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ON THE SINGULAR SET OF A NONLINEAR DEGENERATE PDE ARISING IN TEICHMÜLLER THEORY

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Abstract. Harmonic maps into a Coxeter complex of Teichmüller spaces are described by a certain degenerate elliptic partial differential equation. We analyze the structure of the singular set near a junction of Teichmüller spaces. In particular, we show that the singular set is $(n-1)$-rectifiable.

1. Introduction

In recent years, there has been increasing attention towards variational problems associated with singular spaces. Some noteworthy examples among many others include: (i) the fundamental work of Gromov-Schoen on p-adic superrigidity (cf. [GrSc]) and the development of the theory of harmonic maps to metric spaces (cf. [KS1], [KS2], [Jo], [DM1] and [DaMeVd]); (ii) the work of Eells-Fuglede on harmonic functions or more generally harmonic maps defined on singular domains (cf. [EF], [DM5] and [DM7]); (iii) the theory of degenerations of character varieties and coupled Yang-Mills equations (cf. [DDW], [T]); and finally (iv) the theory of harmonic maps into Teichmüller space related to holomorphic rigidity of Teichmüller space and the rigidity of the mapping class group (cf. [DM3]).

The simplest examples of singular spaces that are not pseudo-manifolds are trees and their generalizations. For example, a Euclidean building (which can be thought of as a higher dimensional version of a tree) can be characterized by the property that any two points lie in an isometrically and totally geodesically embedded copy of Euclidean space. *A common theme in all the work above is to consider harmonic (or energy minimizing) maps to trees or buildings and obtain estimates on the size as well as the structure of their singular set.* From this, one then can conclude important geometric and analytic consequences.

The reason why trees and buildings are amongst the simplest types of singular spaces is because they are made out of Euclidean spaces. On the other hand, in [DM2] and [DM3], we studied harmonic maps into the Weil-Petersson completion of Teichmüller space. This is a space, as explained below, that has significantly worse singularities than buildings. More precisely, let $\mathcal{T}$ denote the Teichmüller space of a genus g Riemann surface with n punctures and $3g-3+n>0$. Endowed with the Weil-Petersson metric, $\mathcal{T}$ is a smooth incomplete Kähler manifold of negative sectional curvature. Its metric completion $\overline{\mathcal{T}}$, called the Weil-Petersson completion of Teichmüller space, is no longer a Riemannian manifold, but an *NPC space*; i.e. a

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complete metric space of non-positive curvature in the sense of Alexandrov (cf. [Ya1]).

A neighborhood $\mathcal{N} \subset \overline{\mathcal{T}}$ of a point in $\partial\mathcal{T}$ is asymptotically a product $\mathcal{U} \times \mathcal{V}$ (cf. [Ya1], [DW], [Wo1], [Wo2], [LSY1], [LSY2] and [DM4]), where the smooth manifold $\mathcal{U}$ is an open subset of a lower dimensional Teichmüller space along with the Weil-Petersson metric and $\mathcal{V}$ is an open subset of $\overline{\mathbf{H}} \times \cdots \times \overline{\mathbf{H}}$ where $\overline{\mathbf{H}}$ (referred to as the *model space*) is the metric completion of the half-plane

$$\mathbf{H} = \{(\rho, \phi) \in \mathbf{R}^2 : \rho > 0\}$$

with respect to the metric $g_{\mathbf{H}} = d\rho^2 + \rho^6 d\phi^2$. The Riemannian manifold $(\mathbf{H}, g_{\mathbf{H}})$ is not complete reflecting also the incompleteness of $\mathcal{T}$ via neck pinching of nodal surfaces (cf. [Wo3], [Ch]). The metric completion of $(\mathbf{H}, g_{\mathbf{H}})$ is the NPC space

$$(\overline{\mathbf{H}}, d_{\mathbf{H}}) = (\mathbf{H} \cup \{P_0\}, d_{\mathbf{H}}) \tag{1}$$

constructed by identifying the axis $\rho = 0$ to a single point P_0 and extending the induced distance function $d_{\mathbf{H}}$ of $g_{\mathbf{H}}$ to $\overline{\mathbf{H}}$ by setting $d_{\mathbf{H}}(Q, P_0) = \rho$ for $Q = (\rho, \phi) \in \mathbf{H}$.

Since each boundary stratum of $\overline{\mathcal{T}}$ is a smooth Riemannian manifold, the singular behavior of the Weil-Petersson geometry is completely captured by the model space $\overline{\mathbf{H}}$. A harmonic map into $\overline{\mathcal{T}}$ can be locally expressed as

$$(V, v^1, \ldots, v^m),$$

where V maps into a lower dimensional Teichmüller space and v^l (for $l = 1, \ldots, m$) maps into $\overline{\mathbf{H}}$. As the Weil-Petersson metric is only asymptotically a product metric near the boundary, the component map v^l is not harmonic. On the other hand, as explained in detail in [DM3], v^l is approximately harmonic and the crucial step in understanding the behavior of harmonic maps into $\overline{\mathcal{T}}$ is understanding the behavior of harmonic maps into $\overline{\mathbf{H}}$.

Since the sectional curvature of $\mathbf{H}$ blows up near P_0, the harmonic map equations become very degenerate. For a map $u : \Omega \to \overline{\mathbf{H}}$, we can write in a neighborhood of a regular point $u = (u_\rho, u_\phi)$ in terms of the coordinates (ρ, ϕ) and write down the harmonic map equations

$$u_\rho \triangle u_\rho = 3u_\rho^6 |\nabla u_\phi|^2 \quad \text{and} \quad u_\rho^4 \triangle u_\phi = -6\nabla u_\rho \cdot u_\rho^3 \nabla u_\phi. \tag{2}$$

Although the right hand side of the above equations is locally bounded by the Lipschitz regularity of harmonic maps (cf. [KS1] Theorem 2.4.6), the left hand side is degenarate since $u_\rho(x)$ is the distance of the image $u(x)$ to P_0 which tends to zero. Thus, from this point of view, it is hard to see why the map should be uniformly regular near a singular point.

An important observation is that, because of the non-local compactness of $\overline{\mathbf{H}}$ near P_0, the Alexandrov tangent space $T_{P_0}\overline{\mathbf{H}}$ of $\overline{\mathbf{H}}$ at P_0 (which is isometric to the interval $[0, \infty)$) does not properly reflect the geometry of $\overline{\mathbf{H}}$ in a neighborhood of P_0. Thus, a tangent map of a harmonic map $u : \Omega \to \overline{\mathbf{H}}$ at a singular point (i.e. a point in $u^{-1}(P_0)$) does not map into $T_{P_0}\overline{\mathbf{H}}$. Indeed, (cf. [DM3, proof of Lemma 3.13] or [W]), a tangent map of a harmonic map u into $\overline{\mathbf{H}}$ at a singular point is a harmonic map u_* whose image is contained in the space

$$\overline{\mathbf{H}}_N = \overline{\mathbf{H}}^{(1)} \cup \overline{\mathbf{H}}^{(2)} \cdots \cup \overline{\mathbf{H}}^{(N)} / \sim \tag{3}$$

defined by taking N copies $\overline{\mathbf{H}}^{(1)}, \ldots, \overline{\mathbf{H}}^{(N)}$ of $\overline{\mathbf{H}}$ and where $\sim$ indicates that the point P_0 from each copy is identified as a single point. The space $\overline{\mathbf{H}}^{(N)}$ should be thought of as a *tree-like N-pod where all the 2-dimensional simplices, in this case copies of $\overline{\mathbf{H}}$, meet at the single vertex P_0.*

In [DM2], we studied how harmonic maps into $\overline{\mathbf{H}}_2$ approximate harmonic maps into $\overline{\mathbf{H}}$ near a point of order 1. The goal of this paper is to investigate the singular set of a harmonic map $u : \Omega \to \overline{\mathbf{H}}_N$. The main theorem is the following:

Theorem 1. *If $u : \Omega \to \overline{\mathbf{H}}_N$ is a non-constant harmonic map from an n-dimensional smooth Riemannian domain, then the singular set $u^{-1}(P_0)$ is $(n-1)$-rectifiable.*

In [Ya2], Yamada constructed a geodesic completion X of the Teichmüller space through the formalism of Coxeter complex with the Teichmüller space as its non-linear non-homogeneous fundamental domain. His main result is that this space X, called the *Teichmüller-Coxeter complex*, is of finite rank (in the sense of [KS2]) which in turn implies an existence theorem of equivariant harmonic maps (cf. [Ya2, Theorem 2]). Given a harmonic map $u : \Omega \to X$ from a n-dimensional Riemannian domain into a Teichmüller-Coxeter complex, we can define a regular point as a point of Ω that maps to the interior of some fundamental domain of X (i.e. an isometric copy of $\mathcal{T}$ in X), the regular set $\mathcal{R}(u)$ as the set of regular points and the singular set $\mathcal{S}(u)$ as the complement of $\mathcal{R}(u)$. By combining [DM3] and Theorem 1, we obtain the following regularity result:

Theorem 2. *If $u : \Omega \to X$ is a harmonic map from a n-dimensional Riemannian domain into a Teichmüller-Coxeter complex, then $\mathcal{S}(u)$ is $(n-1)$-rectifiable.*

2. Preliminaries

Let $(\mathbf{H}, g_{\mathbf{H}})$ and $(\overline{\mathbf{H}}, d_{\mathbf{H}})$ be as above. The *homogeneous cordinates* (ρ, Φ) of $\mathbf{H}$ are defined by setting

$$\Phi = \rho^3 \phi.$$

It can be easily seen that the metric $g_{\mathbf{H}}$ is invariant under the scaling

$$\rho \to \lambda\rho, \ \Phi \to \lambda\Phi.$$

For $\lambda \in (0, \infty)$, we define the map $P \mapsto \lambda P$ using homogeneous coordinates by setting

$$\lambda P = \begin{cases} (\lambda\rho, \lambda\Phi) & \text{for } P = (\rho, \Phi) \in \mathbf{H} \\ P_0 & \text{for } P = P_0. \end{cases} \tag{4}$$

The distance function is homogeneous degree 1 in the sense that

$$d_{\mathbf{H}}(\lambda P, \lambda Q) = \lambda d_{\mathbf{H}}(P, Q).$$

We now let $\overline{\mathbf{H}}_N$ as in (3). The distance function $d_{\mathbf{H}_N}$ on $\overline{\mathbf{H}}_N$ is defined by setting $d_{\mathbf{H}_N}(P_1, P_2) = d_{\mathbf{H}}(P_1, P_2)$ if $P_1, P_2 \in \overline{\mathbf{H}}^{(j)}$ for some $j \in \{1, \ldots, N\}$ and $d_{\mathbf{H}_2}(P_1, P_2) = \rho_1 + \rho_2$ if $P_1 = (\rho_1, \phi_1) \in \mathbf{H}^{(j)} = \overline{\mathbf{H}}^{(j)} \backslash \{P_0\}$ and $P_2 = (\rho_2, \phi_2) \in \mathbf{H}^{(k)} = \overline{\mathbf{H}}^{(k)} \backslash \{P_0\}$ for $j \neq k$. The metric space $(\overline{\mathbf{H}}_N, d_{\mathbf{H}_N})$ is an NPC space (cf. [BH]).

Convention 3. *For $N = 2$, we write*

$$\overline{\mathbf{H}}_2 = \overline{\mathbf{H}}^+ \sqcup \overline{\mathbf{H}}^- / \sim \tag{5}$$

where $\overline{\mathbf{H}}^+ = \overline{\mathbf{H}}^{(1)}$ and $\overline{\mathbf{H}}^- = \overline{\mathbf{H}}^{(2)}$. We will consider $\overline{\mathbf{H}}_2$ as a totally geodesic subset of $\overline{\mathbf{H}}_N$ by the obvious inclusion. Furthermore, we define coordinates on $\overline{\mathbf{H}}_2 \backslash \{P_0\}$ by first applying the change of variables $(\rho, \phi) \mapsto (-\rho, \phi)$ to obtain new coordinates for $\overline{\mathbf{H}}^-$. Thus, we then have coordinates

$$(\rho, \phi) \in \mathbf{R} \backslash \{0\} \times \mathbf{R} \tag{6}$$

for $\overline{\mathbf{H}}_2 \backslash \{P_0\}$ with the property that $\rho > 0$ implies $(\rho, \phi) \in \mathbf{H}^+$ and $\rho < 0$ implies $(\rho, \phi) \in \mathbf{H}^-$. The metric $g_{\mathbf{H}_2}$ at (ρ, ϕ) with $\rho \neq 0$ is given by

$$g_{\mathbf{H}_2}(\rho, \phi) = d\rho^2 + \rho^6 d\phi^2. \tag{7}$$

We also define the homogeneous coordinates (ρ, Φ) on $\overline{\mathbf{H}}_2 \backslash \{P_0\}$.

Convention 4. Given $\overline{\mathbf{H}}_N$ and any two copies $\overline{\mathbf{H}}^{(j)}$ and $\overline{\mathbf{H}}^{(k)}$, there is a totally geodesic isometry $\sigma : \overline{\mathbf{H}}_2 \to \overline{\mathbf{H}}_N$ with image $\overline{\mathbf{H}}^{(j)} \sqcup \overline{\mathbf{H}}^{(k)} / \sim$. In particular (6) and (7) induce coordinates and a metric on the image of σ inside $\overline{\mathbf{H}}_N$.

For a map $v : \Omega \to \overline{\mathbf{H}}_N$ from a bounded Riemannian domain, let the function $|\nabla v|^2$ be the energy density as defined in [KS1]. The *energy* of v is

$$E^v = \int_\Omega |\nabla v|^2 d\mu.$$

Definition 5. *The map $u : \Omega \to \overline{\mathbf{H}}_N$ is said to be* harmonic *if for every $x \in \Omega$, there exists $r > 0$ such that $u\big|_{B_r(x)}$ is energy minimizing with respect to all finite energy maps $v : B_r(x) \to \overline{\mathbf{H}}_N$ with the same trace (cf.* [KS1]*).*

For a harmonic map $u : \Omega \to \overline{\mathbf{H}}_N$, we have the following important monotonicity formula. Given $x_0 \in \Omega$ and $\sigma > 0$ such that $B_\sigma(x_0) \subset \Omega$, let

$$E^u(\sigma) := \int_{B_\sigma(x_0)} |\nabla u|^2 d\mu \quad \text{and} \quad I^u(\sigma) := \int_{\partial B_\sigma(x_0)} d^2(u, u(x)) d\Sigma.$$

There exists a constant $c > 0$ depending only on the C^2 norm of the metric on g (with $c = 0$ when g is the standard Euclidean metric) such that

$$\sigma \mapsto e^{c\sigma^2} \frac{\sigma\ E^u(\sigma)}{I^u(\sigma)}$$

is non-decreasing. As a non-increasing limit of continuous functions,

$$Ord^u(x_0) := \lim_{\sigma \to 0} e^{c\sigma^2} \frac{\sigma\ E^u(\sigma)}{I^u(\sigma)}$$

is an upper semicontinuous function and $Ord^u(x_0) \geq 1$. (See Section 1.2 of [GS] with [KS1] and [KS2] justify various technical steps.)

Definition 6. *The value $Ord^u(x_0)$ is called the* order of u at x_0.

The singular set of a harmonic map $u : \Omega \to \overline{\mathbf{H}}_N$ is defined by

$$\mathcal{S}(u) = \{x \in \Omega : u(x) = P_0\}.$$

The set $\mathcal{S}(u)$ is partitioned into the following two sets

$$\mathcal{S}_0(u) = \{x \in \mathcal{S}(u) : Ord^u(x) > 1\}$$

and

$$\mathcal{S}_1(u) = \{x \in \mathcal{S}(u) : Ord^u(x) = 1\}.$$

The following result follows from [DM2] or [DM3].

Lemma 7. *If $u : B_1(0) \to \overline{\mathbf{H}}_N$ is a harmonic map, then the set of higher order points of u is of Hausdorff codimension at least 2, i.e.*

$$\dim_{\mathcal{H}}(S_0(u)) \leq n - 2.$$

Proof. In [DM2, Theorem 35] or [DM3, Proposition 3.16], we state this result in the case $N = 1$ or $N = 2$. On the other hand, the argument presented there goes through without modification in the case when N is any finite positive integer. □

Lemma 7 implies that we need only consider $S_1(u)$ in order to prove Theorem 1.

We now define the notion of blow-up maps of u at $x \in \Omega$. To do this, we need that the domain metric is expressed with respect to normal coordinates so we make Definition 8.

Definition 8. *A smooth Riemannian metric g on $B_R(0) \subset \mathbf{R}^n$ is said to be* normalized *if the standard Euclidean coordinates $(x^1, \dots, x^n)$ are normal coordinates of g. The metric g_s for $s \in (0, R]$ on $B_1(0)$ is defined by*

$$g_s(x) = g(sx).$$

Given a normalized metric g on $B_R(0)$ and a harmonic map $u : (B_R(0), g) \to \overline{\mathbf{H}}_N$, the homogeneous coordinates can be used to define blow-up maps of u at 0. More precisely, write

$$u = (u_\rho, u_\Phi)$$

in homogeneous coordinates. For $\sigma \in (0, R]$, define a harmonic map (which will be referred to as a *blow-up map*)

$$u_\sigma = (u_{\sigma\rho}, u_{\sigma\Phi}) : (B_1(0), g_\sigma) \to \overline{\mathbf{H}}_N \tag{8}$$

by setting

$$u_{\sigma\rho}(x) = \mu^{-1}(\sigma) u_\rho(\sigma x) \quad \text{and} \quad u_{\sigma\Phi}(x) = \mu^{-1}(\sigma) u_\Phi(\sigma x)$$

where

$$\mu(\sigma) = \sqrt{\frac{I^u(\sigma)}{\sigma^{n-1}}}. \tag{9}$$

The choice of the scaling constant $\mu(\sigma)$ implies that

$$I^{u_\sigma}(1) = \int_{\partial B_1(0)} d^2(u_\sigma, P_0) d\Sigma = 1. \tag{10}$$

By the monotonicity property stated above, $E^{u_\sigma}(1) \leq 2Ord^u(0)$ for $\sigma > 0$ sufficiently small. Thus, by [KS1, Theorem 2.4.6], $\{u_\sigma\}$ has a local uniform modulus of continuity. In turn, this implies that given a sequence u_{σ_i} with $\sigma_i \to 0$, there exists a subsequence converging locally uniformly in the pullback sense to a map $u_* : B_1(0) \to (Y_*, d_*)$ into an NPC space (cf. [KS1, Proposition 3.7]). In particular,

$$d(u_{\sigma_i}(\cdot), u_{\sigma_i}(\cdot)) \to d_*(u_*(\cdot), u_*(\cdot)) \text{ uniformly on compact sets.}$$

Following [GrSc], we have that u_* is a homogeneous map of degree $\alpha = Ord^u(0)$, i.e. $d(u_*(x), u_*(0)) = |x|^\alpha d(u_*(\frac{x}{|x|}, u(0))$ and the curve $t \mapsto u_*(tx)$ is a geodesic in Y_* for each $x \in \partial B_1(0)$.

We now state the qualitative behavior of harmonic maps at order one points. This is already given in [DM2, Lemma 9] when the target is $\overline{\mathbf{H}}_2$, but the same argument yields the following:

Lemma 9. *Let g be a normalized metric on $B_1(0)$ and $u : (B_1(0), g) \to \overline{\mathbf{H}}_N$ a harmonic map with $Ord^u(0) = 1$ and $u(0) = P_0$. Then given a sequence $\sigma_i \to 0$, there exists a subsequence (denoted again by σ_i), a rotation $\mathcal{R} : \mathbf{R}^n \to \mathbf{R}^n$, a sequence of homogeneous degree 1 maps $l_{\sigma_i} : B_1(0) \to \overline{\mathbf{H}}_2 \subset \overline{\mathbf{H}}_N$ defined by (after renumbering the copies of $\overline{\mathbf{H}}$ in $\overline{\mathbf{H}}_N$ if necessary and using Convention 4)*

$$l_{\sigma_i}(x) = \begin{cases} (Ax^1, \phi^+_{\sigma_i}) & x^1 > 0 \\ P_0 & x^1 = 0 \\ (Ax^1, \phi^-_{\sigma_i}) & x^1 < 0 \end{cases} \tag{11}$$

for a constant $A > 0$ and sequences $\{\phi^+_{\sigma_i}\}$, $\{\phi^-_{\sigma_i}\}$ such that

$$\lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i} \circ \mathcal{R}, l_{\sigma_i}) = 0, \ \forall r \in (0,1)$$

where u_{σ_i} are the blow-up maps u at 0.

After rotating the domain if necessary, we may assume in Lemma 9 that

$$\lim_{i\to\infty} d(u_{\sigma_i}, l_{\sigma_i}) = 0.$$

For each i, define an isometry $\iota_{\sigma_i} : \overline{\mathbf{H}}_N \to \overline{\mathbf{H}}_N$ by first defining

$$\iota_{\sigma_i}(P) = \begin{cases} (\rho, \phi - \phi^+_{\sigma_i}) & \text{if } P = (\rho,\phi) \text{ with } \rho > 0 \\ P_0 & \text{if } P = P_0 \\ (\rho, \phi - \phi^-_{\sigma_i}) & \text{if } P = (\rho,\phi) \text{ with } \rho < 0 \end{cases}$$

on $\overline{\mathbf{H}}_2$ and extending it to $\overline{\mathbf{H}}_N$ as an identity map outside of $\overline{\mathbf{H}}_2$. In particular, we then have $l(x) := \iota_{\sigma_i} \circ l_{\sigma_i}(x) = (Ax^1, 0)$ and

$$\lim_{i\to\infty} d(\iota_{\sigma_i} \circ u_{\sigma_i}, l) = 0. \tag{12}$$

3. Order 1 singular points

We start with the following:

Theorem 10. *Let $E_0 > 0$, $A > 0$ and a normalized metric g on $B_1(0)$ be given. There exist $\sigma_0 > 0$, $D_0 \in (0, \frac{1}{\sqrt{8}})$ and $C > 0$ such that if $\sigma \in (0, \sigma_0]$ and $u : (B_1(0), g_\sigma) \to \overline{\mathbf{H}}_N$ is a harmonic map that satisfies*

$$u(0) = P_0, \quad Lip(u\big|_{B_{\frac{1}{2}}(0)}) \le E_0,$$

and

$$\sup_{B_1(0)} d(u, l) < D_0 \quad \text{where } l(x) = (Ax^1, 0), \tag{13}$$

then

$$\sup_{B_s(0)} d(u, l) < CD_0 s, \quad \forall s \in (0, 1].$$

Proof. First notice that the proof of [DM3, Iterative Lemma 5.5] goes through without any changes when we replace the target space $\overline{\mathbf{H}}$ by $\overline{\mathbf{H}}_N$. In the situation when the target space is $\overline{\mathbf{H}}$, we showed that this implies the image of the harmonic map u lies in the interior $\mathbf{H}$ (cf. [DM3, Proposition 3.22]). Below, we argue in the same way as in the proof of [DM3, Proposition 3.22] to show that in the case of $\overline{\mathbf{H}}_N$, we obtain the linear decay asserted in this theorem.

For E_0, A and g given in the statement of the theorem and for $\sigma_0 > 0$ sufficiently small such that g_σ is close to the Euclidean metric, let $\theta \in (0, \frac{1}{24})$, $\epsilon_0 > 0$ and $D_0 \in (0, \frac{1}{\sqrt{8}})$ be as in the [DM3, Iterative Lemma 5.5]. By letting ${}_0l = l$ and ${}_0\delta = D_0$, the assumption implies

$$\begin{cases} \sup_{B_1(0)} d(u, {}_0l) < D_0 \\ \sup_{B_1(0)} |u_\rho - Ax^1| < {}_0\delta. \end{cases}$$

Apply the [DM3, Iterative Lemma 5.7] repeatedly to conclude that for all $i = 0, 1, 2, \dots$,

$$\sup_{B_{\theta^i}(0)} d(v, l) < \theta^{i-1} \left(\frac{2^3 \left(A + 9D_0\right)^3}{\epsilon_0^3} + 10 \right) D_0.$$

For $s \in (0, 1]$, let i be a nonnegative integer such that $s \in (\theta^{i+1}, \theta^i]$. Then

$$\begin{aligned} \sup_{B_s(0)} d(u, l) &\leq \sup_{B_{\theta^i}(0)} d(u, l) \\ &< \theta^{i-1} \left(\frac{2^3 \left(A + 9D_0\right)^3}{\epsilon_0^3} + 10 \right) D_0 \\ &\leq s\theta^{-2} \left(\frac{2^3 \left(A + 9D_0\right)^3}{\epsilon_0^3} + 10 \right) D_0 \\ &< CD_0 s \end{aligned}$$

for some $C > 0$ depending only on E_0, A and g. □

Lemma 11. *Let g be a normalized metric defined on $B_R(0)$ (cf. Definition 8) and $u : (B_R(0), g) \to (\overline{\mathbf{H}}_N, d)$ be a harmonic map with $Ord^u(0) = 1$ and $u(0) = P_0$. Furthermore, let $\mathcal{R}$ and $A > 0$ be as in Lemma 9. Given $\delta_0 > 0$, there exists $\sigma > 0$ such that*

$$s^{-1} \sup_{B_s(0)} d(u_\sigma, l_\sigma \circ \mathcal{R}) < \delta_0, \quad \forall s \in (0, 1)$$

where u_σ is a blow-up map of u at 0 as defined in (8) and $l_\sigma : B_1(0) \to \overline{\mathbf{H}}_2 \subset \overline{\mathbf{H}}_N$ defined by

$$l_\sigma(x) = \begin{cases} (Ax^1, \phi_\sigma^+) & x^1 > 0 \\ P_0 & x^1 = 0 \\ (Ax^1, \phi_\sigma^-) & x^1 < 0 \end{cases} \tag{14}$$

for some fixed constants $\phi_\sigma^+, \phi_\sigma^- \in \mathbf{R}$.

Proof. Let $\sigma_i \to 0$ as in Lemma 9. By the normalization (10) and the fact that $Ord^u(x_0) = 1$, we have that

$$\lim_{\sigma_i \to 0} E^{u_{\sigma_i}}(1) = 1.$$

For $\sigma_i > 0$ sufficiently small such that $E^{u_{\sigma_i}} \leq 2$ there exists $E_0 > 0$ such that $Lip(u_{\sigma_i}\big|_{B_{\frac{1}{2}}(0)}) < \frac{E_0}{2}$. For this choice of $E_0 > 0$, $A > 0$ and g given in the statement of the lemma, let $\sigma_0 > 0$, $D_0 > 0$ and $C > 0$ be as in Theorem 10. Given $\delta_0 > 0$, we can choose $D_0 > 0$ sufficiently small such that $CD_0 < \delta_0$ (cf. comment in [DM3, before equation (5.70)]). Fix $\sigma_i \in (0, \sigma_0]$ sufficiently small such that (after applying a rotation in the domain and an isometry in the target)

$$\sup_{B_{\frac{1}{2}}(0)} d(u_{\sigma_i}, l) < D_0.$$

Set $\sigma = \sigma_i > 0$, $u(x) = u_{\sigma_i}(2x)$ and note that $u(0) = P_0, Lip(u) < E_0$ and $\sup_{B_1(0)} d(u, l) < D_0$. Theorem 10 implies the assertion immediately. □

Lemma 12. *If g is a normalized metric defined on $B_R(0)$ and $u : (B_R(0), g) \to (\overline{\mathbf{H}}_N, d)$ is a harmonic map with $Ord^u(0) = 1$ and $u(0) = P_0$, then*

$$\mathcal{I}_*^u := \lim_{r \to 0} \frac{I^u(r)}{r^{n+1}} \neq 0.$$

Proof. The fact that the limit as $r \to 0$ of the ratio $\frac{I^u(r)}{r^{n+1}}$ exists follows from [GrSc] (also see [DM1, Corollary 60]). Let $\mathcal{R}$ and $A > 0$ be as in Lemma 11. By choosing $\delta_0 \in (0, \frac{A}{2})$ in Lemma 11, there exists $\sigma > 0$ such that

$$\sup_{B_s(0)} |u_{\sigma\rho} - l_{\sigma\rho} \circ \mathcal{R}| \leq \sup_{B_s(0)} d(u_\sigma, l_\sigma \circ \mathcal{R}) < \delta_0 s$$

Applying the triangle inequality, we obtain

$$\frac{As}{2} \leq \sup_{B_s(0)} l_{\sigma\rho} \circ \mathcal{R} - \sup_{B_s(0)} |u_{\sigma\rho} - l_{\sigma\rho} \circ \mathcal{R}| \leq \sup_{B_s(0)} u_{\sigma\rho}.$$

Therefore,

$$0 \neq \frac{A}{2} \leq \lim_{s \to 0} \frac{1}{s} \sup_{B_s(0)} u_{\sigma\rho}.$$

The assertion now follows from the fact that

$$\frac{I^u(r)}{r^{n+1}} = \frac{I^u(\sigma)}{\sigma^{n-1}} \sigma^{-2} \frac{I^{u_\sigma}(\sigma r)}{(\sigma r)^{n+1}}.$$

□

Let g be a normalized metric defined on $B_R(0)$ and $u : (B_R(0), g) \to (\overline{\mathbf{H}}_N, d)$ be a harmonic map with $u(0) = P_0$ and $Ord^u(0) = 1$. By virtue of Lemma 12, there exists a constant $\lambda > 0$ such that

$$\lambda s \leq \mu(s) \leq \lambda^{-1} s$$

where μ is defined in (9). Thus, we will consider blow-up maps of u at x_0 normalized by $\frac{1}{t}$ instead of $\mu^{-1}(t)$.

Definition 13. *The map*

$$u^t : B_1(0) \to \overline{\mathbf{H}}_N, \quad u^t(x) := \frac{1}{t} u(tx) \tag{15}$$

will be referred to as the renormalized blow-up map.

We now prove uniqueness of the tangent map.

Theorem 14. *If g is a normalized metric defined on $B_R(0)$ (cf. Definition 8) and $u : (B_R(0), g) \to (\overline{\mathbf{H}}_N, d)$ is a harmonic map with $Ord^u(0) = 1$ and $u(0) = P_0$. Then there exists a rotation $\mathcal{R}_0 : \mathbf{R}^n \to \mathbf{R}^n$ and constants $A_0, \phi^+, \phi^- \in \mathbf{R}$ such that*

$$\lim_{t\to 0} \sup_{B_1(0)} d(u^t, l \circ \mathcal{R}_0) = 0$$

where $l : B_1(0) \to \overline{\mathbf{H}}_2 \subset \overline{\mathbf{H}}_N$ is defined by

$$l(x) = \begin{cases} (A_0 x^1, \phi^+) & x^1 > 0 \\ P_0 & x^1 = 0 \\ (A_0 x^1, \phi^-) & x^1 < 0. \end{cases} \tag{16}$$

Proof. By Lemma 11, given $\delta_0 > 0$, we can choose $\sigma > 0$ and a homogeneous degree 1 map $l_\sigma : B_1(0) \to \mathbf{H}_N$ given by (14) such that

$$\sup_{B_s(0)} d(u_\sigma, l_\sigma \circ \mathcal{R}_0) < \delta_0 s, \quad \forall s \in (0,1).$$

The lemma now follows immediately since σ is fixed. □

Proposition 15. *If $u : \Omega \to \overline{\mathbf{H}}_N$ is a harmonic map, then the set*

$$\mathcal{S}_1(u) = u^{-1}(P_0) \cap \{x \in \Omega : Ord^u(x) = 1\}$$

is locally a graph of a Lipschitz function over an $(n-1)$-dimensional affine subspace.

Proof. For the sake of simplicity, we will assume in this proof that Ω is a Euclidean domain. Slight modification of the argument below will prove the case when Ω is equipped with an arbitrary Riemannian metric. By [Si, Section 3.8 Corollary 1], it is enough to show that given $\delta \in (0,1)$ and $y_0 \in \mathcal{S}_1(u)$, there exist $\rho_0 > 0$, $\epsilon_0 > 0$ and an $(n-1)$-dimensional affine subset $L_0 \subset \mathbf{R}^n$ such that for any $y \in B_{\epsilon_0}(y_0) \cap \mathcal{S}_1(u)$,

$$\mathcal{S}_1(u) \cap B_\rho(y) \subset \{x : \operatorname{dist}(x, L_0) \leq \delta\rho\}, \qquad \forall \rho < \rho_0. \tag{17}$$

Let $T_0 > 0$ be such that $\overline{B_{2T_0}(y_0)} \subset \Omega$. Theorem 14 implies that (after rotating the domain if necessary) there exists l as in (16) such that

$$\lim_{t\to 0} \sup_{B_{\frac{1}{2}}(0)} d(u^t_{y_0}, l) = 0$$

where

$$u^t_{y_0} : B_1(0) \to \overline{\mathbf{H}}_N, \quad u^t_{y_0}(x) = \frac{1}{t} u(y_0 + tx),$$

Without the loss of generality, we can assume $\phi^+ = \phi^- = 0$ in (16). By the local Lipschitz continuity ([KS1, Theorem 2.4.6]), there exists $E_0 > 0$ such that the Lipschitz constant of u^t_y for $t \in (0, T_0)$ and $y \in B_{T_0}(0)$ is bounded by E_0. For E_0, $A = A_0$ and $\delta_0 = 1$, let $\sigma_0 > 0$, $D_0 > 0$, $C > 0$ be as in Theorem 10. As in the proof of Lemma 11, choose D_0 small such that

$$2CD_0 < A\delta \tag{18}$$

and $t_0 \in (0, T_0]$ such that

$$\sup_{B_{\frac{1}{2}}(0)} d(u^{t_0}_{y_0}, l) < \frac{D_0}{2}.$$

By the continuity of u, there exists $\epsilon_0 > 0$ such that

$$\sup_{B_{\frac{1}{2}}(0)} d(u^{t_0}_{y_0}, u^{t_0}_y) < \frac{D_0}{2}, \qquad \forall y \in B_{\epsilon_0}(y_0).$$

Thus, by the triangle inequality,

$$\sup_{B_{\frac{1}{2}}(0)} d(u_y^{t_0}, l) < D_0, \quad \forall y \in B_{\epsilon_0}(y_0).$$

In other words, assumption (13) of Theorem 10 is satisfied with $u(x) = u^{t_0}(\frac{x}{2})$, and thus by (18) we can conclude

$$\frac{1}{t_0} \sup_{B_{st_0}(y)} d(u, l) = \sup_{B_s(0)} d(u_y^{t_0}, l) < 2CD_0 s < A\delta s, \quad \forall s \in (0, \frac{\sigma_0}{2}].$$

By letting $\rho_0 = \frac{\sigma_0 t_0}{2}$, we obtain

$$\begin{aligned} y \in B_{\epsilon_0}(y_0) \quad &\Rightarrow \quad \sup_{B_{st_0}(y)} d(u, l) < A\delta s t_0, \quad \forall s \in (0, \frac{\sigma_0}{2}] \\ &\Leftrightarrow \quad \sup_{B_\rho(y)} d(u, l) < A\delta\rho, \quad \forall \rho \in (0, \rho_0] \end{aligned}$$

Therefore, assuming $y \in B_{\epsilon_0}(y_0)$ and $\rho \in (0, \rho_0]$, we have

$$x \in \mathcal{S}_1(u) \cap B_\rho(y) \quad \Rightarrow \quad x^1 = \frac{1}{A} d(P_0, l(x)) = \frac{1}{A} d(u(x), l(x)) < \delta\rho.$$

By setting L_0 equal to the hyperplane $\{x_1 = 0\}$, this immediately implies (17). □

4. Proof of Theorem 1 and Theorem 2

We are now ready to prove our main theorems.

Proof of Theorem 1. Combine Lemma 7 and Proposition 15 □

Proof of Theorem 2. Let P be a point in the boundary of a Teichmüller-Coxeter complex. The metric estimates of [DM4] imply that the Weil-Petersson metric is asymptotically a product of a lower dimensional Teichmüller space and copies of $\overline{\mathbf{H}}_N$'s. This is analogous to the situation in [DM1] where we studied harmonic maps to the Weil-Petersson completion $\overline{\mathcal{T}}$ of Teichmüller space. In this case $\overline{\mathcal{T}}$ is, near a point in the boundary, asymptotically a product space of a lower dimensional Teichmüller space and copies of $\overline{\mathbf{H}}$'s. In particular, we showed that the singular component maps (the component maps which map into $\overline{\mathbf{H}}$) have blow-up maps and tangent maps at singular points. Similarly, we can show the same for component maps into $\overline{\mathbf{H}}_N$. Thus, applying an argument as in the proof of Theorem 1, the theorem follows. □

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