LIE BRACKETS ON HOPF ALGEBRA COHOMOLOGY

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ABSTRACT. By work of Farinati, Solotar, and Taillefer, it is known that the Hopf algebra cohomology of a quasi-triangular Hopf algebra, as a graded Lie algebra under the Gerstenhaber bracket, is abelian. Motivated by the question of whether this holds for nonquasi-triangular Hopf algebras, we show that Gerstenhaber brackets on Hopf algebra cohomology can be expressed via an arbitrary projective resolution using Volkov's homotopy liftings as generalized to some exact monoidal categories. This is a special case of our more general result that a bracket operation on cohomology is preserved under exact monoidal functors—one such functor is an embedding of Hopf algebra cohomology into Hochschild cohomology. As a consequence, we show that this Lie structure on Hopf algebra cohomology is abelian in positive degrees for all quantum elementary abelian groups, most of which are nonquasi-triangular.

1. INTRODUCTION

The Ext algebra of a Hopf algebra, that is its Hopf algebra cohomology, has a Lie structure arising from its embedding into Hochschild cohomology in case the antipode is bijective. This Lie structure is known to be abelian when the Hopf algebra is quasi-triangular. More specifically, Taillefer [10] constructed a Lie bracket on cohomology arising from a category of Hopf bimodule extensions of a Hopf algebra and showed that brackets are always zero. In the finite dimensional case, this corresponds to Hopf algebra cohomology of the opposite of the Drinfeld double, whose Lie structure was also investigated by Farinati and Solotar [2] using different techniques. Farinati and Solotar showed this is indeed the bracket arising from an embedding into Hochschild cohomology. Hermann [4] looked at a more general monoidal category setting and showed that the Lie structure is trivial in case the category is braided [4, Theorem 5.2.7]. It is an open question as to what happens more generally.

In this paper, we apply a new technique, the homotopy lifting method of Volkov [11], to shed more light on this question, particularly in the Hopf algebra setting. Homotopy liftings were defined for some exact monoidal categories in [12], and we use them to prove that brackets are preserved under exact monoidal functors. As a consequence, we show that the Lie bracket on Hopf algebra cohomology defined by

homotopy liftings agrees with that induced by its embedding into Hochschild co-homology, in case the antipode is bijective. An advantage of the homotopy lifting method is its ability to handle the Lie structure independently of choice of projective resolution. A good choice of resolution can facilitate understanding of the Lie structure.

We illustrate with the quantum elementary abelian groups. These are tensor products of copies of the Taft algebras T_n that depend on an nth root of unity; when $n > 2$ these are nonquasi-triangular [3, Proposition 2.1]. The first author [5] found that the Lie bracket on the Hopf algebra cohomology of T_n is trivial by using techniques of Negron and the second author [7], a consequence of a larger computation of the bracket on Hochschild cohomology of T_n. This was the first known example of the Lie structure on Hopf algebra cohomology of a nonquasi-triangular Hopf algebra. In this paper we take a different approach, giving a second proof in positive degrees, for Taft algebras T_n via homotopy liftings instead of relying on a computation for the full Hochschild cohomology ring. We extend this result to quantum elementary abelian groups, as iterated tensor products of Taft algebras.

An outline of the contents is as follows. In Section 2, we recall some definitions and the homotopy lifting method for defining a Lie structure on the cohomology of some exact monoidal categories. We use this method in Section 3, proving that the Lie structure is always preserved under an exact monoidal functor. In Section 4, we apply the results of Section 3 to Hopf algebra cohomology, concluding a formula for Gerstenhaber brackets based on homotopy liftings. This formula allows us to find the bracket structure on Hopf algebra cohomology independently of its embedding into Hochschild cohomology, yielding a new and different proof that for a cocommutative Hopf algebra, this Lie structure is abelian in positive degrees. The Lie structure on the Hopf algebra cohomology of a quantum elementary abelian group, in positive degrees, is shown to be abelian in Section 5.

2. Preliminaries

We begin by recalling the definitions of Hopf algebra cohomology and Hochschild cohomology, as well as an embedding of one into the other that provides a connection between their Lie structures.

Let k be a field and abbreviate $\otimes_k$ by $\otimes$ when it will cause no confusion.

Hopf algebra cohomology. Let A be a Hopf algebra over a field k with coproduct $\Delta : A \to A \otimes A$, counit $\varepsilon : A \to k$ and antipode $S : A \to A$. *We will always assume that S is bijective.* Note that, since all finite dimensional and most known infinite dimensional Hopf algebras of interest have bijective antipodes, this assumption is not very restrictive.

Give the field k the structure of an A-module, called the *trivial A-module*, via the counit ε, i.e. $a \cdot 1_k = \varepsilon(a)$ for all $a \in A$. Give A the structure of an A-bimodule by multiplication on both sides; equivalently, A is a left A^e-module where $A^e := A \otimes A^{op}$ and A^{op} is the vector space A with multiplication opposite to A. The *Hopf algebra*

cohomology of A is $\mathrm{H}^*(A, k) := \mathrm{Ext}^*_A(k, k)$. The *Hochschild cohomology* of A is $\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^e}(A, A)$.

We can embed Hopf algebra cohomology into Hochschild cohomology. We will need details, and we recall some lemmas. For proofs, see [13, Section 9.4].

Lemma 2.1. *There is an isomorphism of A^e-modules,*

$$A \cong A^e \otimes_A k,$$

where $A^e \otimes_A k$ is the tensor induced A^e-module under the identification of A with the subalgebra of A^e that is the image of the embedding $\delta : A \to A^e$ defined for all $a \in A$ by

$$\delta(a) = \sum a_1 \otimes S(a_2).$$

Lemma 2.2. *The right A-module A^e, where A acts by right multiplication under its identification with $\delta(A)$, is projective.*

Lemma 2.3 (Eckmann-Shapiro)**.** *Let A be a ring and let B be a subring of A such that A is projective as a right B-module. Let M be an A-module and N be a B-module. Then*

$$\mathrm{Ext}^n_B(N, M) \cong \mathrm{Ext}^n_A(A \otimes_B N, M),$$

where M is considered to be a B-module under restriction.

We will consider A to be a left A-module by the left adjoint action, which is for $a, b \in A$,

$$a \cdot b = \sum a_1 b S(a_2).$$

Denote this A-module by A^{ad}.

For any left A-module M, let $\mathrm{H}^*(A, M) := \mathrm{Ext}^*_A(k, M)$. The following theorem is well known; see, e.g. [13, Theorem 9.4.5]. We sketch a proof since we will need some of the details later.

Theorem 2.4. *There is an isomorphism of graded k-vector spaces*

$$\mathrm{HH}^*(A) \cong \mathrm{H}^*(A, A^{ad}).$$

Proof. By Lemma 2.2, A^e is a projective as a right A-module, so we can apply the Eckmann-Shapiro Lemma. We replace A with A^e, B with A and take $M = A$, $N = k$ in the Eckmann-Shapiro Lemma and obtain the isomorphism $\mathrm{Ext}^n_{A^e}(A^e \otimes_A k, A) \cong \mathrm{Ext}^n_A(k, A^{ad})$ as k-vector spaces. Lastly, we apply Lemma 2.1 and obtain $\mathrm{Ext}^n_{A^e}(A, A) \cong \mathrm{Ext}^n_A(k, A^{ad})$. $\square$

A consequence of the theorem is an embedding of Hopf algebra cohomology $\mathrm{H}^*(A, k)$ into Hochschild cohomology $\mathrm{HH}^*(A)$: Let $P \to k$ be a projective resolution of the A-module k. We can embed $\mathrm{H}^*(A, k)$ into $\mathrm{H}^*(A, A^{ad}) \cong \mathrm{HH}^*(A)$ via the map $\eta_* : \mathrm{Hom}_A(P, k) \to \mathrm{Hom}_A(P, A^{ad})$ induced by the unit map $\eta : k \to A$ (see [13, Corollary 9.4.7]). Equivalently, viewing each space $\mathrm{H}^n(A, k)$ as equivalence classes of n-extensions of k by k, the functor $A^e \otimes_A -$ induces an embedding of $\mathrm{H}^*(A, k)$ into $\mathrm{HH}^*(A)$.

Homotopy liftings and exact monoidal categories. Gerstenhaber defined the bracket on Hochschild cohomology via the bar resolution. We will not need that definition here. Instead, by using homotopy liftings as defined below, a similar operation is constructed for some exact monoidal categories in [12, Section 4] as follows. This turns out to be equivalent to the Gerstenhaber bracket when the category is that of A-bimodules.

We refer to [1] for definitions and properties of monoidal categories. Let $\mathcal{C}$ be an exact monoidal category and let $\mathbf{1}$ be its unit object. As is customary, we will identify $\mathbf{1} \otimes X$ and $X \otimes \mathbf{1}$ with X for all objects X in $\mathcal{C}$, under assumed fixed isomorphisms (for which we will not need notation). One example of an exact monoidal category is the category of left modules for a Hopf algebra over k, with tensor product $\otimes$ of modules, and the unit object given by the trivial module k. Another example is the category of A-bimodules (equivalently left A^e-modules) for an associative algebra A over k, with tensor product $\otimes_A$ and unit object A.

Let $P \to \mathbf{1}$ be a projective resolution of $\mathbf{1}$ with differential d and let $\mu_P : P_0 \to \mathbf{1}$ be the corresponding augmentation map. In [12, Definition 4.3], the resolution (P, d, μ_P) of $\mathbf{1}$ is called *n-power flat* if $(P^{\otimes r}, d^{\otimes r}, \mu_P^{\otimes r})$ is a projective resolution of $\mathbf{1}$ for each r $(1 \leq r \leq n)$. If P is n-power flat for each $n \geq 2$, then we say that P is *power flat*. For the two categories mentioned above, projective resolutions of $\mathbf{1}$ are generally power flat.

Assume $\mathbf{1}$ has a projective power flat resolution P. For a degree l morphism $\psi : P \to P$, its differential in the Hom complex $\mathrm{Hom}_{\mathcal{C}}(P, P)$ is defined to be

$$\partial(\psi) = d\psi - (-1)^l \psi d.$$

Let $f : P \to \mathbf{1}$ be an m-cocycle. Since $\mathbf{1}$ is 2-power flat, $P \otimes P \to \mathbf{1}$ is also a projective resolution. Let $\Delta_P : P \to P \otimes P$ be a diagonal map, i.e. a chain map lifting the isomorphism $\mathbf{1} \xrightarrow{\sim} \mathbf{1} \otimes \mathbf{1}$. A degree $(m-1)$ morphism $\psi_f : P \to P$ is a *homotopy lifting of* (f, Δ_P) if

$$(2.5) \qquad\qquad \partial(\psi_f) = (f \otimes 1_P - 1_P \otimes f)\Delta_P$$

and $\mu_P \psi_f \sim (-1)^{m+1} f\psi$ for some degree -1 map $\psi : P \to P$ such that

$$(2.6) \qquad\qquad \partial(\psi) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P.$$

The cohomology of the monoidal category $\mathcal{C}$ is $\mathrm{H}^*(\mathcal{C}) = \mathrm{H}^*(\mathcal{C}, \mathbf{1}) := \mathrm{Ext}_{\mathcal{C}}^*(\mathbf{1}, \mathbf{1})$. It has a Lie bracket defined as follows [12, Section 4]: For an m-cocycle $f : P_m \to \mathbf{1}$ and an n-cocycle $g : P_n \to \mathbf{1}$, let ψ_f and ψ_g be homotopy liftings of (f, Δ_P) and (g, Δ_P) respectively. Then the cochain $[f, g]$ defined as

$$(2.7) \qquad\qquad [f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f$$

induces a graded Lie bracket on $\mathrm{H}^*(\mathcal{C})$. That is, it induces a well-defined operation on cohomology that is graded alternating and satisfies a graded Jacobi identity (cf. [13, Lemma 1.4.3]).

Remark 2.8. The bracket in (2.7) is independent of choice of the resolution P and Δ_P [12, Theorem 4.17]. The assumption of power flatness is used in the general theory of homotopy liftings in [12, Section 4] that we need here. See [12, Remark 4.18] for more details.

3. Change of exact monoidal categories

Let $\mathcal{C}$, $\mathcal{C}'$ be exact monoidal categories for which there exist power flat resolutions of their unit objects $\mathbf{1}$, $\mathbf{1}'$. Let $F : \mathcal{C} \to \mathcal{C}'$ be an exact monoidal functor [1, Definition 2.4.1], that is, F is exact and there is a natural isomorphism η of functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}'$ given by

$$\eta_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$$

for all X, Y in $\mathcal{C}$, $F(\mathbf{1}) \cong \mathbf{1}'$ and (F, η) satisfies the monoidal structure axiom of [1, Definition 2.4.1], that is, the following diagram commutes for all objects X, Y, Z in $\mathcal{C}$:

(3.1)
$$
\begin{array}{ccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\;\sim\;} & F(X) \otimes (F(Y) \otimes F(Z)) \\
\downarrow{\scriptstyle \eta_{X,Y} \otimes 1_{F(Z)}} & & \downarrow{\scriptstyle 1_{F(X)} \otimes \eta_{Y,Z}} \\
F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\
\downarrow{\scriptstyle \eta_{X \otimes Y, Z}} & & \downarrow{\scriptstyle \eta_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{\;\sim\;} & F(X \otimes (Y \otimes Z))
\end{array}
$$

(The horizontal isomorphisms are given by the associativity constraint for $\mathcal{C}'$ and the image of the associativity constraint for $\mathcal{C}$ under F, respectively. We will not need notation for these isomorphisms.)

Denote the isomorphism from $F(\mathbf{1})$ to $\mathbf{1}'$ by ϕ. Then the following diagrams commute for all objects X by [1, Proposition 2.4.3]; we have chosen to show diagrams involving the inverse maps $\eta_{\mathbf{1},X}^{-1}$ and $\eta_{X,\mathbf{1}}^{-1}$ since we will need these later. The unlabeled isomorphisms in the diagrams are those canonically determined by the fixed isomorphisms given by tensoring with unit objects and the fixed isomorphism $F(\mathbf{1}) \cong \mathbf{1}'$.

$$
\begin{array}{ccc}
F(\mathbf{1} \otimes X) & \xrightarrow{\;\sim\;} & F(X) \\
{\scriptstyle \eta_{\mathbf{1},X}^{-1}} \downarrow & & \downarrow{\scriptstyle \sim} \\
F(\mathbf{1}) \otimes F(X) & \xrightarrow{\;\sim\;} & \mathbf{1}' \otimes F(X)
\end{array}
\qquad
\begin{array}{ccc}
F(X \otimes \mathbf{1}) & \xrightarrow{\;\sim\;} & F(X) \\
{\scriptstyle \eta_{X,\mathbf{1}}^{-1}} \downarrow & & \downarrow{\scriptstyle \sim} \\
F(X) \otimes F(\mathbf{1}) & \xrightarrow{\;\sim\;} & F(X) \otimes \mathbf{1}'
\end{array}
$$

Example 3.2. Let A be a Hopf algebra and B a Hopf subalgebra of A. Let $\mathcal{C}$ be the category of left A-modules and $\mathcal{C}'$ the category of left B-modules. Let $F : \mathcal{C} \to \mathcal{C}'$ be the restriction functor, that is on each A-module X, the action is restricted to B via the inclusion map $B \hookrightarrow A$. The restriction of a tensor product of modules to

B is isomorphic to the tensor product of their restrictions to B, and thus there is a natural transformation η as required.

In the next section we will apply the following theorem to shed light on the connection between the Lie structures on Hopf algebra cohomology and on Hochschild cohomology.

Let P be a projective resolution of $\mathbf{1}$ in $\mathcal{C}$ and write $P' = F(P)$, which is a projective resolution of $F(\mathbf{1}) \cong \mathbf{1}'$ in $\mathcal{C}'$ under our assumptions. Let d denote the differential and $\mu_P : P \to \mathbf{1}$ denote the augmentation map of P. Write $d' = F(d)$ and $\mu_{P'} = F(\mu_P)$. Note that $P \otimes P$ is also a projective resolution of $\mathbf{1}$ in $\mathcal{C}$ with augmentation map $\mu_P \otimes \mu_P$ followed by the canonical isomorphism $\mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$. Let

$$\Delta_{P'} = \eta_{P,P}^{-1} F(\Delta_P),$$

which is a diagonal map on P' under our assumptions.

Theorem 3.3. *Let $\mathcal{C}$, $\mathcal{C}'$ be exact monoidal categories and let $F : \mathcal{C} \to \mathcal{C}'$ be an exact monoidal functor. Assume there exists a power flat resolution P of $\mathbf{1}$ in $\mathcal{C}$. Let $f \in \mathrm{Hom}_{\mathcal{C}}(P_m, \mathbf{1})$, an m-cocycle. Let ψ_f be a homotopy lifting of f with respect to Δ_P. Then $F(\psi_f)$ is a homotopy lifting of $F(f)$ with respect to $\Delta_{P'}$.*

Proof. Since ψ_f is a homotopy lifting of f,

$$d\psi_f - (-1)^{m-1}\psi_f d = (f \otimes 1 - 1 \otimes f)\Delta_P.$$

Set $f' = F(f)$, $\psi_{f'} = F(\psi_f)$, and apply F to each side of this equation to obtain

(3.4) $$d'\psi_{f'} - (-1)^{m-1}\psi_{f'}d' = F(f \otimes 1 - 1 \otimes f)F(\Delta_P).$$

Since η is a natural transformation, under our assumptions (see the above commuting diagrams), the following diagram commutes:

$$
\begin{array}{ccccccc}
F(P) & \xrightarrow{F(\Delta_P)} & F(P \otimes P) & \xrightarrow{F(f \otimes 1)} & F(\mathbf{1} \otimes P) & \xrightarrow{\sim} & F(P) \\
& & \Big\downarrow{\eta_{P,P}^{-1}} & & \Big\downarrow{\eta_{1,P}^{-1}} & & \Big\downarrow{=} \\
& & F(P) \otimes F(P) & \xrightarrow{F(f) \otimes 1} & F(\mathbf{1}) \otimes F(P) & \xrightarrow{\sim} & F(P)
\end{array}
$$

Therefore $F(f \otimes 1)F(\Delta_P)$ can be identified with $(F(f) \otimes 1)\eta_{P,P}^{-1}F(\Delta_P)$, and similarly $F(1 \otimes f)F(\Delta_P)$ with $(1 \otimes F(f))\eta_{P,P}^{-1}F(\Delta_P)$. So the right side of expression (3.4) is equal to

$$(f' \otimes 1 - 1 \otimes f')\eta_{P,P}^{-1}F(\Delta_P),$$

which is in turn equal to $(f' \otimes 1 - 1 \otimes f')\Delta_{P'}$, as desired.

Since ψ_f is a homotopy lifting, $\mu_P\psi_f \sim (-1)^{m+1}f\psi$ for some degree -1 map $\psi : P \to P$ such that

$$\partial(\psi) = d\psi + \psi d = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P.$$

By applying F to both sides of the above equation, we obtain

$$F(d\psi + \psi d) = F(\mu_P \otimes 1_P - 1_P \otimes \mu_P)F(\Delta_P),$$

and under our identifications, letting $\psi' = F(\psi)$, this is

$$d'\psi' + \psi'd' = F(\mu_P \otimes 1)F(\Delta_P) - F(1 \otimes \mu_P)F(\Delta_P).$$

Via a commutative diagram such as that above, we see this is equal to

$$(\mu_{P'} \otimes 1)\Delta_{P'} - (1 \otimes \mu_{P'})\Delta_{P'}.$$

Therefore, $\psi' : P' \to P'$ is a degree -1 map such that

$$\partial(\psi') = d'\psi' + \psi'd' = (\mu_{P'} \otimes 1)\Delta_{P'} - (1 \otimes \mu_{P'})\Delta_{P'},$$

and $\mu_{P'}F(\psi_f) \sim (-1)^{m+1}F(f)\psi'$, that is, $\mu_{P'}\psi_{f'} \sim (-1)^{m+1}f'\psi'$, as desired.

We have shown that $F(\psi_f)$ is a homotopy lifting of $F(f)$ with respect to $\Delta_{P'}$. $\square$

Corollary 3.5. *The functor F induces a graded Lie algebra homomorphism from $\mathrm{H}^*(\mathcal{C})$ to $\mathrm{H}^*(\mathcal{C}')$, in positive degrees.*

Proof. As a consequence of the theorem and formula (2.7), the functor F takes the Lie bracket of two elements of positive degree in the cohomology $\mathrm{H}^*(\mathcal{C})$ to the Lie bracket of their images in $\mathrm{H}^*(\mathcal{C}')$ under F. $\square$

4. Hopf algebra cohomology

Let A be a Hopf algebra with bijective antipode. Let $\mathcal{C}$ be the category of (left) A-modules, and let $\mathcal{C}'$ be the category of (left) A^e-modules. For each A-module U, let

$$F(U) = A^e \otimes_A U,$$

the tensor induced module, where we identify A with the subalgebra $\delta(A)$ of A^e as in Lemma 2.1. Also by Lemma 2.1, F takes the unit object k of $\mathcal{C}$ to an isomorphic copy of the unit object A of $\mathcal{C}'$. It takes projective A-modules to projective A^e-modules since A^e is projective as an A-module by Lemma 2.2. For each A-module homomorphism $f : U \to V$, define $F(f)$ by

$$F(f)((1 \otimes 1) \otimes_A u) = (1 \otimes 1) \otimes_A f(u)$$

for all $u \in U$. Then F may be viewed as the functor providing the embedding of Hopf algebra cohomology $\mathrm{H}^*(A, k)$ into Hochschild cohomology $\mathrm{HH}^*(A)$; see the proof of Theorem 2.4 and the subsequent paragraph.

For each pair of A-modules U, V, we wish to define an A^e-module homomorphism

$$\eta_{U,V} : F(U) \otimes_A F(V) \to F(U \otimes V),$$

that is,

$$\eta_{U,V} : (A^e \otimes_A U) \otimes_A (A^e \otimes_A V) \to A^e \otimes_A (U \otimes V).$$

For all $a, b \in A$, $u \in U$, and $v \in V$, set

$$\eta_{U,V}((a \otimes 1) \otimes_A u) \otimes_A ((1 \otimes b) \otimes_A v) = (a \otimes b) \otimes_A (u \otimes v).$$

Note that all elements of $(A^e \otimes_A U) \otimes_A (A^e \otimes_A V)$ can indeed be written as linear combinations of elements of the indicated forms and that the map is well-defined. For example, for all $a, b \in A$ and $u \in U$, letting $b = S(b')$,

$$
\begin{aligned}
(a \otimes b) \otimes_A u &= \sum (aS(b'_1)b'_2 \otimes S(b'_3)) \otimes_A u \\
&= \sum (aS(b'_1) \otimes 1) \otimes_A ((b'_2 \otimes S(b'_3)) \cdot u).
\end{aligned}
$$

By its definition, $\eta_{U,V}$ is an A^e-module homomorphism.

We check that η is a natural transformation. That is, the following diagram commutes for all objects U, V, U', V' and morphisms $f : U \to U'$, $g : V \to V'$:

$$
\begin{CD}
F(U) \otimes_A F(V) @>{F(f) \otimes_A F(g)}>> F(U') \otimes_A F(V') \\
@V{\eta_{U,V}}VV @VV{\eta_{U',V'}}V \\
F(U \otimes V) @>{F(f \otimes g)}>> F(U' \otimes V')
\end{CD}
$$

Commutativity follows from the definitions of $\eta_{U,V}$, $\eta_{U',V'}$. To see that η is monoidal, that is diagram (3.1) commutes, it is easier to check the corresponding diagram associated to η^{-1}, a straightforward calculation.

For the following theorem, we define the *Gerstenhaber bracket* of two elements in Hopf algebra cohomology $\mathrm{H}^*(A, k)$ via the embedding into Hochschild cohomology followed by the Gerstenhaber bracket on Hochschild cohomology. The theorem states that this is the same as their bracket defined by (2.7) on Hopf algebra cohomology $\mathrm{H}^*(A, k)$ via homotopy liftings. Thus the theorem allows us to bypass the need to work with Hochschild cohomology at all, for questions purely about Hopf algebra cohomology.

Theorem 4.1. *Let A be a Hopf algebra with bijective antipode. Let P be a projective resolution of k and let f, g be cocycles in $\mathrm{Hom}_A(P_m, k)$, $\mathrm{Hom}_A(P_n, k)$, respectively, representing elements of Hopf algebra cohomology $\mathrm{H}^*(A, k)$. Let Δ_P be a diagonal map, and let ψ_f, ψ_g be homotopy liftings of f, g with respect to Δ_P. The Gerstenhaber bracket of the corresponding elements in Hopf algebra cohomology $\mathrm{H}^*(A, k)$ is represented by*

$$
[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f.
$$

Proof. This is an immediate consequence of Theorem 3.3 and expression (2.7), since we showed above that the induction functor F is an exact monoidal functor. $\square$

One consequence Theorem 4.1 is a quick new proof that for a cocommutative Hopf algebra in characteristic not 2, Gerstenhaber brackets on Hopf algebra cohomology in positive degree are always 0. We state this as Corollary 4.2 next. Since cocommutative Hopf algebras are quasi-triangular, this is a small special case of the well known results of Farinati, Solotar, Taillefer, and Hermann, but it highlights our completely different approach.

Corollary 4.2. *Let k be a field of characteristic not 2, and let A be a cocommutative Hopf algebra. The Lie structure on Hopf algebra cohomology $\mathrm{H}^*(A, k)$, given by the Gerstenhaber bracket, is abelian in positive degrees.*

Proof. Let P be a projective resolution of k as an A-module. Let $\Delta' : P \to P \otimes P$ be a diagonal map. Let $\sigma : P \otimes P \to P \otimes P$ be the signed transposition map, i.e. $\sigma(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for all homogeneous $x, y \in P$. Since A is cocommutative, $\sigma \Delta'$ is also an A-module homomorphism, and therefore a diagonal map. Let

$$\Delta = \frac{1}{2}(\Delta' + \sigma \Delta'),$$

a diagonal map as well. Note that Δ is symmetric in the sense that $\sigma \Delta = \Delta$.

Now, by symmetry, $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta \equiv 0$, and so in (2.6), we can take $\psi \equiv 0$. Similarly, in (2.5), we can take $\psi_f \equiv 0$ for any cocycle f. Thus by Theorem 4.1, Gerstenhaber brackets on the Hopf algebra cohomology $\mathrm{H}^*(A, k)$ are always 0 in positive degrees. $\qquad\square$

Remark 4.3. The map σ is special to the setting in Corollary 4.2, and in particular cannot be defined for Hochschild cohomology. This shows the strength of Theorem 4.1: It allows us to work independently of the embedding of Hopf algebra cohomology into Hochschild cohomology, clearing the way for new approaches.

5. TAFT ALGEBRAS AND QUANTUM ELEMENTARY ABELIAN GROUPS

In this section we illustrate our results by showing that the Lie structure on the Hopf algebra cohomology of a Taft algebra, and more generally of a quantum elementary abelian group, is abelian in positive degrees.

Let k be a field of characteristic 0 containing a primitive nth root ω of 1. Let A be the Taft algebra generated by x and g with relations

$$gx = \omega xg, \quad x^n = 0, \quad g^n = 1.$$

We take

$$\Delta(x) = x \otimes 1 + g \otimes x, \quad \Delta(g) = g \otimes g,$$

$\varepsilon(x) = 0$, $\varepsilon(g) = 1$, $S(x) = -g^{-1}x$, $S(g) = g^{-1}$. Let $B = k[x]/(x^n)$, a subalgebra of A, and let P be the following projective resolution of the trivial B-module k:

$$\cdots \xrightarrow{x^{n-1}\cdot} B \xrightarrow{x\cdot} B \xrightarrow{x^{n-1}\cdot} B \xrightarrow{x\cdot} B \xrightarrow{\varepsilon} k \longrightarrow 0$$

The action of B on each term is by multiplication. Give each component B in the resolution the structure of an A-module by letting $g \cdot x^i = \omega^i x^i$ in even degrees and $g \cdot x^i = \omega^{i+1} x^i$ in odd degrees. For clarity of notation, in each degree l, denote the element 1_B of $P_l = B$ by ϵ_l. Note that P_l is projective as an A-module since n is not divisible by the characteristic of k: Specifically, in even degrees, there are A-module homomorphisms $P_l \to A$ $(x^i \mapsto \frac{1}{n}\sum_{j=0}^{n-1} x^i g^j)$ and $A \to P_l$ $(x^i g^j \mapsto x^i)$

whose composition is the identity map. In odd degrees, a similar statement is true of the maps $P_l \to A$ and $A \to P_l$ given respectively by

$$x^i \mapsto \begin{cases} \frac{1}{n} \sum_{j=0}^{n-1} x^{i+1} g^j, & \text{if } i < n-1, \\ \frac{1}{n} \sum_{j=0}^{n-1} g^j, & \text{if } i = n-1 \end{cases} \quad \text{and} \quad x^i g^j \mapsto \begin{cases} x^{i-1}, & \text{if } i \neq 0, \\ x^{n-1}, & \text{if } i = 0. \end{cases}$$

Calculations show that the following formulas yield a diagonal map $\Delta : P \to P \otimes P$, that is for each l, Δ_l is an A-module homomorphism, and Δ is a chain map lifting the canonical isomorphism $k \xrightarrow{\sim} k \otimes k$:

$$\Delta_{2j+1}(\epsilon_{2j+1}) = \sum_{i=0}^{2j+1} \epsilon_i \otimes \epsilon_{2j+1-i},$$

$$\Delta_{2j}(\epsilon_{2j}) = \sum_{i=0}^{j} \epsilon_{2i} \otimes \epsilon_{2j-2i} + \sum_{i=0}^{j-1} \sum_{a=0}^{n-2} \binom{n-1}{a+1}_\omega x^a \epsilon_{2i+1} \otimes x^{n-2-a} \epsilon_{2j-2i-1},$$

where $\binom{n-1}{a+1}_\omega$ is the ω-binomial coefficient defined for all nonnegative integers a, b, c by

$$\binom{b}{c}_\omega = \frac{(b)_\omega (b-1)_\omega \cdots (b-c+1)_\omega}{(c)_\omega (c-1)_\omega \cdots (1)_\omega} \quad \text{where} \quad (a)_\omega = 1 + \omega + \omega^2 + \cdots + \omega^{a-1}.$$

Note that $\Delta \neq \sigma\Delta$ since $\binom{n-1}{a+1}_\omega \neq \binom{n-1}{a}_\omega$ in general. However, symmetry does hold after projection onto even degrees, a key property for the proof of the theorem below since the cohomology is concentrated in even degrees as we see next.

The cohomology of A can be computed directly from the resolution P above, and is

$$\mathrm{H}^*(A, k) \cong \mathrm{H}^*(B, k)^G \cong k[z],$$

where $\deg(z) = 2$. Alternatively, see [9, Corollary 3.4] for the relevant general theory for skew group algebras.

The following theorem was proven in [5] by different techniques, and more generally there, the elements of degree 0 were included. The homotopy lifting method that we use here was designed for positive degree cohomology.

Theorem 5.1. *The Lie structure given by the Gerstenhaber bracket on the cohomology* $\mathrm{H}^*(A, k)$ *of a Taft algebra A is abelian in positive degrees.*

Proof. Let P be the resolution of k given above. Let $f \in \mathrm{Hom}_A(P_2, k)$ denote the cocycle with $f(\epsilon_2) = 1$, a representative of the generator z of the cohomology ring $\mathrm{H}^*(A, k)$, described above. By [13, Lemma 1.4.7], it will suffice to show that the bracket of f with itself is 0 since f represents an algebra generator of cohomology.

We wish to find a homotopy lifting of f. First note that in (2.6), we can take $\psi \equiv 0$, the zero map, by symmetry of the image of the diagonal map under the projection onto $(P_0 \otimes P_i) \oplus (P_i \otimes P_0)$ for each i. Similarly, by symmetry of the image of the diagonal map under the projection onto $(P_{\mathrm{even}} \otimes P) \oplus (P \otimes P_{\mathrm{even}})$, since f has even degree, in (2.5), we can take $\psi_f \equiv 0$ and indeed we can take the

homotopy lifting of any representative element of cohomology in positive degree to be 0. Specifically, the map ψ_f must satisfy

$$d\psi_f + \psi_f d = (f \otimes 1 - 1 \otimes f)\Delta_P.$$

The right side of this equation, evaluated on ϵ_l, is

$$(f \otimes 1 - 1 \otimes f)(\Delta_P(\epsilon_l)),$$

and comparing to the formulas for $\Delta_P(\epsilon_{2j})$ and $\Delta_P(\epsilon_{2j+1})$ above, the only terms that will be nonzero after applying $f \otimes 1 - 1 \otimes f$ are those having ϵ_2 as one of the tensor factors. By symmetry, the resulting terms after applying $f \otimes 1$ and $1 \otimes f$ cancel due to their opposite signs. So we may take $\psi_f \equiv 0$ as claimed. Now, by Theorem 4.1, $[f, f] = 2f\psi_f = 0$. $\qquad\square$

The following theorem is a consequence since the Lie structure of a tensor product of algebras reduces to that on each factor [6]. Quantum elementary abelian groups are defined to be iterated tensor products of Taft algebras [8].

Theorem 5.2. *Let A be a quantum elementary abelian group. The Lie structure of the Hopf algebra cohomology $\mathrm{H}^*(A, k)$, given by Gerstenhaber bracket, is abelian in positive degrees.*

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