Cost of Quantum Entanglement Simplified

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Quantum entanglement is a key physical resource in quantum information processing that allows for performing basic quantum tasks such as teleportation and quantum key distribution, which are impossible in the classical world. Ever since the rise of quantum information theory, it has been an open problem to quantify entanglement in an information-theoretically meaningful way. In particular, every previously defined entanglement measure bearing a precise information-theoretic meaning is not known to be efficiently computable, or if it is efficiently computable, then it is not known to have a precise information-theoretic meaning. In this Letter, we meet this challenge by introducing an entanglement measure that has a precise information-theoretic meaning as the exact cost required to prepare an entangled state when two distant parties are allowed to perform quantum operations that completely preserve the positivity of the partial transpose. Additionally, this entanglement measure is efficiently computable by means of a semidefinite program, and it bears a number of useful properties such as additivity and faithfulness. Our results bring key insights into the fundamental entanglement structure of arbitrary quantum states, and they can be used directly to assess and quantify the entanglement produced in quantum-physical experiments.

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Introduction.—Quantum entanglement is a fundamental property of quantum states that has no classical analog. As famously remarked by Schrödinger [1], it is "the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." Einstein, Podolsky, and Rosen were confounded by entanglement [2], and, based on this, proposed a theory alternative to quantum mechanics, which was later ruled out by a theoretical proposal of Bell [3] and experimental confirmations of Bell's test [4–7].

The aforementioned early work on understanding entanglement ended up being foundational for the modern field of quantum information science [8,9], whose goal is to harness the strange properties of quantum states for information processing tasks that are not possible in the classical world. Due to seminal work by Bennett *et al.*, we now understand quantum entanglement to be the enabling fuel for a variety of quantum protocols such as teleportation [10], dense coding [11], and quantum key distribution [12,13].

In the fundamental protocols mentioned above, it is required for the entangled states being consumed to be in a pure form, known as maximally entangled states. However, in experimental practice, quantum states do not come in this pure variety, but instead are produced as mixtures of pure states. As such, a key goal of the resource theory of entanglement [14] is to understand how well mixed quantum states can be converted to pure maximally entangled states and vice versa, by means of "free" physical operations that do not increase entanglement. Motivated by the "distant laboratories paradigm," in which the two parties holding shares of a quantum state are spatially separated, one set of physical operations that is reasonable to allow for free consists of those that can be implemented by local operations and classical communication (LOCC). The characterization of entanglement as a resource in practical settings is also rooted in this distant laboratories paradigm.

There are two primary operational ways for quantifying entanglement in a two-party quantum state ρ_{AB}: the first is known as distillable entanglement [14] and the second is known as entanglement cost [14,15]. In the first approach, one is interested to know the largest rate at which maximally entangled states can be distilled by means of LOCC from the state ρ_{AB}. In the second, one is interested to know the smallest rate at which maximally entangled states are required to prepare the state ρ_{AB} by means of LOCC. There are a number of technical variations of each task that have been considered [14–17], involving one or multiple copies of the state ρ_{AB}, or for the task to be accomplished exactly or with some error tolerance. So far, beyond the case of pure states [18], it has been a great challenge since

the publication of the seminal work in [14] to characterize the distillability and cost of quantum entanglement. Much of the progress during the past two decades has to do with finding alternative entanglement measures that bound entanglement distillability or cost, while possessing properties that are generally agreed upon to be reasonable [19–28]. Even many of the measures that have been defined are known to be difficult to compute [29].

Due to the aforementioned challenges associated with mixed-state entanglement and the set LOCC in general [30], researchers have looked in other directions in order to understand the nature of entanglement. One approach was pioneered in [21], with the introduction of another set of free operations that "completely preserve the positivity of the partial transpose" [31] (we explain the precise meaning of this term later). This set (abbreviated by C-PPT-P) has been considered in prior work [24,28,32–34] on entanglement theory for at least two reasons. (i) The mathematical structure of LOCC is difficult to work with and so enlarging the set to a more mathematically tractable set allows for providing bounds on what one could accomplish with LOCC. That is, such free operations provide accessible estimates of the capabilities of LOCC in entanglement manipulation. (ii) The free states that correspond to this enlarged set, known as positive partial transpose (PPT) states, in any case do not have any useful entanglement on their own (in the sense that it is impossible to distill maximally entangled states from them at a nontrivial rate via LOCC). As observed in [21], an advantage of the set C-PPT-P over LOCC is that performing optimizations over it allows for incorporating the tools of semi-definite programming.

One key problem that has remained open for many years now is to characterize the exact entanglement cost of a quantum state ρ_{AB} when C-PPT-P operations are allowed for free, which is equal to the minimum rate at which entanglement is required to prepare many perfect and identical copies of ρ_{AB} by means of these free operations. The optimal rate is known as the PPT exact entanglement cost. The problem was formalized in [33], where some bounds on this quantity were given and some partial solutions were presented. The problem was considered further in [35], which however focused mainly on transformations of pure entangled states.

In this Letter, we determine the PPT exact entanglement cost of an arbitrary two-party quantum state, thus closing a longstanding investigation in entanglement theory. We find that the solution is given by a new entanglement measure, which we call κ entanglement. The κ entanglement can be calculated by means of a semidefinite program [36], implying that it can be efficiently calculated in time polynomial in the dimension of the state on which it is being evaluated [37–41]. These two properties single out the κ entanglement as the first entanglement measure that has a concrete information-theoretic meaning while being efficiently calculable. The κ entanglement also bears a number of desirable properties, including additivity, normalization, and faithfulness, which we expand upon later. It is neither convex nor monogamous [42], which calls into question whether it is truly necessary for an entanglement measure to satisfy either of these properties.

Our results on the κ entanglement of quantum states bring new insights regarding the structure of quantum entanglement as a physical resource. For one, they demonstrate that entanglement can be quantified in a precise and physically relevant operational scenario. Furthermore, they call into question whether properties such as monogamy or convexity are really required for entanglement measures; this is in light of the fact that κ entanglement does not have these properties while at the same time having the aforementioned operational meaning.

We now begin the more technical part of our Letter by giving some background, defining the PPT exact entanglement cost and κ entanglement of a quantum state, and justifying how these quantities are equal. We note here that all mathematical proofs of the various statements and properties summarized in this Letter are given in the Supplemental Material [43], which also includes the following references not mentioned in the main text [44–57].

Let us first recall some basic elements of quantum information. A two-party or bipartite quantum state ρ_{AB} is a unit trace, positive semidefinite operator acting on a tensor-product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We say that Alice possesses system A and Bob system B, and we imagine that Alice and Bob are located in distant laboratories. Such a state is separable [58] if there exists a probability distribution p_X and sets of states $\{\sigma_A^x\}_x$ and $\{\tau_B^x\}_x$ such that

$$\rho_{AB} = \sum_x p_X(x)\sigma_A^x \otimes \tau_B^x. \tag{1}$$

If ρ_{AB} cannot be written in the above way, then it is entangled [58].

It is a difficult (NP hard) computational problem to decide whether an arbitrary quantum state is separable or entangled [59,60]. As such, researchers have sought out simpler, "one-way" criteria to classify entanglement of quantum states. Possibly the simplest such criterion is the positive partial transpose criterion [61,62]. To define this, recall that the partial transpose, with respect to a given orthonormal basis $\{|i\rangle_B\}_i$, is defined as the following linear map:

$$T_B(X_{AB}) \coloneqq \sum_{i,j} (I_A \otimes |i\rangle\langle j|_B) X_{AB} (I_A \otimes |i\rangle\langle j|_B), \tag{2}$$

which we also write as $X_{AB}^{T_B} \equiv T_B(X_{AB})$. An operator X_{AB} has positive partial transpose (PPT) if $T_B(X_{AB})$ is positive and semidefinite. By inspecting definitions, we conclude that if a bipartite state is separable, then it is a PPT state. By contrapositive, we conclude that a bipartite state is entangled if it has a negative partial transpose. The PPT criterion is

one-way in the sense that there exist entangled PPT states [63].

A quantum channel is a completely positive trace-preserving map, and a bipartite quantum channel $\mathcal{N}_{AB\to A'B'}$ accepts input systems A and B and outputs A' and B', where one party Alice possesses A and A' and another party Bob possesses B and B'. A bipartite quantum channel $\mathcal{N}_{AB\to A'B'}$ is completely positive-partial-transpose preserving [21], abbreviated as C-PPT-P, if the map $T_{B'}\circ\mathcal{N}_{AB\to A'B'}\circ T_B$ is completely positive.

In the resource theory of NPT (nonpositive partial transpose) entanglement, the free operations allowed are C-PPT-P bipartite channels and the free states are PPT states, and one of the main goals is to determine if one bipartite state can be converted to another either exactly or approximately by means of the free operations. The particular task of interest to us here is the PPT exact entanglement cost.

One-shot exact entanglement cost.—We begin by defining the one-shot PPT exact entanglement cost of a bipartite state ρ_{AB} as the logarithm of the minimum Schmidt rank of a maximally entangled state that is required to prepare ρ_{AB} by means of a C-PPT-P channel:

$$E^{(1)}_{\mathrm{PPT}}(\rho_{AB}) \coloneqq \log_2 \inf_{d\in\mathbb{N},\Lambda\in\mathrm{PPT}}\{d : \rho_{AB} = \Lambda_{\hat{A}\hat{B}\to AB}(\Phi^d_{\hat{A}\hat{B}})\},$$

where $\Lambda\in\mathrm{PPT}$ is a shorthand for Λ being a C-PPT-P bipartite channel and the maximally entangled state $\Phi^d_{\hat{A}\hat{B}}$ is defined as

$$\Phi^d_{\hat{A}\hat{B}} \coloneqq \frac{1}{d}\sum_{i,j}|i\rangle\langle j|_{\hat{A}}\otimes|i\rangle\langle j|_{\hat{B}}, \tag{3}$$

with $\{|i\rangle_{\hat{A}}\}_i$ and $\{|i\rangle_{\hat{B}}\}_i$ orthonormal bases. The (asymptotic) PPT exact entanglement cost of ρ_{AB} is defined as

$$E_{\mathrm{PPT}}(\rho_{AB}) \coloneqq \limsup_{n\to\infty}\frac{1}{n}E^{(1)}_{\mathrm{PPT}}(\rho^{\otimes n}_{AB}). \tag{4}$$

By building on earlier results from [33,35], our first result is for the one-shot PPT exact entanglement cost.

Proposition 1.—For a given bipartite quantum state ρ_{AB}, its one-shot PPT exact entanglement cost is given by

$$E^{(1)}_{\mathrm{PPT}}(\rho_{AB}) = \inf\left\{\begin{array}{c}\log_2 m : G_{AB}\geq 0, \mathrm{Tr}[G_{AB}]=1,\\ -(m-1)G^{T_B}_{AB}\leq\rho^{T_B}_{AB}\leq(m+1)G^{T_B}_{AB}, m\in\mathbb{N}\end{array}\right\}. \tag{5}$$

The proof of this result involves an achievability and optimality part. The achievability part constructs the channel $\Lambda_{\hat{A}\hat{B}\to AB}\in\mathrm{PPT}$ as the following measure-prepare procedure:

$$\Lambda_{\hat{A}\hat{B}\to AB}(\omega_{\hat{A}\hat{B}}) \coloneqq \rho_{AB}\mathrm{Tr}[\Phi^m_{\hat{A}\hat{B}}\omega_{\hat{A}\hat{B}}] + G_{AB}\mathrm{Tr}[(I_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}})\omega_{\hat{A}\hat{B}}], \tag{6}$$

for G_{AB} a quantum state satisfying $-(m-1)G^{T_B}_{AB}\leq\rho^{T_B}_{AB}\leq(m+1)G^{T_B}_{AB}$. That $\Lambda_{\hat{A}\hat{B}\to AB}$ is a quantum channel follows immediately from its construction, and that $\Lambda_{\hat{A}\hat{B}\to AB}\in\mathrm{PPT}$ follows from the constraint on G_{AB}. For the optimality part, we exploit the symmetry of the maximally entangled state $\Phi^d_{\hat{A}\hat{B}}$, that it is invariant under the unitary channel $(U\otimes\bar{U})(\cdot)(U\otimes\bar{U})^\dagger$ for an arbitrary unitary U, in order to constrain the set of channels that we have to consider for the PPT exact entanglement cost. Then by applying the constraint that $\Lambda_{\hat{A}\hat{B}\to AB}\in\mathrm{PPT}$, it follows that the constructed channel is optimal.

κ entanglement.—The bottleneck of solving the PPT entanglement cost of a general bipartite state lies in determining the regularization of the one-shot cost, which involves evaluating the limit of a series of optimization problems. To overcome this difficulty, we introduce an efficiently computable entanglement measure, called κ entanglement, defined as

$$E_\kappa(\rho_{AB}) \coloneqq \log_2 \inf_{S_{AB}\geq 0}\{\mathrm{Tr}[S_{AB}] : -S^{T_B}_{AB}\leq\rho^{T_B}_{AB}\leq S^{T_B}_{AB}\}.$$

In particular, E_κ can be computed by means of a semidefinite program (SDP) [64] (see Section II-B in [43] for details). SDPs can be computed efficiently by polynomial-time algorithms [37–41] and are often applied in quantum information (e.g., [65–72]). The CVX software [73] allows one to compute SDPs in practice.

By observing that κ entanglement is a relaxation of the one-shot cost in Proposition 1 up to small corrections, we arrive at the following bounds on the one-shot exact entanglement cost:

Proposition 2.—For a bipartite state ρ_{AB}, we have

$$\log_2(2^{E_\kappa(\rho_{AB})}-1)\leq E^{(1)}_{\mathrm{PPT}}(\rho_{AB})\leq\log_2(2^{E_\kappa(\rho_{AB})}+2). \tag{7}$$

This result gives a tight and efficiently computable bound for the one-shot PPT exact entanglement cost in terms of κ entanglement. A rigorous proof can be found in [43]. Thus, the inequality in Proposition 2 demonstrates that the κ entanglement is closely related to the one-shot PPT exact entanglement cost, and as both the operational quantity $E^{(1)}_{\mathrm{PPT}}(\rho_{AB})$ and the entanglement measure $E_\kappa(\rho_{AB})$ become larger, the gap between them disappears.

In addition to being efficiently calculable by means of a semidefinite program, the κ entanglement possesses several properties desirable for an entanglement measure, including monotonicity under selective C-PPT-P operations, additivity, faithfulness, and normalization. We elaborate on each of these briefly now. The monotonicity is the following inequality:

$$E_\kappa(\rho_{AB}) \geq \sum_{x:\, p(x)>0} p(x) E_\kappa(\rho^x_{A'B'}), \tag{8}$$

where $p(x) \coloneqq \mathrm{Tr}[\mathcal{P}^x_{AB\to A'B'}(\rho_{AB})]$, the set $\{\mathcal{P}^x_{AB\to A'B'}\}_x$ consists of completely positive, trace nonincreasing, C-PPT-P maps such that $\sum_x \mathcal{P}^x_{AB\to A'B'}$ is trace preserving, and $\rho^x_{A'B'} \coloneqq \mathcal{P}^x_{AB\to A'B'}(\rho_{AB})/p(x)$. The inequality in (8) asserts that κ entanglement does not increase on average under the action of selective C-PPT-P operations, which include selective LOCC operations as a special case. Additivity is the following statement, which is critical for establishing one of the key results of our Letter:

$$E_\kappa(\omega_{A_1A_2:B_1B_2}) = E_\kappa(\rho_{A_1B_1}) + E_\kappa(\theta_{A_2B_2}), \tag{9}$$

where $\omega_{A_1A_2:B_1B_2} \coloneqq \rho_{A_1B_1} \otimes \theta_{A_2B_2}$ and $\rho_{A_1B_1}$ and $\theta_{A_2B_2}$ are quantum states. Faithfulness is that $E_\kappa(\rho_{AB}) = 0$ if and only if ρ_{AB} is a PPT state. Finally, normalization is that $E_\kappa(\Phi^d_{AB}) = \log_2 d$ for Φ^d_{AB} in a maximally entangled state of the form in (3). Proofs of the properties above are provided in [43].

Exact entanglement cost.—The PPT exact entanglement cost E_{PPT} has been a longstanding open question since it was first introduced in [33]. The previously best known upper and lower bounds [33] are tight for general Werner states, but they are not tight in general. The difficulty of determining E_{PPT} comes from the fact that the one-shot cost is not an SDP, and its regularization makes the problem more intractable. However, by utilizing the techniques of semidefinite optimization and relaxation, we prove that the asymptotic exact entanglement cost of a state ρ_{AB} is given by $E_\kappa(\rho_{AB})$. Specifically, by exploiting (7), the definition of PPT exact entanglement cost in (4), and the additivity of κ entanglement in (9), we arrive at one of our core contributions.

Theorem 1.—The PPT exact entanglement cost of an arbitrary bipartite state ρ_{AB} is given by

$$E_{\mathrm{PPT}}(\rho_{AB}) = E_\kappa(\rho_{AB}). \tag{10}$$

This result has two important consequences. First, it demonstrates that κ entanglement precisely determines the PPT exact entanglement cost of an arbitrary quantum state. Notably, this is the first time that an entanglement measure for general bipartite states has been proven not only to possess a direct operational meaning but also to be efficiently computable, thus solving a question that has remained open since the inception of entanglement theory over two decades ago. Second, note that E_κ is additive [cf., Eq. (9)], so that Theorem 1 implies that the PPT exact entanglement cost is additive in general:

$$E_{\mathrm{PPT}}(\rho_{AB} \otimes \omega_{A'B'}) = E_{\mathrm{PPT}}(\rho_{AB}) + E_{\mathrm{PPT}}(\omega_{A'B'}). \tag{11}$$

Based on Theorem 1, we further show that the PPT exact entanglement cost violates the convexity and monogamy inequalities, which gives insight to the fundamental structure of entanglement. Recall that for an entanglement measure E, convexity is the following statement:

$$E(\bar{\rho}_{AB}) \leq \sum_z p(z) E(\rho^z_{AB}), \tag{12}$$

where $p(z)$ is a probability distribution, $\{\rho^z_{AB}\}_z$ is a set of states, and $\bar{\rho}_{AB} \coloneqq \sum_z p(z)\rho^z_{AB}$. This is not true for the PPT exact entanglement cost. In particular, let us choose the two-qubit states $\rho_1 \coloneqq \Phi_2$, $\rho_2 \coloneqq \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, and their average $\rho \coloneqq \frac{1}{2}(\rho_1 + \rho_2)$. By direct calculation, we find that $E_{\mathrm{PPT}}(\rho_1) = 1$, $E_\kappa(\rho_2) = 0$, and $E_\kappa(\rho) = \log_2 \frac{3}{2}$, from which we conclude that

$$E_\kappa(\rho) > \frac{1}{2}[E_\kappa(\rho_1) + E_\kappa(\rho_2)], \tag{13}$$

This implies the following.

Proposition 3: no convexity.—The PPT exact entanglement cost is not generally convex.

As a consequence of the finding above, the exact entanglement cost of preparing the average of two states ρ_1 and ρ_2 can sometimes be strictly larger than the average exact entanglement cost of preparing each state separately. Convexity is sometimes associated with the loss of entanglement under the discarding of classical information. However, this is only sensible for entanglement measures that obey what is known as the "flags" property [24,26,74]. Note that the κ entanglement does not possess this property (if it were to, then it would be convex). We stress here that the κ entanglement is monotone under LOCC, as indicated in (8), which implies that it does not increase when Alice and Bob discard local registers in their possession. Since local registers of course can be classical registers, we conclude that κ entanglement does not increase under the loss of classical information in this sense. The lack of convexity for κ entanglement simply means that, in some cases, the cost of preparing the average of two states can exceed the average cost of preparing the individual states. See [24] for further discussions about this point.

Monogamy of an entanglement measure E is as follows [42]:

$$E(\rho_{A:BC}) \geq E(\rho_{A:B}) + E(\rho_{A:C}), \tag{14}$$

where ρ_{ABC} is a tripartite state. It captures the idea that the sum of the entanglement that Alice shares individually with Bob and Charlie when they are all in separate laboratories cannot exceed the entanglement that she has with them when Bob and Charlie are in the same laboratory. Here, by utilizing κ entanglement, we show that $E_{\mathrm{PPT}}(\psi_{AB}) + E_{\mathrm{PPT}}(\psi_{AC}) > E_{\mathrm{PPT}}(\psi_{A(BC)})$ for the tripartite state $|\psi\rangle_{ABC} = \frac{1}{2}(|000\rangle_{ABC} + |011\rangle_{ABC} + \sqrt{2}|110\rangle_{ABC})$. Thus, we have the following.

Proposition 4: no monogamy.—The PPT exact entanglement cost is not generally monogamous.

In some literature on entanglement (see, e.g., [24]), the properties of convexity and monogamy were thought to be essential features of entanglement, but the fact that κ entanglement is neither convex nor monogamous, while having a clear-cut operational meaning, calls into question whether these properties are really necessary for an entanglement measure. See [24,75] for other discussions questioning the necessity of these two properties.

As another implication of our results, we find by example that exact PPT entanglement manipulation is irreversible. In particular, this example together with several classes of examples in Section V of [43] imply that E_{PPT} is generally not equal to the logarithmic negativity E_N [22,24]. Consider the following rank-two state supported on the 3×3 antisymmetric subspace [28]:

$$\rho_{AB}^{v} = \frac{1}{2}(|v_1\rangle\langle v_1|_{AB} + |v_2\rangle\langle v_2|_{AB}), \tag{15}$$

with

$$|v_1\rangle_{AB} \coloneqq (|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2}, \tag{16}$$

$$|v_2\rangle_{AB} \coloneqq (|02\rangle_{AB} - |20\rangle_{AB})/\sqrt{2}. \tag{17}$$

For the state ρ_{AB}^{v}, it holds that

$$\begin{aligned} E_N(\rho_{AB}^{v}) = \log_2(1 + 1/\sqrt{2}) < E_{\mathrm{PPT}}(\rho_{AB}^{v}) = 1 \\ < \log_2 Z(\rho_{AB}^{v}) = \log_2(1 + 13/4\sqrt{2}), \end{aligned} \tag{18}$$

where $\log_2 Z(\rho_{AB})$ is the previous upper bound on E_{PPT} from [33]. The strict inequalities above also imply that the previously best-known lower and upper bounds from [33] are not tight. Since the logarithmic negativity is known to be an upper bound on PPT exact distillable entanglement [22,76], we conclude that exact PPT entanglement manipulation is irreversible.

Conclusions.—We have shown that the PPT exact entanglement cost is equal to the κ entanglement, a single-letter, efficiently computable entanglement measure. Our results constitute a significant development for entanglement theory, representing the first time that an entanglement measure has been proven to be not only efficiently computable but also to possess a direct information-theoretic meaning. Prior to our work, every other entanglement measure introduced previously possesses only one of these two properties, and thus they were either not accessible computationally or not information-theoretically meaningful. Our work closes this outstanding theoretical gap, because our entanglement measure can be calculated efficiently by semidefinite programming and it has an operational meaning as the cost of maximally entangled states needed to prepare a state. This unique feature improves our understanding of the fundamental structure and power of entanglement.

Furthermore, we have shown that the κ entanglement (or exact PPT entanglement cost) possesses properties such as additivity, monotonicity, faithfulness, normalization, nonconvexity, and nonmonogamy. These results give insight into the structure of quantum entanglement that have not previously been observed in a general operational setting and bring a significant simplification to entanglement theory. In particular, most prior discussions about the structure and properties of entanglement are based on entanglement measures. However, none of these measures, with the exception of the regularized relative entropy of entanglement, possesses a direct operational meaning. Thus, the connection made by Theorem 1 allows for the study of the structure of entanglement via an entanglement measure possessing a direct operational meaning. Given that $E_\kappa = E_{\mathrm{PPT}}$ is neither convex nor monogamous, this raises questions of whether these properties should really be required or necessary for measures of entanglement, in contrast to the discussions put forward in [26,42] based on intuition.

Our results may also shed light on the open question of whether distillable entanglement is convex [77], but this remains the topic of future work. In the multipartite setting, it is known that a version of distillable entanglement is not convex [78].

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