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Diffusion-limited annihilating systems and the increasing convex order*

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Abstract

We consider diffusion-limited annihilating systems with mobile A -particles and stationary B -particles placed throughout a graph. Mutual annihilation occurs whenever an A -particle meets a B -particle. Such systems, when ran in discrete time, are also referred to as parking processes. We show for a broad family of graphs and random walk kernels that augmenting either the size or variability of the initial placements of particles increases the total occupation time by A -particles of a given subset of the graph. A corollary is that the same phenomenon occurs with the total lifespan of all particles in internal diffusion-limited aggregation.

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1 Introduction

We study a class of diffusion-limited annihilating systems (DLAS) in which A -particles diffuse across a graph interspersed with stationary B -particles. Mutual annihilation $A + B \rightarrow \emptyset$ occurs whenever opposite particle types meet. DLAS were introduced by physicists as toy models exhibiting anomalous kinetic behavior observed in more complicated reactions. The overarching finding was that spatial concentration fluctuations of reactants play a significant role in solute decay [GD96, TdALB⁺12]. Such fluctuations arise naturally in physical systems with thermal fluctuations [OZ78, TW83], turbulent flows [Hil76], and porous media [RK00].

For two-type systems, Bramson and Lebowitz gave an extensive rigorous analysis of the setting in which both particle types are mobile and diffuse at the same rate

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[BL88, BL90, BL91a, BL91b]. Cabezas, Rolla, and Sidoravicius made progress on two-type systems with asymmetric diffusion rates [CRS18]. More results concerning the limiting density of particles were obtained in [JJLS20, CJJ⁺21, PRS19].

The extreme case, in which B -particles are stationary, has also been studied by combinatorialists and probabilists under the name *parking* [KW66]. This comes from viewing A -particles as cars in search of B -particle spots. The impetus of our present work comes from results for parking on Galton-Watson trees [GP19, CH19, Con20, BBJ21, CG21]. These articles considered the setting with one B -particle per site along with an independent and identically distributed number of A -particles (any initially overlapping A - and B -particles are cancelled out). The graph is directed so that all A -particles move towards the root in discrete time. The total number of A -particles to visit the root has two phases: *transience* (finitely many visits almost surely) and *recurrence* (infinitely many visits with positive probability). Another version of a DLAS with stationary B -particles was introduced in [RSSS19], which studied simultaneous internal diffusion-limited aggregation on finite graphs. This model corresponds to a DLAS with a finite number of A -particles at the root of a connected graph and one B -particle at each nonroot vertex.

An interesting feature of parking on directed Galton-Watson trees is that the phase behavior depends on more than the average initial particle density. Curien and Hénard gave a precise characterization of the transience/recurrence phase behavior on critical Galton-Watson trees that involves the mean and variance of the number of A -particles as well as the variance of the offspring distribution for the tree [CH19]. The phase transition was later proven to be sharp by Contat [Con20].

There are no known explicit criteria for recurrence and transience for parking on supercritical Galton-Watson trees. However, Collett, Eckmann, Glaser, and Martin gave a precise value of the phase transition in terms of exponential moments for the process on binary trees with A -particles started exclusively from the leaves [CEGM83]. They were interested in these dynamics because of a connection to a spin glass model. For the usual parking process Bahl, Barnet, and Junge demonstrated through an example on d -ary trees that the phase state depends on more than just the mean initial density of particles [BBJ21, Proposition 7]. In an attempt to qualitatively describe this phenomenon [BBJ21, Theorem 8] further showed that the total number of visits to the root increases when the underlying placement of A -particles is made more volatile. An additional definition is needed to describe the result.

The *increasing convex order* (icx order) is a less commonly used stochastic ordering that rewards random variables for being larger or more volatile. We say that $X \preceq_{icx} Y$ for two random variables X and Y supported on the real numbers $\mathbb{R}$, if for all increasing convex functions $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ it holds that $\mathbf{E}\varphi(X) \leq \mathbf{E}\varphi(Y)$ provided the expectations exist. See [SS07] for a thorough discussion of the icx and related stochastic orders.

Note that the icx order is weaker than the *standard order* $X \preceq_{sd} Y$, which is defined as $\mathbf{P}(X \geq a) \leq \mathbf{P}(Y \geq a)$ for all a . An equivalent definition that more closely resembles the definition of the icx order is that $X \preceq_{sd} Y$ if and only if $\mathbf{E}\varphi(X) \leq \mathbf{E}\varphi(Y)$ for all increasing functions φ for which the expectations exist [SS07, (1.A.7)]. Another familiar definition of the standard order [SS07, Theorem 1.A.1] is that there exist two random variables $\hat{X}, \hat{Y}$ defined on the same probability space with $\hat{X} \stackrel{d}{=} X$ and $\hat{Y} \stackrel{d}{=} Y$ and $\mathbf{P}(X \leq Y) = 1$. (Here $\stackrel{d}{=}$ denotes distributional equality.) The icx order has an analogous formulation [SS07, Theorem 4.A.5] with the modification that $\{\hat{X}, \hat{Y}\}$ is a submartingale i.e., $\mathbf{E}[\hat{Y} \mid \hat{X}] \geq \hat{X}$.

As an example, suppose that $X = c$ with probability one for some constant $c \in \mathbb{R}$, while Y is any other random variable with mean c . Then $X \preceq_{icx} Y$ by Jensen's inequality. However, X and Y are not comparable in the standard order. Here are a few more

examples to keep in mind.

Example 1.1. Let X and Y be real-valued random variables with $X \preceq_{icx} Y$.

- (i) Since $x \mapsto x$ is convex and increasing, we have $\mathbf{E}X \leq \mathbf{E}Y$.
- (ii) If X and Y are nonnegative and $\mathbf{E}X = \mathbf{E}Y$, then the fact that $x \mapsto x^2$ is convex and increasing implies that $\text{var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 \leq \mathbf{E}Y^2 - (\mathbf{E}Y)^2 = \text{var}(Y)$.
- (iii) For $t \in [0, 1]$ the function $x \mapsto t^x$ is convex and decreasing. If X and Y are nonnegative, then $\mathbf{E}t^X \geq \mathbf{E}t^Y$. Taking the limit as $t \downarrow 0$ gives $\mathbf{P}(X = 0) \geq \mathbf{P}(Y = 0)$.
- (iv) Let $(X_T)_{T \geq 0}$ and $(Y_T)_{T \geq 0}$ be collections of nonnegative random variables supported on $[0, \infty)$ with pointwise limits $X = \lim_{T \rightarrow \infty} X_T$ and $Y = \lim_{T \rightarrow \infty} Y_T$ that exist almost surely. Suppose further that $X_T \preceq_{icx} Y_T$ for all $T \geq 0$. As in (iii), we have $\mathbf{E}t^{X_T} \geq \mathbf{E}t^{Y_T}$ for all $t \in [0, 1]$. It follows that

$$\mathbf{P}(X < \infty) = \lim_{t \uparrow 1} \lim_{T \rightarrow \infty} \mathbf{E}t^{X_T} \geq \lim_{t \uparrow 1} \lim_{T \rightarrow \infty} \mathbf{E}t^{Y_T} = \mathbf{P}(Y < \infty).$$

Equivalently, $\mathbf{P}(X = \infty) \leq \mathbf{P}(Y = \infty)$.

Let V (and V') be the number of A -particles that arrive to the root for parking on a directed Galton-Watson tree with an i.i.d. η -distributed (i.i.d. η' -distributed) A -particles initially at each site, one of which immediately cancels with the one B -particle also initially at the site. [BBJ21, Theorem 8] proved that if $\eta \preceq_{icx} \eta'$, then $V \preceq_{icx} V'$. Combining this with Example 1.1 (iv) implies that recurrence is preserved after increasing η in the icx order.

There are a few other results concerning particle systems and nonstandard stochastic orders. Johnson and Junge proved that the frog model, with $A + B \rightarrow 2A$ reactions so that mobile particles activate stationary particles, exhibits the opposite phenomena of parking: more volatility reduces root visits [JJ18]. Johnson and Rolla later proved via an example on regular trees that transience and recurrence of that model depends on more than the average density of the initial configuration [JR19]. A similar relationship between concentration and expansion occurs with the limiting size of the ball in first passage percolation [vdBK93, Mar02]. Recently, Hutchcroft studied the effect of a weaker stochastic order called the germ order on transience and recurrence sets for branching random walk [Hut22].

1.1 Statement of results

We extend [BBJ21, Theorem 8] to general graphs, initial particle configurations, and paths. We begin with a more formal construction that follows the notation from [CRS18]. Fix a locally finite graph G . We write $x \in G$ and $H \subseteq G$ to denote that x is an element and H is a subset of the vertex set of G . The systems we consider here are described by time-indexed counts at each vertex $\xi = (\xi_t(x))_{t \geq 0, x \in G}$. When $\xi_t(x) > 0$, it denotes the number of A -particles at x at time t . If $\xi_t(x) < 0$, then there are $|\xi_t(x)|$ many B -particles at x at time t .

We call $\xi_0 = (\xi_0(x))_{x \in G}$ the *initial conditions*. For each $x \in G$ and $j \in \mathbb{Z}^+$ let $S^{x,j} = (S_t^{x,j})_{t \geq 0} : [0, \infty) \rightarrow G$ be a right-continuous path with left limits started at x . We assign to the j th particle counted by $\xi_0(x)$ the path $S^{x,j}$. We further assign to the j th A -particle counted by $\xi_0(x)$ a *braveness* $h^{x,j} \in [0, 1]$ and to the j th B -particle braveness $h^{x,-j} \in [0, 1]$, to be used to break ties in deciding which particles to annihilate when multiple A - and B -particles end up at the same site. We assume that the bravenesses $(h^{x,j})_{x \in G, j \in \mathbb{Z} \setminus \{0\}}$ are distinct. Since B -particles do not have paths or anything else

to distinguish them, there is no real need to determine which B -particle on a site is annihilated, but it will be convenient to do so when we consider variant processes.

We call $(S, h) = (S_t^{x,j}, h^{x,j})_{x \in G, j \in \mathbb{Z} \setminus \{0\}, t \geq 0}$ the *instructions*. Particles follow their assigned trajectories and when one or more A -particles arrive to a site containing B -particles, the bravest of the A -particles mutually annihilate with B -particles until there are no remaining pairs of opposite type particles at the site. We say that (ξ_0, S, h) is *regular* if all of the following hold:

- The initial conditions $(\xi_0(x))$ are independent over $x \in G$; the instructions $(S^{x,j})$ and $(h^{x,j})$ are both independent over $x \in G$ and $j \in \mathbb{Z} \setminus \{0\}$; and the three collections of random variables are independent of each other as well.
- For $j > 0$, the $S^{x,j}$ are random walk paths with the same transition kernel, either entirely in continuous time (i.e., jumping at times given by a unit intensity Poisson process) or entirely in discrete time (i.e., jumping at positive integer times).
- For $j < 0$, the paths $S_t^{x,j} \equiv x$ so that B -particles are stationary.
- Each $h^{x,j}$ is a uniform random variable on $[0, 1]$.
- The instructions and initial conditions are such that the system is well defined and can be locally approximated by a system with finitely many particles, in the sense of Remark 1.2.

Remark 1.2. We say that (ξ_0, S, h) can be locally approximated by a system with finitely many particles if the following holds: Let $G_1 \subseteq G_2 \subseteq \dots \subseteq G$ be any sequence of finite subgraphs whose union is G . Let $H \subseteq G$ be a finite collection of vertices and let $(\xi_t^{(n)}(x))_{t \geq 0, x \in G}$ be DLAS defined by (S, h) with initial conditions $\xi_0(x) \mathbf{1}\{x \in G_n\}$. Then the sequence $(\xi_t^{(n)}(x))_{0 \leq t \leq T, x \in H}$ indexed by n is almost surely eventually constant, and its limit does not depend on the choice of $G_1, G_2, \dots$.

The point of this rather technical condition is simply to avoid pathological examples. A system defined by (ξ_0, S, h) may not yield a well-defined DLAS $(\xi_t(x))_{t \geq 0, x \in G}$ if, for example, blow-up occurs and infinitely many particles move to a single vertex in a finite amount of time. It is proven in [CRS18, Appendix A] that local approximation holds when the initial conditions satisfy $\sup_{x \in G} \mathbf{E} \xi_0(x) < \infty$ and the paths are random walks on any graph with a transitive unimodular group of automorphisms that commutes with the probability kernel of the random walk. It is not hard to show that it holds for a DLAS such that a system of noninteracting random walks with the same initial conditions has finitely many visits in finite time to any site.

Let $H \subseteq G$. Define the *occupation time of H by A -particles up to time $T \geq 0$* as

$$V_T(H, \xi_0, S, h) = V_T = \sum_{x \in H} \int_0^T \mathbf{1}\{\xi_t(x) > 0\} \xi_t(x) dt. \quad (1.1)$$

We set $V_T = \infty$ whenever $\sum_{x \in H} \xi_0(x) \mathbf{1}\{\xi_0(x) > 0\} = \infty$. To avoid this pathology we assume that either H or the number of A -particles in the system is finite:

$$\min \{ |H|, \sum_{x \in G} \xi_0^+(x) \} < \infty \text{ almost surely.} \quad (1.2)$$

The prototypical choice is $H = \{x\}$, so that V_T measures the occupancy time of a single site. Another interesting choice is $H = G$ so that V_T equals the aggregate time all of the A -particles are in motion. This case is only meaningful when (1.2) holds. We further remark that (1.2) along with the requirement $\sup_{x \in G, t \geq 0} \mathbf{E} |\xi_t(x)| < \infty$ ensure that V_T is almost surely finite for all $T > 0$.

Given another DLAS ξ' define V'_T to be the analogue of (1.1) for ξ' . A useful condition for comparing two systems (ξ_0, S, h) and (ξ'_0, S', h') is

$$(\xi_0, S, h) \text{ and } (\xi'_0, S', h') \text{ are regular with } S \stackrel{d}{=} S' \text{ and } h \stackrel{d}{=} h'. \quad (1.3)$$

We write $\xi_0 \preceq_{icx} \xi'_0$ if $\xi_0(x) \preceq_{icx} \xi'_0(x)$ for all $x \in G$.

Theorem 1.3. Fix $H \subseteq G$. Suppose that (1.2) and (1.3) hold. If $\xi_0 \preceq_{icx} \xi'_0$, then $V_T \preceq_{icx} V'_T$ for all $T \geq 0$.

Proof. The result follows when $\sum_{x \in G} |\xi_0(x)| < \infty$ and $\sum_{x \in G} |\xi'_0(x)| < \infty$ from Proposition 2.2 and Proposition 3.1. If there are infinitely many A -particles present, then the finite approximation assumption in the definition of regularity ensures that V_T is realized as the limit as $R \rightarrow \infty$ of the systems that only contain the particles from ξ_0 that lie in the balls $\cup_{x \in H} B(x, R)$. Taking such a limit gives the statement for V_T when the initial configuration contains infinitely many particles. $\square$

Our proof comes in two parts. First we show that V_T is an *icx statistic* (see Definition 2.1 and Proposition 2.2), and then we show a generalized version of Theorem 1.3 that holds for all icx statistics. This opens the door to establishing that other statistics besides V_T respect the icx order; see Section 4 for more discussion.

The analogue of Theorem 1.3 for the standard order holds immediately by monotonicity of the process with respect to its initial conditions [CRS18, Lemma 3]. Because our result uses the weaker icx order, it applies more broadly. In particular, two different distributions with the same mean can be comparable in the icx order but are never comparable in the standard order.

To illustrate our theorem and begin our discussion of why the icx order is natural to consider, we give an example where $\xi_0 \preceq_{icx} \xi'_0 \preceq_{icx} \xi''_0$ and the conclusion $V_T \preceq_{icx} V'_T \preceq_{icx} V''_T$ can be seen by direct calculation.

Example 1.4. Set $G = \mathbb{Z}$ and $H = 0$. Let S consist of discrete symmetric nearest neighbor random walk paths. Now, we define systems in this environment with three different sets of initial conditions. The DLAS ξ starts with one B -particle at position 1 and one A -particle at position 2. For ξ' , we place either zero or two B -particles at position 1 with equal probability and one A -particle at position 2. And ξ'' begins with 0 or 2 B -particles at 1 with equal probability and 0 or 2 A particles at position 1 with equal probability. In all three systems, there are no particles initially outside of positions 1 and 2. To summarize,

- $\xi_0(1) = -1, \xi_0(2) = 1$
- $\xi'_0(1) = \begin{cases} 0 & \text{with probability } 1/2, \\ -2 & \text{with probability } 1/2, \end{cases}$ and $\xi'_0(2) = 1$,
- $\xi''_0(1) = \begin{cases} 0 & \text{with probability } 1/2, \\ -2 & \text{with probability } 1/2, \end{cases}$ and $\xi''_0(2) = \begin{cases} 0 & \text{with probability } 1/2, \\ 2 & \text{with probability } 1/2; \end{cases}$
- and $\xi_0(x) = \xi'_0(x) = \xi''_0(x) = 0$ for $x \notin \{1, 2\}$.

We have $\mathbf{E}\xi_0(x) = \mathbf{E}\xi'_0(x) = \mathbf{E}\xi''_0(x)$ for all $x \in G$, and Jensen's inequality confirms the intuitively obvious fact that ξ_0, ξ'_0 , and ξ''_0 are increasingly volatile, i.e., $\xi_0 \preceq_{icx} \xi'_0 \preceq_{icx} \xi''_0$. Observe that $V_T = 0$ a.s., while

$$V'_T = \begin{cases} 0 & \text{with probability } 1/2, \\ L_T & \text{with probability } 1/2, \end{cases}$$

and

$$V_T'' = \begin{cases} 0 & \text{with probability } 3/4, \\ L_T + L_T' & \text{with probability } 1/4, \end{cases}$$

where L_T and L_T' are the local times at 0 up to time T of two independent random walks started at 2. We leave it as an exercise to compute directly that $V_T \preceq_{icx} V_T' \preceq_{icx} V_T''$ (we note that no knowledge of the distribution of L_T is needed). We also observe that $\mathbf{E}V_T < \mathbf{E}V_T' = \mathbf{E}V_T''$, demonstrating that the expectation of V_T can but need not strictly increase when the initial conditions increase in the icx order.

The reason that V_T is increasing with respect to changes in the initial configuration in the icx order is because of a convexity property inherent to the model. Essentially, the gain to V_T when adding two A -particles to the system is more than the sum of gains from each A -particle alone. As we can see in the previous example, this is because one A -particle can help the other by eliminating obstacles on its behalf. More discussion is presented at the start of Section 1.3.

1.2 Applications

Theorem 1.3 applies to the systems considered in [CH19, DGJ⁺19, CJJ⁺21, JJLS20, BBJ21, CRS18, RSSS19]. This includes a broad class of transitive unimodular graphs and Galton-Watson trees. Such graphs are good to keep in mind, but we note that significantly more general graphs are also covered since we require minimal regularity.

One application of Theorem 1.3 that follows from Example 1.1 (i), (iii), and (iv) is that increasing the initial condition in the icx order has the following effects.

Corollary 1.5. *Fix $H \subseteq G$. If (1.2) and (1.3) hold and $\xi_0 \preceq_{icx} \xi_0'$ then:*

- (i) $\mathbf{E}V_T \leq \mathbf{E}V_T'$ for all $T \geq 0$,
- (ii) $\mathbf{P}(V_\infty = \infty) \leq \mathbf{P}(V'_\infty = \infty)$, and
- (iii) $\mathbf{P}(V_T = 0) \geq \mathbf{P}(V'_T = 0)$ for all $T \geq 0$.

To make Theorem 1.3 more concrete we derive as a consequence that taking $\xi_0(x)$ to be independent ± 1 -valued random variables minimizes V_T in the icx order across all other initial configurations with the same mean number of particles at each site. Note that such initial configurations are used in [DGJ⁺19, PRS19, JJLS20, GP19, BBJ21].

Corollary 1.6. *Fix a graph G , a finite subset $H \subseteq G$, and possibly distinct values $p_x \in [0, 1]$ for each $x \in G$. Let $U(x)$ be independent random variables uniformly distributed on $(0, 1)$. Let $\alpha(x)$ and $\beta(x)$ be independent and identically distributed random variables on the nonnegative integers with mean 1. For all $x \in G$ define*

$$\begin{aligned} \xi_0(x) &= \mathbf{1}\{U(x) \leq p_x\} - \mathbf{1}\{U(x) > p_x\} \\ \xi_0'(x) &= \mathbf{1}\{U(x) \leq p_x\}\alpha(x) - \mathbf{1}\{U(x) > p_x\}\beta(x) \end{aligned}$$

Suppose that (ξ_0, S, h) and (ξ_0', S, h) are regular and let V and V' be the total occupation time of H for the two systems, respectively. It holds that $V \preceq_{icx} V'$.

Proof. The results [JJ18, Proposition 15 (b)] and [SS07, Theorem 4.2.A] are easily adapted to prove that

$$\begin{aligned} \mathbf{1}\{U(x) \leq p_x\} &\preceq_{icx} \mathbf{1}\{U'(x) \leq p_x\}\alpha'(x) \\ -\mathbf{1}\{U(x) > p_x\} &\preceq_{icx} -\mathbf{1}\{U'(x) > p_x\}\beta'(x). \end{aligned}$$

Summing both sides and applying closure under mixtures of the icx order [SS07, Theorem 4.A.8 (b)] gives that $\xi_0(x) \preceq_{icx} \xi_0'(x)$. Theorem 1.3 implies $V \preceq_{icx} V'$. $\square$

As mentioned below (1.1), the case $H = G$ gives the aggregate time A -particles are in motion. Suppose further that G is a rooted graph with all non-root sites initially containing one B -particle and $n \geq 0$ A -particles at the root. Using coupling methods different from ours, the quantity V_T in this setup, referred to as W in [RSSS19], was shown in [RSSS19, Lemma 4.4] to have the same distribution as $L(n)$ the combined path length of the particles in n -iterations of the usual, sequential version of internal diffusion-limited aggregation [LBG92, JLS12]. The authors of [RSSS19] noted that this equivalence “motivates the study of W for general graphs...” In this line, we obtain the following consequence of Theorem 1.3.

Corollary 1.7. *Let $L(n)$ be the combined lifespans of n particles executing internal diffusion-limited aggregation from the root of a given graph. Let η and η' be nonnegative random variables. If $\eta \preceq_{icx} \eta'$, then $L(\eta) \preceq_{icx} L(\eta')$.*

1.3 Proof overview

The first and main part of the proof is Proposition 2.2, where we show V_T is an *icx statistic* (see Definition 2.1). The essence of the definition is that the statistic increases with the addition of a single A -particle (condition (d)), and that the increase from adding two A -particles at once is greater than the increase from adding two A -particles individually in two separate systems (condition (e)). Our proof amounts to confirming the intuition that one of the extra particles may clear space for the other by annihilating with a B -particle that would destroy each particle separately in individual augmented systems. This property helps explain why greater volatility in the initial configuration causes an icx statistic to increase. After all, if the gain from adding two particles is more than twice that of adding one, we would rather have two particles or none at a site with probability $1/2$ than one particle with probability 1 as illustrated in Example 1.4.

To establish these properties formally, we introduce couplings of systems with extra A -particles. Our approach is inspired by the *tracer* construction from [CRS18] but differs slightly. In light of conditions (d) and (e) from Definition 2.1, we need a way to compare what happens in systems with extra A -particles. If we simply add two A -particles to the system and let all other particles proceed with their original instructions, we obtain a coupling satisfying (d), the monotonicity requirement, but not (e), the convexity requirement. To make the convexity requirement hold, we must prioritize one of the added A -particles over the other, which is where our paper diverges from [CRS18]. For example, it may happen that one of the extra A -particles, call it X , destroys a B -particle at x , thus allowing the other extra A -particle, call it Y , to survive a later visit to x . When Y arrives to x the law of the DLAS is preserved whether we allow Y to continue its assigned path, or if we instead have it pick up and extend the path of X . We call this process either the *tracer system* or the *flipped tracer system* depending on which of the two extra A -particles we prioritize. We will leave the details to the next section, but from the tracer system we track the effect of adding Y after X , while from the flipped tracer system we track the effect of adding Y alone. We then prove in Lemma 2.5 that Y traverses more of its preassigned path if it is added after X , thus establishing (e). We note that this comparison fails when B -particles move, as we discuss in Remark 2.8.

The second part of the proof is to show that icx statistics are monotone under changes to the initial conditions in the icx order. We do this under the assumption that the system has only finitely many particles (Proposition 3.1). From there it is straightforward to extend the result to infinite systems.

The overall structure of the proof is similar to the one used in [JJ18] to prove a comparison result for the frog model. Some differences are that: (a) here we consider a two-type particle system, (b) we use the icx order rather than the increasing concave

(icv) order, and (c) the definition of icx statistic is more complicated than the analogous definition in [JJ18].

To address difference (a), we make use of the basic equivalence between removing B -particles as adding A -particles. So, it is sufficient to restrict our focus to the impact of adding A -particles as discussed in the previous paragraph. This makes for no technical difficulty. Difference (b) is a consequence of differences between DLAS and the frog model. Superadditive statistics (see Definition 2.1) behave well for DLAS as opposed to subadditive ones for the frog model. But this difference barely affects the proofs because the icx and icv orders are interchangeable via [SS07, Theorem 3.A.1].

Difference (c) is more substantial. For all icv statistics considered in [JJ18], it is easy to prove that they are so. In contrast, establishing that V_T is an icx statistic is most of the work of this paper. The difference is that statistics in the frog model behave subadditively when adding particles to the system in the most straightforward way. On the other hand, the statistics considered in this paper behave superadditively only when particles are added via the complicated coupling described in Section 2.

We further remark that [JJ18] considers a weaker stochastic order, the *probability generating function order* (pgf order), alongside the icv order. We say that X is dominated by Y in the pgf order if the probability generating function $\mathbf{E}t^X$ of X is pointwise at least as large as that of Y for $t \in [0, 1]$. With more technical difficulty, it is proven in [JJ18] that the number of visits to the root in the frog model also respects this stochastic order. The analogous generalization to the convex version of the pgf order for Theorem 1.3 would be to prove that V_T respects the stochastic order defined by $\mathbf{E}\varphi(X) \leq \mathbf{E}\varphi(Y)$ holding for all smooth functions φ with all derivatives positive. It is quite possible that this is true and could be proven by the methods of this paper, but we have not attempted it since there is no obvious application of the result.

2 Icx statistics and tracer couplings

We begin with a definition that we will relate to the icx order in the next section. Throughout this section we assume that any DLAS under discussion has finitely many A - and B -particles in the initial configuration.

Definition 2.1. Let (ξ_0, S, h) be regular with $\sum_{x \in G} |\xi_0(x)| < \infty$ a.s. Let $\xi_{0,k}$ be the same as ξ_0 except that $\xi_{0,k}(x) = k$ for some given $x \in G$. We call a functional f an icx statistic if for all $x \in G$ and $k \in \mathbb{Z}$ there exists a coupling $(\Phi, \Phi^X, \Phi^Y, \Phi^{X,Y})$ such that

- (a) $\Phi \stackrel{d}{=} f(\xi_{0,k}, S, h)$
- (b) $\Phi^X \stackrel{d}{=} \Phi^Y \stackrel{d}{=} f(\xi_{0,k+1}, S, h)$
- (c) $\Phi^{X,Y} \stackrel{d}{=} f(\xi_{0,k+2}, S, h)$
- (d) $\Phi^X \geq \Phi$ a.s. and $\Phi^Y \geq \Phi$ a.s.
- (e) $\Phi^{X,Y} - \Phi^X - \Phi^Y + \Phi \geq 0$ a.s.

The purpose of this section is to prove the following:

Proposition 2.2. V_T defined at (1.1) is an icx statistic.

2.1 The tracer system and the flipped tracer system

We now introduce the coupling we use to track the effect of adding extra particles at a given vertex x . We start with a regular system (ξ_0, S, h) . Let $k = \max(\xi_0(x) + 1, 1)$, and let $X = S^{x,k}$ and $Y = S^{x,k+1}$. We will describe two modified versions of the underlying DLAS where we add particles with paths X and Y with special behavior.

In both systems, start with the initial conditions given by ξ_0 together with two special particles at a given site x that we call the X -tracer and the Y -tracer. Each tracer has two possible states, A and B . The nontracer particles follow their assigned instructions from (S, h) . The X - and Y -tracers follow paths X and Y , respectively, when they are in state A . When a tracer enters state B , it pauses and remains stationary until it enters state A again, at which point it continues following its path X or Y starting from when it was paused.

In the *tracer system*, we assign the X -tracer braveness -2 and the Y -tracer braveness -1 ; that is, the Y -tracer has lower braveness than all other particles except for the X -tracer, which has the lowest braveness. In the *flipped tracer system*, we assign the X -tracer braveness -1 and the Y -tracer braveness -2 . The tracer system gives the X -tracer priority to be in state A , while the flipped tracer system prioritizes the Y -tracer. In both systems tracers start out in state A .

In the following interaction rules for the system's evolution, we treat a tracer as if were an A -particle when it is in state A and as a B -particle when it is in state B . If A -particles and B -particles finds themselves together on a site, then the bravest A -particle and the bravest B -particle interact as follows:

- (a) If neither particle is a tracer, then they interact as usual by mutual annihilation.
- (b) If the A -particle is a tracer and the B -particle is not, then the B -particle annihilates and the tracer particle switches to state B .
- (c) If the B -particle is a tracer and the A -particle is not, then the A -particle annihilates and the tracer particle switches to state A .
- (d) If both particles are tracers, then the Y -tracer takes priority to be in state A in the tracer system, while the X -tracer takes priority to be in state A in the flipped tracer system. That is, in the tracer system, if the X -tracer is in state A and the Y -tracer is in state B , then both tracers switch states; if the X -tracer is in state B and the Y -tracer is in state A , then no interaction occurs.

The interactions repeat until there are no A - and B -particles together on the site except possibly the two tracers with the prioritized tracer in state A and the nonprioritized tracer in state B . Both tracers start in state A when $\xi_0(x) \geq 0$. If $\xi_0(x) = -1$, then the X -tracer starts in state B and the Y -tracer starts in state A , and the B -particle at x is annihilated at time 0. If $\xi_0(x) \leq -2$, then both tracers start in state B , and the two B -particles at x with the greatest braveness are annihilated at time 0.

In the tracer system, let A_t^X and A_t^Y be the events that the X - and Y -tracers, respectively, are in state A at time t . Let B_t^X and B_t^Y be the complements of A_t^X and A_t^Y . We define the *life* of each tracer up to time T as

$$L_T^X = \int_0^T \mathbf{1}\{A_t^X\} dt \quad \text{and} \quad L_T^Y = \int_0^T \mathbf{1}\{A_t^Y\} dt.$$

These quantities represent the duration of the X and Y paths that each tracer has traveled along up to time T . Observe that the locations of the tracers at time t are $X_{L_t^X}$ and $Y_{L_t^Y}$. Let $\hat{A}_t^X$, $\hat{A}_t^Y$, $\hat{B}_t^X$, and $\hat{B}_t^Y$ be the analogous events for the flipped tracer system, and let the life of each tracer up to time t in the flipped system be defined analogously and denoted by $\hat{L}_t^X$ and $\hat{L}_t^Y$.

In the tracer system let $\alpha_t(z)$ and $\beta_t(z)$ denote the number of nontracer A - and

B -particles, respectively, present on site z at time $t > 0$. We then define

$$\zeta_t(z) = \alpha_t(z) - \beta_t(z) - \mathbf{1}\{B_t^X\}\mathbf{1}\{X_{L_t^X} = z\} - \mathbf{1}\{B_t^Y\}\mathbf{1}\{Y_{L_t^Y} = z\}, \quad (2.1)$$

$$\zeta_t^X(z) = \alpha_t(z) - \beta_t(z) + \mathbf{1}\{A_t^X\}\mathbf{1}\{X_{L_t^X} = z\} - \mathbf{1}\{B_t^Y\}\mathbf{1}\{Y_{L_t^Y} = z\}, \quad (2.2)$$

$$\zeta_t^{X,Y}(z) = \alpha_t(z) - \beta_t(z) + \mathbf{1}\{A_t^X\}\mathbf{1}\{X_{L_t^X} = z\} + \mathbf{1}\{A_t^Y\}\mathbf{1}\{Y_{L_t^Y} = z\}. \quad (2.3)$$

In short, ζ_t gives the particle counts for the tracer system with the tracer particles ignored in state A and counted as B -particles in state B . Then ζ_t^X does the same except it counts the X -tracer only in state A , while $\zeta_t^{X,Y}$ counts both tracers only while in state A .

Finally, with $\hat{\alpha}_t(z)$ and $\hat{\beta}_t(z)$ denoting the counts of nontracer A - and B -particles at site z in the flipped tracer system, we define

$$\hat{\zeta}_t(z) = \hat{\alpha}_t(z) - \hat{\beta}_t(z) - \mathbf{1}\{\hat{B}_t^Y\}\mathbf{1}\{Y_{\hat{L}_t^Y} = z\} - \mathbf{1}\{\hat{B}_t^X\}\mathbf{1}\{X_{\hat{L}_t^X} = z\}, \quad (2.4)$$

$$\hat{\zeta}_t^Y(z) = \hat{\alpha}_t(z) - \hat{\beta}_t(z) + \mathbf{1}\{\hat{A}_t^Y\}\mathbf{1}\{Y_{\hat{L}_t^Y} = z\} - \mathbf{1}\{\hat{B}_t^X\}\mathbf{1}\{X_{\hat{L}_t^X} = z\}, \quad (2.5)$$

$$\hat{\zeta}_t^{Y,X}(z) = \hat{\alpha}_t(z) - \hat{\beta}_t(z) + \mathbf{1}\{\hat{A}_t^Y\}\mathbf{1}\{Y_{\hat{L}_t^Y} = z\} + \mathbf{1}\{\hat{A}_t^X\}\mathbf{1}\{X_{\hat{L}_t^X} = z\}. \quad (2.6)$$

Note the reversal of the roles of X and Y in (2.4)–(2.6) as compared to (2.1)–(2.3).

2.2 Identities

We write ξ^x for the system with the initial condition at x increased by one A -particle so that $\alpha_0^x(x) = \alpha_0(x) + 1$, $\alpha_0^x(z) = \alpha_0(z)$ for $z \neq x$, and $\beta_0^x = \beta_0$. Let $\xi^{x,x} = (\xi^x)^x$. The relevance of the tracer system we have defined is that ζ , ζ^X , $\zeta^{X,Y}$, $\hat{\zeta}$, $\hat{\zeta}^Y$, and $\hat{\zeta}^{Y,X}$ all represent particle counts for DLAS, as we will see in Proposition 2.3. Both ζ and $\hat{\zeta}$ turn out to be identical to ξ . The counts ζ^X and $\hat{\zeta}^Y$ are both instances of DLAS with initial configuration ξ_0^x , but in ζ^X the extra particle compared to ξ follows path X , while in $\hat{\zeta}^Y$ it follows path Y (see Lemma 2.5). Meanwhile $\zeta^{X,Y}$ and $\hat{\zeta}^{Y,X}$ are both instances of DLAS with initial configuration $\xi_0^{x,x}$, but they will differ slightly from each other because of prioritizing paths X and Y differently.

Proposition 2.3.

- (I) $\zeta_t(z) = \hat{\zeta}_t(z) = \xi_t(z)$ for all $t > 0$ and $z \in G$;
- (II) $(\zeta_t^X(z))_{t>0, z \in G} \stackrel{d}{=} (\hat{\zeta}_t^Y(z))_{t>0, z \in G} \stackrel{d}{=} (\xi_t^x(z))_{t>0, z \in G}$;
- (III) $(\zeta_t^{X,Y}(z))_{t>0, z \in G} \stackrel{d}{=} (\hat{\zeta}_t^{Y,X}(z))_{t>0, z \in G} \stackrel{d}{=} (\xi_t^{x,x}(z))_{t>0, z \in G}$.

Proof. We start by proving $\zeta^X \stackrel{d}{=} \xi^x$ and $\zeta^{X,Y} \stackrel{d}{=} \xi^{x,x}$. First, observe that ζ^X , and $\zeta^{X,Y}$ have initial configurations matching ξ^x and $\xi^{x,x}$. Next, we consider the effect of a jump in the tracer system from the perspective of ζ^X and $\zeta^{X,Y}$. All possibilities when a nontracer A -particle jumps are depicted in Figure 1 (located in the Appendix). Note that if the random walk paths are in discrete time and multiple A -particles arrive simultaneously, what is depicted is the last collision to be resolved with the least brave arriving A -particle. Examining the figure, we see that when such a jump occurs, the counts given by ζ_t^X and $\zeta_t^{X,Y}$ evolve according to the rules of DLAS. The possibilities when a tracer particle in state A jumps are shown in Figure 2 (located in the Appendix). Again, in all cases the counts given by ζ_t^X and $\zeta_t^{X,Y}$ evolve according to the rules of DLAS; the only difference with a tracer particle jumping is that in some of the cases for ζ^X , a jump results in no change. Since ζ^X , and $\zeta^{X,Y}$ have initial distributions matching ξ^x and $\xi^{x,y}$ and evolve according to the same rules, their laws match as well.

Next, we show that ζ and ξ are equal a.s. We do so by showing that the particle system counted by ξ at all times matches the nontracer particles in the tracer system along with an additional B -particle at the location of any tracer particle in state B . This is initially true by definition of the tracer process. We claim that after every jump, it continues to hold. Indeed, we just check that this is true for each interaction type (a)–(d) from the definition of the tracer system.

Finally, we observe that $\hat{\zeta}$, $\hat{\zeta}^Y$, and $\hat{\zeta}^{Y,X}$ are defined identically to ζ , ζ^X , and $\zeta^{X,Y}$ except that the roles of X and Y are reversed. Thus $\zeta \stackrel{d}{=} \hat{\zeta}$, $\zeta^X \stackrel{d}{=} \hat{\zeta}^Y$, and $\zeta^{X,Y} \stackrel{d}{=} \hat{\zeta}^{Y,X}$, since the paths X and Y are i.i.d. And the almost sure equality of $\hat{\zeta}$ and ξ holds by the same proof as for ζ and ξ . $\square$

Remark 2.4. The tracer and flipped tracer systems are closely related to the *dragged tracer* construction of [CRS18, Section 4.1]. In the terminology of [CRS18], a tracer in state A is *following an A -particle* and a tracer in state B is *following a B -particle*. The difference between our construction and the one in [CRS18] is rule (d), the prioritization of one tracer over the other. In the dragged tracer construction, a tracer following an A -particle does not interact with one following a B -particle. Using this construction, parts (I) and (III) of Proposition 2.3 hold, but part (II) fails. Essentially, our construction allows us to simultaneously couple the systems with zero, one, and two particles added at a site. Without it, we can couple any two of these systems, but not all three together.

The tracers are so-called because they track the differences between these systems:

Lemma 2.5. For all $t > 0$ and $z \in G$,

$$\begin{aligned}\zeta_t^X(z) - \zeta_t(z) &= \mathbf{1}\{X_{L_t^X} = z\}, & \zeta_t^{X,Y}(z) - \zeta_t(z) &= \mathbf{1}\{X_{L_t^X} = z\} + \mathbf{1}\{Y_{L_t^Y} = z\}, \\ \hat{\zeta}_t^Y(z) - \hat{\zeta}_t(z) &= \mathbf{1}\{Y_{\hat{L}_t^Y} = z\}, & \hat{\zeta}_t^{Y,X}(z) - \hat{\zeta}_t(z) &= \mathbf{1}\{Y_{\hat{L}_t^Y} = z\} + \mathbf{1}\{X_{\hat{L}_t^X} = z\}.\end{aligned}$$

Proof. These facts are direct consequences of definitions (2.1)–(2.3) and (2.4)–(2.6). $\square$

Lemma 2.5 highlights that the tracer system tracks what happens when we add the X -tracer first and the Y -tracer second, while the flipped tracer system tracks what happens when the tracers are added in the opposite order. To prove that the monotonicity condition (e) from Definition 2.1 holds, we use the tracer system to track the effect of adding the Y -tracer after the X -tracer is added, and we use the flipped tracer system to track the effect of adding the Y -tracer alone. The following lemma is the basis of comparison between these two effects:

Lemma 2.6. It holds for all $t > 0$ that $L_t^Y \geq \hat{L}_t^Y$.

Proof. Let $T = \inf\{t: L_t^Y < \hat{L}_t^Y\}$ and suppose by way of contradiction it is finite. Since L_t^Y and $\hat{L}_t^Y$ are continuous in t , we have $L_T^Y = \hat{L}_T^Y$. Let $y = Y_{L_T^Y} = Y_{\hat{L}_T^Y}$, the location of the Y -tracer at time T in both the tracer and flipped tracer systems. By the definition of T , we have $L_T^Y \geq \hat{L}_T^Y$ and that the Y -tracer must be in state B in the tracer system and state A in the flipped tracer system. Now, we argue this is a contradiction. By the dynamics of the tracer system, the Y -tracer in state B may not sit on the same site as a nontracer A -particle. From the definition of ζ given in (2.1), we have $\zeta_T(y) < 0$. On the other hand, in the flipped tracer system, the Y -tracer in state A may not sit on the same site as a nontracer B -particle or the X -tracer in state B . From the definition of $\hat{\zeta}$ given in (2.4), we have $\hat{\zeta}_T(y) \geq 0$. But by Proposition 2.3 (I) and its analogue for the flipped tracer system, we have $\zeta_T(y) = \xi_T(y) = \hat{\zeta}_T(y)$, a contradiction. $\square$

Recalling the definition of the occupation time V_T from (1.1), it follows from Proposition 2.3 (I) that

$$V_T = \sum_{x \in H} \int_0^T \zeta_t(x) \mathbf{1}\{\zeta_t(x) > 0\} dt,$$

using the equality of ξ_t and ζ_t . Define

$$\begin{aligned} V_T^X &= \sum_{x \in H} \int_0^T \zeta_t^X(x) \mathbf{1}\{\zeta_t^X(x) > 0\} dt, \\ V_T^Y &= \sum_{x \in H} \int_0^T \widehat{\zeta}_t^Y(x) \mathbf{1}\{\widehat{\zeta}_t^Y(x) > 0\} dt, \\ V_T^{X,Y} &= \sum_{x \in H} \int_0^T \zeta_t^{X,Y}(x) \mathbf{1}\{\zeta_t^{X,Y}(x) > 0\} dt. \end{aligned} \quad (2.7)$$

The random variables V_T^X and V_T^Y give the occupation time at sites H when the X -tracer and Y -tracer, respectively, are added. Thus V_T^X is defined in terms of the tracer system while V_T^Y is defined in terms of the flipped tracer system. The occupation time $V_T^{X,Y}$ could be defined in terms of either system—if $\widehat{\zeta}_t^{Y,X}(x)$ replaced $\zeta_t^{X,Y}(x)$ in its definition, it would not change its distribution—but its current definition is consistent with the proof strategy given in the paragraph preceding Lemma 2.6.

Lemma 2.7. For all $T > 0$,

$$V_T^X - V_T = \int_0^{L_T^X} \mathbf{1}\{X_t \in H\} dt, \quad (2.8)$$

$$V_T^Y - V_T = \int_0^{\widehat{L}_T^Y} \mathbf{1}\{Y_t \in H\} dt, \quad (2.9)$$

$$V_T^{X,Y} - V_T = \int_0^{L_T^X} \mathbf{1}\{X_t \in H\} dt + \int_0^{L_T^Y} \mathbf{1}\{Y_t \in H\} dt. \quad (2.10)$$

Proof. We claim that for $x \in H$

$$\zeta_t^X(x) \mathbf{1}\{\zeta_t^X(x) > 0\} - \zeta_t(x) \mathbf{1}\{\zeta_t(x) > 0\} = \mathbf{1}\{X_{L_t^X} = x\} \mathbf{1}\{\zeta_t(x) \geq 0\} \quad (2.11)$$

$$= \mathbf{1}\{X_{L_t^X} = x\} \mathbf{1}_{A_t^X}. \quad (2.12)$$

Indeed, by Lemma 2.5 we have $\zeta_t^X(x) - \zeta_t(x) = \mathbf{1}\{X_{L_t^X} = x\}$. If $\zeta_t(x) < 0$, then $\zeta_t^X(x) \leq 0$, and both terms on the left-hand side of (2.11) are zero. If $\zeta_t(x) \geq 0$, then $\zeta_t^X(x) \geq 0$, and (2.11) is equal to $\mathbf{1}\{X_{L_t^X} = x\}$. To show (2.12), assume the X -tracer is at x . If it is in state B , then there can be no nontracer A -particles at x , and from (2.1) we have $\zeta_t(z) < 0$. If it is in state A , then there can be no nontracer B -particles at x , and by (d) the Y -tracer cannot be at x in state B ; hence $\zeta_t(x) \geq 0$ from (2.1). Thus A_t^X occurs if and only if $\zeta_t(x) \geq 0$ under the assumption that $X_{L_t^X} = x$.

Applying (2.11) and (2.12),

$$V_T^X - V_T = \sum_{x \in H} \int_0^T \mathbf{1}\{X_{L_t^X} = x\} \mathbf{1}_{A_t^X} dt.$$

Since L_t^X only increases while the X -tracer is in state A , we arrive at (2.8). Equation (2.9) is proven identically.

The proof of (2.10) is similar. We argue that

$$\begin{aligned} \zeta_t^{X,Y}(x) \mathbf{1}\{\zeta_t^{X,Y}(x) > 0\} - \zeta_t(x) \mathbf{1}\{\zeta_t(x) > 0\} \\ = \mathbf{1}\{X_{L_t^X} = x\} \mathbf{1}_{A_t^X} + \mathbf{1}\{Y_{L_t^Y} = x\} \mathbf{1}_{A_t^Y}. \end{aligned} \quad (2.13)$$

There are three cases to consider. First, if $\zeta_t(x) \geq 0$, then $\zeta_t^{X,Y}(x) \geq 0$ by Lemma 2.5. Thus the left-hand side of (2.13) is equal to $\mathbf{1}\{X_{L_t^X} = x\} + \mathbf{1}\{Y_{L_t^Y} = x\}$ in this case by Lemma 2.5. Next, if $\zeta_t(x) \leq -2$, then both terms on the left-hand side of (2.13) are zero. Last, if $\zeta_t(x) = -1$ then $\zeta_t^{X,Y}$ is one of $-1, 0$, and 1 , and hence the left-hand side of (2.13) to $\mathbf{1}\{\zeta_t^{X,Y}(x) = 1\}$. All together, we have shown that

$$\begin{aligned} \zeta_t^{X,Y}(x) \mathbf{1}\{\zeta_t^{X,Y}(x) > 0\} - \zeta_t(x) \mathbf{1}\{\zeta_t(x) > 0\} \\ = (\mathbf{1}\{X_{L_t^X} = x\} + \mathbf{1}\{Y_{L_t^Y} = x\}) \mathbf{1}\{\zeta_t(x) \geq 0\} + \mathbf{1}\{\zeta_t^{X,Y}(x) = 1, \zeta_t(x) = -1\} \end{aligned}$$

The event that $\zeta_t^{X,Y}(x) = 1$ and $\zeta_t(x) = -1$ can only occur when both tracers are at x by Lemma 2.5. If both tracers are in state A then $\zeta_t(z) \geq 0$, while if both tracers are in state B then $\zeta_t^{X,Y}(x) \leq 0$. Hence this event occurs when both tracers are at x with the X -tracer in state B , the Y -tracer in state A , and (necessarily) no nontracer particles are at x . That is,

$$\mathbf{1}\{\zeta_t^{X,Y}(x) = 1, \zeta_t(x) = -1\} = \mathbf{1}\{X_{L_t^X} = x, Y_{L_t^Y} = x\} \mathbf{1}_{B_t^X} \mathbf{1}_{A_t^Y}.$$

Meanwhile, when proving (2.12), we showed that A_t^X occurs if and only if $\zeta_t(x) \geq 0$ under the assumption that $X_{L_t^X} = x$. All together, the left-hand side of (2.13) is equal to

$$\begin{aligned} \mathbf{1}\{X_{L_t^X} = x\} \mathbf{1}_{A_t^X} + \mathbf{1}\{Y_{L_t^Y} = x\} \mathbf{1}\{\zeta_t(x) \geq 0\} + \mathbf{1}\{X_{L_t^X} = x, Y_{L_t^Y} = x\} \mathbf{1}_{B_t^X} \mathbf{1}_{A_t^Y} \\ = \mathbf{1}\{X_{L_t^X} = x\} \mathbf{1}_{A_t^X} + \mathbf{1}\{Y_{L_t^Y} = x\} (\mathbf{1}\{\zeta_t(x) \geq 0\} + \mathbf{1}\{X_{L_t^X} = x\} \mathbf{1}_{B_t^X} \mathbf{1}_{A_t^Y}) \\ = \mathbf{1}\{X_{L_t^X} = x\} \mathbf{1}_{A_t^X} + \mathbf{1}\{Y_{L_t^Y} = x\} \mathbf{1}_{A_t^Y}. \end{aligned}$$

With (2.13) proven, the rest of the proof of (2.10) goes the same as for (2.8) and (2.9). $\square$

Remark 2.8. If B -particles moved, then similar equations to those in (2.1)–(2.6) could be derived. However the time change would be more involved since parts of X and Y would be traversed by the tracer in state B . This time change would also complicate the formulas in Lemma 2.7, since the integrals would be over disconnected subintervals within $[0, T]$ traversed by the tracers while in state A , rather than a single connected subinterval such as $[0, L_T^X]$ when B -particles do not move. Nonetheless, the approach used in [CRS18, Section 4] to track the difference between systems with different starting configurations when the jump rates of A - and B -particles are possibly distinct ought to apply to our similarly constructed tracers. Hence Proposition 2.3 and by extension Lemma 2.5 should both continue to hold.

Issues arise with Lemma 2.6 and Lemma 2.7. The lives of, say, the Y -tracer in the tracer and flipped tracer systems are given by integrals over possibly different collections of subintervals. Coupling the size of these subintervals and the tracer locations at these times does not seem possible. Consequently, we do not believe that the conclusions of Lemma 2.6 nor Lemma 2.7 generalize to the setting in which B -particle are also mobile.

2.3 Proof of Proposition 2.2

Proof. Suppose that (ξ_0, S, h) is regular with $\sum_{x \in G} |\xi_0(x)| < \infty$. Fix x and k and define the tracer and flipped tracer systems with initial configuration $\xi_{0,k}$. Let V_T be the oc-

cupation time of $H \subseteq G$ for $(\xi_{0,k}, S, h)$. We claim that the coupled random variables $(V_T, V_T^X, V_T^Y, V_T^{X,Y})$ defined at (2.7) satisfy the conditions of Definition 2.1. By Proposition 2.3, conditions (a)–(c) hold. By (2.8) and (2.9) in Lemma 2.7, condition (d) holds. By Lemma 2.7 again,

$$\begin{aligned} V_T^{X,Y} - V_T^X - V_T^Y + V_T &= (V_T^{X,Y} - V_T) - (V_T^X - V_T) - (V_T^Y - V_T) \\ &= \int_0^{L_T^Y} \mathbf{1}\{Y_t \in H\} dt - \int_0^{\widehat{L}_T^Y} \mathbf{1}\{Y_t \in H\} dt, \end{aligned}$$

and this is nonnegative since $L_T^Y \geq \widehat{L}_T^Y$ by Lemma 2.6, which shows that condition (e) holds. $\square$

3 Monotonicity of icx statistics under the icx order

This section connects icx statistics to the icx order. The idea is to increase the initial configuration in the icx order at one site at a time, and to show that at each step the icx statistic increases as well. It is more a technical argument about stochastic orders than it is anything about diffusion-limited annihilating systems, and it uses similar arguments from [JJ18, Lemma 17] as templates.

Proposition 3.1. *Suppose that (1.3) holds. Further assume that $\sum_{z \in G} |\xi_0(z)| < \infty$ and $\sum_{z \in G} |\xi'_0(z)| < \infty$ almost surely. Fix $x \in G$ and assume that $\xi_0(z) = \xi'_0(z)$ for all $z \neq x$. Suppose that f is a nonnegative icx statistic of (ξ_0, S, h) as in Definition 2.1 and let $f' = f(\xi'_0, S, h)$.*

$$\text{If } \xi_0(x) \preceq_{icx} \xi'_0(x), \text{ then } f \preceq_{icx} f'.$$

For a function h on the integers, we define the difference operators

$$\begin{aligned} Dh(k) &= h(k+1) - h(k) \\ D^2h(k) &= D[h(k+1) - h(k)] = h(k+2) - 2h(k+1) + h(k). \end{aligned}$$

The discrete analogue of convexity is that f satisfies $Df(k) \geq 0$ and $D^2f(k) \geq 0$ for all integers k . We would expect that if $X \preceq_{icx} Y$ for integer-valued random variables X and Y , then $\mathbf{E}h(X) \leq \mathbf{E}h(Y)$ when h is convex. Indeed, this is correct:

Lemma 3.2. *Let X and Y be integer-valued random variables and $h: \mathbb{Z} \rightarrow \mathbb{R}$ satisfy $Dh(k) \geq 0$ and $D^2h(k) \geq 0$ for all $k \in \mathbb{Z}$. If $X \preceq_{icx} Y$, then $\mathbf{E}h(X) \leq \mathbf{E}h(Y)$.*

Proof. Let $\bar{h}: (-\infty, \infty) \rightarrow \mathbb{R}$ be the linear interpolation of h between adjacent integer points. Since $Dh(k) \geq 0$ and $D^2h(k) \geq 0$, the function $\bar{h}$ is increasing and convex on $(-\infty, \infty)$. Thus, $X \preceq_{icx} Y$ implies that $\mathbf{E}h(X) = \mathbf{E}\bar{h}(X) \leq \mathbf{E}\bar{h}(Y) = \mathbf{E}h(Y)$. $\square$

Proof of Proposition 3.1. Define $\xi_{0,k}$ to be the same as ξ_0 except that $\xi_{0,k}(x) = k$. Let $W(k) = f(\xi_{0,k}, S, h)$ so that

$$W(\xi_0(x)) \stackrel{d}{=} f \quad \text{and} \quad W(\xi'_0(x)) \stackrel{d}{=} f'.$$

Let $\varphi: [0, \infty) \rightarrow [0, \infty)$ be an increasing convex function and let $h(k) = \mathbf{E}\varphi(W(k))$ for all $k \in \mathbb{Z}$. Then

$$\mathbf{E}h(\xi_0(x)) = \mathbf{E}\varphi(f) \quad \text{and} \quad \mathbf{E}h(\xi'_0(x)) = \mathbf{E}\varphi(f'). \quad (3.1)$$

We will show that $Dh(k) \geq 0$ and $D^2h(k) \geq 0$. Then it will follow from Lemma 3.2 that $\mathbf{E}h(\xi_0(x)) \leq \mathbf{E}h(\xi'_0(x))$, since $\xi_0(x) \preceq_{icx} \xi'_0(x)$ by hypothesis. By (3.1), this shows that $\mathbf{E}\varphi(f) \leq \mathbf{E}\varphi(f')$, which proves $f \preceq_{icx} f'$.

Thus it only remains to show $Dh(k) \geq 0$ and $D^2h(k) \geq 0$. Let $(\Phi, \Phi^X, \Phi^Y, \Phi^{X,Y})$ be a coupling as described in the definition of an icx statistic. Then

$$Dh(k) = \mathbf{E}\varphi(f(\xi_{0,k+1}, S, h)) - \mathbf{E}\varphi(f(\xi_{0,k}, S, h)) = \mathbf{E}[\varphi(\Phi^X) - \varphi(\Phi)].$$

Since $\Phi^X \geq \Phi$ a.s. and φ is increasing, we have $Dh(k) \geq 0$.

For the second order condition, we expand $D^2h(k)$ as

$$\begin{aligned} D^2h(k) &= \mathbf{E}[\varphi(f(\xi_{0,k+2}, S, h))] - 2\mathbf{E}[\varphi(f(\xi_{0,k+1}, S, h))] + \mathbf{E}[\varphi(f(\xi_{0,k}, S, h))] \\ &= \mathbf{E}[\varphi(\Phi^{X,Y}) - \varphi(\Phi^X) - \varphi(\Phi^Y) + \varphi(\Phi)]. \end{aligned} \quad (3.2)$$

We claim that

$$\begin{aligned} \varphi(\Phi^{X,Y}) - \varphi(\Phi^X) &\geq \varphi(\Phi^X + \Phi^Y - \Phi) - \varphi(\Phi^X) \\ &\geq \varphi(\Phi^Y) - \varphi(\Phi). \end{aligned}$$

The first inequality holds because φ is increasing and $\Phi^{X,Y} \geq \Phi^X + \Phi^Y - \Phi$ by item (e) of the definition of icx statistic. For the second inequality, observe that by convexity $\varphi(a+u) - \varphi(a) \geq \varphi(b+u) - \varphi(b)$ for any $a \geq b$ and $u \geq 0$, and then take $a = \Phi^X$, $b = \Phi$, and $u = \Phi^Y - \Phi$. Thus (3.2) is nonnegative, completing the proof. $\square$

4 Further questions

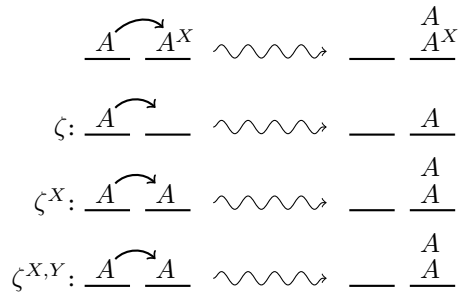
A natural question is whether or not our results extend to DLAS with mobile B -particles. Our intuition is that they do, since it remains the case that two A -particles can assist each other to clear out B -particles that would otherwise destroy an A -particle added individually. But as we discuss in Remark 2.8, if we use our current coupling with moving B -particles, the tracer and flipped tracer systems have a more complex relationship, and we cannot prove that V_T is an icx statistic.

Even for the stationary B -particle case, it would be interesting to find functionals besides occupation time that respect the icx order. For example, we speculate that the lifespan of a distinguished A - or B -particle might be monotonic (increasing or decreasing, respectively) in the icx order as more A -particles are added. Similarly, the time of the first visit to a distinguished vertex by an A -particle as well as the total number of A -particles still alive at time t might respect the icx order.

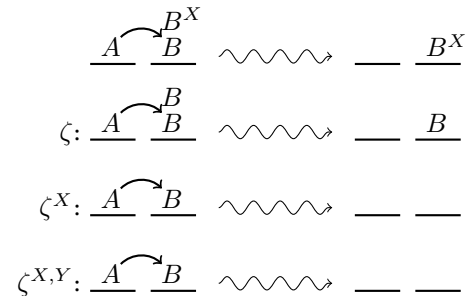
Finally, we are interested in whether there is a general framework for understanding which statistics in different interacting particle systems respect which stochastic orders. This paper demonstrates that DLAS, with interaction rule $A + B \rightarrow \emptyset$, is compatible with the icx order. In [JJ18], it is shown that the frog model, with interaction rule $A + B \rightarrow 2A$, is compatible with the icv order. Is there a systematic explanation across different particle systems?

A Figures from the proof of Proposition 2.3

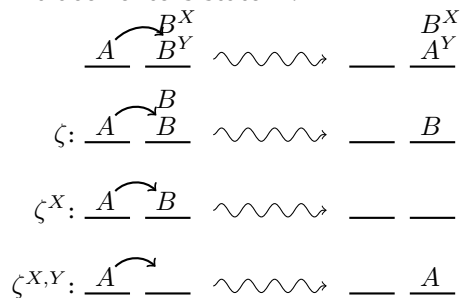
Case 1: The A -particle jumps onto a site containing neither B -particles nor tracers in state B . No interaction occurs.



Case 2: The A -particle jumps onto a site containing nontracer B -particles, mutually annihilating with one of them.



Case 3: The A -particle jumps onto a site containing no nontracer particles, the Y -tracer in state B , and possibly the X -tracer in either state. The A -particle is annihilated and the Y -tracer enters state A .



Case 4: The A -particle jumps onto a site containing only the X -tracer in state B . The A -particle is annihilated and the X -tracer enters state A .

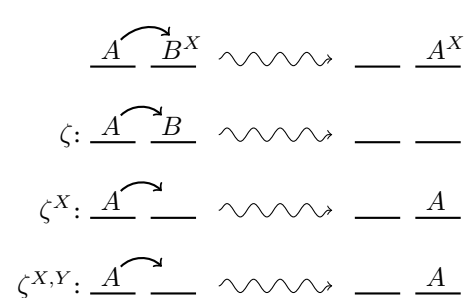
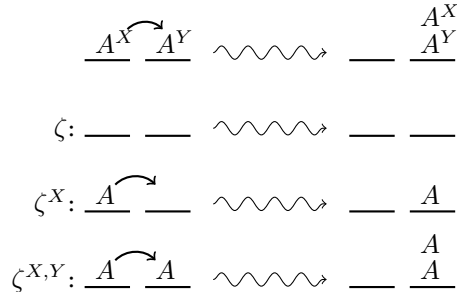
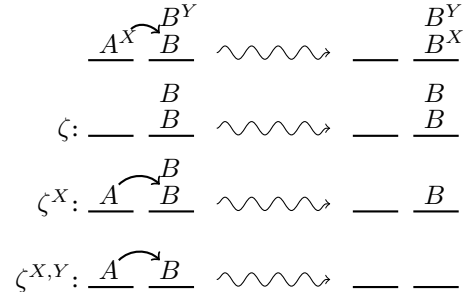


Figure 1: When a nontracer A -particle in the tracer system jumps, there are four cases. The top line in each case shows how the tracer system evolves when the particle jumps. The tracer particles are indicated by A^X , A^Y , B^X , or B^Y , with A and B giving their states and X and Y specifying the tracer. The lines below show how the particles are viewed by ζ , ζ^X , and $\zeta^{X,Y}$.

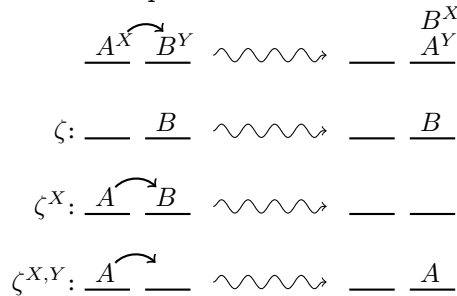
Case 1: The X -tracer in state A jumps onto a site containing no nontracer B -particles nor the Y -tracer in state B . No interaction occurs.



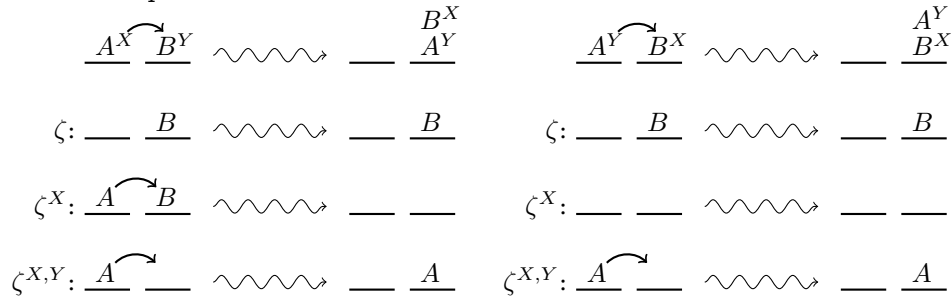
Case 2: The X -tracer in state A jumps onto a site containing nontracer B -particles. It annihilates one of them and then enters state B .



Case 3: The X -tracer in state A jumps onto a site containing the Y -tracer in state B and no other particles. The tracers swap states.



Case 4: The Y -tracer in state A jumps onto a site containing no nontracer B -particles. No interaction occurs.



Case 5: The Y -tracer in state A jumps onto a site containing nontracer B -particles. It annihilates one of them and then enters state B .

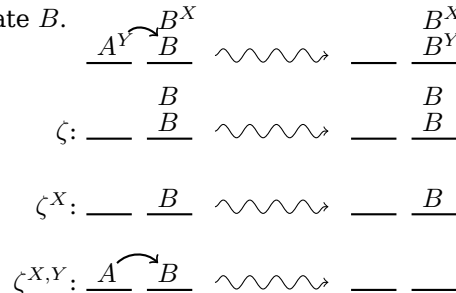


Figure 2: When a tracer particle jumps in the tracer system, there are five cases. The same notation is used here as in Figure 1.

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