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ORIGINAL RESEARCH ARTICLE

Embeddings of Maximal Tori in Classical Groups, Odd Degree Descent and Hasse Principles

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Abstract

The aim of this paper is to revisit the question of local–global principles for embeddings of étale algebras with involution into central simple algebras with involution over global fields of characteristic not 2. A necessary and sufficient condition is given in Bayer-Fluckiger et al. (J Eur Math Soc 20:137–163, 2018). In the present paper, we give a simpler description of the obstruction group. It is also shown that if the étale algebra is a product of pairwise linearly disjoint field extensions, then the Hasse principle holds, and that if an embedding exists after an odd degree extension, then it also exists over the global field itself. An appendix gives a generalization of this later result, in the framework of a question of Burt Totaro.

Keywords Embedding maximal tori · Hasse principle · Algebras with involution

1 Introduction

The aim of this paper is to revisit a topic investigated in [2–4,9,12,15], namely the question of embeddings of maximal tori in classical groups. As in the above references, this is reformulated in terms of embeddings of commutative étale algebras with involution in central simple algebras with involution.

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If the ground field is a global field of characteristic $\neq 2$, a necessary and sufficient condition for the local–global principle to hold is given in [4, Theorem 4.6.1]. This result is formulated in terms of an obstruction group, constructed in [4, §2]. The description of this group is however quite complicated. One of the aims of the present paper is to give a much simpler description of the obstruction group (see Sect. 4), leading to a simpler version of of Theorem 4.6.1 of [4], see Theorem 6.1.

On the other hand, [3] gives a necessary and sufficient criterion for an embedding to exist locally everywhere. However, this result is not enough for some applications of Theorem 4.6.1 of [4]: we need to know when an *oriented* embedding exists locally everywhere. We provide a criterion for this to hold in Theorem 3.1.

We then give two applications:

We show that if the commutative étale algebra is a product of pairwise linearly disjoint fields, then the obstruction group is trivial, and hence the local–global principle holds (see Corollary 7.5).

We also prove an odd degree descent result: if an embedding exists after a finite extension of odd degree, then it also exists over the ground field (see Theorem 8.1). This provides a positive answer to the following question of Totaro (see [18]): if a homogeneous space has a zero-cycle of degree one, does it also have a rational point ? A more general result on a positive answer to Totaro's question for homogeneous spaces under connected linear algebraic groups with connected stabilizers over number fields, using Borovoi's arguments, is proved in Proposition 9.1 of Appendix.

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2 Definitions, Notation and Basic Facts

Let L be a field with $\mathrm{char}(L) \neq 2$, and let K be a subfield of L such that either $K = L$, or L is a quadratic extension of K.

2.1 Étale Algebras with Involution

Let E be a commutative étale algebra of finite rank over L, and let $\sigma : E \to E$ be a K–linear involution. Set $F = \{e \in E | \sigma(e) = e\}$, and $n = \dim_L(E)$. Assume that if $L = K$, then we have $\dim_K(F) = [\frac{n+1}{2}]$. Note that if $L \neq K$, then $\dim_K(F) = n$ (cf. [15, Proposition 2.1]).

2.2 Central Simple Algebras with Involution

Let A be a central simple algebra over L. Let τ be an involution of A, and assume that K is the fixed field of τ in L. Recall that τ is said to be of the *first kind* if $K = L$ and of the *second kind* if $K \neq L$; in this case, L is a quadratic extension of K. After extension to a splitting field of A, any involution of the first kind is induced by a symmetric or by a skew-symmetric form. We say that the involution is of the *orthogonal type* in the first case, and of the *symplectic type* in the second case. An involution of the second kind is also called *unitary involution*.

2.3 Embeddings of Algebras with Involution

Let (E, σ) and (A, τ) be as above, with $n = \dim_L(E)$ and $\dim_L(A) = n^2$; assume moreover that $\sigma|L = \tau|L$.

An *embedding* of (E, σ) in (A, τ) is by definition an injective homomorphism $f : E \to A$ such that $\tau(f(e)) = f(\sigma(e))$ for all $e \in E$. It is well–known that embeddings of maximal tori into classical groups can be described in terms of embeddings of étale algebras with involution into central simple algebras with involution satisfying the above dimension hypothesis (see for instance [15, Proposition 2.3]).

Let $\epsilon : E \to A$ be an L-embedding which may not respect the given involutions. There exists an involution θ of A of the same type (orthogonal, symplectic or unitary) as τ such that $\epsilon(\sigma(e)) = \theta(\epsilon(e))$ for all $e \in E$, in other words $\epsilon : (E, \sigma) \to (A, \theta)$ is an L-embedding of algebras with involution (see [10, §2.5] or [15, Proposition 3.1]).

For all $a \in F^\times$, let $\theta_a : A \to A$ be the involution given by $\theta_a = \theta \circ \mathrm{Int}(\epsilon(a))$. Note that $\epsilon : (E, \sigma) \to (A, \theta_a)$ is an embedding of algebras with involution.

Proposition 2.1 *The following conditions are equivalent:*

(a) *There exists an L–embedding* $\iota : (E, \sigma) \to (A, \tau)$ *of algebras with involution.*
(b) *There exists an* $a \in F^\times$ *such that* $(A, \theta_a) \simeq (A, \tau)$ *as algebras with involution.*

Proof See [15, Theorem 3.2].

Oriented embeddings

Let(A, τ) be an algebra with orthogonal involution. The algebra A is said to be *split* if A is a matrix algebra. Note that if n is odd, then A is split. In the case of a non-split A, we need an additional notion, called *orientation* (see [4, §2]).

Assume that (A, τ) is of orthogonal type and that n is even. We need an additional notion, called an *oriented embedding* (see [4, §2.6]). We redefine this notion in terms of fixing a module structure for the Clifford algebra $\mathcal{C}(A, \tau)$ over a given etale quadratic algebra. We identify this notion with the one given in [4]. We denote by $Z(A, \tau)$ the center of the algebra $\mathcal{C}(A, \tau)$. Then $Z(A, \tau)$ is a quadratic étale algebra over K (cf. [11, Chap II. (8.7)]).

Let $\Delta(E)$ be the discriminant algebra of E (cf. [11, Chapter V, §18, p. 290]). An isomorphism of K–algebras

$$\Delta(E) \to Z(A, \tau)$$

will be called an *orientation* (see [4, §2]).

Let π be the generalized Pfaffian of (A, τ) (see [11, Chap. II (8.22)]). For an embedding $\iota : (E, \sigma) \to (A, \tau)$, we denote by $u_\iota : \Delta(E) \to Z(A, \tau)$ the isomorphism induced by ι via the generalized Pfaffian π (cf. [4, 2.3]). □

Definition 2.2 Fix an orientation $\nu : \Delta(E) \to Z(A, \tau)$. An embedding $\iota : (E, \sigma) \to (A, \tau)$ is said to be an oriented embedding with respect to ν if $u_\iota = \nu$.

Let $u : \Delta(E) \to Z(A, \theta)$ be the orientation of (A, θ) constructed via $\epsilon : (E, \sigma) \to (A, \theta)$ and via the generalized Pfaffian of (A, θ). For all $a \in F^\times$ let $u_a : \Delta(E) \to Z(A, \theta_a)$ be the isomorphism induced by $\epsilon : (E, \sigma) \to (A, \theta_a)$ and by the generalized Pfaffian of (A, θ_a). Note that the orientation u_a defined above coincides with the one defined in [4, 2.5].

Proposition 2.3 *Let (A, τ) be an orthogonal involution with A of even degree, and let $\nu : \Delta(E) \to Z(A, \tau)$ be an orientation. Let $\iota : (E, \sigma) \to (A, \tau)$ be an oriented embedding with respect to ν. Then there exist $a \in F^\times$ and $\alpha \in A^\times$ satisfying the following conditions:*

(a) $\text{Int}(\alpha) : (A, \theta_a) \to (A, \tau)$ *is an isomorphism of algebras with involution such that* $\text{Int}(\alpha) \circ \epsilon = \iota$.
(b) *The induced automorphism $c(\alpha) : Z(A, \theta_a) \to Z(A, \tau)$ satisfies*

$$c(\alpha) \circ u_a = \nu.$$

The elements (ι, a, α, ν) are called parameters of the oriented embedding.

Proof By Proposition 2.1 there are $a \in F^\times$ and $\alpha \in A^\times$ such that $\text{Int}(\alpha) \circ \epsilon = \iota$. Consider the induced map between $\mathcal{C}(A, \theta_a)$ and $\mathcal{C}(A, \tau)$. Then we have $c(\alpha) \circ u_a = u_\iota = \nu$. □

Lemma 2.4 *Let (A, τ) be an orthogonal involution with A of even degree, and let $\nu : \Delta(E) \to Z(A, \tau)$ be an orientation. For $a \in F^\times$, regard $\mathcal{C}(A, \theta_a)$ and $\mathcal{C}(A, \tau)$ as $\Delta(E)$-modules via u_a and ν respectively. Then there exists an oriented embedding of (E, σ) into (A, τ) with respect to the orientation ν if and only if there exist $a \in F^\times$ and $\alpha \in A^\times$ such that* $\text{Int}(\alpha) : (A, \theta_a) \to (A, \tau)$ *is an isomorphism of algebras with involution and the induced isomorphism on the Clifford algebras $c(\alpha) : \mathcal{C}(A, \theta_a) \to \mathcal{C}(A, \tau)$ is an isomorphism of $\Delta(E)$-modules.*

Proof Suppose that there is an oriented embedding ι of (E, σ) into (A, τ) with respect to the orientation ν. By definition there exist parameters (ι, a, α, ν) of the oriented embedding ι. Consider the induced isomorphism $c(\alpha) : \mathcal{C}(A, \theta_a) \to \mathcal{C}(A, \tau)$. Since $c(\alpha) \circ u_a = \nu$, the algebras $\mathcal{C}(A, \theta_a)$ and $\mathcal{C}(A, \tau)$ are isomorphic as $\Delta(E)$-modules.

Suppose the converse. As $c(\alpha)$ is an isomorphism of $\Delta(E)$-modules, we have $c(\alpha) \circ u_a = \nu$. Therefore $\text{Int}(\alpha) \circ \epsilon$ an oriented embedding with respect to ν.

□

Notation 2.5 If M is a field, let $\text{Br}(M)$ be the Brauer group of M. For $a, b \in M^\times$, we denote by (a, b) the class of the quaternion algebra determined by a and b in $\text{Br}(M)$.

We say that (E, σ) is *split* if $E = F \times F$ and that $\sigma(x, y) = (y, x)$ for all $x, y \in F$.

Lemma 2.6 *Assume that (E, σ) is split. Then for all $a \in F^\times$, the algebras with involution (A, θ) and (A, θ_a) are isomorphic. Moreover, there exists an isomorphism* $\text{Int}(\alpha) : (A, \theta) \to (A, \theta_a)$ *such that $c(\alpha) \circ u = u_a$.*

Proof Let $a \in F^\times$. Since $E = F \times F$, there exists $x \in E^\times$ such that $\mathrm{N}_{E/F}(x) = a$. Hence $\mathrm{Int}(\epsilon(x^{-1})) : A \to A$ is an isomorphism between the algebras with involution (A, θ) and (A, θ_a), and $c(\epsilon(x^{-1})) \circ u = u_a$. □

Proposition 2.7 *Assume that K is a local field, that (E, σ) is nonsplit, and that (A, τ) is isomorphic to $(\mathrm{M}_{2n}(H), \tau_h)$, where H is the unique quaternion field over K, and τ_h is induced by the hyperbolic form on H^{2n}. Suppose that there exists an embedding $(E, \sigma) \to (A, \tau)$. Let $\nu : \Delta(E) \to Z(A, \tau)$ be an orientation. Then there exists an oriented embedding of (E, σ) into (A, τ) with respect to the orientation ν.*

Proof Let $\iota : (E, \sigma) \to (A, \tau)$ be an embedding, and let $u_\iota : \Delta(E) \to Z(A, \tau)$ be the isomorphism induced by ι. If $u_\iota = \nu$, we are done. Suppose that $u_\iota \neq \nu$. Let $d \in F^\times$, with $d \notin F^{\times 2}$, such that $E = F(\sqrt{d})$. Then there is some $a \in F^\times$ such that $\mathrm{res}_{\Delta(E)/K}\mathrm{cor}_{F/K}(a, d) \neq 0$ in $\mathrm{Br}_2(\Delta(E))$. Let $\iota_a : (E, \sigma) \to (A, \tau_a)$ be the embedding defined by $\iota_a(x) = \iota(x)$ for all $x \in E$. By [4], Lemma 2.5.4, we have $[\mathcal{C}(A, \tau_a)] = [\mathcal{C}(A, \tau)] + \mathrm{res}_{\Delta(E)/K}\mathrm{cor}_{F/K}(a, d)$ in $\mathrm{Br}_2(\Delta(E))$, where we regard $\mathcal{C}(A, \tau)$ and $\mathcal{C}(A, \tau_a)$ as $\Delta(E)$-modules via u_ι and u_{ι_a} respectively.

Since τ and τ_a are both hyperbolic involutions, there is an $\alpha \in A^\times$ such that $\mathrm{Int}(\alpha) : (A, \tau_a) \to (A, \tau)$ is an isomorphism of algebras with involution. By the choice of a, the algebras $\mathcal{C}(A, \tau_a)$ and $\mathcal{C}(A, \tau)$ are not isomorphic as $\Delta(E)$-modules. Therefore $c(\alpha) \circ u_{\iota_a} \neq u_\iota$. As there are exactly two distinct isomorphisms between $\Delta(E)$ and $Z(A, \tau)$, this implies that $\mathrm{Int}(\alpha) \circ \iota_a$ is an embedding with orientation ν. □

3 Local Criteria for the Existence of Oriented Embeddings

Assume that K is a global field, and let V_K be the set of places of K. If $v \in V_K$, we denote by K_v the completion of K at v. Let (A, τ) and (E, σ) be as in §2. In [3], we gave necessary and sufficient conditions for an embedding $(E, \sigma) \to (A, \tau)$ to exist everywhere locally. The aim of this section is to give such conditions for an *oriented embedding* to exist everywhere locally. The existence of embeddings implies the existence of oriented embeddings unless A is non-split and τ is of orthogonal type. Hence in the rest of this section, we assume that A is non-split and τ is of orthogonal type.

We keep the notation of Sect. 2; in particular, $\epsilon : E \to A$ is an embedding of algebras with involution $(E, \sigma) \to (A, \theta)$, and $u : \Delta(E) \to Z(A, \theta)$ is the isomorphism induced by ϵ as above. For all $v \in V_K$, we denote by H_v the unique quaternion field over K_v, and let τ_h be the involution of $\mathrm{M}_{2n}(H_v)$ induced by the hyperbolic hermitian form on H_v^{2n}. Let us denote by $\mathcal{P}$ the set of $v \in V_K$ such that

- $(A, \tau) \otimes_K K_v \simeq (\mathrm{M}_{2n}(H_v), \tau_h)$;
- $E \otimes K_v = (F \otimes_K K_v) \times (F \otimes_K K_v)$, and for all $x, y \in F \otimes_K K_v$ we have $\sigma(x, y) = (y, x)$.

Theorem 3.1 *Assume that for all $v \in V_K$, there exists an embedding $(E, \sigma) \otimes_K K_v \to (A, \tau) \otimes_K K_v$. Let $\nu : \Delta(E) \to Z(A, \tau)$ be an orientation. The following properties are equivalent*

(i) *There exists an oriented embedding* $(E, \sigma) \otimes_K K_v \to (A, \tau) \otimes_K K_v$ *with respect to* ν *for all* $v \in V_K$*;*
(ii) *View* $\mathcal{C}(A \otimes_K K_v, \theta)$ *and* $\mathcal{C}(A \otimes_K K_v, \tau)$ *as* $\Delta(E)$*-modules via* u *and* ν *respectively. Then*

$$[\mathcal{C}(A \otimes_K K_v, \theta)] = [\mathcal{C}(A \otimes_K K_v, \tau)]$$

in $\mathrm{Br}(\Delta(E \otimes_K K_v))$ *for all* $v \in \mathcal{P}$.

Proof For any $a_v \in F_v^\times$, we view $\mathcal{C}(A, \theta_{a_v})$ as a $\Delta(E \otimes_K K_v)$-module via u_{a_v}. Let us first show that (i) implies (ii). For all $v \in V_K$, set $A^v = A \otimes_K K_v$. By Lemma 2.4, we know that for all $v \in V_K$ there exists $a_v \in F_v^\times$ and an isomorphism $\mathrm{Int}(\alpha^v) : (A^v, \theta_{a_v}) \to (A^v, \tau)$ of algebras with involution such that $c(\alpha) : \mathcal{C}(A^v, \theta_{a_v}) \to \mathcal{C}(A^v, \tau)$ is an isomorphism of $\Delta(E)$-modules. Let $v \in \mathcal{P}$. Then $(E \otimes K_v, \sigma)$ is split. Hence by Lemma 2.6 we have $(A^v, \theta) \simeq (A^v, \theta_{a_v})$ and $\mathcal{C}(A^v, \theta) \simeq \mathcal{C}(A^v, \theta_{a_v})$ as $\Delta(E \otimes_K K_v)$-modules. Therefore $[\mathcal{C}(A^v, \theta)] = [\mathcal{C}(A^v, \tau)]$ in $\mathrm{Br}(\Delta(E \otimes_K K_v))$.

Let us now show that (ii) implies (i). If A^v is split or $\mathrm{disc}(A^v, \tau) \neq 1$ in $K_v/K_v^{\times 2}$, then by [4, Corollary 2.7.3] there exists an oriented embedding $(E \otimes_K K_v, \sigma) \to (A^v, \tau)$. Hence we only have to consider the following two cases.
Case 1 $(E \otimes_K K_v, \sigma)$ non-split, A^v non-split, and $\mathrm{disc}(A^v, \tau) = 1$ in $K_v/K_v^{\times 2}$. Then by Proposition 2.7 there exists an oriented embedding of $(E \otimes_K K_v, \sigma)$ into (A^v, τ) with respect to the orientation ν.
Case 2 Assume that $v \in \mathcal{P}$. In this case, both (A^v, τ) and (A^v, θ) are induced by hyperbolic forms. Hence there exists $\alpha_v \in (A^v)^\times$ such that $\mathrm{Int}(\alpha_v) : (A^v, \tau) \to (A^v, \theta)$ is an isomorphism of algebras with involution. Let $c(\alpha_v) : \mathcal{C}(A^v, \tau) \to \mathcal{C}(A^v, \theta)$ be the isomorphism induced by $\mathrm{Int}(\alpha_v)$. Then $[\mathcal{C}(A^v, \tau)] = [\mathcal{C}(A^v, \theta)]$ in $\mathrm{Br}(\Delta(E \otimes_K K_v))$, where we regard $\mathcal{C}(A^v, \tau)$ as an algebra over $\Delta(E \otimes_K K_v)$ via $c(\alpha_v)^{-1} \circ u$. However, by assumption we have $[\mathcal{C}(A^v, \theta)] = [\mathcal{C}(A^v, \tau)]$ in $\mathrm{Br}(\Delta(E \otimes_K K_v))$, where we regard $\mathcal{C}(A^v, \tau)$ as a $\Delta(E \otimes_K K_v)$-module via ν. Hence $c(\alpha_v)^{-1} \circ u = \nu$, and $\mathrm{Int}(\alpha_v^{-1}) \circ \epsilon$ is an oriented embedding with respect to ν. □

4 The Obstruction Groups

4.1 A General Construction

Recall from [1] the following construction. Let I be a finite set, and let $C(I)$ be the set of maps $I \to \mathbf{Z}/2\mathbf{Z}$. Let $\sim$ be an equivalence relation on I. We denote by $C_\sim(I)$ the set of maps that are constant on the equivalence classes. Note that $C(I)$ and $C_\sim(I)$ are finite elementary abelian 2-groups.

4.2 An Example

This example will be used in Sect. 7. We say that two finite extensions K_1 and K_2 of a field K are *linearly disjoint* if the tensor product $K_1 \otimes_K K_2$ is a field. Let $E = \prod_{i \in I} E_i$

be a product of finite field extensions E_i of K, and let us consider the equivalence relation $\sim$ generated by the elementary equivalence

$$i \sim_e j \iff E_i \text{ and } E_j \text{ are linearly disjoint over } K.$$

Let $C_{\text{indep}}(E)$ be the quotient of $C_\sim(I)$ by the constant maps; this is a finite elementary abelian 2-group.

4.3 Commutative Étale Algebras with Involution

Let (E, σ) be a commutative étale L-algebra with involution such as in Sect. 2. Note that E is a product of fields, some of which are stable by σ, and the others come in pairs, exchanged by σ. Let us write $E = E' \times E''$, where $E' = \prod_{i \in I} E_i$ with E_i a field stable by σ for all $i \in I$, and where E'' is a product of fields exchanged by σ. With the notation of §2, we have $F = F' \times F''$, with $F' = \prod_{i \in I} F_i$, where F_i is the fixed field of σ in E_i for all $i \in I$. Note that $E'' = F'' \times F''$. For all $i \in I$, let $d_i \in F_i^\times$ be such that $E_i = F_i(\sqrt{d_i})$, and let $d = (d_i)$.

4.4 The Subsets $V_{i,j}$

Let V be a set, and for all $i, j \in I$ let $V_{i,j}$ be a subset of V. We consider the equivalence relation $\sim$ on I generated by the elementary equivalence $i \sim_e j \iff V_{i,j} \neq \varnothing$.

4.5 Global Fields

Assume that K is a global field.

For all $i \in I$, let V_i be the set of places $v \in V_K$ such that there exists a place of F_i above v that is inert or ramified in E_i. For all $i, j \in I$, set $V_{i,j} = V_i \cap V_j$, and let $\sim$ be the equivalence relation generated by the elementary equivalence $i \sim_e j \iff V_{i,j} \neq \varnothing$.

Let $C(E, \sigma)$ be the quotient of $C_\sim(I)$ by the constant maps; note that $C(E, \sigma)$ is a finite elementary abelian 2-group.

As a consequence of [4, Theorem 5.2.1], we show the following. Let (A, τ) be as in Sect. 2. For all $v \in V_K$, set $E^v = E \otimes_K K_v$ and $A^v = A \otimes_K K_v$.

Theorem 4.1 *Assume that for all $v \in V_K$, there exists an oriented embedding $(E^v, \sigma) \to (A^v, \tau)$, and that $C(E, \sigma) = 0$. Then there exists an embedding $(E, \sigma) \to (A, \tau)$.*

The proof is given in Sect. 6, as a consequence of Theorem 6.1.

5 Embedding Data

Let K be a global field, and let (E, σ) and (A, τ) be as above. The aim of this section is to recall some notions from [4] that we need in the following section.

We start by recalling from [4] the notion of embedding data. Assume that for all $v \in V_K$ there exists an embedding $(E^v, \sigma) \to (A^v, \tau)$. The set of $(a) = (a^v)$, with $a^v \in (F^v)^\times$, such that for all $v \in V_K$ we have $(A_v, \tau) \simeq (A_v, \theta_{a^v})$, is called a *local embedding datum*. This notion will be sufficient for our purpose if (A, τ) is unitary or split orthogonal; however, when (A, τ) is nonsplit orthogonal, we need the notion of *oriented local embedding data*, as follows.

Let us introduce some notation.

Notation 5.1 For K and F as above, and for $v \in V_K$, set $F^v = F \otimes_K K_v$, and we denote by $\mathrm{cor}_{F^v/K_v} : \mathrm{Br}(F^v) \to \mathrm{Br}(K_v)$ the corestriction map. Recall that we have a homomorphism $\mathrm{inv}_v : \mathrm{Br}(K_v) \to \mathbf{Q}/\mathbf{Z}$.

5.1 Oriented Embedding Data

Assume that (A, τ) is nonsplit orthogonal, and let $\nu : \Delta(E) \to Z(A, \tau)$ be an orientation. Suppose that for all $v \in V_K$ there exists an oriented embedding $(E^v, \sigma) \to (A^v, \tau)$. An *oriented embedding datum* will be an element $(a) = (a^v)$ with $a^v \in (F^v)^\times$ such that for all $v \in V_K$ there exists $\alpha^v \in (A^v)^\times$ such that $(\mathrm{Int}(\alpha) \circ \epsilon, a^v, \alpha^v, \nu)$ are parameters of an oriented embedding, and that moreover the following conditions are satisfied see [4, 4.1]):

- Let V' be the set of places $v \in V_K$ such that $\Delta(E^v) \simeq K_v \times K_v$. Then $\mathrm{cor}_{F^v/K_v}(a^v, d) = 0$ for almost all $v \in V'$.
- We have

$$\sum_{v \in V_K} \mathrm{cor}_{F^v/K_v}(a^v, d) = 0.$$

We denote by $\mathcal{L}(E, A)$ the set of oriented local embedding data (of course, the orientation is only required in the nonsplit orthogonal case - if (A, τ) is unitary or split orthogonal, then $\mathcal{L}(E, A)$ is by definition the set of local embedding data).

6 A Necessary and Sufficient Condition

Let K be a global field, and let (E, σ) and (A, τ) be as in the previous sections. The aim of this section is to reformulate the necessary and sufficient condition for the existence of embeddings in [4]; the only difference is a simpler description of the obstruction group. Suppose that for all $v \in V_K$ there exists an oriented embedding $(E^v, \sigma) \to (A^v, \tau)$, and let $(a) = (a_i^v) \in \mathcal{L}(E, A)$ be an oriented local embedding datum. For all $i \in I$, recall that $d_i \in F_i^\times$ is such that $E_i = F_i(\sqrt{d_i})$.

Let $C(E, \sigma)$ be the group defined in Sect. 4. We define a homomorphism $\rho = \rho_a : C(E, \sigma) \to \mathbf{Q}/\mathbf{Z}$ by setting

$$\rho_a(c) = \sum_{v \in V_K} \sum_{i \in I} c(i)\ \mathrm{inv}_v \mathrm{cor}_{F_i^v/K_v}(a_i^v, d_i).$$

We have the following

Theorem 6.1 (a) *The homomorphism ρ is independent of the choice of* $(a) = (a_i^v) \in \mathcal{L}(E, A)$.

(b) *There exists an embedding of algebras with involution* $(E, \sigma) \to (A, \tau)$ *if and only if* $\rho = 0$.

Proof As we will see, the theorem follows from [4, Theorems 4.4.1 and 4.6.1], and from the fact that the group $C(E, \sigma)$ above and the group $Ш(E', \sigma)$ of [4] are isomorphic, a fact we shall prove now. Note that part (a) of the theorem can also be deduced directly from [1, Proposition 13.6].

Let us recall the definition of $Ш(E', \sigma)$ from [4, 5.1 and §3]. Recall that $E' = \prod_{i \in I} E_i$ with E_i a field stable by σ, and $F' = \prod_{i \in I} F_i$, where F_i is the fixed field of σ in E_i for all $i \in I$. As in [4, §3], let Σ_i be the set of $v \in V_K$ such that all the places of F_i above v split in E_i; in other words, Σ_i is the complement of V_i in V_K. Let $m = |I|$. Given an m-tuple $x = (x_1, \ldots, x_m) \in (\mathbf{Z}/2\mathbf{Z})^m$, set

$$I_0 = I_0(x) = \{i \mid x_i = 0\},$$

$$I_1 = I_1(x) = \{i \mid x_i = 1\}.$$

Let S' be the set

$$S' = \{(x_1, \ldots, x_m) \in (\mathbf{Z}/2\mathbf{Z})^m \mid \left(\bigcap_{i \in I_0} \Sigma_i\right) \cup \left(\bigcap_{j \in I_1} \Sigma_j\right) = V_K\},$$

and let $S = S' \cup (0, \ldots, 0) \cup (1, \ldots, 1)$. Componentwise addition induces a group structure on S (see [4, Lemma 3.1.1]). We denote by $Ш(E', \sigma)$ the quotient of S by the subgroup generated by $(1, \ldots, 1)$.

We next show that the groups $Ш(E', \sigma)$ and $C(E, \sigma)$ are isomorphic. The first remark is that with the above notation, we have

$$S' = \{(x_1, \ldots, x_m) \in (\mathbf{Z}/2\mathbf{Z})^m \mid \left(\bigcup_{i \in I_0} V_i\right) \cap \left(\bigcup_{j \in I_1} V_j\right) = \varnothing\}.$$

Let us consider the map $F : (\mathbf{Z}/2\mathbf{Z})^m \to C(I)$ sending (x_i) to the map $c : I \to \mathbf{Z}/2\mathbf{Z}$ defined by $c(i) = x_i$. We have

$$S' = \{(x_1, \ldots, x_m) \in (\mathbf{Z}/2\mathbf{Z})^m \mid \left(\bigcup_{c(i)=0} V_i\right) \cap \left(\bigcup_{c(j)=1} V_j\right) = \varnothing\}.$$

Note that this shows that the following two properties are equivalent:

(1) $(x_i) \in S$;

(2) If $i, j \in I$ are such that $V_i \cap V_j \neq \varnothing$, then we have $c(i) = c(j)$.

Recall from §3 the definition of the group $C(E, \sigma)$. We consider the equivalence relation $\sim$ on I generated by the elementary equivalence $i \sim_e j \iff V_{i,j} \neq \varnothing$, and we denote by $C_\sim(I)$ the set of $c \in C(I)$ that are constant on the equivalence classes.

Since (1) $\implies$ (2), F sends S to $C_\sim(I)$. Moreover, F is clearly injective. Let us show that $F : S \to C_\sim(I)$ is surjective: this follows from the implication (2) $\implies$ (1). Hence we obtain an isomorphism of groups $S \to C_\sim(I)$, inducing an isomorphism of groups $Ш(E', \sigma) \to C(E, \sigma)$, as claimed.

The isomorphism $F : Ш(E', \sigma) \to C(E, \sigma)$ transforms $\overline{f} : Ш(E', \sigma) \to \mathbf{Q}/\mathbf{Z}$ defined in [4, 5.1 and 4.4] into the homomorphism $\rho : C(E, \sigma) \to \mathbf{Q}/\mathbf{Z}$ defined above. By [4, Theorem 4.4.1] (see also 5.1) the homomorphism $\overline{f}$ is independent of the choice of $(a) = (a_i^v) \in \mathcal{L}(E, A)$, and this implies part (a) of the theorem. Applying Theorem 4.6.1 and Proposition 5.1.1, we obtain part (b).

Proof of Theorem 4.1 This is an immediate consequence of Theorem 6.1. □

Remark 6.2 As part of the proof of Theorem 6.1, we showed that the groups $C(E, \sigma)$ and $Ш(E', \sigma)$ are isomorphic. Using this and [2, Section 2], we obtain a cohomological interpretation of $C(E, \sigma)$.

7 An Application—Linearly Disjoint Extensions

We keep the notation of the previous section; in particular, K is a global field. The following is a consequence of [1, Proposition 14.4]:

Proposition 7.1 *Assume that $E = E_1 \times E_2$, where E_1 and E_2 are linearly disjoint field extensions of K, both stable by σ. Then $C(E, \sigma) = 0$.*

Proof It suffices to show that $V_1 \cap V_2 \neq \varnothing$, and this is done in [1, Proposition 14.4]. We give the proof here for the convenience of the reader. Let Ω/K be a Galois extension containing E_1 and E_2, and set $G = \mathrm{Gal}(\Omega/L)$. Let $H_i \subset G_i$ be subgroups of G such that for $i = 1, 2$, we have $E_i = \Omega^{H_i}$ and $F_i = \Omega^{G_i}$. Since E_i is a quadratic extension of F_i, the subgroup H_i is of index 2 in G_i. By hypothesis, E_1 and E_2 are linearly disjoint over K, therefore $[G : H_1 \cap H_2] = [G : H_1][G : H_2]$. Note that F_1 and F_2 are also linearly disjoint over K, hence $[G : G_1 \cap G_2] = [G : G_1][G : G_2]$. This implies that $[G_1 \cap G_2 : H_1 \cap H_2] = 4$, hence the quotient $G_1 \cap G_2 / H_1 \cap H_2$ is an elementary abelian group of order 4.

The field Ω contains the composite fields F_1F_2 and E_1E_2. By the above argument, the extension E_1E_2/F_1F_2 is biquadratic. Hence there exists a place of F_1F_2 that is inert in both E_1F_2 and E_2F_1. Therefore there exists a place v of K and places w_i of F_i above v that are inert in E_i for $i = 1, 2$. □

Let (A, τ) be a central simple algebra as in the previous sections.

Corollary 7.2 *Assume that $E = E_1 \times E_2$, where E_1 and E_2 are linearly disjoint field extensions of K, both stable by σ, and suppose that for all $v \in V_K$ there exists an oriented embedding $(E^v, \sigma) \to (A^v, \tau)$. Then there exists an embedding of algebras with involution $(E, \sigma) \to (A, \tau)$.*

Proof This follows from Theorems 6.1 and Proposition 7.1. □

To deal with the case where E has more than two factors, we introduce a group $C_{\text{indep}}(E, \sigma)$. As in Sect. 4, we write $E = E' \times E''$, where $E' = \prod_{i \in I} E_i$ with E_i a field stable by σ for all $i \in I$, and where E'' is a product of fields exchanged by σ. Let $\approx$ be the equivalence relation $\approx$ on I generated by the elementary equivalence

$i \approx_e j \iff E_i$ and E_j are linearly disjoint over K.
We denote by $C_{\text{indep}}(E, \sigma) = C_{\text{indep}}(E')$ the group constructed in Sect. 4 using this equivalence relation.

Theorem 7.3 *Assume that $C_{\text{indep}}(E, \sigma) = 0$, and suppose that for all $v \in V_K$ there exists an oriented embedding $(E^v, \sigma) \to (A^v, \tau)$. Then there exists an embedding of algebras with involution $(E, \sigma) \to (A, \tau)$.*

Proof Recall that the group $C(E, \sigma)$ is constructed using the equivalence relation $\sim$ generated by the elementary equivalence $i \sim_e j \iff V_i \cap V_j \neq \varnothing$. Theorem 7.1 implies that if E_i and E_j are linearly disjoint over K, then $V_i \cap V_j \neq \varnothing$, hence $i \approx j \implies i \sim j$. By hypothesis, $C_{\text{indep}}(E, \sigma) = 0$, therefore $C(E, \sigma) = 0$; hence Theorem 4.1 implies the desired result. □

Corollary 7.4 *Assume that there exists $i \in I$ such that E_i and E_j are linearly disjoint over K for all $j \in I$, $j \neq i$. Suppose that for all $v \in V_K$ there exists an oriented embedding $(E^v, \sigma) \to (A^v, \tau)$. Then there exists an embedding of algebras with involution $(E, \sigma) \to (A, \tau)$.*

Proof Since there exists $i \in I$ such that E_i and E_j are linearly disjoint over K for all $j \in I$, $j \neq i$, the group $C_{\text{indep}}(E, \sigma)$ is trivial, hence the result follows from Theorem 7.3. □

Corollary 7.5 *Assume that the fields E_i are pairwise linearly disjoint over K. Suppose that for all $v \in V_K$ there exists an oriented embedding $(E^v, \sigma) \to (A^v, \tau)$. Then there exists an embedding of algebras with involution $(E, \sigma) \to (A, \tau)$.*

Proof This follows immediately from Corollary 7.4. □

8 An Application—Odd Degree Descent

We keep the notation of the previous sections: K is a global field, (E, σ) and (A, τ) are as before. The aim of this section is to prove the following:

Theorem 8.1 *There exists an embedding of (E, σ) into (A, τ) if and only if such an embedding exists over a finite extension of odd degree.*

In other words, if there exists a finite extension of odd degree K'/K such that $(E, \sigma) \otimes_K K'$ can be embedded into $(A, \tau) \otimes_K K'$, then (E, σ) can be embedded into (A, τ).

Remark 8.2 Note that the embeddings of algebras with involution considered here can be viewed as points on oriented embedding functors, which are homogeneous spaces under classical groups with connected stabilizer (see [12] and [2]). Therefore Theorem 8.1 gives an affirmative answer to Totaro's question on zero cycles of degree one (cf. [18, Question 0.2]) in our case. A more general statement is proved in Appendix.

We start by showing that there is a natural injective map from $C(E,\sigma)$ to $C(E\otimes_K K',\sigma)$. For each $i\in I$ let $F_i\otimes_K K'\simeq\prod_{j\in S(i)} F'_{i,j}$ where $F'_{i,j}$'s are extensions of K'. Define

$$I'=\{(i,j)|\ i\in I,\ \ j\in S(i) \text{ and the image of } d_i \text{ in } F'_{i,j} \text{ is not a square.}\}$$

Define a map π from I' to I by sending (i,j) to i.

Lemma 8.3 *The map π induces an injective map*

$$\pi^{\vee}: C(E,\sigma)\to C(E\otimes_K K',\sigma).$$

Proof It is sufficient to show that if $(i_1,k_1)\sim_e(i_2,j_2)$ in I', then $i_1\sim_e i_2$ in I. Suppose that $(i_1,k_1)\sim_e(i_2,j_2)$ in I'. By definition $V_{(i_1,j_1),(i_2,j_2)}\neq\emptyset$. Pick $w\in V_{(i_1,j_1),(i_2,j_2)}$ and let v be the restriction of w on K. Then $v\in V_{i_1,i_2}$. Hence $i_1\sim_e i_2$. □

Remark 8.4 One can take the cohomological point of view of the group $C(E,\sigma)$ (see also Remark 6.2). Then Lemma 8.3 follows immediately from the restriction-corestriction of Galois cohomology.

Lemma 8.5 *Assume that K is a local field. Let M be a local field which is an extension of K with odd degree. If there is an embedding of $(E,\sigma)\otimes_K M$ into $(A,\tau)\otimes_K M$, then there is an embedding of (E,σ) into (A,τ).*

Proof It is a consequence of the criteria of existence of local embeddings in [3]. Namely the conditions in [3, Theorems 2.1.1, 2.1.2, 2.2.1, 2.2.2, 2.3.1, 2.4.1, 2.4.2] hold over an odd degree extension M if and only if they hold over K. Hence if there is an embedding of $(E,\sigma)\otimes_K M$ into $(A,\tau)\otimes_K M$, then there is an embedding of (E,σ) into (A,τ). □

Proof of Theorem 8.1 We only prove the nontrivial direction. Let K'/K be an extension of odd degree. Suppose that there exists an embedding ι of $(E,\sigma)\otimes_K K'$ into $(A,\tau)\otimes_K K'$. By definition, ι is an oriented embedding with respect to the orientation u_ι associated to ι over K'. Since K'/K is of odd degree, u_ι descends to an orientation over K, which we still denote by u_ι. The embedding ι gives rise to an oriented embedding of $(E,\sigma)\otimes_K K'$ into $(A,\tau)\otimes_K K'$ with respect to the orientation u_ι everywhere locally. Since $[K':K]$ is odd, for all $v\in V_K$ there exists a place $w\in V_{K'}$ over v such that $[K'_w:K_v]$ is odd. By Lemma 8.5 there exists an embedding of $(E,\sigma)\otimes_K K_v$ into $(A,\tau)\otimes_K K_v$. Moreover if $[\mathcal{C}(A\otimes_K K'_w,\theta)]=[\mathcal{C}(A\otimes_K K'_w,\tau)]$ in $\mathrm{Br}(\Delta(E\otimes_K K'_w))$, then $[\mathcal{C}(A\otimes_K K_v,\theta)]=[\mathcal{C}(A\otimes_K K_v,\tau)]$ in $\mathrm{Br}(\Delta(E\otimes_K K_v))$. Proposition 3.1 implies that there exists an oriented embedding of (E,σ) into (A,τ)

locally everywhere. Let $a = (a^v)$ be an oriented embedding datum, let $c \in C(E, \sigma)$, and let

$$\rho_a(c) = \sum_{v \in V_K} \sum_{i \in I} c(i)\, \mathrm{inv}_v \mathrm{cor}_{F_i^v/K_v}(a_i^v, d_i)$$

be the associated homomorphism.

Let $v \in V_K$, and w be a place of K' over v. Let us take the extension of a^v in $F^v \otimes_{K_v} K'_w$to define $\rho_{K'}$. If $[K'_w : K_v]$ is odd, then

$$\mathrm{inv}_v \mathrm{cor}_{F_i^v/K_v}(a_i^v, d_i) = \mathrm{inv}_w \mathrm{cor}_{F_i^v \otimes_{K_v} K'_w/K'_w}(a_i^w, d_i).$$

If $[K'_w : K_v]$ is even, then $\mathrm{inv}_w \mathrm{cor}_{F_i^v \otimes_{K_v} K'_w/K'_w}(a_i^w, d_i) = 0$.

As K' is an odd degree extension of K, there is an odd number of places w of K' over v such that $[K'_w : K_v]$ is odd. Hence

$$\mathrm{inv}_v \mathrm{cor}_{F_i^v/K_v}(a_i^v, d_i) = \sum_{w|v} \mathrm{inv}_w \mathrm{cor}_{F_i^v \otimes_{K_v} K'_w/K'_w}(a_i^w, d_i).$$

For $c \in C(E, \sigma)$, let $c' = \pi^\vee(c)$. In other words $c'(i, j) = c(i)$. Let $a_{i,j}^{v'}$ be the image of a_i^v in $F'_{i,j} \otimes_{K'} K'_{v'}$. Then $(a_{i,j}^{v'})$ is an oriented embedding data over K'. Denote by $d_{i,j}$ the image of d_i in $F'_{i,j}$. Note that $F_i \otimes_K K' \otimes_{K'} K'_w \simeq F_i^v \otimes_{K_v} K'_w$. Therefore

$$\begin{aligned}
&\sum_{v' \in V_{K'}} \sum_{(i,j) \in I'} c'(i,j)\, \mathrm{inv}_{v'} \mathrm{cor}_{F_{i,j}^{v'}/K'_{v'}}(a_{i,j}^{v'}, d_{i,j}) \\
&\quad = \sum_{v' \in V_{K'}} \sum_{i \in I} c(i) \sum_{j \in S(i)} \mathrm{inv}_{v'} \mathrm{cor}_{F_{i,j}^{v'}/K'_{v'}}(a_{i,j}^{v'}, d_{i,j}) \\
&\quad = \sum_{v' \in V_{K'}} \sum_{i \in I} c(i)\, \mathrm{inv}_{v'} \mathrm{cor}_{F_i^v \otimes_{K_v} K'_{v'}/K'_{v'}}(a_i^{v'}, d_i) \\
&\quad = \sum_{v \in V_K} \sum_{i \in I} c(i)\, \mathrm{inv}_v \mathrm{cor}_{F^v/K_v}(a_i^v, d_i).
\end{aligned}$$

By hypothesis, $(E, \sigma) \otimes_K K'$ can be embedded into $(A, \tau) \otimes_K K'$, hence by Theorem 6.1, we have $\rho_{K'} = 0$. The above argument show that $\rho_K = 0$ as well, hence applying Theorem 6.1 again, we see that (E, σ) can be embedded into (A, τ). □

Appendix A

In this section, we give a positive answer to Totaro's question for homogeneous spaces under connected linear algebraic groups with connected stabilizers over number fields. Namely we show the following.

Proposition 9.1 *Let X be a homogeneous space over a number field under a connected linear algebraic group G with connected stabilizers. Suppose that X has a zero-cycle of degree one, then it has a rational point.*

The main ingredients to prove the above proposition are flasque resolutions for connected linear algebraic groups, due to Colliot-Thélène [6, Prop. 3.1] and the cohomological obstruction constructed by Borovoi (cf. [5, 1.3–1.5]).

We first fix the notation Let K be a number field and $\overline{K}$ be an algebraic closure of K. For a connected linear algebraic group G defined over K, we denote by G^u its unipotent radical. Let $G^{red} = G/G^u$ and denote by G^{ss} the derived group of G^{red}; $G^{tor} = G^{red}/G^{ss}$; $G^{ssu} = \ker(G \to G^{tor})$.

Recall the cohomological obstruction constructed in [5]. Let G be a connected linear algebraic K-group. Let X be a homogeneous space under G. Pick $x \in X(\overline{K})$ and let $\overline{M} = \mathrm{Stab}_{G_{\overline{K}}}(x)$. Suppose that G^{ss} is semi-simple simply connected and that $\overline{M}$ is connected. We construct a K-form M^{tor} of $\overline{M}^{tor}$ (see [5, 1.2]).

For each σ in $\mathrm{Gal}(\overline{K}/K)$, we choose $g_\sigma \in G(\overline{K})$ such that $g_\sigma \cdot {}^\sigma x = x$. For each σ, τ in $\mathrm{Gal}(\overline{K}/K)$, set $u_{\sigma,\tau} = g_{\sigma\tau}(g_\sigma {}^\sigma g_\tau)^{-1}$. Let $\hat{u}_{\sigma,\tau}$ be the image of $u_{\sigma,\tau}$ in $\overline{M}^{tor}(\overline{K})$, and $\hat{g}_\sigma$ be the image of g_σ in $\mathrm{G}^{tor}(\overline{K})$.

Let $M^{tor} \xrightarrow{i} G^{tor}$ be the complex with M^{tor} in degree -1 and G^{tor} in degree 1, where i is induced by the inclusion $\overline{M} \to G_{\overline{K}}$. We denote by $\eta(X)$ the class $[(\hat{u}, \hat{g})] \in H^1(K, M^{tor} \to G^{tor})$. If X has a K-point, then $\eta(X) = 0$ (see [5, 1.3–1.5]).

Let K' be a finite extension of K. Let (s, t) be a hypercocycle in

$$Z^1(K', M^{tor} \to G^{tor}).$$

Define $\mathrm{cor}_{K'/K}(s, t) = (\mathrm{cor}_{K'/K}(s), \mathrm{cor}_{K'/K}(t))$, where $\mathrm{cor}_{K'/K}(s)$ and $\mathrm{cor}_{K'/K}(t)$ are in the usual sense of corestriction on cochains in Galois cohomology (cf. [13, Chap. I, §5]).

Proposition 9.2 *Keep the notation as above. Then* $\mathrm{cor}_{K'/K}(s, t)$ *is a hypercocyle in* $Z^1(K, M^{tor} \to G^{tor})$.

Proof If s is a 2-cocycle in $Z^2(K', M^{tor})$, then $\mathrm{cor}_{K'/K}(s)$ is a cocycle in $Z^2(K, M^{tor})$. As i is defined over K, we have $i \circ \mathrm{cor}_{K'/K}(s) = \mathrm{cor}_{K'/K}(i \circ s)$. On the other hand, as (s, t) is a hypercocycle, we have $i \circ s = \partial t^{-1}$ and hence $\mathrm{cor}_{K'/K}(i \circ s) = \mathrm{cor}_{K'/K}(\partial t^{-1}) = \partial \mathrm{cor}_{K'/K}(t^{-1})$. The conclusion then follows. □

Since corestriction commutes with coboundary operators in Galois cohomology, by the above proposition we can define $\mathrm{cor}[(s, t)]$ as $[\mathrm{cor}(s, t)]$ in the hypercohomology $H^1(K, M^{tor} \to G^{tor})$.

Lemma 9.3 *Let* $[(s, t)] \in H^1(K, M^{tor} \to G^{tor})$ *and* K' *be a finite extension of* K. *Suppose that* G^{tor} *is quasi-trivial. Then*

$$\mathrm{cor}_{K'/K}\mathrm{res}_{K'/K}[(s, t)] = [K' : K][(s, t)].$$

Proof Consider the exact sequence of complexes

$$1 \longrightarrow (1 \to G^{tor}) \longrightarrow (M^{tor} \to G^{tor}) \longrightarrow (M^{tor} \to 1) \longrightarrow 1 \ .$$

From this we have the exact sequence

$$H^1(K, G^{tor}) = H^1(K, 1 \to G^{tor}) \longrightarrow H^1(K, M^{tor} \to G^{tor}) \\ \longrightarrow H^1(K, M^{tor} \to 1) = H^2(K, M^{tor}).$$

Using the above definition, we have

$$\mathrm{cor}_{K'/K}\mathrm{res}_{K'/K}[(s,t)] = [(\mathrm{cor}_{K'/K}\mathrm{res}_{K'/K}(s), \mathrm{cor}_{K'/K}\mathrm{res}_{K'/K}(t))].$$

From the restriction and corestriction in Galois cohomology (cf. [13, Chap. I, §5, Cor. 1.5.7]), we have that $\mathrm{cor}_{K'/K}\mathrm{res}_{K'/K}[(s,t)]$ and $[K' : K][(s,t)]$ have the same image in $H^2(K, M^{tor})$. Since G^{tor} is quasi-trivial, the group $H^1(K, G^{tor})$ is trivial. Thus $\mathrm{cor}_{K'/K}\mathrm{res}_{K'/K}[(s,t)] = [K' : K][(s,t)]$. □

Proof of Proposition 9.1 Let $1 \to S \to H \to G \to 1$ be a flasque resolution of G (see [6, 3.1]). We can regard X as a homogeneous space under H. Let $\overline{M}$ be the stabilizer of a geometric point $x \in X(\overline{K})$ in $G_{\overline{K}}$. Since $\overline{M}$ and S are connected, the preimage of $\overline{M}$ in $H_{\overline{K}}$ is also connected. By the definition of a flasque resolution we have H^{ss} is a semi-simple simply connected group and H^{tor} is quasi-trivial [6, 2.2]. We may therefore assume that G^{ss} is semi-simple simply connected with connected stabilizer and G^{tor} is quasi-trivial.

Let V_∞ be the set of infinite places of K. Since X has a zero-cycle of degree one, there is a K_v-point of X for all $v \in V_\infty$. By [5, Cor. 2.3], the space X has K-points if and only if $\eta(X) = 0$.

Since X has a zero-cycle of degree one, by applying the restriction and corestriction map on $H^1(K, M^{tor} \to G^{tor})$ and by Lemma 9.3 we have $\eta(X) = 0$. Thus X has a K-point.

Remark 9.4 If we replace the number field K with a global function field and assume that G is a reductive group, then G still has a flasque resolution [6, 3.1] and our arguments above still work in this case [7, §2].

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