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STABILITY FOR A FORMALLY DETERMINED INVERSE PROBLEM FOR A HYPERBOLIC PDE WITH SPACE AND TIME DEPENDENT COEFFICIENTS*

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Abstract. We prove stability for a formally determined inverse problem for a hyperbolic PDE in one or higher space dimensions with the coefficients dependent on space and time variables. The hyperbolic operator has constant wave speed, and we study the recovery of the first-order and zeroth-order coefficients. We use a modification of the Bukhgeim–Klibanov method to obtain our results.

Key words. hyperbolic inverse problem, stability estimates, space and time dependent coefficients

AMS subject classifications. 35R30, 35L05, 35L20

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1. Introduction.

Suppose D is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary and $T > 0$. Let $a(x,t), c(x,t)$ be smooth real valued functions on $\overline{D} \times [0,T]$ and $b(x,t) = (b^1(x,t), \dots, b^n(x,t))$ a smooth n-dimensional real vector field on $\overline{D} \times [0,T]$. Define the hyperbolic operator

$$\mathcal{L}_{a,b,c} := (\partial_t - a)^2 - (\nabla - b)^2 + c \tag{1.1}$$

$$= \square - 2a\partial_t + 2b \cdot \nabla + c - a_t + \nabla \cdot b + a^2 - b^2. \tag{1.2}$$

When it is clear from the context, we use $\mathcal{L}$ instead of $\mathcal{L}_{a,b,c}$.

Let $w(x,t)$ be the solution of the well-posed IBVP

$$\mathcal{L}_{a,b,c} w = 0, \qquad (x,t) \in D \times [0,T], \tag{1.3}$$

$$w(\cdot,0) = f, \quad w_t(\cdot,0) = g \qquad \text{on } D, \tag{1.4}$$

$$w = h \qquad \text{on } \partial D \times [0,T] \tag{1.5}$$

for f, g, h with appropriate regularity.

For a given a, b, c, define the response operator

$$\Lambda_{a,b,c} : (f,g,h) \to \left[w(\cdot,T)|_D, w_t(\cdot,T)|_D,\ \partial_\nu w|_{\partial D \times [0,T]} \right]; \tag{1.6}$$

hence, $\Lambda_{a,b,c}(f,g,h)$ represents the boundary and final time response, of the acoustic medium with acoustic properties (a,b,c), to the initial boundary input (f,g,h). So we have the forward map

$$\Lambda : (a,b,c) \to \Lambda_{a,b,c},$$

whose injectivity and stability has been studied by several authors. This is an overdetermined problem (when $n > 1$) because the distribution kernel of Λ depends on $2n$

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parameters, while a, b, c depend on $n+1$ parameters. Our goal is to study the recovery of a, b, c from less (but slightly different) data than $\Lambda_{a,b,c}$. We study a formally determined problem where the data depend only on $n+1$ parameters. Before we state our goal, we first describe what is known about the injectivity and stability of Λ-type forward maps.

In general, Λ is not injective due to gauge invariance (described later), and in such cases, one hopes to recover $\text{curl}(a, b)$ and c, or one studies special cases when a, b are known or c is known. Below, injectivity and stability results for Λ-type forward maps are to be understood in this sense. We use the term Λ-type forward maps because there are results in the literature with one or more of the following:

- data are collected only on a part of the lateral boundary;
- data are not collected on $t = T$;
- there are no sources on $t = 0$;
- the data are the far-field pattern in the frequency domain, which in some sense is equivalent to Λ but with t varying over $(-\infty, \infty)$;
- the principal part of the operator is not the wave operator but a hyperbolic operator associated with a nonconstant velocity or even a Lorentzian metric.

While the inverse problems associated with Λ-type forward maps are overdetermined problems, there are considerable challenges dealing with some of these problems, either because three coefficients are being determined simultaneously, the data are given only on a part of the lateral boundary, or the wave velocity is nonconstant. The results we obtain are only for the constant velocity case though for a formally determined problem.

From domain of dependence arguments, it is clear that, for hyperbolic operators with coefficients dependent on x, t and measurements over a finite t interval $[0, T]$, to recover the coefficients on $D \times [0, T]$, one needs sources on $D \times \{t = 0\}$ and measurements on $D \times \{t = T\}$, in addition to the lateral boundary sources and measurements. So, for inverse problems with coefficients dependent on x and t, with sources only on the lateral boundary and receivers/measurements only on the lateral boundary of the x, t domain, one must either know the coefficients in appropriate regions contiguous with $t = 0$ and $t = T$, assume analyticity of the coefficients with respect to t, or have data from measurements over infinitely long t intervals. The situation is different when the principal part of the operator is not the wave operator (or coming from a Lorentzian metric) but the Schrödinger operator $i\partial_t + \Delta$ (infinite speed of propagation) or perhaps a fractional differential operator (a nonlocal operator). We do not describe the results for such operators.

For coefficients which depend on x, t, results on the injectivity of Λ-type forward maps, for data on infinite time intervals, may be found in, for example, [28, 29, 33, 34]. For the finite time interval case, the injectivity of Λ-type forward maps but with coefficients known in certain regions near $t = 0$ and $t = T$ or analytic in t, results may be found in, for example, [3, 5, 10, 11, 12, 15, 16, 19, 27, 32]. The stability of Λ has been studied extensively in, for example, [4, 6, 8, 30, 31, 35]. The results mentioned here, for x, t-dependent coefficients, are for overdetermined problems, and the stability results, even for these overdetermined problems, are of log-log type. There are better stability results for the Schrödinger operator (infinite speed of propagation) with Holder stability (but not Lipschitz stability) still for an overdetermined problem; see [18].

We do not survey results for Λ-type maps when the coefficients are independent of t. No sources are needed on $t = 0$, and no measurements are needed on $t = T$. A brief survey of such results may be found in [17]. Most of these results use generalizations of the boundary control method introduced by Belishev (see [1, 2]) or generalizations of geometric optics solutions for hyperbolic PDEs introduced in [25], which were themselves imitations of similar (but harder-to-construct) solutions for elliptic PDEs constructed by Sylvester and Uhlmann in [36].

We now describe results for formally determined inverse problems for hyperbolic PDEs.

For coefficients independent of t, there are uniqueness and stability results for formally determined problems based on the ideas introduced by Bukhgeĭm and Klibanov in [9], which had the first such results in dimension $n > 1$. See [7] for a survey of such results and an exposition of the significant modifications of the important ideas in [9]. The only drawback of these results is that they require the *initial source* to be a positive (or negative) function throughout the domain (in x space). Rakesh and Salo, in [23, 24], for the case $a = 0, b = 0$ (recover c) but for the domain $\mathbb{R}^n \times (-\infty, T]$ instead of $D \times [0, T]$, avoided the use of positive initial sources, using instead the more natural incoming plane wave source, except one needed data from two such experiments, corresponding to incoming plane waves coming from opposite directions. These ideas were extended to obtain similar results for the operator with general a, b, c or the operator associated with a Lorentzian metric (with restrictions) in [21, 22]. The articles [21, 22, 23, 24] contain uniqueness and Lipschitz stability results for these problems.

We mention three results which rely on carefully constructed sources. The article [14] discusses a formally determined inverse problem for the wave operator associated to a Riemannian metric (independent of t), and the goal is the determination of the metric from the space-time boundary response to a single source. They prove a uniqueness result assuming their source is a specially constructed pseudorandom source and the metric is simple and conformally Euclidean. We also note the work in [20], on a coefficient recovery problem for a semilinear hyperbolic PDE, with the coefficient independent of t and the data consisting of a weighted average of lateral boundary measurements. This seems to be an underdetermined inverse problem, but the nonlinearity of the PDE is crucial for this result. Finally, [13] also contains a uniqueness (and reconstruction) result for a formally determined a, b, c recovery problem with the coefficients dependent on x and t. They use a single boundary source h, constructed as the infinite sum of a combination of sources, each generating a solution traveling along a ray for the hyperbolic PDE and the rays associated with these solutions forming a dense subset of the x, t domain. The challenge is to build the source h so that the data from the h source can be separated into the data contributions from the sources in the sum. We believe such a source h on the lateral boundary would have support consisting of the full lateral boundary.

The articles [23, 24] were attempts at (and have come close to) solving the long-standing open *fixed angle scattering* inverse problem. There are other long-standing formally determined open problems for hyperbolic PDEs (with coefficients independent of t) such as the *back-scattering problem*, where the results are much weaker than the result for the fixed angle scattering problem. We do not survey the results for these two problems, as the introductions to [22, 24, 26] have a good survey of the results.

We study a formally determined inverse problem with the coefficients a, b, c dependent on x, t. We prove uniqueness (up to gauge) and Lipschitz stability using modifications of the ideas of Bukhgeĭm and Klibanov in [9], of an idea in [22], and our new idea for problems with coefficients dependent on x, t. Our results have one weakness: The problem must be posed in the full space $\mathbb{R}^n \times (-\infty, T]$ and do not work for space-time cylinders with bounded bases such as $D \times (-\infty, T]$.

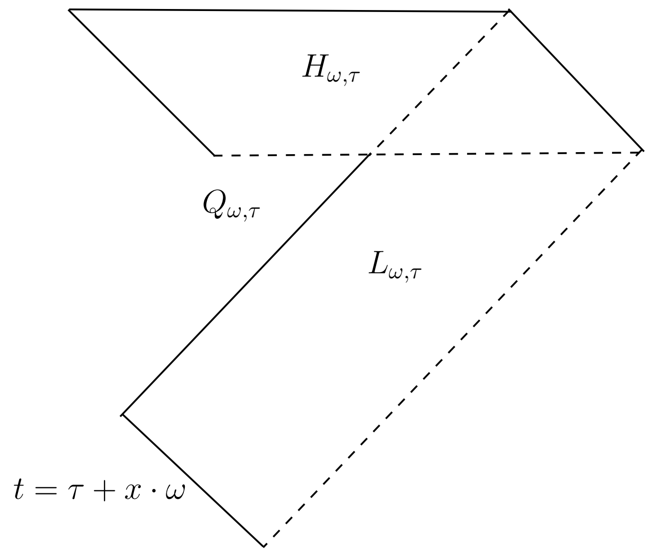

FIG. 1.1. *The wedge-shaped region and its boundary.*

Let B denote the open unit ball in $\mathbb{R}^n$, $n \geq 1$, $T > 0$, and suppose $a(x,t), b^i(x,t), c(x,t)$, $i = 1, \dots, n$ are compactly supported smooth functions on $\mathbb{R}^n \times \mathbb{R}$. If ω is a unit vector in $\mathbb{R}^n$ and $\tau \in \mathbb{R}$, let $U(x,t;\omega,\tau)$ be the solution of the IVP

$$\mathcal{L}U = 0 \qquad \text{on } \mathbb{R}^n \times \mathbb{R}, \tag{1.7}$$

$$U(x,t;\omega,\tau) = H(t - \tau - x \cdot \omega), \qquad x \in \mathbb{R}^n,\ t \ll 0, \tag{1.8}$$

and let $V(x,t;\omega,\tau)$ be the solution of the IVP

$$\mathcal{L}V = 0 \qquad \text{on } \mathbb{R}^n \times \mathbb{R}, \tag{1.9}$$

$$V(x,t;\omega,\tau) = \delta(t - \tau - x \cdot \omega), \qquad x \in \mathbb{R}^n,\ t \ll 0. \tag{1.10}$$

Here $t \ll 0$ means for large negative numbers t. So U, V are the disturbances in the medium caused by two types of impulsive incoming plane waves. Here τ is the time the incoming plane wave reaches the origin; τ may also be regarded as a time delay. Given $T > 0$, define the map

$$\mathcal{F} : (a,b,c) \to [U, U_t, V, V_t](x, T; \omega, \tau)|_{x \in \mathbb{R}^n, \omega \in \Omega, \tau \in (-\infty, T+1]}$$

mapping the medium properties (a,b,c) of the region $\mathbb{R}^n \times (-\infty, T]$, to the final time medium response, to incoming plane waves, coming from a finite set of directions ω in the finite set Ω of unit vectors in $\mathbb{R}^n$, with delays $\tau \in (-\infty, T+1]$. Our goal is to study the injectivity and stability of $\mathcal{F}$. This problem may be considered a variation of the fixed angle scattering problem but for mediums with physical properties dependent on space and time. We note that the data set for our inverse problem (associated with $\mathcal{F}$) depends on $n+1$ parameters and that our unknown functions (a,b,c) depend on $n+1$ parameters; hence, our problem is formally determined.

We introduce definitions used throughout the article. Given a unit vector $\omega \in \mathbb{R}^n$, a $\tau \in \mathbb{R}$, and a $T > 0$, we define the wedge-shaped region (see Figure 1.1)

$$Q_{\omega,\tau} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \tau + x \cdot \omega \leq t \leq T\}$$

and its higher and lower boundary

$$H_{\omega,\tau} = Q_{\omega,\tau} \cap \{t = T\}, \qquad L_{\omega,\tau} = Q_{\omega,\tau} \cap \{t = \tau + x\cdot\omega\}.$$

We suppress the T dependence of these sets, as T will not vary. Given $\sigma > 0$, for any submanifold M of $\mathbb{R}^n \times \mathbb{R}$ and a function w on M, we define the weighted norms

$$\|w\|_{1,M,\sigma} = \left(\int_M e^{2\sigma t}(|\nabla_M w|^2 + \sigma^2 |w|^2)\right)^{1/2}, \qquad \|w\|_{0,M,\sigma} = \left(\int_M e^{2\sigma t}|w|^2\right)^{1/2},$$

where ∇_M is the gradient on the manifold M made up only of derivatives in directions tangential to M. We will also use $\|w\|_{1,M}$, $\|w\|_{0,M}$ for the standard H^1 and L^2 norms on M.

Given compactly supported smooth functions a, b^j, c on $\mathbb{R}^n \times \mathbb{R}$, we define the function

$$\alpha(x,t;\omega) := \exp\left(\int_{-\infty}^0 (a + \omega\cdot b)(x + s\omega, t + s)\, ds\right), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{1.11}$$

Note that $\alpha(x_0, t_0; \omega)$ is determined by the values of a, b in the region $t \leq t_0$.

We start with the well-posedness of the IVP associated with U and V.

PROPOSITION 1.1 (the Heaviside function solution). *Suppose a, b^i, c, $i = 1, \ldots, n$, are compactly supported smooth functions on $\mathbb{R}^n\times\mathbb{R}$, ω a unit vector in $\mathbb{R}^n$, and $\tau \in \mathbb{R}$. The IVP* (1.7)–(1.8) *has a unique distributional solution*

$$U(x,t;\omega,\tau) = u(x,t;\omega,\tau)H(t - \tau - x\cdot\omega), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $u(x,t;\omega,\tau)$ is a smooth function in the region $t \geq \tau + x\cdot\omega$ and is the unique solution of the characteristic IBVP

$$\mathcal{L}_{a,b,c} u = 0, \qquad x \in \mathbb{R}^n,\ \tau + x\cdot\omega \leq t, \tag{1.12}$$
$$u(x,t;\omega,\tau) = \alpha(x,t;\omega), \qquad x \in \mathbb{R}^n,\ t = \tau + x\cdot\omega, \tag{1.13}$$
$$u(x,t;\omega,\tau) = 1, \qquad x \in \mathbb{R}^n,\ \tau + x\cdot\omega \leq t \ll 0. \tag{1.14}$$

Further, given $T > 0$ if $\|[a,b,c]\|_{C^N(Q_{\omega,\tau})} \leq M$ for $N = 5 + [n/2]$, then

$$\|u\|_{C^3(Q_{\omega,\tau})} \leq C,$$

where C depends on M and the support of a, b, c.

A similar result is true for $V(x,t;\omega,\tau)$.

PROPOSITION 1.2 (the delta function solution). *Suppose a, b^i, c, $i = 1, \ldots, n$, are compactly supported smooth functions on $\mathbb{R}^n \times \mathbb{R}$, ω a unit vector in $\mathbb{R}^n$, and $\tau \in \mathbb{R}$. The IVP* (1.9)–(1.10) *has a unique distributional solution in $\mathbb{R}^n \times \mathbb{R}$*

$$V(x,t;\omega,\tau) = \alpha(x, t-\tau;\omega)\,\delta(t - \tau - x\cdot\omega) + v(x,t;\omega,\tau)H(t - \tau - x\cdot\omega)$$

where $v(x,t;\omega,\tau)$ is a smooth function on the region $t \geq \tau + x\cdot\omega$ and is the unique solution of the characteristic IBVP

$$\mathcal{L}_{a,b,c} v = 0, \qquad t \geq \tau + x\cdot\omega, \tag{1.15}$$
$$v(x,t;\omega,\tau) = 0, \qquad t \ll 0 \tag{1.16}$$
$$v_t + \omega\cdot\nabla v - (a + \omega\cdot b)v = -\frac{1}{2}\mathcal{L}_{a,b,c}\alpha, \qquad t = \tau + x\cdot\omega. \tag{1.17}$$

Further, given $T > 0$, if $\|[a,b,c]\|_{C^N(Q_{\omega,\tau})} \leq M$ for $N = 7 + [n/2]$, then

$$\|v\|_{C^3(Q_{\omega,\tau})} \leq C,$$

where C depends on M and the support of a, b, c.

While $V = -\partial_\tau U$, the relationship between u and v may be a little more complicated because the domains of u, v depend on τ.

The inverse problem has a gauge invariance. If $\phi(x,t)$ is a smooth function on $\mathbb{R}^n \times \mathbb{R}$, then, for any smooth function $f(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$, we have

$$(\partial_t - a - \phi_t)(e^\phi f) = e^\phi(\partial_t - a)f, \qquad (\nabla - b - \nabla\phi)(e^\phi f) = e^\phi(\nabla - b)f, \tag{1.18}$$

implying

$$\mathcal{L}_{a+\phi_t, b+\nabla\phi, c}(e^\phi f) = e^\phi \mathcal{L}_{a,b,c} f; \tag{1.19}$$

in particular,

$$\mathcal{L}_{a+\phi_t, b+\nabla\phi, c}(e^\phi U) = e^\phi \mathcal{L}_{a,b,c} U = 0, \qquad \mathcal{L}_{a+\phi_t, b+\nabla\phi, c}(e^\phi V) = e^\phi \mathcal{L}_{a,b,c} V = 0.$$

Hence, if ϕ is compactly supported, then $e^\phi U$ and $e^\phi V$ are the Heavisde function and delta function solutions corresponding to the triple $(a + \phi_t, b + \nabla\phi, c)$. So, if we also have $\phi(\cdot, T) = 0$, then

$$\mathcal{F}(a,b,c) = \mathcal{F}(a + \phi_t, b + \nabla\phi, c).$$

Actually our data on $t = T$ will also involve time derivatives of U, V, so, for gauge invariance, we will also need some time derivatives of ϕ to be zero at $t = T$. We will be specific below.

Noting the expressions for U, V given by Propositions 1.1 and 1.2, our inverse problem may be reformulated as the study of the injectivity, stability, and inversion of the map

$$\mathcal{F} : (a,b,c) \mapsto [u, u_t, v, v_t](x, T; \omega, \tau)|_{x \in H_{\omega,T}, \omega \in \Omega, \tau \in (-\infty, T+1]}$$

for some finite set of directions Ω.

For future use we observe that it is enough to have data only for the solutions corresponding to $\tau \in [-1, T+1]$ given that a, b, c are supported in $\overline{B} \times [0, T]$. For $\tau > T + 1$, the incoming plane wave never hits the support of a, b, c so

$$U(x,t;\omega,\tau) = H(t - \tau - x\cdot\omega), \qquad V(x,t;\omega,\tau) = \delta(t - \tau - x\cdot\omega), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

hence U, V have no information about a, b, c. For $\tau < -1$, the $\delta(t - \tau - x\cdot\omega)$ plane wave never hits the support of a, b, c so

$$V(x,t;\omega,\tau) = \delta(t - \tau - x\cdot\omega), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

and V has no information about a, b, c. For $\tau < -1$, the plane $t = \tau + x\cdot\omega$ never hits the support of a, b, c so one observes that

$$u(x,t;\omega,\tau) = 1 \qquad \text{when } \tau + x\cdot\omega \leq t \leq -1 + x\cdot\omega,$$

and hence

$$u(x,t;\tau,\omega) = u(x,t;-1,\omega), \qquad \text{when } -1 + x\cdot\omega \leq t.$$

This implies that for $\tau < -1$, $U(\cdot,\cdot;\tau,\omega)$ and $U(\cdot,\cdot;-1,\omega)$ have the same information about a, b, c.

We state our principal results next. We have seen in (1.2) that $\mathcal{L}_{a,b,c}$ can also be written in the form

$$\mathcal{L}_{a,b,c} = \Box - 2a\partial_t + 2b \cdot \nabla + q,$$

where

$$q := c - a_t + \nabla \cdot b + a^2 - b^2. \tag{1.20}$$

We can regard the operator $\mathcal{L}_{a,b,c}$ as determined by the functions a, b^i, c or by the functions a, b^i, q. We use both points of view below. The context will clarify the point of view in play.

Our work has two new ideas, perhaps one more significant than the other. Our most significant idea allows us to obtain Lipschitz stability for a formally determined x,t-dependent coefficient problem as compared to the logarithmic stability results for overdetermined problems (though on bounded domains) in the literature. This is showcased in its simplest form in the study of the less complicated problem of recovering q given a, b. Our second idea is about separating the estimates on c from the estimates on a, b when we prove stability for the a, b, c problem.

Below, $T > 0$ will be fixed, and a, b, c, q will be smooth real valued functions on $\mathbb{R}^n \times (-\infty, T]$ with support in $\overline{B} \times [0, T]$. Note this does not imply that a, b, c, q are zero on $t = T$. Further, the U, V solutions are defined on $\mathbb{R}^n \times (-\infty, T]$: The IVP (1.7)–(1.10) are to be solved only on $\mathbb{R}^n \times (-\infty, T]$.

We start with the stability result about recovering q, given a, b.

Theorem 1.1 (stability for the q recovery problem, given a, b). *Suppose $T > 0$ and $a(x,t), b^i(x,t)$, $i = 1, \ldots, n$, are compactly supported smooth functions on $\mathbb{R}^n \times (-\infty, T]$ and ω is a unit vector in $\mathbb{R}^n$. If $q, \acute{q}$ are compactly supported smooth functions on $\mathbb{R}^n \times (-\infty, T]$ with support in $\overline{B} \times [0, T]$ and $\|[q, \acute{q}, a, b]\|_{C^{7+[n/2]}} \leq M$, then*

$$\|q - \acute{q}\|_{L^2} \preccurlyeq \int_{-1}^{T+1} \|(v - \acute{v})(\cdot, T; \omega, \tau)\|_{1,H_{\omega,\tau}} + \|(v_t - \acute{v}_t)(\cdot, T; \omega, \tau)\|_{0,H_{\omega,\tau}} \, d\tau.$$

Here $v, \acute{v}$ are the functions associated with (a, b, q) and $(a, b, \acute{q})$ in Proposition 1.2, and the constant depends on M and the support of $a, b, q, \acute{q}$.

The proof of this theorem presents one of our ideas, uncluttered by the complications appearing in the proofs of the other theorems.

Next we state a stability result about recovering a, b if q is known. Note that there is no gauge invariance if q is known. Below, $e^1, \ldots, e^n$ is the standard basis for $\mathbb{R}^n$.

Theorem 1.2 (stability for the a, b recovery problem, given q). *Suppose $T > 0$ and $q(x,t)$ is a smooth compactly supported smooth function on $\mathbb{R}^n \times [0, T]$. If $a, b, \acute{a}, \acute{b}$ are compactly supported smooth functions on $\mathbb{R}^n \times (-\infty, T]$ with support in $\overline{B} \times [0, T]$ and $\|[a, b, q, \acute{a}, \acute{b}]\|_{C^{7+[n/2]}} \leq M$, then*

$$\|[a - \acute{a}, b - \acute{b}]\|_{L^2} \preccurlyeq \sum_{\omega} \int_{-1}^{T+1} \|(u - \acute{u})(\cdot, T; \omega, \tau)\|_{1,H_{\omega,\tau}} + \|(u_t - \acute{u}_t)(\cdot, T; \omega, \tau)\|_{0,H_{\omega,\tau}},$$

where ω takes the values $-e^n$ and $e^1, \ldots, e^n$. Here $u, \acute{u}$ are the functions associated with (a, b, q) and $(\acute{a}, \acute{b}, q)$ in Proposition 1.1. The constant depends on M and the supports of $a, b, \acute{a}, \acute{b}, q$.

Next we have a uniqueness result about recovering (a, b, c). Noting the gauge invariance mentioned earlier in the introduction, the most we can hope to recover is curl(a, b) and c. However, for $\phi(x, t)$ to be a gauge, we needed $\phi(\cdot, T) = 0$ (and $\phi_t(\cdot, T) = 0$ because of the data we use in our theorems). Hence, one cannot expect to recover curl (a, b) without imposing additional conditions on a, b. We impose a condition on the integrals of $a + b^n$ and $a_t + b^n_t$, dictated by necessity for our argument to go through. These conditions are motivated by similar conditions in Theorems 1.1 and 2.1 in [22]. However, it is not clear whether these conditions are optimal in some sense or the most natural.

THEOREM 1.3 (uniqueness for the curl(a, b) and c recovery problem). *Suppose* $T > 0$ *and* $a, b, c, \acute{a}, \acute{b}, \acute{c}$ *are compactly supported smooth functions on* $\mathbb{R}^n \times (-\infty, T]$ *with support in* $\overline{B} \times [0, T]$*. For* $\tau \in [-1, T + 1]$*, if*

$$
\begin{aligned}
[u, u_t](x, T, \omega, \tau) &= [\acute{u}, \acute{u}_t](x, T; \omega, \tau) \qquad \forall x \in H_{\omega,\tau},\ \omega = \pm e^n, e^i,\ i = 1, \dots, n-1,\\
[v, v_t](x, T, e^n, \tau) &= [\acute{v}, \acute{v}_t](x, T; e^n, \tau) \qquad \forall x \in H_{e_n,\tau}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-\infty}^{T} (a + b^n)(x + se^n, s)\, ds &= \int_{-\infty}^{T} (\acute{a} + \acute{b}^n)(x + se^n, s)\, ds \qquad \forall x \in \mathbb{R}^n,\\
\int_{-\infty}^{T} (a_t + b^n_t)(x + se^n, s)\, ds &= \int_{-\infty}^{T} (\acute{a}_t + \acute{b}^n_t)(x + se^n, s)\, ds \qquad \forall x \in \mathbb{R}^n,
\end{aligned}
$$

then

$$
c = \acute{c}, \qquad d\left(adt + \sum_{i=1}^{n} b^i dx^i\right) = d\left(\acute{a}dt + \sum_{i=1}^{n} \acute{b}^i dx^i\right).
$$

Here $u, v, \acute{u}, \acute{v}$ *are the functions associated with* (a, b, c) *and* $(\acute{a}, \acute{b}, \acute{c})$ *in Propositions* 1.1 *and* 1.2.

This result is obtained by combining our most significant idea with an idea in [22] about a uniqueness problem for a *time-independent* coefficient determination problem. We do not know how to prove a similar uniqueness result when all the three coefficients a, b, q are to be recovered: That problem does not have gauge invariance.

Our final result is a stability result for the (a, b, c) recovery problem. Again, due to the gauge invariance, we can only expect to recover curl(a, b) and c provided we impose additional conditions on a, b, c or use more data directly related to a, b, c, as explained in the paragraph before Theorem 1.3. We define $\psi(x, t)$ to be the solution of the IVP

$$\Box\psi = \nabla \cdot b - a_t + c, \qquad (x, t) \in \mathbb{R}^n \times (-\infty, T] \tag{1.21}$$

$$\psi(\cdot, t) = 0, \qquad t \ll 0. \tag{1.22}$$

Instead of imposing conditions on the integral of $a + b^n$ and $a_t + b^n_t$, as done in Theorem 1.3, we choose to have the value of ψ, ψ_t, ψ_{tt} on $t = T$ as part of our data. Again, it is not clear whether this is the optimum choice or a natural choice.

THEOREM 1.4 (stability for curl(a, b) and c recovery problem). *Suppose* $T > 0$ *and* $a, b, c, \acute{a}, \acute{b}, \acute{c}$ *are compactly supported smooth functions on* $\mathbb{R}^n \times (-\infty, T]$ *with* *support in* $\overline{B} \times [0,T]$. *If* $\|[a,b,c,\acute{a},\acute{b},\acute{c}]\|_{C^{7+[n/2]}} \leq M$, *then*

$$\begin{aligned}\|[c-\acute{c}, d\eta - d\acute{\eta}]\|_{L^2} \preccurlyeq \sum_{\omega} \int_{-1}^{T+1} \|(u-\acute{u})(\cdot,T;\omega,\tau)\|_{2,H_{\omega,\tau}} \\ + \sum_{\omega} \int_{-1}^{T+1} \|(u_t-\acute{u}_t)(\cdot,T;\omega,\tau)\|_{1,H_{\omega,\tau}} + \|(u_{tt}-\acute{u}_{tt})(\cdot,T;\omega,\tau)\|_{0,H_{\omega,\tau}}\, d\tau \\ + \sum_{\omega} \int_{-1}^{T+1} \|(v-\acute{v})(\cdot,T;\omega,\tau)\|_{1,H_{\omega,\tau}} + \|(v_t-\acute{v}_t)(\cdot,T;\omega,\tau)\|_{0,H_{\omega,\tau}}\, d\tau \\ + \|(\psi-\acute{\psi})(\cdot,T)\|_{2,\mathbb{R}^n} + \|\partial_t(\psi-\acute{\psi})(\cdot,T)\|_{1,\mathbb{R}^n} + \|\partial_t^2(\psi-\acute{\psi})(\cdot,T)\|_{0,\mathbb{R}^n}\end{aligned}$$

with ω *taking the values* $e^i, \pm e^n$, $i=1,\dots,n-1$. *Here*

$$\eta = a dt + \sum_{i=1}^{n} b^i dx^i, \qquad \acute{\eta} = \acute{a} dt + \sum_{i=1}^{n} \acute{b}^i dx^i,$$

$u, v, \acute{u}, \acute{v}$ *are the functions associated with* (a,b,c) *and* $(\acute{a},\acute{b},\acute{c})$ *in Propositions* 1.1 *and* 1.2*, and the constant depends on* M *and the supports of* $a,b,c,\acute{a},\acute{b},\acute{c}$.

The information about ψ is needed for the stability result in Theorem 1.4. This information corresponds to having (for odd n) the integral of $\nabla \cdot b - a_t + c$ on all light cones with vertices on $t=T$ and related quantities. For the even n case, it would be a weighted integral on such solid cones.

The above theorems used the traces on $t=T$ of u, v and their time derivatives for τ in the interval $[-1, T+1]$. As explained above, there is no additional information in the u, v solutions corresponding to $\tau < -1$ or $\tau > T+1$.

The values of the solutions $u(x,t;\omega,\tau)$ and $v(x,t;\omega,\tau)$ on the planes $t = \tau + x \cdot \omega$ are explicitly related to the unknown coefficients a, b, c. This fact and that, for a fixed omega, the planes $t = \tau + x\cdot\omega$ foliate the region $\mathbb{R}^n \times \mathbb{R}$ play a crucial role in the proofs of the results. Note that we do not collect any data on these hyperplanes.

The Carleman estimate with explicit boundary terms in Proposition 6.1 (in section 6) plays an important role in the proofs of the theorems. It is perhaps of mild interest that one can use the weight t in the Carleman estimate for the wave operator even though this weight is not strongly pseudoconvex. The proofs of our theorems do not require this particular weight; any increasing function of t, such as the traditional Carleman weight $e^{\lambda t}$ for some large λ, would be sufficient for use in our theorems.

We can obtain similar results if our data consist of the lateral boundary trace and final time trace on a bounded domain, that is, if we study the injectivity and stability of the map

$$(a,b,c) \to \left\{ \left[\partial_{x,t}^{\beta} u, \partial_{x,t}^{\beta} v\right]_{\tilde{H}_{\omega,\tau}}, \left[\partial_{x,t}^{\beta} u, \partial_{x,t}^{\beta} v\right]_{S_{\omega,\tau}} \right\}_{\omega\in\Omega,\, \tau\in(-\infty,T+1],\, |\beta|\leq 2},$$

where $\Omega = \{\pm e^i : i = 1,\dots,n\}$ and (see Figure 1.2)

$$\tilde{H}_{\omega,\tau} = \left(\overline{B} \times \{t=T\}\right) \cap Q_{\omega,\tau}, \qquad S_{\omega,\tau} = (\partial B \times (-\infty, T]) \cap Q_{\omega,\tau}.$$

To accomplish this, we would replace the Carleman estimate for the region $Q_{\omega,\tau}$ in Proposition 6.1 by a Carleman estimate for the region $(\overline{B}\times\mathbb{R}) \cap Q_{\omega,\tau}$, and the revised proofs would be almost identical to the proofs in this article. The proof of the modified Carleman estimate also would be almost identical to the proof of Proposition 6.1.

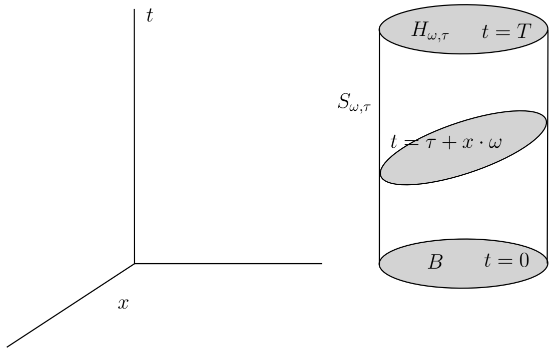

FIG. 1.2. *The cylindrical domain and its boundary.*

We compare the results for the $\Lambda_{a,b,c}$ inverse problem and our inverse problem. The data for the $\Lambda_{a,b,c}$ problem as well as our (modified in the previous paragraph) problem is measured on the lateral part and the top part of the space-time cylinder. We use a finite number of plane wave sources, whereas the $\Lambda_{a,b,c}$ problem requires "all possible sources" on the lateral and the bottom part of the space-time cylinder, so our problem is formally determined, whereas the $\Lambda_{a,b,c}$ problem is an overdetermined problem. We prove Lipschitz stability, whereas the results in the literature for the $\Lambda_{a,b,c}$ problem give logarithmic stability. Our results do have two weaknesses. We need sources which are plane waves in free space for $t \ll 0$, and hence our results do not say anything about the $\Lambda_{a,b,c}$ problem. The $\Lambda_{a,b,c}$ results include results where data are gathered only on a part of the lateral boundary and the top boundary, and we do not have such a result. We conjecture that there is Lipschitz stability for the $\Lambda_{a,b,c}$ problem.

We introduce definitions used throughout the article. We define the differences

$$\bar{u} = u - \acute{u},\ \bar{v} := v - \acute{v},\ \ \bar{a} := a - \acute{a},\ \ \bar{b} := b - \acute{b},\ \ \bar{c} = c - \acute{c},\ \ \bar{q} = q - \acute{q}.$$

Sometimes we suppress writing the a, b, c dependence of $\mathcal{L}_{a,b,c}$ and just use $\mathcal{L}$ and $\acute{\mathcal{L}}$, where $\acute{\mathcal{L}}$ corresponds to $\acute{a}, \acute{b}, \acute{c}$. We also have the corresponding functions α and $\acute{\alpha}$ defined in (1.11).

We also note that

$$(\partial_t + \omega \cdot \nabla - (a + \omega \cdot b))\alpha(x, t; \omega) = 0 \tag{1.23}$$

as seen from

$$\begin{aligned}\alpha^{-1}(\alpha_t + \omega \cdot \nabla\alpha)(x, t; \omega) &= \int_{-\infty}^{0} ((\partial_t + \omega \cdot \nabla)(a + \omega \cdot b))(x + s\omega, t + s)\, ds \\ &= \int_{-\infty}^{0} \frac{d}{ds}\left((a + \omega \cdot b)(x + s\omega, t + s)\right)\, ds \\ &= (a + \omega \cdot b)(x, t).\end{aligned}$$

2. Proof of Theorem 1.1.

In this theorem, $a = \acute{a}$, $b = \acute{b}$. Since ω is fixed, we suppress the dependence on ω.

Using (1.15), (1.17), its version for $\acute{a}, \acute{b}, \acute{c}$, and that $a = \acute{a}$, $b = \acute{b}$, the function $\bar{v}$ satisfies

$$\begin{aligned} \mathcal{L}\bar{v} &= -\bar{q}\acute{v} && \text{on } Q_\tau, \\ \bar{v} &= 0, && t \ll 0, \\ 2(\partial_t + \omega\cdot\nabla - (a+\omega\cdot b))\bar{v} &= -\bar{q}\alpha && \text{on } L_\tau. \end{aligned}$$

Since $\bar{q}$ is compactly supported, a domain of dependence argument shows that $\bar{v}$ is compactly supported when its domain is restricted to Q_τ. Hence, Proposition 6.1 is applicable.

Applying the Carleman estimate in Proposition 6.1 to $\bar{v}$ on the region Q_τ, we have

$$\sigma\|\bar{v}\|^2_{1,\sigma,L_\tau} \preccurlyeq \|\mathcal{L}\bar{v}\|^2_{0,\sigma,Q_\tau} + \sigma\|\bar{v}\|^2_{1,\sigma,H_\tau} + \sigma\|\partial_t\bar{v}\|^2_{0,\sigma,H_\tau}.$$

Since α is positive and bounded away from 0, on L_τ, we have

$$|\bar{q}| \preccurlyeq |\bar{q}\alpha| = 2|(\partial_t + \omega\cdot\nabla - (a+\omega\cdot b))\bar{v}| \preccurlyeq |\bar{v}_t| + |\nabla\bar{v}| + |\bar{v}|,$$

while on Q_τ

$$|\mathcal{L}\bar{v}| = |\bar{q}\acute{v}| \preccurlyeq |\bar{q}|;$$

hence,

$$\sigma\|\bar{q}\|^2_{0,\sigma,L_\tau} \preccurlyeq \|\bar{q}\|^2_{0,\sigma,Q_\tau} + \sigma\|\bar{v}\|^2_{1,\sigma,H_\tau} + \sigma\|\partial_t\bar{v}\|^2_{0,\sigma,H_\tau}. \tag{2.1}$$

We integrate (2.1) w.r.t. τ over $[-1, T+1)$. Noting that $\bar{q}$ is supported on $\overline{B}\times[0,T]$, we define $\bar{q} = 0$ for $t > T$ for convenience and L_τ to be the set $t = \tau + x\cdot\omega \le T$. The left-hand side of (2.1) is

$$\begin{aligned} &\sigma\int_{-1}^{T+1}\int_{\mathbb{R}^n, t=\tau+x\cdot\omega} e^{2\sigma t}\,|\bar{q}(x,t)|^2\,dx\,d\tau \\ &= \sigma\int_{\mathbb{R}}\int_{\mathbb{R}^n\times\mathbb{R}} e^{2\sigma t}\,|\bar{q}(x,t)|^2\delta(t-\tau-x\cdot\omega)\,dx\,dt\,d\tau \\ &= \sigma\int_{\mathbb{R}^n\times\mathbb{R}}\int_{\mathbb{R}} e^{2\sigma t}\,|\bar{q}(x,t)|^2\delta(t-\tau-x\cdot\omega)\,d\tau\,dx\,dt \\ &= \sigma\int_{\mathbb{R}^n\times\mathbb{R}} e^{2\sigma t}\,|\bar{q}(x,t)|^2\,dx\,dt \\ &= \sigma\|\bar{q}\|^2_{0,\sigma,\overline{B}\times[0,T]}. \end{aligned}$$

The integral w.r.t. τ over $[-1, T+1]$, of the right-hand side of (2.1), consists of the "data" part

$$\text{data} = \sigma\int_{-1}^{T+1} \|\bar{v}\|^2_{1,\sigma,H_\tau} + \sigma\|\partial_t\bar{v}\|^2_{0,\sigma,H_\tau}\,d\tau$$

and (using the support of $\bar{q}$)

$$\int_{-1}^{T+1}\|\bar{q}\|^2_{0,\sigma,Q_\tau}\,d\tau \le \int_{-1}^{T+1}\|\bar{q}\|^2_{0,\sigma,\overline{B}\times[0,T]}\,d\tau \preccurlyeq \|\bar{q}\|^2_{0,\sigma,\overline{B}\times[0,T]}.$$

Combining the two pieces, we have

$$\sigma\|\bar{q}\|^2_{0,\sigma,\overline{B}\times[0,T]} \preccurlyeq \|\bar{q}\|^2_{0,\sigma,\overline{B}\times[0,T]} + \text{data},$$

which gives us the estimate in Theorem 1.1 if we choose σ large enough.

3. Proof of Theorem 1.2. Here $q = \acute{q}$. The proof is similar to the proof of Theorem 1.1 except one uses the solution U.

We start with an intermediate estimate for a fixed ω, τ. We suppress the dependence on ω, τ during the derivation of this intermediate estimate. Using (1.12), (1.13), and their analogues for $\acute{a}, \acute{b}$ and that $q = \acute{q}$, $\bar{u}$ satisfies

$$\begin{aligned} \mathcal{L}\bar{u} &= 2\bar{a}\acute{u}_t - 2\bar{b}\cdot\nabla\acute{u} && \text{on } Q,\\ \bar{u} &= 0, && t \ll 0,\\ \bar{u} &= \alpha - \acute{\alpha} && \text{on } L. \end{aligned}$$

Since $\alpha - \acute{\alpha}$ and $\bar{a}, \bar{b}$ are compactly supported in Q, a domain of dependence argument shows that $\bar{u}$ is compactly supported in Q; hence, Proposition 6.1 is applicable.

Applying the Carleman estimate in Proposition 6.1 to $\bar{u}$ on the region Q, we obtain

$$\sigma\|\bar{u}\|^2_{1,\sigma,L} \preccurlyeq \|[\bar{a},\bar{b}]\|^2_{0,\sigma,Q} + \sigma\|\bar{u}\|^2_{1,\sigma,H} + \sigma\|\partial_t\bar{u}\|^2_{0,\sigma,H}. \tag{3.1}$$

Now, on L, using (1.23), we have

$$\begin{aligned} (\partial_t + \omega\cdot\nabla - (a+\omega\cdot b))(\alpha-\acute{\alpha}) &= -(\partial_t + \omega\cdot\nabla - (a+\omega\cdot b))\acute{\alpha}\\ &= -(\partial_t + \omega\cdot\nabla - (\acute{a}+\omega\cdot \acute{b}))\acute{\alpha} + (\bar{a}+\omega\cdot\bar{b})\acute{\alpha}\\ &= (\bar{a}+\omega\cdot\bar{b})\acute{\alpha}. \end{aligned}$$

Since $\acute{\alpha}$ is positive and bounded away from zero, we have

$$|\bar{a}+\omega\cdot\bar{b}| \preccurlyeq |(\partial_t + \omega\cdot\nabla - (a+\omega\cdot b))(\alpha-\acute{\alpha})|.$$

Using this in (3.1), we obtain

$$\sigma\|\bar{a}+\omega\cdot\bar{b}\|^2_{0,\sigma,L_\tau} \preccurlyeq \|[\bar{a},\bar{b}]\|^2_{0,\sigma,Q_\tau} + \sigma\|\bar{u}\|^2_{1,\sigma,H_\tau} + \sigma\|\partial_t\bar{u}\|^2_{0,\sigma,H_\tau}.$$

Integrating this w.r.t. τ over $[-1, T+1]$ and repeating the argument in the proof of Theorem 1.1, we obtain

$$\sigma\|\bar{a}+\omega\cdot\bar{b}\|^2_{0,\sigma,\overline{B}\times[0,T]} \preccurlyeq \|[\bar{a},\bar{b}]\|^2_{0,\sigma,\overline{B}\times[0,T]} + \sigma\int_{-1}^{T+1} \|\bar{u}\|^2_{1,\sigma,H_{\omega,\tau}} + \|\bar{u}_t\|^2_{0,\sigma,H_{\omega,\tau}}\,d\tau. \tag{3.2}$$

Noting that

$$2\bar{a} = (\bar{a} + e^n\cdot\bar{b}) + (\bar{a} - e^n\cdot\bar{b})$$

and

$$e^i\cdot\bar{b} = (\bar{a} + e^i\cdot\bar{b}) - \bar{a},$$

we obtain

$$\sigma\|[\bar{a},\bar{b}]\|^2_{0,\sigma,\overline{B}\times[0,T]} \preccurlyeq \|[\bar{a},\bar{b}]\|^2_{0,\sigma,\overline{B}\times[0,T]} + \sigma\sum_{\omega}\int_{-1}^{T+1} \|\bar{u}\|^2_{1,\sigma,H_{\omega,\tau}} + \|\bar{u}_t\|^2_{0,\sigma,H_{\omega,\tau}}\,d\tau,$$

where ω takes the values $-e^n$ and $e^1, \dots, e^n$. The theorem follows if we choose σ large enough.

4. Proof of Theorem 1.3. The proof proceeds as in the proofs of Theorems 1.1 and 1.2 but using both the U and the V solution. However, we need to add an idea from [22] to separate c from a, b.

We define

$$\phi(x,t) = -\int_{-\infty}^{0} (a+e^n \cdot b)(x+se^n, t+s)\, ds, \ \acute{\phi}(x,t) = -\int_{-\infty}^{0} (\acute{a}+e^n \cdot \acute{b})(x+se^n, t+s)\, ds.$$

Also, we are given that for $\tau \in [-1, T+1]$,

$$\begin{aligned} [u, u_t](\cdot, T; \omega, \tau) &= [\acute{u}, \acute{u}_t](\cdot, T; \omega, \tau) && \text{on } H_{\omega,\tau}, \ \omega = \pm e^n, e^i, i = 1, \dots, n-1, \\ [v, v_t](\cdot, T; e^n, \tau) &= [\acute{v}, \acute{v}_t](\cdot, T; e^n, \tau) && \text{on } H_{e^n,\tau}, \\ [\phi, \phi_t](\cdot, T) &= [\acute{\phi}, \acute{\phi}_t](\cdot, T) && \text{on } \mathbb{R}^n. \end{aligned}$$

From the introduction, we also know that $u, \acute{u}, v, \acute{v}$ are zero when $\tau > T+1$. Hence we have for $\tau \geq -1$ and $\omega = \pm e^n, e^i, \ i = 1, \dots, n-1$,

$$\begin{aligned} [e^{\phi} u, (e^{\phi} u)_t](\cdot, T; \omega, \tau) &= [e^{\acute{\phi}} \acute{u}, (e^{\acute{\phi}} \acute{u})_t](\cdot, T; \omega, \tau) && \text{on } H_{\omega,\tau}, \\ [e^{\phi} v, (e^{\phi} v)_t](\cdot, T; e^n, \tau) &= [e^{\acute{\phi}} \acute{v}, (e^{\acute{\phi}} \acute{v})_t](\cdot, T; e^n, \tau) && \text{on } H_{e^n,\tau}. \end{aligned}$$

The two sides correspond to the data for the coefficients $[a + \phi_t, b + \nabla\phi, c]$ and $[\acute{a} + \acute{\phi}_t, \acute{b} + \nabla\acute{\phi}, \acute{c}]$, so we work with this new set of coefficients. What we gain from this new set of coefficients is that

$$\begin{aligned} &((a + \phi_t) + e^n \cdot (b + \nabla\phi))(x,t) \\ &= (a + e^n \cdot b)(x,t) + (\partial_t + e^n \cdot \nabla)\phi(x,t) \\ &= (a + e^n \cdot b)(x,t) - (\partial_t + e^n \cdot \nabla) \int_{-\infty}^{0} (a + e^n \cdot b)(x + se_n, t+s)\, ds \\ &= (a + e^n \cdot b)(x,t) - \int_{-\infty}^{0} \frac{d}{ds}((a + e^n \cdot b)(x + se_n, t+s))\, ds \\ &= 0. \end{aligned}$$

Further, $[a, b]$ and $[a + \phi_t, b + \nabla\phi]$ have the same curl. So to prove our theorem, it is enough to show that if we have $[a, b, c]$ and $[\acute{a}, \acute{b}, \acute{c}]$ such that for $\tau \in [-1, T+1]$ and $\omega = \pm e^n, e^i, \ i = 1, \dots, n-1$,

$$\begin{aligned} [u, u_t](\cdot, T; \omega, \tau) &= [\acute{u}, \acute{u}_t](\cdot, T; \omega, \tau) \text{ on } H_{\omega,\tau}, \\ [v, v_t](\cdot, T; e^n, \tau) &= [\acute{v}, \acute{v}_t](\cdot, T; e^n, \tau) \text{ on } H_{e^n,\tau}, \end{aligned}$$

and

$$a + e^n \cdot b = 0, \quad \acute{a} + e^n \cdot \acute{b} = 0 \qquad \text{on } \mathbb{R}^n \times (-\infty, T],$$

then

$$[a, b, c] = [\acute{a}, \acute{b}, \acute{c}];$$

actually, we show

$$[a, b, q] = [\acute{a}, \acute{b}, \acute{q}],$$

which then implies $c = \acute{c}$.

Summarizing, we are given that for $\tau \geq -1$ and $\omega = \pm e^n, e^i, i = 1, \dots, n-1$,

$$[\bar{u}, \bar{u}_t](\cdot, T; \omega, \tau) = 0 \qquad \textit{on } H_{\omega,\tau}, \tag{4.1}$$

$$[\bar{v}, \bar{v}_t](\cdot, T; e^n, \tau) = 0 \qquad \textit{on } H_{e^n,\tau}, \tag{4.2}$$

and

$$a + e^n \cdot b = 0, \quad \acute{a} + e^n \cdot \acute{b} = 0 \qquad \text{on } \mathbb{R}^n \times (-\infty, T]. \tag{4.3}$$

We have to show that

$$[a, b, q] = [\acute{a}, \acute{b}, \acute{q}].$$

Note that for $t \leq T$, the supports of the new a, b, c and $\acute{a}, \acute{b}, \acute{c}$ need not be in $\overline{B} \times [0, T]$ but are in the larger region

$$K = \{(x, t) : |x| - 1 \leq t \leq T, t \geq 0\} \tag{4.4}$$

because, for $t \leq T$, ϕ and $\acute{\phi}$ defined at the start of this section are supported in this region. This was the reason why we emphasized the equivalence of working with data for $\tau \in [-1, T+1]$ for the original a, b, c and working with data for $\tau \geq -1$ for the new gauge modfied a, b, c. The support of the old a, b, c was swept out by the planes $t = \tau + x \cdot \omega$ when $\tau \in [-1, T+1]$ but one needs the larger τ interval $[-1, 2T+1]$ for the planes $t = \tau + x \cdot \omega$ to sweep out the support of the gauge modified a, b, c.

Using (1.12), (1.13), and its analogues for $\acute{a}, \acute{b}, \acute{c}$, the function $\bar{u}$ satisfies

$$\begin{aligned} \mathcal{L}\bar{u} = 2\bar{a}\acute{u}_t - 2\bar{b} \cdot \nabla \acute{u} - \bar{q}\acute{u} \qquad & \text{on } Q, \\ \bar{u} = 0, \qquad & t \ll 0, \\ \bar{u} = \alpha - \acute{\alpha} \qquad & \text{on } L. \end{aligned}$$

Repeating the argument in the proof of Theorem 1.2, the only difference being that $\mathcal{L}\bar{u}$ now has a $\bar{q}$ term on the right-hand side and that (4.1) holds, one obtains

$$\sigma \| [\bar{a}, \bar{b}] \|_{0,\sigma,K} \preccurlyeq \| [\bar{a}, \bar{b}, \bar{q}] \|_{0,\sigma,K}. \tag{4.5}$$

Next, we take $\omega = e^n$, and we suppress writing the explicit dependence on e^n. Using (1.15), (1.16), and its analogues for $\acute{a}, \acute{b}, \acute{c}$, the function $\bar{v}$ satisfies

$$\begin{aligned} \mathcal{L}\bar{v} = 2\bar{a}\acute{v}_t - 2\bar{b} \cdot \nabla \acute{v} - \bar{q}\acute{v} \qquad & \text{on } Q, \\ \bar{v} = 0, \qquad & t \ll 0. \end{aligned}$$

Applying the Carleman estimate in Proposition 6.1 to $\bar{v}$ in the region Q and noting (4.2), we have

$$\sigma \| \bar{v} \|^2_{1,\sigma,L_\tau} \preccurlyeq \| \mathcal{L}\bar{v} \|^2_{0,\sigma,Q_\tau} \preccurlyeq \| [\bar{a}, \bar{b}, \bar{q}] \|^2_{0,\sigma,Q_\tau}. \tag{4.6}$$

In this estimate, $\omega = e_n$ and from our discussion above, we know that $\alpha = 1$ and $\acute{\alpha} = 1$ in this case. So, on L, using (1.17) and its equivalent for $[\acute{a}, \acute{b}, \acute{c}]$, we have

$$\begin{aligned} 2(\partial_t + \omega \cdot \nabla - (a + \omega \cdot b))(\bar{v}) &= 2(\partial_t + \omega \cdot \nabla - (a + \omega \cdot b))(v - \acute{v}) \\ &= -\mathcal{L}\alpha - 2(\partial_t + \omega \cdot \nabla - (\acute{a} + \omega \cdot \acute{b}))\acute{v} + (\bar{a} + \omega \cdot \bar{b})\acute{v} \\ &= -\mathcal{L}\alpha + \acute{\mathcal{L}}\acute{\alpha} + (\bar{a} + \omega \cdot \bar{b})\acute{v} \\ &= -\bar{q} + (\bar{a} + \omega \cdot \bar{b})\acute{v} \\ &= -\bar{q}. \end{aligned}$$

Using this in (4.6), we obtain

$$\sigma \| \bar{q} \|^2_{0,\sigma,L_{e_n,\tau}} \preccurlyeq \| [\bar{a}, \bar{b}, \bar{q}] \|^2_{0,\sigma,Q_{e^n,\tau}}. \tag{4.7}$$

Integrating this over $\tau \in [-1, T+1]$ and using the arguments used in the proofs of the earlier theorems, we obtain

$$\sigma \|\bar{q}\|_{0,\sigma,K} \preccurlyeq \|[\bar{a}, \bar{b}, \bar{q}]\|_{0,\sigma,K}.$$

Combining this with (4.5), we obtain

$$\sigma \|[\bar{a}, \bar{b}, \bar{q}]\|_{0,\sigma,K} \preccurlyeq \|[\bar{a}, \bar{b}, \bar{q}]\|_{0,\sigma,K},$$

so taking σ large enough, we obtain $\bar{a} = 0, \bar{b} = 0, \bar{q} = 0$; hence, $(a, b) = (\acute{a}, \acute{b})$ and $c = \acute{c}$. However, these $a, b, \acute{a}, \acute{b}$ are the ϕ modified versions of the old $a, b, \acute{a}, \acute{b}$, so we obtain

$$d\left(a dt + \sum_{i=1}^{n} b^i dx^i\right) = d\left(\acute{a} dt + \sum_{i=1}^{n} \acute{b}^i dx^i\right)$$

for the older $a, b, \acute{a}, \acute{b}$. Of course, we have already shown $c = \acute{c}$.

5. Proof of Theorem 1.4. As shown in the introduction, we have $u(\cdot, \cdot, \tau, \omega) = 0$, $v(\cdot, \cdot, \tau, \omega) = 0$ if $\tau > T + 1$ so, in the statement of Theorem 1.4 we can replace the upper limits $T + 1$, in the integrals, by ∞ (we need just $2T + 1$). This change will be useful when we replace a, b, c by a gauge modfied a, b, c.

From the introduction, we know that if u, v are the solutions associated with the coefficients $[a, b, c]$, then $e^{\psi} u, e^{\psi} v$ are the solutions associated with the coefficients $[a + \psi_t, b + \nabla\psi, c]$. Further, using $|e^s - 1| \leq e^M |s|$ for all $s \in [-M, M]$, we have

$$\begin{aligned} |e^{\psi} w - e^{\acute{\psi}} \acute{w}| &\leq |e^{\psi} w - e^{\psi} \acute{w}| + |e^{\psi} \acute{w} - e^{\acute{\psi}} \acute{w}| \\ &\preccurlyeq |w - \acute{w}| + |e^{\psi - \acute{\psi}} - 1| \\ &\preccurlyeq |w - \acute{w}| + |\psi - \acute{\psi}|. \end{aligned}$$

Similar estimates hold for the first- and second-order derivatives of $e^{\psi} w - e^{\acute{\psi}} \acute{w}$. Further, $[a, b]$ and $[a + \psi_t, b + \nabla\psi]$ have the same curl, so we may assume we are working with the coefficients $[a + \psi_t, b + \nabla\psi, c]$. Now

$$c - (a + \psi_t)_t + \nabla \cdot (b + \nabla\psi) = c - a_t + \nabla \cdot b - \Box\psi = 0.$$

So it is enough to prove Theorem 1.4 with the assumption that

$$c - a_t + \nabla \cdot b = 0, \qquad \acute{c} - \acute{a}_t + \nabla \cdot \acute{b} = 0; \tag{5.1}$$

note this also implies $\psi = 0, \ \acute{\psi} = 0$.

As in the proof of Theorem 1.3, the supports of the new a, b, c and $\acute{a}, \acute{b}, \acute{c}$ (restricted to $t \leq T$) are in the larger region

$$K = \{(x, t) : |x| - 1 \leq t \leq T, t \geq 0\}. \tag{5.2}$$

Note that the support of the new a, b, c is swept out by the planes $t = \tau + x \cdot \omega$ as τ varies over $[-1, 2T + 1]$ - this was the reason, at the beginning of the proof, we justified the increase in the range of τ from $[-1, T + 1]$ to $[-1, 2T + 1]$.

Given the unit vector ω, we define the orthogonal decompositions

$$\nabla = \nabla_\omega + \nabla_\omega^\perp, \qquad b = b_\omega + b_\omega^\perp,$$

where

$$\nabla_\omega := \omega(\omega \cdot \nabla), \quad \nabla_\omega^\perp := \nabla - \omega(\omega \cdot \nabla), \quad b_\omega := (\omega \cdot b)\omega, \quad b_\omega^\perp := b - (\omega \cdot b)\omega.$$

Note that

$$\nabla_\omega \cdot \nabla_\omega^\perp = 0 = \nabla_\omega^\perp \cdot \nabla_\omega, \quad \omega \cdot \nabla_\omega^\perp = 0, \quad b_\omega \cdot \nabla_\omega^\perp = 0 = \nabla_\omega^\perp \cdot b_\omega, \quad b_\omega^\perp \cdot \nabla_\omega = 0 = \nabla_\omega \cdot b_\omega^\perp.$$

We obtain some intermediate estimates, and, for convenience, *temporarily we suppress the dependence on* τ.

Estimate from the U *solution.* Using (5.1), we have

$$\mathcal{L} = \Box - 2a\partial_t + 2b \cdot \nabla + a^2 - b^2, \qquad \acute{\mathcal{L}} = \Box - 2\acute{a}\partial_t + 2\acute{b} \cdot \nabla + \acute{a}^2 - \acute{b}^2;$$

hence, from (1.12), (1.13), we have

$$\mathcal{L}\bar{u} = 2\bar{a}\acute{u}_t - 2\bar{b} \cdot \nabla \acute{u} + ((b + \acute{b})\bar{b} - (a + \acute{a})\bar{a})\acute{u} \qquad \text{on } Q_\omega, \tag{5.3}$$

$$\bar{u} = 0, \qquad t \ll 0, \tag{5.4}$$

$$\bar{u} = \alpha - \acute{\alpha} \qquad \text{on } L_\omega. \tag{5.5}$$

So Proposition 6.1 applied to $\bar{u}$ in the region Q_ω gives us

$$\|\bar{u}\|^2_{1,\sigma,Q_\omega} + \|\alpha - \acute{\alpha}\|^2_{1,\sigma,L_\omega} \preccurlyeq \frac{1}{\sigma}\|[\bar{a}, \bar{b}]\|^2_{0,\sigma,Q_\omega} + \|\bar{u}\|^2_{1,\sigma,H_\omega} + \|\partial_t \bar{u}\|^2_{0,\sigma,H_\omega}. \tag{5.6}$$

Next, we obtain higher-order estimates by differentiating (5.3)–(5.5), keeping in mind that $\nabla_\omega^\perp$ and $\partial_t + \omega \cdot \nabla$ span the tangent space to L_ω.

We have

$$\begin{aligned} \mathcal{L}(\nabla_\omega^\perp \bar{u}) = \nabla_\omega^\perp(2\bar{a}\acute{u}_t - 2\bar{b} \cdot \nabla \acute{u} + ((b + \acute{b})\bar{b} - (a + \acute{a})\bar{a})\acute{u}) + [\mathcal{L}, \nabla_\omega^\perp]\bar{u} \qquad & \text{on } Q_\omega, \\ \nabla_\omega^\perp \bar{u} = 0, \qquad & t \ll 0, \\ \nabla_\omega^\perp \bar{u} = \nabla_\omega^\perp(\alpha - \acute{\alpha}) \qquad & \text{on } L_\omega, \end{aligned}$$

so, in Q_ω,

$$|\mathcal{L}(\nabla_\omega^\perp \bar{u})| \preccurlyeq |[\bar{a}, \bar{b}, \nabla\bar{a}, \nabla\bar{b}, \bar{a}_t, \bar{b}_t]| + |[\bar{u}, \nabla\bar{u}, \partial_t\bar{u}]|.$$

Hence, applying Proposition (6.1) to $\nabla_\omega^\perp \bar{u}$, we obtain

$$\begin{aligned} \|\nabla_\omega^\perp(\alpha - \acute{\alpha})\|^2_{1,\sigma,L_\omega} \preccurlyeq \frac{1}{\sigma}\|[\bar{a}, \bar{b}, \nabla\bar{a}, \nabla\bar{b}, \bar{a}_t, \bar{b}_t]\|^2_{0,\sigma,Q_\omega} + \frac{1}{\sigma}\|\bar{u}\|^2_{1,\sigma,Q_\omega} \\ + \|\nabla_\omega^\perp \bar{u}\|^2_{1,\sigma,H_\omega} + \|\partial_t \nabla_\omega^\perp \bar{u}\|^2_{0,\sigma,H_\omega}. \end{aligned} \tag{5.7}$$

We repeat the argument used to obtain (5.7) with differentiation w.r.t. $\partial_t + \omega \cdot \nabla$ replacing differentiation w.r.t. $\nabla_\omega^\perp$. Noting that $\partial_t + \omega \cdot \nabla$ is also tangential to L_ω, we obtain

$$\begin{aligned} \|(\partial_t + \omega \cdot \nabla)(\alpha - \acute{\alpha})\|^2_{1,\sigma,L_\omega} \preccurlyeq \frac{1}{\sigma}\|[\bar{a}, \bar{b}, \nabla\bar{a}, \nabla\bar{b}, \bar{a}_t, \bar{b}_t]\|^2_{0,\sigma,Q_\omega} + \frac{1}{\sigma}\|\bar{u}\|^2_{1,\sigma,Q_\omega} \\ + \|[\nabla\bar{u}, \partial_t\bar{u}]\|^2_{1,\sigma,H_\omega} + \|\partial_t^2 \bar{u}\|^2_{0,\sigma,H_\omega}. \end{aligned} \tag{5.8}$$

Using (5.6), (5.7), (5.8), for σ large enough, we obtain

$$\|(\partial_t+\omega\cdot\nabla)(\alpha-\acute{\alpha})\|^2_{1,\sigma,L_\omega}+\|\nabla^\perp_\omega(\alpha-\acute{\alpha})\|^2_{1,\sigma,L_\omega}+\|\alpha-\acute{\alpha}\|^2_{1,\sigma,L_\omega} \preccurlyeq \frac{1}{\sigma}\|[\bar{a},\bar{b},\nabla\bar{a},\nabla\bar{b},\bar{a}_t,\bar{b}_t]\|^2_{0,\sigma,Q_\omega}+\|[\bar{u},\nabla\bar{u},\partial_t\bar{u}]\|^2_{1,\sigma,H_\omega}+\|\partial_t^2\bar{u}\|^2_{0,\sigma,H_\omega}. \tag{5.9}$$

We use (5.9) to estimate $\bar{a},\bar{b}$. From (1.23),

$$\begin{aligned}(\partial_t+\omega\cdot\nabla-(a+\omega\cdot b))(\alpha-\acute{\alpha}) &= -(\partial_t+\omega\cdot\nabla-(a+\omega\cdot b))\acute{\alpha}\\ &= -(\partial_t+\omega\cdot\nabla-(\acute{a}+\omega\cdot\acute{b}))\acute{\alpha}+(\bar{a}+\omega\cdot\bar{b})\acute{\alpha}\\ &= (\bar{a}+\omega\cdot\bar{b})\acute{\alpha},\end{aligned} \tag{5.10}$$

and $\acute{\alpha}$ is positive and bounded away from zero. Hence,

$$|\bar{a}+\omega\cdot\bar{b}| \preccurlyeq |(\partial_t+\omega\cdot\nabla)(\alpha-\acute{\alpha})|+|\alpha-\acute{\alpha}|. \tag{5.11}$$

Differentiating (5.10) w.r.t. $\nabla^\perp_\omega$ and noting that $\nabla^\perp_\omega$ commutes with $\partial_t+\omega\cdot\nabla$, we obtain

$$|\nabla^\perp_\omega(\bar{a}+\omega\cdot\bar{b})| \preccurlyeq |\bar{a}+\omega\cdot\bar{b}|+|\nabla^\perp_\omega(\partial_t+\omega\cdot\nabla)(\alpha-\acute{\alpha})|+|\nabla^\perp_\omega(\alpha-\acute{\alpha})| + |\alpha-\acute{\alpha}|. \tag{5.12}$$

Differentiating (5.10) w.r.t. $\partial_t+\omega\cdot\nabla$, we obtain

$$|(\partial_t+\omega\cdot\nabla)(\bar{a}+\omega\cdot\bar{b})| \preccurlyeq |\bar{a}+\omega\cdot\bar{b}|+|(\partial_t+\omega\cdot\nabla)^2(\alpha-\acute{\alpha})| + |(\partial_t+\omega\cdot\nabla)(\alpha-\acute{\alpha})|+|\alpha-\acute{\alpha}|. \tag{5.13}$$

Since $\nabla^\perp_\omega$ and $\partial_t+\omega\cdot\nabla$ are tangential to L_ω, using (5.11), (5.12), (5.13), and (5.9), we conclude

$$\begin{aligned}&\|[(\partial_t+\omega\cdot\nabla)(\bar{a}+\omega\cdot\bar{b}),\,\nabla^\perp_\omega(\bar{a}+\omega\cdot\bar{b}),\bar{a}+\omega\cdot\bar{b}]\|^2_{0,\sigma,L_\omega}\\ &\quad+\|[\nabla^\perp_\omega(\alpha-\acute{\alpha}),\alpha-\acute{\alpha}]\|^2_{1,\sigma,L_\omega} \preccurlyeq \frac{1}{\sigma}\|[\bar{a},\bar{b},\nabla\bar{a},\nabla\bar{b},\bar{a}_t,\bar{b}_t]\|^2_{0,\sigma,Q_\omega}\\ &\qquad+\|[\bar{u},\nabla\bar{u},\partial_t\bar{u}]\|^2_{1,\sigma,H_\omega}+\|\bar{u}_{tt}\|^2_{0,\sigma,H_\omega}\end{aligned} \tag{5.14}$$

for large enough σ.

Estimate from the V solution. Using (1.15), (1.16), and its version for $\acute{a},\acute{b},\acute{c}$, the function $\bar{v}$ satisfies

$$\begin{aligned}\mathcal{L}\bar{v} = 2\bar{a}\acute{v}_t-2\bar{b}\cdot\nabla\acute{v}+((b+\acute{b})\bar{b}-(a+\acute{a})\bar{a})\acute{v} &\qquad \text{on } Q_\omega,\\ \bar{v}=0, &\qquad t\ll 0.\end{aligned}$$

Hence, applying Proposition (6.1) to $\bar{v}$ over the region Q_ω, we obtain

$$\|\bar{v}\|^2_{1,\sigma,L_\omega} \preccurlyeq \frac{1}{\sigma}\|[\bar{a},\bar{b}]\|^2_{0,\sigma,Q_\omega}+\|\bar{v}\|^2_{1,\sigma,H_\omega}+\|\partial_t\bar{v}\|^2_{0,\sigma,H_\omega}. \tag{5.15}$$

On L_ω,

$$2(\partial_t+\omega\cdot\nabla-(a+\omega\cdot b))v=-\mathcal{L}\alpha, \qquad 2(\partial_t+\omega\cdot\nabla-(\acute{a}+\omega\cdot\acute{b}))\acute{v}=-\acute{\mathcal{L}}\acute{\alpha};$$

hence,

$$2(\partial_t + \omega\cdot\nabla - (a+\omega\cdot b))(\bar v) = 2(\partial_t + \omega\cdot\nabla - (a+\omega\cdot b))(v-\acute v)$$
$$= -\mathcal{L}\alpha - 2(\partial_t + \omega\cdot\nabla - (\acute a+\omega\cdot\acute b))\acute v + 2(\bar a+\omega\cdot\bar b)\acute v$$
$$= -\mathcal{L}\alpha + \acute{\mathcal{L}}\acute\alpha + 2(\bar a+\omega\cdot\bar b)\acute v,$$

implying

$$|(\partial_t + \omega\cdot\nabla - (a+\omega\cdot b))(\bar v)| \succcurlyeq |\mathcal{L}\alpha - \acute{\mathcal{L}}\acute\alpha| - |[\bar a,\bar b]|,$$

which used in (5.15) gives us

$$\|\mathcal{L}\alpha - \acute{\mathcal{L}}\acute\alpha\|^2_{0,\sigma,L_\omega} \preccurlyeq \|[\bar a,\bar b]\|^2_{0,\sigma,L_\omega} + \frac{1}{\sigma}\|[\bar a,\bar b]\|^2_{0,\sigma,Q_\omega} + \|\bar v\|^2_{1,\sigma,H_\omega} + \|\partial_t\bar v\|^2_{0,\sigma,H_\omega}. \tag{5.16}$$

We need a different representation for $\mathcal{L}\alpha - \acute{\mathcal{L}}\acute\alpha$. We claim

$$\mathcal{L}\alpha = \alpha(\partial_t - \omega\cdot\nabla)(a+\omega\cdot b) - \left({\nabla^\perp_\omega}^2 - 2b^\perp_\omega\cdot\nabla^\perp_\omega + {b^\perp_\omega}^2\right)\alpha, \tag{5.17}$$

provided $c - a_t + \nabla\cdot b = 0$. We postpone the proof of (5.17) to the end of this section. Then

$$\mathcal{L}\alpha - \acute{\mathcal{L}}\acute\alpha = \alpha\,(\partial_t - \omega\cdot\nabla)\,(\bar a+\omega\cdot\bar b) + (\alpha-\acute\alpha)\,(\partial_t - \omega\cdot\nabla)\,(\acute a+\omega\cdot\acute b) - {\nabla^\perp_\omega}^2(\alpha-\acute\alpha)$$
$$+ 2b^\perp_\omega\cdot\nabla^\perp_\omega(\alpha-\acute\alpha) - {b^\perp_\omega}^2(\alpha-\acute\alpha) + 2\bar b^\perp_\omega\acute\alpha - \bar b^\perp_\omega\cdot(b+\acute b)^\perp_\omega\acute\alpha.$$

Using this and that α is bounded away from zero, we have

$$|\mathcal{L}\alpha - \acute{\mathcal{L}}\acute\alpha| \succcurlyeq \left|(\partial_t - \omega\cdot\nabla)\,(\bar a+\omega\cdot\bar b)\right| - |{\nabla^\perp_\omega}^2(\alpha-\acute\alpha)| - |\nabla^\perp_\omega(\alpha-\acute\alpha)| - |\alpha-\acute\alpha| - |\bar b|,$$

which used in (5.16) gives us

$$\|\,(\partial_t - \omega\cdot\nabla)\,(\bar a+\omega\cdot\bar b)\|^2_{0,\sigma,L_\omega} \preccurlyeq \|[\bar a,\bar b]\|^2_{0,\sigma,L_\omega} + \|{\nabla^\perp_\omega}^2(\alpha-\acute\alpha)\|^2_{0,\sigma,L_\omega} \tag{5.18}$$
$$+ \|\nabla^\perp_\omega(\alpha-\acute\alpha)\|^2_{0,\sigma,L_\omega} + \|\alpha-\acute\alpha\|^2_{0,\sigma,L_\omega} + \frac{1}{\sigma}\|[\bar a,\bar b]\|^2_{0,\sigma,Q_\omega} + \|\bar v\|^2_{1,\sigma,H_\omega} + \|\partial_t\bar v\|^2_{0,\sigma,H_\omega}.$$

Combining the U, V estimates. Multiplying the V-based estimate (5.18) by a small ϵ (independent of σ) in $(0,1)$ and adding it to the combined U based estimate (5.14), we obtain

$$\epsilon\|\,(\partial_t - \omega\cdot\nabla)\,(\bar a+\omega\cdot\bar b)\|^2_{0,\sigma,L_{\omega,\tau}} + \|\bar a+\omega\cdot\bar b\|^2_{0,\sigma,L_{\omega,\tau}} \tag{5.19}$$
$$+ \|\nabla^\perp_\omega(\bar a+\omega\cdot\bar b)\|^2_{0,\sigma,L_{\omega,\tau}} + \|(\partial_t + \omega\cdot\nabla)(\bar a+\omega\cdot\bar b)\|^2_{0,\sigma,L_{\omega,\tau}}$$
$$\preccurlyeq \epsilon\|[\bar a,\bar b]\|^2_{0,\sigma,L_{\omega,\tau}} + \frac{1}{\sigma}\|[\bar a,\bar b,\nabla\bar a,\nabla\bar b,\bar a_t,\bar b_t]\|^2_{0,\sigma,Q_{\omega,\tau}} + \mathrm{data}_{\omega,\tau,\sigma},$$

where

$$\mathrm{data}_{\omega,\tau,\sigma} = \|[\nabla\bar u,\bar u_t,\bar u]\|^2_{1,\sigma,H_{\omega,\tau}} + \|\bar u_{tt}\|^2_{0,\sigma,H_{\omega,\tau}} + \|\bar v\|^2_{1,\sigma,H_{\omega,\tau}} + \|\bar v_t\|^2_{0,\sigma,H_{\omega,\tau}}.$$

Expanding the left-hand side of (5.19), we get (noting that $\epsilon < 1$)

$$\|\bar a+\omega\cdot\bar b\|^2_{0,\sigma,L_{\omega,\tau}} + \epsilon\|\nabla^\perp_\omega(\bar a+\omega\cdot\bar b)\|^2_{0,\sigma,L_{\omega,\tau}}$$
$$+ \epsilon\|\partial_t(\bar a+\omega\cdot\bar b)\|^2_{0,\sigma,L_{\omega,\tau}} + \epsilon\|(\omega\cdot\nabla)(\bar a+\omega\cdot\bar b)\|^2_{0,\sigma,L_{\omega,\tau}}$$
$$\preccurlyeq \epsilon\|[\bar a,\bar b]\|^2_{0,\sigma,L_{\omega,\tau}} + \frac{1}{\sigma}\|[\bar a,\bar b,\nabla\bar a,\nabla\bar b,\bar a_t,\bar b_t]\|^2_{0,\sigma,Q_{\omega,\tau}} + \mathrm{data}_{\omega,\tau,\sigma},$$

which immediately gives

$$\|\bar{a}+\omega\cdot\bar{b}\|^2_{0,\sigma,L_{\omega,\tau}}+\epsilon\|\nabla_{x,t}(\bar{a}+\omega\cdot\bar{b})\|^2_{0,\sigma,L_{\omega,\tau}} \preccurlyeq \epsilon\|[\bar{a},\bar{b}]\|^2_{0,\sigma,L_{\omega,\tau}}+\frac{1}{\sigma}\|[\bar{a},\bar{b},\nabla\bar{a},\nabla\bar{b},\bar{a}_t,\bar{b}_t]\|^2_{0,\sigma,Q_{\omega,\tau}}+\text{data}_{\omega,\tau,\sigma}.$$

Integrating this w.r.t. τ over $[-1,2T+1]$ (note that the supports of the gauge modified a,b,c are swept out by the planes $t=\tau+x\cdot\omega$ as τ varies in $[-1,2T+1]$) and repeating the argument used at the end of the proof of Theorem 1.1, we obtain

$$\|\bar{a}+\omega\cdot\bar{b}\|^2_{0,\sigma,\mathbb{R}^n\times[0,T]}+\epsilon\|\nabla_{x,t}(\bar{a}+\omega\cdot\bar{b})\|^2_{0,\sigma,\mathbb{R}^n\times[0,T]} \preccurlyeq \epsilon\|[\bar{a},\bar{b}]\|^2_{0,\sigma,\mathbb{R}^n\times[0,T]}+\frac{1}{\sigma}\|[\bar{a},\bar{b},\nabla\bar{a},\nabla\bar{b},\bar{a}_t,\bar{b}_t]\|^2_{0,\sigma,\mathbb{R}^n\times[0,T]}+\int_{-1}^{2T+1}\text{data}_{\omega,\tau,\sigma}\,d\tau.$$

All norms below are $\|\cdot\|_{0,\sigma,\mathbb{R}^n\times[0,T]}$ unless noted otherwise. To complete the proof, we repeat the argument in the proof of Theorem 1.4. That is, we vary ω in the set $\{\pm e^n, e^i\ ; 1\le i\le n-1\}$ and finally obtain

$$\|[\bar{a},\bar{b}]\|^2+\epsilon\|[\nabla_{x,t}\bar{a},\nabla_{x,t}\bar{b}]\|^2 \preccurlyeq \epsilon\|[\bar{a},\bar{b}]\|^2+\frac{1}{\sigma}\|[\nabla_{x,t}\bar{a},\nabla_{x,t}\bar{b}]\|^2 + \sum_{\omega}\int_{-1}^{2T+1}\text{data}_{\omega,\tau,\sigma}\,d\tau.$$

So taking ϵ small enough and then fixing a σ large enough,

$$\|[\bar{a},\bar{b},\nabla_{x,t}\bar{a},\nabla_{x,t}\bar{b}]\|^2 \preccurlyeq \sum_{\omega}\int_{-1}^{2T+1}\text{data}_{\omega,\tau,\sigma}\,d\tau.$$

Since $\bar{c}=\bar{a}_t-\nabla\cdot\bar{b}$, we conclude

$$\|[\bar{a},\bar{b},\bar{c},\nabla_{x,t}\bar{a},\nabla_{x,t}\bar{b}]\|^2 \preccurlyeq \sum_{\omega}\int_{-1}^{2T+1}\text{data}_{\omega,\tau,\sigma}\,d\tau$$

for the fixed large enough σ.

For a fixed σ, on a compact set, the weighted and unweighted norms are equivalent, so the theorem is proved. It remains to show (5.17) when $c=a_t-\nabla\cdot b$.

Proof of (5.17). We note that

$$\begin{aligned}\mathcal{L}\alpha &= \left(\partial_t^2-\Delta-2a\partial_t+2b\cdot\nabla-a_t+\nabla\cdot b+a^2-b^2+c\right)\alpha\\ &= \left(\partial_t^2-(\omega\cdot\nabla)^2-{\nabla_\omega^\perp}^2-2a\partial_t+2b\cdot\nabla+a^2-b^2\right)\alpha\\ &= \left((\partial_t-\omega\cdot\nabla)(\partial_t+\omega\cdot\nabla)-{\nabla_\omega^\perp}^2-2a\partial_t+2b\cdot\nabla+a^2-b^2\right)\alpha.\end{aligned}$$

Hence, using (1.23),

$$\begin{aligned}
\mathcal{L}\alpha &= (\partial_t - \omega\cdot\nabla)\left((a+\omega\cdot b)\alpha\right) - \left({\nabla_\omega^\perp}^2 + 2a\partial_t - 2b\cdot\nabla - a^2 + b^2\right)\alpha \\
&= \alpha(\partial_t - \omega\cdot\nabla)(a+\omega\cdot b) - \left(-(a+\omega\cdot b)(\partial_t - \omega\cdot\nabla) + {\nabla_\omega^\perp}^2 + 2a\partial_t - 2b\cdot\nabla\right)\alpha \\
&\qquad - \left(-a^2+b^2\right)\alpha \\
&= \alpha(\partial_t - \omega\cdot\nabla)(a+\omega\cdot b) - \left((a-\omega\cdot b)(\partial_t + \omega\cdot\nabla) + {\nabla_\omega^\perp}^2 - 2(b-\omega(\omega\cdot b))\cdot\nabla\right)\alpha \\
&\qquad - \left(-a^2+b^2\right)\alpha \\
&= \alpha(\partial_t - \omega\cdot\nabla)(a+\omega\cdot b) - \left((a-\omega\cdot b)(a+\omega\cdot b) + {\nabla_\omega^\perp}^2 - 2(b-\omega(\omega\cdot b))\cdot\nabla\right)\alpha \\
&\qquad - \left(-a^2+b^2\right)\alpha \\
&= \alpha(\partial_t - \omega\cdot\nabla)(a+\omega\cdot b) - \left({\nabla_\omega^\perp}^2 - 2b_\omega^\perp\cdot\nabla_\omega^\perp + {b_\omega^\perp}^2\right)\alpha. \qquad \square
\end{aligned}$$

6. The Carleman estimate. We show that the standard Carleman estimate with boundary terms holds for the operator $\mathcal{L}_{a,b,c}$ with the Carleman weight t over the region $Q_{\omega,\tau}$. We need the explicit boundary terms in the proofs of our theorems. Here a, b^i, c are compactly supported smooth functions on $\mathbb{R}^n\times[0,T]$.

PROPOSITION 6.1. *If $w(x,t)$ is a compactly supported C^3 function on $Q_{\omega,\tau}$, then for large enough σ, we have*

$$\sigma\int_{Q_{\omega,\tau}} e^{2\sigma t}(|\nabla_{x,t}w|^2+\sigma^2|w|^2) + \sigma\int_{L_{\omega,\tau}} e^{2\sigma t}(|\nabla_L w|^2+\sigma^2|w|^2) \preccurlyeq \int_{Q_{\omega,\tau}} e^{2\sigma t}|\mathcal{L}_{a,b,c}w|^2 + \sigma\int_{H_{\omega,\tau}} e^{2\sigma t}(|\nabla_{x,t}w|^2+\sigma^2|w|^2) \tag{6.1}$$

with the constant independent of w and σ. Here ∇_L is the gradient operator on the plane $L_{\omega,\tau}$.

Proof. This proposition could probably be proved by using energy estimates coming from standard multipliers, but we use Carleman estimates since we have already calculated the boundary terms in [23] for a general situation. Below, we use the notation used for Theorem A.7 in [23].

We appeal to Theorem A.7 of [23]. The hypothesis of Theorem A.7 requires the strong pseudoconvexity of ϕ, but the proof of Theorem A.7 just needs that the relation (A.9) (in Lemma A.6) holds. One can verify that (A.9) holds for the wave operator and $\phi(x,t)=t$. In fact, (A.9) holds because there are no *"$(x,\xi,\sigma)\in\bar{\Omega}\times S$ with $p(x,\xi)-\sigma^2p(x,\partial\phi)=0$ and $\{p,\phi\}(x,\xi)=0$"* as we show next. In [23], S is the set of $(\xi,\sigma)\in\mathbb{R}^n\times\mathbb{R}$ with $|\xi|^2+\sigma^2=1$. We have $p(x,t,\xi,\tau)=-\tau^2+|\xi|^2$ and $\phi(x,t)=t$. Hence,

$$0=\{p,\phi\}(x,t,\xi,\tau)=p_\tau\phi_t=-2\tau$$

and

$$0=p(x,t,\xi,\tau)-\sigma^2p(x,t,\nabla\phi,\phi_t)=-\tau^2+|\xi|^2+\sigma^2$$

imply $\tau=0,\xi=0,\sigma=0$; hence, there are no points "$(x,\xi,\sigma)\in\bar{\Omega}\times S$ with $p(x,\xi)-\sigma^2p(x,\partial\phi)=0$ and $\{p,\phi\}(x,\xi)=0$."

To apply Theorem A.7 in [23], one needs to work in a bounded domain, but $Q_{\omega,\tau}$ is unbounded. However, w has compact support, so apply Theorem A.7 in [23] to a region consisting of the intersection of $Q_{\omega,\tau}$ with a cylinder $D\times(-\infty,T]$ large enough to contain the support of w. Since w and its derivatives will have zero traces on the part of the lateral boundary of $Q_{\omega,\tau}\cap D\times(-\infty,T]$ that lies on $(\partial D)\times(-\infty,T]$, we may apply Theorem A.7 in [23] to the region $Q_{\omega,\tau}$.

The proposition will follow from an analysis of the boundary terms in the statement of Theorem A.7. The principal part of $\mathcal{L}_{a,b,c}$ is the wave operator, and without loss of generality, we assume that $\tau=0$, $x=(y,z)$ with $y\in\mathbb{R}^{n-1}$, $z\in\mathbb{R}$ and ω is the unit vector in the direction of the positive z axis; hence, $L_{\omega,\tau}$ is the plane $t=z$.

The boundary term on $t=z$ has been computed in subsection A.2 in [23] and is given by

$$\begin{aligned}\frac{1}{\sqrt{2}}\nu_j E_j &= (\phi_t-\phi_z)(u_z+u_t)^2+(\phi_z+\phi_t)|u_y|^2-2(u_z+u_t)(u_y\cdot\phi_y)\\ &\quad-\sigma^2(\phi_z+\phi_t)(|\phi_x|^2-\phi_t^2)u^2-(u_z+u_t)g(x,t)u,\end{aligned}$$

where $u=we^{\sigma\phi}$ and g is some smooth function independent of σ. Hence, on $t=z$ for $\phi=t$, we have $u=we^{\sigma t}$ and

$$\begin{aligned}\frac{1}{\sqrt{2}}\nu_j E_j &= (u_z+u_t)^2+|u_y|^2+\sigma^2u^2-(u_z+u_t)g(x,t)u\\ &\geq (u_z+u_t)^2+|u_y|^2+\sigma^2u^2-\frac{1}{2}(u_z+u_t)^2-ku^2\\ &=\frac{1}{2}(u_z+u_t)^2+|u_y|^2+(\sigma^2-k)u^2 \qquad k \text{ independent of } \sigma\\ &\succcurlyeq e^{2\sigma t}((w_z+w_t)^2+|w_y|^2+\sigma^2w^2) \qquad \text{using a standard argument}\end{aligned}$$

for σ large enough.

To get the boundary terms on $t=T$, we again go to the expressions in subsection A.2 on [23] for the wave operator. Here $\nu_x=0$ and $\nu_t=(0,0,\ldots,0,1)$; hence, $\nu_jE_j=0$ for $j=1,\ldots,n$ and

$$\nu_tE_t=-\phi_t(|u_x|^2-u_t^2)+\sigma^2\phi_t(|\phi_x|^2-\phi_t^2)u^2+2u_t(u_x\cdot\phi_x-u_t\phi_t)+g(x,t)u_tu.$$

Hence, on $t=T$, for $\phi=t$, we have

$$\begin{aligned}\nu_tE_t &= -(|u_x|^2-u_t^2)-\sigma^2u^2-2u_t^2+g(x,t)u_tu\\ &=-(|u_x|^2+u_t^2)-\sigma^2u^2+g(x,t)u_tu,\end{aligned}$$

which implies

$$|\nu_tE_t|\preccurlyeq e^{2\sigma t}\left(|\nabla_{x,t}w|^2+\sigma^2w^2\right)$$

by a standard argument. The proposition now follows from (A.11) of Theorem A.7 in [23]. ∎

7. The forward problems.

7.1. Proof of Proposition 1.1. The existence, uniqueness, and regularity may be proved in a fashion similar to the proof of Proposition 1.1 in [24]. The only part which is new is the progressing wave expansion, which we show below. Below, $\mathcal{L}$ will mean $\mathcal{L}_{a,b,c}$.

We seek U in the form

$$U(x,t;\omega,\tau)=u(x,t;\omega,\tau)H(t-\tau-x\cdot\omega)$$

for some function $u(x,t;\omega,\tau)$ defined on the region $t \geq \tau + x\cdot\omega$. To describe $u(x,t;\omega,\tau)$ in detail, we work with the special case when $\tau = 0$; the general τ result will be inferred easily from this special case. Below, we denote $U(x,t;\omega,0), u(x,t;\omega,0)$ and $\alpha(x;\omega,0)$ by $U(x,t;\omega), u(x,t;\omega)$ and $\alpha(x;\omega)$.

Since $U(x,t;\omega) = u(x,t;\omega)H(t - x\cdot\omega)$, the initial condition (1.8) (note $\tau = 0$) forces

$$u(x,t;\omega) = 1 \qquad \text{when } t \ll 0.$$

Also, observe that

$$\begin{aligned}
(\partial_t - a)(f(x,t)F(t - x\cdot\omega)) &= fF'(t - x\cdot\omega) + ((\partial_t - a)f)F(t - x\cdot\omega)\\
(\partial_t - a)^2(f(x,t)F(t - x\cdot\omega)) &= fF''(t - x\cdot\omega) + 2((\partial_t - a)f)F'(t - x\cdot\omega)\\
&\quad + ((\partial_t - a)^2 f)F(t - x\cdot\omega)\\
(\nabla - b)(f(x,t)F(t - x\cdot\omega)) &= -\omega fF'(t - x\cdot\omega) + ((\nabla - b)f)F(t - x\cdot\omega)\\
((\nabla - b)^2)F(t - x\cdot\omega)) &= fF''(t - x\cdot\omega) - 2(\omega\cdot(\nabla - b)f)F'(t - x\cdot\omega)\\
&\quad + ((\nabla - b)^2 f)F(t - x\cdot\omega),
\end{aligned}$$

so

$$\begin{aligned}
\mathcal{L}(f(x,t)F(t - x\cdot\omega)) &= 2(f_t + \omega\cdot\nabla f - (a + \omega\cdot b)f)F'(t - x\cdot\omega)\\
&\quad + (\mathcal{L}f)F(t - x\cdot\omega).
\end{aligned} \tag{7.1}$$

Hence,

$$\mathcal{L}U = 2(u_t + \omega\cdot\nabla u - (a + \omega\cdot b)u)\delta(t - x\cdot\omega) + (\mathcal{L}u)H(t - x\cdot\omega).$$

This forces $\mathcal{L}u = 0$ on the region $t \geq x\cdot\omega$, and, on $t = x\cdot\omega$, u must satisfy the transport equation

$$(u_t + \omega\cdot\nabla u - (a + \omega\cdot b)u)(x, x\cdot\omega;\omega) = 0.$$

Since $u(x,t;\omega) = 1$ for $t \ll 0$, we have $u(x, x\cdot\omega;\omega) = 1$ if $x\cdot\omega \ll 0$. Hence, solving the transport equation with this "initial condition," we obtain $u(x, x\cdot\omega;\omega) = \alpha(x, x\cdot\omega;\omega)$ proving (1.13). Hence,

$$U(x,t;\xi,\tau) = u(x,t;\omega,\tau)H(t - \tau - x\cdot\omega),$$

where $u(x,t;\omega,\tau)$ is the solution of the characteristic IVP (1.12)–(1.14).

7.2. Proof of Proposition 1.2. The existence, uniqueness, and regularity may be proved in a fashion similar to the proof of Proposition 1.1 in [24]. The only part which is new is the progressing wave expansion, which we show below. Below, $\mathcal{L}$ will mean $\mathcal{L}_{a,b,c}$.

We seek V in the form

$$V(x,t;\omega,\tau) = f(x,t;\omega,\tau)\delta(t - \tau - x\cdot\omega) + v(x,t;\omega,\tau)H(t - \tau - x\cdot\omega)$$

with $v(x,t;\omega,\tau)$ supported in the region $t \geq \tau + x\cdot\omega$, and, for $t \ll 0$, we have $f(x,t;\omega,\tau) = 1$ and $v(x,t;\omega,\tau) = 0$. There are many choices for $f(x,t;\omega,\tau)$ but a unique choice for $f(x,\tau + x\cdot\omega;\omega,\tau)$. To describe $V(x,t;\omega,\tau)$ in detail, we work with the special case when $\tau = 0$; the general τ result will be inferred easily from this special case. Below, we denote $V(x,t;\omega,0), f(x,;\omega,0)$ and $v(x,t;\omega,0)$ by $V(x,t;\omega), f(x;\omega)$ and $v(x,t;\omega)$.

We seek V in the form

$$V(x,t;\omega) = f(x,t;\omega)\delta(t - x\cdot\omega) + v(x,t;\omega)H(t - x\cdot\omega);$$

hence, using (7.1),

$$\begin{aligned}(\mathcal{L}V)(x,t;\omega) = 2(f_t + \omega\cdot\nabla f - (a+\omega\cdot b)f)(x,t;\omega)\delta'(t - x\cdot\omega)\\ + (2v_t + 2\omega\cdot\nabla v - 2(a+\omega\cdot b)v + \mathcal{L}f)(x, x\cdot\omega;\omega)\delta(t - x\cdot\omega)\\ + (\mathcal{L}v)(x,t;\omega)H(t - x\cdot\omega).\end{aligned}$$

Amongst the many choices for f to zero out the first term in the above expansion of $\mathcal{L}V$, we choose one for which

$$f_t + \omega\cdot\nabla f - (a+\omega\cdot b)f = 0 \qquad \text{on } \mathbb{R}^n\times\mathbb{R}.$$

Hence, we choose $f(x,t;\omega) = \alpha(x,t;\omega)$, so we must now require

$$\mathcal{L}v = 0 \qquad \text{on } t \geq x\cdot\omega,$$

and, on $t = x\cdot\omega$, v must satisfy the transport equation

$$2(v_t + \omega\cdot\nabla v - (a+\omega\cdot b)v)(x, x\cdot\omega;\omega) = -(\mathcal{L}\alpha)(x, x\cdot\omega;\omega), \qquad x\in\mathbb{R}^n.$$

So for a general τ,

$$V(x,t;\omega,\tau) = \alpha(x, t-\tau;\omega)\delta(t-\tau-x\cdot\omega) + v(x,t;\omega,\tau)H(t-\tau-x\cdot\omega),$$

where $v(x,t;\omega,\tau)$ is the solution of the characteristic IVP.

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