On the Robustness of Nominally Well-Posed Event-Triggered Controllers

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***Abstract*—In this paper, we show that: 1) if the Krasovskii regularization of a hybrid system $\mathcal{H}$ has complete and discrete solutions, then $\mathcal{H}$ has solutions with arbitrarily small separation between jumps under the influence of admissible state perturbations; 2) if $\mathcal{H}$ is nominally well-posed and does not have complete discrete solutions, then it does not have solutions with vanishing time between jumps (such as Zeno solutions); 3) if, in addition, there exists a compact set $\mathcal{A}$ such that all maximal solutions to $\mathcal{H}$ from $\mathcal{A}$ are complete, discrete and remain in $\mathcal{A}$, then all solutions converging to $\mathcal{A}$ have vanishing time between jumps. The results in this paper demonstrate that a good practice to avoid solutions with arbitrarily fast sampling in Event-Triggered Control (ETC) is to ensure that the closed-loop system is nominally well-posed and that it does not have complete discrete solutions.**

I. Introduction

Event-Triggered Control (ETC) refers to a feedback strategy in which sensors and actuators are sampled "only if needed". The main goal of this approach is to improve the efficiency of control tasks by reducing the average sampling frequency with respect to standard periodic sampling approaches. Crucially, one loses direct control over the sampling frequency, thus there is a lingering possibility that the minimum time between samples – the minimum intersampling time – does not satisfy hardware requirements. For this reason, the design of event-triggered controllers must demonstrate that such requirements are satisfied. In this paper, we study the existence of solutions to ETC systems that have arbitrarily fast sampling and, for that reason, cannot meet any hardware requirements.

A. Hybrid Dynamical Systems

A large part of ETC has been developed within a model of dynamical systems with impulsive dynamics that describes the behavior of dynamical systems using discontinuous functions of continuous-time (see e.g. [1]–[10]).

On the other hand, the framework of hybrid dynamical systems presented in [11] describes dynamical systems with impulsive dynamics as solutions to systems $\mathcal{H}$ of the form:

$$\begin{aligned} \dot{\xi} &\in F(\xi) \quad \xi \in C \\ \xi^+ &\in G(\xi) \quad \xi \in D \end{aligned} \tag{1}$$

where $\xi \in \mathbb{R}^p$ is the state, the set $C \subset \mathbb{R}^p$ and the set-valued map $F : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ describe the continuous-time dynamics and are therefore called the flow set and the flow map, respectively, whereas the set $D \subset \mathbb{R}^p$ and the set-valued map $G : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ describe the discrete-time dynamics and are called the jump set and the jump map, respectively.[1] For any given initial condition ξ in C, solutions can be extended in continuous-time if there is any vector in $F(\xi)$ that is tangent to C. If, on the other hand, ξ belongs to D then it may be extended in discrete-time by jumping. Solutions to hybrid dynamical systems are therefore described using a combination of continuous-time and discrete-time domains: the hybrid time domain, which is defined below.

Definition 1. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\}$ is a compact hybrid time domain.

Hybrid systems are able to represent any combination of continuous and discrete time solutions, including purely discrete solutions (which are characterized by having a hybrid time domain that is a subset of $\{0\} \times \mathbb{N}$). For example, Zeno solutions are solutions with infinite amount of discontinuities (jumps) in finite continuous-time and they can be represented either in continuous-time or in hybrid time. When described on a hybrid time domain, the slice of the hybrid time domain at the tail end of a Zeno solution is converging to $\{0\} \times \mathbb{N}$, i.e., the same hybrid time domain of a complete discrete solution. Suppose that we consider a sequence of solutions to the hybrid system that is obtained from the tail end of a Zeno solution in a way that the sequence converges. It depends on the data of $\mathcal{H}$ whether the sequence converges to a solution of $\mathcal{H}$ or not but, if $\mathcal{H}$ is *nominally well-posed*, then all bounded

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[1] A set-valued map $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ maps points in $\mathbb{R}^p$ to subsets of $\mathbb{R}^p$. The fact that the flow and jump dynamics of a hybrid dynamical system are presented in this way is done for greater generality, but in many cases these are just differential and difference equations, respectively.

sequences of convergent solutions to $\mathcal{H}$ tend to a solution of $\mathcal{H}$, and, in that case, we can say that the sequence we just constructed out of a Zeno solution converges to a complete discrete solution to $\mathcal{H}$. This is an intuitive explanation of the results presented in Section III. The implication of this result is that, if the hybrid system is nominally well-posed and it does not have complete discrete solutions, then its bounded solutions are not Zeno solutions. More generally, we can say that it does not have solutions with time domain converging to $\{0\}\times\mathbb{N}$, which we call *solutions with vanishing time between jumps*.

Remark. Since it is fairly cumbersome to check nominal well-posedness of a hybrid system through the convergence of sequences of solutions, it is often preferable to check that the hybrid system satisfies the so-called *hybrid basic conditions* given in [11, Assumption 6.5], which imply nominal well-posedness (c.f. [11, Theorem 6.8]).

B. ETC Systems as Hybrid Systems

It is instructive to look at existing ETC systems and classify them according to the existence of discrete solutions. In this direction, we borrow the definition of an ETC system that is given in [12], where $\xi := (x, e, \eta)$ is a state variable comprised of the state of the plant x, the sampling error e and an auxiliary variable η. The flow and jump maps are given by

$$F(\xi) := \begin{pmatrix} f(x,e) \\ g(x,e) \\ h(x,e,\eta) \end{pmatrix} \qquad G(\xi) := \begin{pmatrix} x \\ 0 \\ \ell(x,e,\eta) \end{pmatrix} \tag{2}$$

for each $\xi \in \mathbb{R}^p$. Under the assumption that F and G are continuous and that C and D are closed, the ETC system (2) is nominally well-posed. The paper [12] presents five different ETC strategies that are encompassed by the hybrid system (2). Each of the five ETC systems in [12] has a semiglobal uniform positive lower bound to the intersampling time outside of the attractor $\mathcal{A}$, but in three of the five ETC systems presented in [12] this property does not extend to solutions from $\mathcal{A}$, due to the existence of complete discrete solutions from $\mathcal{A}$. Another work that addresses the inter-event separation properties in ETC systems is [13], and it considers ETC systems of the form (2) without the auxiliary variable η and with C and D given by:

$$C := \{\xi \in \mathbb{R}^p : \rho_1(|e|) \le \sigma\,|x| + \beta\}$$
$$D := \{\xi \in \mathbb{R}^p : \rho_1(|e|) \ge \sigma\,|x| + \beta\}$$

where ρ_1 is a class-$\mathcal{K}$ function, $\sigma \in [0, 1)$ and $\beta \ge 0$. It is possible to verify that when $\beta > 0$,

$$G(D) \cap D = \emptyset \tag{4}$$

thus there are no complete discrete solutions to (2). This implies that there is a positive lower bound to the intersampling time and that this property is robust to arbitrarily small noise, which constitutes one of the main results in [13]. A thorough analysis of (4) and its implications in the design of event-triggered controllers is also provided in [14, Proposition 3].

Condition (4) provides an easy way to check that there are no complete discrete solutions to (1). In fact, when a nominally well-posed hybrid system satisfies condition (4), each of its bounded solutions has a positive lower bound to the intersampling time, as proved in [15, Lemma 2.7]. There are a few strategies to ensure that (4) is satisfied in ETC: state-space regularization (see e.g. [12, Proposition 2]) and temporal regularization (see e.g. [14, Section 3.5]). However, these strategies often sacrifice asymptotic stability for practical stability. A notable exception is the case of [16], which provides a set of assumptions on the system data that allow for a explicit computation of the intersampling time, thus enabling the use of temporal regularization to remove solutions with vanishing time between jumps without sacrificing asymptotic stability. Due to this advantage, the approach in [16] has become a pivotal tool in the design of ETC systems as evidenced by the recent contributions in [17], [18], and [19], for example.

However, the condition (4) is fairly restrictive and not necessary to show that there are no solutions to (1) with vanishing time between jumps.

C. When removing complete discrete solutions is not a good idea

A common approach to circumvent the existence of complete discrete solutions to (2) consists in removing $\mathcal{A}$ from the jump set (c.f. [1], [6]). However, this breaks nominal well-posedness of the hybrid system.

In Section IV, we show that if a hybrid system has a Krasovskii solution that is complete and discrete, then it has arbitrarily small separation between jumps under the influence of admissible state perturbations. Therefore, we prescribe the following procedure in order to check if inter-event separation properties are robust to small perturbations: 1) apply a Krasovskii regularization to the ETC system; 2) verify that the regularized system has no complete discrete solutions. Unlike [13, Theorem IV], our result applies to hybrid systems in general, so it is not tied to any particular ETC system.

In Section V, we show that there is a particularly pathological case of nonrobustness of inter-event separation properties: if there are forward invariant sets from which all solutions are discrete, then solutions with vanishing time between jumps are unavoidable.

D. Summary of Contributions and Organization of the Paper

The remainder of the paper is devoted to making precise the points that have been highlighted before. Section II presents the definition of solution to a hybrid system (1) and of solutions with vanishing time between jumps. Section III presents the formal definition of nominal well-posedness and follows that definition with a result on the necessity of complete discrete solutions to nominally well-posed hybrid systems in the presence of bounded solutions with vanishing time between jumps. In Section IV, we show that the existence of complete discrete Krasovskii solutions to (1) implies that there exist admissible state perturbations that induce arbitrarily small separation between jumps. In Section V, we show that if there exists a set from which all maximal solutions to (1) are complete and discrete, then arbitrarily small separation between jumps is unavoidable.

Each of the contributions in this paper is accompanied by an illustrative numerical example. In Section VI, we present some concluding remarks.

Notation. Given a topological space S, $\operatorname{cl}(S)$, $\operatorname{int}(S)$ and $\operatorname{co}(S)$ denote the closure, the interior and the convex hull of S, respectively. The symbols $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ denote the set of natural numbers and zero and the set of nonnegative real numbers, respectively. Given $\xi \in \mathbb{R}^p$, $|\xi| := \sqrt{\langle \xi, \xi \rangle}$, where $\langle a, b \rangle$ denotes the inner product between $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^p$. The unitary ball in $\mathbb{R}^p$ is given by $\mathbb{B} := \{\xi \in \mathbb{R}^p : |\xi| \leq 1\}$ and $c + \delta\mathbb{B} := \{\xi \in \mathbb{R}^p : |\xi - c| \leq \delta\}$. Given $S \subset \mathbb{R}^p$ and a set-valued map $M : S \rightrightarrows \mathbb{R}^k$, its domain is given by $\operatorname{dom} M := \{\xi \in \mathbb{R}^p : M(\xi) \neq \emptyset\}$ and its graph is given by $\operatorname{gph} M := \{(\xi, y) \in \mathbb{R}^p \times \mathbb{R}^k : y \in M(\xi)\}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be class-$\mathcal{K}$, denoted by $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and zero at zero.

II. Solutions to Hybrid Dynamical Systems

In this section, we introduce various kinds of solutions to the hybrid dynamical system $\mathcal{H}$ defined in (1) which are of interest to the developments of this paper. With the exception of solutions with vanishing time between jumps, most of the concepts presented in this paper are directly taken from [11].

A function $\phi : E \to \mathbb{R}^p$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j := \{t \in \mathbb{R}_{\geq 0} : (t, j) \in E\}$. A solution ϕ to $\mathcal{H}$ is a hybrid arc that satisfies $\phi(0, 0) \in \operatorname{cl}(C) \cup D$, $\phi(t, j) \in C$ for all $t \in \operatorname{int} I^j$, $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in I^j$, $\phi(t, j) \in D$ and $\phi(t, j + 1) \in G(\phi(t, j))$ for all $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$.

A solution ϕ to a hybrid system is said to be *nontrivial* if it has at least two points, *maximal* if it cannot be extended by flowing nor jumping, *complete* if its domain is unbounded, *discrete* if it is nontrivial and $\operatorname{dom} \phi \subset \{0\} \times \mathbb{N}$, *Zeno* if it is complete and

$$\sup_t \operatorname{dom} \phi := \sup\{t \in \mathbb{R}_{\geq 0} : \exists_{j \in \mathbb{N}} (t, j) \in \operatorname{dom} \phi\} < +\infty,$$

eventually discrete if $\sup_t \operatorname{dom} \phi = T < +\infty$ and $\operatorname{dom} \phi \cap \{T\} \times \mathbb{N}$ contains at least two points. Given $S \subset \mathbb{R}^p$, the set of maximal solutions to $\mathcal{H}$ satisfying $\phi(0, 0) \in S$ is denoted by $\mathcal{S}_{\mathcal{H}}(S)$.

Next, we formally introduce the concept of solutions with vanishing time between jumps. Given a solution ϕ to (1), let $t_0 = 0$ and, for each $j \in \mathbb{N} \backslash \{0\}$, $t_j \in \mathbb{R}_{\geq 0}$ is such that $(t_j, j - 1)$ and (t_j, j) belong to $\operatorname{dom} \phi$. We say that ϕ has *vanishing time between jumps* if: 1) for each $j \in \mathbb{N}$, there exists $t \in \mathbb{R}_{\geq 0}$ such that $(t, j) \in \operatorname{dom} \phi$, and; 2) for each $\tau > 0$, there exists $J \in \mathbb{N}$ such that $t_{j+1} - t_j < \tau$ for all $j \geq J$. Roughly speaking, solutions with vanishing time between jumps have an infinite amount of jumps and the time between jumps tends to zero. Examples of solutions that have vanishing time between jumps include: 1) eventually discrete solutions, where the time between jumps becomes zero; 2) Zeno solutions, which have a infinite amount of jumps in finite continuous time, but also; 3) solutions where the time between jumps converges to 0 and $\sup_t \operatorname{dom} \phi = +\infty$. For example, $E = \bigcup_{j=0}^{\infty} [t_{j+1}, t_j] \times \{j\}$ with $t_0 = 0$ and $t_{j+1} = t_j + 1/j$ for each $j \in \mathbb{N} \backslash \{0\}$ is a hybrid time domain where, for each $\tau > 0$, there exists J such that $t_{j+1} - t_j < \tau$ for all $j \geq J$, and $\sup_t E = +\infty$.

To understand why solutions with vanishing time between jumps are important, let ϕ denote a solution to (1) with $t_i \in \mathbb{R}_{\geq 0}$ satisfying $(t_i, i-1), (t_i, i) \in \operatorname{dom} \phi$ for each $i \in \mathbb{N} \backslash \{0\}$, and let

$$\phi_i(t, j) := \phi(t + t_i, j + i) \quad \forall (t, j) \in \operatorname{dom} \phi_i \tag{5}$$

for each $i \in \mathbb{N}$ with $\operatorname{dom} \phi_i = \{(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} : (t + t_i, j + i) \in \operatorname{dom} \phi\}$. In other words, for each $i \in \mathbb{N}$, the hybrid arc ϕ_i is a solution to (1) that is extracted from the tail end of ϕ. Vanishing time between jumps is a necessary and sufficient condition for the convergence of $\{\operatorname{dom} \phi_i\}_{i=0}^{\infty}$ to $\{0\} \times \mathbb{N}$, as shown next.

Lemma 1. *Given a complete solution ϕ to* (1)*, $\{\operatorname{dom} \phi_i\}_{i=0}^{\infty}$ (with ϕ_i given in* (5)*) converges to $\{0\} \times \mathbb{N}$ if and only if ϕ has vanishing time between jumps.*

Proof. Suppose that ϕ is a solution to (1) with vanishing time between jumps. Defining ϕ_i as in (5), we have that $\lim_{i \to +\infty} \operatorname{dom} \phi_i = \{0\} \times \mathbb{N}$ if and only if, for each $J \in \mathbb{N}$, $\operatorname{dom} \phi_i \cap (\mathbb{R}_{\geq 0} \times \{J\})$ converges to $\{0\} \times \{J\}$ (c.f. [11, Example 5.3]). Let t_k denote the continuous-time associated with the jump k of ϕ for each $k \in \mathbb{N}$, i.e., $t_0 := 0$ by definition and $(t_k, k - 1), (t_k, k) \in \operatorname{dom} \phi$. Since

$$\operatorname{dom} \phi_i \cap (\mathbb{R}_{\geq 0} \times \{J\}) = [t_{J+i} - t_i, t_{J+i+1} - t_i] \times \{J\},$$

for each $i \in \mathbb{N}$, it follows that $\operatorname{dom} \phi_i \cap (\mathbb{R}_{\geq 0} \times \{J\})$ converges to $\{0\} \times \{J\}$ if and only if $t_{J+i+1} - t_i$ converges to 0 as i tends to ∞. Since ϕ has vanishing time between jumps, we have that, for each $\tau > 0$, there exists $i_0 \in \mathbb{N}$ such that $t_{i+1} - t_i < \tau$ for each $i \geq i_0$. For each $\epsilon > 0$, selecting $\tau = \epsilon/(J+1)$ yields $t_{J+i+1} - t_i = \sum_{j=0}^{J} t_{j+i+1} - t_{j+i} < \epsilon$, allowing us to conclude that $\{\operatorname{dom} \phi_i\}_{i=0}^{\infty}$ converges to $\{0\} \times \mathbb{N}$.

If $\{\operatorname{dom} \phi_i\}_{i=0}^{\infty}$ converges to $\{0\} \times \mathbb{N}$, then, for each $J \in \mathbb{N}$, $\operatorname{dom} \phi_i \cap (\mathbb{R}_{\geq 0} \times \{J\})$ converges to $\{0\} \times \{J\}$. This implies that, for each $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that $t_{J+i+1} - t_i < \epsilon$ for all $i \geq i_0$. Selecting $\tau = \epsilon$ and $J = 0$ we conclude that $t_{i+1} - t_i < \tau$ for each $i \geq i_0$, thus ϕ has vanishing time between jumps. □

Furthermore, if there are no complete solutions with vanishing time between jumps then the continuous time can be extended indefinitely, as shown in the following lemma.

Lemma 2. *If* (1) *does not have solutions with vanishing time between jumps, then each complete solution ϕ to* (1) *satisfies $\sup_t \operatorname{dom} \phi = +\infty$.*

Proof. Proceeding by contradiction, suppose that $\mathcal{H}$ does not have solutions with vanishing time between jumps and that there exists a complete solution to $\mathcal{H}$ such that $t_f := \sup_t \operatorname{dom} \phi < +\infty$. If ϕ has a finite number of jumps then it follows from completeness of ϕ that $\sup_t \operatorname{dom} \phi = +\infty$ which contradicts the assumption. It follows that ϕ must have an infinite number of jumps, and $\lim_{j \to \infty} t_j = t_f$ with $(t_j, j) \in \operatorname{dom} \phi$ satisfying $(t_j, j - 1), (t_j, j) \in \operatorname{dom} \phi$ for each $j \in \mathbb{N} \backslash \{0\}$. Since $t_{j+1} - t_j \leq t_f - t_j$ for each $j \in \mathbb{N}$, it follows that, for each $\tau > 0$, there exists $J \in \mathbb{N}$ such that $t_{j+1} - t_j < \tau$ for all $j \geq J$. However, this is a contradiction because we assumed that there were no solutions to $\mathcal{H}$ with vanishing time between jumps. $\square$

III. A Necessary Condition for Solutions With Vanishing Time Between Jumps

One of the most important properties of hybrid systems is that of nominal well-posedness, since it has important implications in the robustness of stability of a compact set for a hybrid system as discussed in [11, Chapter 6]. To understand nominal well-posedness, we introduce the following preliminary definitions.

A sequence $\{\phi\}_{i=1}^{\infty}$ of hybrid arcs $\phi_i : \operatorname{dom}\phi_i \to \mathbb{R}^p$ converges graphically if the sequence of sets $\{\operatorname{gph}\phi_i\}_{i=1}^{\infty}$ converges and its limit ϕ is $\phi := \operatorname{gph-lim}_{i\to\infty}\phi_i$. A sequence of hybrid arcs $\{\phi_i\}_{i=1}^{\infty}$ is locally eventually bounded if, for any $m > 0$, there exists $i_0 > 0$ and a compact set $K \subset \mathbb{R}^p$ such that, for all $i > i_0$, $(t, j) \in \operatorname{dom}\phi_i$ with $t + j < m$, $\phi_i(t, j) \in K$.

The study of nominally well-posed hybrid dynamical systems has one major advantage over other system models with impulsive dynamics: locally eventually bounded convergent sequences of solutions to nominally well-posed hybrid systems converge to solutions of the hybrid system.

Definition 2. A hybrid system is called nominally well-posed if the following property holds: for every graphically convergent sequence $\{\phi_i\}_{i=1}^{\infty}$ of solutions to (1) with $\lim_{i\to\infty}\phi_i(0,0) = \xi$ for some $\xi \in \mathbb{R}^p$, 1) if the sequence $\{\phi_i\}_{i=1}^{\infty}$ is locally eventually bounded then the sequence $\{\operatorname{length}(\operatorname{dom}\phi_i)\}_{i}^{\infty} := \{\sup_t \operatorname{dom}\phi_i + \sup_j \operatorname{dom}\phi_i\}_{i=1}^{\infty}$ is either convergent or properly divergent to ∞ and $\phi = \operatorname{gph-lim}_{i\to\infty}\phi_i$ is a solution to (1) with $\phi(0,0) = \xi$ and $\operatorname{length}(\operatorname{dom}\phi) = \lim_{i\to\infty}\operatorname{length}(\operatorname{dom}\phi_i)$; 2) if the sequence is not locally eventually bounded, then there exists a number $m > 0$ for which there exist $(t_i, j_i) \in \operatorname{dom}\phi_i$, $i \in \mathbb{N}\backslash\{0\}$ such that $\lim_{i\to\infty}|\phi_i(t_i, j_i)| = \infty$ and $\phi = \operatorname{gph-lim}_{i\to\infty}\phi_i|_{t+j<m}$ is a maximal solution to (1) with $\operatorname{length}(\operatorname{dom}\phi) = m$ and $\lim_{t\to\sup_t \operatorname{dom}\phi}|\phi(t, \sup_j \operatorname{dom}\phi)| = +\infty$.

Using the previous definitions, we are able to state the main result of this section.

Theorem 1. *If* (1) *is nominally well-posed and if there exists a bounded solution ϕ to* (1) *with vanishing time between jumps, then there exists a complete discrete solution to* (1).

Proof. Let ϕ_i be given by (5) for each $i \in \mathbb{N}$. Note that, for each $i \in \mathbb{N}$, ϕ_i is a solution to (1), since it is a tail of the solution ϕ. Moreover, $\{\phi_i\}_{i=0}^{\infty}$ is locally eventually bounded because ϕ is bounded and it has a subsequence $\{\phi_{i_k}\}_{k=0}^{\infty}$ that is graphically convergent because it does not escape to the horizon (c.f. [11, Theorem 5.7]). Since there exists a subsequence of $\{\phi_i\}_{i=0}^{\infty}$ that is locally eventually bounded and graphically convergent, it follows from nominal well-posedness of (1) that its limit is a solution to (1).

Since ϕ has vanishing time between jumps, it follows from Lemma 1 that $\operatorname{dom}\phi_{i_k}$ converges to $\{0\} \times \mathbb{N}$ as k tends to ∞, which concludes the proof. $\square$

If a hybrid system (1) does not have complete discrete solutions, but it is not nominally well-posed, then the implication that it does not have solutions with vanishing time between jumps cannot be drawn from Theorem 1. In that case, one should look for Krasovskii solutions to (1) that are complete and discrete, as demonstrated in the next section.

Example 1. In this example, we illustrate the usefulness of Theorem 1 in the design of an event-triggered controller. Consider an LTI plant with the following state-space description:

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^k$ denote the plant state, control input, and measured output, respectively. Furthermore, the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{k\times n}$ are known and the pair (A, C) is observable.

The design of the finite-time observer of [20] requires two parallel Luenberger observers with hybrid dynamics

$$\left.\begin{aligned} \dot{\widehat{x}}_1 &= A\widehat{x}_1 + Bu + L_1(y - C\widehat{x}_1) \\ \dot{\widehat{x}}_2 &= A\widehat{x}_2 + Bu + L_2(y - C\widehat{x}_2) \\ \dot{\tau} &= 1 \end{aligned}\right\} \quad \xi \in C_1 \tag{6a}$$

$$\left.\begin{aligned} \widehat{x}_1^+ &= H_1\widehat{x}_1 + H_2\widehat{x}_2 \\ \widehat{x}_2^+ &= H_1\widehat{x}_1 + H_2\widehat{x}_2 \\ \tau^+ &= 0 \end{aligned}\right\} \quad \xi \in D_1 \tag{6b}$$

where $\xi := (x, \widehat{x}_1, \widehat{x}_2, x_c, \tau) \in \Xi := \mathbb{R}^{4n} \times [0, \bar{\tau}]$ is the full state, $\widehat{x}_1, \widehat{x}_2, \in \mathbb{R}^n$ are observer states, $C_1 := \Xi$, $D_1 := \{\xi \in \Xi : \tau = \bar{\tau}\}$, $\tau \in [0, \bar{\tau}]$ is a timer state that triggers updates of $\widehat{x}_1$ and $\widehat{x}_2$ with period $\bar{\tau} > 0$, $L_1, L_2 \in \mathbb{R}^{n\times k}$ are observer gains, $H_1, H_2 \in \mathbb{R}^{n\times n}$ are given by $H_i := (I - e^{F_i\bar{\tau}}e^{-F_{3-i}\bar{\tau}})^{-1}$ with $F_i := A - L_iC$ for each $i \in \{1, 2\}$, and $x_c \in \mathbb{R}^n$ is a memory variable that stores the information on $\widehat{x}_1$ and is updated at events according to the dynamics:

$$\dot{x}_c = 0 \qquad \xi \in C_2 := \{\xi \in \Xi : |\widehat{x}_1 - x_c| \le \delta\} \tag{7a}$$

$$x_c^+ = \widehat{x}_1 \qquad \xi \in D_2 := \{\xi \in \Xi : |\widehat{x}_1 - x_c| \ge \delta\}. \tag{7b}$$

In order to guarantee that the finite-time observer works as intended, the parameters $L_1, L_2 \in \mathbb{R}^{n\times k}$ and $\bar{\tau} > 0$ are chosen such that F_1 and F_2 are Hurwitz and $I - e^{F_1\bar{\tau}}e^{-F_2\bar{\tau}}$ is invertible. In order for the observer (6) to achieve finite-time estimation, it requires at most two updates of $\widehat{x}_1$ and $\widehat{x}_2$, thus, for each solution $\phi = (x, \widehat{x}_1, \widehat{x}_2, x_c, \tau)$ to the closed-loop system, there must be $(t, j) \in \operatorname{dom}\phi$ such that $t \ge 2\bar{\tau} - \tau(0, 0)$. Note that the jump set and jump map of the closed-loop systems are given by $D := D_1 \cup D_2$ and $G(\xi) := G_1(\xi) \cup G_2(\xi)$ for each $\xi \in D$ with

$$G_1(\xi) := (x, H_1\widehat{x}_1 + H_2\widehat{x}_2, H_1\widehat{x}_1 + H_2\widehat{x}_2, x_c, 0) \quad \forall \xi \in D_1,$$
$$G_2(\xi) := (x, \widehat{x}_1, \widehat{x}_2, \widehat{x}_1, \tau) \quad \forall \xi \in D_2,$$

hence the condition $G(D) \cap D = \emptyset$ is not verified, because for $\xi \in D_1 \cap D_2$ we have $G_2(\xi) \subset D_1$. However, since there are no complete discrete solutions and the closed-loop system satisfies [11, Assumption 6.5], it follows from Theorem 1 that there are no bounded solutions with vanishing time between jumps (c.f. Remark I-A). It follows from Lemma 2 that all bounded complete solutions to the closed-loop system satisfy $\sup_t \operatorname{dom}\phi = +\infty > 2\bar{\tau} - \tau(0, 0)$, thus by proving that all solutions are bounded (using Lyapunov analysis, for example), one is able to prove that there are no solutions with vanishing time between jumps.

IV. Complete Discrete Solutions imply Arbitrarily Small Separation Between Jumps

By definition, Krasovskii solutions to a hybrid system $\mathcal{H} := (C, F, D, G)$ are solutions of the Krasovskii regularization of $\mathcal{H}$, which is a hybrid system $\widehat{\mathcal{H}} := (\widehat{C}, \widehat{F}, \widehat{D}, \widehat{G})$ given by

$$\dot{\xi} \in \widehat{F}(\xi) := \bigcap_{\delta>0} \mathrm{cl}\,(\mathrm{co}\, F((x+\delta\mathbb{B}) \cap C)) \quad \xi \in \widehat{C} \tag{8a}$$

$$\xi \in \widehat{G}(\xi) := \bigcap_{\delta>0} \mathrm{cl}\,(G((x+\delta\mathbb{B}) \cap D)) \quad \xi \in \widehat{D} \tag{8b}$$

where the flow and jump sets are given by $\widehat{C} := \mathrm{cl}(C)$ and $\widehat{D} := \mathrm{cl}(D)$, respectively. The regularized system (8) is nominally well-posed and, if it does have complete discrete solutions, then arbitrarily small perturbations of $\mathcal{H}$ induce arbitrarily small separation between jumps.

To be more precise, the perturbation of $\mathcal{H}$ is defined as follows:

$$\dot{\xi} \in F(\xi + n) \qquad \xi + n \in C \tag{9a}$$

$$\xi^+ \in G(\xi + n) \qquad \xi + n \in D, \tag{9b}$$

where n is an *admissible state perturbation*, i.e., $\mathrm{dom}\, n$ is a hybrid time domain and the function $t \mapsto n(t,j)$ is measurable on $\mathrm{dom}\, n \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$. The next result follows directly from the equivalence between solutions to (9) and solutions to (8).

Theorem 2. *Given a hybrid system $\mathcal{H} := (C, F, D, G)$ as in* (1)*, suppose that F and G are locally bounded. If there is a Krasovskii solution ϕ_z to* (1) *that is complete and discrete, then, for each $\epsilon > 0$ and each $J \in \mathbb{N}\backslash\{0\}$, there exists an admissible state perturbation n and a solution ϕ_n to* (9) *such that $t_j - t_{j-1} < \epsilon$ for each $j \in \{1, \ldots, J\}$, where $t_j \in \mathbb{R}_{\geq 0}$ satisfies $(t_j, j-1), (t_j, j) \in \mathrm{dom}\, \phi_n$ for each $j \in \{1, \ldots, J\}$. Moreover,*

$$\sup_{(t,j)\in E} |n(t,j)| \to 0 \tag{10}$$

as $\epsilon \to 0$, where $E := \mathrm{dom}\, \phi_n \cap (\mathbb{R}_{\geq 0} \times \{0, \ldots, J\})$ is a compact hybrid time domain.

Proof. Since ϕ_z is a Krasovskii solution to (1), then it is also a Hermes solution to (1) (c.f. [11, Theorem 4.17]). For each $J \in \mathbb{N}\backslash\{0\}$, the restriction of ϕ_z to

$$\mathrm{dom}\, \phi_z \cap \left(\mathbb{R}_{\geq 0} \times \{0, \ldots, J\}\right) \tag{11}$$

is a compact Hermes solution to (1) as in [11, Definition 4.12]. Hence, there exists a sequence $\{\phi_i\}_{i\in\mathbb{N}}$ of hybrid arcs and a sequence $\{n_i\}_{i\in\mathbb{N}}$ of admissible state perturbations such that ϕ_i is a solution to (9) with admissible state perturbation n_i for each $i \in \mathbb{N}$. Furthermore, for each $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$, such that, for each $i > i_0$, ϕ_i is ϵ-close to the restriction of ϕ_z to (11). From the definition of ϵ-closeness in [11, Definition 4.11], it follows that $\sup_t \mathrm{dom}\, \phi_i < \epsilon$ for each $i > i_0$, thus $t_j - t_{j-1} < \epsilon$ for each $j \in \{1, \ldots, J\}$. The desired result follows from the fact that (10) hold by deifinition of a compact Hermes solution and because we can choose $\phi_n = \phi_i$ and $n = n_i$ with $i > i_0$. □

It is often the case that ETC systems verify $G(D) \cap D = \emptyset$ (see e.g. [1] and [6]), which precludes the existence of complete discrete solutions. However, Theorem 2 demonstrates that, if such ETC systems have Krasovskii solutions that are complete and discrete, then there are admissible state perturbations that induce arbitrarily fast sampling in the presence of arbitrarily small state perturbations. This is illustrated in the following example.

Example 2. Let us consider the stabilization of the integrator $\dot{x} = u$ with state $x \in \mathbb{R}$ and input $u \in \mathbb{R}$ following the approach of [1]. In this direction, we define the nominal feedback law $\kappa(x) = -x$ for each $x \in \mathbb{R}$ and use it to define the following event-triggered controller:

$$\dot{\widehat{u}} = 0 \qquad (x, \widehat{u}) \in C := \{(x, \widehat{u}) \in \mathbb{R}^2 : \gamma(x, \widehat{u}) \leq 0\}$$

$$\widehat{u}^+ = \kappa(x) \quad (x, \widehat{u}) \in D := \{(x, \widehat{u}) \in \mathbb{R}^2 : \gamma(x, \widehat{u}) \geq 0\}$$

with $\gamma(x, \widehat{u}) := |\widehat{u} - \kappa(x)| - \sigma\,|x|$ and $\sigma \in (0,1)$. Let us consider a solution $\phi_n = (x, \widehat{x})$ to the perturbed closed-loop system with initial condition $\phi_n(0,0) = (x_0, -x_0)$ satisfying $x_0 \neq 0$ under the influence of an admissible state perturbation $n = (n_1, n_2)$ satisfying $n_2(t,j) = 0$ and

$$n_1(t,j) := -(2^{j+1} - 1)x(t_j, j)(t - t_j) \tag{12}$$

for each $(t,j) \in \mathrm{dom}\, \phi_n$. It is possible to show that the intersampling time is given by $t_{j+1} - t_j = \frac{\sigma}{1+\sigma}\frac{1}{2^{j+1}}$ for all $j \in \mathbb{N}$ and that the magnitude of the noise signal is $|n_1(t,j)| = (2^{j+1} - 1)\left(\frac{\sigma}{2(1+\sigma)}\right)^j \frac{\sigma}{1+\sigma}\frac{1}{2^j}$, for each $(t,j) \in \mathrm{dom}\, \phi_n$. We conclude that the jumps of the perturbed solution accumulate at $t_f := \sum_{j=0}^{+\infty} t_{j+1} - t_j = \frac{\sigma}{1+\sigma}$ while the noise signal (12) converges to 0. In particular, this implies that there does not exist a positive lower bound to the inter-sampling time in the presence of arbitrarily small noise.

A similar result to Theorem 2 is given in [13, Theorem IV.1], the main differences between the two being that: the latter applies solely to linear event-triggered systems while the former applies to hybrid systems in general, and; admissible state perturbations constitute a broader class of perturbations than those that are considered in [13, Theorem IV.1].

V. When Solutions with Vanishing time between jumps are Unavoidable

The final result that we present in this paper reveals a particularly pathological case that may happen in ETC systems (see e.g. [21]).

Theorem 3. *Suppose that $\mathcal{H}$ in* (1) *is nominally well-posed and that there exists a compact subset $\mathcal{A}$ of $\mathbb{R}^p$ such that each maximal solution ϕ_0 to $\mathcal{H}$ from $\mathcal{A}$ is complete, discrete, and satisfies $\mathrm{rge}\, \phi_0 := \{\phi_0(t,j) : (t,j) \in \mathrm{dom}\, \phi_0\} \subset \mathcal{A}$. Then each complete solution ϕ to $\mathcal{H}$ satisfying $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{A}} = 0$ has vanishing time between jumps.*

Proof. Similarly to the proof of Theorem 1, consider the sequence $\{\phi_i\}_{i=0}^{\infty}$ defined in (5). Since ϕ converges to $\mathcal{A}$ by assumption, it follows from compactness of $\mathcal{A}$ that $\{\phi_i\}_{i=0}^{\infty}$ is locally eventually bounded, hence it has a convergent subsequence that converges graphically. Since $\lim_{i\to\infty} |\phi_i(0,0)|_{\mathcal{A}} = 0$, it follows from nominal well-posedness that $\{\phi_i\}_{i=0}^{\infty}$ converges to a solution to $\mathcal{H}$ from

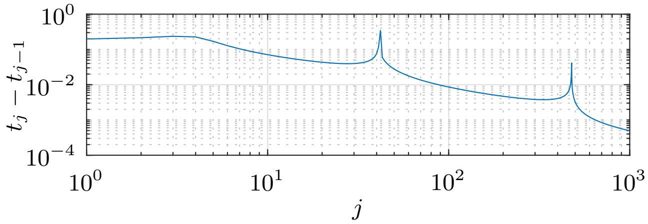

Fig. 1. Representation of the time between jumps for a simulation of the system in Example 3.

$\mathcal{A}$. Since all maximal solutions to $\mathcal{H}$ from $\mathcal{A}$ are complete and discrete, we have that $\{\operatorname{dom}\phi_i\}_{i=0}^{\infty}$ converges to $\{0\}\times\mathbb{N}$. It follows from Lemma 1 that ϕ has vanishing time between jumps. □

It follows directly from Theorem 3, that, if $\mathcal{A}$ is attractive for $\mathcal{H}$, then complete solutions converging to $\mathcal{A}$ have vanishing time between jumps, necessarily. We illustrate this phenomenon in an ETC system in the next example.

Example 3. Let us consider the single integrator system of Example 2, but instead of stabilizing the origin, suppose that we wish to track a reference $t \mapsto x_d(t)$ generated by the system $(\dot{x}_d, \dot{v}_d) = (v_d, -x_d)$ with $(x_d, v_d) \in \mathbb{S} := \{(x_d, v_d) \in \mathbb{R}^2 : |(x_d, v_d)| = 1\}$. Using the feedback law

$$\kappa(x_d, v_d, x) := -(x - x_d) + v_d \quad \forall (x_d, v_d, x) \in \mathbb{S} \times \mathbb{R},$$

the closed-loop system is given by (1) with data

$$\begin{aligned} F(\xi) &:= \big(v_d, -x_d, \widehat{u}, 0\big) && \forall \xi \in C \\ G(\xi) &:= \big(x_d, v_d, x, \kappa(x_d, v_d, x)\big) && \forall \xi \in D \end{aligned}$$

and state has state $\xi := (x_d, v_d, x, \widehat{u}) \in \Xi := \mathbb{S} \times \mathbb{R} \times \mathbb{R}$, and flow set and jump set given by $C := \{\xi \in \Xi : |\widehat{u} - \kappa(x_d, v_d, x)| \leq \sigma |x - x_d|\}$, $D := \{\xi \in \Xi : |\widehat{u} - \kappa(x_d, v_d, x)| \geq \sigma |x - x_d|\}$. It is possible to verify that $\mathcal{A} := \{\xi \in \Xi : x = x_d, \widehat{u} = \kappa(x_d, v_d, x)\}$ is globally asymptotically stable. However, each maximal solution from $\mathcal{A}$ is complete and discrete, thus all solutions have vanishing time between jumps (c.f. Fig. 1).

VI. Conclusion

Motivated by applications in Event-Triggered Control (ETC), we analysed conditions for the existence of solutions to hybrid dynamical systems with arbitrarily small separation between jumps. In particular, we introduced the concept of solutions with vanishing time between jumps – which are solutions whose time between jumps converges to zero, – and we demonstrated that, if a nominally well-posed hybrid system does not have complete discrete solutions then it does not have solutions with vanishing time between jumps. We also demonstrated that the existence of Krasovskii solutions to a hybrid system that are complete and discrete implies that there are admissible state perturbations that generate arbitrarily small separation between jumps. Finally, we proved that hybrid systems with a compact set from which all solutions are complete and discrete, the existence of solutions with vanishing time between jumps might be unavoidable. These results were illustrated with applications in ETC.

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