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A ν Supersymmetric Anomaly-free Atlas

B.C. Allanach,[a] **Maeve Madigan,**[1a] **Joseph Tooby-Smith**[b]

[a] *DAMTP, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom*

[b] *Cavendish Laboratory, University of Cambridge, J J Thomson Ave, Cambridge, CB3 0HE, United Kingdom*

E-mail: B.C.Allanach@damtp.cam.ac.uk, mum20@cam.ac.uk, jss85@cam.ac.uk

Abstract: Extensions of the minimal supersymmetric standard model (MSSM) gauge group abound in the literature. Several of these include an additional $U(1)_X$ gauge group. Chiral fermions' charge assignments under $U(1)_X$ are constrained to cancel local anomalies in the extension and they determine the structure and phenomenology of it. We provide all anomaly-free charge assignments up to a maximum absolute charge of $Q_{\max} = 10$, assuming that the chiral superfield content of the model is that of the MSSM plus up to three Standard Model (SM) singlet superfields. The fermionic components of these SM singlets may play the rôle of right-handed neutrinos, whereas one of the scalar components may play the rôle of the flavon, spontaneously breaking $U(1)_X$. Easily scanned lists of the charge assignments are made publicly available on `Zenodo`. For the case where no restriction is placed upon $Q_{\max}$, we also provide an analytic parameterisation of the general solution using simple techniques from algebraic geometry.

[1] Corresponding author.

Contents

1 Introduction

Quantum field theories of vector bosons are notoriously problematic unless they arise from gauge symmetries, whence non-renormalisability and non-unitarity can be tamed. It is thus imperative that the gauge symmetry of the renormalisable ultra-violet completion of any such model should not contain any quantum field theoretic gauge *anomalies*, where quantum corrections spoil the gauge symmetry that was imposed upon the tree-level theory. The

Standard Model (SM) itself is anomaly-free and can thus remain a self-consistent theory up to very large renormalisation scales. Despite this, there are good reasons to expect the SM to be an effective field theory resulting from decoupling other fields. Many reasons have been invoked to motivate extending the Lie algebra[1] $\mathfrak{sm} := \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_Y$ of the Standard Model (SM) by a spontaneously broken gauged $\mathfrak{u}(1)_X$ summand, for example. Such extensions have been used to explain measurements of the anomalous magnetic moment of the muon [1], to provide axions [2] or leptogenesis [3], to provide fermion masses through the Froggatt-Neilsen mechanism [4], or explain measurements of the $b \to sl^+l^-$ transition which are currently in tension with SM predictions [5–13]. In general, the X charge assignments of the models can be family dependent, resulting in family-dependent couplings of a resulting massive Z' vector boson. Indeed, in several applications (the last two in our aforementioned list) it is a necessary requirement that the X charges are family dependent, since the symmetry and the Z' are respectively used to explain family non-universal effects.

In $\mathfrak{u}(1)_X$ extensions, the phenomenology of the Z' is often key and is dictated by the integer X charges of the other fields in the model (integer X charges results from an implicit assumption that the extension is compact). The X charges of the chiral fermions in particular dictate the contribution to perturbative local anomalies of such models. There is therefore a non-trivial cross-over between the extensions' phenomenology and anomaly cancellation via the chiral fermions' charge assignments. Unfortunately, in general, with a fixed chiral fermion content, anomaly cancellation conditions (ACCs) are difficult to solve, the number theory state-of-the art being the solution of a single cubic in three unknown integer parameters [14].

Some recent progress has been made in this direction, however. In Ref. [15], the gravitational and gauge anomalies of a pure $U(1)$ gauge symmetry (i.e. with no SM gauge group but with charged chiral fermionic fields) were solved analytically for the charges of *a priori* fixed numbers of chiral fermions via an ingenious algebraic method[2]; this was soon understood from a geometric perspective [17] by using a theorem due to Mordell [14]. Similar geometric methods were employed to find an analytic solution to the more difficult problem of $\mathfrak{sm} \oplus \mathfrak{u}(1)$ anomaly-free charge assignments in the specific case of SM fermion content, plus three right-handed (RH) neutrinos (i.e. SM-singlet chiral fermion fields which may carry X charge) [18]. The number of solutions is formally infinite,[3] unlike the case of semi-simple SM extensions with identical fermionic field content, where there is a list of 340 [21]. Unfortunately, the geometric methods employed only solve a small family of similar cases and cannot be deployed on general chiral fermionic contents. Furthermore, the analytic solution, whilst of intrinsic interest in and of itself, comes with a significant drawback for model-builders interested in

[1] We shall refer to the Lie algebra (as opposed to the Lie group) in $\mathfrak{mathfrak}$ script.

[2] The algebraic approach was partially extended to $U(1)^n$ gauge symmetries in Ref. [16].

[3] One way of seeing this is to set the X charges of the first family of particles to be equal to their hypercharges, the second family to be equal to some integer multiplied by baryon number minus lepton number $B - L$, and the third family to have zero charge. Any such charge assignment solves the anomaly cancellation conditions. Since there are an infinite number of constants we can multiply the second family by, each of which leads to a distinct chiral solution, there are an infinite number of solutions.

using it: each charge is parameterised in terms of a fourth-order polynomial of integer parameters. Whilst it is easy to input these parameters and achieve anomaly-free charges, model builders often want to fix a function of them to certain values for phenomenological purposes, but this is a difficult and currently unsolved problem, because it involves solving a system of coupled fourth-order diophantine equations.

Fortunately, when appropriately employed, computers come to the rescue of the reverse-engineering model builder. In an $\mathfrak{sm} \oplus \mathfrak{u}(1)$ 'anomaly-free atlas' [22], all solutions of the ACCs for integer charges between -10 and 10 for 18 chiral fermion gauge representations in the SM plus three RH neutrinos were found by a scan.[4] Cases which are in a sense equivalent (where the charges differ by a common multiple which can be absorbed into the $\mathfrak{u}(1)_X$ gauge coupling, or which differ by a permutation of the family indices within a *species* - fields which have identical SM representations) were only counted once (and aside from some rare cases, only scanned over once). Anomaly-free solutions are scarce: only roughly one in 10^9 was anomaly-free from the whole sample. The list of anomaly-free fermionic charge assignments was made publicly available. It is a list of over 21 000 000 solutions that is easy and quick to search through and filter with the aid of a simple computer program. As such, it is user friendly for would-be $U(1)_X$ gauge extension model builders who can search through the list and filter for charge assignments with various desired properties. The charges are limited in *height* (the maximum absolute value of a charge in any solution), but have the advantage of being easily useable provided one can adapt or write a simple computer program that reads the list in and filters it.

Heretofore, there has been no similar list made for supersymmetric (SUSY) models. SUSY model building has several motivations, the primary one being that it does not suffer from the technical hierarchy problem, where radiative corrections to the Higgs mass tend to drag it up to the largest fundamental energy scale (for example the Planck mass $\sim 10^{19}$ GeV) divided by a loop factor. There are other motivations for supersymmetry too, for example, in an $\mathcal{N} = 1$ supersymmetrisation of the SM (the MSSM), the experimental measurements of the gauge couplings agree with the gauge coupling unification condition predicted by SUSY grand unified theories. When one includes an extra multiplicative discrete symmetry such as $R-$parity or matter parity[5] the MSSM possesses a stable particle which, depending upon parameters, has the correct properties to constitute the universe's dark matter and potentially dangerous proton decay processes are suppressed. Particular examples of $\mathfrak{u}(1)_X$ gauge extensions of the MSSM can combine the aforementioned phenomenological benefits of a Z' with those of SUSY models. Some of these have appeared in the literature, for example see Refs. [24–29].

It is our intention here to extend the original non-SUSY anomaly-free atlas to the SUSY case and make a new list (a 'ν SUSY anomaly-free atlas') available to interested SUSY $\mathfrak{u}(1)_X$-extension model builders and others. We shall include the addition of up to three

[4] This strategy has also recently been used for the case of $U(1)$ gauge theory with different numbers of Weyl fermions, in a search for scotogenic models [23].

[5] Matter parity is defined as $(-1)^{3(B-L)}$, where B is baryon number and L is lepton number, whereas $R-$parity is defined as $(-1)^{3(B-L)+2s}$, where s is spin.

MSSM-singlet chiral superfields: the fermionic components of all or some of these can play the rôle of RH neutrinos, resulting in tiny neutrino masses via the see-saw mechanism (below, we call this model the νMSSM). The scalar component of one of these MSSM-singlet chiral superfields is expected to play the rôle of the flavon, which has a necessarily non-zero X charge and acquires a vacuum expectation value, spontaneously breaking $U(1)_X$. One might expect that one of the SM-singlet fields must therefore have a non-zero $\mathfrak{u}(1)_X$ charge, unlike the non-SUSY case, where the charges of the flavon and all SM-singlet fermions were *a priori* unconstrained. However, we won't impose this condition because the field content of the model can easily be extended in a way that does not change the ACCs but which effectively removes the condition, as we shall explain below. A functional difference to the original non-SUSY anomaly-free atlas is the appearance of the Higgsino partners of the two MSSM Higgs doublets, augmenting the number of Weyl fermion $SU(2)$ gauge representations by two. This therefore extends the original list of 18 X charges to 20. In case a height larger than 10 is required, we will also provide a general analytic solution to the anomaly cancellation conditions. This relies on using the same geometric framing in which the SM-plus-3 RH neutrino case was solved [18]; we take the opportunity to demonstrate a new technique to solve such problems, although the technique used in Ref. [18] would also have worked.

The paper proceeds as follows: in §2, we describe the anomaly cancellation conditions relevant for the Lie algebra $\mathfrak{mssm} \oplus \mathfrak{u}(1)_X$, and a chiral superfield content of the νMSSM. In §3, we describe the computational scan and how the solutions are listed and ordered, giving the number of solutions found up to a height of 10. We provide an analytic method of solution in §4, along with a parameterisation of the solution. Various consistency checks of the solutions are described in §5: some are checks solely of the numerical solutions, some are of the analytic solution and some are checks of the analytic solution versus the numeric solutions. Some initial filters of the numerical solutions (chosen for specific phenomenological reasons) are explored in §6. We provide a summary of the paper and a discussion in §7.

We list chiral fermionic fields in the representations displayed in Table 1. As previously mentioned, the left-handed fermionic fields contained within the two Higgs chiral superfields provide a new feature as regards the ACCs. We note here that the fermionic components of the chiral superfields H_d and L_i have identical representations under the SM gauge Lie algebra, but the fermionic component of H_d may or may not be discriminated by a different quantum number under an imposed symmetry such as matter parity or $R-$parity.

We have thus augmented the MSSM, as far as the fermionic X charges go, by 20 parameters which we write in a 20-tuple

$$\begin{aligned}\mathbf{X} := \{&X_{Q_1}, X_{Q_2}, X_{Q_3}, X_{n_1}, X_{n_2}, X_{n_3}, X_{e_1}, X_{e_2}, X_{e_3}, X_{u_1}, X_{u_2}, X_{u_3}, X_{d_1},\\ &X_{d_2}, X_{d_3}, X_{L_1}, X_{L_2}, X_{L_3}, X_{H_d}, X_{H_u}\}.\end{aligned} \tag{1.1}$$

We take it as understood that, for the case where $R-$parity is *not* a symmetry of the theory, we modify (1.1) such that X_{H_d} is merged with X_{L_i} to form X_{L_α}, where $\alpha \in \{1, 2, 3, 4\}$. For now though, we shall continue the discussion where H_d is discriminated from L_i by a discrete

symmetry. Since the gauge extension is here assumed to be compact, $\mathbf{X}$ is *a priori* valued in $\mathbb{Z}^{20}$.

Fermions				
	$\mathfrak{su}(3)$	$\mathfrak{su}(2)_L$	$\mathfrak{u}(1)_Y$	$\mathfrak{u}(1)_X$
LH quark doublets Q_i	3	2	1	X_{Q_i}
RH neutrinos n_i	1	1	0	X_{n_i}
RH charged leptons e_i	1	1	-6	X_{e_i}
RH up quarks u_i	3	1	4	X_{u_i}
RH down quarks d_i	3	1	-2	X_{d_i}
LH lepton doublets L_i	1	2	-3	X_{L_i}
LH down-type Higgsino $\tilde{H}_d$	1	2	-3	X_{H_d}
LH up-type Higgsino $\tilde{H}_d$	1	2	3	X_{H_u}
Chiral superfields				
$\hat{Q}_i$	3	2	1	X_{Q_i}
$\hat{N}_i^c$	1	1	0	$-X_{n_i}$
$\hat{E}_i^c$	1	1	6	$-X_{e_i}$
$\hat{U}_i^c$	$\bar{3}$	1	-4	$-X_{u_i}$
$\hat{D}_i^c$	$\bar{3}$	1	2	$-X_{d_i}$
$\hat{L}_i$	1	2	-3	X_{L_i}
$\hat{H}_d$	1	2	-3	X_{H_d}
$\hat{H}_d$	1	2	3	X_{H_u}

Table 1. Conventions for field content with representations under the gauge Lie algebra. RH stands for right-handed and LH stands for left-handed. $i \in \{1, 2, 3\}$ is a family index. Note that we have re-scaled a more conventional hypercharge assignment by a factor of 6 to make all hypercharges setwise coprime integers. Such a re-scaling can be absorbed into the hypercharge gauge coupling. ^{c} denotes charge conjugation on the scalar and fermionic components of the chiral superfield.

2 $\mathfrak{u}(1)_X$ Extension of the MSSM Lie Algebra

2.1 Anomaly cancellation conditions

The MSSM *per se* is anomaly free. With the addition of $\mathfrak{u}(1)_X$, local anomalies persist unless $\mathbf{X}$ satisfies the ACCs[6]

$$\mathfrak{su}(3)^2 \oplus \mathfrak{u}(1)_X : \sum_{i=1}^{3}(2X_{Q_i} - X_{u_i} - X_{d_i}) = 0, \tag{2.1}$$

$$\mathfrak{su}(2)^2 \oplus \mathfrak{u}(1)_X : \sum_{i=1}^{3}(3X_{Q_i} + X_{L_i}) + X_{H_d} + X_{H_u} = 0, \tag{2.2}$$

$$\mathfrak{u}(1)_X\text{-gravity} : \sum_{i=1}^{3}(6X_{Q_i} - X_{n_i} - X_{e_i} - 3X_{u_i} - 3X_{d_i} + 2X_{L_i}) + 2X_{H_d} + 2X_{H_u} = 0, \tag{2.3}$$

$$\mathfrak{u}(1)_X^3 : \sum_{i=1}^{3}(6X_{Q_i}^3 - X_{n_i}^3 - X_{e_i}^3 - 3X_{u_i}^3 - 3X_{d_i}^3 + 2X_{L_i}^3) + 2X_{H_d}^3 + 2X_{H_u}^3 = 0, \tag{2.4}$$

$$\mathfrak{u}(1)_X^2 \oplus \mathfrak{u}(1)_Y : \sum_{i=1}^{3}(X_{Q_i}^2 - 2X_{u_i}^2 + X_{d_i}^2 + X_{e_i}^2 + X_{L_i}^2) + X_{H_d}^2 - X_{H_u}^2 = 0, \tag{2.5}$$

$$\mathfrak{u}(1)_Y^2 \oplus \mathfrak{u}(1)_X : \sum_{i=1}^{3}(X_{Q_i} - 6X_{e_i} - 8X_{u_i} - 2X_{d_i} + 3X_{L_i}) + 3X_{H_d} + 3X_{H_u} = 0. \tag{2.6}$$

These ACCs inherit some in-practice physical equivalences between $\mathfrak{u}(1)_X$ extensions related by the following operations:

(i) Permutation of family indices within each species, since this is really just a change of basis.

(ii) $\mathbf{X} \to a\mathbf{X}$, where $a \in \mathbb{Q}\backslash\{0\}$, when the gauge coupling only appears in the Lagrangian multiplied by a $U(1)_X$ charge, since the $U(1)_X$ gauge coupling may be simultaneously re-scaled by $1/a$ resulting in no substantive change. This is displayed by the fact that the ACCs are homogeneous.

(iii) $\mathbf{X} \to \mathbf{X} + y\mathbf{Y}$, where $\mathbf{Y}$ is the 20-tuple of fermionic field hypercharges (in the same field ordering as $\mathbf{X}$) and $y \in \mathbb{Z}$. Resulting from a group outer automorphism, this change in fermionic representations can be accounted for by a redefinition of gauge fields [16].

Ideally, we wish to record exactly one entry in a list for each physically inequivalent charge assignment. Together, (ii) with (iii) imply that we should regard $\mathbf{X} \to x\mathbf{X} + y\mathbf{Y}$ as an

[6] Note that where necessary, we discriminate between the gauge Lie algebra, which is equivalent to $\mathfrak{sm}\oplus\mathfrak{u}(1)_X$ and the MSSM$\times U(1)_X$ gauge group, which is strictly only determined up to certain quotients, but this does not affect any of our discussion.

equivalent theory, where $x \in \mathbb{Q}\backslash\{0\}$ and $y \in \mathbb{Q}$. Unfortunately, we have not found an easy enough and fast enough method of incorporating this, implying that there will remain a few physically equivalent charge assignments in any anomaly-free list that we produce. The necessary existence of these will end up providing us with a check of our computer program in §5. In any case, such equivalent charge assignments are rare, and we do not foresee particular problems resulting from their presence in our list. From now on, we refer to 'inequivalent' solutions to implicitly mean inequivalent under conditions (i) and (ii) only.

To incorporate (i), we take the convention that the family indices in $\mathbf{X}$ are such that, for each species $S \in \{Q, n, e, u, d, L\}$, $X_{S_1} \leq X_{S_2} \leq X_{S_3}$ (for the case without additional discrete symmetries to distinguish H_d and $S = L$, $X_{S_3} \leq X_{S_4}$ as well). To take (ii) into account, all integers in the tuple must be setwise coprime but note that this still does not implement the equivalence with $\mathbf{X}' := \{-X_{Q_3}, -X_{Q_2}, -X_{Q_1}, \ldots, -X_{H_d}, -X_{H_u}\}$. In order to only list one instance of $\mathbf{X}, \mathbf{X}'$, we must define a condition that unambiguously picks one of them: here, we use the *lexicographically smaller* tuple.[7] An $n-$tuple $a = \{a_1, \ldots, a_n\}$ is lexicographically smaller than another $n-$tuple $b = \{b_1, \ldots, b_n\}$ (written as $a < b$) if and only if an $i \in \{1, \cdots, n\}$ exists such that $a_i < b_i$ and $a_j = b_j$ for all $j \in \{1, \cdots, i-1\}$.

2.2 Symmetry breaking

Since we are not empirically aware of a long-range force that can be attributed to an unbroken $U(1)_X$ gauge symmetry, we suppose that it must be spontaneously broken. We further assume that it is broken by (at least) one of the scalars $\tilde{\nu}_{R_i}$ contained in the SM-singlet chiral superfields N_i^c, so that it does not break the SM gauge symmetry. In order for a $\tilde{\nu}_{R_i}$ field to play this rôle, by Goldstone's theorem it must possess a non-zero X charge. Typically, such a field is called a *flavon*. Let us denote it for the purposes of the current discussion, as θ. Contrary to the non-SUSY case, we obtain a contribution to the ACCs through its fermionic superpartner $\tilde{\theta}$, the flavino. However, we will still solve the ACCs as given above assuming three SM-singlet chiral superfields only: N_i^c, where $i \in \{1, 2, 3\}$. The reasons for not explicitly adding to this number (for example by adding one more SM singlet chiral superfield) are twofold: firstly, we find practical barriers with four (or more) SM-singlets; the ν SUSY anomaly-free atlas would take too long to compute and would take up too much disk space to store for the desired height of 10. Secondly, by sticking to three SM-singlet chiral superfields, we are able to find an analytic solution to the ACCs.

In principle, the requirement that at least one $X_{n_i} \neq 0$ would allow us to reduce the domain of X charges considered in our computational search below, although not by much. We choose not to restrict the domain of X charges in this way however, since one could augment our model by *two* additional scalar singlets θ_1 and θ_2 such that $X_{\theta_1} + X_{\theta_2} = 0$. The contributions from X_{θ_1} and X_{θ_2} would cancel in the ACCs, leaving the ACCs above unmodified. This type of extension is commonly used, for example, in $U(1)_{B-L}$ extensions of

[7] Lexicographical ordering is a much simpler condition than the one used in the original anomaly-free atlas [22].

the MSSM [25]. More generally one can add several SM-singlet chiral superfields which satisfy the pure $\mathfrak{u}(1)$ anomaly equations and thus cancel out of the ACCs [15, 17]. We also note that to set a superfield's charge to zero has the same effect *on the ACCs* as would removing the superfield (or at least its fermionic component) entirely from the model.

With the constraints (or lack thereof) listed above, our inequivalent numerical set of solutions will have the following subsets:

- The SM plus three RH neutrinos corresponds to the subset with $X_{H_u} = X_{H_d} = 0$ (since this is equivalent to the model obtained by removing the H_u superfield and the H_d Higgsino from our current set-up).
- The MSSM with up to three $U(1)_X$-charged RH neutrino chiral superfields, where the $U(1)_X$ is broken by (at least) one of the RH sneutrinos, is the subset where at least one $X_{n_i} \neq 0$.
- The MSSM with three RH neutrinos and two additional scalars θ_1 and θ_2 charged such that $X_{\theta_1} + X_{\theta_2} = 0$. This is a possibility for the subset with $X_{n_1} = X_{n_2} = X_{n_3} = 0$. In particular, this option includes $U(1)_X = U(1)_{B-L}$ extensions of the MSSM.

3 Numerical Solutions up to a Height of 10

We produce a list of solutions to the ACCs using a modification of the computer program that was used to produce the original anomaly-free atlas [22]. To search for solutions, we scan over integer values in the domain $|X_i| \leq Q_{\max}$. Each set of solutions can then be classified by $Q_{\max}$, its maximum possible height. Note that in this definition, a list of solutions $Q_{\max} = N$ also contains all solutions with $Q_{\max} < N$.

There are *a priori* $(2Q_{\max}+1)^{20}$ solutions to be checked as solutions to the ACCs. Time is the biggest limiting factor in our search for solutions. The original computer program is described in detail in §3 of Ref. [22], where we direct the curious reader. Here, we find it expedient to not discriminate *a priori* between H_d and L_i: within the program, we therefore remove the explicit H_d charge, replacing it by L_4, equivalent to considering the theory without a discrete symmetry to distinguish them. The search for solutions is sped up by removing some equivalent solutions from the scan, and by using the four linear ACCs to directly fix the values of four of the charges. This still leaves us with a large solution space to consider, as compared to the original non-supersymmetric anomaly-free atlas. We further improve the speed by parallelising the three outer loops (i.e. over X_{Q_1}, X_{Q_2} and X_{Q_3}). For $Q_{\max} \leq 4$ the parallelisation improvement is minimal, but for $Q_{\max} \geq 5$ we find that this step is necessary to produce solutions in a reasonable amount of time.

3.1 Binary search algorithm

The output of the computer program for $Q_{\max} = 10$ is a large lexicographically ordered list (the ASCII file is around 125 Gb in size) of inequivalent solutions which solve the ACCs, each

$Q_{\max}$	# νMSSM	# $R_p\nu$MSSM	# Non-SUSY	# No ACC condition
1	111	267	37	4.5×10^6
2	2 321	6 882	357	2.3×10^{10}
3	44 212	143 707	4 115	8.6×10^{12}
4	401 129	1367 991	24 551	8.2×10^{14}
5	2 582 166	9 063 191	111 151	3.3×10^{16}
6	13 553 325	48 681 027	435 304	7.6×10^{17}
7	54 699 483	199 275 965	1 358 387	1.1×10^{19}
8	185 454 955	682 827 818	3 612 733	1.2×10^{20}
9	598 267 488	2 224 178 673	9 587 084	1.0×10^{21}
10	1 628 002 737	6 094 894 134	21 546 919	6.8×10^{21}

Table 2. Number of inequivalent anomaly-free charge assignments found for $U(1)_X$ extensions of the νMSSM where L_i and H_d are not discriminated (νMSSM), or where they are ($R_p\nu$MSSM) or for the original non-supersymmetric anomaly-free atlas (Non-SUSY). The rightmost column lists the number of potential inequivalent solutions in the SUSY case before ACCs are applied and where a discrete symmetry distinguishes L_i from H_d.

one comprised of a line made of the 20 integers which form **X**. We have not assumed a discrete symmetry that distinguishes L_i from H_d in the output and so we have four X_{L_α} charges listed. This file forms one of the two most important outputs of the present paper (the other being the analytic solution for any height described in §4). As mentioned in §1, we envisage that our output file may be used by supersymmetric model builders by scanning through it with a computer program and filtering the results. Since the file is so large though, we have facilitated the decrease of the complexity of algorithms used to analyse the file in order to speed them up. The fact that our list of solutions is ordered lexicographically means that one can take advantage of the binary search algorithm. This reduces the complexity of finding a solution in the list from $\mathcal{O}(n)$ to $\mathcal{O}(\log n)$. One usually has an intuitive understanding of the binary search algorithm since it is roughly how one usually finds numbers in a phone book or words in a dictionary, as follows. Let us say we have a solution we want to find in our list or show that it does not exist in the list. The binary search algorithm goes half-way down the list and determines if our solution is less then or equal to the solution at the half-way point. If it is present in the first half then we throw away the second half and keep the first, and if it isn't we discard the first half and keep the second. This is then repeated until a list with a single item which will (if it exists) match the one we are trying to find.

3.2 Output

We display some basic statistics characterising the number of solutions found in Table 2. The number of inequivalent solutions increases rapidly as a function of the maximum height searched over, $Q_{\max}$. In all, we find over 1.6 billion solutions for $Q_{\max} = 10$ in the case where

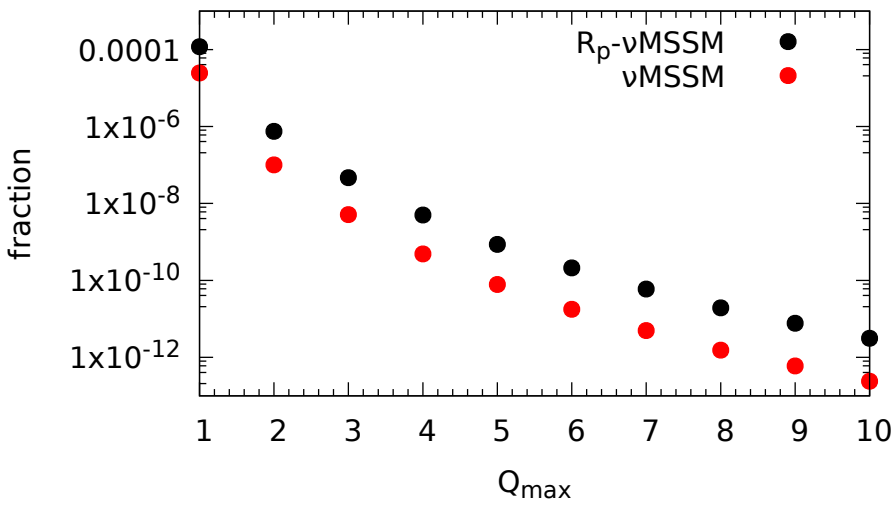

Figure 1. Fraction of otherwise possible solutions that are anomaly free as a function of $Q_{\max}$. Specifically, the fraction is equal to the number of inequivalent solutions divided by the number of inequivalent possible assignments before the local anomaly cancellation requirements are imposed.

a discrete symmetry does not pick out one of the X_{L_α} charges to be X_{H_d}, a far larger number than the original anomaly-free atlas (which counts under 22 million inequivalent solutions). As expected, this increases almost four-fold for the case where one does pick an X_{L_α} to be H_d. Solutions to the ACCs are scarce; their density decreases with increasing height. For a height of 10, for example, only approximately 1 in 10^{12} possible inequivalent charge assignments are anomaly free. We display this fraction for various different values of $Q_{\max}$ in Fig. 1 for the case where no symmetry discriminates between H_d and L_i (νMSSM) and the case where it does ($R_p\nu$MSSM).

In Table 3, we display some solutions that appear in the literature and in our list. All of the solutions shown were found using the binary search algorithm sketched in §3.1. Their presence in the list is a check of some expected and found solutions. Two solutions (Y_3' and B_3') will be useful for our analytic solution, which we turn to now.

4 Analytic Solution

In this section, we will first frame our problem in a geometric language that will facilitate our analytic solution of the ACCs. We shall then go on to sketch the geometric method by which the solution is obtained. Then we shall derive the solution in detail algebraically, eventually providing an explicit parameterisation of the 20 integer charges of the νMSSM chiral superfields in terms of some integer parameters. We then provide a right inverse, which, given a solution to the ACCs, returns parameters which will lead to that solution.

Model	Q	Q	Q	n	n	n	e	e	e	u	u	u	d	d	d	L	L	L	L	$\tilde{H}_u$
Y_3'	-1	-1	1	0	0	0	-6	6	6	-4	-4	4	-2	2	2	-3	3	3	3	-3
B_3'	-1	-1	1	-3	3	3	-3	3	3	-1	-1	1	-1	-1	1	-3	3	3	3	-3
Ref. [24], Table 3	0	0	0	-3	0	0	-1	1	3	-1	-1	-1	1	1	1	-2	0	1	2	-1
Ref. [24], Table 4	-1	-1	-1	-9	0	0	-9	0	0	1	1	1	1	1	1	0	0	0	9	0
Ref. [26]	-1	0	0	-1	0	4	-1	0	4	-1	0	0	-1	0	0	-1	0	0	4	0
Ref. [27]	-1	0	0	1	1	1	1	1	1	-1	0	0	-1	0	0	0	1	1	1	0
SUSY B-L [25]	-1	-1	-1	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	0	3	3	3	0
TFHM [10]	-1	0	0	0	0	0	0	0	6	-4	0	0	0	0	2	0	0	0	3	0

Table 3. Some examples of anomaly-free charge assignments found. Here, we list the $\mathfrak{u}(1)_X$ charges of the (left-handed or right-handed) chiral fermions of each model. These include the solutions Y_3', B_3' used to derive the analytic solution of §4 as well as the non-SUSY Third Family Hypercharge (TFHM) solution which we expect to be contained within our list. Note that all solutions have been rescaled and reordered to satisfy the format of our list as detailed in §3.

Such an inverse has the dual purpose of facilitating checks between the numerical and analytic solutions and of providing an additional proof that our solution is generic.

4.1 Geometric framing of the problem

The ACCs form a set of polynomial equations in the integers - otherwise called *diophantine equations*. Suppose we take account of only the physical equivalence defined by scaling (point (ii) in §2). It then does not matter, from a mathematical point of view, whether we use the label L_4 or H_d for the relevant chiral superfield; here we shall choose the latter. We can view the unknown charges as corresponding to points in the projective space $\mathrm{P}\mathbb{Q}^{19}$. This is formed by considering the charges as living in the rationals $\mathbb{Q}^{20}$, removing the origin and providing an equivalence relation between points in $\mathbb{Q}^{20}$ differing by rational multiples. The points satisfying the ACCs in $\mathrm{P}\mathbb{Q}^{19}$ are said to form a *projective variety*.

One might expect our solution to be parameterised by 14 independent integer-valued parameters (starting with 20 and subtracting 6 for the ACCs). However, as we shall see, we shall have to add 9 parameters to cover exceptional cases, making the total number of integer parameters 23. Our solution will then take the form of a map from $\mathbb{Z}^{23}$ to $\mathrm{P}\mathbb{Q}^{19}$ which satisfies the following properties: its image is completely within the projective variety, it surjects onto the projective variety and its value depends only on the projection onto a $\mathbb{Z}^{14}$ subspace of $\mathbb{Z}^{23}$ in all but a few classes of exceptional cases.

Within $\mathrm{P}\mathbb{Q}^{19}$, the ACCs (2.4) and (2.5) define a cubic and a quadratic hypersurface, respectively (we shall below refer to these as 'the cubic' and 'the quadratic', respectively, for brevity).

To solve systems of diophantine equations, number theorists often use a small set of solutions as a tool for finding all solutions. Given our extensive numerical scan we are in

a position to make an attempt in this manner. For a generic set of equations this is not guaranteed to be possible, however we are lucky in that for our particular set of ACCs, at least two distinct methods exist. The first mirrors the method of Ref. [18] which exploits a special point of $\mathrm{P}\mathbb{Q}^{19}$ that is a 'double point' of both the cubic and quadratic. A second new method is presented here.

4.2 Sketch of the method

The linear ACCs are easy to deal with. Their solution defines a projective subspace PL of $\mathrm{P}\mathbb{Q}^{19}$. It is in PL that we must discuss the quadratic and the cubic.

Given a single solution to the quadratic ACC it is possible to find all solutions to the quadratic ACC by constructing all possible lines through this known solution: along each line there must be one further solution to the quadratic, since every rational quadratic in one dimension has either two or zero rational roots. In a similar vein, given a single solution to the cubic, with the special property that all first order partial derivatives vanish at this point, it is possible to find all solutions to the cubic ACC by constructing lines through this point. Such a point is called a *double point* of the cubic.

To solve both the quadratic and the cubic simultaneously it is sufficient to have a *line* on which every point is a solution to the quadratic and every point is a double point of the cubic (although as noted above, other methods do exist). In fact for us there is only one such line (up to permutations of charges within the $\mathfrak{su}(2)$-doublet, $\mathfrak{su}(3)$-singlet sector, and other species), which is the one between the points Y_3 and B_3 given in Table 4. These two points are a reordering of the charges within Y_3' and B_3', respectively, from Table 2. The first point, Y_3, corresponds to hypercharge except for the third family, which has had its charges sign changed. The second point, B_3, corresponds to $B-L$ where the third family has had its charge's sign changed and the charges X_{H_u} and X_{H_d} are modified from their usual values of zero. We will denote the line between them Y_3B_3.

To see how Y_3B_3 will enable us to find all solutions, let us first define the space PL$'$, defined to be the subspace of PL whose points are orthogonal to Y_3 and B_3 with respect to the standard scalar product on $\mathbb{Q}^{20}$. Every point in PL lies on a plane Y_3B_3R formed by Y_3B_3 and a point $R \in \mathrm{PL}'$. Thus, we can restrict our attention to looking at such planes, and the points within them which satisfy the ACCs.

Generically (we will look at the few exceptions shortly), the intersection of the quadratic with Y_3B_3R consists of the union of Y_3B_3 and another line L_q, as we will see explicitly in the next subsection. In a similar way, the intersection of the cubic with Y_3B_3R consists of the union of Y_3B_3 and another line L_c. The intersection of the projective variety defined by the ACCs, and Y_3B_3R then consists of the line Y_3B_3 and a single point which is the intersection of L_q and L_c as shown in the top left-hand panel of Fig. 2. Finding this point, which is a new solution to the ACCs, is a trivial task, as we shall shortly see.

Let us now look at the exceptional cases, all of which are illustrated in Fig. 2. They correspond to the following situations: (a) the whole plane lies in the quadratic but not the cubic; (b) the whole plane lies in both the quadratic and the cubic; (c) the whole line L_q lies in

	Q	Q	Q	n	n	n	e	e	e	u	u	u	d	d	d	L	L	L	$\tilde{H}_d$	$\tilde{H}_u$
Y_3	1	1	-1	0	0	0	-6	-6	6	4	4	-4	-2	-2	2	-3	-3	3	-3	3
B_3	1	1	-1	-3	-3	3	-3	-3	3	1	1	-1	1	1	-1	-3	-3	3	-3	3

Table 4. A new ordering for the anomaly-free charge assignments Y_3' and B_3' given in Table 2, adapted for the analytic solution. Each row lists the $\mathfrak{u}(1)_X$ charges of the (left-handed or right-handed) chiral fermions of a model.

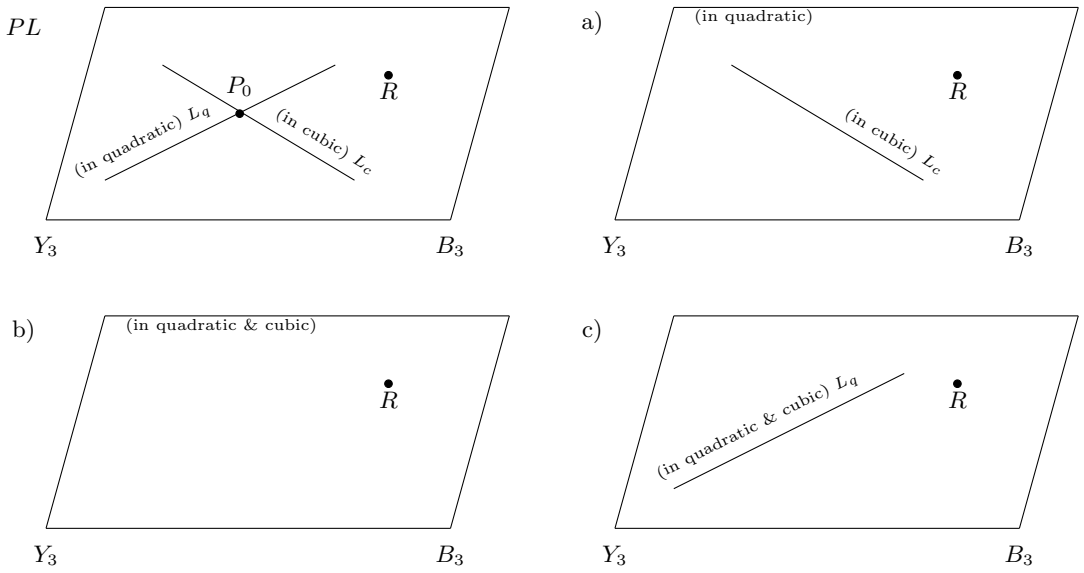

Figure 2. A schematic of our method. In the generic case (top left), the plane contains one further line in the quadratic and one further line in the cubic. These intercept at a point P_0, which is our new solution to all ACCs. The exceptional cases are shown in diagrams (a), (b) and (c). In (a), L_c is a line of solutions to all ACCs. In (b), the Y_3B_3R is a whole plane of solutions to all ACCs and in (c), the line L_q is a line of solutions to all ACCs.

the cubic. The asymmetry between the quadratic and the cubic here is simply a manifestation of the ordering in which we will do our manipulations in the next subsection, and nothing more subtle.

We reiterate that since every point in PL lies in a plane Y_3B_3R, by finding all solutions in all such planes (considering either the generic case or the exceptional cases) we can find every point in the projective variety. In §4.4 we will give an explicit parameterisation of the solution formed by such considerations. In §4.5 a right-inverse to this parametrisation will be given explicitly demonstrating its full generality.

4.3 Derivation of the analytic solution

We now give a more detailed description of our solution. To this end we define

$$\begin{aligned} q(X,Z) &:= \sum_{i=1}^{3}(X_{Q_i}Z_{Q_i} - X_{L_i}Z_{L_i} - 2X_{u_i}Z_{u_i} + X_{d_i}Z_{d_i} + X_{e_i}Z_{e_i}) + X_{H_u}Z_{H_u} - X_{H_d}Z_{H_d}, \\ c(W,X,Z) &:= \sum_{i=1}^{3}(6W_{Q_i}X_{Q_i}Z_{Q_i} + 2W_{L_i}X_{L_i}Z_{L_i} - 3W_{u_i}X_{u_i}Z_{u_i} - 3W_{d_i}X_{d_i}Z_{d_i} - W_{e_i}X_{e_i}Z_{e_i} \\ &\quad - W_{n_i}X_{n_i}Z_{n_i}) + 2W_{H_u}X_{H_u}Z_{H_u} + 2W_{H_d}X_{H_d}Z_{H_d}, \end{aligned} \tag{4.1}$$

which are respectively derived from the quadratic and cubic ACCs with e.g. $X_{Q_1}^3$ replaced with $W_{Q_1}X_{Q_1}Z_{Q_1}$. The maps q and c are the unique trilinear forms which return, respectively, the quadratic (2.5) ACC and the cubic ACC (2.4), when all inputs coincide.

A point $R \in \mathrm{PL}'$ can be parameterised by the 12 charges R_{S_1} for $S \in \{Q, n, e, u, L, d\}$, R_{S_2} for $S \in \{e, L, d\}$, R_{d_3}, R_{H_u} and R_{H_d} as well as an extra two parameters R_1 and R_2. The remaining charges are given by

$$\begin{aligned} R_{Q_2} &= 3(R_1 + R_{H_d} + R_{L_1} + R_{L_2} + R_{Q_1}) + 4R_2 + 2R_{d_3} + R_{e_1} + R_{e_2}, \\ R_{Q_3} &= -4(R_1 + R_2 + R_{Q_1}) - 2(R_{d_3} + R_{e_1} + R_{e_2}) - 3(R_{H_d} + R_{L_1} + R_{L_2}), \\ R_{n_2} &= 4R_1 + R_2 + 3(R_{e_1} + R_{e_2}) - R_{n_1} + R_{Q_1}, \\ R_{n_3} &= -(2R_1 + R_2 + R_{d_1} + R_{d_2} + R_{d_3} + R_{e_1} + R_{e_2} + R_{Q_1}), \\ R_{e_3} &= (R_{d_1} + R_{d_2} + R_{d_3}) + 3(R_{e_1} + R_{e_2}) + 4R_1, \\ R_{u_2} &= -(6R_1 + R_2 + R_{d_1} + R_{d_2} + 2R_{d_3} + 4(R_{e_1} + R_{e_2}) + R_{Q_1} + R_{u_1}), \\ R_{u_3} &= 4R_1 + R_2 + R_{d_3} + 2(R_{e_1} + R_{e_2}) + R_{Q_1}, \\ R_{L_3} &= 3(R_1 + R_{e_1} + R_{e_2}) - (R_{L_1} + R_{L_2}) - (R_{H_u} + R_{H_d}). \end{aligned} \tag{4.2}$$

Substituting the generic point, $\alpha Y_3 + \beta B_3 + \gamma R$, on the plane Y_3B_3R into the quadratic gives

$$\gamma(2\alpha q(Y_3, R) + 2\beta q(B_3, R) + \gamma q(R, R)) = 0. \tag{4.3}$$

Putting the exceptional cases to one side for now, this equation generically has two lines of solutions: one specified by $\gamma = 0$, namely Y_3B_3, and a new line L_q, a general point of which is given by

$$\begin{aligned} P^{(c)}_{c_1,c_2,c_3} =& \{c_2 q(R,R) - 2c_3 q(B_3,R)\}Y_3 + \{2c_3 q(Y_3,R) - c_1 q(R,R)\}B_3 + \\ &+ \{c_1 q(B_3,R) - c_2 q(Y_3,R)\}R, \end{aligned} \tag{4.4}$$

where $c_1, c_2, c_3 \in \mathbb{Z}$ (over-)parameterise the line.[8]

[8] Our use of projective space allows us to use, by clearing denominators, $\mathbb{Z}$ for these parameters rather than $\mathbb{Q}$.

On making the same substitution into the cubic we would get a similar line. However, since we are only interested in the intersection of these two lines, it is sufficient to substitute $P^{(c)}_{c_1,c_2,c_3}$ into the cubic. This yields

$$\begin{aligned}4\{c_1q(B_3,R)-c_2q(Y_3,S)\}^2\big\{&\{2q(B_3,R)c(R,R,R)3q(R,R)c(B_3,R,R)\}c_1+\\&\{3q(R,R)c(Y_3,R,R)-2q(Y_3,R)c(R,R,R)\}c_2+\\&6\{q(Y_3,R)c(B_3,R,R)-q(B_3,R)c(Y_3,R,R)\}c_3\big\}=0.\end{aligned}\tag{4.5}$$

Solving for c_1, c_2, c_3 generically gives the new solution to the ACCs

$$\begin{aligned}P_0=&\{3q(R,R)c(B_3,R,R)-2q(B_3,R)c(R,R,R)\}Y_3+\\&\{2q(Y_3,R)c(R,R,R)-3q(R,R)c(Y_3,R,R)\}B_3+\\&6\{q(B_3,R)c(Y_3,R,R)-q(Y_3,R)c(B_3,R,R)\}R.\end{aligned}\tag{4.6}$$

Let us now return to the exceptional cases.

(a) The plane lies entirely in the quadratic, but not in the cubic: This occurs when $q(Y_3,R)=0$, $q(B_3,R)=0$ and $q(Y_3,R)=0$ but at least one of $c(Y_3,R,R)$, $c(B_3,R,R)$ and $c(R,R,R)$ is non-zero. In this case, we have a line of solutions (over-)parameterised by a_1, a_2, and $a_3\in\mathbb{Z}$ and given by

$$\begin{aligned}P^{(a)}_{a_1,a_2,a_3}=&\{a_2c(R,R,R)-3a_3c(B_3,R,R)\}Y_3+\{3a_3c(Y_3,R,R)-a_1c(R,R,R)\}B_3+\\&3\{a_1c(B_3,R,R)-a_2c(Y_3,R,R)\}R.\end{aligned}\tag{4.7}$$

(b) The plane lies entirely within the quadratic and the cubic: This occurs when $q(Y_3,R)=0$, $q(B_3,R)$, $q(Y_3,R)=0$, $c(Y_3,R,R)=0$, $c(B_3,R,R)=0$ and $c(R,R,R)=0$. In this case, every point on the plane lies in the variety. We then parameterise the plane with$b_1,b_2,b_3\in\mathbb{Z}$:

$$P^{(b)}_{b_1,b_2,b_3}=b_1Y_3+b_2B_3+b_3R\tag{4.8}$$

(c) The line in the quadratic and the cubic are the same lines: this occurs (excluding the case where the line is just $\alpha Y_3+\beta B_3$) when

$$\begin{aligned}2q(B_3,R)c(R,R,R)&=3q(R,R)c(B_3,R,R),\\2q(Y_3,R)c(R,R,R)&=3q(R,R)c(Y_3,R,R),\\q(B_3,R)c(Y_3,R,R)&=q(Y_3,R)c(B_3,R,R).\end{aligned}\tag{4.9}$$

In this case our solution is the line $P^{(c)}_{c_1,c_2,c_3}$.

It is possible to combine these exceptional cases and the generic case into one parameterisation of the solution using Kronecker delta functions. This overall parameterisation is

given by

$$\begin{aligned}P =& P_0 + \delta_{q(Y_3,R),0}\delta_{q(B_3,R),0}\delta_{q(R,R),0}\{P^{(a)}_{a_1,a_2,a_3} + \delta_{c(Y_3,R,R),0}\delta_{c(B_3,R,R),0}\delta_{c(R,R,R),0}P^{(b)}_{b_1,b_2,b_3}\}+\\ &\delta_{2q(B_3,R)c(R,R,R),3q(R,R)c(B_3,R,R)}\delta_{2q(Y_3,R)c(R,R,R),3q(R,R)c(Y_3,R,R)}\times\\ &\delta_{q(B_3,R)c(Y_3,R,R),q(Y_3,R)c(B_3,R,R)}P^{(c)}_{c_1,c_2,c_3}.\end{aligned} \tag{4.10}$$

This parameterisation is written in terms of the 12 charges and two extra parameters specifying R, as well as the parameters a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2 and c_3, which are needed in the exceptional cases. Taking these parameters to be integers returns an integer-valued solution.

4.4 Explicit parameterisation

To write the parameterisation more explicitly, we define

$$\begin{aligned}\Gamma :=& \{3q(R,R)c(B_3,R,R) - 2q(B_3,R)c(R,R,R)\}+\\ &\delta_{q(Y_3,R),0}\delta_{q(B_3,R),0}\delta_{q(R,R),0}\\ &(a_2c(R,R,R) - 3a_3c(B_3,R,R) + \delta_{c(Y_3,R,R),0}\delta_{c(B_3,R,R),0}\delta_{c(R,R,R),0}b_1)\\ &+ \delta_{2q(B_3,R)c(R,R,R),3q(R,R)c(B_3,R,R)}\delta_{2q(Y_3,R)c(R,R,R),3q(R,R)c(Y_3,R,R)}\\ &\delta_{q(B_3,R)c(Y_3,R,R),q(Y_3,R)c(B_3,R,R)}(c_2q(R,R) - 2c_3q(B_3,R)),\\ \Sigma :=& \{2q(Y_3,R)c(R,R,R) - 3q(R,R)c(Y_3,R,R)\}+\\ &\delta_{q(Y_3,R),0}\delta_{q(B_3,R),0}\delta_{q(R,R),0}\times\\ &\{3a_3c(Y_3,R,R) - a_1c(R,R,R) + \delta_{c(Y_3,R,R),0}\delta_{c(B_3,R,R),0}\delta_{c(R,R,R),0}b_2\}+\\ &\delta_{2q(B_3,R)c(R,R,R),3q(R,R)c(B_3,R,R)}\delta_{2q(Y_3,R)c(R,R,R),3q(R,R)c(Y_3,R,R)}\\ &\delta_{q(B_3,R)c(Y_3,R,R),q(Y_3,R)c(B_3,R,R)}\{2c_3q(Y_3,R) - c_1q(R,R)\},\\ \Lambda :=& 6\{q(B_3,R)c(Y_3,R,R) - q(Y_3,R)c(B_3,R,R)\}+\\ &\delta_{q(Y_3,R),0}\delta_{q(B_3,R),0}\delta_{q(R,R),0}\times\\ &\{3\{a_1c(B_3,R,R) - a_2c(Y_3,R,R)\} + \delta_{c(Y_3,R,R),0}\delta_{c(B_3,R,R),0}\delta_{c(R,R,R),0}b_3\}+\\ &\delta_{2q(B_3,R)c(R,R,R),3q(R,R)c(B_3,R,R)}\delta_{2q(Y_3,R)c(R,R,R),3q(R,R)c(Y_3,R,R)}\\ &\delta_{q(B_3,R)c(Y_3,R,R),q(Y_3,R)c(B_3,R,R)}\{2c_1q(B_3,R) - 2c_2q(Y_3,R)\}.\end{aligned} \tag{4.11}$$

Then the charges are given explicitly by fourth order polynomials in the coordinates of R:

$$\begin{aligned}
X_{Q_1} &= \Gamma + \Sigma + \Lambda R_{Q_1}, & X_{Q_2} &= \Gamma + \Sigma + \Lambda R_{Q_2}, & X_{Q_3} &= -\Gamma - \Sigma + \Lambda R_{Q_3},\\
X_{n_1} &= -3\Sigma + \Lambda R_{n_1}, & X_{n_2} &= -3\Sigma + \Lambda R_{n_2}, & X_{n_3} &= 3\Sigma + \Lambda R_{n_3},\\
X_{e_1} &= -6\Gamma - 3\Sigma + \Lambda R_{e_1}, & X_{e_2} &= -6\Gamma - 3\Sigma + \Lambda R_{e_2}, & X_{e_3} &= 6\Gamma + 3\Sigma + \Lambda R_{e_3},\\
X_{u_1} &= 4\Gamma + \Sigma + \Lambda R_{u_1}, & X_{u_2} &= 4\Gamma + \Sigma + \Lambda R_{u_2}, & X_{u_3} &= -4\Gamma - \Sigma + \Lambda R_{u_3},\\
X_{L_1} &= -3\Gamma - 3\Sigma + \Lambda R_{L_1}, & X_{L_2} &= -3\Gamma - 3\Sigma + \Lambda R_{L_2}, & X_{L_3} &= 3\Gamma + 3\Sigma + \Lambda R_{L_3},\\
X_{d_1} &= -2\Gamma + \Sigma + \Lambda R_{d_1}, & X_{d_2} &= -2\Gamma + \Sigma + \Lambda R_{d_2}, & X_{d_3} &= 2\Gamma - \Sigma + \Lambda R_{d_3},\\
X_{H_u} &= 3\Gamma + 3\Sigma + \Lambda R_{H_u}, & X_{H_d} &= -3\Gamma - 3\Sigma + \Lambda R_{H_d}.
\end{aligned} \tag{4.12}$$

4.5 Right inverse

As previously mentioned, this analytic solution has a right inverse, demonstrating its complete generality. Specifically, let T be a known solution and define the point $G = 108T - (Y_3 \cdot T - B_3 \cdot T)Y_3 - (2B_3 \cdot T - Y_3 \cdot T)B_3$, where '$\cdot$' is the usual scalar product. The point G can be thought of as T with its components in the line $\alpha Y_3 + \beta B_3$ projected out. The parameters $R_{X_j} = G_{X_j}$ (for X_j as above), and

$$\begin{aligned}
R_1 &= -\sum_{i=1}^{3}(G_{d_i} + G_{L_i}) + G_{e_3} - G_{H_u} - G_{H_d},\\
R_2 &= \sum_{i=1}^{3}(G_{d_i} - G_{e_i} + 2G_{L_i}) - G_{e_3} + 2G_{H_u} + 2G_{H_d} - G_{Q_1} - G_{n_2},\\
a_1 &= c(B_3, T, T), \quad a_2 = -c(Y_3, T, T),\\
a_3 &= -c(B_3, T, T)(Y_3 \cdot T - B_3 \cdot T) + c(Y_3, T, T)(2B_3 \cdot T - Y_3 \cdot T).\\
b_1 &= (Y_3 \cdot T - B_3 \cdot T) \quad b_2 = (2B_3 \cdot T - Y_3 \cdot T) \quad b_3 = 1\\
c_1 &= q(B_3, T), \quad c_2 = -q(Y_3, T),\\
c_3 &= -q(B_3, T)(Y_3 \cdot T - B_3 \cdot T) + q(Y_3, T)(2B_3 \cdot T - Y_3 \cdot T),
\end{aligned} \tag{4.13}$$

return the point T when substituted into the above analytic solution. In fact, they return T up to a multiplicative constant given by

$$\begin{aligned}
&6 \times 108^4(q(B_3, T)c(Y_3, T, T) - q(Y_3, T)c(B_3, T, T))\\
&+ \delta_{q(Y_3,T),0}\delta_{q(B_3,T),0}(3 \times 108^3(c(B_3, T, T)^2 + c(Y_3, T, T)^2) + 108\delta_{c(Y_3,T,T),0}\delta_{c(B_3,T,T),0})\\
&+ 2 \times 108^2\delta_{q(B_3,T)c(Y_3,T,T),q(Y_3,T)c(B_3,T,T)}(q(B_3, T)^2 + q(Y_3, T)^2)
\end{aligned} \tag{4.14}$$

but given that our discussion above has been implicitly in projective space, such multiplicative factors are not relevant.

In the `Zenodo` repository [30] we provide a `Mathematica`™ script containing the analytic solution, allowing one to generate solutions at will.

5 Checks of the solutions

The material content of §3 is a list of all inequivalent anomaly-free charge assignments up to a fixed $Q_{\max}$. A skeptic could justly ask the question: how does one know this list is complete without redundancies? The algorithm used does guarantee it, but one wishes to mitigate potential errors involved in its computer implementation. A similar level of scrutiny can be applied to the analytic solution of §4. Although here one might hope the correctness of the solution is mathematically clear-cut, due diligence requires that we should try to ensure that no fallacies have been committed. Happily, several checks can be carried out to satisfy all but the most fastidious skeptic. These checks work in three different modes: consistency checks within the numerical solutions, consistency checks within the analytic solution alone,

and cross-checks between the two. The ability to do cross-checks between the two is one of several advantages for providing both. Let us discuss the checks performed for each mode in turn. We note in passing that all checks were carried out successfully.

For any computer program, one useful check is to make a second structurally different program but with the same expected outcome. To this end, we produced a second different program (this one did not use lexicographic ordering, but instead used an ordering similar to that in Ref. [10]). The two outputs where then compared and found to agree.

The addition of hypercharge to any solution also leads to a solution, as stated in (iii) of §2.1. This provides a check of the computer program as follows: each solution for a given $Q_{\max}$ had multiples of hypercharge added or subtracted from it up to three times. If the resulting charges had a height less than or equal to 10, the binary search method discussed in §3 was used to confirm that the solution was present in our $Q_{\max} = 10$ list.

Turning to the analytic solution, the most primitive check is to randomly choose parameters, generate the corresponding charges and confirm that they satisfy the ACCs. This check was carried out on 10^5 randomly generated solutions.

The fact that we have a right inverse for our parameterisation means that we can take a solution, apply the inverse and then the parameterisation to return another solution. If our analysis is correct this new solution should agree with the one we started with (up to a scaling). This was carried out on, again, 10^5 randomly generated solutions. It was also carried out on all the scanned solutions in our list for $Q_{\max}= 10$, thereby providing the first cross check between the numerical and analytic solutions.

The second cross-check between the numerical and analytic solutions was to generate random solutions using the analytic solution, then to identify those of height less than or equal to 10 and confirm that these appear in the numerical solution via the binary search algorithm.

6 Examples of Filters

In this section, we now turn to examples of how our list of solutions to ACCs (2.1)-(2.5) might be filtered in order to identify sets of charge assignments with various possible desirable phenomenological properties or uses.[9] Note that in what follows, as in §4, we will distinguish H_d from L_i, and the number of solutions satisfying each constraint is therefore to be compared with the second column of Table 2.

[9] Computer programs implementing these filters are available on `Zenodo` [30].

6.1 The superpotential

In general, interactions between the chiral supermultiplets of the MSSM are given by the superpotential $W = W_{R_p} + W_{LV} + W_{BV}$, where

$$\begin{aligned} W_{R_p} &= \mu \hat{H}_u \hat{H}_d + (y_u)_{ij} \hat{U}^c_i \hat{Q}_j \hat{H}_u + (y_d)_{ij} \hat{D}^c_i \hat{Q}_j \hat{H}_d + (y_e)_{ij} \hat{E}^c_i \hat{L}_j \hat{H}_d, \\ W_{LV} &= \frac{1}{2} \lambda^{ijk} \hat{L}_i \hat{L}_j \hat{E}^c_k + \lambda'^{ijk} \hat{L}_i \hat{Q}_j \hat{D}^c_k + \mu'^i \hat{L}_i \hat{H}_u, \\ W_{BV} &= \frac{1}{2} \lambda''^{ijk} \hat{U}^c_i \hat{D}^c_j \hat{D}^c_k. \end{aligned} \tag{6.1}$$

$\hat{U}^c_i$, $\hat{D}^c_i$, $\hat{Q}_i$, $\hat{L}_i$, $\hat{E}^c_i$, $\hat{H}_u$ and $\hat{H}_d$ denote the chiral supermultiplets containing of Table 3, and we denote flavour indices by $i, j, k \in \{1, 2, 3\}$. λ^{ijk}, λ'^{ijk}, λ''^{ijk}, $(y_{u,d,e})_{ij}$ are all dimensionless coupling constants and μ, μ'^i each have mass dimension 1. Gauge indices have been suppressed. Note that here we ignore the neutrino chiral supermultiplets $\hat{N}^c_i$, postponing their discussion until §6.3. Here W_{R_p} denotes terms invariant under $R-$parity, whereas $R-$parity is violated in the L and B-violating terms W_{LV} and W_{BV} respectively.

6.1.1 The μ problem

The MSSM has a fine tuning problem associated with the $\mu \hat{H}_u \hat{H}_d$ term. Given that this term respects supersymmetry and gauge symmetry, there is no explicitly stated reason for the scale of μ to be small. The gauge group can be extended by $U(1)_X$ to provide a solution to this so-called μ problem [31]. This is achieved by charging $\hat{H}_u$ and $\hat{H}_d$ under $U(1)_X$ such that the μ term above is forbidden by the $U(1)_X$ symmetry. Instead, the flavon θ is charged, allowing a term of the form (where h is a dimensionless coupling constant)

$$W \supset h\theta \hat{H}_u \hat{H}_d \to h\langle\theta\rangle \hat{H}_u \hat{H}_d, \tag{6.2}$$

such that when the $U(1)_X$ symmetry is spontaneously broken, the scalar component of θ acquires a vacuum expectation value $\langle\theta\rangle$ at the TeV scale i.e. the μ term is dynamically generated.[10] The $\mu\nu$SSM [32–35] also solves the μ problem in precisely this manner. Any model with such a dynamically generated μ term is often referred to as the next-to-minimal supersymmetric standard model (NMSSM). The NMSSM has received much attention in the literature [36–39].

Remembering that we shall pick one of the $\hat{N}^c_i$ chiral superfields with a non-zero charge to be the flavon chiral superfield θ^c, which has a non-zero X charge out of necessity, we search for such solutions in our list of charges by applying the conditions

$$\exists i \in \{1, 2, 3\} : \quad X_{H_u} + X_{H_d} = X_{n_i} \neq 0, \tag{6.3}$$

where we take θ^c to be the $\hat{N}^c_i$ superfield which satisfies this condition.[11] We find a total of 77 solutions satisfying these constraints with $Q_{\max} = 1$, constituting $\sim 30\%$ of the full $Q_{\max} = 1$

[10] Further detailed model building is required to make sure that $\langle\theta\rangle \sim \mathcal{O}(\text{TeV})$, but we shall merely assume here that this is possible.

[11] The U$\mu\nu$SSM [40] uses (6.3) in a certain $U(1)'$ extension of the νMSSM (involving additional quark fields) to solve the μ problem, also.

list. This percentage reduces to 20% when $Q_{\max} = 4$, and 11% when $Q_{\max} = 10$, providing in this case a total of 649 831 168 options for a dynamically generated μ term.

6.1.2 A renormalisable Yukawa sector

In contrast to the rather weak constraints of (6.3), we may place strong conditions on the Yukawa sector by requiring that all renormalisable Yukawa couplings of charged fermions are allowed in the superpotential W_{R_p} by being $U(1)_X$ gauge invariant, i.e. they must satisfy the following equations $\forall i, j \in \{1, 2, 3\}$:

$$X_{Q_i} + X_{H_u} - X_{u_j} = 0, \qquad X_{Q_i} + X_{H_d} - X_{d_j} = 0, \qquad X_{L_i} + X_{H_d} - X_{e_j} = 0. \tag{6.4}$$

(6.4) implies family universality for the species Q, e, u, L and d. For the non-supersymmetric case, it has been shown that anomaly-free charge assignments exist which allow all of the renormalisable Yukawa terms [22]. One can show that in the νMSSM, we obtain one solution for each non-supersymmetric solution of [22], where we must additionally fix H_u and H_d to satisfy

$$3X_{H_u} = -3X_{H_d} = -3\sum_{i=1}^{3} X_{Q_i} - \sum_{i=1}^{3} X_{n_i}. \tag{6.5}$$

(6.5) means that there cannot be any overlap with the solutions satisfying (6.3), i.e. none of these solutions can simultaneously solve the μ problem. By filtering through our list of charges, we find 2 solutions allowing a fully renormalisable Yukawa sector with $Q_{\max} = 1$ and 5 with $Q_{\max} = 4$, as shown in Table 5. The full list of $Q_{\max} = 10$ solutions comprises 38 such solutions.

Q	Q	Q	n	n	n	e	e	e	u	u	u	d	d	d	L	L	L	$\tilde{H}_d$	$\tilde{H}_u$
0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	0	0	0	1	-1
-1	-1	-1	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	3	3	3	0	0
-1	-1	-1	2	2	2	4	4	4	-2	-2	-2	0	0	0	3	3	3	1	-1
-1	-1	-1	4	4	4	2	2	2	0	0	0	-2	-2	-2	3	3	3	-1	1

Table 5. Anomaly-free charge assignments with $Q_{\max} = 4$ allowing all Yukawa terms at the renormalisable level. Each row lists the $\mathfrak{u}(1)_X$ charges of the (left-handed or right-handed) chiral fermions of a model. Note that all listed solutions satisfy $X_{H_u} + X_{H_d} = 0$, reducing the ACCs to those of the SM after substitution.

We will now relax the assumption that all Yukawa terms must be present in the Lagrangian at the renormalisable level. We will enforce that the top and bottom quark and the tau lepton tree-level Yukawa terms can be present (since they are closer to order 1 and so more difficult to explain by non-renormalisable or loop interactions, which imply a suppression

below order 1) by applying the constraints

$$\begin{aligned} \exists \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \in S_3 : \quad & X_{Q_{\sigma_1(3)}} + X_{H_u} - X_{u_{\sigma_2(3)}} = 0, \\ & X_{Q_{\sigma_1(3)}} + X_{H_d} - X_{d_{\sigma_3(3)}} = 0, \\ & X_{L_{\sigma_4(3)}} + X_{H_d} - X_{e_{\sigma_5(3)}} = 0, \end{aligned} \tag{6.6}$$

where S_3 is the group of permutations of 3 objects. We expect $Q_{\sigma_1(3)}$, $u_{\sigma_2(3)}$ and $d_{\sigma_3(3)}$ to be predominantly third generation quarks, and similarly $L_{\sigma_4(3)}$ and $e_{\sigma_5(3)}$ to be predominantly composed of third generation leptons. We will further assume that tree-level renormalisable Yukawa terms are not present for the first and second generation fermions by forbidding all other terms in the Yukawa matrices. We can express these constraints by first defining

$$P_{ijklmn} := (X_{Q_i} + X_{H_u} = X_{u_j}) \wedge (X_{Q_n} + X_{H_d} = X_{d_k}) \wedge (X_{L_l} + X_{H_d} = X_{e_m}), \tag{6.7}$$

(where $\wedge$ means logical 'and') and then imposing

$$(\exists ! \, i, j, k, l, m, n \in \{1, 2, 3\} : \, P_{ijklmn}) \wedge (\forall i, j, k, l, m, n \in \{1, 2, 3\} \; P_{ijklmn} \Rightarrow n = i), \tag{6.8}$$

where, in standard logic notation, $\exists !$ means 'there exists a unique'.

This choice is made with the fermion mass problem in mind: it allows larger masses to be generated for the top, bottom and tau through the standard Yukawa terms, but forbids them for the light quarks, producing a mass hierarchy between the light and heavy fermions. In Ref. [24] it was shown that the chiral fermions can obtain their masses at loop level through the interactions with their superpartners by including non-holomorphic soft terms in the Lagrangian density. Alternatively, light fermion masses may be acquired through non-renormalisable operators after the flavon θ breaks $U(1)_X$. Either of these mechanisms require the Lagrangian density to contain terms which will further constrain the charges. We shall assume that all first and second generation fermions acquire their masses through some mechanism such as one of these two, but leave the more model dependent effect of any additional constraints to future investigations.

We find that when $Q_{\max} = 1$, the list contains 2 solutions satisfying the constraints of (6.8). At $Q_{\max} = 4$ a total of 15 818 solutions pass these constraints, and at $Q_{\max} = 10$ this number grows to 34 646 735. This makes clear that by imposing these constraints, not only do we begin to address the fermion mass problem, but we make way for a larger number of options for model-building compared to those of a fully allowed renormalisable Yukawa sector. For example, when $Q_{\max} = 2$ there are 8 solutions which simultaneously solve the μ problem and satisfy (6.8). This overlap grows to 2 954 solutions when $Q_{\max} = 4$ and 4 088 200 solutions when $Q_{\max} = 10$. Furthermore, the constraints (6.8) are inherently flavour non-universal, and thus have the potential to address the B anomalies. This overlap will be discussed in more detail in §6.2.

6.1.3 $R-$parity violation

In contrast to the SM, L and B violating terms are allowed by the field content and gauge symmetries of the MSSM, as shown in (6.1). The simultaneous presence of both B and L

violating terms will lead to proton decay in contravention to experimental bounds unless one introduces a large degree of fine tuning. Usually, all terms in W_{LV} and W_{BV} are forbidden by the imposition of $R-$parity. In the case that $R-$parity is not imposed though, we may ask that our $U(1)_X$ symmetry maintains the stability of the proton instead. We can form three broad sets of solution within this requirement: where *all* $R-$parity violating terms are banned (this will also maintain the stability of the lightest supersymmetric particle, which may have the properties to constitute cold dark matter), where all terms in W_{BV} are banned but where at least one term in W_{LV} is allowed, and those where all terms in W_{LV} are banned but at least one in W_{BV} is allowed. Terms such as those in W_{LV} give a Majorana mass term to left-handed neutrinos (sometimes through loop diagrams) without the need for right-handed neutrinos [41]. Terms in W_{BV}, on the other hand, can assist in baryogenesis [42].

We may ban terms in W_{BV} by imposing $\forall i, j, k \in \{1, 2, 3\}$

$$X_{u_i} + X_{d_j} + X_{d_k} \neq 0 \tag{6.9}$$

where $j \neq k$ since the antisymmetry of λ''^{ijk} in j, k forbids the $j = k$ terms from appearing in the superpotential. Similarly, we may ban all terms in W_{LV} by imposing the conditions $\forall i, j, k, l, m, n, p \in \{1, 2, 3\}$

$$X_{L_i} + X_{L_j} - X_{e_k} \neq 0, \qquad X_{L_l} + X_{Q_m} - X_{d_n} \neq 0, \qquad X_{L_p} + X_{H_u} \neq 0, \tag{6.10}$$

where $i \neq j$ because λ^{ijk} is antisymmetric in i, j. At $Q_{max} = 1$ we find 8 solutions which ban all $R-$parity violating terms. These solutions are listed in Table 6. We find a total of 51 solutions which ban W_{BV} while allowing terms in W_{LV}. We find no solutions which ban W_{LV} while allowing terms in W_{BV}, i.e. the only solutions which ban W_{LV} are those which ban all $R-$parity violation.

Q	Q	Q	n	n	n	e	e	e	u	u	u	d	d	d	L	L	L	$\tilde{H}_d$	$\tilde{H}_u$
-1	-1	1	1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	1	1	1	1
-1	-1	1	1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	1	1	0	0
-1	-1	1	1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	1	1	1	0
-1	-1	1	1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	1	1	1	-1
0	0	0	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	0	0	0	-1	1
0	0	0	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	0	0	1	1	-1
0	0	0	-1	-1	1	-1	1	1	-1	-1	1	-1	1	1	0	0	0	-1	1
0	0	0	-1	0	0	-1	1	1	-1	-1	1	-1	1	1	0	0	0	-1	1

Table 6. At $Q_{max} = 1$, we find 8 anomaly-free charge assignments in our list banning all $R-$parity violating terms in the MSSM superpotential. Each row lists the $\mathfrak{u}(1)_X$ charges of the (left-handed or right-handed) chiral fermions of a model.

By increasing the maximum charge Q_{max}, we find solutions which ban W_{LV} while allowing B-violation. At $Q_{max} = 10$, we find 444 357 847 solutions which forbid W_{LV} while allowing

terms in W_{BV}. We find 2 916 984 840 solutions which forbid W_{BV} while allowing for terms in W_{LV} at $Q_{max} = 10$, and a total of 885 951 137 solutions which ban all $R-$parity violating solutions, constituting 14% of the list of charge assignments.

6.2 B anomalies

Family-dependent charges in the quark and lepton sectors are well-motivated by the recent hints at lepton flavour non-universality associated with $b \to s\ell^+\ell^-$ transitions [5–13], also known as 'B anomalies'. Global fits incorporating angular distributions and branching fractions point towards new physics contributions to the Wilson coefficients C_9, C_{10} of weak effective theory Hamiltonian operators O_9, O_{10}, respectively, where

$$O_9 = (\bar{s}'_L\gamma_\mu b'_L)(\bar{\mu}'\gamma^\mu\mu') \quad O_{10} = (\bar{s}'_L\gamma_\mu b'_L)(\bar{\mu}'\gamma^\mu\gamma^5\mu'). \tag{6.11}$$

Here the primes denote that the fermionic fields are in the mass eigenbasis. A vector-like new physics contribution to C_9 with $C_{10} = 0$, or a new physics coupling to left-handed muons through the combination $C_9 = -C_{10}$, are both favoured by global fits [43] in comparison to the SM.

We will filter through our list in search of solutions potentially capable of explaining the so-called B *anomalies* via the mediation of flavour-changing Z' interactions, resulting from the spontaneously broken $U(1)_X$ symmetry. We will begin by searching for solutions for which there exists $i, j \in \{1, 2, 3\}$ with Q_i and L_j charged. These will play the role of the left-handed bottom/top quark doublet and muon respectively, contributing to the effective operator $(\bar{b}_L\gamma^\mu b_L)(\mu_L\gamma_\mu\mu_L) + \ldots$ once the heavy Z' is integrated out of the effective field theory. We will assume that a rotation to the mass eigenbasis will mix the down-type quarks such that the necessary $\bar{b}'_L\gamma^\mu s'_L$ coupling is produced. As well as this, we will require that the left-handed leptons are not completely flavour universal, i.e. $\exists k \in \{1, 2, 3\}$ such that $X_{L_k} \neq X_{L_\mu}$. This will ensure we can have the necessary $\mu - e$ flavour non-universality to explain the $b \to s\ell^+\ell^-$ data.

We find 114 solutions with $Q_{\max} = 1$ satisfying these conditions, constituting approximately 43% of the total list. When $Q_{\max} = 10$ this number grows to 1 567 142 472, roughly 25% of the full list of charge assignments. Such large numbers indicate that these conditions leave the charges quite unconstrained, and thus we query the list further for interesting solutions. Firstly, there are solutions within this set which can simultaneously address the μ problem and allow only renormalisable tree-level Yukawa terms for the top, bottom and tau. The overlap between each set of constraints is depicted in Figure 3. Only 2 solutions can account for all three conditions when $Q_{\max} = 2$, and are shown in Table 7. This overlap grows when $Q_{\max} = 4$, with 1 556 solutions solving all three conditions.

Secondly, we will filter through the list for solutions that aren't obviously in danger of violating experimental constraints. Following the motivation of Refs. [9, 44], we search for solutions with uniform light quark charges so as to avoid constraints on flavour-violation in the light quark sector. Additionally, we will search for solutions which feature zero coupling

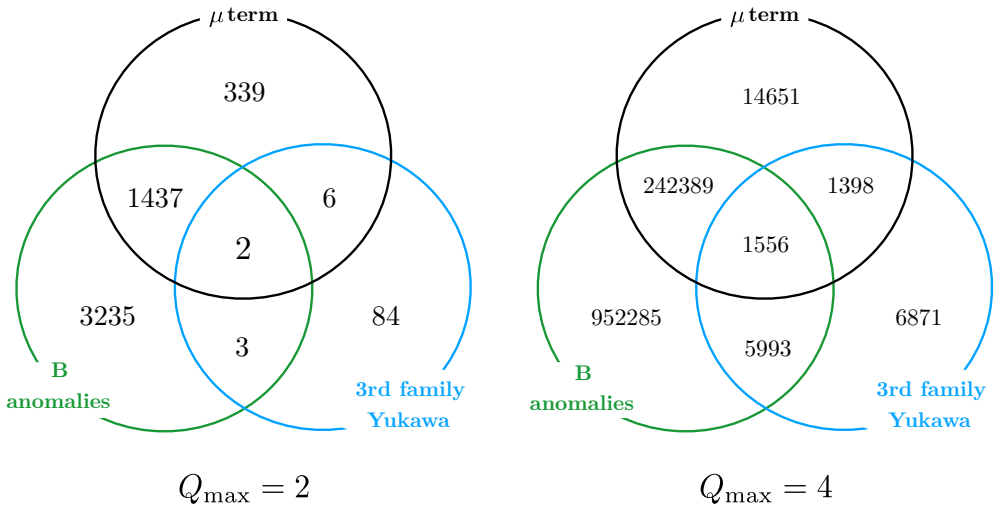

Figure 3. We filter through our list to determine the number of solutions capable of solving the μ problem and the B-anomalies, as well as those allowing only 3rd family Yukawa terms. We find an overlap between these applications, with 2 solutions at $Q_{\max} = 2$ satisfying all constraints and 1 556 at $Q_{\max} = 4$.

Q	Q	Q	n	n	n	e	e	e	u	u	u	d	d	d	L	L	L	$\tilde{H}_d$	$\tilde{H}_u$
-1	0	0	-2	1	2	1	2	2	-2	-1	0	-1	0	2	-1	0	2	1	1
-1	0	0	1	2	2	-2	1	2	-1	0	2	-2	-1	0	-1	0	2	1	1

Table 7. As depicted in Figure 3, at $Q_{\max} = 2$, only 2 anomaly-free charge assignments satisfy the constraints required to solve the μ problem and the B-anomalies while allowing 3rd family Yukawa terms in the Lagrangian. Each row lists the $\mathfrak{u}(1)_X$ charges of the (left-handed or right-handed) chiral fermions of a model.

of the electron to the associated Z' i.e. $\exists\, i, j \in \{1, 2, 3\}$ such that $X_{L_i} = 0$ and $X_{e_j} = 0$. This is motivated by the strong experimental constraints originating from e^+e^- collisions at LEP. We find 21 such solutions that also allow only third family Yukawa terms and address the μ problem in our list with $Q_{\max} = 10$. A selection of 10 of these solutions are listed in Table 8. Here, in contrast to other tables, the index on each fermion denotes the family number (since these are used in the constraints), and we use $\tilde{\theta}$ to denote the RH neutrino that plays the role of the (RH) flavino.

The 10 solutions shown all feature suppressed couplings of the Z' to the light quarks, either because the light RH down-type quarks have zero charge, as in solution (a), or because the light LH quarks have zero charge as in solutions (b)-(k). In solutions (a) and (b), the muon has equal RH and LH charge i.e. $L_2 = e_2$. This results in a purely vector-like coupling with $C_{10} = 0$. Similarly, solutions (c), (d) and (e) are particularly interesting in that they

	Q_1	Q_2	Q_3	$\tilde{\theta}$	n_1	n_2	e_1	e_2	e_3	u_1	u_2	u_3	d_1	d_2	d_3	L_1	L_2	L_3	$\tilde{H}_d$	$\tilde{H}_u$
(a)	-3	-3	3	-6	0	10	0	9	5	-3	-3	-2	0	0	2	0	9	6	-1	-5
(b)	0	0	-2	6	-3	4	0	3	2	0	0	-1	-3	-3	3	0	3	-3	5	1
(c)	0	0	-2	10	-1	10	0	0	-7	3	3	5	-8	-8	1	0	6	-10	3	7
(d)	0	0	-1	8	-9	1	0	0	6	-2	-2	0	-2	-2	6	0	-4	-1	7	1
(e)	0	0	-5	6	8	10	0	-1	7	4	4	-4	-7	-7	0	0	7	2	5	1
(f)	0	0	-3	10	-9	3	0	8	6	-5	-5	2	0	0	2	0	-2	1	5	5
(g)	0	0	-3	2	0	3	0	7	6	-1	-1	-5	0	0	1	0	5	2	4	-2
(h)	0	0	-2	6	-6	-3	0	8	7	-5	-5	-1	2	2	3	0	-2	2	5	1
(i)	0	0	-2	6	-6	3	0	4	5	-3	-3	1	0	0	1	0	-2	2	3	3
(j)	0	0	-1	-4	-6	0	0	7	9	-5	-5	-4	7	7	-2	0	-3	10	-1	-3
(k)	0	0	-1	2	2	8	0	-5	-1	4	4	0	-5	-5	0	0	3	-2	1	1

Table 8. At $Q_{\max} = 10$ we find 21 solutions which simultaneously solve the μ problem and B anomalies, allow 3rd family Yukawa terms and are well-suited to avoid strong experimental constraints from LEP and quark flavour violation between the first two families. A selection of 10 of these are shown here. Each row lists the $\mathfrak{u}(1)_X$ charges of the (left-handed or right-handed) chiral fermions of a model.

all produce negative values of the ratio C_9/C_{10}, with (c) and (d) giving exactly $C_9 = -C_{10}$ and solution (e) satisfying $C_9 = -\frac{3}{4}C_{10}$. In §3 we queried the full list of charge assignments in search of known solutions in the literature, listed in Table 3. None of these solutions are found in the list of 21 solutions passing our constraints: either because they cannot solve the μ problem and address the 3rd family Yukawa terms simultaneously, or because they do not satisfy the constraints we impose to facilitate solving the B anomalies.

6.3 Neutrino masses

Finally, we turn to the neutrinos. The inclusion of RH neutrinos has allowed us the flexibility to solve the ACCs while simultaneously addressing the phenomenological constraints of §6.1 and §6.2, as evidenced by the fact that these solutions often have nonzero charges for the RH neutrinos. In particular, this can be seen from Table 8 in which all of the solutions feature nonzero charges for at least one of the RH neutrinos. It is then useful to ask what these charge assignments imply for the neutrino masses and mixings.

In order to describe neutrino masses and mixings, we extend the superpotential to include the following terms,

$$W = W_{R_p} + (y_\nu)_{ij} N_i^c L_j H_u + (M_{\nu_R})_{ij} N_i^c N_j^c, \tag{6.12}$$

where $(y_\nu)_{ij}$ is a 3 by 3 matrix of dimensionless Dirac Yukawa coupling constants and $(M_{\nu_R})_{ij}$ is a 3 by 3 matrix of Majorana mass terms (of mass dimension 1) for the RH neutrinos. Neutrino masses are then produced through a Type-1 see-saw mechanism. Many alternative

mechanisms for producing neutrino masses in the MSSM exist in the literature. Bilinear $R-$parity violating models extend the superpotential to include the L-violating $\mu_i' L_i H_u$ terms which produce neutrino masses through mixing with the neutralinos [45, 46]. In [47], a suppressed Dirac mass term is produced after $U(1)_X$-breaking, through the flavon's vacuum expectation value $\langle\theta\rangle$. The $\mu\nu SSM$ extends the MSSM to produce neutrino masses through the inclusion of the trilinear term $\kappa_{ijk} N_i^c N_j^c N_k^c$ in the superpotential [32–35]. While an investigation into each of these mechanisms and models is beyond the scope of this paper, we will filter through our list in search of solutions which allow all of the terms of (6.12), allowing all possible neutrino masses and mixings via the see-saw mechanism. These solutions must satisfy the following constraints $\forall i, j \in \{1, 2, 3\}$

$$X_{L_i} + X_{H_u} - X_{n_j} = 0, \qquad X_{n_i} + X_{n_j} = 0, \tag{6.13}$$

implying $X_{n_i} = 0$ and $X_{L_i} = -X_{H_u}$. We find a total of 3 solutions with $Q_{\max} = 1$ in our list satisfying these constraints. At $Q_{\max} = 4$ a total of 118 solutions exist, and at $Q_{\max} = 10$ the list contains 4 878 of these solutions.

6.4 Summary of constraints

We summarise the phenomenological constraints of this section in Table 9. We emphasise that the filters used throughout this section provide an initial exploration into the constraints we expect will be most commonly needed by model builders. We expect that the scope of this list is much broader than the phenomenological applications dealt with here, and by making the list of charge assignments publicly available on `Zenodo` [30] we encourage model builders to search for charge assignments of more specific interest.

7 Summary

Specific models incorporating the MSSM with an additional $U(1)_X$ gauge group can combine the phenomenological advantages of supersymmetry with potential uses of the additional gauge factor and they have received quite some attention in the literature, particularly for the case where the $U(1)_X$ charges are family dependent. We have found, for the first time, all charge assignments of the MSSM plus three SM-singlet chiral superfields which are free of local anomalies (the SM-singlets can produce neutrino masses as well as spontaneously break the $U(1)_X$ symmetry). Chiral superfields in real representations can be added to any anomaly-free matter content and result in an anomaly-free solution, since the additional fermionic content will be in a vector-like representation of the gauge group and so its effects cancel in the anomalies. The local anomaly cancellation conditions described in §2 constitute a system of six homogeneous coupled diophantine equations (2.1)-(2.6), the like of which are notoriously difficult to solve, in general.

Global anomalies are beyond the scope of our work; however, for the case of $U(1)$ extensions of the usual SM gauge group, there are none [48]. One may question whether a quantum field theory absolutely *has* to be free from anomalies; after all, in an infra-red effective field

	Proposition	# $Q_{\max} = 10$
μ problem	$\exists i \in \{1,2,3\} : X_{H_u} + X_{H_d} = X_{n_i} \wedge X_{n_i} \neq 0$	649 831 168
All renormalisable charged fermion Yukawas	$(\forall i,j,k,l,m,n \in \{1,2,3\}\, P_{ijklmn})$	38
Only 3rd family renormalisable charged fermion Yukawas	$(\exists!\, i,j,k,l,m,n \in \{1,2,3\} : P_{ijklmn}) \wedge$ $(\forall i,j,k,l,m,n \in \{1,2,3\} P_{ijklmn} \Rightarrow n = i)$	34 646 735
L-conservation & B-violation	$P_L \wedge \neg P_B$	444 357 847
B-conservation & L-violation	$P_B \wedge \neg P_L$	2 916 984 840
L & B-conservation	$P_L \wedge P_B$	885 951 137
B anomalies	$\exists i,j,k \in \{1,2,3\} : X_{Q_i} \neq 0 \wedge X_{L_j} \neq 0 \wedge X_{L_k} \neq X_{L_j}$	1 567 142 472
B anomalies, μ problem, 3rd family Yukawa terms & experimental constraints	See §6.2	21
See-saw ν masses	$\forall i,j \in \{1,2,3\}\ X_{L_i} + X_{H_u} = X_{n_j} \wedge X_{n_i} = -X_{n_j}$	4 878

Table 9. Summary of the phenomenological conditions applied in this paper, along with the number of inequivalent $Q_{\max} = 10$ solutions which satisfy them. In the above we have used standard logic notation in which $\forall$ reads as 'for all', $\wedge$ as 'and', $\vee$ as 'or', $\exists$ as 'there exists', $\exists!$ as 'there exists a unique', $\Rightarrow$ as 'implies', : as 'such that', $\neg$ as 'not'. For the condition of allowing all renormalisable charged fermion Yukawa terms, we have used the proposition P_{ijklm} defined as $P_{ijklmn} := (X_{Q_i} + X_{H_u} = X_{u_j} \wedge X_{Q_n} + X_{H_d} = X_{d_k} \wedge X_{L_l} + X_{H_d} = X_{e_m})$. For the R-parity related conditions we have used the propositions $P_L := \forall i,j,k,l,m,n,p \in \{1,2,3\} \quad i = j \vee (X_{L_i} + X_{L_j} - X_{e_k} \neq 0 \wedge\ X_{L_l} + X_{Q_m} - X_{d_n} \neq 0 \wedge\ X_{L_p} + X_{H_u} \neq 0)$, and $P_B := \forall i,j,k \in \{1,2,3\} \quad i = j \vee X_{u_i} + X_{d_j} + X_{d_k} \neq 0$.

theory (such as we might expect the MSSM$\times U(1)_X$ to be) one can in principle add Wess-Zumino terms to the Lagrangian density in order to cancel them. Such terms can result from decoupling a heavy state from the effective field theory. In order to contribute to the anomaly though, the additional heavy state must be a chiral fermion of non-zero $U(1)_X$ charge. It is then not *a priori* obvious how such a state may acquire a large mass, unless it is linked to the scale of $U(1)_X$ breaking.[12] One recent non-supersymmetric $U(1)$ gauge extension of the SM [13] has achieved this with some additional fermions that under the SM are in vector-like representations, but which are chiral with respect to $U(1)_X$. However, it is far from obvious whether this is possible in general model set-ups, particularly when several mixed anomalies do not cancel. From the model builder's point of view therefore, it is safer to begin with an

[12]Integrating the top quark out of the SM yields apparent gauge anomalies, but when one includes effective operators resulting from integrating it out, gauge symmetry is restored [49]. This is precisely a case where the heavy mass is linked to the symmetry breaking scale (in this case, of the electroweak symmetry).

anomaly-free effective field theory rather than having to worry about how such anomalies are cancelled.

We have provided the general analytic solution for the charges via a new geometric method (a different geometric method was previously employed to solve the anomaly cancellation conditions for non-supersymmetric $U(1)_X$ extensions of the SM [18]) described in §4. One inputs 23 integer parameters for each anomaly-free charge assignment. A `Mathematica`™ program has been made publicly available which, given the input parameters, produces one such assignment. The general analytic solution passed various internal consistency checks. Whilst the general analytic solution can be difficult for model builders to use, it is useful for (among other things) providing non-trivial checks of any list of numerical solutions.

Anomaly-free charge assignments are scarce: for example, for heights up to 10, as Fig. 1 shows, only one out of some 10^{12} (or so) inequivalent assignments is anomaly free. Despite their scarcity, the different assignments are still legion (we have identified over 1.6 billion up to a height of 10). The model builder is therefore faced with an enormous haystack in which to find the proverbial needle.

An explicit list of all of these 1.6 billion inequivalent charge assignments up to a maximum absolute value of 10 has been produced via a computer program described in §3 and made publicly available [30]. Each entry in the list comprises 20 integers, the $U(1)_X$ charge assignments of 20 chiral superfields of the model. Extensive checks of the list have been made using the analytic solution as well as those of internal consistency. With the aid of a computer, such a list is easily and quickly searched and filtered, looking for charge assignments with various desirable properties. For example, if fewer than three SM-singlets are required for the model, one can filter the list and find all solutions where one of the SM-singlet $U(1)_X$ charges is zero. As far as anomalies go, having a zero charge for the superfield is equivalent to removing it from the model. We have shown some simple example filters, looking for different desirable properties of the charge assignments in §6 as a tutorial in their implementation. We hope that the list will be of use for beyond-the-MSSM builders in terms of inspiration and phenomenology.

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