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Canonical Embeddings of Pairs of Arcs

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Abstract
We show that for given four points in the Riemann sphere and a given isotopy class of two disjoint arcs connecting these points in two pairs, there exists a unique configuration with the property that each arc is a hyperbolic geodesic segment in the complement of the other arc.

Keywords Conformal map · Hyperbolic metric · Conic singularity

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In the recent paper [7], Peltola and Wang made a remarkable restatement of results in [3,4] about the existence and uniqueness of a real rational function with prescribed real critical points. To formulate this precisely, we consider *chord diagrams* in the closure $\overline{\mathbb{D}}$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Such a chord diagram has prescribed points $a_1, \ldots, a_{2d-2}$, $d \geq 2$, on the unit circle and $d-1$ disjoint crosscuts $e_1, \ldots, e_{d-1}$ in $\overline{\mathbb{D}}$ connecting pairs of these points. We call such a chord diagram *canonical* if every crosscut e_k is a hyperbolic geodesic in the unique component of $\mathbb{D} \setminus \bigcup_{j \neq k} e_j$ that contains the interior points of e_k.

Theorem 1.1 in [7] states that for any prescribed points $a_1, \ldots, a_{2d-2}$ there is a unique canonical chord diagram in every combinatorial class. This canonical chord

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diagram can be obtained from the preimage of the real line under a real rational function of degree d with critical points at $a_1, \dots, a_{2d-2}$. This theorem has important applications for the study of the Stochastic Loewner Evolution (SLE).

The number of combinatorial classes of chord diagrams with prescribed vertices is finite: it is the Catalan number. In this note we give a simple example of a similar problem with four prescribed points and infinitely many canonical configurations.

Our configurations consist of four distinct points a_0, a_1, a_2, a_3 in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and two disjoint arcs γ_0 and γ_1, where γ_0 has the endpoints a_0 and a_1, and γ_1 has the endpoints a_2 and a_3. We say that two such configurations are *equivalent* if the points are the same, and the arcs of the first configuration can be deformed into the arcs of the second configuration by an isotopy of the sphere that keeps the endpoints of the arcs fixed. A configuration is called *canonical* if for each $k \in \{0, 1\}$ the arc γ_k is a hyperbolic geodesic segment in the simply connected hyperbolic region $\widehat{\mathbb{C}} \backslash \gamma_{1-k}$.

Theorem *For every equivalence class of configurations, there exists a unique canonical configuration.*

Our proof will show that one can obtain an explicit description of canonical configurations as follows. We may assume without loss of generality that

$$(a_0, a_1, a_2, a_3) = (\infty, e_1, e_2, e_3), \tag{1}$$

where $e_1, e_2, e_3 \in \mathbb{C}$ and $e_1+e_2+e_3 = 0$. Let $\wp$ be the Weierstrass function satisfying

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3). \tag{2}$$

We denote the line segment joining two points $z, w \in \mathbb{C}$ by $[z, w]$. Then each canonical configuration for the points as in (1) has the form

$$\gamma_0 = \wp([0, \omega_1/2]), \quad \gamma_1 = \wp([\omega_2/2, (\omega_1 + \omega_2)/2]), \tag{3}$$

where the pair ω_1 and ω_2 generates the period lattice of $\wp$, and $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$, and $e_3 = \wp((\omega_1 + \omega_2)/2)$.

In order to prove our theorem, we first state some auxiliary facts. An anti-conformal *involution* of a region D in the Riemann sphere $\widehat{\mathbb{C}}$ is an anti-conformal homeomorphism σ of D onto itself such that $\sigma \circ \sigma = \mathrm{id}_D$, where id_D denotes the identity map on D.

We will use the following well-known facts about anti-conformal involutions.

Lemma 1 *Let D be a simply connected hyperbolic region in $\widehat{\mathbb{C}}$, and σ be an anti-conformal involution of D. Then the set of fixed points of σ is a hyperbolic geodesic. Conversely, for every hyperbolic geodesic there exists a unique anti-conformal involution of D that fixes all points on this geodesic.*

Proof By the Riemann mapping theorem, we may assume that D is the unit disk $\mathbb{D}$. If C is a hyperbolic geodesic in $\mathbb{D}$, then C is an arc of a circle or a line segment that is orthogonal to the unit circle $\partial\mathbb{D}$. Then reflection in C is an anti-conformal involution σ_C of $\mathbb{D}$ that fixes every point of C.

Note that if $C = I = (-1, 1)$, then $\sigma_I(z) = \bar{z}$ and so the anti-conformal involution σ_I is a hyperbolic isometry.

Now let σ be an arbitrary anti-conformal involution of $\mathbb{D}$. Then σ is also a hyperbolic isometry. Indeed, $\tau = \sigma \circ \sigma_I$ is a conformal automorphism of $\mathbb{D}$ and hence a hyperbolic isometry. This implies that $\sigma = \tau \circ \sigma_I$ is a hyperbolic isometry as well.

This in turn implies that σ has a fixed point $w_0 \in \mathbb{D}$, namely, for w_0 we can take the midpoint of the hyperbolic geodesic segment S joining some point $z_0 \in \mathbb{D}$ with $\sigma(z_0) \in \mathbb{D}$. To see this, note that σ is an isometry on $S = \sigma(S)$ that interchanges the endpoints of S.

By conjugating with an auxiliary automorphism, we may assume that $w_0 = 0$. Then $\tau = \sigma \circ \sigma_I$ is an automorphism of $\mathbb{D}$ that fixes 0. Hence $\tau(z) = e^{i\theta} z$ with $\theta \in \mathbb{R}$. It follows that $\sigma(z) = (\tau \circ \sigma_I)(z) = e^{i\theta}\overline{z}$, and so σ is equal to the reflection σ_C in the hyperbolic geodesic $C = \{e^{i\theta/2} t : -1 < t < 1\}$. In particular, σ fixes the points in C and no other points.

The argument also shows that each anti-conformal involution of σ of $\mathbb{D}$ has the form $\sigma = \sigma_C$ for some hyperbolic geodesic C. This implies that the fixed point set of σ uniquely determines σ. □

Lemma 2 *An anti-conformal involution σ of an annulus $A = \{z \in \mathbb{C} : 1 < |z| < R\}$ with $R > 1$ that leaves each boundary component invariant is of the form $\sigma(z) = e^{i\theta}\overline{z}$ with $\theta \in \mathbb{R}$.*

Note that a priori the involution σ is not defined on the boundary of A; so by invariance of the boundary components we mean that $\sigma(z) \to \partial_k A$ as $z \in A \to \partial_k A$ for each boundary component $\partial_k A = \{z \in \mathbb{C} : |z| = R^k\}$ of A for $k = 0, 1$.

Proof Let $\tau(z) = \overline{z}$. Then $\rho = \sigma \circ \tau$ is a conformal automorphism of A that preserves the boundary components of A. It is well-known that then $\rho(z) = e^{i\theta} z$ with $\theta \in \mathbb{R}$. Hence $\sigma(z) = (\rho \circ \tau)(z) = e^{i\theta}\overline{z}$. □

Proof of the Theorem We assume that we have some canonical configuration. We will analyze the situation and will obtain an explicit description from which existence and uniqueness will be evident.

So suppose the disjoint arcs γ_1 and γ_2 in $\widehat{\mathbb{C}}$ form a canonical configuration. Then by Lemma 1, there exists an anti-conformal involution $\sigma_k : \widehat{\mathbb{C}}\backslash\gamma_{1-k} \to \widehat{\mathbb{C}}\backslash\gamma_{1-k}$ fixing the points in γ_k for $k \in \{0, 1\}$.

If we restrict these maps to the ring domain $D = \widehat{\mathbb{C}}\backslash(\gamma_0 \cup \gamma_1)$, then we obtain anti-conformal involutions of D fixing the boundary components. Now D is conformally equivalent to an annulus $A = \{z : 1 < |z| < R\}$ with $R > 1$. Then by Lemma 2, each anti-conformal involution σ_k on D corresponds to a reflection τ_k in a line through the origin on A.

Conversely, suppose that τ_0 and τ_1 are two reflections in lines through the origin. Then we can identify or "weld" the points on each boundary component $\partial_k A$ of A together by using the map τ_k for $k = 0, 1$. The quotient space carries a natural conformal structure, and is hence conformally equivalent to the Riemann sphere by the uniformization theorem. This sphere will carry two distinguished arcs $\gamma_k, k = 0, 1$, corresponding to each boundary $\partial_k A$ after the welding. Note that each reflection τ_k passes to the quotient of $A \cup \partial_k A$ as an anti-conformal involution fixing the points corresponding to points on $\partial_k A$. This induces an anti-conformal involution of $\widehat{\mathbb{C}}\backslash\gamma_{1-k}$ fixing the points on γ_k. By Lemma 1, the arc γ_k is a hyperbolic geodesic segment in the hyperbolic region $\widehat{\mathbb{C}}\backslash\gamma_{1-k}$. It follows that γ_0 and γ_1 form a canonical configuration.

We have shown that the canonical configurations are precisely those that can be obtained from an annulus $A = \{z \in \mathbb{C} : 1 < |z| < R\}$, $R > 1$, by welding the points in each boundary component $\partial_k A$ together by using a reflection τ_k in a line through the origin for $k = 0, 1$.

Now a (closed) annulus $\overline{A} = \{z \in \mathbb{C} : 1 \leq |z| \leq R\}$ carries an essentially unique (up to scaling) flat conformal metric in which the circles $\{z \in \mathbb{C} : |z| = R^k\}, k = 0, 1$, are geodesics. It has length element

$$\frac{|dz|}{|z|}. \tag{4}$$

The annulus $\overline{A}$ equipped with this metric is isometric to the cylinder

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, \quad 0 \leq x_3 \leq \log R\}$$

with the Riemannian metric induced from $\mathbb{R}^3$.

For the proof of the essential uniqueness of such a metric on $\overline{A}$, one extends it to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by successive reflections and then lifts it to the universal cover $\mathbb{C}$ by the exponential map. The resulting conformal metric on $\mathbb{C}$ will be complete and flat, and hence equal to the Euclidean metric up to a scaling (see Huber [6] for an analytic approach). A representation of the conformal metric on $\overline{A}$ as in (4) follows.

After the welding of the boundary components $\partial_k A$ by each reflection τ_k, the flat metric in (4) descends to a flat metric on the quotient space with possible singularities in the points of $\gamma_0 \cup \gamma_1$. Since $\partial_k A$ is a geodesic in the flat metric (4), γ_k will be a geodesic arc in this metric with conic singularities at the endpoints and the angles at these singularities are π.

So we obtain the following geometric description of canonical configurations: to each canonical configuration corresponds a flat metric on the sphere with four conic singularities at a_0, a_1, a_2, a_3 with cone angle π such that γ_0 and γ_1 are geodesic segments.

The converse is also true: given a flat metric on the sphere with four conic singularities a_0, a_1, a_2, a_3 with cone angles π, any pair of disjoint geodesics γ_0 connecting a_0 with a_1, and γ_1 connecting a_2 with a_3 is a canonical configuration.

To see this, note that we can cut open the sphere along the arcs γ_0 and γ_1. Then up to scaling, $D = \widehat{\mathbb{C}}\backslash(\gamma_0 \cup \gamma_1)$ equipped with the flat metric is isometric to an annulus $A = \{z \in \mathbb{C} : 1 < |z| < R\}$ with $R > 1$, equipped with the metric (4). Here each geodesic γ_k is doubled and represented by two circular arcs α_k and α'_k of equal length that have common endpoints and whose union is a boundary component of A. We may assume that γ_k corresponds to $\partial_k A = \alpha_k \cup \alpha'_k$ for $k = 0, 1$. The sphere $\widehat{\mathbb{C}}$ with the flat metric and the geodesic arcs γ_0 and γ_1 can be recovered from A if we identify correspond points on α_k and α'_k by an isometry fixing the common endpoints of α_k

and α'_k. But such an isometry is necessarily given by a reflection τ_k in a line though the origin. So we are back to our first description of canonical configurations as a quotient space of an annulus A.

A flat metric on the sphere $\widehat{\mathbb{C}}$ with four prescribed conic singularities with angles π gives $\widehat{\mathbb{C}}$ the structure of a parabolic orbifold. The corresponding flat metric is unique up to scaling, and obtained by pushing the Euclidean metric in the plane forward by the universal orbifold covering map $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$. For a parabolic orbifold with four conic singularities with angles π this universal orbifold covering map Θ is a Weierstrass $\wp$-function followed by a Möbius transformation. With the normalization (1), we actually have $\Theta = \wp$, where $\wp$ is as in (2). Then the length element of the flat metric is given by $ds = |\wp'(z)||dz|$ and geodesic segments on $\widehat{\mathbb{C}}$ in the flat metric are given by images of Euclidean geodesic segments under $\wp$ (for a thorough discussion of the relevant facts about orbifolds see [2, Sections 3.5, A.9, A.10]).

For the given normalization (1), the arc γ_0 in a canonical configuration lifts under $\wp$ to a Euclidean line segment $[z_0, z_1] \subseteq \mathbb{C}$ such that $\wp$ is an isometry of $[z_0, z_1]$ onto γ_0. Here we may assume that $\wp(z_0) = \infty$ and $\wp(z_1) = e_1$.

Let $\Gamma \subseteq \mathbb{C}$ be the period lattice of $\wp$. Since we have translation invariance of $\wp$ under Γ and $\wp^{-1}(\infty) = \Gamma$, we may further assume that $z_0 = 0$. Since $\wp(z_1) = e_1$, the point z_1 must be a half-period of $\wp$, i.e., $z_1 \in \frac{1}{2}\Gamma$. Now $\wp$ is injective on $[z_0, z_1] = [0, z_1]$, and so the point z_1 must be of the form $z_1 = \omega_1/2$, where $\omega_1 \neq 0$ is a *primitive* element of Γ, i.e., ω_1 cannot be represented in the form $\omega_1 = n\gamma$ with $n \in \mathbb{N}$, $n \geq 2$, and $\gamma \in \Gamma$. It follows that $\gamma_0 = \wp([0, \omega_1/2)]$ as in (3).

Since ω_1 is a primitive element of Γ, there exists an element $\omega_2 \in \Gamma$ such that ω_1 and ω_2 form a basis of Γ. For given ω_1, the choice of ω_2 is not unique, but if we make one choice for ω_2, then all other choices ω'_2 are of the form

$$\omega'_2 = \pm\omega_2 + n\omega_1 \tag{5}$$

with $n \in \mathbb{Z}$. Note that $\wp$ maps the half-periods $\omega'_2/2$ to e_2 or e_3. With suitable choice of ω_2 we may assume that $\wp(\omega_2/2) = e_2$.

We now lift the second arc γ_1 in our canonical configuration under $\wp$ to a line segment $[z_2, z_3] \subseteq \mathbb{C}$ starting at $z_2 = \omega_2/2$. Here $\wp(z_3) = e_3$, and so $z_3 \in \frac{1}{2}\Gamma$ is a half-period.

We can say more here. Since the $\wp$-function satisfies

$$\wp(\pm z + \alpha) = \wp(z) \quad \text{for } z \in \mathbb{C} \text{ and } \alpha \in \Gamma, \tag{6}$$

it is invariant under reflections in half-periods. This implies that $\wp^{-1}(\gamma_0)$ contains the full line passing through $z_0 = 0$ and $z_1 = \omega_1/2$. Similarly, $\wp^{-1}(\gamma_1)$ contains the full line passing through $z_2 = \omega_2/2$ and the half-period z_3. Since γ_0 and γ_1 are disjoint, these lines cannot meet and hence must be parallel. It follows that z_3 necessarily has the form $z_3 = (n\omega_1 + \omega_2)/2$ with $n \in \mathbb{Z}$. Since $\wp(z_3) = e_3$, the integer n must be odd, and since $\wp$ is injective on the lift $[z_2, z_3]$, we must have $z_3 = (\pm\omega_1 + \omega_2)/2$. By reflection symmetry of $\wp$ in the half-period $z_2 = \omega_2/2$ and replacing the original lift $[z_2, z_3]$ by its reflection image in z_2, we may assume that $z_3 = (\omega_1 + \omega_2)/2$.

We conclude that under the normalization (1), arcs in a canonical configuration γ_0 and γ_1 have the form (3). Conversely, arcs as in (3) form a canonical configuration as follows from our geometric description of canonical configuration in terms of flat metrics on $\widehat{\mathbb{C}}$.

It remains to show that under the assumption (1) we obtain exactly one canonical configuration of arcs in each isotopy class.

Our previous analysis shows that in (3) the arcs γ_0 and γ_1 are uniquely determined once we know the primitive element ω_1 up to sign. Indeed, it is clear that this determines γ_0. Moreover, the choice of ω_1 up to sign does not determine ω_2 uniquely, but it follows from (5) and (6) that the arc $\gamma_2 = [\wp(\omega_2/2, (\omega_1 + \omega_2)/2]$ is uniquely determined independent of the choice of the sign of ω_1 and the choice of ω_2.

Now suppose we have chosen a fixed basis ω_1^0 and ω_2^0 of the period lattice Γ. Then the primitive elements ω_1 of Γ are precisely the elements of the form

$$\omega_1 = r\omega_1^0 + s\omega_2^0,$$

where $r, s \in \mathbb{Z}$ are relatively prime. By choosing the appropriate sign of ω_1, we may assume that $s \geq 0$ and that $r = 1$ if $s = 0$. The ratio $r/s \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ describes the slope of the line in $\mathbb{C}$ passing through 0 and ω_1 and we have a bijective correspondence between slopes $r/s \in \widehat{\mathbb{Q}}$ and primitive elements $\pm\omega_1$ of Γ up to sign.

Now it is a well known fact that isotopy classes of arcs in a sphere with four marked points are in one-to-one correspondence with these rational slopes $r/s \in \widehat{\mathbb{Q}}$ (see [5, Ch. 2] for a related discussion). In our case, $\pm\omega_1 \leftrightarrow \wp([0, \omega_1/2])$ induces a bijective correspondence between primitive elements $\pm\omega_1$ of Γ up to sign and isotopy classes of arcs γ_0 in $\widehat{\mathbb{C}}$ with marked points as (1) (see [1, Sect. 2.6] for a thorough discussion in the spirit of the present considerations). Note that the isotopy class of γ_0 uniquely determines the isotopy class of the pair (γ_0, γ_1). The statement follows. □

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