Parameter Estimation for Hybrid Dynamical Systems using Hybrid Gradient Descent

Ryan S. Johnson, Stefano Di Cairano, and Ricardo G. Sanfelice

***Abstract*— We consider the problem of estimating a vector of unknown constant parameters for a hybrid system whose flow and jump dynamics are affine in the unknown parameter. Using a hybrid systems framework, a hybrid algorithm is proposed and sufficient conditions are established to guarantee exponential stability of the parameter estimate. Examples are provided showing the merits of the proposed approach.**

I. Introduction

Estimating the unknown parameters of a system is critical in many engineering applications [1]. A popular estimation application is the classical model-reference adaptive control (MRAC) problem, which has been a topic of research since the 1960s [2] and has seen a recent resurgence with the advent of machine learning [3]. For such models, the estimation algorithm is typically based on the gradient descent algorithm [1], [4]. This algorithm consists of exploiting the information about the structure of the system along with the available input signals to compute online an estimate of the unknown parameters. Analyzing the convergence rate of the gradient algorithm can be translated into showing exponential stability of the origin for a linear time-varying system. This problem has been studied in [1], [5] for the continuous-time case. It is well-known since [6] that a persistence of excitation condition is necessary and sufficient for exponential stability of such systems. The aforementioned approaches translate naturally to the discrete-time case [7].

In this paper, we consider the problem of estimating an unknown vector of constant parameters for an MRAC-type system whose inputs and dynamics are hybrid; namely, its state and inputs exhibit both continuous and discrete evolution. As we show in Section III, for such systems, the purely continuous-time gradient algorithm fails to converge and the purely discrete-time gradient algorithm converges, but disregards relevant information. To resolve this issue, in Section IV we combine both algorithms into one (hybrid) algorithm that addresses the estimation problem. In Section VI we provide sufficient conditions to guarantee exponential convergence of the parameter estimate and provide a lower bound on the convergence rate. The recently developed tools for robust stability in hybrid systems [8] and the hybrid gradient descent algorithm in [9] form the enabling techniques to achieve these results. Due to space constraints, the proofs of some results are sketched or omitted and will be published elsewhere.

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II. Preliminaries

A. Notation

We denote the set of real, nonnegative, positive, and natural numbers (including zero) as $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, and $\mathbb{N}$, respectively. The matrix I denotes the identity matrix of appropriate dimension. The Euclidean norm of vectors and the induced matrix norm is denoted $|\cdot|$, and the infinity norm is denoted $|\cdot|_\infty$. The distance of a point x to a nonempty set S is denoted $|x|_S = \inf_{y\in S}|y-x|$. Given a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the domain of M is the set $\operatorname{dom} M = \{x \in \mathbb{R}^m : M(x) \neq \varnothing\}$.

B. Continuous and discrete-time gradient algorithms

Before introducing the proposed hybrid gradient descent algorithm, we review the classical continuous-time and discrete-time gradient algorithms commonly used in MRAC applications [1], [7].

- Suppose that the signal $t \mapsto x(t)$ is generated by a continuous-time system of the form

$$\dot{x} = A_c x + b_c \left(r(t) + \phi^\top(t)\theta\right)$$

where $A_c \in \mathbb{R}^{n\times n}$ and $b_c \in \mathbb{R}^n$ are known constant matrices, $x \in \mathbb{R}^n$ is the known state vector, $t \mapsto r(t) \in \mathbb{R}$ is a known exogenous input (e.g., measured disturbance, reference, or feedforward signal), $t \mapsto \phi(t) \in \mathbb{R}^p$ is the known regressor, and $\theta \in \mathbb{R}^p$ is an unknown vector of constant parameters.

We estimate the parameter vector θ using a gradient algorithm [1] of the form

$$\begin{aligned} \dot{\hat{x}} &= A_c x - A(x-\hat{x}) + b_c(r(t) + \phi^\top(t)\hat{\theta}) \\ \dot{\hat{\theta}} &= \gamma_c \phi(t) b_c^\top P(x-\hat{x}) \end{aligned} \tag{1}$$

where $\gamma_c > 0$ is a design parameter, $A \in \mathbb{R}^{n\times n}$ is a user-selected Hurwitz matrix, $P = P^\top \in \mathbb{R}^{n\times n}$ is a positive definite matrix that solves $A^\top P + PA = -Q$, and $Q = Q^\top \in \mathbb{R}^{n\times n}$ is a user-selected positive definite matrix.

Denote the state estimation error as $e := x - \hat{x}$ and the parameter estimation error as $\tilde{\theta} := \theta - \hat{\theta}$, then the error dynamics can be written as follows:

$$\dot{e} = Ae + b_c\phi^\top(t)\tilde{\theta}, \qquad \dot{\tilde{\theta}} = -\gamma_c\phi(t)b_c^\top Pe. \tag{2}$$

- Suppose that the signal $j \mapsto x(j) \in \mathbb{R}^n$ is generated by a discrete-time system of the form

$$x^+ = A_d x + b_d(r(j) + \phi(j)^\top\theta) \tag{3}$$

where $A_d \in \mathbb{R}^{n\times n}$ and $b_d \in \mathbb{R}^n$ are known constant matrices, $x \in \mathbb{R}^n$ is the known state vector, $j \mapsto r(j) \in \mathbb{R}$ is a known exogenous input (e.g., measured disturbance, reference, or feedforward signal), $j \mapsto \phi(j) \in \mathbb{R}^p$ is the known regressor, and $\theta \in \mathbb{R}^p$ is an unknown vector of constant parameters.

We develop a gradient algorithm for θ by first rewriting (3) as $x^+ - A_d x - b_d r(j) = b_d\phi^\top(j)\theta$. Then, pre-multiplying both sides by $b_d^\top \neq 0$ yields

$$y = \theta^\top\phi_d(j) \tag{4}$$

where $y := b_d^\top(x^+ - A_d x - b_d r(j)) \in \mathbb{R}$ and $\phi_d := \phi(j)b_d^\top b_d \in \mathbb{R}^p$. Note that, to compute y, we require measurements of x for two consecutive discrete steps. Omitting the first discrete step included in computing y, we have expressed the jump dynamics of (3) in the form of a linear regression model, and the gradient algorithm for $\hat{\theta}$ [7] is given by

$$\hat{\theta}^+ = \hat{\theta} + \frac{\phi_d(j)}{\gamma_d + |\phi_d(j)|^2}(y^\top - \phi_d^\top(j)\hat{\theta}) \tag{5}$$

where $\gamma_d > 0$ is a design parameter. Then, the parameter estimation error, $\tilde{\theta}$, has dynamics

$$\tilde{\theta}^+ = \tilde{\theta} - \frac{\phi_d(j)\phi_d^\top(j)}{\gamma_d + |\phi_d(j)|^2}\tilde{\theta}. \tag{6}$$

Analyzing the convergence of the gradient algorithms can be translated into showing exponential stability of the origin for the systems (2) and (6). It is shown in [10] that the following persistence of excitation condition on the regressor ϕ is necessary and sufficient for exponential stability of (2):

(C1) There exist $\mu_1, \mu_2 > 0$ and $\phi_M > 0$ such that, for each $t \geq 0$,

$$\int_t^{t+\mu_1} \phi(\tau)\phi(\tau)^\top d\tau \geq \mu_2 I$$

and ess sup $\{|\phi(t)|, |\dot{\phi}(t)| : t \geq 0\} \leq \phi_M$.

Similarly, following [7], the persistence of excitation condition for the discrete-time case is:

(C2) There exist $\eta_1 \in \mathbb{N}_{>0}$, $\eta_2 > 0$, and $\phi_M > 0$ such that, for each $j \in \mathbb{N}$,

$$\sum_{k=j}^{j+\eta_1} \phi(k)\phi(k)^\top \geq \eta_2 I$$

and sup $\{|\phi(j)| : j \in \mathbb{N}\} \leq \phi_M$.

C. Hybrid dynamical systems

In this paper, a hybrid system $\mathcal{H}$ is defined as in [11] by (C, F, D, G) as

$$\mathcal{H} = \begin{cases} \dot{\xi} = F(\xi, u) & (\xi, u) \in C \\ \xi^+ = G(\xi, u) & (\xi, u) \in D \end{cases} \tag{7}$$

where $\xi \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the flow map defining a differential equation capturing the continuous dynamics, and $C \subset \mathbb{R}^n$ defines the flow set on which flows are permitted. The mapping $G : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the jump map defining the law resetting ξ at jumps, and $D \subset \mathbb{R}^n$ is the jump set on which jumps are permitted.

A solution ξ to $\mathcal{H}$ is a hybrid arc that is parameterized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t is the elapsed ordinary time and j is the number of jumps that have occurred. The domain of ξ, denoted $\operatorname{dom}\xi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, is a *hybrid time domain*, in the sense that for every $(t', j') \in \operatorname{dom}\xi$, there exists a nondecreasing sequence $\{t_j\}_{j=0}^{j'+1}$ with $t_0 = 0$ such that $\operatorname{dom}\xi \cap ([0, t'] \times \{0, 1, \ldots, j'\}) = \bigcup_{j=0}^{j'} ([t_j, t_{j+1}], \{j\})$. A solution ξ to $\mathcal{H}$ is said to be

- *nontrivial* if $\operatorname{dom}\xi$ contains at least two points;
- *eventually continuous* if $J = \sup_j \operatorname{dom}\xi < \infty$ and $\operatorname{dom}\xi \cap (\mathbb{R}_{\geq 0} \times \{J\})$ contains at least two points;
- *eventually discrete* if $T = \sup_t \operatorname{dom}\xi < \infty$ and $\operatorname{dom}\xi \cap (\{T\} \times \mathbb{N})$ contains at least two points;
- *continuous* if nontrivial and $\operatorname{dom}\xi \subset \mathbb{R}_{\geq 0} \times \{0\}$;
- *discrete* if nontrivial and $\operatorname{dom}\xi \subset \{0\} \times \mathbb{N}$.

A solution is called *maximal* if it cannot be extended further, and is called *complete* if its domain is unbounded.

We employ the following notion of exponential stability for hybrid systems [11, Definition 3.11].

Definition 2.1: Given a hybrid system $\mathcal{H}$ with data as in (7), the origin is said to be *globally pre-exponentially stable* for $\mathcal{H}$ if there exist $\kappa > 0$ and $\lambda > 0$ such that each solution ξ to $\mathcal{H}$ satisfies

$$|\xi(t, j)| \leq \kappa e^{-\lambda(t+j)}|\xi(0, 0)| \quad \forall (t, j) \in \operatorname{dom}\xi. \tag{8}$$

When, additionally, every maximal solution to $\mathcal{H}$ is complete, we say that the origin is *globally exponentially stable* for $\mathcal{H}$.

III. Motivational example

To motivate the proposed algorithm for estimation of parameters in hybrid systems, consider a system with dynamics

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= -10\theta & x &\in C_P \\ x_1^+ &= x_1, & x_2^+ &= -x_2\theta & x &\in D_P \end{aligned} \tag{9}$$

where $C_P := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$ and $D_P := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\}$ are the flow and jump sets, respectively, and $\theta \in \mathbb{R}_{\geq 0}$ is an unknown parameter. We apply the estimation algorithms in Section II-B to estimate θ. The continuous-time algorithm receives state measurements during flows while the discrete-time algorithm receives measurements immediately before and after each jump. Figure 1 shows simulation results with initial conditions $x(0, 0) = \hat{x}(0, 0) = (1, 0)$ and

$\hat{\theta}(0,0) = 0$, with parameters $\theta = 1$, $A = -5I$, $Q = I$, $\gamma_c = 0.4$, and $\gamma_d = 0.5$, using the regressors $\phi = -10$ during flows and $\phi = -x_2$ at each jump.[1]

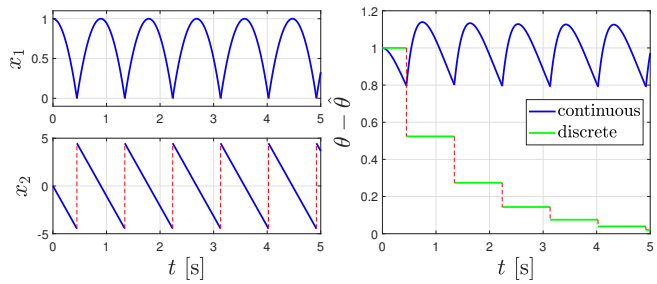

Fig. 1: Trajectories of x_1 and x_2 (left) and parameter estimation error (right).

From Figure 1, we see that the continuous-time gradient descent algorithm fails to converge even though condition (C1) is satisfied over each interval of flow. The reason it does not converge is that the algorithm does not account for the resets of the state that occur at each jump. Moreover, while the discrete-time gradient descent algorithm successfully converges, it disregards the information available to estimate θ during each interval of flow. On the other hand, if we combine the continuous-time update map in (1) during flows and the discrete-time reset map in (5) at jumps, the resulting hybrid algorithm leverages the information available during both flows and jumps to estimate the unknown parameter.

IV. Problem Statement

Motivated by the example in Section III, we extend the continuous-time and discrete-time gradient algorithms in Section II-B to hybrid dynamical systems of the form

$$\begin{aligned} \dot{x} &= A_c x + b_c \left(r(t,j) + \phi^\top(t,j)\theta\right) \quad x \in C_P \\ x^+ &= A_d x + b_d \left(r(t,j) + \phi^\top(t,j)\theta\right) \quad x \in D_P \end{aligned} \tag{10}$$

where $C_P \subset \mathbb{R}^n$ is the flow set, $D_P \subset \mathbb{R}^n$ is the jump set, and the inputs $r \in \mathbb{R}$ and $\phi \in \mathbb{R}^p$ are now hybrid.

Since the regressor ϕ may exhibit both flows and jumps, it is important to update $\hat{x}$ and $\hat{\theta}$ according to (2) when ϕ flows and to update $\hat{\theta}$ according to (6) each time ϕ jumps, under the assumption that jumps in ϕ are detected instantaneously. Due to the fact that the reset map for $\hat{\theta}$ in (6) does not depend on the value of $\hat{x}^+$, we are free to choose the reset map for $\hat{x}$ so that the resulting hybrid system exhibits the desired stability properties. In this paper, we choose the update map for $\hat{x}$ at jumps as $\hat{x}^+ = x^+ + \left(1 - \left|\frac{\phi_d \phi_d^\top}{\gamma_d + |\phi_d|^2}\right|\right)(x - \hat{x})$.

Given a hybrid arc $(t,j) \mapsto \phi(t,j)$ representing the regressor, we express the dynamics of $\hat{x}$ and $\hat{\theta}$ as a hybrid system that flows when ϕ flows and jumps when ϕ jumps. We denote this hybrid system as $\mathcal{H}_g$, with state $\xi_g := (x, \hat{x}, \hat{\theta}) \in \mathcal{X}_g := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$, inputs $\phi : \operatorname{dom}\phi \mapsto \mathbb{R}^p$ and $r : \operatorname{dom}\phi \mapsto \mathbb{R}$, and data

$$\mathcal{H}_g : \begin{cases} \dot{\xi}_g = F_g(\xi_g, \phi(t,j), r(t,j)) & (t,j) \in C_g \\ \xi_g^+ = G_g(\xi_g, \phi(t,j), r(t,j)) & (t,j) \in D_g \end{cases} \tag{11}$$

[1]Code at https://github.com/HybridSystemsLab/HybridGradient

where $C_g := \operatorname{dom}\phi \setminus D_g$, $D_g := \{(t,j) \in \operatorname{dom}\phi : (t, j+1) \in \operatorname{dom}\phi\}$, and

$$F_g(\xi_g, \phi(t,j), r(t,j)) := \begin{bmatrix} A_c x + b_c\left(r(t,j) + \phi^\top(t,j)\theta\right) \\ A_c x - A(x - \hat{x}) + b_c(r(t,j) + \phi^\top(t,j)\hat{\theta}) \\ \gamma_c \phi(t,j) b_c^\top P(x - \hat{x}) \end{bmatrix}$$

$$G_g(\xi_g, \phi(t,j), r(t,j)) := \begin{bmatrix} A_d x + b_d\left(r(t,j) + \phi^\top(t,j)\theta\right) \\ x^+ + \left(1 - \left|\frac{\phi_d(t,j)\phi_d^\top(t,j)}{\gamma_d + |\phi_d(t,j)|^2}\right|\right)(x - \hat{x}) \\ \hat{\theta} + \frac{\phi_d(t,j)}{\gamma_d + |\phi_d(t,j)|^2}(y^\top - \phi_d^\top(t,j)\hat{\theta}) \end{bmatrix}.$$

where x^+ is given in (10).

Remark 4.1: We assume for simplicity that the state x, the regressor ϕ, and the input r have the same hybrid time domains. The proposed algorithm can be extended to the case where x, ϕ, and r have different hybrid time domains through the inclusion of the flow set C_P and jump set D_P of the plant. In this case, we need to reparameterize the domain of ϕ and r to express x, ϕ, and r on a common hybrid time domain, for example, as in [12].

Remark 4.2: For simplicity, the hybrid algorithm in (11) is expressed such that jumps in the estimator state coincide with jumps in ϕ. In practice, since measurements of x^+ are not available until after a jump in ϕ, the corresponding jump in the estimator state will occur at a time instant after a jump in ϕ. Section VII presents a numerical example that demonstrates the effects of including this delay in the closed-loop dynamics. A formal study of the effects of this delay is left as future research.

V. A General Class of Hybrid Gradient Algorithms

Recall the error coordinates $e = x - \hat{x}$ and $\tilde{\theta} = \theta - \hat{\theta}$ corresponding to the state and parameter estimation error, respectively. The system resulting from expressing the hybrid system $\mathcal{H}_g$ in error coordinates belongs to a class of hybrid systems, denoted $\mathcal{H}$, with state $\xi := (e, \tilde{\theta}) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^p$, inputs $\Phi_c : E \mapsto \mathbb{R}^{p \times n}$ and $\Phi_d : E \mapsto \mathbb{R}^{p \times p}$ with $E := \operatorname{dom}\Phi_c = \operatorname{dom}\Phi_d$, and data

$$\mathcal{H} : \begin{cases} \dot{\xi} = \begin{bmatrix} Ae + \Phi_c(t,j)\tilde{\theta} \\ -\gamma \Phi_c^\top(t,j) P e \end{bmatrix} =: F(\xi, \Phi_c(t,j)) & (t,j) \in C \\ \xi^+ = \begin{bmatrix} e - |\Phi_d(t,j)|e \\ \tilde{\theta} - \Phi_d(t,j)\tilde{\theta} \end{bmatrix} =: G(\xi, \Phi_d(t,j)) & (t,j) \in D \end{cases} \tag{12}$$

where $C := E \setminus D$, $D := \{(t,j) \in E : (t, j+1) \in E\}$. The matrix functions $\Phi_c : E \mapsto \mathbb{R}^{p \times n}$ and $\Phi_d : E \mapsto \mathbb{R}^{p \times p}$ are called the continuous and discrete regression matrices, respectively, and $\gamma > 0$ is a design parameter that modifies the convergence rate of $\hat{\theta}$ during flows.

To study the stability properties induced by $\mathcal{H}_g$, we focus on providing sufficient conditions on the hybrid regressors Φ_c and Φ_d that guarantee global pre-exponential stability of the origin for $\mathcal{H}$ in the sense of Definition 2.1.

The following remark relates the hybrid systems $\mathcal{H}$ in (12) and $\mathcal{H}_g$ in (11).

Remark 5.1: The hybrid system $\mathcal{H}$ in (12) reduces to the hybrid gradient system $\mathcal{H}_g$ in (11) (expressed in error coordinates) when $\gamma = \gamma_c$ and

$$\Phi_c = b_c \phi^\top, \qquad \Phi_d = \frac{\phi_d \phi_d^\top}{\gamma_d + |\phi_d|^2}. \tag{13}$$

We assume the following structural properties for the matrices Φ_c and Φ_d to match those required in the continuous-time and discrete-time gradient algorithms, respectively.

Assumption 5.2: The matrix functions Φ_c and Φ_d satisfy the following properties:

1. There exists $\phi_M > 0$ such that
$$\text{ess sup}\{|\Phi_c(t,j)|, |\dot{\Phi}_c(t,j)| : (t,j) \in E\} \leq \phi_M;$$
2. For each $(t,j) \in E$,
$$\Phi_d(t,j) = \Phi_d(t,j)^\top \geq 0, \qquad |\Phi_d(t,j)| \leq 1;$$
3. $\operatorname{dom} \Phi_c = \operatorname{dom} \Phi_d =: E$.

Finally, inspired from the conditions in (C1) and (C2), we assume the following persistence of excitation conditions that will enable us to guarantee convergence of the parameter estimate using the proposed hybrid algorithm.

Assumption 5.3: The matrix functions Φ_c and Φ_d satisfy the following properties:

1. There exist $\mu_1, \mu_2 > 0$ such that, for each solution ξ to $\mathcal{H}$ and each hybrid time window $[t, t+\mu_1] \times \{j, j+1, \cdots, j^*\} \subset \operatorname{dom} \xi$, the following holds:
$$\begin{aligned} &\int_t^{t_{j+1}} \Phi_c^\top(\tau, j) \Phi_c(\tau, j) d\tau \\ &\quad + \sum_{k=j+1}^{j^*-1} \int_{t_k}^{t_{k+1}} \Phi_c^\top(\tau, k) \Phi_c(\tau, k) d\tau \\ &\quad + \int_{t_{j^*}}^{t+\mu_1} \Phi_c^\top(\tau, j^*) \Phi_c(\tau, j^*) d\tau \geq \mu_2 I. \end{aligned} \tag{14}$$
2. There exist $\eta_1 \in \mathbb{N}_{\geq 1}$, $\eta_2 > 0$ such that for each solution ξ to $\mathcal{H}$ and each hybrid time window $[t_{j+1}, t_{j+\eta_1+1}] \times \{j, j+1, \cdots, j+\eta_1\} \subset \operatorname{dom} \xi$, the following holds:
$$\sum_{k=j}^{j+\eta_1} \Phi_d(t_{k+1}, k) \geq \eta_2 I. \tag{15}$$

The excitation conditions in Assumption 5.3 are similar in form to the hybrid persistence of excitation condition proposed in [9] for linear regression models. However, compared to the condition in [9], the conditions in this paper are more restrictive since they require the regression matrices Φ_c and Φ_d to be persistently exciting during flows and jumps, respectively. Relaxing Assumption 5.3 to the case of hybrid persistence of excitation is left as future research.

VI. Stability Analysis

To analyze the stability properties induced by $\mathcal{H}$, we first establish the following two propositions that study the rate of descent for solutions of $\mathcal{H}$ during only flows and jumps, respectively. These results will be used to show convergence of solutions of in the hybrid case.

Proposition 6.1: Given a hybrid system $\mathcal{H}$ with data as in (12)*, where the matrix A is Hurwitz and $\gamma > 0$, suppose Assumption 5.2 and item 1 of Assumption 5.3 hold. Then, there exist $\kappa_c, \lambda_c > 0$ such that for each solution ξ to $\mathcal{H}$, the following holds:*

$$|\xi(t,j)| \leq \kappa_c \mathrm{e}^{-\lambda_c (t - t_j)} |\xi(t_j, j)| \tag{16}$$

for all $(t,j) \in \operatorname{dom} \xi$ such that $(t,j) \in [t_j, t_{j+1}] \times \{j\}$.

Sketch of Proof: We consider the following cases:

- Using the function $V_1(\xi) := e^\top P e + \gamma^{-1} |\tilde{\theta}|^2$, where P is given below (1), it can be shown that there exists $\sigma_0 > 0$ such that each solution ξ to $\mathcal{H}$ with $\sup_t \operatorname{dom} \xi < \mu_1$ satisfies $|\xi(t,j)| \leq \sigma_0 |\xi(0,0)|$ for all $(t,j) \in \operatorname{dom} \xi$.
- Inspried by [10], consider the function
$$V(\xi, t, j) = c V_1(\xi) + W_1(\xi, t, j) + \tfrac{1}{4} W_2(\xi, t, j)$$
where $c > 0$ is a design parameter, $W_1(\xi, t, j) := -e^\top \Phi_c(t,j) \tilde{\theta}$, and $W_2(\xi, t, j) := -\tilde{\theta}^\top M(t,j) \tilde{\theta}$ with
$$\begin{aligned} M(t,j) := &\int_t^{t_{j+1}} \mathrm{e}^{t-\tau} \Phi_c^\top(\tau, j) \Phi_c(\tau, j) d\tau \\ &+ \sum_{k=j+1}^{J-1} \int_{t_k}^{t_{k+1}} \mathrm{e}^{t-\tau} \Phi_c^\top(\tau, k) \Phi_c(\tau, k) d\tau \\ &+ \int_{t_J}^{T} \mathrm{e}^{t-\tau} \Phi_c^\top(\tau, J) \Phi_c(\tau, J) d\tau \end{aligned}$$
where $T := \sup_t \operatorname{dom} \xi$ and $J := \sup_j \operatorname{dom} \xi$. Using the fact that $(t,j) \mapsto \Phi_c(t,j)$ is persistently exciting as in item 1 of Assumption 5.3, it can be shown that there exist constants $\sigma_1, \sigma_2, \sigma_3 > 0$ with $\sqrt{\frac{\sigma_2}{\sigma_1}} \mathrm{e}^{-\frac{\sigma_3}{2\sigma_2} \mu_1} \geq \sigma_0$ such that each solution ξ to $\mathcal{H}$ with $\sup_t \operatorname{dom} \xi \geq \mu_1$ satisfies
$$|\xi(t,j)| \leq \sqrt{\frac{\sigma_2}{\sigma_1}} \mathrm{e}^{-\frac{\sigma_3}{2\sigma_2} t} |\xi(0,0)|$$
for all $(t,j) \in \operatorname{dom} \xi$. Hence, (16) holds. □

Proposition 6.2: Given a hybrid system $\mathcal{H}$ with data as in (12)*, suppose item 2 of Assumptions 5.2 and 5.3 hold. Denote the matrix $(t,j) \mapsto R(t,j)$ as*

$$R(t,j) := I - \begin{bmatrix} |\Phi_d(t,j)| I & 0 \\ 0 & \Phi_d(t,j) \end{bmatrix}.$$

Then, for each solution ξ to $\mathcal{H}$ and each $(t,j) \in \operatorname{dom} \xi$ such that $(t, j+1) \in \operatorname{dom} \xi$, the following holds:

$$|\xi(t, j+1)| \leq |R(t,j)| |\xi(t,j)| \tag{17}$$

where $|R(t,j)| \leq 1$. Furthermore, there exist $\kappa_d, \lambda_d > 0$ such that for any $(t,j) \in \operatorname{dom} \xi$,

$$\prod_{k=0}^{j-1} |R(t_{k+1}, k)| \leq \kappa_d \mathrm{e}^{-\lambda_d j}. \tag{18}$$

Sketch of Proof: For each solution ξ to $\mathcal{H}$ and each $(t,j) \in \operatorname{dom}\xi$ such that $(t,j+1) \in \operatorname{dom}\xi$, according to the jump map we have $\xi(t,j+1) = R(t,j)\xi(t,j)$, and (17) follows from the triangle inequality. Next, since for all $(t,j) \in \operatorname{dom}\xi$, $\Phi_d(t,j) \geq 0$ and $|\Phi_d(t,j)| \leq 1$, we have $|R(t,j)| \leq 1$, and (18) follows from the fact that $(t,j) \mapsto \Phi_d(t,j)$ is persistently exciting as in item 2 of Assumption 5.3. □

A. Main Result

We are now ready to establish our main result stating the stability properties induced by proposed hybrid algorithm.

Theorem 6.3: Given a hybrid system $\mathcal{H}$ with data as in (12) *where the matrix A is Hurwitz and $\gamma > 0$, suppose that Assumptions 5.2 and 5.3 hold. Then, for each solution ξ to $\mathcal{H}$ and each $(t,j) \in \operatorname{dom}\xi$, we have the following:*

1. *If ξ is eventually continuous (or continuous), then there exists $\kappa_J > 0$ such that*
$$|\xi(t,j)| \leq \kappa_J \mathrm{e}^{-\lambda_c t}|\xi(0,0)| \tag{19}$$
with $\lambda_c > 0$ given in Proposition 6.1.
2. *If ξ is eventually discrete (or discrete), then there exists $\kappa_T > 0$ such that*
$$|\xi(t,j)| \leq \kappa_T \mathrm{e}^{-\lambda_d j}|\xi(0,0)| \tag{20}$$
with $\lambda_d > 0$ given in Proposition 6.2.
3. *If ξ is neither eventually continuous nor eventually discrete, then*
$$|\xi(t,j)| \leq \kappa_c\kappa_d \mathrm{e}^{-\lambda_c t-(\lambda_d-\ln(\kappa_c))j}|\xi(0,0)| \tag{21}$$
with $\kappa_c, \lambda_c > 0$ given in Proposition 6.1 and $\kappa_d, \lambda_d > 0$ given in Proposition 6.2.

Hence, with $\kappa_c, \lambda_c > 0$ coming from Proposition 6.1 and $\kappa_d, \lambda_d > 0$ from Proposition 6.2, the origin is globally pre-exponentially stable for $\mathcal{H}$ if

I. *$\lambda_d > \ln(\kappa_c)$, or*
II. *there exist $\gamma > 0$ and $M > 0$ such that every solution ξ to $\mathcal{H}$ is such that $-\lambda_c t-(\lambda_d-\ln(\kappa_c))j \leq M-\gamma(t+j)$ for all $(t,j) \in \operatorname{dom}\xi$.*

Sketch of Proof: Pick a solution ξ to $\mathcal{H}$ and a hybrid time $(t,j) \in \operatorname{dom}\xi$ and define $W := \operatorname{dom}\xi \cap [0,t] \times \{0,\cdots,j\}$. Since, W is a compact hybrid time domain, there exists a finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_j$ such that $W = \cup_{j'=0}^{j}([t_{j'}, t_{j'+1}] \times \{j'\})$. Then, at each time $(t',j') \in W$, let $(t',j') \mapsto i(t',j') \in \mathbb{N}$ denote the number intervals of flow with nonempty interior between hybrid times $(0,0)$ and (t',j').[2] That is, given $(t',j') \in W$,

$$i(t',j') := \begin{cases} 0 & \text{if} \quad t' = j' = 0 \\ \sum_{k=0}^{j'-1} \beta(I^k) & \text{if} \quad t' = t'_{j'}, j' \geq 1 \\ \sum_{k=0}^{j'} \beta(I^k) & \text{if} \quad t' > t'_{j'} \end{cases}$$

[2]For example, if ξ is discrete, then $i(t',j') = 0$ for all $(t',j') \in W$. If ξ is continuous, then $i(t',j') = 1$ for all $(t',j') \in W$ with $t' > 0$. If ξ is neither continuous nor discrete, then $i(t',j') \geq 0$ for all $(t',j') \in W$.

where $I^k := \{t' : (t',k) \in W\}$ and $\beta(I^k) := 0$ if $\operatorname{int}(I^k) = \varnothing$ and $\beta(I^k) := 1$ if $\operatorname{int}(I^k) \neq \varnothing$.

By induction on $j' \in \{0,1,\cdots,j\}$ and, using the fact that Assumptions 5.2 and 5.3 hold, it can be shown that

$$|\xi(t,j)| \leq \kappa_c^{i(t,j)}\kappa_d \mathrm{e}^{-\lambda_c t-\lambda_d j}|\xi(0,0)| \tag{22}$$

for all $(t,j) \in \operatorname{dom}\xi$. Then, we consider the following cases:

1. If ξ is eventually continuous (or continuous), then there exists $(t_J, J) \in \operatorname{dom}\xi$ such that $\sup_j \operatorname{dom}\xi = J \geq 0$ and thus (19) holds with
$$\kappa_J := \kappa_c^{i(t_J,J)+1}\kappa_d \mathrm{e}^{-\lambda_d J}.$$
2. If ξ is eventually discrete (or discrete), then there exists $(T,j') \in \operatorname{dom}\xi$ such that $\sup_t \operatorname{dom}\xi = T \geq 0$ and thus (20) holds with
$$\kappa_T := \kappa_c^{i(T,j')}\kappa_d \mathrm{e}^{-\lambda_c T}.$$
3. If ξ is neither eventually continuous nor eventually discrete, then using the fact that for each $(t,j) \in \operatorname{dom}\xi$, $i(t,j) \leq j+1$, we have from (22) that (21) holds and we consider the following cases:
   a) If $\lambda_d > \ln(\kappa_c)$, we define $\rho := \lambda_d - \ln(\kappa_c) > 0$. By substituting this expression into (21), we have
   $$|\xi(t,j)| \leq \kappa_c\kappa_d \mathrm{e}^{-\lambda_c t-\rho j}|\xi(0,0)|.$$
   b) If there exist $\gamma > 0$ and $M > 0$ such that $(t,j) \in \xi$ implies $-\lambda_c t - (\lambda_d - \ln(\kappa_c))j \leq M - \gamma(t+j)$, by substituting this expression into (21), we have
   $$|\xi(t,j)| \leq \mathrm{e}^{M-\gamma(t+j)}|\xi(0,0)|.$$

Combining the conditions from items 1, 2, and 3 of the list above, it follows that the origin is globally pre-exponentially stable for $\mathcal{H}$ in the sense of Definition 2.1 if item I or item II of Theorem 6.3 hold. □

VII. Examples

In this section, we present simulation results that demonstrate the practicality of the proposed algorithm. Simulations are performed using the Hybrid Equations Toolbox [13].

A. Motivational example

First, we briefly revisit the motivational example from Section III. The proposed hybrid algorithm $\mathcal{H}_g$ in 11 is applied to estimate the unknown parameter θ in (9). Using the system parameters and initial conditions given in Section III, it can be shown that the conditions of Theorem 6.3 are satisfied. The parameter estimate from $\mathcal{H}_g$ is shown in Figure 2 alongside the estimates from the purely continuous-time and discrete-time gradient algorithms for comparison. The parameter estimate for $\mathcal{H}_g$ converges exponentially to the true value in accordance with Theorem 6.3. Additionally, the proposed hybrid algorithm converges more quickly than the discrete-time algorithm due to the ability of the hybrid algorithm to leverage information available during both flows and jumps.

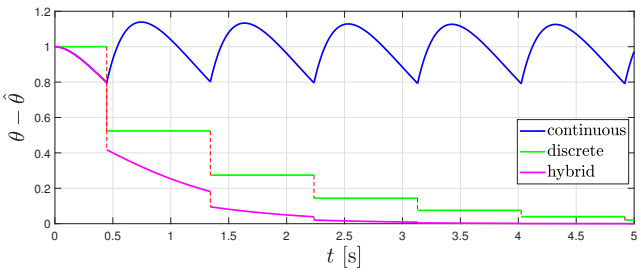

Fig. 2: The projection onto t of the parameter estimation error for the proposed hybrid algorithm.

B. Pressure mounter machine

Consider the problem of estimating the friction coefficient $c \in \mathbb{R}_{>0}$ for the vertical dynamics of the main shaft of a pressure mounter machine. In this work, we consider a simplified model, akin to a mass-spring-damper, which is obtained by adding appropriate compensators to the open-loop dynamics. Let $x_1 \in \mathbb{R}$ denote the vertical position of the shaft ($x_1 = 0$ at rest, $x_1 = x_{\max} > 0$ while in contact with the workbench), and $x_2 \in \mathbb{R}$ the vertical velocity of the machine. Then, during flows, the dynamics are given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u$$

where m is the mass of the machine and k is the spring constant. The input u is provided by a full-state feedback controller of the form $u = -Kx + F_{\text{cl}}v$, where F_{cl} is chosen to achieve unitary dc-gain and v is the reference command.

Jumps occur each time the machine impacts a plate at position $x_{\max}$. Following each impact, the machine rebounds from the plate with a velocity that is inversely proportional to the friction coefficient, as follows:

$$x_1^+ = x_1, \qquad x_2^+ = -\alpha x_2 \frac{1}{c}$$

where $\alpha \in \mathbb{R}_{>0}$ is the known proportionality constant. Assuming the variation in c is small, we linearize the jump dynamics about a nominal value of c, denoted $\bar{c}$ as

$$x_1^+ = x_1, \qquad x_2^+ = -\alpha x_2 \left(\frac{1}{\bar{c}} - \frac{1}{\bar{c}^2}(c - \bar{c})\right).$$

Denoting $\theta := c$, the dynamics of the pressure mounter machine can be written in the form of (10) as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\left(u + \phi^\top\theta\right) \qquad x \in C_P$$

$$\begin{bmatrix}x_1^+\\ x_2^+\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{\alpha}{\bar{c}}\end{bmatrix}\left(-2x_2 + \phi^\top\theta\right) \qquad x \in D_P$$

where $\phi(t,j) := x_2(t,j)$ if $x \in C_P$ and $\phi(t,j) := x_2(t,j)/\bar{c}$ if $x \in D_P$, with $C_P := \{x \in \mathbb{R}^2 : x_1 \leq x_{\max}\}$, $D_P := \{x \in \mathbb{R}^2 : x_1 = x_{\max}, x_2 \geq 0\}$, and $x_{\max} > 0$.

The proposed hybrid gradient descent algorithm is applied to estimate θ. In accordance with Remark 4.2, the delay between each jump of the pressure mounter machine and the corresponding jump of the estimator is explicitly represented in this model. The system has parameters $m = 0.5$, $k = 25$, $\alpha = 0.5$, $x_{\max} = 3$, and $\theta = 1.5$ and the proposed hybrid estimator has parameters $A = -50I$, $Q = I$, $\gamma_c = 10$, $\gamma_d = 0.5$, and $\bar{c} = 1$. The reference command v is chosen such that trajectories of the pressure mounter machine achieve a limit cycle in steady-state, thereby assuring (numerically) that the conditions in Theorem 6.3 are satisfied. The simulation has initial conditions $x(0,0) = \hat{x}(0,0) = (0,0)$, $\hat{\theta}(0,0) = \bar{c}$, producing the results in Figure 3.[3] The parameter estimate converges exponentially to the true value in accordance with Theorem 6.3. The instantaneous increases in the state estimation error resulting from the delay between jumps in the plant state and jumps in the estimator state are visible in the upper right subplot of Figure 3.

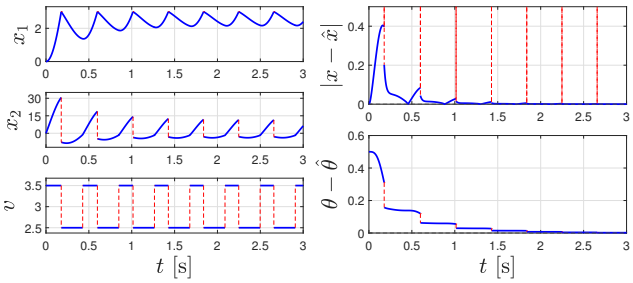

Fig. 3: The projection onto t of the pressure mounter machine position, velocity, and reference input (left), and the norm of the state and parameter estimation error (right).

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[3] Code at https://github.com/HybridSystemsLab/PressureMounter