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# UNIFORMLY BRANCHING TREES

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Abstract. A quasiconformal tree $T$ is a (compact) metric tree that is doubling and of bounded turning. We call $T$ trivalent if every branch point of $T$ has exactly three branches. If the set of branch points is uniformly relatively separated and uniformly relatively dense, we say that $T$ is uniformly branching. We prove that a metric space $T$ is quasisymmetrically equivalent to the continuum self-similar tree if and only if it is a trivalent quasiconformal tree that is uniformly branching. In particular, any two trees of this type are quasisymmetrically equivalent.

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## 1. Introduction

In this paper we study the quasiconformal geometry of a certain fractal space and solve the related quasisymmetric uniformization problem. In general, *quasiconformal geometry* refers to geometric properties of a metric space that are of a robust scale-invariant nature. These properties are mostly characterized by conditions that do not involve *absolute* distances, but rather *relative* distances, that is, ratios of distances. These conditions appear naturally when one investigates the geometry of self-similar fractals, in particular fractals that arise in some dynamical settings such as limits sets of Kleinian groups, Julia sets of rational maps, or attractors of iterated function systems. One wants to characterize these spaces up to a natural equivalence given by homeomorphisms with good geometric control.

A relevant class of mappings in this context are homeomorphisms that distort relative distances in a controlled way, namely *quasisymmetric* homeomorphisms (for

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a review of the relevant definition see Section 2; general background on quasisymmetries and related concepts can be found in [He01]). Properties of a space related to its quasiconformal geometry are often invariant under quasisymmetries.

We call two metric spaces $S$ and $T$ *quasisymmetrically equivalent* if there exists a quasisymmetric homeomorphism from $S$ onto $T$. This notion of equivalence is stronger than topological equivalence, since it respects features of quasiconformal geometry.

The problem of quasisymmetric equivalence has been studied for various types of spaces (see [Bo06] for a general overview). For example, David and Semmes [DS97] gave a characterization of the standard 1/3-Cantor set up to quasisymmetric equivalence. A similar characterization for the unit interval $[0,1]$ or the unit circle is due to Tukia and Väisälä [TV80] (see also [Ro01,Me11,HM12]). The problem of characterizing the standard 2-sphere up to quasisymmetric equivalence is particularly interesting because of the connection of this problem to questions in complex dynamics and geometric group theory (see [BM17] for more discussion on the background and references to the literature).

One can see the underlying questions as instances of the *quasisymmetric uniformization problem:* Suppose we consider a metric space $S$ that is homeomorphic to a "standard" space $T$. Under what conditions is $S$ actually quasisymmetrically equivalent to $T$?

The goal of the present paper is to investigate the quasiconformal geometry of a particular tree-like space with a self-similar structure and very regular branching behavior, namely the *continuum self-similar tree* $\mathbb{T}$ (abbreviated as CSST). We will characterize $\mathbb{T}$ up to quasisymmetric equivalence. This can be seen as a solution to the quasisymmetric uniformization problem where $\mathbb{T}$ plays the role of the "standard" space.

Under this name, the CSST was introduced in [BT21] and defined as the attractor of an iterated function system in the complex plane (see Figure 1 for an illustration). One motivation to take $\mathbb{T}$ as a standard space for the quasisymmetric uniformization problem is that sets homeomorphic to $\mathbb{T}$ were considered in the literature before in various contexts (see [BT21] for more discussion and references). For example, the Julia set $\mathcal{J}(P)$ of the quadratic polynomial $P(z) = z^2 + i$ is homeomorphic to $\mathbb{T}$. Actually, one can show the stronger result that $\mathcal{J}(P)$ is even quasisymmetricaly equivalent to $\mathbb{T}$. The original proof of this fact (unpublished) was based on an ad hoc approach by H. Tran and the first author, but can be streamlined by the results in this paper.

There are other interesting tree-like fractals whose characterization up quasisymmetric equivalence may be interesting. We refer to Section 9 for more discussion.

In order to characterize the CSST up to quasisymmetry, we need a way to quantitatively describe the separation and density of branch points in a (metric) tree. We formulate the relevant conditions precisely in Definitions 1.1 and 1.2. Trees satisfying these conditions are called *uniformly branching* (see Definition 1.3). A major theme of this paper is the study of uniformly branching trees from the perspective of quasiconformal geometry.

To state our main result, we will now discuss some concepts in more detail. We first turn to the CSST. Up to bi-Lipschitz equivalence, one can define an abstract version of $\mathbb{T}$ as follows. We start with a line segment $J_0$ of length 2. Its midpoint $c$ subdivides $J_0$ into two line segments of length 1. We glue to $c$ one of the endpoints

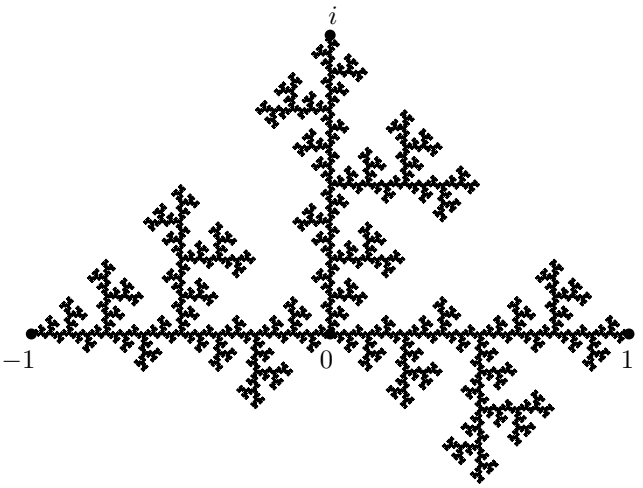


FIGURE 1. The continuum self-similar tree $\mathbb{T}$

of another line segment $s$ of the same length. Then we obtain a tripod-like set $J_1$ consisting of three line segments of length 1. The set $J_1$ carries its natural path metric. We now repeat this procedure inductively. At the $n$-th step we obtain a simplicial tree $J_n$ consisting of $3^n$ line segments of length $2^{1-n}$. To pass to $J_{n+1}$, each of these line segments $s$ is subdivided by its midpoint $c_s$ into two line segment of length $2^{-n}$ and we glue to $c_s$ one endpoint of another line segment of length $2^{-n}$. One can actually realize $J_n$ as a subset of the complex plane (equipped with the induced path metric). See [BT21] for more discussion and Figure 2 for an illustration of $J_5$.

In this way, we obtain an ascending sequence $J_0 \subset J_1 \subset \dots$ of (simplicial) trees equipped with a geodesic metric, i.e., a metric such that any two points in the underlying space can be joined by an arc whose length is equal to the distance of the points (see (3.1) below). The union $J = \bigcup_{n\in\mathbb{N}_0} J_n$ carries a natural path metric $\varrho$ that agrees with the metric on $J_n$ for each $n \in \mathbb{N}_0$. As an abstract space one can now define $\mathbb{T}$ as the completion of the metric space $(J, \varrho)$. Taking the completion here is necessary to obtain a compact space and strictly enlarges $J$. Indeed, the Hausdorff dimension of $J$ is 1 (as a countable union of intervals), while the Hausdorff dimension of $\mathbb{T}$ is $\log 3/\log 2 > 1$ (this easily follows from [Fa03, Theorem 9.3]).

There is an alternative way to define the CSST as the attractor of a suitable iterated function system in the complex plane $\mathbb{C}$ (see Section 5). Then it inherits the Euclidean metric as a subset of $\mathbb{C}$. The two versions of the CSST are bi-Lipschitz, and hence quasisymmetrically, equivalent. This means that for our given goal, namely the characterization of the CSST up to quasisysmmetric equivalence, we may choose any version of the CSST. For concreteness, we prefer to consider $\mathbb{T}$ (the CSST) as the attractor of an iterated function system and a subset of $\mathbb{C}$.

In order to describe the topological properties of $\mathbb{T}$, we have to recall some terminology. By definition a *metric tree* is a compact, connected, and locally connected metric space $T$ that contains at least two distinct points and has the following

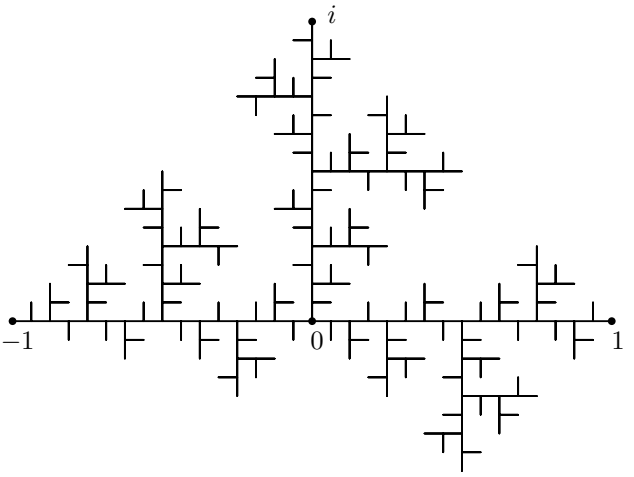


FIGURE 2. The set $J_5$

property: if $x, y \in T$, then there exists a unique (possibly degenerate) arc in $T$ with endpoints $x$ and $y$. We denote this unique arc in $T$ by $[x, y]$.

Let $T$ be a metric tree, and $p \in T$. The closure $B$ of a connected component $U$ of $T \setminus \{p\}$ has the form $B = \overline{U} = U \cup \{p\}$ and is called a *branch* of $p$ in $T$. A point $p \in T$ can have at most countably many branches. If there are at least three branches of $p$ in $T$, then $p$ is called a *branch point* and a *triple point* if there are exactly three branches. We say that the metric tree $T$ is *trivalent* if every branch point of $T$ is a triple point.

In [BT21] it was shown that the CSST is a trivalent metric tree with a dense set of triple points. This seems obvious from the abstract definition of $\mathbb{T}$ outlined above, but is harder to show if one introduces $\mathbb{T}$ as the attractor of an iterated function system. The CSST $\mathbb{T}$ is characterized by these properties up to homeomorphism [BT21, Theorem 1.7]: *a metric space $T$ is homeomorphic to $\mathbb{T}$ if and only if $T$ is a trivalent metric tree with a dense set of triple points.* This gives an essentially complete understanding of the topological properties of the CSST.

In order to describe the quasiconformal geometry of the CSST, we have to recall some relevant concepts. The following two are well known.

We say that a metric tree $T$ is of *bounded turning* if there exists a constant $K \geq 1$ such that

$$\operatorname{diam}[x, y] \leq K|x - y| \tag{1.1}$$

for all $x, y \in T$. Here and elsewhere in this paper we use Polish notation $|x - y|$ to denote the distance of two points $x$ and $y$ in a space with a given underlying metric.

A metric space $T$ is *doubling* if there exists a constant $N \in \mathbb{N}$ such that every ball in $T$ of radius $r > 0$ can be covered by $N$ or fewer balls of radius $r/2$.

The CSST has these properties (see Section 5), and they are invariant under quasisymmetries. Accordingly, they are necessary conditions for a tree $T$ to be quasisymmetrically equivalent to the CSST. It is useful to introduce a name for such trees. We call a metric tree $T$ a *quasiconformal tree* if it is doubling and of bounded turning.

Quasiconformal trees were considered before in our paper [BM20]. There we showed that every quasiconformal tree is quasisymmetrically equivalent to a tree with a geodesic metric.

Our discussion shows that a necessary condition for a metric space $T$ to be quasisymmetrically equivalent to the CSST is that $T$ is a trivalent quasiconformal tree with a dense set of branch points. It is easy to see that this is only necessary, but in general not sufficient. For a sufficient condition one would expect a more quantitative version of the density of branch points. As we will see momentarily, one also has to stipulate that branch points are quantitatively separated in a suitable sense.

To formulate this precisely, we need a quantitative measure of the size of a branch point $p$ in a (not necessarily trivalent) metric tree $T$. Only finitely many of the branches of $p$ can have a diameter exceeding a given positive number (see [BT21, Section 3] for more details). This implies that we can label the branches $B_n$ of $p$ by numbers $n = 1, 2, 3, \dots$ so that

$$\operatorname{diam}(B_1) \geq \operatorname{diam}(B_2) \geq \operatorname{diam}(B_3) \geq \dots .$$

Then we define the *height* $H_T(p)$ of $p$ in $T$ as

$$H_T(p) = \operatorname{diam}(B_3). \tag{1.2}$$

So $H_T(p)$ is the diameter of the third largest branch of $p$.

Note that when $T$ is trivalent, we have

$$H_T(p) = \min\{\operatorname{diam}(B) : B \text{ is a branch of } p \text{ in } T\}. \tag{1.3}$$

We can now define the relevant concepts.

**Definition 1.1** (Uniform relative separation of branch points)**.** A tree $T$ is said to have *uniformly relatively separated* branch points if

$$|p - q| \gtrsim \min\{H_T(p), H_T(q)\}$$

for all distinct branch points $p, q \in T$ with $C(\gtrsim)$ independent of $p$ and $q$.

The reference $C(\gtrsim)$ to the implicit multiplicative constant here is explained in the subsection on notation (see (1.5)).

**Definition 1.2** (Uniform relative density of branch points)**.** A tree $T$ is said to have *uniformly relatively dense* branch points if for all $x, y \in T$, $x \neq y$, there exists a branch point $p \in [x, y]$ with

$$H_T(p) \gtrsim |x - y|,$$

where $C(\gtrsim)$ independent of $x$ and $y$.

These definitions lead to the most important concept of this paper.

**Definition 1.3** (Uniformly branching trees)**.** A tree is called *uniformly branching* if its branch points are uniformly relatively separated and uniformly relatively dense.

Now our main result can be formulated as follows.

**Theorem 1.4.** *A metric space $T$ is quasisymmetrically equivalent to the continuum self-similar tree $\mathbb{T}$ if and only if $T$ is a trivalent quasiconformal tree that is uniformly branching.*

An immediate consequence is the following fact.

**Corollary 1.5.** *Let $S$ and $T$ be two trivalent quasiconformal trees that are uniformly branching. Then $S$ and $T$ are quasisymmetrically equivalent.*

*Proof.* Indeed, by Theorem 1.4 there exist quasisymmetries $\varphi\colon S \to \mathbb{T}$ and $\psi\colon T \to \mathbb{T}$. Then $\psi^{-1} \circ \varphi$ is a quasisymmetry of $S$ onto $T$, and so $S$ and $T$ are quasisymmetrically equivalent. □

One can outline the proof of Theorem 1.4 as follows. The "only if" part of the statement is not hard to show, because the CSST is a trivalent quasiconformal tree that is uniformly branching (see Proposition 5.4). Moreover, the relevant conditions are invariant under quasisymmetries. This is well-known for the doubling and the bounded turning conditions which are the basis for the definition of a quasiconformal tree. It is also true that uniform relative separation and uniform relative density of branch points are invariant under quasisymmetries (see Lemmas 4.3 and 4.5). This is not surprising, because these are conditions in terms of relative distances: heights of branch points versus distances of points, both of which are measured in "units" of length. A quasisymmetry gives good control for the distortion of such relative distances. As we will see in Section 4, it is completely straightforward to implement these ideas rigorously.

It is much more difficult to prove the "if" part of Theorem 1.4. The basic strategy is very similar to ideas in the recent papers [BT21] and [BM20]. For the given tree $T$ one wants to create a "subdivision" by finer and finer decompositions of $T$. This subdivision is represented by a sequence $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ of decompositions of $T$ given by finite covers $\mathbf{X}^n$, $n \in \mathbb{N}_0$, of $T$ by compact subsets. We call the sets $X \in \mathbf{X}^n$ the *tiles* of level $n$ of the decomposition. We will obtain these tiles by cutting $T$ at the set $\mathbf{V}^n$ of all branch points with heights $\geq \delta^n$, where $\delta \in (0,1)$ is a small parameter. More precisely, the tiles in $\mathbf{X}^n$ are the closures of the components of $T \setminus \mathbf{V}^n$. Since $\mathbf{V}^n \subset \mathbf{V}^{n+1}$, the tiles of level $n$ are subdivided into smaller tiles of level $n+1$.

If we assume that $T$ is a uniformly branching trivalent quasiconformal tree, then the tiles in our subdivision have very good geometric properties. For example, if $X$ is a tile of level $n$, then $\operatorname{diam}(X) \asymp \delta^n$. Moreover, $\mathbf{X}^n$ for $n \in \mathbb{N}_0$ is an *edge-like* decomposition of $T$ in the sense that each tile $X \in \mathbf{X}^n$ has at most two boundary points (see Proposition 6.1). It implies that the incidence relations of the tiles $X \in \mathbf{X}^n$ are the same as the incidence relations of the edges in a suitable finite trivalent simplicial tree (which explains our terminology).

One wants to realize a similar subdivision with the same combinatorics for the CSST $\mathbb{T}$. Once this is achieved, then under some mild extra conditions there exists a homeomorphism $F\colon T \to \mathbb{T}$ that maps the tiles in the subdivision of $T$ to corresponding tiles in the subdivision of $\mathbb{T}$. One can show that this homeomorphism is a quasisymmetry if the tiles in the corresponding subdivisions satisfy suitable geometric conditions. We condense the relevant conditions into the concept of a *quasi-visual approximation* and *quasi-visual subdivision* of a space (see Section 2 for precise definitions).

Roughly speaking, if we have quasi-visual approximations for two spaces that correspond under a homeomorphism, then this homeomorphism is a quasisymmetry. Related ideas have been used by us and other authors before to prove that a homeomorphism is a quasisymmetry (see for example [BM17, Proposition 18.8 and Lemma 18.10], [Me02, Lemmas 4.4 and 4.5], [Ki14, Theorem 3.4]). We formulate this in a general setting in Section 2. We hope that it clarifies some of the earlier approaches and may be of independent interest.

In view of this, one would like our subdivision $\{\mathbf{X}^n\}$ of $T$ to be quasi-visual. This is always the case under the given hypotheses (see Proposition 6.1 (ii)). In order to find a quasi-visual subdivision of $\mathbb{T}$ corresponding to the subdivision $\{\mathbf{X}^n\}$ of $T$, one uses an inductive process. The decomposition of $\mathbb{T}$ on level $n$ is given by the image sets $F^n(X)$, $X \in \mathbf{X}^n$, of an auxiliary homeomorphism $F^n\colon T \to \mathbb{T}$. Note that we use the superscript $n$ here to indicate the level (and not the $n$-th iterate of some map $F$). For the inductive step, we subdivide the tile $F^n(X)$ of $\mathbb{T}$ for each $X \in \mathbf{X}^n$ in the same way as $X$ is subdivided by tiles in $\mathbf{X}^{n+1}$.

In this way, one obtains a subdivision $\{F^n(\mathbf{X}^n)\}$ of $\mathbb{T}$ with the same combinatorics as the subdivision $\{\mathbf{X}^n\}$ of $T$. As we will see, one can realize the isomorphism between $\{\mathbf{X}^n\}$ and $\{F^n(\mathbf{X}^n)\}$ by a single homeomorphism $F\colon T \to \mathbb{T}$ such that $F(X) = F^n(X)$ for all $X \in \mathbf{X}^n$. Then $\{\mathbf{X}^n\}$ and $\{F^n(\mathbf{X}^n)\} = \{F(\mathbf{X}^n)\}$ are in a suitable sense isomorphic subdivisions of $T$ and $\mathbb{T}$, respectively. The subdivision $\{\mathbf{X}^n\}$ is quasi-visual, and one can show that with careful choices in the inductive process $\{F(\mathbf{X}^n)\}$ is a quasi-visual subdivision as well (for more details see Proposition 7.1 and its proof). Hence the homeomorphism $F$ is a quasisymmetry, and $T$ and $\mathbb{T}$ are quasisymmetrically equivalent.

**Organization of this paper.** We first introduce *quasi-visual approximations* and *quasi-visual subdivisions* in Section 2. These are sequences of decompositions of a space that approximate it in a geometrically controlled way. They are closely connected to quasisymmetries (see Proposition 2.7 and Proposition 2.13). In Section 3 we collect some facts about the topology of trees and their decompositions. In Section 4 we show that quasisymmetries preserve the property of a trivalent tree to be uniformly branching.

The CSST and some of its properties are reviewed in Section 5. There we show that the CSST is uniformly branching (Proposition 5.4). In Section 6 we consider specific subdivisions of trees and introduce *edge-like subdivisions*. We will show that each uniformly branching quasiconformal tree admits edge-like subdivisions with good geometric control (see Proposition 6.1).

Section 7 is the technical core of the paper. There we show that if a uniformly branching quasiconformal tree $T$ has an edge-like subdivision as in Proposition 6.1, then there exists a subdivision of the CSST with the same combinatorics (see Proposition 7.1). For the proof we use an inductive process based on an existence result for homeomorphisms $F\colon T \to \mathbb{T}$ with good properties (see Lemma 7.3).

Our argument is wrapped up in Section 8, where we assemble the considerations of the earlier sections for the proof of Theorem 1.4. We conclude with Section 9 where we make some additional comments and formulate open problems related to the investigations in this paper.

**Notation.** We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We write $\#X \in \mathbb{N}_0$ for the cardinality of a finite set $X$.

Given two non-negative quantities $a$ and $b$, we write

$$a \asymp b \tag{1.4}$$

if there is a constant $C > 0$ depending on some ambient parameters such that $a/C \le b \le Ca$. In this case, we refer to the constant as $C = C(\asymp)$. Similarly, we write

$$a \lesssim b \quad \text{or} \quad b \gtrsim a \tag{1.5}$$

if there is a constant $C > 0$ such that $a \le Cb$. We refer to the constant as $C = C(\lesssim) = C(\gtrsim)$.

Let $(X, d)$ be a metric space. If $A \subset X$, we denote by $\overline{A}$ the closure, by $\operatorname{int}(A)$ the interior, and by $\partial A$ the boundary of $A$. Moreover,

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in X\}$$

is the diameter of $A$. If $B \subset X$ is another set, then we use the notation

$$\operatorname{dist}(A, B) = \inf\{d(x, y) : x \in A,\, y \in B\}$$

for the distance of $A$ and $B$.

Let $T$ be a metric tree, $x, y \in T$, and $[x, y]$ be the unique arc in $T$ joining $x$ and $y$. Then we write $\operatorname{diam}[x, y]$ instead of $\operatorname{diam}([x, y])$ for the diameter of the arc $[x, y]$; so we omit the parentheses in our notation for better readability. Similarly, we denote by $\operatorname{length}[x, y]$ the length of $[x, y]$ with respect to the given metric on $T$.

If $F\colon X \to Y$ is a map between sets $X$, $Y$, and $A \subset X$ is a subset of $X$, then we write $F|A$ for the restriction of $F$ to $A$. The identity map on $X$ is denoted by $\mathrm{id}_X$, or simply by id if $X$ is understood.

## 2. Quasi-visual approximations and subdivisions

In this section, we provide a general framework to describe a space by *approximations* and *subdivisions*. Our main result here gives a method to prove quasisymmetric equivalence of two metric spaces based on these concepts (see Proposition 2.13). Very similar ideas have recently been formulated by Kigami [Ki20, Section 3.6]. We will first recall the definition of a quasisymmetric homeomorphism and then consider *quasi-visual approximations* of metric spaces.

Let $(S, d)$ and $(T, \rho)$ be metric spaces. A homeomorphism $F\colon S \to T$ is called a *quasisymmetric homeomorphism* or a *quasisymmetry* if there exists a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that for all $t > 0$ and all $x, y, z \in S$ the following implication holds:

$$d(x, y) \le t d(x, z) \Rightarrow \rho(F(x), F(y)) \le \eta(t)\rho(F(x), F(z)). \tag{2.1}$$

If we want to emphasize the distortion function $\eta$ here, then we call $F$ an *$\eta$-quasisymmetry*.

If a bijection $F\colon S \to T$ satisfies the implication (2.1), then $F$ is continuous, and $F^{-1}$ satisfies a similar implication. Hence $F^{-1}$ is also continuous, and $F$ is a homeomorphism. So once we have (2.1) for a bijection $F\colon S \to T$, it is a quasisymmetric homeomorphism.

If $F\colon S \to T$ is a quasisymmetry, then $F^{-1}\colon T \to S$ is also a quasisymmetry. If $R$ is another metric space, and $F\colon S \to T$ and $G\colon T \to R$ are quasisymmetries, then $G \circ F\colon S \to R$ is a quasisymmetry (see [He01, Proposition 10.6]).

As we will see, the following concept is relevant in connection with quasisymmetries.

**Quasi-visual approximations.**

**Definition 2.1** (Quasi-visual approximations)**.** Let $S$ be a bounded metric space. A *quasi-visual approximation* of $S$ is a sequence $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ of finite covers $\mathbf{X}^n$, $n \in \mathbb{N}_0$, of $S$ by some of its non-empty subsets. Here we assume $\mathbf{X}^0 = \{S\}$ and make the following requirements for all $n \in \mathbb{N}_0$ with implicit constants independent of $n$, $X$, $Y$:

(i) $\operatorname{diam}(X) \asymp \operatorname{diam}(Y)$ for all $X, Y \in \mathbf{X}^n$ with $X \cap Y \neq \emptyset$.

(ii) $\operatorname{dist}(X, Y) \gtrsim \operatorname{diam}(X)$ for all $X, Y \in \mathbf{X}^n$ with $X \cap Y = \emptyset$.

(iii) $\operatorname{diam}(X) \asymp \operatorname{diam}(Y)$ for all $X \in \mathbf{X}^n$, $Y \in \mathbf{X}^{n+1}$ with $X \cap Y \neq \emptyset$.

(iv) For some constants $k_0 \in \mathbb{N}$ and $\lambda \in (0, 1)$ independent of $n$ we have $\operatorname{diam}(Y) \leq \lambda \operatorname{diam}(X)$ for all $X \in \mathbf{X}^n$ and $Y \in \mathbf{X}^{n+k_0}$ with $X \cap Y \neq \emptyset$.

For simplicity, we will usually just write $\{\mathbf{X}^n\}$ instead of $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ with the index set $\mathbb{N}_0$ for $n$ understood. We call the elements of $\mathbf{X}^n$ the *tiles of level* $n$, or simply the *$n$-tiles* of the sequence $\{\mathbf{X}^n\}$.

Condition (iv) in the previous definition can be expressed in a slightly different form.

**Lemma 2.2.** *Let $\{\mathbf{X}^n\}$ be a sequence of finite covers of a bounded metric space $S$ satisfying condition (iii) in Definition 2.1. Then $\{\mathbf{X}^n\}$ satisfies (iv) if and only if there exist constants $C > 0$ and $\rho \in (0, 1)$ such that*

$$\operatorname{diam}(X^{n+k}) \leq C\rho^k \operatorname{diam}(X^n) \tag{iv'}$$

*for all $k, n \in \mathbb{N}_0$, $X^n \in \mathbf{X}^n$, $X^{n+k} \in \mathbf{X}^{n+k}$ with $X^n \cap X^{n+k} \neq \emptyset$.*

*Proof.* Assume first that condition (iv) in Definition 2.1 holds with $k_0 \in \mathbb{N}$ and $\lambda \in (0, 1)$. Let $k, n \in \mathbb{N}_0$, $X^n \in \mathbf{X}^n$, and $X^{n+k} \in \mathbf{X}^{n+k}$ with $X^n \cap X^{n+k} \neq \emptyset$ be arbitrary. We pick a point $x \in X^n \cap X^{n+k}$. For $i = n + 1, \ldots, n + k - 1$, we choose $X^i \in \mathbf{X}^i$ with $x \in X^i$. We write $k$ in the form $k = jk_0 + \ell$ with $j \in \mathbb{N}_0$ and $\ell \in \{0, \ldots, k_0 - 1\}$. Finally, let $C_1 = C(\asymp) \geq 1$ be the constant in condition (iii) of Definition 2.1. Then by repeated applications of (iii) and (iv) we see that

$$\begin{aligned}\operatorname{diam}(X^{n+k}) &= \operatorname{diam}(X^{n+jk_0+\ell}) \leq C_1^{\ell} \operatorname{diam}(X^{n+jk_0}) \\ &\leq C_1^{\ell}\lambda^j \operatorname{diam}(X^n) \leq C_1^{k_0-1}(\lambda^{1/k_0})^{jk_0} \operatorname{diam}(X^n).\end{aligned}$$

If we define $\rho := \lambda^{1/k_0} \in (0, 1)$ and $C := C_1^{k_0-1}\lambda^{-(k_0-1)/k_0} \geq C_1^{k_0-1}\rho^{-\ell}$, we can write this as

$$\begin{aligned}\operatorname{diam}(X^{n+k}) &\leq C_1^{k_0-1}\rho^{jk_0} \operatorname{diam}(X^n) \leq C\rho^{jk_0+\ell} \operatorname{diam}(X^n) \\ &= C\rho^k \operatorname{diam}(X^n),\end{aligned}$$

as desired.

To show the reverse implication, assume that (iv') holds for constants $C > 0$ and $\rho \in (0, 1)$. We can then choose $k_0 \in \mathbb{N}$ sufficiently large such that $\lambda := C\rho^{k_0} < 1$. With these choices, condition (iv) in Definition 2.1 is clearly satisfied. □

A very similar argument shows that condition (iii) in Definition 2.1 implies an inequality opposite to (iv'). Namely, there exists a constant $\tau \in (0, 1)$ such that

$$\operatorname{diam}(X^{n+k}) \geq \tau^k \operatorname{diam}(X^n) \tag{2.2}$$

for all $k, n \in \mathbb{N}_0$, $X^n \in \mathbf{X}^n$, $X^{n+k} \in \mathbf{X}^{n+k}$ with $X^n \cap X^{n+k} \neq \emptyset$.

The previous lemma implies that the diameters of tiles in a quasi-visual approximation approach 0 uniformly as their levels tend to infinity.

**Corollary 2.3.** *Let $\{\mathbf{X}^n\}$ be a quasi-visual approximation of a bounded metric space $S$. Then*

$$\max\{\operatorname{diam}(X) : X \in \mathbf{X}^n\} \to 0 \text{ as } n \to \infty.$$

*Proof.* Indeed, by Lemma 2.2 we know that there exist constants $C > 0$ and $\rho \in (0,1)$ with

$$\operatorname{diam}(X^n) \le C\rho^n \operatorname{diam}(X^0)$$

for all $n \in \mathbb{N}_0$ and $X^n \in \mathbf{X}^n$. Here $X^0 = S$ is the only 0-tile. The statement follows. □

We record a special situation when a sequence of coverings forms a quasi-visual approximation.

**Lemma 2.4.** *Let $S$ be a bounded metric space, and let $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ be a sequence of finite covers $\mathbf{X}^n$, $n \in \mathbb{N}_0$, of $S$ by some of its subsets. Suppose that $\mathbf{X}^0 = \{S\}$, and that there exists a constant $\delta \in (0,1)$ such that the following conditions are true:*

(i) $\operatorname{diam}(X) \asymp \delta^n$ *for all* $n \in \mathbb{N}_0$ *and* $X \in \mathbf{X}^n$.

(ii) $\operatorname{dist}(X,Y) \gtrsim \delta^n$ *for all* $n \in \mathbb{N}_0$ *and* $X, Y \in \mathbf{X}^n$ *with* $X \cap Y = \emptyset$.

*Here we require that the implicit constants are independent of $n$, $X$, and $Y$. Then $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ is a quasi-visual approximation of $S$.*

It is natural to call $\{\mathbf{X}^n\}$ as in this lemma a *visual approximation* of $S$, because its tiles have properties that are analogous to properties of tiles of an expanding Thurston map with respect to a *visual metric* (see [BM17, Proposition 8.4]). We chose the term *quasi-visual approximation* in Definition 2.1, because there the conditions are more relaxed in comparison to the more stringent ones for a visual approximation.

*Proof of Lemma 2.4.* We have to verify conditions (i)–(iv) in Definition 2.1. In the following, all implicit constants are independent of the tiles under consideration and their levels.

Fix $n \in \mathbb{N}_0$, and consider arbitrary tiles $X, Y \in \mathbf{X}^n$. Then

$$\operatorname{diam}(X) \asymp \delta^n \asymp \operatorname{diam}(Y),$$

by (i). Thus condition (i) in Definition 2.1 is satisfied.

If $X \cap Y = \emptyset$, then by (ii) we have

$$\operatorname{dist}(X,Y) \gtrsim \delta^n \asymp \operatorname{diam}(X),$$

and so condition (ii) in Definition 2.1 is satisfied.

Let $Z \in \mathbf{X}^{n+1}$ be arbitrary. Then

$$\operatorname{diam}(Z) \asymp \delta^{n+1} \asymp \delta \operatorname{diam}(X),$$

by (i). Condition (iii) in Definition 2.1 immediately follows (with a constant depending on $\delta$ in addition to the other ambient parameters).

Similarly, if $Z \in \mathbf{X}^{n+k}$ for some $k \in \mathbb{N}_0$, then

$$\operatorname{diam}(Z) \asymp \delta^{n+k} \asymp \delta^k \operatorname{diam}(X)$$

by (i). This implies that condition (iv′) in Lemma 2.2 is true, which is equivalent to (iv) in Definition 2.1.

The statement follows. □

The covers $\mathbf{X}^n$ in a quasi-visual approximation provide a discrete approximation of $S$ that improves with larger $n$. We introduce a quantity that records at which level this approximation allows us to distinguish two distinct points.

**Definition 2.5.** Let $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ be a quasi-visual approximation of the bounded metric space $S$. For two distinct points $x, y \in S$ we define

$$m(x,y) \coloneqq \max\{n \in \mathbb{N}_0 : \text{there exist sets } X, Y \in \mathbf{X}^n \text{ with } x \in X,\ y \in Y, \text{ and } X \cap Y \neq \emptyset\}.$$

Note that by Corollary 2.3 we have

$$\lim_{n\to\infty} \max_{X\in\mathbf{X}^n} \operatorname{diam}(X) = 0,$$

which implies that $m(x,y) \in \mathbb{N}_0$ is well defined for $x, y \in S$ with $x \neq y$.

**Lemma 2.6.** *Let $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ be a quasi-visual approximation of the bounded metric space $(S,d)$. Then there is a constant $C(\asymp)$ such that for all distinct $x, y \in S$ we have*

$$d(x,y) \asymp \operatorname{diam}(X^m),$$

*where $m = m(x,y)$ and $X^m \in \mathbf{X}^m$ is an arbitrary $m$-tile that contains $x$.*

*Proof.* Let $x, y \in S$ with $x \neq y$ be arbitrary, and $m = m(x,y) \in \mathbb{N}_0$ be given as in Definition 2.5. This means that there are $m$-tiles $X, Y \in \mathbf{X}^m$ with $x \in X$, $y \in Y$, and $X \cap Y \neq \emptyset$. Let $X^m \in \mathbf{X}^m$ be an arbitrary $m$-tile that contains $x$. Then

$$d(x,y) \le \operatorname{diam}(X) + \operatorname{diam}(Y) \asymp \operatorname{diam}(X) \asymp \operatorname{diam}(X^m),$$

where we used condition (i) in Definition 2.1.

On the other hand, we can find $X', Y' \in \mathbf{X}^{m+1}$ with $x \in X'$ and $y \in Y'$. The definition of $m(x,y)$ now implies that $X'$ and $Y'$ must be disjoint. Thus

$$d(x,y) \ge \operatorname{dist}(X', Y') \gtrsim \operatorname{diam}(X') \asymp \operatorname{diam}(X^m).$$

Here we used (ii) and (iii) in Definition 2.1. The statement follows. □

**Quasi-visual approximations and quasisymmetries.** The following proposition relates quasi-visual approximations and quasisymmetries. This is the main reason why we consider such approximations.

**Proposition 2.7.** *Let $(S,d)$ and $(T,\rho)$ be bounded metric spaces, the map $F\colon S \to T$ be a bijection, and $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ be a quasi-visual approximation of $(S,d)$. Then $F\colon S \to T$ is a quasisymmetry if and only if $\{F(\mathbf{X}^n)\}_{n\in\mathbb{N}_0}$ is a quasi-visual approximation of $(T,\rho)$.*

Here we use the obvious notation $F(\mathbf{X}^n) \coloneqq \{F(X) : X \in \mathbf{X}^n\}$.

The proof of Proposition 2.7 requires some preparation. We first remind the reader of the following well-known fact (it follows immediately from [He01, Proposition 10.8]). We include a proof for the sake of completeness.

**Lemma 2.8.** *Let $(S,d)$ and $(T,\rho)$ be bounded metric spaces, and the map $F\colon S \to T$ be an $\eta$-quasisymmetry. Let $A \subset S$ be a subset of $S$, and $x, y \in A$ be points satisfying $d(x,y) \asymp \operatorname{diam}(A)$ with a constant $C_1 = C(\asymp)$. Then*

$$\operatorname{diam}(F(A)) \asymp \rho(F(x), F(y)),$$

*where $C(\asymp) = C(C_1, \eta)$.*

*Proof.* Under the given assumptions, let $z \in A$ be arbitrary. Then

$$d(x,z) \le \operatorname{diam}(A) \le C_1 d(x,y).$$

We write $x' = F(x)$, $y' = F(y)$, $z' = F(z)$, and $A' = F(A)$. Since $F$ is an $\eta$-quasisymmetry, we have $\rho(x',z') \le \eta(C_1)\rho(x',y')$. It follows that

$$\operatorname{diam}(A') \le 2 \sup_{z' \in A'} \rho(x',z') \le 2\eta(C_1)\rho(x',y').$$

On the other hand, $\rho(x',y') \le \operatorname{diam}(A')$. The statement follows with $C_2 = C(\asymp) = \max\{1, 2\eta(C_1)\}$. □

Quasi-visual approximations are invariant under "quasisymmetric changes" of the metric.

**Lemma 2.9.** *Let $d_1$ and $d_2$ be bounded metrics on a set $S$. Suppose that a sequence of finite covers $\{\mathbf{X}^n\}$ of $S$ is a quasi-visual approximation of the metric space $(S,d_1)$ and that the identity map $\mathrm{id}\colon (S,d_1) \to (S,d_2)$ is a quasisymmetric homeomorphism. Then $\{\mathbf{X}^n\}$ is also a quasi-visual approximation of $(S,d_2)$.*

*Proof.* Under the given assumptions, we need to show that $\{\mathbf{X}^n\}$ is a quasi-visual approximation of $(S,d_2)$, meaning that it satisfies the conditions (i)–(iv) in Definition 2.1. For $i = 1,2$ we denote the diameter of a set $M \subset S$ with respect to the metric $d_i$ by $\operatorname{diam}_i(M)$.

(i) Let $n \in \mathbb{N}_0$ and $X, Y \in \mathbf{X}^n$ with $X \cap Y \neq \emptyset$ be arbitrary. Then there are points $z \in X \cap Y$, $x \in X$, and $y \in Y$ with $d_1(x,z) \asymp \operatorname{diam}_1(X)$ and $d_1(y,z) \asymp \operatorname{diam}_1(Y)$, where $C(\asymp) = 3$. Since condition (i) of Definition 2.1 is satisfied for $d_1$, we know that

$$d_1(x,z) \asymp \operatorname{diam}_1(X) \asymp \operatorname{diam}_1(Y) \asymp d_1(y,z).$$

The quasisymmetric equivalence of $d_1$ and $d_2$ together with Lemma 2.8 now implies that

$$\operatorname{diam}_2(X) \asymp d_2(x,z) \asymp d_2(y,z) \asymp \operatorname{diam}_2(Y).$$

So $\{\mathbf{X}^n\}$ satisfies (i) in Definition 2.1 for $(S,d_2)$.

(iii) The verification of this condition is very similar to the one for (i) and so we will skip the details.

(ii) Let $n \in \mathbb{N}_0$ and $X, Y \in \mathbf{X}^n$ with $X \cap Y = \emptyset$, as well as $x \in X$ and $y \in Y$ be arbitrary. We denote the distance of $X$ and $Y$ with respect to the metric $d_i$ by $\operatorname{dist}_i(X,Y)$ for $i = 1,2$. We can choose $x' \in X$ with $d_1(x,x') \asymp \operatorname{diam}_1(X)$, where $C(\asymp) = 3$. Since (ii) holds for $d_1$, we know that

$$d_1(x,x') \le \operatorname{diam}_1(X) \lesssim \operatorname{dist}_1(X,Y) \le d_1(x,y).$$

Quasisymmetric equivalence of $d_1$ and $d_2$ together with Lemma 2.8 now implies

$$\operatorname{diam}_2(X) \asymp d_2(x,x') \lesssim d_2(x,y).$$

Taking the infimum over $x \in X$ and $y \in Y$ yields

$$\operatorname{diam}_2(X) \lesssim \operatorname{dist}_2(X,Y),$$

which is condition (ii) for $d_2$.

(iv) Let $\lambda \in (0,1)$ and $k_0 \in \mathbb{N}$ be constants as in condition (iv) of Definition 2.1 for the metric space $(S,d_1)$. Let $n, \ell \in \mathbb{N}_0$, $X^n \in \mathbf{X}^n$ and $X^{n+\ell k_0} \in \mathbf{X}^{n+\ell k_0}$ with $X^n \cap X^{n+\ell k_0} \neq \emptyset$ be arbitrary. Then we can choose $x \in X^n \cap X^{n+\ell k_0}$ and $x' \in X^n$ with $\operatorname{diam}_1(X^n) \le 3 d_1(x,x')$. Finally, let $y \in X^{n+\ell k_0}$ be arbitrary.

By repeated application of (iv) in Definition 2.1 for $(S, d_1)$, we see that

$$\begin{aligned} d_1(x,y) &\le \operatorname{diam}_1(X^{n+\ell k_0}) \\ &\le \lambda^\ell \operatorname{diam}_1(X^n) \le 3\lambda^\ell d_1(x,x'). \end{aligned}$$

If we assume that $\mathrm{id}\colon (S,d_1) \to (S,d_2)$ is an $\eta$-quasisymmetry, then it follows that

$$\begin{aligned} \operatorname{diam}_2(X^{n+\ell k_0}) &\le 2\sup\{d_2(x,y) : y \in X^{n+\ell k_0}\} \\ &\le 2\eta(3\lambda^\ell) d_2(x,x') \le 2\eta(3\lambda^\ell)\operatorname{diam}_2(X^n). \end{aligned}$$

Since $\eta\colon [0,\infty) \to [0,\infty)$ is a homeomorphism, there exist $\ell_0 \in \mathbb{N}$ such that $\widetilde{\lambda} := 2\eta(3\lambda^{\ell_0}) < 1$. It follows that with this number $\widetilde{\lambda}$ and $\widetilde{k}_0 := \ell_0 k_0$ condition (iv) in Definition 2.1 holds for $(S, d_2)$. □

The following result is a converse of the previous lemma.

**Lemma 2.10.** *Let $d_1$ and $d_2$ be bounded metrics on a set $S$. Suppose that a sequence of finite covers $\{\mathbf{X}^n\}$ of $S$ is a quasi-visual approximation of both metric spaces $(S,d_1)$ and $(S,d_2)$. Then the identity map $\mathrm{id}\colon (S,d_1) \to (S,d_2)$ is a quasisymmetric homeomorphism.*

*Proof.* We have to find a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that for all $t > 0$ and $x, y, z \in S$ we have the implication

$$d_1(x,y) \le t d_1(x,z) \Rightarrow d_2(x,y) \le \eta(t) d_2(x,z). \tag{2.3}$$

So let $t > 0$ and $x, y, z \in S$ with $d_1(x,y) \le t d_1(x,z)$ be arbitrary. We may assume that $x \ne y$. Then also $x \ne z$.

In the following, we denote by $X^k \in \mathbf{X}^k$ $k$-tiles that contain the point $x$ for various $k \in \mathbb{N}_0$. We also use the notation $\operatorname{diam}_1$ and $\operatorname{diam}_2$ for the diameters of sets for the metrics $d_1$ and $d_2$, respectively.

By condition (iv′) in Lemma 2.2 and by (2.2) there are constants $\rho, \tau \in (0,1)$ and $C > 0$ independent of $t, x, y, z$ such that

$$\operatorname{diam}_i(X^{k+\ell}) \le C\rho^\ell \operatorname{diam}_i(X^k) \tag{2.4}$$

and

$$\operatorname{diam}_i(X^{k+\ell}) \ge \tau^\ell \operatorname{diam}_i(X^k) \tag{2.5}$$

for all $k, \ell \in \mathbb{N}_0$ and $i \in \{1,2\}$.

Now let $m = m(x,y) \in \mathbb{N}_0$ and $n = m(x,z) \in \mathbb{N}_0$ be as in Definition 2.5. By Lemma 2.6 there exists a constant $C(\asymp) > 0$ independent of $x, y, z$ such that

$$d_i(x,y) \asymp \operatorname{diam}_i(X^m) \text{ and } d_i(x,z) \asymp \operatorname{diam}_i(X^n) \tag{2.6}$$

for $i = 1, 2$. This means that by enlarging the original constant $C$ in (2.4) if necessary (and thus avoiding introducing new constants) we may assume that

$$\frac{1}{C} d_i(x,y) \le \operatorname{diam}_i(X^m) \le C d_i(x,y) \tag{2.7}$$

and

$$\frac{1}{C} d_i(x,z) \le \operatorname{diam}_i(X^n) \le C d_i(x,z) \tag{2.8}$$

for $i = 1, 2$. Together with our assumption $d_1(x,y) \le t d_1(x,z)$, this implies in particular that

$$\operatorname{diam}_1(X^m) \le t C^2 \operatorname{diam}_1(X^n). \tag{2.9}$$

The idea of the proof is now to translate the previous inequalities for the metric $d_1$ to a relation between $m$, $n$, and $t$. Using this for the metric $d_2$, we will obtain a bound for $\operatorname{diam}_2(X^m)$ in terms of $\operatorname{diam}_2(X^n)$. This will lead to the desired estimate by (2.7) and (2.8). We consider two cases.

*Case* 1. $m \le n$. Then by (2.9) and (2.4) we have

$$\begin{aligned}\operatorname{diam}_1(X^m) &\le tC^2 \operatorname{diam}_1(X^n) = tC^2 \operatorname{diam}_1(X^{m+(n-m)}) \\ &\le tC^3 \rho^{n-m} \operatorname{diam}_1(X^m),\end{aligned}$$

and so $tC^3\rho^{n-m} \ge 1$. This implies

$$n - m \le r(t) := \frac{\log(tC^3)}{\log(1/\rho)}. \tag{2.10}$$

Then it follows that

$$\begin{aligned}
d_2(x,y) &\le C \operatorname{diam}_2(X^m) && \text{by (2.7)} \\
&\le C(1/\tau)^{(n-m)} \operatorname{diam}_2(X^{m+(n-m)}) && \text{by (2.5)} \\
&= C(1/\tau)^{n-m} \operatorname{diam}_2(X^n) \\
&\le C(1/\tau)^{r(t)} \operatorname{diam}_2(X^n) && \text{by (2.10)} \\
&\le C^2(1/\tau)^{r(t)} d_2(x,z) && \text{by (2.8)} \\
&= \eta_1(t) d_2(x,z),
\end{aligned}$$

where we set $\eta_1(t) = C^2(1/\tau)^{r(t)}$. If we define $\eta_1(0) = 0$, then $\eta_1\colon [0,\infty) \to [0,\infty)$ is a homeomorphism as follows from the expression for $r(t)$ in (2.10).

*Case* 2. $m > n$. Then by (2.9) and (2.5) we have

$$\begin{aligned}tC^2 \operatorname{diam}_1(X^n) &\ge \operatorname{diam}_1(X^m) = \operatorname{diam}_1(X^{n+(m-n)}) \\ &\ge \tau^{m-n} \operatorname{diam}_1(X^n),\end{aligned}$$

and so $tC^2 \ge \tau^{m-n}$. This implies

$$m - n \ge s(t) := -\frac{\log(tC^2)}{\log(1/\tau)}. \tag{2.11}$$

Then it follows that

$$\begin{aligned}
d_2(x,y) &\le C \operatorname{diam}_2(X^m) = C \operatorname{diam}_2(X^{n+(m-n)}) && \text{by (2.7)} \\
&\le C^2 \rho^{m-n} \operatorname{diam}_2(X^n) && \text{by (2.4)} \\
&\le C^2 \rho^{s(t)} \operatorname{diam}_2(X^n) && \text{by (2.11)} \\
&\le C^3 \rho^{s(t)} d_2(x,z) && \text{by (2.8)} \\
&= \eta_2(t) d_2(x,z),
\end{aligned}$$

where we set $\eta_2(t) = C^3\rho^{s(t)}$. If we define $\eta_2(0) = 0$, then again $\eta_2\colon [0,\infty) \to [0,\infty)$ is a homeomorphism.

If we now set $\eta(t) = \max\{\eta_1(t), \eta_2(t)\}$ for $t \ge 0$, then the previous considerations show that $\eta\colon [0,\infty) \to [0,\infty)$ is a homeomorphism for which implication (2.3) is true. The statement follows. □

Parts of the previous argument follow ideas from the proof of [BM17, Lemma 18.10] which go back to [Me02, Theorem 4.2]. In an earlier version of this paper, we only claimed *weak quasisymmetry* of the identity map (corresponding to Case 1 in the previous proof). Angela Wu observed that one can actually show that it is a quasisymmetry.

If one carefully traces through the proof, then one can see that the distortion function $\eta$ can be chosen to have the form

$$\eta(t) = K \max\{t^\alpha, t^{1/\alpha}\}$$

with a constant $K \geq 1$ and $\alpha = \log(\rho)/\log(\tau) \in (0, 1]$ (note that $0 < \tau \leq \rho$ as follows from (2.4) and (2.5)). Quasisymmetries with such a distortion function are called *power quasisymmetries*. It is no surprise that our map is of this type; essentially, this is implied by [He01, Theorem 11.3].

Combining Lemma 2.9 and Lemma 2.10, we can now establish the main result of this subsection.

*Proof of Proposition 2.7.* We define a distance function $\tilde{d}$ on $S$ by setting

$$\tilde{d}(x, y) = \rho(F(x), F(y))$$

for $x, y \in S$. Since $F\colon S \to T$ is a bijection and $\rho$ is a bounded metric on $T$, it is clear that $\tilde{d}$ is a bounded metric on $S$. Moreover, the map $F\colon (S, \tilde{d}) \to (T, \rho)$ is an isometry.

Now suppose $F$ is a quasisymmetry from $(S, d)$ onto $(T, \rho)$. If we postcompose this with the isometry $F^{-1}\colon (T, \rho) \to (S, \tilde{d})$, then we obtain the identity map $\mathrm{id}\colon (S, d) \to (S, \tilde{d})$ and see that it is a quasisymmetry. Lemma 2.9 implies that $\{\mathbf{X}^n\}$ is a quasi-visual approximation of $(S, \tilde{d})$. If we map $\{\mathbf{X}^n\}$ to $T$ by the isometry $F\colon (S, \tilde{d}) \to (T, \rho)$, then it follows that $\{F(\mathbf{X}^n)\}$ is a quasi-visual approximation of $(T, \rho)$.

Conversely, suppose $\{F(\mathbf{X}^n)\}$ is a quasi-visual approximation of $(T, \rho)$. Applying the isometry $F^{-1}\colon (T, \rho) \to (S, \tilde{d})$, we see that $\{\mathbf{X}^n\} = \{F^{-1}(F(\mathbf{X}^n))\}$ is a quasi-visual approximation of both metric spaces $(S, \tilde{d})$ and $(S, d)$. By Lemma 2.10 the identity map $\mathrm{id}\colon (S, d) \to (S, \tilde{d})$ is a quasisymmetry. If we compose this with the isometry $F\colon (S, \tilde{d}) \to (T, \rho)$, then we obtain the map $F\colon (S, d) \to (T, \rho)$ and it follows that this map is a quasisymmetry. □

**Subdivisions.** We now consider quasi-visual approximations of different metric spaces $S$ and $T$ that are "combinatorially isomorphic" in a suitable sense and hope to construct a quasisymmetry $F\colon S \to T$ that establishes the correspondence between these approximations. In order to obtain a positive result in this direction, we have to impose somewhat stronger assumptions on the approximations. This is the motivation for the following concept.

**Definition 2.11** (Subdivisions)**.** A *subdivision* of a compact metric space $S$ is a sequence $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ with the following properties:

(i) $\mathbf{X}^0 = \{S\}$, and $\mathbf{X}^n$ is a finite collection of compact subsets of $S$ for each $n \in \mathbb{N}$.

(ii) For each $n \in \mathbb{N}_0$ and $Y \in \mathbf{X}^{n+1}$, there exists $X \in \mathbf{X}^n$ with $Y \subset X$.

(iii) For each $n \in \mathbb{N}_0$ and $X \in \mathbf{X}^n$, we have

$$X = \bigcup \{Y \in \mathbf{X}^{n+1} : Y \subset X\}.$$

Let $\{\mathbf{X}^n\}$ be a subdivision of $S$. Since $\mathbf{X}^0 = \{S\}$, it follows from induction based on (iii) that

$$S = \bigcup_{X \in \mathbf{X}^n} X,$$

for each $n \in \mathbb{N}_0$. In particular, each $\mathbf{X}^n$ is a finite cover of $S$ by some of its compact subsets. As before, we call a set $X \in \mathbf{X}^n$, where $n \in \mathbb{N}_0$, a *tile of level* $n$ or an $n$*-tile* of the subdivision; we refer to $X$ as a *tile* if its level is irrelevant.

If $k, n \in \mathbb{N}_0$ and $X \in \mathbf{X}^n$, then (iii) also implies that

$$X = \bigcup \{Y \in \mathbf{X}^{n+k} : Y \subset X\}. \tag{2.12}$$

So each $n$-tile $X$ is "subdivided" by tiles of a higher level $n + k$.

The following concept is of key importance for this paper.

**Definition 2.12** (Quasi-visual subdivisions)**.** A subdivision $\{\mathbf{X}^n\}$ of a compact metric space $S$ is called a *quasi-visual subdivision* of $S$ if it is a quasi-visual approximation according to Definition 2.1.

A sequence $\{X^n\}_{n\in\mathbb{N}_0}$ of tiles in a subdivision $\{\mathbf{X}^n\}$ is called *descending* if $X^n \in \mathbf{X}^n$ for $n \in \mathbb{N}_0$ and

$$X^0 \supset X^1 \supset X^2 \supset \dots . \tag{2.13}$$

It easily follows from (2.12) that for each $x \in S$ there is a descending sequence $\{X^n\}_{n\in\mathbb{N}_0}$ of tiles such that $x \in \bigcap_n X^n$. Note that this sequence is not unique in general.

Let $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$ be subdivisions of compact metric spaces $S$ and $T$, respectively. We say $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$ are *isomorphic subdivisions* if there exist bijections $F^n \colon \mathbf{X}^n \to \mathbf{Y}^n$, $n \in \mathbb{N}_0$, such that for all $n \in \mathbb{N}_0$, $X, Y \in \mathbf{X}^n$, and $X' \in \mathbf{X}^{n+1}$ we have

$$X \cap Y \neq \emptyset \quad \text{if and only if} \quad F^n(X) \cap F^n(Y) \neq \emptyset \tag{2.14}$$

and

$$X' \subset X \quad \text{if and only if} \quad F^{n+1}(X') \subset F^n(X). \tag{2.15}$$

We say that the isomorphism between $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$ is *given by* the family $\{F^n\}$. We say that an isomorphism $\{F^n\}$ is *induced* by a homeomorphism $F \colon S \to T$ if $F^n(X) = F(X)$ for all $n \in \mathbb{N}_0$ and $X \in \mathbf{X}^n$.

**Proposition 2.13.** *Let $S$ and $T$ be compact metric spaces with quasi-visual subdivisions $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$, respectively. If $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$ are isomorphic, then there exists a unique quasisymmetric homeomorphism $F \colon S \to T$ that induces the isomorphism between $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$.*

*Proof.* Since $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$ are quasi-visual subdivisions of $S$ and $T$, respectively, by Corollary 2.3 we know that

$$\lim_{n\to\infty} \max_{X\in\mathbf{X}^n} \operatorname{diam}(X) = 0 \text{ and } \lim_{n\to\infty} \max_{Y\in\mathbf{Y}^n} \operatorname{diam}(Y) = 0. \tag{2.16}$$

Now if the isomorphism between $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$ is given by the sequence $\{F^n\}$ of bijections $F^n \colon \mathbf{X}^n \to \mathbf{Y}^n$, then there exists a unique homeomorphism $F \colon S \to T$ such that $F(X) = F^n(X)$ for all $n \in \mathbb{N}_0$ and $X \in \mathbf{X}^n$. This follows from [BT21, Proposition 2.1]. The idea for the proof is straightforward: for each $x \in S$ one finds a descending sequence $\{X^n\}$ with $X^n \in \mathbf{X}^n$ for $n \in \mathbb{N}_0$ and $\{x\} = \bigcap_n X^n$.

Then $\{F^n(X^n)\}$ is a descending sequence so that $\bigcap_n F^n(X^n)$ contains a single point $y \in T$. Here (2.16) is important, because it guarantees that the intersection of a descending sequence of tiles is a singleton set. One now sets $F(x) = y$ and shows that $F$ is a well-defined homeomorphism that induces the isomorphism given by $\{F^n\}$.

We then have $\mathbf{Y}^n = F^n(\mathbf{X}^n) = F(\mathbf{X}^n)$ for $n \in \mathbb{N}_0$, and so $\{\mathbf{Y}^n\} = \{F(\mathbf{X}^n)\}$. Our hypotheses and Proposition 2.7 now imply that $F$ is a quasisymmetry. □

## 3. Topological facts about trees

In this section we collect some general facts about metric trees that will be useful later on. There is a rich literature on the underlying topological spaces, usually called dendrites (a metric space is a tree if and only if it is a non-degenerate dendrite; see [BM20, Proposition 2.2]). We refer to [Wh63, Chapter V], [Ku68, Section §51 VI], [Na92, Chapter X], and the references in these sources for more on the subject.

Let $T$ be a (metric) tree (as defined in the introduction), and $x, y \in T$. Then $[x, y]$ denotes the unique arc in $T$ with endpoints $x$ and $y$. If $x = y$, then this arc is degenerate and $[x, y] = \{x\}$. We also consider the half-open and open arcs joining $x$ and $y$ in $T$ defined as

$$(x, y] \coloneqq [x, y] \setminus \{x\}, \quad [x, y) \coloneqq [x, y] \setminus \{y\}, \quad (x, y) \coloneqq [x, y] \setminus \{x, y\}.$$

Note that $[x, y] = [y, x]$, $[x, y) = (y, x]$, etc. The given metric on the tree $T$ (or $T$ itself) is called *geodesic* if

$$|x - y| = \operatorname{length}[x, y] \tag{3.1}$$

for all $x, y \in T$. This is a strong assumption that is in general *not* true for the trees we consider.

**Subtrees.** A subset $X$ of a tree $T$ is called a *subtree* of $T$ if $X$ equipped with the restriction of the metric on $T$ is also a tree. One can show that $X \subset T$ is a subtree of $T$ if and only if $X$ contains at least two points and is closed and connected (see [BT21, Lemma 3.3]). If $X$ is a subtree of $T$, then $[x, y] \subset X$ for all $x, y \in X$.

The following statement is [BM20, Lemma 2.3].

**Lemma 3.1.** *Let $T$ be a tree and $\mathbf{V} \subset T$ be a finite set. Then the following statements are true:*

(i) *Two points $x, y \in T \setminus \mathbf{V}$ lie in the same component of $T \setminus \mathbf{V}$ if and only if $[x, y] \cap \mathbf{V} = \emptyset$.*

(ii) *If $U$ is a component of $T \setminus \mathbf{V}$, then $U$ is an open set and $\overline{U}$ is a subtree of $T$ with $\partial\overline{U} \subset \partial U \subset \mathbf{V}$.*

(iii) *If $U$ and $W$ are two distinct components of $T \setminus \mathbf{V}$, then $\overline{U}$ and $\overline{W}$ have at most one point in common. Such a common point belongs to $\mathbf{V}$, and is a boundary point of both $\overline{U}$ and $\overline{W}$.*

Let $T$ be a tree and $p \in T$. If $U$ is a component of $T \setminus \{p\}$, then

$$B \coloneqq \overline{U} = U \cup \{p\}$$

is called a *branch* of $p$ (in $T$). This is a subtree of $T$ (see [BT21, Lemma 3.2 (ii) and Lemma 3.4]). In particular, only the point $p$ is added as we pass from $U$ to $\overline{U}$.

The set $T \setminus \{p\}$ has at least one and at most countably many distinct complementary components (see [BT21, Lemma 3.8 and the discussion after its proof]).

We denote the number of these components by $\deg_T(p) \in \mathbb{N} \cup \{\infty\}$, where we set $\deg_T(p) = \infty$ if $T \setminus \{p\}$ has infinitely many components. Note that $\deg_T(p)$ is equal to the number of branches of $p$ in $T$.

If $\deg_T(p) = 1$, then we call $p$ a *leaf* of $T$; so $p$ is a leaf of $T$ precisely when $T \setminus \{p\}$ has one component or, equivalently, if $T \setminus \{p\}$ is connected.

If $\deg_T(p) \geq 3$, or equivalently, if $T \setminus \{p\}$ has at least three components, we call $p$ a *branch point* of $T$. If $\deg_T(p) = 3$, or equivalently, if $T \setminus \{p\}$ has exactly three components, $p$ is called a *triple point* of $T$. Finally, we call $T$ *trivalent*, if each branch point of $T$ is a triple point, or equivalently, if $\deg_T(p) \leq 3$ for all $p \in T$.

**Lemma 3.2.** *Let $T$ be a tree, $S \subset T$ be a subtree of $T$, and $p \in S$. Then the following statements are true:*

(i) *Every branch $B$ of $p$ in $S$ is contained in a unique branch $B'$ of $p$ in $T$. The assignment $B \mapsto B'$ is an injective map between the sets of branches of $p$ in $S$ and in $T$. If $p$ is an interior point of $S$, then this map is a bijection. In particular, $\deg_S(p) \leq \deg_T(p)$ with equality if $p \in \operatorname{int}(S)$.*

(ii) *If $p$ is a leaf of $T$, then $p$ is a leaf of $S$. Conversely, if $p$ is a leaf of $S$ with $p \notin \partial S$, then $p$ is a leaf of $T$.*

(iii) *If $B \subset S$ is a branch of $p$ in $S$ with $B \cap \partial S = \emptyset$, then $B$ is a branch of $p$ in $T$.*

*Proof.* (i) This is [BT21, Lemma 3.5].

(ii) Suppose $p$ is a leaf of $T$. Then by (i) we have $1 \leq \deg_S(p) \leq \deg_T(p) = 1$. Hence $\deg_S(p) = 1$ and $p$ is a leaf of $S$.

Conversely, suppose $p$ is a leaf of $S$ with $p \notin \partial S$. Then $p \in S \setminus \partial S = \operatorname{int}(S)$, and so $\deg_T(p) = \deg_S(p) = 1$ by (i). Hence $p$ is a leaf of $T$.

(iii) The set $U = B \setminus \{p\}$ is a connected subset of $S \setminus \{p\} \subset T \setminus \{p\}$. This set is relatively open in $S$, and hence an open subset of $T$, because $U \cap \partial S = \emptyset$.

The set $U = \overline{U} \cap (S \setminus \{p\})$ is also relatively closed in $S \setminus \{p\}$. Since $S$ is closed, every limit point of $U$ in $T \setminus \{p\}$ belongs to $S \setminus \{p\}$, and hence to $U$. This shows that $U$ is relatively closed in $T \setminus \{p\}$. Since $U$ is also open and connected, $U$ must be a component of $T \setminus \{p\}$, i.e., a maximal connected subset of $T \setminus \{p\}$. Indeed, if $U \subset M \subset T \setminus \{p\}$ with $U \neq M$, then $M$ has $U$ as a non-trivial open and relatively closed subset, and so $M$ cannot be connected. It follows that $B = \overline{U} = U \cup \{p\}$ is a branch of $p$ in $T$. □

**Decompositions and subdivisions of trees.** We now assume that $T$ is a tree such that each branch point $p$ of $T$ has only finitely many branches (i.e., $\deg_T(p) < \infty$ or, equivalently, $T \setminus \{p\}$ has only finitely many components). This condition is trivially satisfied in trivalent trees that we will consider later. It is also satisfied in quasiconformal trees (see Lemma 3.5).

Let $\mathbf{V}$ be a finite (possibly empty) set of points in $T$ that does not contain any leaf of $T$. We want to decompose $T$ into pieces by "cutting" $T$ at the points in $\mathbf{V}$. A *tile* $X$ (in the decomposition induced by $\mathbf{V}$) is the closure of a component of $T \setminus \mathbf{V}$. It follows from Lemma 3.1 (ii) that each tile is a subtree of $T$. The set of all tiles obtained in this way from $\mathbf{V}$ is denoted by $\mathbf{X}$ and called the *decomposition induced by* $\mathbf{V}$. We will see momentarily that the set $\mathbf{X}$ is always a finite cover of $T$ (see Lemma 3.3 (vii)).

Most of the following statements are intuitively clear, but we will include full proofs for the sake of completeness.

**Lemma 3.3.** *Let $T$ be a tree such that every branch point of $T$ has only finitely many branches, and let $\mathbf{V} \subset T$ be a finite (possibly empty) subset of $T$ that contains no leaf of $T$. Let $\mathbf{X}$ denote the decomposition of $T$ induced by $\mathbf{V}$. Then the following statements are true:*

(i) *If $X \in \mathbf{X}$ and $v \in \mathbf{V}$, then $X$ is contained in the closure of one of the components of $T \setminus \{v\}$ and disjoint from the other components of $T \setminus \{v\}$.*

(ii) *Each tile $X \in \mathbf{X}$ is a subtree of $T$ with $\partial X = \mathbf{V} \cap X$ and $\operatorname{int}(X) = X \setminus \mathbf{V}$. The closure of $\operatorname{int}(X)$ is equal to $X$.*

(iii) *If $\mathbf{V} \neq \emptyset$ and $X \in \mathbf{X}$, then $\partial X \neq \emptyset$ and each point $v \in \partial X$ is a leaf of the subtree $X$.*

(iv) *Two distinct tiles $X, Y \in \mathbf{X}$ have at most one point in common. Such a common point of $X$ and $Y$ belongs to $\mathbf{V}$ and is a boundary point of both $X$ and $Y$.*

(v) *If $v \in \mathbf{V}$ and $n = \deg_T(v) \in \mathbb{N}$, $n \geq 2$, is the number of branches of $v$ in $T$, then $v$ is contained in precisely $n$ distinct tiles in $\mathbf{X}$. Each branch of $v$ in $T$ contains precisely one of these tiles.*

(vi) *If $X \in \mathbf{X}$ and $\partial X = \{v\} \subset \mathbf{V}$ is a singleton set, then $X$ is a branch of $v$ in $T$.*

(vii) *The set of tiles $\mathbf{X}$ is a finite cover of $T$.*

(viii) *Suppose, in addition, that $T$ is a trivalent tree with a dense set of branch points. Then each $X \in \mathbf{X}$ is also a trivalent tree with a dense set of branch points.*

Note that if $\mathbf{V} = \emptyset$, then $\mathbf{X} = \{T\}$; so $X = T$ is the only tile in $\mathbf{X}$ and $\partial X = \partial T = \emptyset$. In this case, the statements in the previous lemma are vacuously or trivially true.

*Proof.* (i) We have $X = \overline{U}$, where $U$ is a component of $T \setminus \mathbf{V}$. Moreover, since $v \in \mathbf{V}$ is not a leaf of $T$, our hypotheses imply that the set $T \setminus \{v\}$ has $n$ distinct components $W_1, \ldots, W_n$, where $n = \deg_T(v) \in \mathbb{N}$, $n \geq 2$. Since $U$ is a connected subset of $T \setminus \mathbf{V} \subset T \setminus \{v\}$, it is contained in one of these components, say $U \subset W_1$. Then $X = \overline{U} \subset \overline{W_1} = W_1 \cup \{v\}$, and $X$ is disjoint from $W_2 \cup \cdots \cup W_n = T \setminus \overline{W_1}$.

(ii) If $X \in \mathbf{X}$ is a tile, then $X = \overline{U}$, where $U$ is a component of $T \setminus \mathbf{V}$. It then follows from Lemma 3.1 (ii) that $X = \overline{U}$ is a subtree of $T$ with $\partial X = \partial \overline{U} \subset \partial U \subset \mathbf{V}$. Hence $\partial X \subset \mathbf{V} \cap X$, because $X$ is a closed subset of $T$.

Conversely, if $v \in \mathbf{V} \cap X$, then by what we have seen in (i), $v$ is in the closure $\overline{W} = W \cup \{v\}$ of at least one component $W$ of $T \setminus \{v\}$ disjoint from $X$. This implies that $v \in \partial X$, and so $\partial X = \mathbf{V} \cap X$. The last identity also shows that $\operatorname{int}(X) = X \setminus \partial X = X \setminus \mathbf{V}$.

Note that $U$ is disjoint from $\mathbf{V}$, and $\partial U \subset \mathbf{V}$ by Lemma 3.1 (ii). We also have $X = \overline{U} = U \cup \partial U$. This implies that

$$\operatorname{int}(X) = X \setminus \mathbf{V} = (U \cup \partial U) \setminus \mathbf{V} = U.$$

Since $\overline{U} = X$ and $\operatorname{int}(X) = U$, the closure of $\operatorname{int}(X)$ is equal to $X$.

(iii) If $\mathbf{V} \neq \emptyset$, then it follows from (i) that $X \neq T$. Since $T$ is connected, this implies that $\partial X \neq \emptyset$; otherwise, the non-empty set $X \neq T$ would be an open and closed subset of the connected space $T$. This is impossible.

We have $X = \overline{U}$, where $U$ is a component of $T \setminus \mathbf{V}$. If $v \in \partial X \subset \mathbf{V}$, then $X \setminus \{v\}$ is connected, because $U \subset X \setminus \{v\} \subset \overline{U} = X$ (every set squeezed between a connected set and its closure is connected). Hence $v$ is a leaf of $X$.

(iv) This immediately follows from Lemma 3.1 (iii).

(v) For $n = \deg_T(v)$ let $W_1, \dots, W_n$ denote the distinct components of $T \setminus \{v\}$. Here $n \in \mathbb{N}$, $n \geq 2$. We have $\overline{W_k} = W_k \cup \{v\}$ and so $v \in \overline{W_k}$ for $k = 1, \dots, n$.

For each $k \in \{1, \dots, n\}$ we choose a point $x_k \in W_k \subset T \setminus \{v\}$. Then by Lemma 3.1 (i) we have

$$v \in [x_k, x_\ell] \text{ for all } k, \ell \in \{1, \dots, n\} \text{ with } k \neq \ell. \tag{3.2}$$

We may assume that each point $x_k$ is so close to $v$ (by replacing the original point $x_k$ with a point on $[x_k, v) \subset W_k$ sufficiently close to $v$ if necessary) that $[x_k, v)$ contains no point in the finite set $\mathbf{V} \setminus \{v\}$. Then $[x_k, v)$ is a non-empty connected subset of $T \setminus \mathbf{V}$ for each $k \in \{1, \dots, n\}$, and so it must be contained in a unique component $U_k$ of $T \setminus \mathbf{V}$. These components $U_1, \dots, U_n$ of $T \setminus \mathbf{V}$ are pairwise disjoint open sets as follows from (3.2) and Lemma 3.1 (i) and (ii). Then $X_k := \overline{U_k}$ for $k = 1, \dots, n$ is a tile in $\mathbf{X}$ that contains the arc $[x_k, v]$, and so $v \in X_k$. Moreover, the tiles $X_1, \dots, X_n$ are all distinct, because each tile $X_k$ contains the non-empty open set $U_k$ that is disjoint from all the other tiles in this list. So $v$ is contained in at least $n$ distinct tiles in $\mathbf{X}$.

Suppose $Z = \overline{U} \in \mathbf{X}$ is any tile with $v \in Z$, where $U$ is a component of $T \setminus \mathbf{V}$. We choose a point $z \in U$. Then $z$ must be contained in one of the components $W_1, \dots, W_n$ of $T \setminus \{v\}$, say $z \in W_1$. Then $[x_1, z] \subset T \setminus \{v\}$ by Lemma 3.1 (i). We may assume that $x_1$ and $z$ are so close to $v$ that $[x_1, z]$ contains no point in $\mathbf{V} \setminus \{v\}$. Then $[x_1, z] \cap \mathbf{V} = \emptyset$, and so $x_1$ and $z$ are contained in the same component of $T \setminus \mathbf{V}$. It follows that $U = U_1$, and so $Z = \overline{U} = \overline{U_1} = X_1$. This shows that $X_1, \dots, X_n$ are the only tiles in $\mathbf{X}$ that contain $v$. So $v$ is contained in $n$ distinct tiles $X_1, \dots, X_n \in \mathbf{X}$, and in no other tiles in $\mathbf{X}$.

Let $k \in \{1, \dots, n\}$. Then the set $U_k$ is a connected subset of $T \setminus \mathbf{V} \subset T \setminus \{v\}$, and so it is contained in a unique component of $T \setminus \{v\}$. This component must be $W_k$, because $x_k \in U_k \cap W_k$. Hence $X_k = \overline{U_k} \subset \overline{W_k}$, which shows that $X_k$ is contained in the branch $\overline{W_k}$ of $v$ in $T$. The tile $X_k$ cannot be contained in any other branch $\overline{W_\ell} = W_\ell \cup \{v\}$ with $\ell \neq k$, because $X_k$ is disjoint from $W_\ell$ as follows from (i).

(vi) Suppose that $\partial X = \{v\} \subset \mathbf{V}$. We have $X = \overline{U}$, where $U$ is a component of $T \setminus \mathbf{V}$. As we have seen in the proof of (i), there is a component $W$ of $\mathbf{T} \setminus \{v\}$ with $U \subset W$.

*Claim.* $U = W$. To see this, we argue by contradiction and assume that $U \neq W$. Then there exists a point $x \in U \subset W$, as well as a point $y \in W \setminus U$. Then $[x, y] \cap \mathbf{V} \neq \emptyset$, because otherwise $y \in U$. So as we travel from $x$ to $y$ along $[x, y]$, there must be a first point $u \in [x, y]$ that belongs to $\mathbf{V}$. Then $[x, u) \subset U$, and so $[x, u] \subset \overline{U} = X$.

Now (ii) implies that $u \in \mathbf{V} \cap X = \partial X = \{v\}$ and so $u = v$. Since $v = u \in [x, y]$, Lemma 3.1 (i) shows that $x$ and $y$ lie in different components of $T \setminus \{v\}$; but $x$ and $y$ lie in the same component $W$ of $T \setminus \{v\}$, and so this is a contradiction. The Claim follows.

Since $U = W$ by the Claim, we conclude that $X = \overline{U} = \overline{W}$ is the closure of the component $W$ of $T \setminus \{v\}$, and so a branch of $v$ in $T$.

(vii) If $\mathbf{V} = \emptyset$, this is obvious, because then we have $\mathbf{X} = \{T\}$.

Now suppose $\mathbf{V} \neq \emptyset$. Then each tile in $\mathbf{X}$ contains some point in $\mathbf{V}$ as follows from (ii) and (iii), and each point in $\mathbf{V}$ is contained in at most $N$ tiles by (v), where for $N \in \mathbb{N}$ we choose some upper bound for the number of branches for the finitely many points in $\mathbf{V}$. This implies that the number of tiles in $\mathbf{X}$ is bounded above by $\#\mathbf{V} \cdot N$, and so $\mathbf{X}$ is a finite set.

Each point in $T \setminus \mathbf{V}$ lies in a tile in $\mathbf{X}$ by definition of tiles; this is also true for the points in $\mathbf{V}$ by (v). Hence $\mathbf{X}$ is a finite cover of $T$.

(viii) Under the given additional assumptions on $T$, let $B$ denote the set of branch points of $T$ and $B_X \subset X$ denote the set of branch points of its subtree $X$. If $p \in \partial X = \mathbf{V} \cap X$, then $p$ is a leaf of $X$ by (iii).

If $p \in \operatorname{int}(X) = X \setminus \partial X$, then the number of branches of $p$ in $X$ is the same as the number of branches of $p$ in $T$ (see Lemma 3.2 (i)). This shows that $p \in X$ is a branch point of $X$ if and only if $p \in \operatorname{int}(X)$ and $p$ is a branch point of $T$; in this case, $p$ is a triple point of $X$. In particular, $X$ is a trivalent tree and $B_X = B \cap \operatorname{int}(X)$.

Since $B$ is dense in $T$, the set $B_X = B \cap \operatorname{int}(X)$ is dense in $\operatorname{int}(X)$, and hence dense in $X$, because the closure of $\operatorname{int}(X)$ is equal to $X$ by (ii). It follows that $X$ is a trivalent tree with a dense set $B_X$ of branch points. □

We will now record some relations of two tile decompositions of a tree $T$ induced by a set $\mathbf{V}$ and a larger set $\mathbf{V}'$.

**Lemma 3.4.** *Let $T$ be a tree such that every branch point of $T$ has only finitely many branches. Let $\mathbf{V} \subset \mathbf{V}' \subset T$ be finite (possibly empty) subsets of $T$ that contain no leaf of $T$. Let $\mathbf{X}$ and $\mathbf{X}'$ denote the decompositions of $T$ induced by $\mathbf{V}$ and $\mathbf{V}'$, respectively. Then the following statements are true:*

(i) *Each tile $X' \in \mathbf{X}'$ is contained in a unique tile $X \in \mathbf{X}$.*

(ii) *Each tile $X \in \mathbf{X}$ is equal to the union of all tiles $X' \in \mathbf{X}'$ with $X' \subset X$.*

(iii) *Let $X \in \mathbf{X}$, $\mathbf{V}_X := \mathbf{V}' \cap \operatorname{int}(X)$, and $\mathbf{X}_X$ be the decomposition of $X$ induced by $\mathbf{V}_X$. Then $\mathbf{X}_X = \{X' \in \mathbf{X}' : X' \subset X\}$.*

(iv) *Let $X' \in \mathbf{X}'$, and $X \in \mathbf{X}$ be the unique tile with $X' \subset X$. If $\partial_X X'$ denotes the relative boundary of $X'$ in $X$, then $\partial_X X' = \partial X' \setminus \mathbf{V}$.*

*Proof.* (i) If $X' \in \mathbf{X}'$, then there exists a component $W$ of $T \setminus \mathbf{V}'$ with $X' = \overline{W}$. Since $\mathbf{V} \subset \mathbf{V}'$, the set $W$ is a connected subset of $T \setminus \mathbf{V} \supset T \setminus \mathbf{V}'$ and so contained in a unique component $U$ of $T \setminus \mathbf{V}$. Then $X'$ is contained in the tile $X := \overline{U} \in \mathbf{X}$, because $X' = \overline{W} \subset \overline{U} = X$. There can be no other tile in $\mathbf{X}$ containing $X'$, because by Lemma 3.3 (ii) the set $X'$ is a subtree of $T$ and hence an infinite set, but distinct tiles in $\mathbf{X}$ can have at most one point in common by Lemma 3.3 (iv).

(ii) If $X \in \mathbf{X}$, then $X = \overline{U}$, where $U$ is a component of $T \setminus \mathbf{V}$. Since $U$ is connected, this set cannot contain isolated points. This implies that the set $U \setminus \mathbf{V}' \subset X$ is dense in $U$ and hence also dense in $X = \overline{U}$.

If $x \in U \setminus \mathbf{V}'$ is arbitrary, then there exists a component $W$ of $T \setminus \mathbf{V}'$ with $x \in W$. Since $W$ is a connected subset of $T \setminus \mathbf{V}' \subset T \setminus \mathbf{V}$, this set must be contained in a component of $T \setminus \mathbf{V}$. Since $x \in U \cap W$, it follows that $W \subset U$. Then $X' := \overline{W}$ is a tile in $\mathbf{X}'$ with $x \in X'$ and $X' = \overline{W} \subset \overline{U} = X$.

This shows that if we denote by $Y$ the union of all tiles $X' \in \mathbf{X}'$ with $X' \subset X$, then $Y \subset X$ contains the set $U \setminus \mathbf{V}'$. By Lemma 3.3 (vii) applied to $\mathbf{V}'$ there are only finitely many tiles in $\mathbf{X}'$, and so $Y$ is closed. Since $U \setminus \mathbf{V}'$ is dense in $X$ and $U \setminus \mathbf{V}' \subset Y$, it follows that $X = Y$ as desired.

(iii) We first verify the hypotheses of Lemma 3.3 for the tree $X$ and its finite subset $\mathbf{V}_X$. First note that $X$ is a subtree of $T$, and hence a tree. If $p \in X$ is arbitrary, then the number of branches of $p$ in $X$ is bounded by the number of branches of $p$ in $T$ (see Lemma 3.2 (i)), and hence finite.

If $v \in \mathbf{V}_X \subset \operatorname{int}(X) = X \setminus \partial X$, then $v$ is not a leaf of $X$; otherwise, $v \in \mathbf{V}_X$ is a leaf of $T$ by Lemma 3.2 (ii), but $\mathbf{V}_X \subset \mathbf{V}'$ contains no leaves of $T$ by our hypotheses. This shows that $\mathbf{V}_X \subset X$ contains no leaf of $X$; so we can apply all the statements from Lemma 3.3 for the decomposition $\mathbf{X}_X$ of the tree $X$ induced by $\mathbf{V}_X$.

We first show that $\{X' \in \mathbf{X}' : X' \subset X\} \subset \mathbf{X}_X$. To this end, let $X' \in \mathbf{X}'$ with $X' \subset X$ be arbitrary. Then there exists a component $U$ of $T \setminus \mathbf{V}'$ with $X' = \overline{U}$. Since $U$ is a connected subset of $X \setminus \mathbf{V}' \subset X \setminus \mathbf{V}_X$, there exists a component $W$ of $X \setminus \mathbf{V}_X$ with $U \subset W \subset X$.

*Claim.* $W \subset \overline{U}$. In order to see this, we argue by contradiction and assume that there exists a point $x \in W \setminus \overline{U}$. Since $W$ is a relatively open subset of the tree $X$ (as follows from Lemma 3.1 (ii)), we may assume (by moving $x$ slightly if necessary) that $x$ does not belong to the finite set $\mathbf{V}'$.

We choose $y \in U$, and consider the arc $[x, y] \subset X$. Since $x \in W \setminus (U \cup \mathbf{V}')$ and $y \in U$ lie in different components of $T \setminus \mathbf{V}'$, there exists a point $v \in [x, y] \cap \mathbf{V}' \subset X$ by Lemma 3.1 (i). Here $v \neq x, y$. We cannot have $v \in \operatorname{int}(X)$, because then $v \in \mathbf{V}' \cap \operatorname{int}(X) = \mathbf{V}_X$; so $x$ and $y$ would lie in different components of $X \setminus \mathbf{V}_X$, but these points lie in the same component $W$ of $X \setminus \mathbf{V}_X$. It follows that $v \in \partial X$. By Lemma 3.3 (iii) this implies that $v$ is a leaf of $X$. This leads to a contradiction, because we have $v \in [x, y]$ and so $x$ and $y$ lie in different components of $X \setminus \{v\}$ by Lemma 3.1 (i), but $X \setminus \{v\}$ is connected, because $v$ is a leaf of $X$. The Claim follows.

By the Claim, we have $U \subset W \subset \overline{U}$, and so $X' = \overline{U} = \overline{W}$. This shows that $X'$ is the closure of the component $W$ of $X \setminus \mathbf{V}_X$, and so $X'$ belongs to the set of tiles $\mathbf{X}_X$ of the decomposition of $X$ induced by $\mathbf{V}_X$. We have proved $\{X' \in \mathbf{X}' : X' \subset X\} \subset \mathbf{X}_X$.

To show the other inclusion $\mathbf{X}_X \subset \{X' \in \mathbf{X}' : X' \subset X\}$, first note that $\mathbf{X}_X$ is a finite cover of $X$ by Lemma 3.3 (vii). Moreover, each tile in $\mathbf{X}_X$ has non-empty interior relative to $X$ (as follows from the definition of tiles in combination with Lemma 3.1 (ii)) and this interior of a tile is disjoint from all the other tiles in $\mathbf{X}_X$ (as follows from Lemma 3.3 (iv)).

Now let $Y \in \mathbf{X}_X$ be arbitrary. Then we can choose a point $x$ in the interior of $Y$ relative to $X$. By (ii) there exists $X' \in \mathbf{X}'$ with $X' \subset X$ and $x \in X'$. By what we have seen, $X' \in \mathbf{X}_X$. Since $x$ cannot belong to any tile in $\mathbf{X}_X$ except $Y$, we conclude that $Y = X'$. This shows the other inclusion $\mathbf{X}_X \subset \{X' \in \mathbf{X} : X' \subset X\}$. It follows that these two sets are equal, as desired.

(iv) Let $X' \in \mathbf{X}'$ be arbitrary. Then by (i) there exists a unique tile $X \in \mathbf{X}$ with $X' \subset X$. To identify the relative boundary $\partial_X X'$, we can apply the results from Lemma 3.3 to $X$ and its decomposition $\mathbf{X}_X$ induced by $\mathbf{V}_X = \mathbf{V}' \cap \operatorname{int}(X)$, because the relevant hypotheses of Lemma 3.3 are true as was pointed out in the proof of (iii). In particular, by repeated application of Lemma 3.3 (ii) we see that

$$\begin{aligned} \partial_X X' &= X' \cap \mathbf{V}_X = X' \cap \mathbf{V}' \cap \operatorname{int}(X) \\ &= \partial X' \cap \operatorname{int}(X) = \partial X' \cap (X \setminus \mathbf{V}) = \partial X' \setminus \mathbf{V}. \end{aligned}$$

In the last equality, we used that $\partial X' \subset X' \subset X$. The statement follows. □

Let $T$ be a tree such that every point in $T$ has only finitely many branches (as in Lemmas 3.3 and 3.4). Let $\{\mathbf{V}^n\}_{n\in\mathbb{N}_0}$ be an increasing sequence of finite subsets of $T$, meaning that

$$\mathbf{V}^0 \subset \mathbf{V}^1 \subset \mathbf{V}^2 \subset \dots . \tag{3.3}$$

In addition, we require that $\mathbf{V}^0 = \emptyset$, and that no set $\mathbf{V}^n$ for $n \in \mathbb{N}$ contains a leaf of $T$. We denote by $\mathbf{X}^n$ the decomposition of $T$ induced by $\mathbf{V}^n$. Then it immediately follows from Lemma 3.3 (vii) and Lemma 3.4 (i) and (ii) that the sequence $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ is a subdivision of the tree $T$ according to Definition 2.11. We call $\{\mathbf{X}^n\}$ the *subdivision of $T$ induced by* $\{\mathbf{V}^n\}$. Subdivisions of $T$ obtained in this way are one of the main ingredients in the proof of the "if" implication in Theorem 1.4. We will discuss this in detail in Sections 6 and 7.

We now show that in a quasiconformal tree $T$ there is a uniform bound on $\deg_T(p)$ for $p \in T$. Recall from the introduction that a quasiconformal tree is a metric tree that is doubling and of bounded turning.

**Lemma 3.5.** *Let $T$ be a quasiconformal tree. Then there is a constant $M \in \mathbb{N}$ such that*

$$\deg_T(p) \le M$$

*for all $p \in T$.*

As the proof will show, $M$ depends only on the doubling constant $N$ of $T$ and the constant $K$ appearing in the definition of bounded turning (see (1.1)).

*Proof.* Let $p \in T$ be arbitrary, and suppose $B_1, \dots, B_n$ are distinct branches of $p$ in $T$, where $n \in \mathbb{N}$. It is enough to find a uniform upper bound on $n$ independent of $p$. For this we choose a point $x_k' \in B_k \setminus \{p\}$ for $k = 1, \dots, n$. By considering points on $(p, x_k'] \subset B_k$ sufficiently close to $p$, we can then find a (small) constant $\delta > 0$ and $x_k \in B_k$ with $|x_k - p| = \delta$ for $k = 1, \dots, n$.

Then for $k, \ell \in \{1, \dots, n\}$ with $k \neq \ell$ we have $p \in [x_k, x_\ell]$, which implies $\operatorname{diam}[x_k, x_\ell] \ge \delta$. Since $T$ is of bounded turning, it follows that

$$|x_k - x_\ell| \ge \frac{1}{K} \operatorname{diam}[x_k, x_\ell] \ge \delta/K,$$

where $K > 0$ is the constant from (1.1). Hence the set $\{x_1, \dots, x_n\}$ is $\delta/K$-separated and contained in the closed ball of radius $\delta$ centered at $p$. It is well-known that in a doubling metric space, the number of such points is bounded by a number $M \in \mathbb{N}$ depending only on the doubling constant $N$ and the ratio $\delta/(\delta/K) = K$. Thus $n \le M$ with $M$ only depending on $N$ and $K$, and independent of $p$. This implies $\deg_T(p) \le M$ and the statement follows. □

**Height and center.** Let $T$ be a tree and $p \in T$ be a branch point of $T$. Then there are points $x, y, z \in T$ that lie in distinct branches of $p$. Conversely, we often want to find $p$ when $x, y, z \in T$ are given. In addition, when $x, y, z$ are branch points, we want to relate their heights $H_T(x), H_T(y), H_T(z)$ to the height $H_T(p)$ of $p$. We remind the reader that height was defined in (1.2).

**Lemma 3.6.** *Let $T$ be a tree, and $x, y, z \in T$.*

(i) *Then $[x, y] \cap [y, z] \cap [z, x]$ is a singleton set.*

(ii) *If $x, y, z$ are branch points of $T$, and $\{c\} = [x, y] \cap [y, z] \cap [z, x]$, then $c$ is a branch point of $T$ with*

$$H_T(c) \geq \min\{H_T(x), H_T(y), H_T(z)\}.$$

We call the point $c \in [x, y] \cap [y, z] \cap [z, x]$ the *center* of $x, y, z$ (in $T$).

*Proof.* (i) As we travel from $z$ to $x$ along the arc $[z, x]$, there exists a first point $c \in [z, x]$ that lies on $[x, y]$. Then $c \in [x, y] \cap [z, x]$ and $[x, y] \cap [z, c) = \emptyset$. Moreover,

$$[x, y] = [x, c) \cup \{c\} \cup (c, y] \tag{3.4}$$

is a decomposition of $[x, y]$ into pairwise disjoint sets. Note that $[x, c)$ or $(c, y]$ could be empty, but this does not affect this statement. It follows that the sets

$$[x, c),\ [y, c),\ [z, c),\ \{c\} \text{ are pairwise disjoint.} \tag{3.5}$$

This implies that $[x, c) \cup \{c\} \cup (c, z]$ is an arc in $T$ joining $x$ and $z$. Since $[z, x]$ is the unique such arc, we conclude that

$$[z, x] = [x, c) \cup \{c\} \cup (c, z],$$

and similarly,

$$[y, z] = [y, c) \cup \{c\} \cup (c, z].$$

Together with (3.4) and (3.5), this implies that the arcs $[x, y]$, $[y, z]$, $[z, x]$ have the point $c$ in common, but no other point.

(ii) We may assume that $c \neq x, y, z$, because otherwise the statement is obvious. Then the sets $[x, c)$, $[y, c)$, $[z, c)$ are non-empty, connected, and pairwise disjoint by (3.5). It follows from Lemma 3.1 (i) that each of these sets must lie in a different component of $T \setminus \{c\}$. In particular, $T \setminus \{c\}$ has at least three such components, and so $c$ is a branch point of $T$. Let $B_x$, $B_y$, $B_z$ be the distinct branches of $c$ that contain $[x, c)$, $[y, c)$, $[z, c)$, respectively.

Since $(x, c]$ is a connected subset of $T \setminus \{x\}$, it must be contained in a branch of $x$ in $T$. Since $x$ is a branch point of $T$, there exists another branch $\widetilde{B}_x$ of $x$ in $T$ distinct from the branch of $x$ containing $(x, c]$. We can choose $\widetilde{B}_x$ so that $\operatorname{diam}(\widetilde{B}_x) \geq H_T(x)$ (by definition of $H_T(x)$, there are actually at least two such choices).

Then $\widetilde{B}_x \cap [x, c] = \{x\}$, and so $\widetilde{B}_x \cup [x, c)$ is a connected subset of $T \setminus \{c\}$. Hence

$$B_x \supset \widetilde{B}_x \cup [x, c) \supset \widetilde{B}_x,$$

which implies that

$$\operatorname{diam}(B_x) \geq \operatorname{diam}(\widetilde{B}_x) \geq H_T(x).$$

Similarly, $\operatorname{diam}(B_y) \geq H_T(y)$ and $\operatorname{diam}(B_z) \geq H_T(z)$. It follows that

$$\begin{aligned} H_T(c) &\geq \min\{\operatorname{diam}(B_x), \operatorname{diam}(B_y), \operatorname{diam}(B_z)\} \\ &\geq \min\{H_T(x), H_T(y), H_T(z)\}, \end{aligned}$$

as desired. □

## 4. Quasisymmetries preserve uniform branching

After the preparations in the previous two sections, we now start the proof of Theorem 1.4. We will show that uniform relative separation and uniform relative density of branch points of a tree are preserved under quasisymmetries. Note that we do not assume that the trees under consideration are trivalent, doubling, or of bounded turning.

We first need some auxiliary facts.

**Lemma 4.1.** *Let $F\colon T \to T'$ be a quasisymmetry between trees $T$ and $T'$. Suppose $B_1$ and $B_2$ are two branches in $T$ of a point $x \in T$, and*

$$\operatorname{diam}(B_1) \geq \operatorname{diam}(B_2).$$

*Let $B_1' := F(B_1)$ and $B_2' := F(B_2)$ be their images in $T'$. Then*

$$\operatorname{diam}(B_1') \gtrsim \operatorname{diam}(B_2'),$$

*where $C(\gtrsim)$ depends only on the distortion function $\eta$ of $F$.*

If $x$ is a leaf, this is trivially true, because then $B_1 = B_2$. In general, we will not necessarily have $\operatorname{diam}(B_1') \geq \operatorname{diam}(B_2')$, but the lemma says that this is true "up to a fixed constant".

*Proof.* Under the given assumptions, we can find a point $z_1 \in B_1$ with $|z_1 - x| \geq \frac{1}{2}\operatorname{diam}(B_1)$. It follows that for each $z_2 \in B_2$,

$$|z_1 - x| \geq \tfrac{1}{2}\operatorname{diam}(B_1) \geq \tfrac{1}{2}\operatorname{diam}(B_2) \geq \tfrac{1}{2}|z_2 - x|.$$

Let $x' := F(x)$, $z_1' := F(z_1)$, $z_2' := F(z_2) \in T'$. Since $F$ is a quasisymmetry, we conclude that

$$|z_1' - x'| \gtrsim |z_2' - x'|,$$

where $C(\gtrsim)$ depends only on the distortion function $\eta$ of $F$, but not on the other choices. Now $z_2 \in B_2$ was arbitrary, and so we obtain

$$\operatorname{diam}(B_1') \geq |z_1' - x'| \gtrsim \sup_{z_2' \in B_2'} |z_2' - x'| \geq \tfrac{1}{2}\operatorname{diam}(B_2').$$

The statement follows. □

We record the following immediate consequence.

**Corollary 4.2.** *Let $F\colon T \to T'$ be a quasisymmetry between trees $T$ and $T'$, let $x \in T$ be a branch point of $T$, and $B$ be a branch of $x$ in $T$ with $H_T(x) = \operatorname{diam}(B)$. If we set $x' := F(x)$ and $B' := F(B)$, then*

$$H_{T'}(x') \asymp \operatorname{diam}(B').$$

*Here $C(\asymp)$ depends only on the distortion function $\eta$ of $F$.*

*Proof.* The branches $B_1, B_2, B_3, \dots$ of $x$ in $T$ can be labeled such that

$$\operatorname{diam}(B_1) \geq \operatorname{diam}(B_2) \geq \operatorname{diam}(B_3) \geq \dots$$

and $B_3 = B$.

Let $B'_n := F(B_n)$ for all $n \in \mathbb{N}$ for which $B_n$ is defined. Since $F\colon T \to T'$ is a quasisymmetry, and in particular a homeomorphism from $T$ onto $T'$, the point $x' = F(x)$ is a branch point of $T'$, and $B'_1, B'_2, B'_3, \dots$ are the branches of $x'$ in $T'$. Note that $B' = F(B) = F(B_3) = B'_3$.

By Lemma 4.1 we know that

$$\operatorname{diam}(B'_1) \gtrsim \operatorname{diam}(B'_3), \ \operatorname{diam}(B'_2) \gtrsim \operatorname{diam}(B'_3), \text{ and}$$
$$\operatorname{diam}(B'_3) \gtrsim \operatorname{diam}(B'_n) \text{ for all } n \geq 4,$$

where $C(\gtrsim)$ depends only on the distortion function $\eta$ of $F$.

Therefore, the three branches $B'_1$, $B'_2$, $B'_3$ of $x'$ in $T'$ have diameter $\gtrsim \operatorname{diam}(B'_3)$, and so $H_{T'}(x') \gtrsim \operatorname{diam}(B'_3)$. On the other hand, we also see that all other branches have diameter $\lesssim \operatorname{diam}(B'_3)$, and so $H_{T'}(x') \lesssim \operatorname{diam}(B'_3)$. Hence $H_{T'}(x') \asymp \operatorname{diam}(B'_3) = \operatorname{diam}(B')$, and the statement follows. □

We now consider uniform relative separation of branch points.

**Lemma 4.3** (Qs-invariance of uniform relative separation)**.** *Suppose $F\colon T \to T'$ is a quasisymmetry between trees $T$ and $T'$. If $T$ has uniformly relatively separated branch points, then the same is true for $T'$.*

*Proof.* Suppose $T$ has uniformly relatively separated branch points. By definition, this means that

$$|x - y| \gtrsim \min\{H_T(x), H_T(y)\} \tag{4.1}$$

for all distinct branch points $x, y \in T$ with $C(\gtrsim)$ independent of $x$ and $y$. Since $F$ is a quasisymmetry and hence a homeomorphism, the branch points of $T$ and $T'$ correspond to each other under the map $F$. So in order to establish the statement, it is enough to show an estimate analogous to (4.1) for the image points of $x$ and $y$ under $F$. Without loss of generality, we may assume that $H_T(x) \leq H_T(y)$, i.e., that the minimum in (4.1) is attained for $x$.

Let $B$ be a branch of $x$ in $T$ with $\operatorname{diam}(B) = H_T(x)$, and consider an arbitrary point $z \in B$. Then

$$|x - y| \gtrsim H_T(x) = \operatorname{diam}(B) \geq |x - z|.$$

Let $x' := F(x), y' := F(y), z' := F(z) \in T'$, and $B' := F(B)$ be the images under $F$. Since $F$ is quasisymmetry, the previous inequality implies that

$$|x' - y'| \gtrsim |x' - z'|,$$

where $C(\gtrsim)$ only depends on the implicit constant in (4.1) and the distortion function $\eta$ of $F$. Now $z \in B$ was arbitrary, and so

$$\begin{aligned} |x' - y'| &\gtrsim \sup_{z' \in B'} |x' - z'| \geq \tfrac{1}{2} \operatorname{diam}(B') \\ &\asymp \tfrac{1}{2} H_{T'}(x'), && \text{by Corollary 4.2} \\ &\geq \tfrac{1}{2} \min\{H_{T'}(x'), H_{T'}(y')\}. \end{aligned}$$

Here the implicit constants are independent of $x'$ and $y'$. The statement follows. □

Let us now consider uniform relative density of branch points. This condition is equivalent to the following, seemingly stronger, property.

**Lemma 4.4.** *A tree $T$ has uniformly relatively dense branch points if and only if for all $x, y \in T$, $x \neq y$, there is a branch point $z \in [x, y]$ with*

$$H_T(z) \gtrsim \operatorname{diam}[x, y],$$

*where $C(\gtrsim)$ is independent of $x$ and $y$.*

*Proof.* Since we always have $\operatorname{diam}[x, y] \geq |x - y|$, it is obvious that the above condition implies uniform relative density of branch points.

Conversely, assume that $T$ has uniformly relatively dense branch points. Let $x, y \in T$ with $x \neq y$ be arbitrary. Then there is a point $w \in [x, y]$ with $|x - w| = \frac{1}{2} \operatorname{diam}[x, y]$. Uniform relative density of branch points now implies that there is a branch point $z \in [x, w] \subset [x, y]$ with

$$H_T(z) \gtrsim |x - w| = \tfrac{1}{2} \operatorname{diam}[x, y].$$

The statement follows. □

After these preparations, we are now ready for the second main result of this section.

**Lemma 4.5** (Qs-invariance of uniform relative density)**.** *Let $F\colon T \to T'$ be a quasisymmetry between trees $T$ and $T'$. If $T$ has uniformly relatively dense branch points, then the same is true for $T'$.*

*Proof.* Suppose $T$ has uniformly relatively dense branch points. Let $x, y \in T$ with $x \neq y$ be arbitrary. Then by Lemma 4.4 there is a branch point $z \in [x, y]$ of $T$ with

$$H_T(z) \gtrsim \operatorname{diam}[x, y]. \tag{4.2}$$

Since $F$ is a homeomorphism, the branch points and arcs in $T$ and $T'$ correspond to each other under $F$. This implies that in order to establish the statement, it is enough to show an inequality analogous to (4.2) for the image points of $x$, $y$, and $z$ under $F$.

To see this, let $B \subset T$ be a branch of $z$ in $T$ with $H_T(z) = \operatorname{diam}(B)$. Then there is a point $w \in B$ with

$$|z - w| \geq \tfrac{1}{2} \operatorname{diam}(B) = \tfrac{1}{2} H_T(z) \gtrsim \operatorname{diam}[x, y].$$

Thus, for each point $q \in [x, y]$ we obtain

$$|z - w| \gtrsim \operatorname{diam}[x, y] \geq |z - q|.$$

Let $x' := F(x)$, $y' := F(y)$, $z' := F(z)$, $w' := F(w)$, $q' := F(q)$, and $B' := F(B)$. Note that $z' \in F([x, y]) = [x', y']$. Using Corollary 4.2 and the fact that $F$ is a quasisymmetry, we see that

$$H_{T'}(z') \asymp \operatorname{diam}(B') \geq |z' - w'| \gtrsim |z' - q'|.$$

Since $q \in [x, y]$ was arbitrary, this gives

$$H_{T'}(z') \gtrsim \sup_{q' \in [x', y']} |z' - q'| \geq \tfrac{1}{2} \operatorname{diam}[x', y'] \geq \tfrac{1}{2} |x' - y'|.$$

Here $z' = F(z) \in [x', y']$ in a branch point of $T'$, and $C(\gtrsim)$ is independent of $x'$, $y'$, and $z'$. The statement follows. □

Our considerations show that uniform branching of a tree is invariant under quasisymmetric equivalence. To complete the proof of the "only if" implication in Theorem 1.4, we need to show that the continuum self-similar tree is uniformly branching. Naturally, this means that first we have to define it precisely. We address this in the next section.

## 5. The continuum self-similar tree

The continuum self-similar tree (abbreviated as CSST) is a standard model for a quasiconformal tree that is trivalent and uniformly branching. It was studied in the paper [BT21], to which we refer for more details and references to the literature.

For the definition of the CSST we consider the maps $g_k\colon \mathbb{C} \to \mathbb{C}$, $k = 1, 2, 3$, given by

$$g_1(z) = \tfrac{1}{2}z - \tfrac{1}{2}, \quad g_2(z) = \tfrac{1}{2}\bar{z} + \tfrac{1}{2}, \quad g_3(z) = \tfrac{i}{2}\bar{z} + \tfrac{i}{2}. \tag{5.1}$$

Then the *continuum self-similar tree* is the attractor $\mathbb{T}$ of the iterated function system $\{g_1, g_2, g_3\}$ (see [Fa03, Theorem 9.1] for a general result in this direction). In other words, it is the unique non-empty compact set $\mathbb{T} \subset \mathbb{C}$ satisfying

$$\mathbb{T} = g_1(\mathbb{T}) \cup g_2(\mathbb{T}) \cup g_3(\mathbb{T}). \tag{5.2}$$

See Figure 1 for an illustration of the set $\mathbb{T}$.

We summarize some properties of the CSST. Since $\mathbb{T} \subset \mathbb{C}$, the CSST inherits the Euclidean metric from $\mathbb{C}$. Then $\mathbb{T}$ is a trivalent metric tree [BT21, Propositions 1.4 and 1.5]. It is a *quasi-convex* subset of $\mathbb{C}$ in the following sense: there exists a constant $L > 0$ such that if $x, y \in \mathbb{T}$ are arbitrary and $[x, y]$ is the unique arc in $\mathbb{T}$ joining $x$ and $y$, then

$$|x - y| \le \operatorname{length}[x, y] \le L|x - y| \tag{5.3}$$

(see [BT21, Proposition 1.6]). In particular, $\mathbb{T}$ is of bounded turning, since

$$\operatorname{diam}[x, y] \le \operatorname{length}[x, y] \le L|x - y|.$$

As a subset of $\mathbb{C}$, the space $\mathbb{T}$ is also doubling. Thus, $\mathbb{T}$ is a quasiconformal tree. In Proposition 5.4 below, we will see that $\mathbb{T}$ is uniformly branching.

Inequality (5.3) implies that if we define

$$\varrho(x, y) := \operatorname{length}[x, y] \tag{5.4}$$

for $x, y \in \mathbb{T}$, then $(\mathbb{T}, \varrho)$ is a geodesic tree that is bi-Lipschitz equivalent to $\mathbb{T}$ equipped with the Euclidean metric (by the identity map on $\mathbb{T}$). The space $(\mathbb{T}, \varrho)$ is isometric to the abstract version of the CSST as defined in the introduction (see the discussion at the end of Section 4 in [BT21]).

Considering $\mathbb{T}$ as a subset of $\mathbb{C}$ has the advantage that we have the simple characterization (5.2) of $\mathbb{T}$ with explicit maps $g_k$ as in (5.1). On the other hand, we never consider points in $\mathbb{C} \setminus \mathbb{T}$. Therefore, we think of $\mathbb{T}$ as the ambient space when we discuss topological properties of a set $M \subset \mathbb{T}$. For example, we denote by $\partial M$ the (relative) boundary of $M$ in $\mathbb{T}$. Naturally, this is in general different from the boundary of $M$ as a subset of $\mathbb{C}$. Similarly, the interior of $M$ is always understood to be relative to $\mathbb{T}$.

The CSST can be decomposed into subtrees in a natural way. These subtrees will be labeled by words in an alphabet. Here we use the set $\mathcal{A} := \{1, 2, 3\}$ as our *alphabet*. It consists of the *letters* $1, 2, 3$. For each $n \in \mathbb{N}$, let $\mathcal{A}^n := \{w_1 \dots w_n : w_j \in \mathcal{A} \text{ for } j = 1, \dots, n\}$ be the set of all *words* of *length* $n$. We set $\mathcal{A}^0 := \{\emptyset\}$; so

the empty word $\emptyset$ is the only word of length 0. If $n \in \mathbb{N}_0$ and $w \in \mathcal{A}^n$, we denote by $\ell(w) := n$ the length of $w$. Finally, we denote by $\mathcal{A}^* := \bigcup_{n\in\mathbb{N}_0} \mathcal{A}^n$ the set of all *finite* words.

If $u = u_1 \dots u_k \in \mathcal{A}^k$ and $w = w_1 \dots w_n \in \mathcal{A}^n$ with $k, n \in \mathbb{N}_0$, then

$$uw := u_1 \dots u_k w_1 \dots w_n \in \mathcal{A}^{k+n}$$

denotes the concatenation of the words $u$ and $w$.

For a finite word $w = w_1 \dots w_n \in \mathcal{A}^n$ of length $n \in \mathbb{N}_0$, we define the map $g_w \colon \mathbb{C} \to \mathbb{C}$ by

$$g_w := g_{w_1} \circ g_{w_2} \circ \cdots \circ g_{w_n},$$

where $g_{w_i}$ for $i = 1, \dots, n$ is as in (5.1) and it is understood that $g_\emptyset := \mathrm{id}_{\mathbb{C}}$ is the identity map on $\mathbb{C}$ for the empty word $w = \emptyset$. The map $g_w$ is a Euclidean similarity that scales distances by the factor $2^{-\ell(w)}$.

For each $w \in \mathcal{A}^*$ we now set

$$\mathbb{T}_w := g_w(\mathbb{T}). \tag{5.5}$$

It follows from repeated application of (5.2) that $\mathbb{T}_w \subset \mathbb{T}$. Hence $\mathbb{T}_w = g_w(\mathbb{T})$ is a subtree of $\mathbb{T}$, because $\mathbb{T}_w$ is a tree as the image of $\mathbb{T}$ under the homeomorphism $g_w$. We use the same notation for the restriction of $g_w$ to $\mathbb{T}$ and consider this as a map $g_w \colon \mathbb{T} \to g_w(\mathbb{T}) = \mathbb{T}_w \subset \mathbb{T}$.

Since $\operatorname{diam}(\mathbb{T}) = 2$, we have

$$\operatorname{diam}(\mathbb{T}_w) = 2^{-\ell(w)+1} \tag{5.6}$$

(see [BT21, (4.15)]). We call the sets of the form $\mathbb{T}_w = g_w(\mathbb{T})$ the *tiles* of $\mathbb{T}$ and denote by

$$\mathbb{X}^* := \{\mathbb{T}_w : w \in \mathcal{A}^*\} \tag{5.7}$$

the set of all tiles. One can easily show (based on (5.10) below) that these are tiles in a subdivision of $\mathbb{T}$. We will not discuss this in more detail here as other subdivisions of $\mathbb{T}$ will be important in the proof of Theorem 1.4.

If $\mathbb{T}_w \in \mathbb{X}^*$, then we call $n = \ell(w) \in \mathbb{N}_0$ the *level* of the tile, which we also denote by $\ell(\mathbb{T}_w) := n$. We then say that $\mathbb{T}_w$ is a tile of *level* $n$, or simply an *$n$-tile*. By (5.6) the level of a tile is well defined. The set $\mathbb{T}_\emptyset = \mathbb{T}$ is the only 0-tile, and there are three 1-tiles, namely $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$. For an illustration see Figure 3 which shows the 1-tiles, and Figure 4 which shows some of the 2-tiles (note that here the dotted lines indicate tiles, but are not part of the CSST).

We record some properties of 1-tiles for later use.

**Lemma 5.1.** *The following statements are true:*

(i) *The point $0 \in \mathbb{C}$ is a branch point of $\mathbb{T}$ with the three branches $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$.*

(ii) *The points $-1$ and $1$ are leaves of $\mathbb{T}$ with*

$$-1 \in \mathbb{T}_1 \quad \text{and} \quad -1 \notin \mathbb{T}_2, \mathbb{T}_3;$$
$$1 \in \mathbb{T}_2 \quad \text{and} \quad 1 \notin \mathbb{T}_1, \mathbb{T}_3.$$

(iii) *Let $x, y \in \mathbb{T}$ be contained in distinct 1-tiles. Then*

$$|x - y| \gtrsim |x| + |y|,$$

*where $C(\gtrsim)$ is independent of $x$ and $y$.*

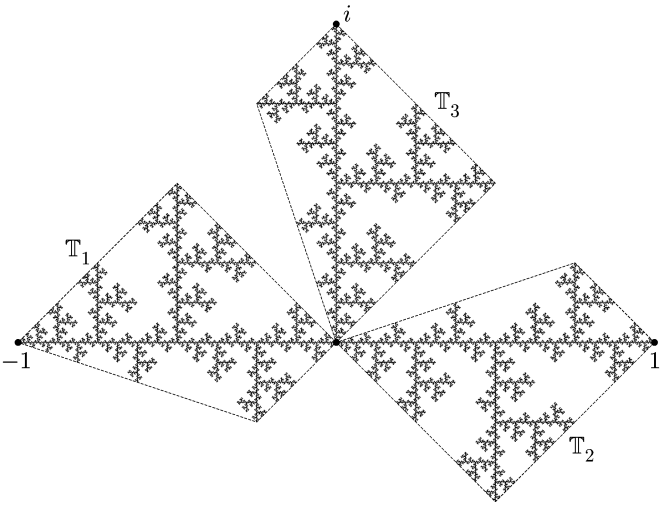


FIGURE 3. The CSST and its subtrees $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$

In particular, (i) means that the set $\mathbb{T} \setminus \{0\}$ has precisely three distinct components $U_k$, $k = 1, 2, 3$, and with suitable labeling we have $\overline{U_k} = U_k \cup \{0\} = \mathbb{T}_k$. Hence Lemma 3.3 (ii) implies that

$$\partial \mathbb{T}_1 = \partial \mathbb{T}_2 = \partial \mathbb{T}_3 = \{0\}. \tag{5.8}$$

Moreover, the subtrees $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$ of $\mathbb{T}$ are pairwise disjoint except for the common point 0. In other words,

$$\mathbb{T}_k \cap \mathbb{T}_\ell = \{0\} \text{ for } k, \ell \in \{1, 2, 3\} \text{ with } k \neq \ell. \tag{5.9}$$

*Proof of Lemma 5.1.*

(i) This follows from [BT21, Lemma 4.5 (i)].

(ii) By [BT21, Proposition 4.2 and Lemma 4.4 (ii)] the points $-1$ and 1 belong to $\mathbb{T}$ and are leaves of $\mathbb{T}$. Moreover, $-1 = g_1(-1) \in \mathbb{T}_1$ and $1 = g_2(1) \in \mathbb{T}_2$ (see (5.1)). Since 0 is the only common point of two distinct 1-tiles, we also know that $1 \notin \mathbb{T}_2, \mathbb{T}_3$ and $-1 \notin \mathbb{T}_1, \mathbb{T}_3$.

(iii) Let $x, y \in \mathbb{T}$ be contained in two distinct 1-tiles. Then $0 \in [x, y]$ by (i). If $L > 0$ is the constant in (5.3), then we obtain

$$L|x - y| \geq \operatorname{length}[x, y] = \operatorname{length}[x, 0] + \operatorname{length}[0, y] \geq |x| + |y|,$$

and (iii) follows. □

If $w \in \mathcal{A}^*$ and $n = \ell(w) \in \mathbb{N}_0$, then $g_w \colon \mathbb{T} \to \mathbb{T}_w$ is a homeomorphism that maps the 1-tiles $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3 \subset \mathbb{T}$ onto the $(n+1)$-tiles $\mathbb{T}_{w1}, \mathbb{T}_{w2}, \mathbb{T}_{w3} \subset \mathbb{T}_w$. So the statements in Lemma 5.1 can be translated via the map $g_w$ to statements about $\mathbb{T}_w$. For example, $g_w(0)$ is a branch point of $\mathbb{T}_w$ with three branches given by the $(n+1)$-tiles $\mathbb{T}_{w1}, \mathbb{T}_{w2}, \mathbb{T}_{w3}$. In particular,

$$\mathbb{T}_w = \mathbb{T}_{w1} \cup \mathbb{T}_{w2} \cup \mathbb{T}_{w3}. \tag{5.10}$$

Since this is true for all $w \in \mathcal{A}^*$, it is easy to see that if $m, n \in \mathbb{N}_0$ with $n \geq m$ and $X$ is an $n$-tile, then there exists an $m$-tile $Y$ with $X \subset Y$.

**Branch points of $\mathbb{T}$ and boundaries of tiles.** In order to show that $\mathbb{T}$ is uniformly branching (see Proposition 5.4), we need to locate the branch points of $\mathbb{T}$ and understand their height. The key observation is that a point $p$ is a branch point of $\mathbb{T}$ if and only if it is a (relative) boundary point of some tile in $\mathbb{T}$ (see Lemma 5.2 (ii) and Lemma 5.3 (i)).

As we will see, there are two ways to describe (relative) boundary points of tiles: as images of the leaves $-1$ or $1$ of $\mathbb{T}$ under some map $g_w$ or as images of the branch point $0$ of $\mathbb{T}$.

**Lemma 5.2** (Boundaries of tiles)**.** *The following statements are true:*

(i) *Let $\mathbb{T}_w$ with $w \in \mathcal{A}^n$ be a tile of level $n \in \mathbb{N}$. Then $\mathbb{T}_w$ has one or two boundary points and $\partial\mathbb{T}_w \subset \{g_w(-1), g_w(1)\}$.*
*Moreover, if $w = u3$ with $u \in \mathcal{A}^{n-1}$, then $\partial\mathbb{T}_w$ consists of a single point.*

(ii) *For $n \in \mathbb{N}$ the set of all boundary points of all $n$-tiles is equal to*

$$\mathbb{V}^n := \{g_u(0) \colon u \in \mathcal{A}^m,\ 0 \le m \le n-1\}.$$

*Proof.* (i) We prove this by induction on $n = \ell(w) \in \mathbb{N}$.

For $n = 1$ the statement follows from (5.8) and

$$0 = g_1(1) = g_2(-1) = g_3(-1). \tag{5.11}$$

Suppose the statement is true for all words of length $n \in \mathbb{N}$, and let $w \in \mathcal{A}^{n+1}$ be arbitrary. Then $w = uk$ with $u \in \mathcal{A}^n$ and $k \in \mathcal{A} = \{1, 2, 3\}$. The map $g_u$ is a homeomorphism that sends $\mathbb{T}$ onto $\mathbb{T}_u$ and the 1-tile $\mathbb{T}_\ell$ onto the $(n+1)$-tile $\mathbb{T}_{u\ell}$ for $\ell = 1, 2, 3$. It follows from (5.8), (5.9), and (5.10) that the sets $\mathbb{T}_{u\ell} \setminus \{g_u(0)\}$, $\ell \in \{1, 2, 3\}$, are pairwise disjoint, but that every neighborhood of $g_u(0)$ contains points from each of these sets. This implies that $g_u(0)$ is a boundary point of $\mathbb{T}_{u\ell}$ for $\ell = 1, 2, 3$. In particular, $g_u(0) \in \partial\mathbb{T}_w = \partial\mathbb{T}_{uk}$, and so $\partial\mathbb{T}_w$ contains at least one point.

Let $p \in \partial\mathbb{T}_w$ be arbitrary. Then $p \in \mathbb{T}_w \subset \mathbb{T}_u$ is contained either in the interior of $\mathbb{T}_u$ or in $\partial\mathbb{T}_u$. In the first case, each sufficiently small neighborhood $U$ of $p$ is contained in $\mathbb{T}_u = \mathbb{T}_{u1} \cup \mathbb{T}_{u2} \cup \mathbb{T}_{u3}$; so $U$ must contain a point in $\mathbb{T}_u \setminus \mathbb{T}_w$, because $p \in \partial\mathbb{T}_w$. Applying the homeomorphism $g_u^{-1}$, we see that every neighborhood of $g_u^{-1}(p) \in \mathbb{T}_k$ contains points from the set $\mathbb{T} \setminus \mathbb{T}_k$, and so $g_u^{-1}(p) \in \partial\mathbb{T}_k$. Hence $g_u^{-1}(p) = 0$ by (5.8), and so $p = g_u(0)$.

In the second case, $p \in \partial\mathbb{T}_u \subset \{g_u(-1), g_u(1)\}$, since (i) is true for $\mathbb{T}_u$ by induction hypothesis. Lemma 5.1 (ii) shows that each 1-tile contains at most one of the points $-1$ and $1$; namely, $\mathbb{T}_1$ contains only $-1$, $\mathbb{T}_2$ contains only $1$, and $\mathbb{T}_3$ contains neither $-1$ nor $1$. We conclude that $\mathbb{T}_w = g_u(\mathbb{T}_k)$ contains at most one of the points $g_u(-1)$ and $g_u(1)$ (in its boundary). So $\partial\mathbb{T}_w$ consists of at most two points, namely the point $g_u(0)$ and at most one of the points $g_u(-1)$ and $g_u(1)$. More precisely, our discussion shows that we have the following alternatives for $k \in \{1, 2, 3\}$.

If $k = 1$, then we have $\partial\mathbb{T}_w = \partial\mathbb{T}_{u1} \subset \{g_u(-1), g_u(0)\}$. Since

$$g_u(-1) = g_u(g_1(-1)) = g_w(-1) \text{ and } g_u(0) = g_u(g_1(1)) = g_w(1),$$

it follows that $\partial\mathbb{T}_w = \partial\mathbb{T}_{u1} \subset \{g_u(-1), g_u(0)\} = \{g_w(-1), g_w(1)\}$.

If $k = 2$, then $\partial\mathbb{T}_w = \partial\mathbb{T}_{u2} \subset \{g_u(0), g_u(1)\}$. Now

$$g_u(0) = g_u(g_2(-1)) = g_w(-1) \text{ and } g_u(1) = g_u(g_2(1)) = g_w(1),$$

which implies $\partial\mathbb{T}_w = \partial\mathbb{T}_{u2} \subset \{g_u(0), g_u(1)\} = \{g_w(-1), g_w(1)\}$.

If $k = 3$, then $\partial\mathbb{T}_w = \partial\mathbb{T}_{u3} = \{g_u(0)\}$. In particular, $\mathbb{T}_w$ has precisely one boundary point in this case. Moreover, we have

$$g_u(0) = g_u(g_3(-1)) = g_w(-1),$$

and so $\partial\mathbb{T}_w = \partial\mathbb{T}_{u3} = \{g_u(0)\} = \{g_w(-1)\}$.

This shows that the statement is true for the tile $\mathbb{T}_w$. This completes the inductive step and (i) follows.

(ii) For $n \in \mathbb{N}$ we denote by

$$B^n := \bigcup_{w\in\mathcal{A}^n} \partial\mathbb{T}_w$$

the set of all boundary points of all $n$-tiles. We have to show that $B^n = \mathbb{V}^n$. We prove this by induction on $n \in \mathbb{N}$.

Let $n = 1$. Then (5.8) shows that $B^1 = \{0\} = \{g_\emptyset(0)\} = \mathbb{V}^1$ as desired.

Suppose the statement is true for some $n \in \mathbb{N}$. To show that it is also true for $n + 1$, we first verify that $\mathbb{V}^{n+1} \subset B^{n+1}$. To this end, let $p \in \mathbb{V}^{n+1}$ be arbitrary. Then $p = g_u(0)$ for some $u \in \mathcal{A}^m$ with $0 \le m \le n$. If $m = n$, then $p$ is in the boundary of the $(n + 1)$-tiles $\mathbb{T}_{u1}, \mathbb{T}_{u2}, \mathbb{T}_{u3}$, as have seen in the proof of (i). So $p \in B^{n+1}$.

If $m \le n - 1$, we have $p \in \mathbb{V}^n$, and so $p \in \partial\mathbb{T}_w \subset \mathbb{T}_w$ for some $w \in \mathcal{A}^n$ by induction hypothesis. By (5.10) there exists $k \in \{1, 2, 3\}$ such that $p \in \mathbb{T}_{wk} \subset \mathbb{T}_w$. Then every neighborhood of $p$ contains points in $\mathbb{T} \setminus \mathbb{T}_w \subset \mathbb{T} \setminus \mathbb{T}_{wk}$. This implies that $p \in \partial\mathbb{T}_{wk}$, and so $p \in B^{n+1}$ also in this case. Thus $\mathbb{V}^{n+1} \subset B^{n+1}$.

To see the reverse inclusion $B^{n+1} \subset \mathbb{V}^{n+1}$, let $p \in B^{n+1}$ be arbitrary. Then $p$ is a boundary point of an $(n + 1)$-tile, say $p \in \partial\mathbb{T}_{uk}$, where $u \in \mathcal{A}^n$ and $k \in \{1, 2, 3\}$. In the proof of (i) we have seen that either $p = g_u(0)$ or $p \in \partial\mathbb{T}_u$. In the first case, $p \in \mathbb{V}^{n+1}$, while in the second case, $p \in \mathbb{V}^n \subset \mathbb{V}^{n+1}$ by induction hypothesis. This shows $B^{n+1} \subset \mathbb{V}^{n+1}$, and so $B^{n+1} = \mathbb{V}^{n+1}$.

This completes the inductive step and the statement follows. □

For $n \in \mathbb{N}$ let $\mathbb{V}^n$ be as in Lemma 5.2 (ii), and define

$$\mathbb{V}^* := \bigcup_{n\in\mathbb{N}} \mathbb{V}^n = \{g_w(0) : w \in \mathcal{A}^*\}.$$

Then $\mathbb{V}^* = \bigcup_{X\in\mathbb{X}^*} \partial X$ (note that for the only 0-tile $X = \mathbb{T}$, we have $\partial X = \emptyset$). So $\mathbb{V}^*$ is equal to the set of all boundary points of all tiles of $\mathbb{T}$.

**Lemma 5.3** (Branch points of $\mathbb{T}$)**.** *The following statements are true:*

(i) *The set $\mathbb{V}^*$ is equal to the set of all branch points of $\mathbb{T}$.*

(ii) *The points $g_w(0)$, $w \in \mathcal{A}^*$, are all distinct. More precisely, if $v, w \in \mathcal{A}^*$, $v \ne w$, then*

$$|g_v(0) - g_w(0)| \gtrsim \min\{2^{-\ell(v)}, 2^{-\ell(w)}\} > 0$$

*with $C(\gtrsim)$ independent of $v$ and $w$.*

(iii) *If $w \in \mathcal{A}^*$ and $B_1$, $B_2$, $B_3$ are the three branches of the branch point $p = g_w(0)$ of $\mathbb{T}$, then with suitable labeling we have*

$$g_{w1}(\mathbb{T}) \subset B_1,\ g_{w2}(\mathbb{T}) \subset B_2,\ g_{w3}(\mathbb{T}) = B_3.$$

*In particular, $H_{\mathbb{T}}(p) = 2^{-\ell(w)}$.*

*Proof.* (i) This was proved in [BT21, Lemma 4.7].

(ii) We have $g_1(0) = -\frac{1}{2}$, $g_2(0) = \frac{1}{2}$, $g_3(0) = \frac{i}{2}$, and so $|g_k(0)| = \frac{1}{2}$ for $k \in \mathcal{A} = \{1,2,3\}$. It follows that if $u \in \mathcal{A}^*$ and $k \in \mathcal{A}$ are arbitrary, then

$$\begin{aligned} |g_u(0) - g_{uk}(0)| &= |g_u(0) - g_u(g_k(0))| = 2^{-\ell(u)}|g_k(0)| \\ &= 2^{-\ell(u)-1}. \end{aligned}$$

So if $u = a_1 \dots a_n \in \mathcal{A}^n$ with $n \in \mathbb{N}$, then

$$\begin{aligned} |g_u(0)| &\geq |g_{a_1}(0)| - |g_{a_1}(0) - g_{a_1\dots a_n}(0)| \\ &\geq \frac{1}{2} - \sum_{j=1}^{n-1} |g_{a_1\dots a_j}(0) - g_{a_1\dots a_j a_{j+1}}(0)| \\ &= \frac{1}{2} - \sum_{j=1}^{n-1} 2^{-j-1} = 2^{-n} = 2^{-\ell(u)}. \end{aligned} \tag{5.12}$$

Now let $v, w \in \mathcal{A}^*$ with $v \neq w$ be arbitrary, and let $u \in \mathcal{A}^*$ be the longest common initial word of $v$ and $w$. Then $v = uv'$ and $w = uw'$ with $v', w' \in \mathcal{A}^*$, where either $v'$ or $w'$ is the empty word, or both $v'$ and $w'$ are non-empty, but have a different initial letter. Accordingly, we consider two cases.

*Case* 1. Either $v' = \emptyset$ or $w' = \emptyset$. We may assume $w' = \emptyset$. Then $v' \neq \emptyset$, and so $\ell(v) \geq \ell(w)$. Moreover, by (5.12) we have

$$\begin{aligned} |g_v(0) - g_w(0)| &= |g_u(g_{v'}(0)) - g_u(g_{w'}(0))| \\ &= 2^{-\ell(u)}|g_{v'}(0)| \geq 2^{-\ell(u)-\ell(v')} \\ &= 2^{-\ell(v)} = \min\{2^{-\ell(v)}, 2^{-\ell(w)}\}. \end{aligned}$$

*Case* 2. $v', w' \neq \emptyset$. Then $v'$ and $w'$ necessarily start with different letters, say $v' = kv''$ and $w' = mw''$ with $k, m \in \mathcal{A} = \{1,2,3\}$, $k \neq m$, and $v'', w'' \in \mathcal{A}^*$. Then $g_{v'}(0) \in \mathbb{T}_k$ and $g_{w'}(0) \in \mathbb{T}_m$, i.e., these points are contained in distinct 1-tiles. Then Lemma 5.1 (iii) and (5.12) imply that

$$\begin{aligned} |g_{v'}(0) - g_{w'}(0)| &\gtrsim |g_{v'}(0)| + |g_{w'}(0)| \\ &\geq 2^{-\ell(v')} + 2^{-\ell(w')} \geq \min\{2^{-\ell(v')}, 2^{-\ell(w')}\} \end{aligned}$$

with $C(\gtrsim)$ independent of $v$ and $w$. We conclude that

$$\begin{aligned} |g_v(0) - g_w(0)| &= |g_u(g_{v'}(0)) - g_u(g_{w'}(0))| \\ &= 2^{-\ell(u)}|g_{v'}(0) - g_{w'}(0)| \\ &\gtrsim 2^{-\ell(u)} \min\{2^{-\ell(v')}, 2^{-\ell(w')}\} \\ &= \min\{2^{-\ell(v)}, 2^{-\ell(w)}\}. \end{aligned}$$

The statement follows.

(iii) If $w \in \mathcal{A}^*$, then $p = g_w(0)$ is a branch point of $\mathbb{T}_w$ with the three (distinct) branches $\mathbb{T}_{w1}$, $\mathbb{T}_{w2}$, $\mathbb{T}_{w3}$. Hence there exist unique distinct branches $B_1$, $B_2$, $B_3$ of $g_w(0)$ in $\mathbb{T}$ such that $\mathbb{T}_{wk} \subset B_k$ for $k = 1,2,3$ (see Lemma 3.2 (i)). By Lemma 5.2 (i) we have $\partial\mathbb{T}_w \subset \{g_w(-1), g_w(1)\}$. Moreover, $g_w(-1), g_w(1) \notin \mathbb{T}_{w3}$, since $-1, 1 \notin \mathbb{T}_3$. Thus $\mathbb{T}_{w3} \cap \partial\mathbb{T}_w = \emptyset$. Hence it follows from Lemma 3.2 (iii) that $\mathbb{T}_{w3}$ is actually a branch of $p = g_w(0)$ in $\mathbb{T}$, and so $\mathbb{T}_{w3} = B_3$. The first part of the statement follows.

For the second statement note that

$$\operatorname{diam}(B_k) \geq \operatorname{diam}(\mathbb{T}_{wk}) = 2^{-\ell(w)}$$

for $k = 1, 2$, while

$$\operatorname{diam}(B_3) = \operatorname{diam}(\mathbb{T}_{w3}) = 2^{-\ell(w)}.$$

It follows that

$$H_{\mathbb{T}}(p) = \min\{\operatorname{diam}(B_k) : k \in \{1, 2, 3\}\} = 2^{-\ell(w)},$$

as desired. □

**Proposition 5.4.** *The continuous self-similar tree $\mathbb{T}$ is uniformly branching, i.e., its branch points are uniformly relatively separated and uniformly relatively dense.*

*Proof.* Let $p, q \in \mathbb{T}$ be two distinct branch points of $\mathbb{T}$. Then $p = g_v(0)$ and $q = g_w(0)$ with $v, w \in \mathcal{A}^*$, $v \neq w$, by Lemma 5.3 (i). Moreover, we have

$$\begin{aligned} |p - q| &= |g_v(0) - g_w(0)| \\ &\gtrsim \min\{2^{-\ell(v)}, 2^{-\ell(w)}\} && \text{by Lemma 5.3 (ii)} \\ &= \min\{H_{\mathbb{T}}(g_v(0)), H_{\mathbb{T}}(g_w(0))\} && \text{by Lemma 5.3 (iii)} \\ &= \min\{H_{\mathbb{T}}(p), H_{\mathbb{T}}(q)\}. \end{aligned}$$

Here the implicit constant is independent of $p$ and $q$. This shows that $\mathbb{T}$ has branch points that are uniformly relatively separated.

To establish the other property, let $x, y \in \mathbb{T}$, $x \neq y$, be arbitrary. Since $\operatorname{diam}(\mathbb{T}) = 2$, and so $2 \geq |x - y| > 0$, there exists a unique number $n \in \mathbb{N}$ such that $2^{-n+2} \geq |x - y| > 2^{-n+1}$. Then we can find $w \in \mathcal{A}^*$ with $\ell(w) = n$ such that $x \in \mathbb{T}_w$. Since

$$\operatorname{diam}(\mathbb{T}_w) = \operatorname{diam}(g_w(\mathbb{T})) = 2^{-n+1} < |x - y|,$$

we then have $y \notin \mathbb{T}_w$. So as we travel from $x$ to $y$ along the arc $[x, y]$ that joins $x$ and $y$ in $\mathbb{T}$, there is a last point $p \in [x, y] \cap \mathbb{T}_w$. Then $(p, y]$ is non-empty and disjoint from $\mathbb{T}_w$, and so $p \in \partial\mathbb{T}_w$. By Lemma 5.2 (ii) we have $p \in \mathbb{V}^n \subset \mathbb{V}^*$, and so $p$ is a branch point of $\mathbb{T}$ by Lemma 5.3 (i).

Since $p \in \mathbb{V}^n$, we know that $p = g_v(0)$ for some $v \in \mathcal{A}^*$ with $0 \leq \ell(v) \leq n - 1$. Now Lemma 5.3 (iii) implies that

$$H_{\mathbb{T}}(p) = H_{\mathbb{T}}(g_v(0)) = 2^{-\ell(v)} \geq 2^{-\ell(w)} = 2^{-n} \asymp |x - y|.$$

Again the implicit multiplicative constant is independent of the choices here. This shows that the branch points of $\mathbb{T}$ are also uniformly relatively dense. The statement follows. □

Note that Proposition 5.4 together with Lemma 4.3 and Lemma 4.5 essentially shows the "only if" implication in Theorem 1.4.

**Miscellaneous results on tiles in $\mathbb{T}$.** We conclude this section with some auxiliary results, which will be used in the proof of the other implication of Theorem 1.4 (namely in the proof of Lemma 7.3).

Let $\mathbb{V} \subset \mathbb{T}$ be a finite set of branch points of $\mathbb{T}$. Recall from Section 3 that $\mathbb{V}$ then decomposes $\mathbb{T}$ into a set of tiles $\mathbb{X}$. Note that such a tile $X \in \mathbb{X}$ is not necessarily in $\mathbb{X}^*$, i.e., a tile of $\mathbb{T}$ as defined in (5.5). For example, if $\mathbb{V} = \{-1/2\}$, then the decomposition $\mathbb{X}$ induced by $\mathbb{V}$ consists of the sets $X_1 = \mathbb{T}_{11}$, $X_2 = \mathbb{T}_{12} \cup \mathbb{T}_2 \cup \mathbb{T}_3$,

and $X_3 = \mathbb{T}_{13}$. Here $X_2$ is not a tile of $\mathbb{T}$. This can be seen from Figure 4 which shows the relevant 2-tiles.

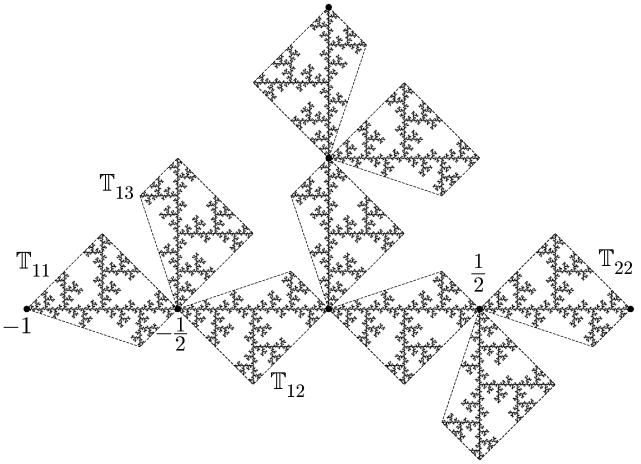


FIGURE 4. The CSST and some of its 2-tiles

We now consider the special case when such a decomposition of $\mathbb{T}$ produces tiles as in (5.7).

**Lemma 5.5.** *Let $\mathbb{V} \subset \mathbb{T}$ be a finite set of branch points of $\mathbb{T}$, and let $\mathbb{X}$ be the set of tiles in the decomposition of $\mathbb{T}$ induced by $\mathbb{V}$.*

*Suppose that $\mathbb{X} \subset \mathbb{X}^*$. Then the levels of tiles in $\mathbb{X}$ satisfy*

$$\ell(X) \leq \#\mathbb{V} \text{ for each } X \in \mathbb{X}.$$

*If, in addition, $\mathbb{V} \neq \emptyset$, then we have $\ell(X) \geq 1$ for each $X \in \mathbb{X}$.*

The requirement $\mathbb{X} \subset \mathbb{X}^*$ means that each $X \in \mathbb{X}$ is a set of the form $X = \mathbb{T}_w$ for some $w \in \mathcal{A}^*$ as in (5.5). Accordingly, the level $\ell(X) = \ell(w) \in \mathbb{N}_0$ of $X$ is well defined.

In this context, the concept of a *tile* has two different meanings. Namely, a tile can be an element of the set of tiles $\mathbb{X}$ in the decomposition of $\mathbb{T}$ induced by the given finite set $\mathbb{V}$, or a tile can be an element of $\mathbb{X}^*$ as defined in (5.7). If we require $\mathbb{X} \subset \mathbb{X}^*$, then every tile in $\mathbb{X}$ is a tile in $\mathbb{X}^*$, but not every tile in $\mathbb{X}^*$ is necessarily a tile in $\mathbb{X}$. An $n$-tile, or a tile (without further specification), is always understood to be in $\mathbb{X}^*$. It will be stated explicitly if we refer to a tile in $\mathbb{X}$.

*Proof.* The second statement is immediate. Indeed, the only tile of level 0 is $X = \mathbb{T}$. If $\mathbb{V} \neq \emptyset$, then for each tile $X \in \mathbb{X} \subset \mathbb{X}^*$ we have $X \neq \mathbb{T}$ as follows from Lemma 3.3 (i), and so $\ell(X) \geq 1$.

Let us now prepare the proof of the first statement. Recall that if $m, n \in \mathbb{N}_0$ with $n \geq m$, then each $n$-tile $X \in \mathbb{X}^*$ is contained in an $m$-tile $Y \in \mathbb{X}^*$. We also know by (5.8) that the point $0 \in \mathbb{T}$ belongs to the boundary of each 1-tile, namely $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$.

*Claim* 1. Let $X \in \mathbb{X}^*$ be a tile with $0 \in X$. Then either $0 \in \partial X$ or $X = \mathbb{T}$.

Indeed, suppose $n = \ell(X) \in \mathbb{N}_0$ is the level of a tile $X \in \mathbb{X}^*$ with $0 \in X$. Assume that 0 is contained in the interior of $X$. Recall that we regard $\mathbb{T}$ as the ambient

space here, as before. If $n \geq 1$, there exists a 1-tile $Y \supset X$. Then 0 is an interior point of $Y$, but we know that $0 \in \partial Y$. This is a contradiction. So $0 \in \partial X$ unless $n = 0$. In the latter case, $X = \mathbb{T}$ and $0 \notin \partial \mathbb{T} = \emptyset$. Claim 1 follows.

We are now ready to prove the first (i.e., the main) statement in the lemma by induction on $\#\mathbb{V} \in \mathbb{N}_0$.

To start the induction, assume $\#\mathbb{V} = 0$. Then $\mathbb{V} = \emptyset$ and $X = \mathbb{T}$ is the only tile in $\mathbb{X}$. Then clearly $\ell(X) = 0 \leq \#\mathbb{V} = 0$, as desired.

For the inductive step we assume that $\#\mathbb{V} \geq 1$ and that the statement is true for all decompositions of $\mathbb{T}$ into tiles induced by sets of branch points with fewer than $\#\mathbb{V}$ points.

*Claim* 2. We have $0 \in \mathbb{V}$.

To see this, note that $\mathbb{X}$ is a covering of $\mathbb{T}$ (see Lemma 3.3 (vii)). Hence there exists a tile $X \in \mathbb{X}$ with $0 \in X$. By Claim 1, there are two possible cases. If $X = \mathbb{T}$, we have $\mathbb{V} = \emptyset$ contradicting our assumption $\#\mathbb{V} \geq 1$. Thus we must have $0 \in \partial X \subset \mathbb{V}$ (see Lemma 3.3 (ii)), which implies $0 \in \mathbb{V}$, as desired. Claim 2 follows.

We know that 0 splits $\mathbb{T}$ into the three branches $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$. Since $0 \in \mathbb{V}$ by Claim 2, each tile $X \in \mathbb{X}$ is contained in one such branch $\mathbb{T}_k$ (see Lemma 3.3 (i)). We now want to apply the induction hypothesis to these subtrees $\mathbb{T}_k$.

For this we fix $k \in \{1, 2, 3\}$, and let

$$\mathbb{V}_k := \mathbb{V} \cap \operatorname{int}(\mathbb{T}_k) = (\mathbb{V} \cap \mathbb{T}_k) \setminus \{0\}.$$

This set consists of branch points of $\mathbb{T}_k$ as follows from Lemma 3.2 (i), and induces a decomposition $\mathbb{X}_k$ of the tree $\mathbb{T}_k$ into tiles. Indeed, $X \in \mathbb{X}_k$ if and only if $X \in \mathbb{X} \subset \mathbb{X}^*$ and $X \subset \mathbb{T}_k$ (see Lemma 3.4 (iii)). In particular, $\mathbb{X}_k \subset \mathbb{X}^*$. Moreover, $\#\mathbb{V}_k < \#\mathbb{V}$, since $0 \in \mathbb{V}$ is not contained in $\mathbb{V}_k$.

The map $g_k^{-1} \colon \mathbb{T}_k \to \mathbb{T}$ is a homeomorphism that sends $(n+1)$-tiles contained in $\mathbb{T}_k$ to $n$-tiles of $\mathbb{T}$ (see (5.1) and (5.5)). Let $\widetilde{\mathbb{V}}_k := g_k^{-1}(\mathbb{V}_k)$. The set $\widetilde{\mathbb{V}}_k$ consists of branch points of $\mathbb{T}$ and induces the decomposition $\widetilde{\mathbb{X}}_k = g_k^{-1}(\mathbb{X}_k) = \{g_k^{-1}(X) : X \in \mathbb{X}_k\}$ of $\mathbb{T}$ into tiles. Then $\widetilde{\mathbb{X}}_k \subset \mathbb{X}^*$. Since $\#\widetilde{\mathbb{V}}_k = \#\mathbb{V}_k < \#\mathbb{V}$, we may apply the induction hypothesis to $\widetilde{\mathbb{V}}_k$ and conclude

$$\ell(\widetilde{X}) \leq \#\widetilde{\mathbb{V}}_k = \#\mathbb{V}_k \leq \#\mathbb{V} - 1$$

for each $\widetilde{X} \in \widetilde{\mathbb{X}}_k$. If $X \in \mathbb{X}_k$, then $g_k^{-1}(X) \in \widetilde{\mathbb{X}}_k$ and so

$$\ell(X) = \ell(g_k^{-1}(X)) + 1 \leq \#\mathbb{V}.$$

Since each $X \in \mathbb{X}$ is contained in one of the sets $\mathbb{X}_k$ with $k \in \{1, 2, 3\}$, the inductive step follows. This completes the proof. □

In our construction in Section 7, the 2-tiles containing one of the leaves $-1$ or 1 of $\mathbb{T}$ will play a special role. We record the following statement related to this.

**Lemma 5.6.** *The tile $X = \mathbb{T}_{11}$ is the only tile $X \in \mathbb{X}^*$ with $-1 \in X$ and $-1/2 \in \partial X$. The tile $X = \mathbb{T}_{22}$ is the only tile $X \in \mathbb{X}^*$ with $1 \in X$ and $1/2 \in \partial X$.*

See Figure 4 for an illustration.

*Proof.* We will only prove the first statement. The second statement is proved along very similar lines and we omit the details.

First we consider $\mathbb{T}_{11}$. Note that $-1 = g_{11}(-1) \in g_{11}(\mathbb{T}) = \mathbb{T}_{11}$. Since $-1$ is a leaf and not a branch point of $\mathbb{T}$, it is not a boundary point of $\mathbb{T}_{11}$, because by

Lemma 5.2 (ii) and Lemma 5.3 (i) the boundary point of any tile is a branch point of $\mathbb{T}$. So $g_{11}(-1) = -1 \notin \partial\mathbb{T}_{11}$. Hence $-1/2 = g_{11}(1) \in \partial\mathbb{T}_{11}$, as follows from Lemma 5.2 (i). Thus $\mathbb{T}_{11}$ is a tile with $-1 \in \mathbb{T}_{11}$ and $-1/2 \in \partial\mathbb{T}_{11}$.

Now suppose $X \in \mathbb{X}^*$ is any tile with $-1 \in X$ and $-1/2 \in \partial X$, and let $n \in \mathbb{N}_0$ be the level of $X$. Then $n \neq 0, 1$, because the 0-tile $\mathbb{T}$ has empty boundary and the 1-tiles $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$ do not contain $-1/2$ in their boundaries (see (5.8)).

We have $\operatorname{diam}(X) = 2^{-n+1} \geq |-1/2 - (-1)| = 1/2$, which rules out $n \geq 3$. It follows that $n = 2$, and so $X$ must be a 2-tile, say $X = \mathbb{T}_{k\ell}$ with $k, \ell \in \{1, 2, 3\}$. Then $-1 \in X = \mathbb{T}_{k\ell} \subset \mathbb{T}_k$. Since $\mathbb{T}_2$ and $\mathbb{T}_3$ do not contain $-1$ (see Lemma 5.1 (i)), we have $k = 1$, and so $X$ must be one of the 2-tiles $\mathbb{T}_{11}, \mathbb{T}_{12}, \mathbb{T}_{13}$.

The homeomorphism $g_1^{-1}\colon \mathbb{T}_1 \to \mathbb{T}$ sends these 2-tiles that are all contained in $\mathbb{T}_1$ to $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$, respectively. Since $-1 = g_1^{-1}(-1) \in g_1^{-1}(X)$ and $-1 \notin \mathbb{T}_2, \mathbb{T}_3$, we must have $g_1^{-1}(X) = \mathbb{T}_1$ and so $X = \mathbb{T}_{11}$. Hence $X = \mathbb{T}_{11}$ is indeed the only tile $X \in \mathbb{X}^*$ with $-1 \in X$ and $-1/2 \in \partial X$. □

## 6. Subdivisions of quasiconformal trees

The results in the previous two sections give the "only if" implication in Theorem 1.4.

In the following, we assume that $T$ is a uniformly branching trivalent quasiconformal tree. We need to show that such a tree $T$ is quasisymmetrically equivalent to $\mathbb{T}$. By rescaling the metric on $T$ if necessary, we may assume that $\operatorname{diam}(T) = 1$ for convenience.

Recall from Section 3 that a finite set $\mathbf{V} \subset T$ that does not contain any leaf of $T$ decomposes $T$ into a set of tiles $\mathbf{X}$. These tiles are subtrees of $T$. We want to map them to tiles of $\mathbb{T}$, i.e., elements of $\mathbb{X}^*$ (see (5.7)). By Lemma 5.2 (i) each tile in $\mathbb{T}$ distinct from $\mathbb{T}$ itself has one or two boundary points. For this reason, we are interested in decompositions such that every tile $X \in \mathbf{X}$ has one or two boundary points. Accordingly, we call $\mathbf{X}$ an *edge-like decomposition* of $T$ if $\mathbf{V} = \emptyset$ and $\mathbf{X} = \{T\}$ (as a degenerate case), or if $\mathbf{V} \neq \emptyset$ and $\#\partial X \leq 2$ for each $X \in \mathbf{X}$. Note that in the latter case $1 \leq \#\partial X \leq 2$ by Lemma 3.3 (iii). We say that a tile $X \in \mathbf{X}$ in an edge-like decomposition $\mathbf{X}$ of $T$ is a *leaf-tile* if $\#\partial X = 1$ and an *edge-tile* if $\#\partial X = 2$.

The reason for this terminology is that in an edge-like decomposition $\mathbf{X}$ of $T$ the tiles in $\mathbf{X}$ satisfy the same incidence relations as the edges in a finite simplicial tree. Here the leaf-tiles in $\mathbf{X}$ correspond to edges of the simplicial tree that contain a leaf and the edge-tiles in $\mathbf{X}$ to edges that do not contain a leaf of the simplicial tree.

We now set $\mathbf{V}^0 = \emptyset$, fix $\delta \in (0, 1)$, and for $n \in \mathbb{N}$ define

$$\mathbf{V}^n = \{p \in T : \ p \text{ is a branch point of } T \text{ with } H_T(p) \geq \delta^n\}. \tag{6.1}$$

Recall that the height $H_T$ was defined in (1.2). Clearly, $\{\mathbf{V}^n\}$ is an increasing sequence as in (3.3) and none of the sets $\mathbf{V}^n$ contains a leaf of $T$. We will momentarily see that each set $\mathbf{V}^n$ is finite. We denote by $\mathbf{X}^n$ the set of tiles in the decomposition of $T$ induced by $\mathbf{V}^n$. Then the sequence $\{\mathbf{X}^n\}$ forms a subdivision of $T$ in the sense of Definition 2.11 (see the discussion after (3.3)).

As we will see, this subdivision $\{\mathbf{X}^n\}$ has some good properties, in particular it is a quasi-visual subdivision of $T$ (see Definition 2.12). By choosing $\delta \in (0, 1)$

sufficiently small, we can also ensure that the points in $\mathbf{V}^{n+1}$ separate the points in $\mathbf{V}^n$ in a suitable way. We summarize this in the following statement.

**Proposition 6.1.** *Let $T$ be a uniformly branching trivalent quasiconformal tree with $\operatorname{diam}(T) = 1$. Let $\delta \in (0,1)$, and $\{\mathbf{V}^n\}_{n\in\mathbb{N}_0}$ be as in (6.1). Then the following statements are true:*

(i) *$\mathbf{V}^n$ is a finite set for each $n \in \mathbb{N}_0$.*

(ii) *$\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ is a quasi-visual subdivision of $T$.*

*Let $n \in \mathbb{N}_0$, $X \in \mathbf{X}^n$, and $\mathbf{V}_X := \mathbf{V}^{n+1} \cap \operatorname{int}(X)$. Then we have:*

(iii) *$\#\partial X \le 2$, and if we denote by $\mathbf{X}_X$ the decomposition of $X$ induced by $\mathbf{V}_X$, then $\mathbf{X}_X$ is edge-like.*

(iv) *There exists $N = N(\delta) \in \mathbb{N}$ depending on $\delta$, but independent of $n$ and $X$, such that $\#\mathbf{V}_X \le N$.*

*If $\delta \in (0,1)$ is sufficiently small (independent of $n$ and $X$), then we also have:*

(v) *$\#\mathbf{V}_X \ge 2$.*

(vi) *If $\#\partial X = 2$ and $\partial X = \{u, v\} \subset T$, then $(u, v) \cap \mathbf{V}_X$ contains at least three elements.*

We will see in the proof that $\{\mathbf{X}^n\}$ is in fact a quasi-visual subdivision that satisfies the (stronger) conditions of Lemma 2.4. The first part of (iii) implies that $\mathbf{X}^n$ is an edge-like decomposition of the whole tree $T$ for each $n \in \mathbb{N}_0$. It follows from (v) that if $\delta \in (0,1)$ is small enough, then we have $\mathbf{V}^n \ne \emptyset$ for each $n \in \mathbb{N}$. One can show based on (iv) that in (i) one has in fact a bound of the form $\#\mathbf{V}^n \le C^n$ for all $n \in \mathbb{N}$ with some constant $C > 1$.

*Proof.* (i) This is clear for $n = 0$, since $\mathbf{V}^0 = \emptyset$.

Let $n \in \mathbb{N}$, and $u, v \in \mathbf{V}^n$ be distinct points. Since $T$ has uniformly relatively separated branch points, by definition of $\mathbf{V}^n$ we have

$$|u - v| \gtrsim \min\{H_T(u), H_T(v)\} \ge \delta^n, \tag{6.2}$$

where $C(\gtrsim)$ is the constant in Definition 1.1 for the tree $T$. Since $T$ is compact, (6.2) implies that $\mathbf{V}^n$ is finite.

We now first prove (iii) before we establish (ii). To this end, consider $n \in \mathbb{N}_0$, $X \in \mathbf{X}^n$, $\mathbf{V}_X = \mathbf{V}^{n+1} \cap \operatorname{int}(X)$, and, as in the statement of (iii), let $\mathbf{X}_X$ be the decomposition of $X$ induced by $\mathbf{V}_X$.

(iii) To show that $\#\partial X \le 2$, we argue by contradiction and assume that there are three distinct points $x, y, z \in \partial X \subset X$. Then $x, y, z \in \partial X \subset \mathbf{V}^n$ by Lemma 3.3 (ii). We now consider the center of $x, y, z$ (see Lemma 3.6 (i)), namely the point $c \in T$ with

$$\{c\} = [x, y] \cap [y, z] \cap [z, x].$$

Since $X$ is a subtree of $T$, we have $c \in X$. By Lemma 3.6 (ii) this is a branch point of $T$ with

$$H_T(c) \ge \min\{H_T(x), H_T(y), H_T(z)\} \ge \delta^n.$$

So $c$ belongs to $\mathbf{V}^n$. At least two of the three points $x$, $y$, $z$ are distinct from $c$, say $x, y \ne c$. Then $c \in (x, y)$ and so $x$ and $y$ cannot lie in the same tile $X$ as follows from Lemma 3.1 (i) and Lemma 3.3 (i). This is a contradiction, showing that indeed $\#\partial X \le 2$.

Now consider a tile $Y \in \mathbf{X}_X$. Then $Y \subset X$, and $Y \in \mathbf{X}^{n+1}$ as follows from Lemma 3.4 (iii). If $\partial_X Y$ denotes the relative boundary of $Y$ as a subset of $X$, then

$\partial_X Y \subset \partial Y$ (see Lemma 3.4 (iv); the latter inclusion is actually true in every topological space). Since $Y \in \mathbf{X}^{n+1}$ we have $\#\partial Y \leq 2$ by the previous considerations (applied to $Y$), and so $\#\partial_X Y \leq 2$. This means that the decomposition $\mathbf{X}_X$ of $X$ induced by $\mathbf{V}_X$ is edge-like, and (iii) follows.

(ii) We now show that $\{\mathbf{X}^n\}$ is a quasi-visual subdivision of $T$. As we have already discussed in the beginning of this section, $\{\mathbf{X}^n\}$ is a subdivision of $T$ in the sense of Definition 2.11. It remains to show that it is a quasi-visual approximation (as in Definition 2.1). For this we will verify conditions (i) and (ii) in Lemma 2.4. Fix some $n \in \mathbb{N}_0$.

*Claim* 1. $\operatorname{diam}(X) \asymp \delta^n$ for all $X \in \mathbf{X}^n$, where $C(\asymp)$ is independent of $n$, $X$, and $\delta$.

Let $X \in \mathbf{X}^n$ be arbitrary. If $n = 0$, then $X = T$ and $\operatorname{diam}(T) = 1 = \delta^0$. So the statement is obviously true in this case.

Now assume $n \geq 1$, and let $u, v \in X$ be arbitrary distinct points. Then $(u, v) \cap \mathbf{V}^n = \emptyset$; otherwise, $u$ and $v$ could not be contained in the same tile $X$ as follows from Lemma 3.3 (i). Since $T$ has uniformly relatively dense branch points, there is a branch point $p$ of $T$ with $p \in (u, v)$ and

$$H_T(p) \gtrsim |u - v|,$$

where $C(\gtrsim)$ only depends on the constant in Definition 1.2 for $T$ (to see this, choose a branch point $p \in [u', v'] \subset (u, v)$ with $u', v' \in (u, v)$ very close to $u, v$, respectively). Then $p \notin \mathbf{V}^n$, and so

$$\delta^n > H_T(p) \gtrsim |u - v|.$$

It follows that $\operatorname{diam}(X) \lesssim \delta^n$ with $C(\lesssim)$ independent of $n$, $X$, and $\delta$.

In order to show an inequality in the opposite direction, recall that $\#\partial X \leq 2$ by (iii) and that $\partial X \subset \mathbf{V}^n$ by Lemma 3.3 (ii).

If $\partial X = \emptyset$, then $X = T$ and so $\operatorname{diam}(X) = 1 \geq \delta^n$.

Assume $\partial X \neq \emptyset$. If $\partial X$ consists of one point $p \in T$, then $p \in \mathbf{V}^n$ and so $p$ is a branch point of $T$ with $H_T(p) \geq \delta^n$. In this case, $X$ is a branch of $p$ in $T$ (see Lemma 3.3 (vi)), and so

$$\operatorname{diam}(X) \geq H_T(p) \geq \delta^n.$$

If $\partial X$ consists of two distinct points $u, v \in \mathbf{V}^n$, then by (6.2) we have

$$\operatorname{diam}(X) \geq |u - v| \gtrsim \delta^n,$$

where $C(\gtrsim)$ is the constant from Definition 1.1.

Claim 1 follows, and so condition (i) in Lemma 2.4 is true.

*Claim* 2. $\operatorname{dist}(X, Y) \gtrsim \delta^n$ for all $X, Y \in \mathbf{X}^n$ with $X \cap Y = \emptyset$, where $C(\gtrsim)$ is independent of $n$, $X$, $Y$, and $\delta$.

To see this second claim, fix arbitrary tiles $X, Y \in \mathbf{X}^n$ with $X \cap Y = \emptyset$. We can find points $x \in X$ and $y \in Y$ with $|x - y| = \operatorname{dist}(X, Y)$. Now consider the unique arc $[x, y]$ joining $x$ and $y$ in $T$. As we travel from $x \in X$ to $y \notin X$ along $[x, y]$, there is a last point $u \in [x, y]$ with $u \in X$. Then $(u, y]$ is non-empty and disjoint from $X$ which implies that $u \in \partial X \subset \mathbf{V}^n$. As we travel from $u \notin Y$ to $y \in Y$ along $[u, y] \subset [x, y]$, there is a first point $v \in [u, y]$ with $v \in Y$. Then $v \in \partial Y \subset \mathbf{V}^n$ by a similar reasoning.

The points $u, v \in \mathbf{V}^n$ are distinct, because $u \in X$, $v \in Y$, and $X \cap Y = \emptyset$. It follows that $|u - v| \gtrsim \delta^n$ by (6.2). Since $T$ is a quasiconformal tree and hence of

bounded turning, we have $\operatorname{diam}[x,y] \asymp |x-y|$. This implies that

$$\operatorname{dist}(X,Y) = |x-y| \asymp \operatorname{diam}[x,y] \geq |u-v| \gtrsim \delta^n.$$

Here all implicit multiplicative constants are independent of $n$, $X$, $Y$, and $\delta$. We have shown Claim 2, which is condition (ii) in Lemma 2.4.

Lemma 2.4 now implies that $\{\mathbf{X}^n\}$ is indeed a quasi-visual subdivision of $T$, and (ii) follows.

(iv) By (6.2) two distinct points $u, v \in \mathbf{V}_X \subset \mathbf{V}^{n+1}$ have separation $|u-v| \gtrsim \delta^{n+1}$. On the other hand, $\mathbf{V}_X$ is contained in $X$ with $\operatorname{diam}(X) \asymp \delta^n$, as we saw in Claim 1. Since $T$ is doubling, it follows that there is a constant $N \in \mathbb{N}$ with

$$\#\mathbf{V}_X \leq N.$$

Here $N$ depends on $\delta$, the doubling constant of $T$, and the constants in (6.2) and Claim 1 (which depend on the constants in Definition 1.1 and Definition 1.2), but not on $n$ and $X$.

Having verified (i)–(iv) for any $\delta \in (0,1)$, we now prove the remaining statements that require us to choose $\delta$ sufficiently small.

(v) By Claim 1 we have $\operatorname{diam}(X) \asymp \delta^n$ and $\operatorname{diam}(Y) \asymp \delta^{n+1}$ for each $Y \in \mathbf{X}^{n+1}$. Here the constant $C(\asymp)$ depends only on the constants in Definition 1.1 and Definition 1.2, as we have seen. This implies that there exists a constant $\delta_1 \in (0,1)$ independent of $n$ and $X$ with the following property: if $0 < \delta \leq \delta_1$ (as we now assume), then

$$\operatorname{diam}(Y) \leq \tfrac{1}{4} \operatorname{diam}(X) \tag{6.3}$$

for all $Y \in \mathbf{X}^{n+1}$.

It follows from Lemma 3.4 (iii) that the decomposition $\mathbf{X}_X$ of $X$ induced by $\mathbf{V}_X$ is given by all tiles $Y \in \mathbf{X}^{n+1}$ with $Y \subset X$. Then clearly $\mathbf{V}_X \neq \emptyset$, because otherwise $\mathbf{X}_X = \{X\} \subset \mathbf{X}^{n+1}$ which is impossible by (6.3).

To show that $\#\mathbf{V}_X \geq 2$, assume on the contrary that $\mathbf{V}_X = \mathbf{V}^{n+1} \cap \operatorname{int}(X)$ contains only one point $p$. This is a branch point of $T$ and hence a branch point of $X$, because $p \in \operatorname{int}(X)$ (see Lemma 3.2 (i)). Then the decomposition $\mathbf{X}_X$ of $X$ induced by $\mathbf{V}_X$ consists precisely of the three branches $Y_1, Y_2, Y_3$ of $p$ in $X$. Moreover, these branches are tiles in $\mathbf{X}^{n+1}$ and share the common point $p$. Then

$$\operatorname{diam}(X) \leq 2 \max\{\operatorname{diam}(Y_k) : k \in \{1,2,3\}\},$$

but this is impossible by (6.3).

We conclude that if $0 < \delta \leq \delta_1$, then necessarily $\#\mathbf{V}_X \geq 2$, and (v) follows.

(vi) Now assume that $\#\partial X = 2$ and $\partial X = \{u, v\}$. Then $u$ and $v$ are distinct points in $\partial X \subset \mathbf{V}^n$, and so $|u-v| \gtrsim \delta^n$ by (6.2). On the other hand, the points in $\mathbf{V}^{n+1} \cap [u,v]$ divide $[u,v] \subset X$ into non-overlapping arcs. Each of these arcs lies in a tile in $\mathbf{X}^{n+1}$, and so has diameter $\lesssim \delta^{n+1}$ by Claim 1. If there are $k \in \mathbb{N}$ of these arcs it follows that

$$\delta^n \lesssim |u-v| \lesssim k\delta^{n+1},$$

where the constants $C(\lesssim)$ do not depend on $n$, $X$, or $\delta$. If $\delta$ is small enough, say $0 < \delta \leq \delta_2$, where $\delta_2 \in (0,1)$ can be chosen independently of $n$ and $X$, then necessarily $k \geq 4$, and so $\mathbf{V}^{n+1} \cap (u,v)$ contains at least three elements. Note that $(u,v) \subset X \setminus \partial X = \operatorname{int}(X)$. Thus $\mathbf{V}_X \cap (u,v) = \mathbf{V}^{n+1} \cap (u,v)$ contains at least three elements, and (vi) follows.

If we choose $0 < \delta \le \min\{\delta_1, \delta_2\}$, where $\delta_1, \delta_2 \in (0,1)$ are the constants from (v) and (vi), respectively, then (v) and (vi) are both true. This finishes the proof. □

## 7. Quasi-visual subdivisions of the CSST

Let $T$ be a uniformly branching trivalent quasiconformal tree. Then it is possible to show that for every quasi-visual subdivision of $T$ there is an isomorphic quasi-visual subdivision of the CSST $\mathbb{T}$. From this one can deduce the quasisymmetric equivalence of $T$ and $\mathbb{T}$ by Proposition 2.13.

We will actually prove a slightly less general statement, namely we restrict ourselves to a quasi-visual subdivision $\{\mathbf{X}^n\}$ of $T$ as in Proposition 6.1. This is formulated in the following proposition, which is the main result of this section.

**Proposition 7.1.** *Let $T$ be a uniformly branching trivalent quasiconformal tree with $\operatorname{diam}(T) = 1$. Suppose $\mathbf{V}^n$, for $n \in \mathbb{N}_0$, is as in (6.1), with $\delta \in (0,1)$ so small that all the statements in Proposition 6.1 are true for the decompositions $\mathbf{X}^n$ of $T$ induced by the sets $\mathbf{V}^n$. Then there exists a quasi-visual subdivision $\{\mathbf{Y}^n\}_{n\in\mathbb{N}_0}$ of $\mathbb{T}$ that is isomorphic to the quasi-visual subdivision of $\{\mathbf{X}^n\}_{n\in\mathbb{N}_0}$ of $T$.*

Recall that the notion of isomorphic subdivisions was based on (2.14) and (2.15). By applying Proposition 2.13, we see that there exists a quasisymmetric homeomorphism $F\colon T \to \mathbb{T}$ that induces the isomorphism between the quasi-visual subdivision $\{\mathbf{X}^n\}$ of $T$ and the quasi-visual subdivision $\{\mathbf{Y}^n\}$ of $\mathbb{T}$. In particular, $T$ and $\mathbb{T}$ are quasisymmetrically equivalent. This gives the "if" implication in Theorem 1.4.

The proof of Proposition 7.1 requires some preparation. We first study homeomorphisms that send tiles in a decomposition of $T$ to tiles in $\mathbb{T}$.

**Decompositions of trees and tile-homeomorphisms.** By Proposition 6.1 (iii) the decompositions $\mathbf{X}^n$ of $T$ as in Proposition 7.1 are edge-like in the sense that for each tile $X \in \mathbf{X}^n$ its boundary is the empty set (only if $X = T$), or contains one point (then $X$ is a leaf-tile), or two points (then $X$ is an edge-tile). The points in $\partial X$ play a special role. They are leaves of $X$ and the only points where $X$ intersects other tiles of the same level (see Lemma 3.3 (iii) and (iv)). Accordingly, we say that the points in $\partial X$ are *marked leaves* of $X$.

Recall that $X \in \mathbf{X}^n$ (viewed as a tree) has an edge-like decomposition $\mathbf{X}_X$ induced by $\mathbf{V}_X = \mathbf{V}^{n+1} \cap \operatorname{int}(X)$ (see Proposition 6.1 (iii)). We want to find a decomposition of $\mathbb{T}$ into tiles in $\mathbb{X}^*$ that is isomorphic to $\mathbf{X}_X$ and respects the marked leaves. To this end, we consider $-1$ or $1$ as a marked leaf of $\mathbb{T}$ if $\#\partial X = 1$, or both $-1$ and $1$ as marked leaves of $\mathbb{T}$ if $\#\partial X = 2$.

To formulate a corresponding statement, we consider an arbitrary trivalent tree $T$ (instead of $X$) with a dense set of branch points. The following theorem allows us to construct homeomorphisms between $T$ and $\mathbb{T}$ that map leaves in a prescribed way.

**Theorem 7.2.** *Let $T$ be a trivalent metric tree whose branch points are dense in $T$. Then $T$ is homeomorphic to $\mathbb{T}$. Moreover, if $p$, $q$, $r$ are three distinct leaves of $T$, then there exists a homeomorphism $F\colon T \to \mathbb{T}$ such that $F(p) = -1$, $F(q) = 1$, $F(r) = i$.*

The first part follows from [BT21, Theorem 1.7] and the second part from [BT21, Theorem 5.4]. Note that $-1$, $1$, $i$ are leaves of $\mathbb{T}$ (see the discussion after [BT21, Theorem 5.4]; that $-1$ and $1$ are leaves of $\mathbb{T}$ was already pointed out in

Lemma 5.1 (ii)). If $T$ is as in Theorem 7.2, then $T$ is homeomorphic to $\mathbb{T}$, and so $T$ has at least three leaves (actually infinitely many). The second part of the theorem says that any three leaves of $T$ can be sent to the prescribed leaves $-1$, $1$, $i$ of the CSST by a homeomorphism between $T$ and $\mathbb{T}$. We will apply this in a weaker form where we send at most two leaves of $T$ to $-1$ and $1$.

Now let $\mathbf{X}$ be an edge-like decomposition of $T$ induced by a finite set of branch points $\mathbf{V}$ of $T$. We say that a homeomorphism $F\colon T \to \mathbb{T}$ is a *tile-homeomorphism* (for $\mathbf{X}$) if $F(X) \in \mathbb{X}^*$ (see (5.7)) for each $X \in \mathbf{X}$. Then $F$ maps tiles in $T$ to tiles in $\mathbb{T}$, and so the level $\ell(F(X))$ (of $F(X)$ as a tile of $\mathbb{T}$) is defined for each $X \in \mathbf{X}$. The following statement is a basic existence result for tile-homeomorphisms.

**Lemma 7.3.** *Let $T$ be a trivalent metric tree whose branch points are dense in $T$. Suppose $\mathbf{V} \subset T$ is a finite set of branch points of $T$ that induces an edge-like decomposition of $T$ into the set of tiles $\mathbf{X}$.*

*We assume that either $T$ has no marked leaves, one marked leaf $p \in T$, or two marked leaves $p, q \in T$, $p \neq q$. If $\mathbf{V} \neq \emptyset$, we also assume that the marked leaf $p$ (when present) lies in a leaf-tile $P \in \mathbf{X}$, and the other marked leaf $q$ (when present) lies in a leaf-tile $Q \in \mathbf{X}$ distinct from $P$.*

*Then there exists a tile-homeomorphism $F\colon T \to \mathbb{T}$ for $\mathbf{X}$ such that the following statements are true:*

(i) *$F(p) = -1$, or alternatively, $F(p) = 1$, when $T$ has one marked leaf $p$; or $F(p) = -1$ and $F(q) = 1$, when $T$ has two marked leaves $p$ and $q$.*

(ii) *If $T$ has one marked leaf $p$ and $\#\mathbf{V} \geq 2$, then we may also assume that $F$ satisfies $\ell(F(P)) = 2$.*

(iii) *If $T$ has two marked leaves $p, q \in T$ and $[p, q] \cap \mathbf{V}$ contains at least three points, then we may also assume that $F$ satisfies $\ell(F(P)) = \ell(F(Q) = 2$.*

(iv) *For each $X \in \mathbf{X}$ we have $\ell(F(X)) \leq \#\mathbf{V}$, and, if $\mathbf{V} \neq \emptyset$, also $\ell(F(X)) \geq 1$.*

If we are in the situation of (ii) or (iii), the homeomorphism $F$ sends the leaf-tiles containing the marked leaves of $T$ to 2-tiles in $\mathbb{T}$. Actually, in (ii) we have $F(P) = \mathbb{T}_{11}$ or $F(P) = \mathbb{T}_{22}$ depending on whether $F(p) = -1$ or $F(p) = 1$, and in (iii) we have $F(P) = \mathbb{T}_{11}$ and $F(Q) = \mathbb{T}_{22}$. It is important that these images of $P$ and $Q$ have fixed diameter $1/2$. This (seemingly technical) condition will be crucial to ensure that the subdivision of $\mathbb{T}$ given in Proposition 7.1 is quasi-visual. In particular, we use it to show that neighboring tiles have comparable diameter, as required by Definition 2.1 (i). This is the reason why we constructed decompositions of $T$ satisfying the corresponding conditions (v) and (vi) in Proposition 6.1.

The tile-homeomorphism $F\colon T \to \mathbb{T}$ in Lemma 7.3 is merely a convenient device that allows us to transfer information on leaves and tiles from $T$ to $\mathbb{T}$. The main point is that we can find a decomposition of $\mathbb{T}$ into tiles (in $\mathbb{X}^*$) isomorphic to the given edge-like decomposition of $T$.

*Proof of Lemma 7.3.*

(i) Depending on the number of marked leaves of $T$ and the desired normalization for $F$, we distinguish several cases in the ensuing discussion:

*Case* 1. $T$ has no marked leaf.

*Case* 2a. $T$ has one marked leaf $p$, and the desired normalization is $F(p) = -1$.

*Case* 2b. $T$ has one marked leaf $p$, and the desired normalization is $F(p) = 1$.

*Case* 3. $T$ has two marked leaves $p$ and $q$, and the desired normalization is $F(p) = -1$ and $F(q) = 1$.

We now prove the statement by induction on $\#\mathbf{V} \geq 0$. In the proof, Case 1 is the easiest to handle, since we do not have to worry about normalizations, but we will have to make some careful choices in Cases 2a, 2b, 3 due to the presence of marked leaves.

Suppose first that $\#\mathbf{V} = 0$, and so $\mathbf{V} = \emptyset$. We invoke Theorem 7.2 to obtain a homeomorphism $F\colon T \to \mathbb{T}$, where we can impose the normalization $F(p) = -1$ or $F(p) = 1$ when $T$ has one marked leaf $p$ (Cases 2a and 2b), or the normalization $F(p) = -1$ and $F(q) = 1$ when $T$ has two marked leaves $p$ and $q$ (Case 3).

Then $F$ is a tile-homeomorphism for $\mathbf{X}$, since there is only one tile $X = T$; it is mapped to $F(T) = \mathbb{T}$, which is a tile of $\mathbb{T}$ (the 0-tile).

For the inductive step suppose $\#\mathbf{V} \geq 1$ and that the statement is true for all marked trees homeomorphic to $\mathbb{T}$ with decompositions induced by sets of branch points with fewer than $\#\mathbf{V}$ elements.

Since $\#\mathbf{V} \geq 1$, we have $\mathbf{V} \neq \emptyset$. We choose a (branch) point $x \in \mathbf{V}$. If $T$ has marked leaves, then they are distinct from the branch point $x$. Moreover, if $T$ is marked by two leaves $p$ and $q$ (Case 3), we may assume in addition that they lie in distinct components of $T \setminus \{x\}$. Indeed, we may then choose any point $x \in (p, q) \cap \mathbf{V}$. The latter set is non-empty as follows from the fact that $p$ and $q$ are contained in distinct leaf-tiles $P, Q \in \mathbf{X}$ by our assumptions.

Let $B_1$, $B_2$, $B_3$ be the branches of $x$ in $T$. Here we may assume that $p \in B_1$ in Case 2a, $p \in B_2$ in Case 2b, and $p \in B_1$, $q \in B_2$ in Case 3. Note that these branches are precisely the tiles in the decomposition of $T$ induced by $\{x\}$.

The idea for the proof is now to apply the induction hypothesis to each of the trees $B_k$ with suitable markings and copy the image of the resulting homeomorphism into the subtree $\mathbb{T}_k$ of $\mathbb{T}$ by the map $g_k$ (as in (5.1)). If we assemble these maps into one, we obtain the desired homeomorphism $F$.

First, note that Lemma 3.3 (viii) implies that each branch $B_k$ is also a trivalent metric tree with a dense set of branch points. The (possibly empty) set

$$\mathbf{V}_k \coloneqq \mathbf{V} \cap \operatorname{int}(B_k) = \mathbf{V} \cap (B_k \setminus \{x\})$$

consists of branch points of $B_k$, and $\#\mathbf{V}_k < \#\mathbf{V}$ (since $\mathbf{V}_k \subset \mathbf{V} \setminus \{x\}$).

The set $\mathbf{V}_k$ induces a decomposition of $B_k$. Its set of tiles $\mathbf{X}_k$ consists precisely of the tiles $X \in \mathbf{X}$ with $X \subset B_k$ (see Lemma 3.4 (iii)). We denote the relative boundary of $X \in \mathbf{X}_k$ in $B_k$ by $\partial_k X$. Then, using Lemma 3.4 (iv), we have

$$\partial_k X = \partial X \setminus \{x\}, \tag{7.1}$$

and in particular, $\#\partial_k X \leq \#\partial X \leq 2$ for each $X \in \mathbf{X}_k$. This implies that $\mathbf{X}_k$ is an edge-like decomposition of $B_k$.

Since $x \in \partial B_k$, this point is a leaf of $B_k$ by Lemma 3.3 (iii). Moreover, each marked leaf of $T$ (when present) is a leaf of the unique branch $B_k$ that contains it, as follows from Lemma 3.2 (ii). We now mark:

$B_1$ by $x$ and, in addition, by $p$ in Cases 2a and 3;

$B_2$ by $x$ and, in addition, by $p$ in Case 2b, and by $q$ in Case 3;

$B_3$ by $x$.

This means that each marked branch $B_k$ is marked by its leaf $x$ and in addition by any marked leaf of $T$ that $B_k$ may contain.

*Claim.* Let $k \in \{1,2,3\}$ and suppose $B_k$ is marked by its leaf $x$ and possibly one other of its leaves as indicated. If $\mathbf{V}_k \neq \emptyset$, then $x$ lies in a leaf-tile $X \in \mathbf{X}_k$. If $B_k$ has another marked leaf, then this leaf lies in a leaf-tile $Y \in \mathbf{X}_k$ distinct from $X$.

In other words, each marked branch $B_k$ with the decomposition $\mathbf{X}_k$ induced by $\mathbf{V}_k$ satisfies the hypotheses of the lemma and we can apply the induction hypothesis.

To prove the Claim, first observe that there exist precisely three distinct tiles $X_1, X_2, X_3 \in \mathbf{X}$ that contain the triple point $x$. With suitable labeling we have $X_k \subset B_k$ for $k = 1,2,3$ (see Lemma 3.3 (v)).

Fix $k \in \{1,2,3\}$ and suppose $\mathbf{V}_k \neq \emptyset$ as in the Claim. We now set $X := X_k \in \mathbf{X}_k$. Then $\partial X \setminus \{x\} = \partial_k X \neq \emptyset$ by (7.1) and Lemma 3.3 (iii). On the other hand, $x \in \partial X$ by Lemma 3.3 (iv), and $\#\partial X \leq 2$, because $\mathbf{X}$ is an edge-like decomposition of $T$. We conclude that $\partial X$ contains precisely two distinct points, one of which is $x$. So $X$ is an edge-tile in $\mathbf{X}$. Then $\partial_k X = \partial X \setminus \{x\}$ is a singleton set, and so $X$ is a leaf-tile in $\mathbf{X}_k$ that contains $x$. This shows the first part of the Claim.

Suppose $B_k$ has another marked leaf apart from $x$. This is only possible if $k \in \{1,2\}$. We assume $k = 1$; the case $k = 2$ is very similar and we skip the details.

Then $p$ is a marked leaf of $B_k = B_1$. By our assumptions, $p$ is contained in a leaf tile $P \in \mathbf{X}$. By Lemma 3.3 (i) the tile $P$ is contained in precisely one of the branches of $x$ in $T$. This branch must be $B_1$, because $p \in P$ is belongs to $B_1$, but not to $B_2$ or $B_3$. We conclude that $P \subset B_1$ and so $P \in \mathbf{X}_1$.

Since $\partial_k P \neq \emptyset$ and $\#\partial_k P \leq \#\partial P = 1$, the set $\partial_k P$ consists of a single point. This means $P$ is a leaf-tile in $\mathbf{X}_1$ that contains $p$. We have $P \neq X$, because $X$ is an edge-tile and $P$ is a leaf-tile in $\mathbf{X}$. So if we set $Y = P$, then we see that the second part of the Claim is also true.

By the Claim we can apply the induction hypothesis to $B_k$, marked by leaves as above, and its edge-like decomposition $\mathbf{X}_k$ induced by $\mathbf{V}_k$ (recall that $\#\mathbf{V}_k < \#\mathbf{V}$). Then we can find tile-homeomorphisms $F_k \colon B_k \to \mathbb{T}$ for $\mathbf{X}_k$ that are normalized by

$$\begin{aligned}
F_1(x) &= \phantom{-}1, \text{ and, in addition, } F_1(p) = -1 \text{ in Cases 2a and 3,}\\
F_2(x) &= -1, \text{ and, in addition, } F_2(p) = \phantom{-}1 \text{ in Case 2b,}\\
F_2(x) &= -1, \text{ and, in addition, } F_2(q) = \phantom{-}1 \text{ in Case 3,}\\
F_3(x) &= -1.
\end{aligned}$$

Recall that these cases were defined at the beginning of the proof. They refer to the markings of $T$ (and not of $B_k$). Case 1 is included here, because then the statements about $p$ and $q$ are void. These normalizations are modeled on how $g_1^{-1}, g_2^{-1}, g_3^{-1}$ map $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$, marked by 0 and possibly $-1$ and/or 1, to $\mathbb{T}$ (see Figure 3).

We now define

$$F \colon T = B_1 \cup B_2 \cup B_3 \to \mathbb{T} \text{ by setting } F(z) = g_k(F_k(z)) \tag{7.2}$$

if $z \in B_k$ for $k \in \{1,2,3\}$. This is well-defined, because $z = x$ is the only point contained in more than one branch and for which multiple definitions apply; but

we have

$$\begin{aligned} g_1(F_1(x)) &= g_1(1) \quad = 0, \\ g_2(F_2(x)) &= g_2(-1) = 0, \\ g_3(F_3(x)) &= g_3(-1) = 0. \end{aligned}$$

So the definitions are consistent and $F(x) = 0$.

The map $F$ sends $T$ onto $\mathbb{T}$, because

$$F(T) = \bigcup_{k=1}^{3} g_k(F_k(B_k)) = \bigcup_{k=1}^{3} g_k(\mathbb{T}) = \mathbb{T}.$$

Since $F|B_k = g_k \circ F_k$ is continuous for $k = 1, 2, 3$, the map $F$ is continuous. On each set $B_k$ the map $F|B_k = g_k \circ F_k$ is injective. Moreover, $F$ sends the pairwise disjoint sets

$$\{x\},\ B_1 \setminus \{x\},\ B_2 \setminus \{x\},\ B_3 \setminus \{x\}$$

onto the pairwise disjoint sets

$$\{0\},\ \mathbb{T}_1 \setminus \{0\},\ \mathbb{T}_2 \setminus \{0\},\ \mathbb{T}_3 \setminus \{0\},$$

respectively.

This implies that $F$ is injective, and hence a continuous bijection $F\colon T \to \mathbb{T}$. Since $T$ is compact, $F$ is actually a homeomorphism. It has the desired normalization, because

$$\begin{aligned} F(p) &= g_1(F_1(p)) = g_1(-1) = -1 \text{ in Cases 2a and 3}, \\ F(p) &= g_2(F_2(p)) = g_2(1) = 1 \text{ in Case 2b}, \\ F(q) &= g_2(F_2(q)) = g_2(1) = 1 \text{ in Case 3}. \end{aligned}$$

It remains to show that $F$ is a tile-homeomorphism for $\mathbf{X}$. Let $X \in \mathbf{X}$ be arbitrary. Then $X$ is contained in one of the branches $B_k$, and so is a tile in $\mathbf{X}_k$. Since $F_k$ is a tile-homeomorphism for $\mathbf{X}_k$, we know that $F_k(X)$ is a tile of $\mathbb{T}$. This implies that $F(X) = g_k(F_k(X))$ is a tile of $\mathbb{T}$, because $g_k$ sends tiles of $\mathbb{T}$ to tiles of $\mathbb{T}$.

This completes the inductive step and the statement follows. The argument actually shows that the tile-homeomorphism $F\colon T \to \mathbb{T}$ with the desired normalization can be constructed so that $F(x) = 0$ with $x \in \mathbf{V}$ arbitrary, when $T$ has one marked leaf, or with $x \in (p, q) \cap \mathbf{V}$ arbitrary, when $T$ has two marked leaves $p$ and $q$.

We now first establish (iii), before we show (ii).

(iii) Let $p'$ and $q'$ be the first, respectively the last, point in $(p, q) \cap \mathbf{V} \neq \emptyset$ as we travel from $p$ to $q$ along $[p, q]$. Then $[p, p']$ is contained in a tile in $\mathbf{X}$; this tile must be $P$, because it is the only tile in $\mathbf{X}$ that contains $p$ (this follows from Lemma 3.3 (iv)). We see that $p' \in \mathbf{V} \cap P = \partial P$. Since $P$ is a leaf-tile, we have $\partial P = \{p'\}$. Similarly, $\partial Q = \{q'\}$.

There exists a point $x \in (p', q') \cap \mathbf{V}$ by our assumptions. As in the inductive step in (i), we decompose $T$ into the three branches $B_1$, $B_1$, $B_3$ with this choice of $x$. Again we may assume $p \in B_1$ and $q \in B_2$. Note that then $p' \in (p, x) \subset B_1$, and so $p' \in \mathbf{V}_1 = \mathbf{V} \cap (B_1 \setminus \{x\})$. Similarly, $q' \in (x, q)$ and $q' \in \mathbf{V}_2 = \mathbf{V} \cap (B_2 \setminus \{x\})$. We now choose tile-homeomorphisms $F_k\colon B_k \to \mathbb{T}$ for $\mathbf{X}_k$ as before, but by the remark

at the end of the proof of (i), we can do this so that $F_1(p') = 0$ and $F_2(q') = 0$. Then the map $F$ as defined in (7.2) satisfies

$$F(p') = g_1(F_1(p')) = g_1(0) = -1/2,$$
$$F(q') = g_2(F_2(q')) = g_2(0) = 1/2.$$

Since $p \in P$, $p' \in \partial P$, and $F$ is a tile-homeomorphism, we have $Z := F(P) \in \mathbb{X}^*$, $-1 = F(p) \in Z$, and $-1/2 = F(p') \in \partial Z$. Lemma 5.6 implies that $Z = F(P) = \mathbb{T}_{11}$, which is a tile of level 2. By a similar reasoning, $F(Q) = \mathbb{T}_{22}$, which is again of level 2. The statement follows.

(ii) This is a slight variant of the argument for (iii). The leaf-tile $P \in X$ with $p \in P$ is the unique tile in $\mathbf{X}$ that contains $p$. Its boundary $\partial P \subset \mathbf{V}$ consists of a single point $p' \in \mathbf{V}$. Now we consider the three tiles containing $p' \in \mathbf{V}$. One of them is $P$, but not all three can be leaf-tiles, because then $\#\mathbf{V} = 1$. So there exists an edge-tile $X$ with $p' \in \partial X$. Then $\partial X \subset \mathbf{V}$ contains another point $x \in \mathbf{V}$ distinct from $p'$. As one travels along the arc $[p, x]$ starting from $p$, one exits $P$, but this is only possible through $p'$ which is the only boundary point of $P$. Hence $p' \in (p, x)$. We now use a construction as in (i) for the branch point $x \in \mathbf{V}$ and its three branches $B_1$, $B_2$, $B_3$.

To obtain the first normalization, we may assume that $p \in B_1$. Since $p' \in (p, x)$, we can choose the homeomorphism $F_1 \colon B_1 \to \mathbb{T}$ so that $F_1(p) = -1$ and $F_1(p') = 0$. Then, as before, we obtain a tile-homeomorphism $F \colon T \to \mathbb{T}$ for $\mathbf{X}$. It satisfies

$$F(p) = g_1(F_1(p)) = g_1(-1) = -1,$$
$$F(p') = g_1(F_1(p')) = g_1(0) = -1/2.$$

Using again Lemma 5.6, we see that $F(P) = \mathbb{T}_{11}$, and so $F(P)$ is a tile in $\mathbb{T}$ of level 2.

To obtain the second normalization, we may assume that $p \in B_2$ and now choose $F_2 \colon B_2 \to \mathbb{T}$ so that $F_2(p) = 1$ and $F_2(p') = 0$. Then

$$F(p) = g_2(F_2(p)) = g_2(1) = 1,$$
$$F(p') = g_2(F_2(p')) = g_2(0) = 1/2.$$

From Lemma 5.6 we see that $F(P) = \mathbb{T}_{22}$, which is a tile of level 2. We have proved (ii).

(iv) Let $\mathbb{V} := F(\mathbf{V}) \subset \mathbb{T}$. Since $F \colon T \to \mathbb{T}$ is a homeomorphism, $\mathbb{V}$ is a set of branch points of $\mathbb{T}$ that induces a decomposition of $\mathbb{T}$ into the set of tiles $\mathbb{X} = F(\mathbf{X})$. Since $F$ is a tile-homeomorphism, we have $\mathbb{X} \subset \mathbb{X}^*$. The statement now immediately follows from Lemma 5.5.

The proof is complete. □

**Subdividing $\mathbb{T}$.** After these preparations, we are now ready to prove Proposition 7.1, the main result in this section.

The desired quasi-visual subdivision $\{\mathbf{Y}^n\}$ will be constructed inductively from auxiliary tile-homeomorphisms $F^n \colon T \to \mathbb{T}$ for $\mathbf{X}^n$, $n \in \mathbb{N}_0$. We then set

$$\mathbf{Y}^n := F^n(\mathbf{X}^n) = \{F^n(X) : X \in \mathbf{X}^n\}.$$

Recall that if $F^n$ is a tile-homeomorphism, then for each tile $X \in \mathbf{X}^n$ we have $F^n(X) \in \mathbb{X}^*$, and so $\mathbf{Y}^n \subset \mathbb{X}^*$ (meaning that $\mathbf{Y}^n$ is a set of tiles in $\mathbb{T}$).

The next map $F^{n+1} \colon T \to \mathbb{T}$ "refines" $F^n$ in the sense that for $X \in \mathbf{X}^n$ we have $F^{n+1}(X) = F^n(X)$. Thus, for $X \in \mathbf{X}^n$ and $X' \in \mathbf{X}^{n+1}$ with $X' \subset X$, we have

$F^{n+1}(X') \subset F^{n+1}(X) = F^n(X)$ in $\mathbb{T}$. This will ensure that $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$ are isomorphic subdivisions according to (2.14) and (2.15).

We will show that $\{\mathbf{Y}^n\}$ is actually a quasi-visual subdivision of $\mathbb{T}$. Then by Proposition 2.13 there exists a quasisymmetry $F\colon T \to \mathbb{T}$ that induces the isomorphism between $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$. In fact, one can show that the sequence $\{F^n\}$ converges uniformly to this quasisymmetry $F$; we will leave the easy argument for this to the reader.

Note that for a given tile $X \in \mathbf{X}^n$, the level of $F^n(X) \in \mathbb{X}^*$, i.e., $\ell(F^n(X))$, will in general be different from $n$ (the level of $X$ in $T$). Moreover, if $X, Y \in \mathbf{X}^n$, the levels of the tiles $F(X)$ and $F(Y)$ (in $\mathbb{T}$) may be different.

*Proof of Proposition 7.1.* By our assumptions $T$ is a uniformly branching trivalent quasiconformal tree, and the set $\mathbf{V}^n$, for $n \in \mathbb{N}_0$, is as in (6.1), with $\delta \in (0,1)$ so small that all the statements in Proposition 6.1 are true for the edge-like decompositions $\mathbf{X}^n$ of $T$ induced by $\mathbf{V}^n$. In particular, $\mathbf{V}^0 = \emptyset$, and the sets $\mathbf{V}^n$ form an increasing sequence of branch points as in (3.3). The sequence $\{\mathbf{X}^n\}$ is a subdivision of $T$. Condition (v) in Proposition 6.1 implies that $\mathbf{V}^n \neq \emptyset$ for $n \in \mathbb{N}$.

For $n \in \mathbb{N}_0$ we will now inductively construct homeomorphisms $F^n\colon T \to \mathbb{T}$ with the following properties:

(A) For each $n \in \mathbb{N}_0$ the map $F^n\colon T \to \mathbb{T}$ is a tile-homeomorphism for $\mathbf{X}^n$, i.e., $F^n$ is a homeomorphism from $T$ onto $\mathbb{T}$ such that $F^n(X) \in \mathbb{X}^*$ for $X \in \mathbf{X}^n$.

In addition, for all $n \in \mathbb{N}$ and $X \in \mathbf{X}^{n-1}$, the following statements are true:

(B) The maps $F^{n-1}$ and $F^n$ are compatible in the sense that
$$F^{n-1}(X) = F^n(X).$$

(C) There exists a constant $N \in \mathbb{N}$ independent of $n$ and $X$ with the following property: if $Y \in \mathbf{X}^n$ with $Y \subset X$, then
$$\ell(F^{n-1}(X)) + 1 \le \ell(F^n(Y)) \le \ell(F^{n-1}(X)) + N.$$

(D) If $Y \in \mathbf{X}^n$ with $Y \subset X$ and $Y \cap \partial X \neq \emptyset$, then
$$\ell(F^n(Y)) = \ell(F^{n-1}(X)) + 2.$$

The constant $N \in \mathbb{N}$ in (C) will be the same as in statement (iv) in Proposition 6.1, which is true by our assumptions. Statement (C) means that the level of the tile $Y' = F^n(Y) \in \mathbb{X}^*$ is strictly, but only by a bounded amount, larger than the level of $X' = F^{n-1}(X) \in \mathbb{X}^*$. In particular, $2^{-N}\operatorname{diam}(X') \le \operatorname{diam}(Y') \le \frac{1}{2}\operatorname{diam}(X')$. In (D) we actually have $\operatorname{diam}(Y') = \frac{1}{4}\operatorname{diam}(X')$.

*Construction of $F^0$ and $F^1$.* Since $\mathbf{V}^0 = \emptyset$, we have $\mathbf{X}^0 = \{T\}$. We know that $T$ is homeomorphic to $\mathbb{T}$ (by Theorem 7.2) and so we can choose a homeomorphism $F^0\colon T \to \mathbb{T}$. Obviously, $F^0$ is a tile-homeomorphism for $\mathbf{X}^0$, since $F^0(T) = \mathbb{T} = \mathbb{T}_\emptyset$ is a tile of $\mathbb{T}$.

Note that $\mathbf{V}_T = \mathbf{V}^1 \cap \operatorname{int}(T) = \mathbf{V}^1$ and the decomposition $\mathbf{X}^1$ of $T$ induced by $\mathbf{V}^1$ is edge-like according to Proposition 6.1 (iii).

So we may apply Lemma 7.3 to $T$ and $\mathbf{V}^1$, where no leaves of $T$ are marked. This gives us a tile-homeomorphism $F^1\colon T \to \mathbb{T}$ for $\mathbf{X}^1$. Then (A) is true for $n = 0$ and $n = 1$. To verify (B)–(D) for $n = 1$ note that $X = T \in \mathbf{X}^0$ is the only 0-tile. Since $F^0(T) = \mathbb{T} = F^1(T)$, we obviously have (B).

Recall that $\mathbf{V}^1 \neq \emptyset$ and let $N \in \mathbb{N}$ be the number from Proposition 6.1 (iv). By definition of $N$ we then have $\#\mathbf{V}^1 \leq N$ for the number of 1-vertices in the 0-tile $T$. Together with Lemma 7.3 (iv) this implies

$$1 \leq \ell(F^1(Y)) \leq \#\mathbf{V}^1 \leq N$$

for all $Y \in \mathbf{X}^1$. Since $\ell(F^0(X)) = \ell(\mathbb{T}) = 0$, (C) holds for $n = 1$. Moreover, (D) is vacuously true for $n = 1$, since $\partial T = \emptyset$. Thus, with the maps $F^0$ and $F^1$ as above, conditions (A)–(D) are true for $n = 1$.

For the inductive step, suppose that for some $n \in \mathbb{N}$ maps $F^{n-1}$ and $F^n$ with the properties (A)–(D) have been defined. The map $F^{n+1}$ will be constructed separately on each tile $X \in \mathbf{X}^n$. The idea is to modify the map $F^n|X$ to a homeomorphism $F_X$ of $X$ onto the same image $F^n(X)$, and "glue" the maps $\{F_X\}$, $X \in \mathbf{X}^n$, together to obtain $F^{n+1}$.

*Constructing $F^{n+1}$ on an $n$-tile $X$.* Fix a tile $X \in \mathbf{X}^n$. Define $\mathbf{V}_X := \mathbf{V}^{n+1} \cap \operatorname{int}(X)$. Then $2 \leq \#\mathbf{V}_X \leq N$ and $\mathbf{V}_X$ induces an edge-like decomposition of $X$ into tiles, denoted by $\mathbf{X}_X$ (see (iii), (iv), and (v) in Proposition 6.1). The set $\mathbf{X}_X$ consists precisely of the tiles $Y \in \mathbf{X}^{n+1}$ with $Y \subset X$ (see Lemma 3.4 (iii)). We want to construct a homeomorphism $F_X \colon X \to F^n(X)$ with the following properties:

(a) The set $F_X(Y)$ is a tile of $\mathbb{T}$ for each $Y \in \mathbf{X}_X$.

(b) $F_X(X) = F^n(X)$ and $F_X|\partial X = F^n|\partial X$.

(c) For each $Y \in \mathbf{X}_X$, we have

$$\ell(F^n(X)) + 1 \leq \ell(F_X(Y)) \leq \ell(F^n(X)) + N.$$

(d) If $Y \in \mathbf{X}_X$ with $Y \cap \partial X \neq \emptyset$, then

$$\ell(F_X(Y)) = \ell(F^n(X)) + 2.$$

Note that the properties (a)–(d) correspond to the properties (A)–(D). The construction of $F_X$ is slightly different depending on whether $X$ is an edge-tile or a leaf-tile. We will discuss the details when $X$ is an edge-tile and will comment on the modifications when it is a leaf-tile.

*Case* 1. $X$ is an edge-tile. Then $\#\partial X = 2$, say $\partial X = \{p, q\} \subset \mathbf{V}^n$. Since $F^n$ is a tile-homeomorphism for $\mathbf{X}^n$, the set $F^n(X)$ is a tile of $\mathbb{T}$, say $F^n(X) = \mathbb{T}_w = g_w(\mathbb{T})$ with $w \in \mathcal{A}^*$ (see (5.5)). Since $F^n$ is a homeomorphism, we have

$$\partial \mathbb{T}_w = \partial F^n(X) = F^n(\partial X) = F^n(\{p, q\}).$$

In particular, $\mathbb{T}_w = g_w(\mathbb{T})$ has two boundary points and so $F^n(\{p, q\}) = \partial \mathbb{T}_w = \{g_w(-1), g_w(1)\}$ by Lemma 5.2 (i). We may assume that

$$F^n(p) = g_w(-1) \text{ and } F^n(q) = g_w(1). \tag{7.3}$$

By our assumptions, the open arc $(p, q)$ contains at least three points in $\mathbf{V}_X$ (see Proposition 6.1 (vi)). In particular, $p$ and $q$ lie in distinct tiles $P, Q \in \mathbf{X}^{n+1}$, respectively, with $P, Q \subset X$. These are leaf-tiles in $\mathbf{X}_X$. Indeed, $P$ has exactly two boundary points in $T$, namely $p$ and one boundary point in $\mathbf{V}_X$ (if not, then $X = P$, which is impossible since $\mathbf{V}_X \neq \emptyset$). It follows that the relative boundary of $P$ in $X$, which is a subset of $\mathbf{V}_X$, contains precisely one point, and so $P$ is a leaf-tile in $\mathbf{X}_X$. Similarly, $Q$ is a leaf-tile in $\mathbf{X}_X$.

Therefore, we may apply Lemma 7.3 to the tree $X$, marked by $p$ and $q$ (which are leaves of $X$), and the set $\mathbf{V}_X$. Thus there exists a tile-homeomorphism $h_X \colon X \to \mathbb{T}$

for $\mathbf{X}_X$ with $h_X(p) = -1$ and $h_X(q) = 1$ that satisfies (iii) and (iv) in Lemma 7.3 as well.

We now define $F_X \colon X \to F^n(X) = g_w(\mathbb{T}) = \mathbb{T}_w$ as $F_X = g_w \circ h_X$. This is a homeomorphism of $X$ onto $F^n(X)$. Since $g_w$ sends tiles of $\mathbb{T}$ to tiles of $\mathbb{T}$, we have (a).

Clearly, we have $F_X(X) = F^n(X)$ as well. Moreover,

$$F_X(p) = g_w(h_X(p)) = g_w(-1) = F^n(p),$$

by (7.3) and similarly $F_X(q) = F^n(q)$. Since $\partial X = \{p, q\}$, we have shown (b).

Recall that $2 \le \#\mathbf{V}_X \le N$. Thus, using Lemma 7.3 (iv), we obtain for any $Y \in \mathbf{X}_X$ that

$$1 \le \ell(h_X(Y)) \le \#\mathbf{V}_X \le N.$$

Together with

$$\begin{aligned}\ell(F_X(Y)) &= \ell(g_w(h_X(Y))) = \ell(w) + \ell(h_X(Y)) \\ &= \ell(F^n(X)) + \ell(h_X(Y)),\end{aligned}$$

this implies (c).

Suppose $Y \in \mathbf{X}_X$ and $Y \cap \partial X \neq \emptyset$. Then $Y$ contains one of the marked leaves $p$ or $q$ of $X$; so $Y$ is identical with one of the tiles $P$ or $Q$, because each leaf of $X$ is contained in a unique tile in $\mathbf{X}_X$ as follows from Lemma 3.3 (iv). Since $h_X$ satisfies condition (iii) in Lemma 7.3, we have $\ell(h_X(Y)) = 2$. Thus

$$\ell(F_X(Y)) = \ell(F^n(X)) + \ell(h_X(Y)) = \ell(F^n(X)) + 2,$$

and (d) follows. We have constructed $F_X$, satisfying (a)–(d), when $X \in \mathbf{X}^n$ is an edge-tile.

*Case* 2. $X$ is a leaf-tile. Then $\#\partial X = 1$, say $\partial X = \{p\} \subset \mathbf{V}^n$. Again there exists $w \in \mathcal{A}^*$ with $F^n(X) = \mathbb{T}_w = g_w(\mathbb{T})$. Then $\partial\mathbb{T}_w = F^n(\partial X) = \{F^n(p)\}$ is a singleton set and so $\partial\mathbb{T}_w = \{g_w(-1)\}$ or $\partial\mathbb{T}_w = \{g_w(1)\}$ by Lemma 5.2 (i).

The point $p$ is a leaf of $X$. Moreover, it is contained in a tile $P \in \mathbf{X}_X$ which is a leaf-tile in $\mathbf{X}_X$ (by the exact same argument used when $X$ is an edge-tile). We know $\#\mathbf{V}_X \ge 2$ and that the decomposition $\mathbf{X}_X$ of $X$ induced by $\mathbf{V}_X$ is edge-like.

Thus we may apply Lemma 7.3 to the tree $X$ marked by its leaf $p$ and the decomposition $\mathbf{X}_X$ of $X$ induced by the set $\mathbf{V}_X$. So there exists a tile-homeomorphism $h_X \colon X \to \mathbb{T}$ for $\mathbf{X}_X$ satisfying the (applicable) conditions in Lemma 7.3. We may normalize $h_X$ so that

$$h_X(p) = -1 \text{ if } \partial\mathbb{T}_w = \{g_w(-1)\}, \text{ or } h_X(p) = 1 \text{ if } \partial\mathbb{T}_w = \{g_w(1)\}.$$

The homeomorphism $F_X \colon X \to F^n(X) = g_w(\mathbb{T}) = \mathbb{T}_w$ is now defined by $F_X = g_w \circ h_X$ as before. Then again (a) and the first part of (b) are true. The normalization of $h_X$ ensures that $F_X$ maps $p$ to the unique point in $\partial\mathbb{T}_w$. Since $F^n$ does the same, we have $F^n(p) = F_X(p)$, which is the second part of (b), because $\partial X = \{p\}$.

Finally, statements (c) and (d) follow from similar arguments as in the case of an edge-tile $X$. For (d) one uses the fact that $\ell(h_X(P)) = 2$ by Lemma 7.3 (ii).

This finishes the construction of the homeomorphism $F_X \colon X \to F^n(X)$ with the properties (a)–(d) for a leaf-tile $X \in \mathbf{X}^n$. Hence we have constructed such $F_X$ for every tile $X \in \mathbf{X}^n$.

We now define $F^{n+1}\colon T \to \mathbb{T}$ by setting $F^{n+1}(z) = F_X(z)$ if $z \in X \in \mathbf{X}^n$. Note that the tiles $X \in \mathbf{X}^n$ cover $T$, and so each point $z \in T$ lies in one of the tiles $X \in \mathbf{X}^n$. Moreover, $F^{n+1}$ is well-defined; indeed, suppose that $z \in X \cap Y$ with $X, Y \in \mathbf{X}^n$, $X \neq Y$. Then $z \in \partial X \cap \partial Y$ (see Lemma 3.3 (iv)), and so (b) shows that

$$F_X(z) = F^n(z) = F_Y(z).$$

The map $F^{n+1}$ is continuous, because $F^{n+1}|X = F_X$ is continuous for each tile $X \in \mathbf{X}^n$ and these finitely many tiles cover $T$. The map $F^{n+1}$ is also surjective, because

$$F^{n+1}(T) = \bigcup_{X \in \mathbf{X}^n} F_X(X) = \bigcup_{X \in \mathbf{X}^n} F^n(X) = F^n(T) = \mathbb{T}.$$

To show that it is injective, suppose that $x, y \in T$ and $F^{n+1}(x) = F^{n+1}(y)$. Then there exist $X, Y \in \mathbf{X}^n$ such that $x \in X$ and $y \in Y$. Now if $X = Y$ and so $x, y \in X$, then $x = y$, because $F^{n+1}|X = F_X$ is injective.

If $X \neq Y$, then

$$u \coloneqq F^{n+1}(x) \subset F^{n+1}(X) = F_X(X) = F^n(X)$$

and similarly, $u = F^{n+1}(y) \in F^n(Y) \neq F^n(X)$. So $u$ lies in different tiles of the decomposition of $\mathbb{T}$ by the set of tiles $F^n(\mathbf{X}^n)$. Then

$$u \in \partial F^n(X) \cap \partial F^n(Y) = F^n(\partial X \cap \partial Y).$$

It follows that there exists a point $v \in \partial X \cap \partial Y \subset \mathbf{V}^n$ with $F^n(v) = u$. Then by (b) we have

$$F^{n+1}(v) = F_X(v) = F^n(v) = u = F^{n+1}(x).$$

Since $v, x \in X$, and $F^{n+1}|X = F_X$ is injective, we see that $x = v$. Similarly, $y = v$, and so $x = v = y$, as desired.

We have shown that $F^{n+1}\colon T \to \mathbb{T}$ is a continuous bijection. Since $T$ is compact, it follows that $F^{n+1}\colon T \to \mathbb{T}$ is a homeomorphism. It remains to verify conditions (A)–(D) for $n+1$.

First, if $Y \in \mathbf{X}^{n+1}$ is arbitrary, then there exists a unique tile $X \in \mathbf{X}^n$ such that $Y \subset X$. It now follows from (a) that $F^{n+1}(Y)$ is a tile of $\mathbb{T}$, because $F^{n+1}(Y) = F_X(Y)$. So we have (A).

Condition (B) is also clear, because if $X \in \mathbf{X}^n$ is arbitrary, then by (b) we have

$$F^n(X) = F_X(X) = F^{n+1}(X).$$

Finally, (C) and (D) (for $n+1$) immediately follow from (c) and (d), respectively, and the definition of $F^{n+1}$.

So we obtain a homeomorphism $F^{n+1}$ as desired, and the inductive step is complete. We conclude that there exists a sequence $\{F^n\}$ of maps with the properties (A)–(D).

Having completed the construction of the sequence $\{F^n\}$, we now show that $\{F^n(\mathbf{X}^n)\}$ is a quasi-visual subdivision of $\mathbb{T}$ isomorphic to $\{\mathbf{X}^n\}$.

*$\{F^n(\mathbf{X}^n)\}$ is a subdivision of $\mathbb{T}$.* We know that the sequence $\{\mathbf{X}^n\}$ is a subdivision of the space $T$ as in Definition 2.11. To see that $\{F^n(\mathbf{X}^n)\}$ is a subdivision of $\mathbb{T}$, first note that for each $n \in \mathbb{N}_0$ the map $F^n$ is a homeomorphism by (A). Thus, the set $F^n(\mathbf{X}^n) = \{F^n(X) : X \in \mathbf{X}^n\}$ is a finite collection of compact subsets of $\mathbb{T}$ that cover $\mathbb{T}$. We have that $\{F^0(\mathbf{X}^0)\} = \{\mathbb{T}\}$. It follows that the condition in Definition 2.11 (i) is true.

Let $n \in \mathbb{N}_0$. If $Y' \in F^{n+1}(\mathbf{X}^{n+1})$ is arbitrary, then $Y' = F^{n+1}(Y)$ for some $Y \in \mathbf{X}^{n+1}$. Then there exists $X \in \mathbf{X}^n$ with $Y \subset X$ and so

$$Y' = F^{n+1}(Y) \subset F^{n+1}(X) = F^n(X) \in F^n(\mathbf{X}^n),$$

where (B) was used. Thus the condition in Definition 2.11 (ii) holds.

Finally, if $X' \in F^n(\mathbf{X}^n)$ is arbitrary, then $X' = F^n(X)$ with $X \in \mathbf{X}^n$. Since

$$X = \bigcup \{Y \in \mathbf{X}^{n+1} : Y \subset X\},$$

it follows that $X'$ can be written as a union of sets in $F^{n+1}(\mathbf{X}^{n+1})$, because (using (B) once more)

$$X' = F^n(X) = F^{n+1}(X) = \bigcup \{F^{n+1}(Y) : Y \in \mathbf{X}^{n+1},\, Y \subset X\}.$$

We see that the condition in Definition 2.11 (iii) is also true, and conclude that $\{F^n(\mathbf{X}^n)\}$ is indeed a subdivision of $\mathbb{T}$.

*The subdivisions* $\{\mathbf{X}^n\}$ *and* $\{F^n(\mathbf{X}^n)\}$ *are isomorphic.* We claim that the bijective correspondence given by $X \leftrightarrow F^n(X)$ for $X \in \mathbf{X}^n$ on each level $n \in \mathbb{N}_0$ is an isomorphism of $\{\mathbf{X}^n\}$ and $\{F^n(\mathbf{X}^n)\}$.

To see this, fix $n \in \mathbb{N}_0$, and let $X, Y \in \mathbf{X}^n$ be arbitrary. Then

$$X \cap Y \neq \emptyset \quad \text{if and only if} \quad F^n(X) \cap F^n(Y) \neq \emptyset,$$

since $F^n$ is a homeomorphism by (A). Thus (2.14) is satisfied. Moreover, let $X \in \mathbf{X}^n$ and $X' \in \mathbf{X}^{n+1}$ be arbitrary. Then $F^{n+1}(X) = F^n(X)$ by (B). Thus

$$X' \subset X \quad \text{if and only if} \quad F^{n+1}(X') \subset F^{n+1}(X) = F^n(X),$$

since $F^{n+1}$ is a homeomorphism. We have shown (2.15); so $\{\mathbf{X}^n\}$ and $\{F^n(\mathbf{X}^n)\}$ are indeed isomorphic.

$\{F^n(\mathbf{X}^n)\}$ *is a quasi-visual subdivision of* $\mathbb{T}$. Since we already know that $\{F^n(\mathbf{X}^n)\}$ is a subdivision of $\mathbb{T}$, we have to show that $\{F^n(\mathbf{X}^n)\}$ is a quasi-visual approximation of $\mathbb{T}$. For this we will verify conditions (i)–(iv) in Definition 2.1.

To see (i), let $n \in \mathbb{N}_0$ be arbitrary and consider two tiles in $F^n(\mathbf{X}^n)$ with nonempty intersection. These tiles can be represented in the form $F^n(X)$ and $F^n(Y)$ with $X, Y \in \mathbf{X}^n$, where $X \cap Y \neq \emptyset$. We want to show that $\operatorname{diam}(F^n(X)) \asymp \operatorname{diam}(F^n(Y))$, with a constant $C(\asymp)$ independent of $n$, $X$, and $Y$.

We can find unique tiles $X^i, Y^i \in \mathbf{X}^i$ for $i = 0, \dots, n$ with

$$X = X^n \subset X^{n-1} \subset \cdots \subset X^0 = T \text{ and}$$
$$Y = Y^n \subset Y^{n-1} \subset \cdots \subset Y^0 = T.$$

Let $k \in \mathbb{N}_0$, $0 \le k \le n$, be the largest number with $X^k = Y^k$. If $k = n$, there is nothing to prove, because then $X = X^n = Y^n = Y$, and so $\operatorname{diam}(F^n(X)) = \operatorname{diam}(F^n(Y))$.

Otherwise, $0 \le k < n$, and $X^{k+1}, Y^{k+1} \subset X^k = Y^k$ with $X^{k+1} \neq Y^{k+1}$. Let $L := \ell(F^k(X^k)) = \ell(F^k(Y^k))$. By (C) we know that

$$\begin{aligned} L + 1 \le{}& \ell(F^{k+1}(X^{k+1})) \le L + N, \text{ and similarly,} \\ L + 1 \le{}& \ell(F^{k+1}(Y^{k+1})) \le L + N. \end{aligned} \tag{7.4}$$

Note that $X^i \cap Y^i \supset X^n \cap Y^n \neq \emptyset$ and $X^i \neq Y^i$ for $i = k+1, \dots, n$. By Lemma 3.3 (iv) the tiles $X^i$ and $Y^i$ intersect in a single point, which is the unique point $v \in X^n \cap Y^n$. Moreover, $v \in \partial X^i \cap \partial Y^i$ for $i = k+1, \dots, n$.

Then (D) implies that

$$\begin{aligned}\ell(F^n(X)) &= \ell(F^{n-1}(X^{n-1})) + 2 = \dots \\ &= \ell(F^{k+1}(X^{k+1})) + 2(n-k-1),\end{aligned}$$

and similarly

$$\ell(F^n(Y)) = \ell(F^{k+1}(Y^{k+1})) + 2(n-k-1).$$

Combining this with (7.4), we see that

$$\begin{aligned}|\ell(F^n(X)) - \ell(F^n(Y))| &= |\ell(F^{k+1}(X^{k+1})) - \ell(F^{k+1}(Y^{k+1}))| \\ &\le N-1.\end{aligned}$$

Hence

$$\operatorname{diam}(F^n(X)) = 2^{-\ell(F^n(Y))+1} \asymp 2^{-\ell(F^n(X))+1} = \operatorname{diam}(F^n(Y))$$

with $C(\asymp) = 2^{N-1}$. Since $N$ is a fixed number, condition (i) in Definition 2.1 follows.

To verify condition (ii) in Definition 2.1, let $n \in \mathbb{N}_0$ and $X', Y' \in F^n(\mathbf{X}^n)$ with $X' \cap Y' = \emptyset$ be arbitrary. We choose points $x \in X'$ and $y \in Y'$ with

$$|x-y| = \operatorname{dist}(X', Y'),$$

and consider the unique arc $[x,y]$ joining $x$ and $y$ in $\mathbb{T}$. Then the points in $F^n(\mathbf{V}^n)$ that lie on $[x,y]$ decompose this arc into non-overlapping subarcs that lie in distinct tiles of the edge-like decomposition $F^n(\mathbf{X}^n)$ of $\mathbb{T}$. At least one of these subarcs must have interior disjoint from $X'$ and $Y'$, because $X' \cap Y' = \emptyset$. Let $\alpha$ be the first of these arcs as we travel along $[x,y]$ from $x$ to $y$, say $\alpha = [p,q] \subset [x,y]$. Then there exists a tile $Z' \in F^n(\mathbf{X}^n)$ such that $\alpha \subset Z'$. Obviously, $Z' \ne X', Y'$. The two endpoints $p$ and $q$ of $\alpha$ belong to $\partial Z'$, and so $Z'$ is an edge-tile. Moreover, one of these endpoints also belongs to $\partial X'$, and so $X' \cap Z' \ne \emptyset$.

The set $Z'$ is a tile of $\mathbb{T}$, and so it can be written in the form $Z' = g_w(\mathbb{T})$ with $w \in \mathcal{A}^*$. By Lemma 5.2 (i) we have $\partial Z' = \partial g_w(\mathbb{T}) = \{g_w(-1), g_w(1)\} = \{p,q\}$. It follows that

$$|p-q| = |g_w(-1) - g_w(1)| = 2^{-\ell(w)+1} = \operatorname{diam}(g_w(\mathbb{T})) = \operatorname{diam}(Z').$$

By condition (i) in Definition 2.1, which we proved already, we also have $\operatorname{diam}(X') \asymp \operatorname{diam}(Z')$. Combining this with the quasi-convexity of $\mathbb{T}$ (see (5.3)), we arrive at

$$\begin{aligned}\operatorname{dist}(X', Y') = |x-y| &\asymp \operatorname{length}[x,y] \ge \operatorname{length}[p,q] \\ &\ge |p-q| = \operatorname{diam}(Z') \asymp \operatorname{diam}(X').\end{aligned}$$

Since the implicit multiplicative constants here are independent of $n$, $X'$, and $Y'$, condition (ii) in Definition 2.1 follows.

To verify the remaining conditions, suppose $k, n \in \mathbb{N}_0$, $X' \in F^n(\mathbf{X}^n)$, $Y' \in F^{n+k}(\mathbf{X}^{n+k})$, and $X' \cap Y' \ne \emptyset$. There exists a tile $Z' \in F^n(\mathbf{X}^n)$ with $Y' \subset Z'$. Then $X' \cap Z' \ne \emptyset$, and so $\operatorname{diam}(X') \asymp \operatorname{diam}(Z')$ by what we have seen.

If $k = 1$, then it follows from (C) that

$$\ell(Z') + 1 \le \ell(Y') \le \ell(Z') + N.$$

Since $\operatorname{diam}(Y') = 2^{-\ell(Y')+1}$ and $\operatorname{diam}(Z') = 2^{-\ell(Z')+1}$, we conclude that

$$\operatorname{diam}(Y') \asymp \operatorname{diam}(Z') \asymp \operatorname{diam}(X')$$

with implicit constants independent of $n$ and the tiles. Condition (iii) in Definition 2.1 follows.

For arbitrary $k \geq 0$ note that repeated application of (C) leads to

$$\ell(Y') \geq \ell(Z') + k.$$

This gives

$$\begin{aligned}\operatorname{diam}(Y') &= 2^{-\ell(Y')+1} \leq 2^{-k} 2^{-\ell(Z')+1} = 2^{-k} \operatorname{diam}(Z') \\ &\asymp 2^{-k} \operatorname{diam}(X').\end{aligned}$$

Again the implicit constants here are independent of $k$, $n$, and the tiles. Condition (iv') in Lemma 2.2 follows from this, which, in our setting, is equivalent to condition (iv) in Definition 2.1. We have shown that $\{F^n(\mathbf{X}^n)\}$ is a quasi-visual subdivision of $\mathbb{T}$ as in Definition 2.12.

Now define $\mathbf{Y}^n = F^n(\mathbf{X}^n)$ for $n \in \mathbb{N}_0$. Then our previous considerations show that $\{\mathbf{Y}^n\} = \{F^n(\mathbf{X}^n)\}$ is a quasi-visual subdivision of $\mathbb{T}$ isomorphic to $\{\mathbf{X}^n\}$. The statement follows. □

## 8. Proof of Theorem 1.4

We are now ready to prove Theorem 1.4. Assume first that the metric space $T$ is quasisymmetrically equivalent to the continuum self-similar tree $\mathbb{T}$. In particular, $T$ is homeomorphic to $\mathbb{T}$. Then $T$ is a trivalent metric tree, because $\mathbb{T}$ is of this type.

The CSST $\mathbb{T}$ is doubling as a subset of $\mathbb{C}$, and of bounded turning, because it is even quasi-convex. Since a quasisymmetry preserves the doubling property (see [He01, Theorem 10.18]) and the bounded turning property (see [TV80, Theorem 2.11]), the tree $T$ is doubling and of bounded turning as well. So $T$ is a quasiconformal tree.

We know from Proposition 5.4 that $\mathbb{T}$ is uniformly branching. It follows from Lemma 4.3 and Lemma 4.5 that $T$ is also uniformly branching. The first implication of the statement follows.

Conversely, assume that $T$ is a trivalent quasiconformal tree that is uniformly branching. By rescaling $T$ is necessary, we may assume that $\operatorname{diam}(T) = 1$. Then by Proposition 6.1 we can choose $\delta > 0$ small enough so that the subdivision $\{\mathbf{X}^n\}$ induced by the sets $\mathbf{V}^n$ as defined in (6.1) has the stated properties in this proposition. In particular, $\{\mathbf{X}^n\}$ is a quasi-visual subdivision of $T$. By Proposition 7.1 there exists a quasi-visual subdivision $\{\mathbf{Y}^n\}$ of $\mathbb{T}$ isomorphic to $\{\mathbf{X}^n\}$. By Proposition 2.13 there exists a quasisymmetry $F\colon T \to \mathbb{T}$ that induces the isomorphism between $\{\mathbf{X}^n\}$ and $\{\mathbf{Y}^n\}$; so $T$ and $\mathbb{T}$ are quasisymmetrically equivalent. The proof is complete.

## 9. Concluding remarks and open problems

An important numerical invariant for the quasiconformal geometry of a metric space $X$ is its *conformal dimension* $\operatorname{confdim}(X)$ defined as the infimum of all Hausdorff dimensions $\dim_H(Y)$ of metric spaces $Y$ that are quasisymmetrically equivalent to $X$ (see [MT10] for more background on this concept). It is known that for every quasiconformal tree $T$ we have $\operatorname{confdim}(T) = 1$ (see [Kin17, Proposition 2.4] and [BM20, Corollary 1.4]). In particular, for the CSST $\mathbb{T}$ we have $\operatorname{confdim}(\mathbb{T}) = 1$. One can show though that the infimum underlying the definition

of the conformal dimension is not achieved here as a minimum: if $T$ is any tree that is quasisymmetrically equivalent to $\mathbb{T}$, then actually $\dim_H(T) > 1$. This follows from [Az15, Theorem 1.6].

The CSST is related to various other metric trees appearing in probabilistic models or in complex dynamics. One of these objects is the *(Brownian) continuum random tree* (CRT) as introduced by Aldous [Al91]. One can define it as a random quotient space of the unit interval $[0,1]$ as follows (see [Al93, Corollary 22]). We consider a sample of Brownian excursion $e = (e(t))_{0\le t\le 1}$ on the interval $[0,1]$. For $s,t \in [0,1]$, we set

$$d_e(s,t) = e(s) + e(t) - 2\inf\{e(r) : \min(s,t) \le r \le \max(s,t)\}.$$

Then $d_e$ is a pseudo-metric on $[0,1]$. We define an equivalence relation $\sim$ on $[0,1]$ by setting $s \sim t$ if $d_e(s,t) = 0$. Then $d_e$ descends to a metric on the quotient space $T_e = [0,1]/\sim$. The metric space $T = (T_e, d_e)$ is almost surely a metric tree (see [LG06, Sections 2 and 3]). Moreover, it is known that a sample $T$ of the CRT is almost surely homeomorphic to the CSST (see [BT21, Corollary 1.9]).

Roughly speaking, this means that the topology of the CRT is completely understood. One can ask whether one can say more about the geometry of the CRT, in particular its quasiconformal geometry. A natural question is whether a sample $T$ of the CRT is almost surely quasisymmetrically equivalent to the CSST. Actually, this is not true, because almost surely a sample $T$ of the CRT is neither doubling nor has uniformly relatively separated branch points, which are necessary conditions for a tree to be quasisymmetrically equivalent to the CSST. So even though the CSST serves as a natural deterministic model for the topology of the CRT, it is not so clear whether there is deterministic model for its quasiconformal geometry. One can ask though whether the quasiconformal geometry of the CRT is uniquely determined almost surely. This leads to the following question.

**Problem 9.1.** Suppose $S$ and $T$ are two independent samples of the CRT. Are $S$ and $T$ almost surely quasisymmetrically equivalent?

This seems to be a difficult problem, because the techniques developed in Section 2 do not apply here.

Since the CRT is almost surely homeomorphic, but not quasisymmetrically equivalent to the CSST, one can ask if they are equivalent with respect to another notion (stronger than topological equivalence, but weaker than quasisymmetric equivalence). A closely related result was recently obtained by Lin and Rohde. They show that almost surely the CRT can be represented by a quasiconformal tree in the plane obtained from a (random) conformal welding (see [LR18]).

An important source of trees is given by Julia sets of postcritically-finite polynomials without periodic critical points in $\mathbb{C}$. For example, $P(z) = z^2 + i$ is such a polynomial. As we indicated in the introduction, one can use Theorem 1.4 to show that $\mathcal{J}(P)$ is quasisymmetrically equivalent to the CSST. For some related results about quasisymmetric equivalence of Julia sets and attractors of iterated functions systems see [ERS10].

The CSST has some interesting universality properties. For example, it is not hard to see that every trivalent tree with only finitely many branch points admits a topological embedding into the CSST. One can ask similar questions for quasisymmetric embeddings (i.e., quasisymmetric homeomorphisms onto the image of the map).

**Problem 9.2.** Suppose $T$ is a trivalent quasiconformal tree with uniformly relatively separated branch points. Does there exist a quasisymmetric embedding $F\colon T \to \mathbb{T}$?

There are several other "model trees" whose characterization up to quasisymmetric equivalence may be interesting. For example, the abstract construction of the CSST as outlined in the introduction can easily be modified to obtain canonical trees $\mathbb{T}_m$ with branch points of a fixed higher order $m \in \mathbb{N}$, $m \geq 3$: instead of gluing one line segment of length $2^{-n}$ to the midpoint $c_s$ of a line segment $s$ of length $2^{1-n}$ obtained in the $n$-th step, we glue an endpoint of $m-2$ such segments to $c_s$. Of course, $\mathbb{T} = \mathbb{T}_3$. A tree homeomorphic to $\mathbb{T}_4$ has been proposed as a model for diffusion controlled aggregation (see [Vi83]).

By a modification of the procedure we just described, one can actually define a tree $\mathbb{T}_\infty$ all of whose branch points have infinitely many branches. For the construction of $\mathbb{T}_\infty$ we again start with a line segment $J_0$ of length 2. The midpoint $c$ of $J_0$ divides $J_0$ into two line segments of length 1. We glue to $c$ one of the endpoints of line segments of lengths $2^{-0}, 2^{-1}, 2^{-2}, \ldots$. We equip the resulting space $J_1$ with the obvious path metric. Then $J_0 \subset J_1$. We repeat this construction for each of the line segments obtained in this way, that is, a line segment $s$ of length $L$ is divided into two at its midpoint $c_s$ and segments of lengths $2^{-1}L, 2^{-2}L, 2^{-3}L, \ldots$ are glued in at $c_s$. Proceeding in this way, we obtain an ascending sequence of compact geodesic metric spaces $J_0 \subset J_1 \subset \ldots$. Then $J = \bigcup_{n\in\mathbb{N}_0} J_n$ carries a geodesic metric that agrees with the metric on $J_n$ for each $n \in \mathbb{N}_0$. Now $\mathbb{T}_\infty$ is defined as the completion of $J$ with respect to this metric.

Lemma 3.5 implies that the tree $\mathbb{T}_\infty$ cannot be a quasiconformal tree, because every branch point in $\mathbb{T}_\infty$ has infinitely many branches. In fact, $\mathbb{T}_\infty$ is of bounded turning, but not doubling.

The tree $\mathbb{T}_\infty$ admits a topological characterization similar to the CSST: a metric tree $T$ is homeomorphic to $\mathbb{T}_\infty$ if and only if branch points are dense in $T$ and each branch point of $T$ has infinitely many branches (this follows from results in the first version of [BT21] that is available on arXiv).

Moreover, $\mathbb{T}_\infty$ has an interesting topological universality property: every metric tree $T$ admits a topological embedding $F\colon T \to \mathbb{T}_\infty$ (see [Na92, Section 10.4]).

It would be interesting to establish a similar universality property for the quasiconformal geometry of $\mathbb{T}_\infty$. The following question seems very natural in this context.

**Problem 9.3.** Suppose $T$ is a quasiconformal tree with uniformly relatively separated branch points. Is there a quasisymmetric embedding $F\colon T \to \mathbb{T}_\infty$?

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