, some were physicist centered, such as "*I needed plus y in the z basis*" (A6) and "*you have it in a different basis*" (A11); these utterances position the physicist (student) as an agent. Ochs, Gonzales, and Jacoby [16] described this as "[The physicists] refer to themselves as agents in relation to physical phenomenon, acting upon physical entities through manipulations of experimental conditions" (pp. 335–336). The students referred to themselves as agents in relation to the vectors by needing or having the vectors be in a certain basis.

3. A vector is written in a basis

Nine students spoke utterances referring to a vector being written in a basis (see Table I). The verbs used in their explanations that cued us to include their utterance in this category include a vector "being written," "expressed," "given," or "expanded" in, or "in terms of," a basis. The utterances in this category are linguistically different from those in the vector is in a basis category due to their inclusion of phrases that focus on the symbolic expression of the vector. These utterances focus on the way a vector is symbolically written as a linear combination of vectors, whereas those in the vector is in a basis category focus on the vector existing or abstractly being a linear combination of basis vectors. The utterances in this category all had a situated meaning related to a particular vector being written as a linear combination of vectors from a certain basis (of which that vector was not an element). Students spoke utterances with this meaning in various contexts, such as in their justifications for needing a change of basis, descriptions of their action of changing basis,[8] and descriptions of the result of the change of basis.

Three students used this discourse as they justified the need for a change of basis. For example, C5 said, "*I need that to be written in the z basis* or to do those inner products to be nice," which meant they needed $|+\rangle_y$ to be written as a linear combination of z-basis vectors. A21 explained, "If *I'm expressing plus y in the z basis*, then I can make all those assumptions about one, you know, the pluses and the minuses, the cross terms are gonna be zero." The situated meaning of this utterance is that A21 was writing the y basis ket $|+\rangle_y$ as a linear combination of z-basis elements. C5 and A21 justified their need to write or express $|+\rangle_y$ in the z basis by explaining how doing so allowed them to make assumptions about the orthogonality of the basis vectors, yielding the inner products $\langle +|+\rangle = 1$ and $\langle +|-\rangle = 0$.

Four students used utterances coded as "written in a basis" as they described the action of changing basis. For example, C6 explained, "But most of the work is in *writing psi in terms of y vectors*, *y-basis vectors*." We posit that the situated meaning within this context and category seem consistent with students thinking about the process of change of basis as merely an algebraic substitution. Finally, two students referred to a vector being written in a basis as they described the result of the change of basis. After A25 changed basis, they said, "that's *y written in the z basis* as a bra." A32 exhibited similar discourse, saying, "*this is written in the z basis*. So it's gonna be the z basis, the *y written in the z basis*… now *I have both of them written, uh, both the bras of the y 's written in terms of the x* (sic) *components*." A32's utterance referred to the result of the change of basis as the vectors both being written in the z basis. Consistent with the aforementioned utterances in this category, when A25 and A32 claimed a vector is "written in the z basis," their utterance had a situated meaning that the vector is written symbolically as a linear combination of basis vectors.

Students' utterances referring to a vector being written in a basis were physics centered or physicist centered [16]. Thus, the students grammatically positioned themselves and the vectors as separate entities. The students' use of physics-centered grammar or physicist-centered grammar seemed to be related to the context of the part of the problem they were referring to. Students' utterances were primarily physicist centered as they justified the need to perform a change of basis or described the action of changing basis, and their utterances were primarily physics centered as they described the result of the change of basis. For example, A21's utterance, "If I'm expressing plus y in the z basis" was physicist centered, for they positioned themself as the subject acting on the vector. The phrases describing the action of changing basis, such as "to *rewrite up y in the z basis*, you can substitute…" (C3) and "you have to *expand out the plus y ket* [$|+\rangle_y$] *in terms of the z*" (A25), referred to a human (e.g., I, you, we) carrying out the action of rewriting or expanding a vector in a basis and were thereby physicist centered. Lastly, the students' utterances spoken in the context of describing the result of the change of basis were primarily physics centered. For example, A25's utterance, "that's *y written in the z basis* as a bra," was physics centered because the subject is $|+\rangle_y$. We find these results sensible, in that students were portraying themselves as active participants (subjects of the sentence) performing the action of changing basis and focusing on the emerging vector (the subject of the sentence) in the result of changing basis.

4. A calculation is in a basis

The last category of utterances referring to entities "in a basis" pertained to a calculation being in a basis. Phrases that cued us to include an utterance in this category are similar to "we do this in a basis" or "we are working in a basis." The use of an action verb, such as "do" or "are working" helped identify phrases in this category. Three students referred to a calculation in a basis (see Table I). For example, A21 explained how performing the change of basis by expressing $|+\rangle_y$ in terms of the z basis allowed them to make assumptions about the inner products $\langle +|-\rangle$ and $\langle -|+\rangle$ being 0. Then they claimed, "But if *I were to do this in the y basis*… then I can't make any assumptions about that [orthogonality], so I don't really know how to calculate that in bra-ket language." In their phrase, A21 referred to performing the probability calculation in the y basis, meaning that if they were to not change ${}_y\langle +|$ to be expressed in terms of z-basis vectors and instead leave ${}_y\langle +|$ as a y-basis vector, they would not know how to calculate the inner product ${}_y\langle +|\psi\rangle$. A8 referred to performing a calculation in a basis as they performed part (a) of the interview task. They explained, "I'm assuming that *we're working in the z basis*… you literally just square the components… It's super simple if *you're working in the basis*." They twice referred to working or performing the calculation in a basis. A8's utterance meant they were performing the probability calculation in a setting where the vectors in the inner product were each expressed in terms of the z basis. The utterances in this category were primarily physicist centered [16]. For example, phrases such as "*we're working in the z basis*" (A8) and "if *I were to do this in the y basis*" (A21) have a human subject

[8]The coded utterances in this category that students used to describe their action of changing basis were also coded with the change of basis category of writing a vector in another form, as described in Sec. V B 2.

acting on a mathematical object and are thereby physicist centered.

5. Instructor A's discourse involving entities in a basis

Instructor A spoke utterances during class that were classified with each of the four utterance categories involving entities in a basis (see Fig. 2 and Table I),[9] with vector is in a basis being the most frequent. First, Instructor A's lone reference to a person being in a basis was as she described the Born rule procedure for finding probabilities. She said, "probabilities are, take the squares of the norms of the coefficients if *you are in the right basis*." This latter phrase meant that the state vector is expressed in terms of the basis coinciding with the axis of spin measurement, as the Born rule is only valid when the vectors in the calculation are in terms of the same basis. Five of Instructor A's students used this discourse category as they calculated the probabilities in the interview tasks (see Table I); for example, A6 stated "because *we're in the z basis*, I already know that the probabilities—my plus z is going to be the square of the norm of my first component." Thus, there was a similarity in Instructor A's discourse and that of her students given that they spoke utterances referring to a person in a basis.

Next, vector is in a basis was the most common utterance category for Instructor A, with just over half of her utterances being categorized this way. Sometimes, the meaning referred to a vector being an element of a basis, while other times, the meaning referred to vectors that were linear combinations of basis vectors. For example, as Instructor A described the probability calculation of $|_{\hat{n}}\langle -|+\rangle|^2$, she said, "*This is a nice easy plus in the z basis. This is a basis vector in its own basis*." In this utterance, Instructor A referred to $|+\rangle$ as being in the z basis, which she did to justify her choice to change the basis used to express $_{\hat{n}}\langle -|$ instead of $|+\rangle$. As she discussed this change of basis, she asked the class, "Now how do you find out what *n hat is in the z basis*?" By this utterance, she meant what $_{\hat{n}}\langle -|$ is as a linear combination of z-basis vectors. There are some similarities between Instructor A's discourse in the vector is in a basis category and her students' discourse. For instance, A6 and A13 referred to vectors in a basis as they justified their choice of procedure for changing basis, just as Instructor A did. They explained, "I needed *plus y in the z basis* because *this was in the z basis*" (A6) and "*Psi* $[|\psi\rangle]$ *is in a completely different basis*, so you can't just multiply out in—when *they're in different bases*" (A13). Furthermore, all of her students spoke utterances that were classified in this category (see Table I). This gives evidence to the similarities in the instructor's and students' discourse.

The second most common usage for Instructor A was the utterance category vector is written in a basis. Her discourse was similar to that of her students, as five of Instructor A's students used this discourse category, as well (see Table I). As she described the action of changing basis, she said, "You are actually going to take a bra-ket of plus y with psi. Make sure that you are *expanding them in terms of the same basis*. Probably here you want to use the z basis." The italicized phrase meant one is expanding the written form of the kets by writing them as linear combinations of the same basis kets. The prominent situated meaning for Instructor A's utterances within this category was that the state was written as a linear combination of the two elements of the z basis (plus z and minus z). As she discussed a different change of basis problem, she explained, "If I take the state plus x, I have to be able to *write that as a linear combination of plus z and minus z*." When justifying why a change of basis was necessary, she said, "as soon as *we wrote plus in the x direction in terms of the z basis*, that guarantees that the probabilities are going to come out right." When describing the result of the change of basis, she claimed, "Now that *I have the minus n hat written in terms of plus z 's and minus z 's*, I can use my nice rule [the Born rule]." In these examples, there is evidence of some similarities in Instructor A's discourse and her students' discourse; her students spoke similar utterances, such as "you have to *expand out the plus y ket* $[|+\rangle_y]$ *in terms of the z*" (A25), while describing the action of, result of, and reason for the change of basis. Her discourse slightly differed from that of her students, for she sometimes used syntax that more explicitly conveyed the precise meaning of her utterances.

Within the context of probability problems like those in Fig. 1, we did not find any instances of Instructor A speaking utterances that could be categorized as a calculation is in a basis, and this was the least frequent utterance category for her students, too (see Table I). We do note, however, that within the class days in which those types of problems were discussed, utterances in this categorization (but a different context) did occur twice. Both occurred while discussing the calculation of what is called an outer product, which results in a matrix. She explained, "Instead of *doing this in the x basis*, let's do it, we want to know what matrix it is in the z basis." Her phrase, "*doing this in the x basis*" referred to performing the outer product of vectors in the x basis.

Finally, there were also some similarities in Instructor A's utterances and her students' utterances regarding their use of indeterminate, physics-centered, and physicist-centered grammar [16]. As indicated in Fig. 2, when Instructor A and her students spoke utterances referring to a person in a basis, all of their utterances were indeterminate, in that they used a personal pronominal subject (e.g., I, you, we) with an inanimate predicate. Instructor A's utterances referring to a vector in a basis were

[9]Instructor A's utterances in the "calculation is in a basis" category were not spoken within the spin probability context and were thus not included in Fig. 2.

all physics centered, for the subject was often a vector or a pronominal referent of a vector. Similarly, her students' utterances referring to a vector in a basis were primarily physics centered. Whenever Instructor A referred to a vector being written in a basis, her utterances were physicist centered. Her students' utterances, however, were primarily physics centered, which indicated a difference in Instructor A and her students' discourse. Instructor A's few utterances referring to a calculation in a basis were physicist centered, and her students' utterances were primarily physicist centered in this category, as well. Overall, our characterizations of discourse pertaining to entities in a basis and Ochs, Gonzales, and Jacoby's [16] categories of grammatical constructions allowed us to identify similarities and differences in Class A and their instructor's discourse.

6. Instructor C's discourse involving entities in a basis

Instructor C spoke utterances during class that were classified with three of the four utterance categories involving entities in a basis (see Fig. 2). Instructor C and his students' discourse were both classified with the categories of a person is in a basis, a vector is in a basis, and a vector is written in a basis, but neither Instructor C nor Class C students spoke utterances referring to a calculation being in a basis (see Table I).

We present the following transcript excerpt from class discussion to illustrate Instructor C's discourse. Instructor C wrote the change of basis formulas $|+\rangle_x = (1/\sqrt{2})[\begin{smallmatrix}1\\1\end{smallmatrix}]$ and $|-\rangle_x = (1/\sqrt{2})[\begin{smallmatrix}1\\-1\end{smallmatrix}]$ on the classroom whiteboard:

> *Instructor C:* In what basis did I just write that? *What is one and one- [points to* $|+\rangle_x = (1/\sqrt{2})[\begin{smallmatrix}1\\1\end{smallmatrix}]$ *written on the board] what are these, the coefficients in?*
> *Student: Those are the z.*
> *Instructor C: The z right, so again our z, yeah I think it's a good phrase to use, it's our home basis right. So* this is in the z basis, right*?*
> *Student: Is there cases in which we won't do that?*
> *Instructor C: Well, so* I could write, I could write z states in the x basis *and then they look the same right? And so that's where it starts to get messy, … but what if we want to go from the z to the x? Right like you can do this algebra, but you can also do the vector math a lot faster. You guys know how to do those kinds of things. So changing the basis, if you're going from one kind of measurement to a different one, that's where we want to play with it. Because* the basis you're in*, so I'll give you a little sneak preview, right the key is, whatever I'm measuring, that quantity is* the basis I'm going to be in*, right, so however many possible outcomes there are, whatever I'm measuring tells me the dimension and the number of basis states I have… the final state and the set of whatever your basis is, is whatever you want to measure it in terms of and* express it in terms of.

This transcript gives evidence of Instructor C using discourse from all three utterance categories. For example, his utterances, *"In what basis did I just write that?"*, "I could *write z states in the x basis*," and "…*express it in terms of*" all had situated meanings pertaining to writing a ket as a linear combination of vectors from a certain basis. We characterized Instructor C's utterance, "So *this* [referring to $|+\rangle_x$] *is in the z basis*" as a vector is in a basis utterance. This utterance meant the numerical values in the vector representation $(1/\sqrt{2})[\begin{smallmatrix}1\\1\end{smallmatrix}]$ of the ket $|+\rangle_x$ indicate that the ket is a linear combination of z basis kets. Instructor C's discourse also referred to a person being in a basis. He said, "Because *the basis you're in*… whatever I'm measuring, that quantity is *the basis I'm going to be in*." This meant the probability of measuring the spin along a particular axis of rotation and direction (in the positive z direction, for instance) would coincide with the basis ket needed for that calculation (the plus z ket).

We identified some similarities between Instructor C's utterances and those of his students. Throughout the class episodes we analyzed, Instructor C most commonly referred to vectors being written in a basis and vectors being in a basis, and the same was true for his students. Instructor C's use of person is in a basis stands out as noticeably higher than the one documented instance from his students. Finally, we found additional similarities between Instructor C and his students' discourse according to Ochs, Gonzales, and Jacoby's [16] types of grammatical constructions. All utterances referring to a person in a basis were indeterminate grammatical constructions, given that they referred to a personal pronominal subject being in a basis. Whenever Instructor C referred to a vector in a basis, all of his utterances were physics centered, as the object of focus was a ket. Similarly, his students' utterances in this category were primarily physics centered. Furthermore, both Instructor C and his students' utterances referring to a vector being written in a basis were primarily physicist centered, as they all commonly positioned themselves as performing the action of writing a vector in or in terms of a basis. Thus, there were several commonalities in the types of grammatical constructions used by Instructor C and his students.

7. Summary

In this section, we presented our findings regarding how the students and instructors discussed entities in a basis, along with the utterances' associated situated meanings [13] and grammatical structure [16]. We found that utterances referring to entities in a basis had similar situated meanings that were mutually understood, and these discourse types were used similarly by the students and their instructors (see Fig. 2). We identified four ways in which the students and instructors referred to entities in a basis: a person in a basis, a vector in a basis, a vector written in a basis, or a calculation in a basis. Utterances in these categories had similar situated meanings [13] related to either a vector being an element of a basis or a vector being expressed as a linear combination of vectors from a certain basis. For instance, the utterances, "we are in the z basis" (A25), "the kets are both in the z basis" (A13), "*I need that to be written in the z basis*" (C5), and "*we're working in the z basis*" (A8), are each from a different discourse category, yet they have a similar situated meaning of vectors being linear combinations of vectors from the z basis. The students and instructors used the different discourse types interchangeably to refer to the same entity. The students and instructors seemed to have a mutual understanding of the situated meanings of these different utterances because they all used them interchangeably in their discourse.

The grammatical constructions were used similarly by the students and their instructors (see Fig. 2). Both instructors and their students spoke utterances categorized as person is in a basis, and they used only indeterminate grammatical constructions [16] when speaking those utterances. The University A students and Instructor A most often spoke utterances categorized as vector is in a basis, and these were primarily physics-centered grammatical constructions [16]. The students at University C and Instructor C most often spoke utterances categorized as vector is written in a basis, which were primarily physicist-centered grammatical constructions. Only students at University A spoke utterances categorized as a calculation is in a basis, and Instructor A spoke utterances classified in that category as well, just in a different context of performing an outer product. Utterances in this category had mostly physicist-centered grammatical constructions [16]. Overall, the instructors and students both spoke utterances classified in the four discourse categories referring to entities in a basis, and they used similar grammatical constructions. The students' and instructors' similar use of discourse types, grammatical constructions, and situated meanings demonstrates the systematicity of their shared social language.

B. Students and instructors' discourse involving change of basis

We next focused our analysis on students and instructors' utterances pertaining to change of basis. We characterized these utterances according to four categories: changing the form of a vector, writing the vector in another form, changing the vector into another vector, and switching bases. These categories were identified by verbs indicating some sort of alteration (e.g., turn, go, rewrite, switch, etc.) with an object of a vector or basis. The assignment of these categories to each student's and instructor's discourse pertaining to change of basis are presented in Table II. Throughout this section, we elucidate our claim that utterances in these discourse categories referring to change of basis have varied situated meanings that are mutually understood by the students and their instructors, and these discourse types were used similarly by the students and their instructors. We elaborate on utterances within each category by explaining their situated meanings given the context in which they were spoken. We first present our analysis of the students' discourse and then present our analysis of the instructors' discourse involving change of basis. We describe how the instructors' discourse was similar to or different from their students' discourse.

We summarize our analysis of the students and instructors' discourse involving change of basis in Table II and Fig. 3. Table II demonstrates our assignment of the change of basis discourse categories for each participant. An "X" indicates that a discourse category was assigned to at least one of a participant's utterances. We reference this table throughout Sec. V B to show how many participants spoke utterances in each of the discourse categories and to

TABLE II. Each participant's overall use of discourse categorized with the change of basis categories.

	Changing the form of a vector	Writing a vector in another form	Changing a vector into another vector	Switching bases
Instructor A	X	X		X
A6	X			
A8		X	X	
A11	X	X		
A13	X			X
A21	X	X		
A25		X		
A30	X			X
A32	X	X		
Instructor C		X		X
C3	X	X		
C5		X		
C6		X		
C12		X		

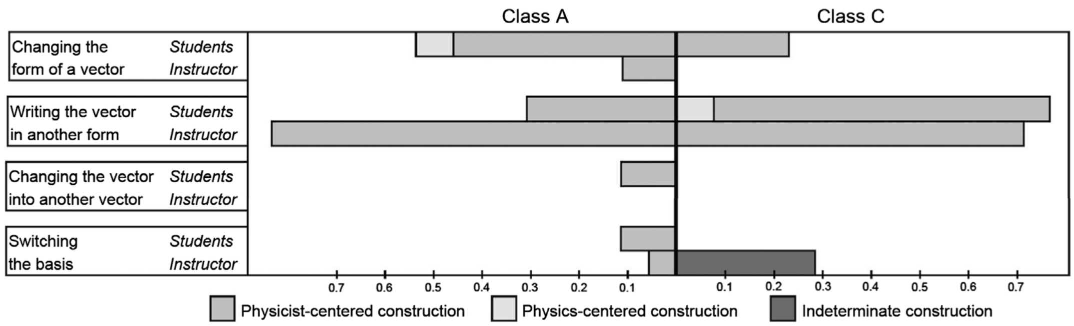

FIG. 3. Proportions of students' and instructors' discourse referring to change of basis.

illustrate the common use of these discourse categories across the participants. We use Table II to demonstrate that the change of basis discourse categories were present in the discourse of several students and that multiple categories were often used to characterize each participant's discourse.

Figure 3 provides a visual to indicate the general proportion of utterances coded within each category for both classes and both instructors. In total, we coded 28 utterances from Class A, 14 from Class C, 18 from Instructor A, and 7 from Instructor C. Class A predominantly referred to change of basis as changing the form of a vector, whereas Class C predominantly referred to change of basis as writing a vector in another form. Instructor A's utterances are indicated in the left column, aligned as the second row per utterance category, with students from Class A in the first row per utterance category; Instructor C's information is similarly conveyed in the rightmost column.

Students' utterances in all of these discourse categories were primarily physicist centered [16]. For instance, the utterances, "*I'm going to convert the y plus or minus bra into the z basis*" (A30), "*you have to expand out the plus y ket in terms of the z*" (A25), "*we can represent our φ going to the φ' in the y basis as being…*" (A8), and "*you have to switch bases*" (A13), all have a human subject performing an action on a vector or a basis. The students and instructors' discourse thus positioned them as agents who were actively involved in performing a change of basis. One of two exceptions of utterances discussing change of basis that were not physicist centered was C3's utterance, "*plus y psi ket [writes ${}_y\langle+|\psi\rangle$]… which can be rewritten as one over root two plus plus i minus*." This utterance was physics centered, for the subjects were the vectors in the inner product.

1. Changing the form of a vector

A student's utterance was classified in this category of changing a form of a vector when the student's discourse indicated the vector was "change[d]," "convert[ed]," or altered as a result of the basis change. Utterances in this category encompassed phrases in which the verb's connotation deemed its object (the vector) as concrete or (figuratively) tangible, and the verb implied a change in its form. Seven students' discourse was coded with this category (see Table II), as they described changing a vector to another basis, converting a vector to another basis, or putting a vector in the right basis. Within the first, A6 said "*I changed this from the y basis to the z basis*," and A11 explained, "it is better to *change the psi* because then you have it in a different basis." Other students referred to conversion, such as "*I'm going to convert the y plus or minus bra into the z basis*" (A30) and "*I would have to convert these back to the z basis*" (A32). Finally, students' utterances included other verbs indicating a change in the form of the vector, such as "*I could've turned this into the y basis*" (A11) and "*I'll put them in the right bases*" (A32). These students often referred to the vector being changed to or into a basis, not another vector. These phrases have a situated meaning that the vector is being changed to be a linear combination of vectors from a different basis, rather than the vector changing into a basis set.

2. Writing a vector in another form

Nine students' utterances referred to change of basis as writing or expressing a particular vector in another form (see Table II). An utterance was classified in this category when the discourse indicated a titular change in the vector. These students spoke phrases that described expanding a vector in terms of another basis, substituting, rewriting the vector, and writing the vector in terms of another basis. The most common verbs used in this category were "rewrite" and "express"; the objects of these verbs were always vectors, never bases. For example, as A32 described their procedure for change of basis, they said "I'll usually *rewrite it…* I take this ket and *I'm gonna rewrite that*," which meant they were rewriting the vector in a different form. C6 similarly referred to changing basis as "*writing psi in terms of y vectors, y-basis vectors*," which meant they were writing $|\psi\rangle$ in a different form as a linear combination of y-basis vectors. Other utterances in this category used other verbs, such as "expand," "substitute," or "express." A25 explained, "For this one you have to *expand out the plus y ket in terms of the z*," which meant that to change the basis, one has to write the vector $|+\rangle_y$ in a different expanded form as a linear combination of z-basis kets. Furthermore, A11 said, "change of basis is what you're doing when you're *substituting it in*. You're *substituting each of these components*." A11 meant change of basis involves algebraically substituting $(1/\sqrt{2})\langle+| - (i/\sqrt{2})\langle-|$ in for ${}_y\langle+|$. This substitution entails writing the vector in another form. These students referred to changing basis as writing the vector in another form, which we consider to be a titular change in the vector, rather than a physical change.

3. Changing a vector into another vector

A student's utterance was classified in this category of changing a vector into another vector when the student's discourse indicated the prechange of basis and postchange of basis vectors were two distinct vectors. Only A8 spoke utterances in this category (see Table II). As A8 described two possible approaches for changing basis, they explained, "one of them is to *put this vector in some phi prime that's in the y basis*, and then just do y plus phi prime y [${}_y\langle+|\varphi'\rangle_y$]," which meant changing the vector φ into another vector that is a linear combination of y-basis vectors. As A8 performed the change of basis, they said, "we can represent our φ *going to the* φ' *in the y basis* as being… [wrote out vector]." A8 gave the pre- and post-change of basis vectors two different names, φ and φ', indicating the two vectors were distinct. Their utterances thus referred to change of basis as changing a vector into another vector. Unlike the first category of utterances referring to changing the form of a vector, these utterances do not imply the postchange of basis vector is the prechange of basis vector with some alterations, but rather imply the postchange vector is a different vector entirely.

4. Switching bases

A student's phrase was classified in this category of switching bases when the discourse positioned the basis set, not an individual vector, as the object of the verb. Two students, A13 and A30, spoke utterances in this category (see Table II). As A13 explained how this problem was related to change of basis, they said, "You… have to… *change your y basis to z basis*." A13 explained they could not calculate the inner product of vectors when they are in different bases and claimed, "you actually have to *switch the basis*." These utterances seemed to have literal meanings, as A13 referred to changing basis by switching from the y basis to the z basis. Unlike the three aforementioned categories of utterances referring to change of basis, the object undergoing the change here is the set of basis kets instead of an individual vector. We see this again in A30's utterance, "I'm going to *convert y into the z basis* before I multiply it out," which described changing basis as converting the y basis into the z basis. Overall, phrases in this category contained verbs such as change, switch, or convert, and they position the basis, not the vector, as the object undergoing the change.

5. Instructor A's discourse involving change of basis

We found similarities in the ways Instructor A and her students discussed change of basis. Instructor A and her students referred to changing the form of a vector, writing a vector in another form, and switching bases (see Table II). During class Instructor A explained, "When you sit down to do a calculation, choose a basis and be consistent… The safest thing to do in this class… is to *transfer everything to the z basis*." Transferring to the z basis meant changing the form of vectors to be linear combinations of z-basis kets. Instructor A's students spoke similar utterances but used different verbs, such as convert or turn, to specify the change. Instructor A most often referred to writing a vector in another form when she discussed change of basis, such as in her utterance, "we can see how to change basis. If *we want to write plus x in the z basis*, this is what its components are. If we want to *write minus x in the z basis*, this is what its components are." Other times, she used the verb "expand" to describe the change of basis, as in her explanation: "To do the matrix multiplication that's implied by a bra with a ket, you have to put these, *you have to expand these two things* (pointing to the bra and ket) *in terms of the same basis*." Her students similarly referred to writing and expanding vectors in a basis. Instructor A's discourse also referred to switching bases, for example, "I probably wanna *switch to the z basis*." Her utterances in this category were similar to her students' utterances that focused on switching the basis instead of changing an individual vector. Instructor A's utterances differed from those of her student A8, who referred to changing a vector into another vector, because Instructor A's discourse did not suggest that connotation. Furthermore, all of Instructor A's utterances referring to change of basis were physicist-centered because the subject was a human performing an action on a vector or basis. This was similar to her students' utterances about change of basis, which were primarily physicist centered as well.

6. Instructor C's discourse involving change of basis

We found similarities in Instructor C's and his students' discourse about change of basis. Instructor C primarily spoke utterances pertaining to writing a vector in another form as he described change of basis, and his students all used similar discourse as they performed the interview tasks (see Table II). Some examples of Instructor C's utterances include, "I could *write z states in the x-basis*" and "I *write the plus x state as one over square root of two, that* [wrote $(|+\rangle + |-\rangle)$]." He exemplified how to write a vector as a linear combination of vectors from another basis. Instructor C's utterances referring to writing a vector in another basis were all physicist centered, which is similar to the characterization of his students' utterances in this category. Instructor C occasionally spoke utterances with a connotation of switching bases. For instance, as he discussed change of basis, he asked, "What if we want to *go from the z to the x*?", implying the meaning of change of basis was switching from one basis to another. This utterance includes an indeterminate grammatical construction [16], for it referred to a personal pronominal subject (we) going from one basis to another, which is typically an experience of an inanimate vector. His students, however, did not refer to switching bases (see Table II). Instead, one of his students referred to changing the form of a vector, which Instructor C did not refer to during the classes we observed. Neither he nor his students discussed changing a vector into another vector. Overall, we identified some similarities in Instructor C and his students' discourse, for they both predominantly referred to writing a vector in a basis when they discussed change of basis. However, there were some differences in the utterances from categories that were infrequently used.

7. Summary

In this section, we presented our findings regarding how the students and instructors discussed change of basis, along with associated situated meanings of their utterances [13] and grammatical structure [16]. We found that utterances referring to change of basis had varied situated meanings that were mutually understood, and these discourse types were used similarly by the students and their instructors. We identified four ways in which the students and instructors referred to change of basis: changing the form of a vector, writing a vector in another form, changing a vector into another vector, and switching bases. Utterances in these categories had varied situated meanings [13]. For instance, the utterance "*I would have to convert these back to the z basis*" (A32) refers to changing the form of a vector and has a situated meaning that the student is changing the vectors to each be a linear combination of vectors from the z basis. This differs from the utterance, "you have to *expand out the plus y ket in terms of the z*" (A25) from the "writing a vector in another form" category, which has a situated meaning of writing the vector $|+\rangle_y$ in a different expanded form as a linear combination of z-basis vectors. Utterances in this category implied a titular change in the vector, rather than a physical change. Utterances in the "changing a vector into another vector" category indicated the prechange-of-basis and postchange-of-basis vectors were two distinct vectors. Utterances in the switching basis category had literal meanings of changing or switching bases, and they positioned the basis, not the vector, as the object undergoing the change, unlike utterances in the prior discourse categories. Although utterances in these discourse categories had varied situated meanings, the students and instructors used them interchangeably when referring to change of basis. The students and instructors seemed to have a mutual understanding of the situated meanings of these different utterances because they used them interchangeably in their discourse.

The discourse types and grammatical constructions were used similarly by the students and their instructors (see Fig. 3). Both Instructor A and her students spoke utterances classified in the "changing the form of a vector" category (see Table II), and they primarily used physicist-centered grammatical constructions [16] when speaking those utterances. The students and instructors from both schools often spoke utterances classified in the writing a vector in another form category (see Table II), and those were primarily physicist-centered grammatical constructions [16]. Only one student from University A spoke utterances classified in the changing a vector into another vector category (see Table II). Instructor A and her students also spoke utterances classified in the "switching bases" category (see Table II), which were all physicist-centered grammatical constructions [16]. Overall, the students and instructors' similar use of discourse types, grammatical constructions, and situated meanings of phrases referring to change of basis demonstrates the systematicity of their shared social language.

C. Two students' discourse regarding basis and change of basis

In this section, we use data from two students to exemplify the finding that students often switched between different discourse types and grammatical constructions in quick succession. We present a detailed analysis of their utterances regarding basis and change of basis, through a more holistic approach to a longer piece of transcript for each student. We chose these students in particular because their utterances exemplify several discourse categories referring to entities in a basis and change of basis, as well as all three grammatical constructions [16]. We demonstrate how they switch between different discourse categories and describe their utterances' situated meanings.

1. Student A8

A8's discourse referred to entities in a basis and change of basis in a variety of ways as they reasoned through the interview questions. As they engaged in question (a), their utterances exemplified three of our in a basis categories: a vector is written in a basis, a calculation is in a basis, and a person is in a basis.

> Because this [points to $|\psi\rangle$] is written in *z, oh and this along the z-axis [points to question (a)], so because this is written along the z axis [points to question (a)], I'm assuming that* we're working in the z basis here, standard representation…*Then you do… norm squared of plus with psi, and by the same rule I talked about earlier…you get* $4/13$ *and* $9/13$. *So you literally just*

square the components if you want it like that. It's super simple if you're working in the basis that you're in.

In saying "*this is written in z*" [coded as "written in a basis"], A8 meant the given $|\psi\rangle$ is written in terms of z-basis kets. Given this and that the question asked for the probability that the spin component of the state vector was up along the z axis, they assumed "*we're working in the z basis*" [coded as calculation is in a basis], which meant every vector in the inner product was expressed in terms of the z basis. This justified their procedure choice of squaring coefficients because they knew that having each vector in terms of the same basis allowed them to use the Born rule. After performing this procedure, they said, "It's super simple if *you're working in the basis that you're in*." They referred to "you" as working in the basis and "you" as being in a basis. Thus, this utterance was double coded as calculation is in a basis and person is in a basis. Their utterance meant the computation is simple if the vectors in the inner product are both linear combinations of the same basis vectors. A8's discourse referred to a basis as a form a vector could be written in, a setting one can perform a calculation in, and a situation a person can be in.

As A8 worked on question (b), their explanation exemplified two of our change of basis categories and the fourth in a basis category:

> *There are two ways to go about it, um, one of them is to* put this vector in some phi prime that's in the y basis*, and then just do y plus phi prime y'cause it makes calculations, and it follows the same rules as this. Um, the other possibility is to do, is to* take the spin up y and go to whatever it is in the *z*, in the *z* basis*, cause we have this in the z basis.*

By stating "put this vector in some phi prime that's in the y basis" [coded as changing a vector into another vector], A8 referred to change of basis as the action of changing the vector $|\psi\rangle$ into another vector $|\psi'\rangle_y$. By stating "take the spin up y and go to whatever it is in the z" [also coded as changing a vector into another vector], A8 similarly referred to change of basis as the action of changing the vector $|+\rangle_y$ into another vector, $(1/\sqrt{2})(|+\rangle + i|-\rangle)$. They referred to the postchange of basis vector as a different vector than the original. A8's discourse here was also indicative of the fourth in a basis category because they referred to a vector in a basis. Particularly, A8 referred to the result of the change of basis as either "some phi prime that's in the y basis" or "whatever it is in the z, in the z basis." These utterances were each coded as "a vector in a basis" because they referred to the vectors $|\psi'\rangle_y$ and $|+\rangle_y = (1/\sqrt{2})(|+\rangle + i|-\rangle)$ being in a basis.

A8's discourse was indicative of other discourse categories as they continued to work on question (b). As they performed this change of basis, A8 said, "*I'm just gonna sub in this minus and plus*" [coded as "writing a vector in another form"], by which they referred to change of basis as a substitution involving writing a vector in another form. A8 later said, "you can't actually perform this calculation *when you're… in the different bases*" [coded as "a person is in a basis"]. A8 referenced "you" being in a basis as a situation in which one can perform the calculation. This utterance meant one cannot perform the calculation when the vectors in the inner product are not each a linear combination of the same basis vectors. Overall, as A8 explained their reasoning while responding to the interview questions, they switched between various discourse categories. A8's discourse exhibited all four categories of utterances referring to entities in a basis. A8's utterances also exhibited two categories of change of basis: changing a vector into another vector and writing a vector in another form.

Overall, A8's work on question (b) entailed determining the expression of the z-basis kets in terms of the y-basis kets, using those to express $|\psi\rangle$ as a linear combination of the y-basis kets, and using the Born rule to calculate the desired probabilities. Their grammatical constructions [16] included the indeterminate form ("in the basis that you're in"), the physics-centered form (e.g., "what this plus up ket actually is in y", "phi going to phi prime in the y basis"), and the physicist-centered form (e.g., "we're working in the z basis," "we can really easily pull out the probabilities"), which was most prevalent.

2. Student C3

C3's discourse was also indicative of several discourse categories. Here we focus on their response to question (b), and their utterances exemplified three of the in a basis categories (vector in a basis, person in a basis, and a vector being written in a basis) and two of the change of basis categories (changing the form of the vector and writing the vector in another form). C3's utterances also illustrated all three of Ochs, Gonzales, and Jacoby's grammatical constructions.

Unique to C3 in our dataset, they first attempted to use matrix notation to solve part (b) of the problem. As C3 approached the task, they said,

> I have psi in z and I have everything in the *z, which was very convenient in the previous problem, but now* I need to convert this, the states, to the y axis… *I know that plus y equals one over root 2… I know that, minus y equals 1 over root* 2*i minus i [writes vectors in Fig. 4]. So, and that's assuming that* these [points to written vectors] are in the z-basis.

C3's utterances "*I have psi in z*," "*I have everything in the z*," and "assuming *these are in the z basis*" were all coded as vector is in a basis. Those statements meant, respectively, that the state vector psi is a linear combination of z-basis kets, that the given kets in the linear combination

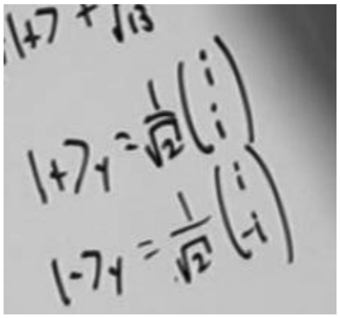

FIG. 4. Student C3's initial change of basis work in question (b).

are elements of the z basis, and the vector representations of the up and down y kets were determined according to the z basis.

Next, the interviewer followed up by asking C3 about their vectors in Fig. 4; C3 consulted a reference sheet that had the change of basis equations written in ket notation, and the interviewer helped C3 notice that the top component in their vectors should be 1 rather than i. C3 began their first attempt to solve the problem by saying "now I can use this [the corrected vectors] to convert the z into y state, into the y basis." This was coded as "changing the form of a vector." C3 then wrote the row-column vector product $[3/\sqrt{13}\;\;0][\begin{smallmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{smallmatrix}]$ but did not complete the computation. Anticipating that their procedure would not produce the result they expected, they explained, "*we're in the plus z state* and therefore *to convert*, I can just multiply by 1 over root 2 and i over root 2. And that would give me the positive. But now I'm thinking if will, if that would work." We coded "*we're in the plus z state"* with "person is in a basis." As C3 attempted to change the $|+\rangle$ portion of $|\psi\rangle$ into the y basis, their focus on *conversion* of the vector was classified as changing the form of a vector.

After a few more minutes trying other matrix-oriented computations to express $|\psi\rangle$ in terms of the y basis, C3 said, "Oh, wait a second. I think I made this way more complicated" and proceeded with the procedure using Dirac notation that expressed $|+\rangle_y$ in terms of the z-basis kets. As C3 decided to use this procedure, they explained, "This [the question] is asking the probability that it's up in the y axis. So, equals plus y psi ket, and this is squared, *which can be rewritten* as one over root two plus plus i minus." This discourse was indicative of the category of writing a vector in another form. After C3 completed this approach (with some algebraic errors), the interviewer asked C3 how this problem related to change of basis: "*to rewrite up y in the z -basis,* you can *substitute* positive y for one over root 2 *up in the z* plus i *down in the z*. So, from that *you just rewrite it like that, and that's essentially changing the basis of plus y into the z basis*." The first sentence was double coded as written in a basis and writing a vector in another form and the latter as writing a vector in another form. C3 detailed what they meant by rewriting $|+\rangle_y$ in the z basis as substituting a linear combination of z-basis vectors (i.e., $(1/\sqrt{2})(|+\rangle + i|-\rangle)$) in place of the vector $|+\rangle_y$. The situated meaning for the change of basis process was as an algebraic substitution.

Overall, C3's work on question (b) began with attempts to use matrix notation to express $|\psi\rangle$ in terms of the y basis before transitioning to expressing $|+\rangle_y$ in terms of the z basis and attempting to calculate the probability in Dirac notation. Their grammatical constructions [16] included the indeterminate form ("we're in the plus z state"), the physics-centered form (e.g., "these are in the z basis"), and the physicist-centered form (e.g., "I have psi in z," "I can use this to convert"), which was most prevalent.

D. Textbook's discourse involving basis and change of basis

After investigating the discourse of these physics students and their instructors, we found some similarities and differences between their discourse and that of their textbook [26]. The textbook discourse was classified with the categories of a vector is in a basis and a vector is written in a basis, as well as the change of basis category of writing a vector in another form. Within these, we found similarities in the situated meanings of the textbook statements and the students' and instructors' utterances, yet there were differences in the syntax used. There were also differences between the discourse in the written text and that of the participants' spoken utterances, for none of the statements in the text were coded with these discourse categories: a person is in a basis, a calculation is in a basis, changing the form of a vector, changing a vector into another vector, and switching bases. We also identified grammatical similarities between the textbook's statements and the students and instructors' utterances evident in their use of physics-centered and physicist-centered grammar [16], and we found a difference in the text's and participants' use of indeterminate grammatical constructions.

Recall that the two situated meanings associated with vector is in a basis utterances were either the vector being an element of the basis set or the vector being a linear combination of basis vectors. These meanings were consistent with what we inferred as meanings within the text as well. For example, the textbook referred to vectors forming or comprising a basis, as in the following quotes: "In the two-dimensional vector space of a spin-$1/2$ system, the *two kets* $|\pm\rangle$ *form a basis*" (p. 11), and "For the S_z measurement, there are only two possible results, corresponding to the states $|+\rangle$ and $|-\rangle$, so these *two states comprise a complete set of basis vectors*" (p. 11). These statements refer to $|+\rangle$ and $|-\rangle$ as the elements that constitute a basis, and we posit that McIntyre, Manogue, and Tate's [26] use of the verbs "form" and "comprise" help explicitly convey the meaning of the statements. Whenever the text conveyed the other meaning of a vector being in a basis—a vector being a linear combination of basis vectors—the meaning also seemed explicitly evident from the syntax in the text. For instance: "The completeness of the basis kets $|+\rangle$ and $|-\rangle$, implies that *a general quantum state vector* $|\psi\rangle$ *is a linear combination of the two basis kets*: $|\psi\rangle = a|+\rangle + b|-\rangle$, where a and b are complex scalar numbers multiplying each ket" (p. 11). By first mentioning the basis elements and writing out the symbolic equation with the basis elements, this text statement explicitly conveyed the meaning that $|\psi\rangle$ was a linear combination of basis elements $|+\rangle$ and $|-\rangle$. Overall, we found that the textbook statements conveyed situated meanings similar to the instructors and students' utterances that referred to a vector in a basis but used more seemingly precise syntax in doing so.

The statements in the textbook chapter commonly referred to a vector being written in a basis, with situated meanings similar to those of the students and instructors' utterances. Consider the statement, "*A general spin*-1/2 *state vector* $|\psi\rangle$ *can be expressed as a combination of the basis kets* $|+\rangle$ and $|-\rangle$: $|\psi\rangle = a|+\rangle + b|-\rangle$" (p. 19). Here, the situated meaning is more explicitly evident given the syntax of the phrase "$|\psi\rangle$ *can be expressed as a combination of the basis kets*." This is similar to the instructors' syntax (e.g., with "I have to be able to *write that as a linear combination of plus z and minus z*, " Instructor A referred to writing a vector as a linear combination of basis vectors). This syntax, however, differs from the common syntax of the students' utterances (e.g., "*I'm expressing plus y in the z basis*," A21) pertaining to a vector being written in or in terms of a basis.

When addressing change of basis, the textbook typically emphasized writing vectors in terms of basis kets. The text states, "Because the $|+\rangle$ and $|-\rangle$ form a complete basis, the kets describing the S_x measurement, $|+\rangle_x$ and $|-\rangle_x$, *can be written in terms of them*" (p. 17), which means that the kets describing the S_x measurement can be written as a linear combination of the z-basis kets. The text's emphasis on describing change of basis as writing kets in terms of basis kets is further illustrated in the following quotes: "We can *express the* S_x *basis kets in terms of the* S_z *basis* kets as $|+\rangle_x = (1/\sqrt{2})(|+\rangle + |-\rangle)$ and $|-\rangle_x = (1/\sqrt{2})(|+\rangle - |-\rangle)$" (p. 18), and "If we were to use the S_x basis, then *we could write the* $|\pm\rangle$ *as general states in terms of the kets* $|\pm\rangle_x$" (p. 19). These statements all had the same situated meaning of related utterances spoken by the students and instructors as a vector being written as a linear combination of basis vectors. However, there were subtle differences in the syntax. The syntax in the textbook explicitly referred to writing kets *in terms of basis kets* instead of referring to writing vectors *in a basis* or *in terms of a basis*. This latter syntax was prevalent in the students and instructors' utterances, such as "I'm *expressing plus y in the z basis*" (A21). Furthermore, the students and instructors used other verbs to connote writing a vector in terms of a basis, such as expand, rewrite, or substitute, which were not used in the text.

We identified other similarities and differences between the spoken utterances and the text in their use of physics-centered, physicist-centered, and indeterminate grammatical constructions. Within the text referring to a vector being in a basis, the grammar was always physics centered because vectors were the subjects of the statements. This is similar to our characterization of the students and instructors' utterances in this category, which were also primarily physics centered. However, the text's grammar differs from the students and instructors' grammar because their utterances referring to a vector in a basis were sometimes physicist centered. Furthermore, the text referring to a vector being written in terms of a basis was both physics centered and physicist centered. For instance, "$|\psi\rangle$ can be expressed as a combination of the basis kets" (p. 19) is physics centered, whereas "we have chosen to write the kets in terms of the $|+\rangle$ and $|-\rangle$ basis kets" (p. 22) is physicist centered. This grammar is similar to the students and instructors' utterances referring to a vector being written in a basis, which were also both physics centered and physicist centered. The text describing change of basis as writing a vector in another form was primarily physicist centered, for it referenced "we" writing or expressing a vector in terms of basis kets. This is similar to the students and instructors' discourse referring to change of basis, which were also primarily physicist centered. A noticeable difference in the students', instructors', and textbook's discourse was that the students and instructors sometimes spoke utterances containing indeterminate grammatical constructions, whereas this type of grammar was not evident in the text. Overall, we identified some similarities in the instructors', students', and textbook's discourse, given that the discourse in the text was categorized with three of the categories observed in the students and instructors' utterances. However, there were some differences with regard to syntax and use of indeterminate grammatical constructions.

V. DISCUSSION

In this study, we investigated two class communities' uses of a social language [13,14] by performing a discourse analysis of utterances made by quantum mechanics students and their instructors related to basis and change of basis and of statements from a portion of their quantum mechanics textbook pertaining to basis and change of basis. We characterized the discourse in these utterances and text using categories we found through a grounded analysis of the students' interview data, as well as by using Ochs, Gonzales, and Jacoby's [16] categories of physics-centered, physicist-centered, and indeterminate grammatical constructions. We first identified the ways in which quantum mechanics students discussed basis and change of basis as they performed probability tasks. We then leveraged the grounded categories and Ochs, Gonzales, and Jacoby's characterizations to analyze the instructors' utterances spoken in class and the statements in a chapter of the course textbook. We used these characterizations to identify similarities and differences among the instructors', students', and textbook's discourse. The students' utterances referred to a person, calculation, or vector being in a basis, or a vector being written in a basis. These in a basis discourse types had similar meanings that were mutually understood and used similarly by students and instructors. The students also discussed change of basis as changing the form of a vector, writing the vector in another way, changing the vector into another vector, and switching bases. Utterances in these change of basis discourse categories had different meanings that were mutually understood and used similarly by students and instructors. We exhibited students' use of their idiosyncratic language as they solved a quantum mechanics probability problem and showed how they often switched between different forms of discourse in quick succession. Through our discussion of the students', instructors', and textbook's discourse, we provided an exposition of a systematic use of socially acquired language in two class communities.

Our nuanced characterizations of student and instructor discourse allowed us to observe similarities and differences in their discourse. We found similar uses of certain utterance categories and similarities in their situated meanings. For example, the instructors commonly referred to a vector in a basis or a vector being written in a basis, as did their students. These utterances contained phrases seemingly particular to their social language [13], such as "the kets are both in the z basis" (A13) and "I need that to be written in the z basis" (C5). Students and instructors also exhibited similarities in the ways in which they discussed change of basis, both in utterance categorization and in grammatical constructions. The instructors most often spoke utterances that referred to change of basis as writing a vector in another form, and their students commonly used this discourse as well. These similarities in the idiosyncratic discourse, situated meanings, and grammatical constructions give evidence to the systematicity of their class community's social language.

We additionally found some similarities between the students and instructors' utterances and the textbook statements, particularly in the categories of "a vector is in a basis," "a vector is written in a basis," and writing the vector in another form. The textbook, however, did not have language consistent with the other discourse categories used by the students and instructors. The textbook also did not use indeterminate grammatical constructions like the students and instructors did, and the syntax in the textbook differed from that of the spoken utterances. The textbook's precise syntax explicitly conveyed the meaning of statements referring to vectors as basis elements or as linear combinations of basis elements, as well as vectors being written in terms of a basis. The syntax in the instructors' utterances sometimes conveyed this precise explicitness, but the students typically spoke utterances containing the idiosyncrasies of their community's social language. The noticeable differences in the textbook's discourse and the participants' discourse give evidence to our claim that the students' discourse shared more commonalities with that of their instructors than the discourse used in their textbook. These observed differences in the textbook's discourse and the participants' discourse could be due to the fact that the written words of a published textbook are edited. We acknowledge that analyzing different data sources (e.g., students' utterances from class or office hours, instructor utterances from additional class sessions, additional textbook sections) or a different concept (e.g., eigenvector) would possibly highlight different phenomena in the social language.

Our analysis demonstrates the systematicity of the social language used by these students and their instructors through our identification of similarities in their idiosyncratic discourse. It is noteworthy that the students in this study were enrolled at two different universities in distinct geographical regions, yet they used similar discourse when referring to entities in a basis or change of basis. Students from both universities spoke utterances that were captured by the same discourse categories. The similarity in their utterances gives evidence to their shared social language with a broader community of physicists. Because the students and instructors spoke utterances using idiosyncrasies seemingly specific to their community's social language, we examined the situated meanings [13] of utterances referring to basis and change of basis. The situated meanings of the exemplified utterances seem to be shared among members of each community of quantum physics students and their instructors at University A and University C. Gee [13] suggested that phrases' situated meanings are negotiated by people in their social language as they converse. Given the systematic use of the language among the students and instructors, we speculate that an implicit meaning-negotiation process likely occurred. For instance, the students and instructors likely negotiated the meanings of utterances such as, "we're in the z basis" as they conversed throughout the course. However, our data do not illustrate how or when this meaning-negotiation process occurred, which is a limitation to our study. We pose investigating the mechanism of this meaning-negotiation process in physics classes as a direction for future research.

Through our results, we exhibited the use of a social language that seems to be shared among instructors and their students. Learning the language and norms in a community of practice is part of students' enculturation into a community of practice [3]. We posit that learning to recognize which potential utterance meanings are relevant according to the context and culture of the community is part of this enculturation process. As students learn to recognize and co-construct meanings of phrases from and consistent with a community's social language, students can learn to recognize contexts in which it is appropriate to use that social language. This contributes to students' development of the social language and thereby their enculturation into the community. Learning to speak a social language involves recognizing grammatical patterns particular to the social language [13,14]. Each social language has its own "rules by which grammatical units like nouns and verbs, phrases and clauses, are used to create patterns which signal or index characteristic social identities and social activities" [14] (p. 158). Speakers of social languages use distinctive word choices and grammatical patterns particular to the social group to which the speakers belong. Members of a social group or community of practice know which strings of grammatical objects typically pattern together. Ochs, Gonzales, and Jacoby [16] identified distinctive grammar that physicists used as they communicated with each other. These included physics-centered, physicist-centered, and indeterminate grammatical constructions. These types of grammar were prevalent in the students' utterances spoken during interviews, the instructors' utterances spoken during class, and the textbook's statements. Our data illustrated systematicity in the grammatical constructions used by the students and instructors. We posit that as students participate in class discussion and listen to their instructors' utterances, they can recognize grammatical patterns typical of the community's social language and learn to use those grammatical patterns in their own utterances. This hypothesis can be explored further in future research.

We offer directions for future research related to how students learn to speak in ways consistent with a broader community's social language. Our analysis revealed similarities in the meanings, grammatical structure, and syntax of different students' utterances. Future research can investigate how students come to understand what certain utterances mean and how those meanings are socially negotiated during class or other settings where students and instructors converse, such as in office hours. Our study highlighted how students' and their instructors' utterances exhibited similar syntax and conveyed similar meanings. Future research could investigate how students' discourse concerning topics covered in the course changes over time and how the instructors' utterances spoken in class may influence any observed changes in the students' discourse. Furthermore, undergraduate science, technology, engineering, and mathematics (STEM) students likely encounter various social languages in their courses from different disciplines (such as mathematics or physics). As they participate in these courses, they may experience enculturation into several different communities by becoming accustomed to the different social languages associated with those communities. We suggest future research could investigate how STEM students reconcile differences in social languages they encounter in their mathematics and science courses.

We conclude with some implications for research on student understanding and for instruction. First, we note that our analysis provides information relevant to understanding how students make sense of and understand the mathematical concepts of basis and change of basis. Within change of basis, for instance, the four utterance categories distinguish according to the subject of the phrase (i.e., a vector or a basis) and the nature of the change (e.g., a change in form or a titular change). We cared about this type of nuance between the discourse categories because the utterances within those categories often carried different meanings, which could be indicative of different ways of understanding change of basis. For example, students who use discourse referring to changing the form of a vector might perceive change of basis as changing a vector, where the vector is still the same vector before and after the change. This understanding would differ from a student perceiving change of basis as changing a vector into a different vector. Students who refer to change of basis as writing a vector in another way might only understand the change of basis procedure as an algebraic substitution that involves rewriting symbols. Furthermore, utterances classified in these three categories position a vector as the object undergoing the change, whereas discourse from the switching the basis category positions the basis set as the object undergoing the change. These different discourse categories may be associated with different ways of understanding change of basis.

Second, as central members of a broader professional community of practice (e.g., physicists), instructors have a primary role in facilitating students' enculturation into their community. As students participate in the classroom and communicate with their instructor, they become accustomed to the norms and practices of the community [3]. Through this participation, students may learn the social language of the community of practice [13]. As students listen to and converse with their instructors, they may acquire the idiosyncrasies of the social language spoken by their instructors. As evident in our data, the students used discourse from their social language as they solved problems, which was similar to their instructors' utterances. Other researchers have suggested that instructors should aim to use only precise, formal language in class (e.g., Refs. [7,11]). We, however, encourage instructors to integrate the social language (e.g., laboratory vernacular [12]) of the broader community of physicists into their classroom, as it could help students make sense of concepts [24] and facilitate their enculturation into the community.

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